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We give a new, effective proof of the separability of cubically convex-cocompact subgroups of special groups. As a consequence, we show that if G is a virtually compact special hyperbolic group, and $Q \leq G$ is a K -quasiconvex subgroup, then any $g \in G - Q$ of word length at most n is separated from Q by a subgroup whose index is polynomial in n and exponential in K . This generalizes a result of Bou-Rabee and the authors on residual finiteness growth (*Math. Z.* 279 (2015), 297–310) and a result of Patel on surface groups (*Proc. Amer. Math. Soc.* 142 (2014), 2891–2906).

Introduction

Early motivation for studying residual finiteness and subgroup separability was a result of the relevance of these properties to decision problems in group theory. An observation of Dyson [1964] and Mostowski [1966], related to earlier ideas of McKinsey [1943], states that finitely presented residually finite groups have a solvable word problem. The word problem is a special case of the membership problem, i.e., the problem of determining whether a given $g \in G$ belongs to a particular subgroup H of G . *Separability* can produce a solution to the membership problem in essentially the same way that a solution to the word problem is provided by residual finiteness; see, e.g., the discussion in [Aschenbrenner et al. 2015]. A subgroup $H \leq G$ is *separable* in G if, for all $g \in G - H$, there exists $G' \leq_{\text{f.i.}} G$ with $H \leq G'$ and $g \notin G'$. Producing an upper bound, in terms of g and H , on the minimal index of such a subgroup G' is what we mean by *quantifying separability* of H in G . Quantifying separability is related to the membership problem; see [Remark D](#) below.

Recently, separability has played a crucial role in low-dimensional topology, namely in the resolutions of the virtually Haken and virtually fibered conjectures [Agol 2013; Wise 2011]. Its influence in topology is a consequence of the seminal paper of Scott [1978], which establishes a topological reformulation of subgroup separability. Roughly, Scott's criterion allows one to use separability to promote

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(appropriately construed) immersions to embeddings in finite covers. Agol [2013] proved the virtually special conjecture of Wise, an outstanding component of the proofs of the above conjectures. Agol’s theorem shows that every word hyperbolic cubical group virtually embeds in a right-angled Artin group (hereafter, RAAG). Cubically convex-cocompact subgroups of RAAGs are separable [Hsu and Wise 2002; Haglund 2008] and Agol’s theorem demonstrates that word hyperbolic cubical groups inherit this property via the virtual embeddings (separability properties are preserved under passing to subgroups and finite index supergroups). In fact, since quasiconvex subgroups of hyperbolic cubical groups are cubically convex-cocompact [Haglund 2008; Sageev and Wise 2015], all quasiconvex subgroups of such groups are separable. In this paper, we give a new, effective proof of the separability of cubically convex-cocompact subgroups of special groups. Our main technical result is:

Theorem A. *Let Γ be a simplicial graph and let Z be a compact connected cube complex, based at a 0-cube x , with a based local isometry $Z \rightarrow S_\Gamma$. For all $g \in A_\Gamma - \pi_1 Z$, there is a cube complex (Y, x) such that*

- (1) $Z \subset Y$,
- (2) *there is a based local isometry $Y \rightarrow S_\Gamma$ such that $Z \rightarrow S_\Gamma$ factors as*

$$Z \hookrightarrow Y \rightarrow S_\Gamma,$$

- (3) *any closed based path representing g lifts to a nonclosed path at x in Y ,*
- (4) $|Y^{(0)}| \leq |Z^{(0)}|(|g| + 1)$,

where $|g|$ is the word length of g with respect to the standard generators.

Via Haglund–Wise’s canonical completion [2008], Theorem A provides the following bounds on the separability growth function (defined in Section 1) of the class of cubically convex-cocompact subgroups of a (virtually) special group. Roughly, separability growth quantifies separability of all subgroups in a given class.

Corollary B. *Let $G \cong \pi_1 X$, with X a compact special cube complex, and let \mathcal{Q}_R be the class of subgroups represented by compact local isometries to X whose domains have at most R vertices. Then*

$$\text{Sep}_{G,S}^{\mathcal{Q}_R}(Q, n) \leq PRn$$

for all $Q \in \mathcal{Q}_R$ and $n \in \mathbb{N}$, where the constant P depends only on the generating set S . Hence, letting \mathcal{Q}'_K be the class of subgroups $Q \leq G$ such that the convex hull of $Q\tilde{x}$ lies in $\mathcal{N}_K(Q\tilde{x})$ and $\tilde{x} \in \tilde{X}^{(0)}$,

$$\text{Sep}_{G,S}^{\mathcal{Q}'_K}(Q, n) \leq P' \text{gr}_{\tilde{X}}(K)n,$$

where P' depends only on G, \tilde{X}, S , and where $\text{gr}_{\tilde{X}}$ is the growth function of $\tilde{X}^{(0)}$.

In the hyperbolic case, where cubically convex-cocompactness is equivalent to quasiconvexity, we obtain a bound that is polynomial in the length of the word and exponential in the quasiconvexity constant:

Corollary C. *Let G be a group with an index- J special subgroup. Fixing a word length $\| - \|_S$ on G , suppose that $(G, \| - \|_S)$ is δ -hyperbolic. For each $K \geq 1$, let \mathcal{Q}_K be the set of subgroups $Q \leq G$ such that Q is K -quasiconvex with respect to $\| - \|_S$. Then there exists a constant $P = P(G, S)$ such that for all $K \geq 0$, $Q \in \mathcal{Q}_K$, and $n \geq 0$,*

$$\mathbf{Sep}_{G,S}^{\mathcal{Q}_K}(Q, n) \leq P \operatorname{gr}_G(PK)^{J!} n^{J!},$$

where gr_G is the growth function of G .

Corollary C says that if G is a hyperbolic cubical group, the subgroup $Q \leq G$ is K -quasiconvex, and $g \in G - Q$, then g is separated from Q by a subgroup of index bounded by a function polynomial in $\|g\|_S$ and exponential in K .

The above results fit into a larger body of work dedicated to quantifying residual finiteness and subgroup separability of various classes of groups; see, e.g., [Bou-Rabee and Kaletha 2012; Bou-Rabee and McReynolds 2015; Kassabov and Matucci 2011; Buskin 2009; Patel 2014, 2013; Rivin 2012; Bou-Rabee 2011; Bou-Rabee and McReynolds 2014; Kozma and Thom 2016]. When G is the fundamental group of a hyperbolic surface, compare Corollary C to [Patel 2014, Theorem 7.1]. Combining various cubulation results with [Agol 2013], the groups covered by Corollary C include fundamental groups of hyperbolic 3-manifolds [Bergeron and Wise 2012; Kahn and Markovic 2012], hyperbolic Coxeter groups [Haglund and Wise 2010], simple-type arithmetic hyperbolic lattices [Bergeron et al. 2011], hyperbolic free-by-cyclic groups [Hagen and Wise 2015], hyperbolic ascending HNN extensions of free groups with irreducible monodromy [Hagen and Wise 2013], hyperbolic groups with a quasiconvex hierarchy [Wise 2011], $C'(\frac{1}{6})$ small cancellation groups [Wise 2004], and hence random groups at low enough density [Ollivier and Wise 2011], among many others.

Bou-Rabee, Hagen and Patel [2015] quantified residual finiteness for virtually special groups, by working in RAAGs and appealing to the fact that upper bounds on residual finiteness growth are inherited by finitely generated subgroups and finite index supergroups. Theorem A generalizes a main theorem of [loc. cit.], and accordingly the proof is reminiscent of the one in that reference. However, residual finiteness is equivalent to separability of the trivial subgroup, and thus it is not surprising that quantifying separability for an arbitrary convex-cocompact subgroup of a RAAG entails engagement with a more complex geometric situation. Our techniques thus significantly generalize those of [loc. cit.].

Remark D (membership problem). If H is a finitely generated separable subgroup of the finitely presented group G , and one has an upper bound on $\mathbf{Sep}_{G,S}^{\{H\}}(|g|)$ for

some finite generating set S of G , then the following procedure decides if $g \in H$: first, enumerate all subgroups of G of index at most $\mathbf{Sep}_{G,S}^{\{H\}}(|g|)$ using a finite presentation of G . Second, for each such subgroup, test whether it contains g ; if so, ignore it, and if not, proceed to the third step. Third, for each finite index subgroup not containing g , test whether it contains each of the finitely many generators of H ; if so, we have produced a finite index subgroup containing H but not g , whence $g \notin H$. If we exhaust the subgroups of index at most $\mathbf{Sep}_{G,S}^{\{H\}}(|g|)$ without finding such a subgroup, then $g \in H$. In particular, [Corollary C](#) gives an effective solution to the membership problem for quasiconvex subgroups of hyperbolic cubical groups, though it does not appear to be any more efficient than the more general solution to the membership problem for quasiconvex subgroups of (arbitrary) hyperbolic groups recently given by Kharlampovich, Myasnikov and Weil [[Kharlampovich et al. 2014](#)].

The paper is organized as follows. In [Section 1](#), we define the separability growth of a group with respect to a class \mathcal{Q} of subgroups, which generalizes the residual finiteness growth introduced in [[Bou-Rabee 2010](#)]. We also provide some necessary background on RAAGs and cubical geometry. In [Section 2](#), we discuss corollaries to the main technical result, including [Corollary C](#), before concluding with a proof of [Theorem A](#) in [Section 3](#).

1. Background

Separability growth. Let G be a group generated by a finite set S and let $H \leq G$ be a subgroup. Let $\Omega_H = \{\Delta \leq G : H \leq \Delta\}$, and define a map $D_G^{\Omega_H} : G - H \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$D_G^{\Omega_H}(g) = \min\{[G : \Delta] : \Delta \in \Omega_H, g \notin \Delta\}.$$

This is a special case of the notion of a *divisibility function* defined in [[Bou-Rabee 2010](#)] and discussed in [[Bou-Rabee and McReynolds 2015](#)]. Note that H is a separable subgroup of G if and only if $D_G^{\Omega_H}$ takes only finite values.

The *separability growth* of G with respect to a class \mathcal{Q} of subgroups is a function $\mathbf{Sep}_{G,S}^{\mathcal{Q}} : \mathcal{Q} \times \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ given by

$$\mathbf{Sep}_{G,S}^{\mathcal{Q}}(Q, n) = \max\{D_G^{\Omega_Q}(g) : g \in G - Q, \|g\|_S \leq n\}.$$

If \mathcal{Q} is a class of separable subgroups of G , then the separability growth measures the index of the subgroup to which one must pass in order to separate Q from an element of $G - Q$ of length at most n . For example, when G is residually finite and $\mathcal{Q} = \{\{1\}\}$, then $\mathbf{Sep}_{G,S}^{\mathcal{Q}}$ is the residual finiteness growth function. The following fact is explained in greater generality in [[Bou-Rabee et al. 2015](#), Section 2]. (In the notation of that reference, $\mathbf{Sep}_{G,S}^{\mathcal{Q}}(Q, n) = \mathbf{RF}_{G,S}^{\Omega_Q}(n)$ for all $Q \in \mathcal{Q}$ and $n \in \mathbb{N}$.)

Proposition 1.1. *Let G be a finitely generated group and let \mathcal{Q} be a class of subgroups of G . If S, S' are finite generating sets of G , then there exists a constant $C > 0$ with*

$$\mathbf{Sep}_{G,S'}^{\mathcal{Q}}(Q, n) \leq C \cdot \mathbf{Sep}_{G,S}^{\mathcal{Q}}(Q, Cn)$$

for $Q \in \mathcal{Q}, n \in \mathbb{N}$. Hence the asymptotic growth rate of $\mathbf{Sep}_{G,S}^{\mathcal{Q}}$ is independent of S .

(Similar statements assert that upper bounds on separability growth are inherited by finite index supergroups and arbitrary finitely generated subgroups but we do not use, and thus omit, these.)

Nonpositively curved cube complexes. We assume familiarity with nonpositively curved and CAT(0) cube complexes and refer the reader to, e.g., [Hagen 2014; Haglund 2008; Wise 2012; 2011] for background. We now make explicit some additional notions and terminology, related to convex subcomplexes, which are discussed in greater depth in [Behrstock et al. 2014]. We also discuss some basic facts about RAAGs and Salvetti complexes. Finally, we will use the method of *canonical completion*, introduced in [Haglund and Wise 2008], and refer the reader to [Bou-Rabee et al. 2015, Lemma 2.8] for the exact statement needed here.

Local isometries, convexity, and gates. A local isometry $\phi : Y \rightarrow X$ of cube complexes is a locally injective combinatorial map with the property that, if e_1, \dots, e_n are 1-cubes of Y all incident to a 0-cube y , and the (necessarily distinct) 1-cubes $\phi(e_1), \dots, \phi(e_n)$ all lie in a common n -cube c (containing $\phi(y)$), then e_1, \dots, e_n span an n -cube c' in Y with $\phi(c') = c$. If $\phi : Y \rightarrow X$ is a local isometry and X is nonpositively curved, then Y is as well. Moreover, ϕ lifts to an embedding $\tilde{\phi} : \tilde{Y} \rightarrow \tilde{X}$ of universal covers, and $\tilde{\phi}(\tilde{Y})$ is *convex* in \tilde{X} in the following sense.

Let \tilde{X} be a CAT(0) cube complex. The subcomplex $K \subseteq \tilde{X}$ is *full* if K contains each n -cube of \tilde{X} whose 1-skeleton appears in K . If K is full, then K is *isometrically embedded* if $K \cap \bigcap_i H_i$ is connected whenever $\{H_i\}$ is a set of pairwise-intersecting hyperplanes of \tilde{X} . Equivalently, the inclusion $K^{(1)} \hookrightarrow \tilde{X}^{(1)}$ is an isometric embedding with respect to the graph-metric. If the inclusion $K \hookrightarrow \tilde{X}$ of the full subcomplex K is a local isometry, then K is *convex*. Note that a convex subcomplex is necessarily isometrically embedded, and in fact K is convex if and only if $K^{(1)}$ is metrically convex in $\tilde{X}^{(1)}$. A convex subcomplex K is a CAT(0) cube complex in its own right, and its hyperplanes have the form $H \cap K$, where H is a hyperplane of \tilde{X} . Moreover, if K is convex, then hyperplanes $H_1 \cap K, H_2 \cap K$ of K intersect if and only if $H_1 \cap H_2 \neq \emptyset$. We often say that the hyperplane H *crosses* the convex subcomplex K to mean that $H \cap K \neq \emptyset$ and we say the hyperplanes H, H' *cross* if they intersect.

Hyperplanes are an important source of convex subcomplexes, in two related ways. First, recall that for all hyperplanes H of \tilde{X} , the carrier $\mathcal{N}(H)$ is a convex

subcomplex. Second, $\mathcal{N}(H) \cong H \times [-\frac{1}{2}, \frac{1}{2}]$, and the subcomplexes $H \times \{\pm\frac{1}{2}\}$ of \tilde{X} “bounding” $\mathcal{N}(H)$ are convex subcomplexes isomorphic to H (when H is given the cubical structure in which its n -cubes are midcubes of $(n+1)$ -cubes of \tilde{X}). A subcomplex of the form $H \times \{\pm\frac{1}{2}\}$ is a *combinatorial hyperplane*. The *convex hull* of a subcomplex $\mathcal{S} \subset \tilde{X}$ is the intersection of all convex subcomplexes that contain \mathcal{S} ; see [Haglund 2008].

Let $K \subseteq \tilde{X}$ be a convex subcomplex. Then there is a map $\mathfrak{g}_K : \tilde{X}^{(0)} \rightarrow K$ such that for all $x \in \tilde{X}^{(0)}$, the point $\mathfrak{g}_K(x)$ is the unique closest point of K to x . (This point is often called the *gate* of x in K ; gates are discussed further in [Chepoi 2000; Bandelt and Chepoi 2008].) This map extends to a cubical map $\mathfrak{g}_K : \tilde{X} \rightarrow K$, the *gate map*. See, e.g., [Behrstock et al. 2014] for a detailed discussion of the gate map in the language used here; we use only that it extends the map on 0-cubes and has the property that for all x, y , if $\mathfrak{g}_K(x), \mathfrak{g}_K(y)$ are separated by a hyperplane H , then the same H separates x from y . Finally, the hyperplane H separates x from $\mathfrak{g}_K(x)$ if and only if H separates x from K . The gate map allows us to define the *projection* of the convex subcomplex K' of \tilde{X} onto K to be $\mathfrak{g}_{K'}(K')$, which is the convex hull of the set $\{\mathfrak{g}_K(x) \in K : x \in K'^{(0)}\}$. Convex subcomplexes K, K' are *parallel* if $\mathfrak{g}_{K'}(K) = K'$ and $\mathfrak{g}_K(K') = K$. Equivalently, K, K' are *parallel* if and only if, for each hyperplane H , we have $H \cap K \neq \emptyset$ if and only if $H \cap K' \neq \emptyset$. Note that parallel subcomplexes are isomorphic.

Remark 1.2. We often use the following facts. Let K, K' be convex subcomplexes of \tilde{X} . Then the convex hull C of $K \cup K'$ contains the union of K, K' and a convex subcomplex of the form $G_K(K') \times \hat{\gamma}$, where $G_K(K')$ is the image of the gate map discussed above and $\hat{\gamma}$ is the convex hull of a geodesic segment γ joining a closest pair of 0-cubes in K, K' , by [Behrstock et al. 2014, Lemma 2.4]. A hyperplane H crosses K and K' if and only if H crosses $G_K(K')$; the hyperplane H separates K, K' if and only if H crosses $\hat{\gamma}$. All remaining hyperplanes either cross exactly one of K, K' or fail to cross C . Observe that the set of hyperplanes separating K, K' contains no triple H, H', H'' of disjoint hyperplanes, none of which separates the other two. (Such a configuration is called a *facing triple*.)

Salvetti complexes and special cube complexes. Let Γ be a simplicial graph and let A_Γ be the corresponding right-angled Artin group (RAAG), i.e., the group presented by

$$\langle V(\Gamma) \mid [v, w], \{v, w\} \in E(\Gamma) \rangle,$$

where $V(\Gamma)$ and $E(\Gamma)$ respectively denote the vertex- and edge-sets of Γ . The phrase *generator of Γ* refers to this presentation; we denote each generator of A_Γ by the corresponding vertex of Γ .

The RAAG A_Γ is isomorphic to the fundamental group of the *Salvetti complex* S_Γ , introduced in [Charney and Davis 1995], which is a nonpositively curved cube

complex with one 0-cube x , an oriented 1-cube for each $v \in V(\Gamma)$, labeled by v , and an n -torus (an n -cube with opposite faces identified) for every n -clique in Γ .

A cube complex X is *special* if there exists a simplicial graph Γ and a local isometry $X \rightarrow S_\Gamma$ inducing a monomorphism $\pi_1 X \rightarrow A_\Gamma$ and a $\pi_1 X$ -equivariant embedding $\tilde{X} \rightarrow \tilde{S}_\Gamma$ of universal covers whose image is a convex subcomplex. Specialness allows one to study geometric features of $\pi_1 X$ by working inside of \tilde{S}_Γ , which has useful structure not necessarily present in general CAT(0) cube complexes; see the next section. Following Haglund and Wise [2008], a group G is (virtually) [compact] special if G is (virtually) isomorphic to the fundamental group of a [compact] special cube complex.

Cubical features particular to Salvetti complexes. Let Γ be a finite simplicial graph and let Λ be an induced subgraph of Γ . The inclusion $\Lambda \hookrightarrow \Gamma$ induces a monomorphism $A_\Lambda \rightarrow A_\Gamma$. In fact, there is an injective local isometry $S_\Lambda \rightarrow S_\Gamma$ inducing $A_\Lambda \rightarrow A_\Gamma$. Hence each conjugate A_Λ^g of A_Λ in A_Γ is the stabilizer of a convex subcomplex $g\tilde{S}_\Lambda \subseteq \tilde{S}_\Gamma$. A few special cases warrant extra consideration.

When $\Lambda \subset \Gamma$ is an n -clique, for some $n \geq 1$, then $S_\Lambda \subseteq S_\Gamma$ is an n -torus, which is the Salvetti complex of the sub-RAAG isomorphic to \mathbb{Z}^n generated by n pairwise-commuting generators. In this case, S_Λ is a *standard n -torus* in S_Γ . (When $n = 1$, S_Λ is a *standard circle*.) Each lift of \tilde{S}_Λ to \tilde{S}_Γ is a *standard flat*; when $n = 1$, we use the term *standard line*; a compact connected subcomplex of a standard line is a *standard segment*. The labels and orientations of 1-cubes in S_Γ pull back to \tilde{S}_Γ ; a standard line is a convex subcomplex isometric to \mathbb{R} , all of whose 1-cubes have the same label, such that each 0-cube has one incoming and one outgoing 1-cube.

When $\text{Lk}(v)$ is the link of a vertex v of Γ , the subcomplex $S_{\text{Lk}(v)}$ is an immersed combinatorial hyperplane in the sense that $\tilde{S}_{\text{Lk}(v)}$ is a combinatorial hyperplane of \tilde{S}_Γ . There is a corresponding hyperplane, whose carrier is bounded by $\tilde{S}_{\text{Lk}(v)}$ and $v\tilde{S}_{\text{Lk}(v)}$, that intersects only 1-cubes labeled by v . Moreover, $\tilde{S}_{\text{Lk}(v)}$ is contained in $\tilde{S}_{\text{St}(v)}$, where $\text{St}(v)$ is the star of v , i.e., the join of v and $\text{Lk}(v)$. It follows that

$$\tilde{S}_{\text{St}(v)} \cong \tilde{S}_{\text{Lk}(v)} \times \tilde{S}_v,$$

where \tilde{S}_v is a standard line. Note that the combinatorial hyperplane $\tilde{S}_{\text{Lk}(v)}$ is parallel to $v^k\tilde{S}_{\text{Lk}(v)}$ for all $k \in \mathbb{Z}$. Likewise, \tilde{S}_v is parallel to $g\tilde{S}_v$ exactly when $g \in A_\Lambda$, and parallel standard lines have the same labels. We say \tilde{S}_v is a *standard line dual to $\tilde{S}_{\text{Lk}(v)}$* , and is a standard line dual to any hyperplane H such that $N(H)$ has $\tilde{S}_{\text{Lk}(v)}$ as one of its bounding combinatorial hyperplanes.

Remark 1.3. We warn the reader that a given combinatorial hyperplane may correspond to distinct hyperplanes whose dual standard lines have different labels; this occurs exactly when there exist multiple vertices in Γ whose links are the

same subgraph. However, the standard line dual to a genuine (noncombinatorial) hyperplane is uniquely determined up to parallelism.

Definition 1.4 (frame). Let $K \subseteq \tilde{S}_\Gamma$ be a convex subcomplex and let H be a hyperplane. Let L be a standard line dual to H . The *frame* of H is the convex subcomplex $H' \times L \subseteq \tilde{S}_\Gamma$ described above, where H' is a combinatorial hyperplane bounding $N(H)$. If $K \subseteq \tilde{S}_\Gamma$ is a convex subcomplex, and H intersects K , then the *frame of H in K* is the complex $K \cap (H \times L)$. It is shown in [Bou-Rabee et al. 2015] that the frame of H in K has the form $(H \cap K) \times (L \cap K)$, provided that L is chosen in its parallelism class to intersect K . Note that the frame of H is in fact well-defined, since all possible choices of L are parallel.

2. Consequences of Theorem A

Assuming Theorem A, we quantify separability of cubically convex-cocompact subgroups of special groups with the proofs of Corollaries B and C, before proving Theorem A in the next section.

Proof of Corollary B. Let Γ be a finite simplicial graph so that there is a local isometry $X \rightarrow S_\Gamma$. Let $Q \in \mathcal{Q}_R$ be represented by a local isometry $Z \rightarrow X$. Then for all $g \in \pi_1 X - \pi_1 Z$, by Theorem A, there is a local isometry $Y \rightarrow S_\Gamma$ such that Y contains Z as a locally convex subcomplex, $g \notin \pi_1 Y$, and $|Y^{(0)}| \leq |Z^{(0)}|(|g| + 1)$. Applying canonical completion [Haglund and Wise 2008] to $Y \rightarrow S_\Gamma$ yields a cover $\hat{S}_\Gamma \rightarrow S_\Gamma$ in which Y embeds; this cover has degree $|Y^{(0)}|$ by [Bou-Rabee et al. 2015, Lemma 2.8]. Let $H' = \pi_1 \hat{S}_\Gamma \cap \pi_1 X$, so that $\pi_1 Z \leq H'$, $g \notin H'$, and $[\pi_1 X : H'] \leq |Z^{(0)}|(|g| + 1)$. The first claim follows.

Let $G \cong \pi_1 X$ with X compact special, $Q \leq G$, and let the convex hull of $Q\tilde{x}$ in \tilde{X} lie in $\mathcal{N}_K(Q\tilde{X})$. Then the second claim follows since we can choose Z to be the quotient of the hull of $Q\tilde{x}$ by the action of Q , and $|Z^{(0)}| \leq \text{gr}_{\tilde{X}}(K)$. \square

In general, the number of 0-cubes in Z is computable from the quasiconvexity constant of a Q -orbit in $\tilde{X}^{(1)}$ by [Haglund 2008, Theorem 2.28]. In the hyperbolic case, we obtain Corollary C in terms of the quasiconvexity constant, without reference to any particular cube complex:

Proof of Corollary C. We use Corollary B when $J = 1$, and promote the result to a polynomial bound when $J \geq 1$. Let $Q \in \mathcal{Q}_K$ and let $g \in G - Q$.

The special case: Suppose $J = 1$ and let X be a compact special cube complex with $G \cong \pi_1 X$. Let $Z \rightarrow X$ be a compact local isometry representing the inclusion $Q \rightarrow G$. Such a complex exists by quasiconvexity of Q and [Haglund 2008, Theorem 2.28], although we shall use the slightly more computationally explicit proof in [Sageev and Wise 2015]. Let $A' \geq 1$, $B' \geq 0$ be constants such that an orbit map $(G, \|\cdot\|_S) \rightarrow (\tilde{X}^{(1)}, d)$ is an (A', B') -quasi-isometric embedding, where d is

the graph-metric. Then there exist constants A, B , depending only on A', B' and hence on $\| \cdot \|_S$, such that Qx is $(AK + B)$ -quasiconvex, where x is a 0-cube in $\tilde{Z} \subset \tilde{X}$. By the proof of [op. cit., Proposition 3.3], the convex hull \tilde{Z} of Qx lies in the ρ -neighborhood of Qx , where

$$\rho = AK + B + \sqrt{\dim \bar{X}} + \delta' \left(\csc \left(\frac{1}{2} \sin^{-1} \frac{1}{\sqrt{\dim \bar{X}}} \right) + 1 \right)$$

and $\delta' = \delta'(\delta, A', B')$. **Corollary B** provides $G' \leq G$ with $g \notin G'$, and the bound $[G : G'] \leq |Z^{(0)}|(|g| + 1)$. But $|g| + 1 \leq A' \|g\|_S + B' + 1$, while $|Z^{(0)}| \leq \text{gr}_{\tilde{X}}(\rho)$. Thus $[G : G'] \leq \text{gr}_{\tilde{X}}(\rho) A' \|g\|_S + \text{gr}_{\tilde{X}}(\rho) B' + \text{gr}_{\tilde{X}}(\rho)$, so there exists P_1 such that

$$\text{Sep}_{G,S}^{Q,K}(Q, n) \leq P_1 \text{gr}_{\tilde{X}}(P_1 K) n$$

for all $K, Q \in \mathcal{Q}_K, n \in \mathbb{N}$, where P_1 depends only on X .

The virtually special case: Now suppose that $J \geq 1$. We have a compact special cube complex X , and $[G : G'] \leq J!$, where $G' \cong \pi_1 X$ and $G' \triangleleft G$. Let $Q \leq G$ be a K -quasiconvex subgroup. By **Lemma 2.1**, there exists $C = C(G, S)$ such that $Q \cap G'$ is $CJ!(K + 1)$ -quasiconvex in G , and thus is $P_2 C J!(K + 1)$ -quasiconvex in G' , where P_2 depends only on G and S .

Let $g \in G - Q$. Since $G' \triangleleft G$, the product QG' is a subgroup of G of index at most $J!$ that contains Q . Hence, if $g \notin QG'$, then we are done. We thus assume $g \in QG'$. Hence we can choose a left transversal $\{q_1, \dots, q_s\}$ for $Q \cap G'$ in Q , with $s \leq J!$ and $q_1 = 1$. Write $g = q_i g'$ for some $i \leq s$, with $g' \in G'$. Suppose that we have chosen each q_i to minimize $\|q_i\|_S$ among all elements of $q_i(Q \cap G')$, so that, by **Lemma 2.3**, $\|q_i\|_S \leq J!$ for all i . Hence $\|g'\|_S \leq (\|g\|_S + J!)$.

By the first part of the proof, there exists a constant P_1 , depending only on G, G', S , and a subgroup $G'' \leq G'$ such that $Q \cap G' \leq G''$ and $g' \notin G''$, and

$$[G' : G''] \leq P_1 \text{gr}_{G'}(P_1 P_2 C J!(K + 1)) \|g'\|_S \leq P_1 \text{gr}_G(P_1 P_2 C J!(K + 1)) \|g'\|_S.$$

Let $G''' = \bigcap_{i=1}^s q_i G'' q_i^{-1}$, so that $g' \notin G'''$ and $Q \cap G' \leq G'''$ (since G' is normal), and

$$[G' : G'''] \leq (P_1 \text{gr}_G(P_1 P_2 C J!(K + 1)) \|g'\|_S)^s.$$

Finally, let $H = QG'''$. This subgroup clearly contains Q . Suppose that $g = q_i g' \in H$. Then $g' \in QG'''$, i.e., $g' = a g'''$ for some $a \in Q$ and $g''' \in G'''$. Since $g' \in G'$ and $G''' \leq G'$, we have $a \in Q \cap G'$, whence $a \in G'''$, by construction. This implies that $g' \in G''' \leq G''$, a contradiction. Hence H is a subgroup of G separating g from Q . Finally,

$$[G : H] \leq [G : G'''] \leq J! [P_1 \text{gr}_G(P_1 P_2 C J!(K + 1)) (\|g\|_S + J!)]^J,$$

and the proof is complete. \square

Lemma 2.1. *Let the group G be generated by a finite set S and let $(G, \|\cdot\|_S)$ be δ -hyperbolic. Let $Q \leq G$ be K -quasiconvex, and let $G' \leq G$ be an index- I subgroup. Then $Q \cap G'$ is $CI(K+1)$ -quasiconvex in G for some C depending only on δ and S .*

Proof. Since Q is K -quasiconvex in G , it is generated by a set \mathcal{T} of $q \in Q$ with $\|q\|_S \leq 2K+1$ by [Bridson and Haefliger 1999, Lemma III.Γ.3.5]. A standard argument shows $(Q, \|\cdot\|_{\mathcal{T}}) \hookrightarrow (G, \|\cdot\|_S)$ is a $(2K+1, 0)$ -quasi-isometric embedding. Lemma 2.3 shows that $Q \cap G'$ is I -quasiconvex in $(Q, \|\cdot\|_{\mathcal{T}})$, since $[Q : Q \cap G'] \leq I$. Hence $Q \cap G'$ has a generating set making it $((2I+1)(2K+1), 0)$ -quasi-isometrically embedded in $(G, \|\cdot\|_S)$. Apply Lemma 2.2 to conclude. \square

The next lemma is standard, but we include it to highlight the constants involved:

Lemma 2.2. *Let G be a group generated by a finite set S and suppose that $(G, \|\cdot\|_S)$ is δ -hyperbolic. Then there exists a (sub)linear function $f : \mathbb{N} \rightarrow \mathbb{N}$, depending on S and δ , such that $\sigma \subseteq \mathcal{N}_{f(\lambda)}(\gamma)$ whenever $\gamma : [0, L] \rightarrow G$ is a $(\lambda, 0)$ -quasigeodesic and σ is a geodesic joining $\gamma(0)$ to $\gamma(L)$.*

Proof. See, e.g., the proof of [Bridson and Haefliger 1999, Theorem III.H.1.7]. \square

Lemma 2.3. *Let Q be a group generated by a finite set S and let $Q' \leq Q$ be a subgroup with $[Q : Q'] = s < \infty$. Then there exists a left transversal $\{q_1, \dots, q_s\}$ for Q' such that $\|q_i\|_S \leq s$ for $1 \leq i \leq s$. Hence Q' is s -quasiconvex in Q .*

Proof. Suppose that $q_k = s_{i_k} \cdots s_{i_1}$ is a geodesic word in $S \cup S^{-1}$ and that q_k is a shortest representative of $q_k Q'$. Let $q_j = s_{i_j} \cdots s_{i_1}$ be the word in Q consisting of the last j letters of q_k for all $1 < j < k$, and let $q_1 = 1$. We claim that each q_j is a shortest representative for $q_j Q'$. Otherwise, there would exist p with $\|p\|_S < j$ such that $q_j Q' = p Q'$. But then $s_k \cdots s_{j+1} p Q' = q_k Q'$, and thus q_k was not a shortest representative. It also follows immediately that $q_j Q' \neq q_{j'} Q'$ for $j \neq j'$. Thus, q_1, q_2, \dots, q_k represent distinct left cosets of Q' provided $k \leq s$, and the claim follows. \square

Remark 2.4 (embeddings in finite covers). Given a compact special cube complex X and a compact local isometry $Z \rightarrow X$, Theorem A gives an upper bound on the minimal degree of a finite cover in which Z embeds; indeed, producing such an embedding entails separating $\pi_1 Z$ from finitely many elements in $\pi_1 X$. However, it is observed in [Bou-Rabee et al. 2015, Lemma 2.8] that the Haglund–Wise canonical completion construction [2008] produces a cover $\hat{X} \rightarrow X$ of degree $|Z^{(0)}|$ in which Z embeds.

3. Proof of Theorem A

In this section, we give a proof of the main technical result.

Definition 3.1. Let S_{Γ} be a Salvetti complex and let \tilde{S}_{Γ} be its universal cover. The hyperplanes H, H' of \tilde{S}_{Γ} are *collateral* if they have a common dual standard line

(equivalently, the same frame). Clearly collateralism is an equivalence relation, and collateral hyperplanes are isomorphic and have the same stabilizer.

Being collateral implies that the combinatorial hyperplanes bounding the carrier of H are parallel to those bounding the carrier of H' . However, the converse is not true when Γ contains multiple vertices whose links coincide. In the proof of [Theorem A](#), we will always work with hyperplanes, rather than combinatorial hyperplanes, unless we explicitly state otherwise.

Proof of Theorem A. Let $\tilde{x} \in \tilde{S}_\Gamma$ be a lift of the base 0-cube x in S_Γ , and let $\tilde{Z} \subseteq \tilde{S}_\Gamma$ be the lift of the universal cover of Z containing \tilde{x} . Since $Z \rightarrow S_\Gamma$ is a local isometry, \tilde{Z} is convex. Let $\hat{Z} \subset \tilde{Z}$ be the convex hull of a compact connected fundamental domain for the action of $\pi_1 Z \leq A_\Gamma$ on \tilde{Z} . Denote by K the convex hull of $\hat{Z} \cup \{g\tilde{x}\}$ and let \mathfrak{S} be the set of hyperplanes of \tilde{S}_Γ intersecting K . We will form a quotient of K , restricting to $\hat{Z} \rightarrow Z$ on \hat{Z} , whose image admits a local isometry to S_Γ .

The subcomplex $[\hat{Z}]$. Let \mathfrak{L} be the collection of standard segments ℓ in K that map to standard circles in S_Γ with the property that $\ell \cap \hat{Z}$ has noncontractible image in Z . Let $[\hat{Z}]$ be the convex hull of $\hat{Z} \cup \bigcup_{\ell \in \mathfrak{L}} \ell$, so that $\hat{Z} \subseteq [\hat{Z}] \subseteq K$.

Partitioning \mathfrak{S} . We now partition \mathfrak{S} according to the various types of frames in K . First, let \mathfrak{J} be the set of hyperplanes intersecting \hat{Z} . Second, let \mathfrak{N} be the set of $N \in \mathfrak{S} - \mathfrak{J}$ such that the frame $(N \cap K) \times (L \cap K)$ of N in K has the property that for some choice of $x_0 \in N^{(0)}$, the segment $(\{x_0\} \times L) \cap \hat{Z}$ maps to a nontrivial cycle of 1-cubes in Z . Let $n_N \geq 1$ be the length of that cycle. By convexity of \hat{Z} , the number n_N is independent of the choice of the segment L within its parallelism class. Note that \mathfrak{N} is the set of hyperplanes that cross $[\hat{Z}]$, but do not cross \hat{Z} . Hence each $N \in \mathfrak{N}$ is collateral to some $W \in \mathfrak{J}$. Third, fix a collection $\{H_1, \dots, H_k\} \subset \mathfrak{S} - \mathfrak{J}$ such that:

- (1) For $1 \leq i \leq k - 1$, the hyperplane H_i separates H_{i+1} from $[\hat{Z}]$.
- (2) For $1 \leq i < j \leq k$, if a hyperplane H separates H_i from H_j , then H is collateral to H_ℓ for some $\ell \in [i, j]$. Similarly, if H separates H_1 from $[\hat{Z}]$, then H is collateral to H_1 , and if H separates H_k from $g\tilde{x}$, then H is collateral to H_k .
- (3) For each i , the frame $(H_i \cap K) \times L_i$ of H_i in K has the property that for every $h \in H_i^{(0)}$, the image in Z of the segment $(\{h\} \times L_i) \cap \hat{Z}$ is empty or contractible. (Here, L_i is a standard segment of a standard line dual to H_i .)

Let \mathfrak{H} be the set of all hyperplanes of $\mathfrak{S} - \mathfrak{J}$ that are collateral to H_i for some i . Condition (3) above ensures that $\mathfrak{H} \cap \mathfrak{N} = \emptyset$, while $\mathfrak{H} = \emptyset$ only if $K = [\hat{Z}]$. Finally, let $\mathfrak{B} = \mathfrak{S} - (\mathfrak{J} \cup \mathfrak{N} \cup \mathfrak{H})$. Note that each $B \in \mathfrak{B}$ crosses some H_i . [Figure 1](#) shows a possible K and various families of hyperplanes crossing it.

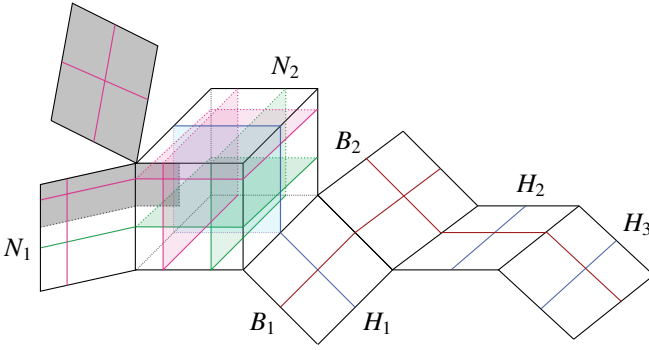


Figure 1. Hyperplanes crossing K (the dark shaded area on the left is \hat{Z}).

Mapping $[\hat{Z}]$ to Z . We now define a quotient map $q : [\hat{Z}] \rightarrow Z$ extending the restriction $\hat{Z} \rightarrow Z$ of $\tilde{Z} \rightarrow Z$. Note that if $\mathfrak{N} = \emptyset$, then $[\hat{Z}] = \hat{Z}$, and q is just the map $\hat{Z} \rightarrow Z$. Hence suppose $\mathfrak{N} \neq \emptyset$ and let $\mathfrak{N}_1, \dots, \mathfrak{N}_s$ be the collateralism classes of hyperplanes in \mathfrak{N} , and for $1 \leq i \leq s$, let \mathfrak{N}'_i be the collateralism class of \mathfrak{N}_i in \mathfrak{S} , i.e., \mathfrak{N}'_i together with a nonempty set of collateral hyperplanes in \mathfrak{Z} . For each i , let L_i be a maximal standard line segment of $[\hat{Z}]$, each of whose 1-cubes is dual to a hyperplane in \mathfrak{N}'_i and which crosses each element of \mathfrak{N}'_i . For each i , let $N_i \in \mathfrak{N}_i$ be a hyperplane separating \hat{Z} from $g\tilde{x}$. Then $N_i \cap N_j \neq \emptyset$ for $i \neq j$, since neither separates the other from \hat{Z} . We can choose the L_i so that there is an isometric embedding $\prod_{i=1}^k L_i \rightarrow [\hat{Z}]$, since whether or not two hyperplanes of $\tilde{\mathfrak{S}}_\Gamma$ cross depends only on their collateralism classes.

For each nonempty $I \subseteq \{1, \dots, k\}$, a hyperplane $W \in \mathfrak{Z}$ crosses some $U \in \bigcup_{i \in I} \mathfrak{N}'_i$ if and only if W crosses each hyperplane collateral to U . Hence, by Lemma 7.11 of [Hagen 2014], there is a maximal convex subcomplex $Y(I) \subset \hat{Z}$, defined up to parallelism, such that a hyperplane W crosses each $U \in \bigcup_{i \in I} \mathfrak{N}'_i$ if and only if $W \cap Y(I) \neq \emptyset$. Let $\mathfrak{A}(I)$ be the set of hyperplanes crossing $Y(I)$. By the definition of $Y(I)$ and the lemma just cited, there is a combinatorial isometric embedding $Y(I) \times \prod_{i \in I} L_i \rightarrow [\hat{Z}]$, whose image we denote by $F(I)$ and refer to as a *generalized frame*. Moreover, for any 0-cube $z \in [\hat{Z}]$ that is not separated from a hyperplane in $\bigcup_{i \in I} \mathfrak{N}'_i \cup \mathfrak{A}(I)$ by a hyperplane not in that set, we can choose $F(I)$ to contain z ; this follows from the proof of the same lemma of Hagen. Figures 2 and 3 show possible collateralism classes \mathfrak{N}'_i and generalized hyperplane frames.

To build q , we will express $[\hat{Z}]$ as the union of \hat{Z} and a collection of generalized frames, define q on each generalized frame, and check that the definition is compatible where multiple generalized frames intersect. Let $z \in [\hat{Z}]$ be a 0-cube. Either $z \in \hat{Z}$, or there is a nonempty set $I \subset \{1, \dots, k\}$ such that the set of hyperplanes separating z from \hat{Z} is contained in $\bigcup_{i \in I} \mathfrak{N}'_i$, and each \mathfrak{N}'_i contains a hyperplane

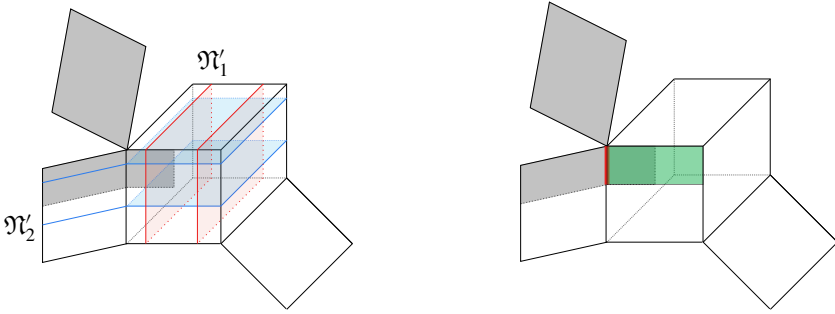


Figure 2. Collateral families \mathfrak{N}'_1 and \mathfrak{N}'_2 (left) and $Y(\{1\}) \times L_1$ (right).

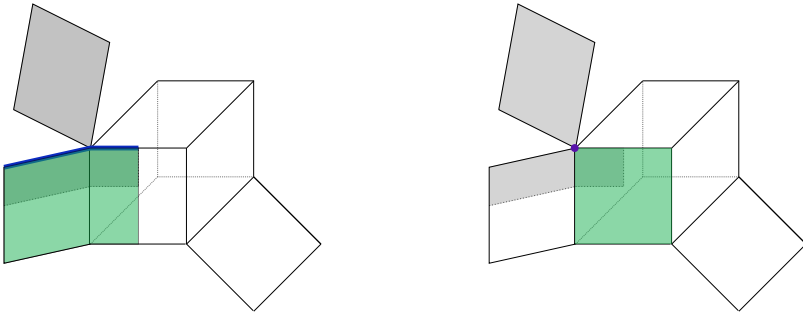


Figure 3. $Y(\{2\}) \times L_2$ (left) and $Y(\{1, 2\}) \times (L_1 \times L_2)$ (right).

separating z from \hat{Z} . If $H \in \mathfrak{A}(I) \cup \bigcup_{i \in I} \mathfrak{N}'_i$ is separated from z by a hyperplane U , then $U \in \mathfrak{A}(I) \cup \bigcup_{i \in I} \mathfrak{N}'_i$, whence we can choose $F(I)$ to contain z . Hence $[\hat{Z}]$ is the union of \hat{Z} and a finite collection of generalized frames $F(I_1), \dots, F(I_t)$.

For any $p \in \{1, \dots, t\}$, we have $F(I_p) = Y(I_p) \times \prod_{j \in I_p} L_j$ and we define $\bar{Y}(I_p) = \text{im}(Y(I_p) \rightarrow Z)$. Also, let $\bar{L}_j = \text{im}(L_j \cap \hat{Z} \rightarrow Z)$ be the cycle of length n_{N_j} to which L_j maps, for each $j \in I_p$. Note that Z contains $\bar{F}(I_p) = \bar{Y}(I_p) \times \prod_{j \in I_p} \bar{L}_j$ and so we define the quotient map $q_p : F(I_p) \rightarrow Z$ as the product of the above combinatorial quotient maps, namely, $q_p(y, (r_j)_{j \in I_p}) = (\bar{y}, (r_j \bmod n_{N_j})_{j \in I_p})$ for $y \in Y(I_p)$ and $r_j \in L_j$.

To ensure that $q_p(F(I_p) \cap F(I_j)) = q_j(F(I_p) \cap F(I_j))$ for all $i, j \leq t$, it suffices to show that

$$F(I_p) \cap F(I_j) := \left(Y(I_p) \times \prod_{k \in I_p} L_k \right) \cap \left(Y(I_j) \times \prod_{\ell \in I_j} L_\ell \right) = [Y(I_p) \cap Y(I_j)] \times \prod_{k \in I_p \cap I_j} L_k.$$

This in turn follows from [Caprace and Sageev 2011, Proposition 2.5]. Hence, the quotient maps q_p are compatible and thus define a combinatorial quotient map $q : [\hat{Z}] \rightarrow Z$ extending the maps q_p .

Observe that if $\mathfrak{H} = \emptyset$, i.e., $K = \lfloor \hat{Z} \rfloor$, then we take $Y = Z$. By hypothesis, Z admits a local isometry to S_Γ and has the desired cardinality. Moreover, our hypothesis on g ensures that $g \notin \pi_1 Y$, but the map q shows that any closed combinatorial path in S_Γ representing g lifts to a (nonclosed) path in Z , so the proof of the theorem is complete. Thus we can and shall assume that $\mathfrak{H} \neq \emptyset$.

Quotients of \mathfrak{H} -frames. To extend q to the rest of K , we now describe quotient maps, compatible with the map $\hat{Z} \rightarrow Z$, on frames associated to hyperplanes in \mathfrak{H} . An *isolated \mathfrak{H} -frame* is a frame $(H \cap K) \times L$, where $H \in \mathfrak{H}$ and H crosses no hyperplane of \hat{Z} (and hence crosses no hyperplane of $\lfloor \hat{Z} \rfloor$). An *interfered \mathfrak{H} -frame* is a frame $(H \cap K) \times L$, where $H \in \mathfrak{H}$ and H crosses an element of \mathfrak{J} . Equivalently, $(H \cap K) \times L$ is interfered if $\mathfrak{g}_{N(H)}(\hat{Z})$ contains a 1-cube and is isolated otherwise.

Define quotient maps on isolated \mathfrak{H} -frames by the same means as was used for arbitrary frames in [Bou-Rabee et al. 2015]. Let $(H \cap K) \times L$ be an isolated \mathfrak{H} -frame. Let \bar{H} be the immersed hyperplane in S_Γ to which H is sent by $\tilde{S}_\Gamma \rightarrow S_\Gamma$, and let $\bar{H} \cap \bar{K}$ be the image of $H \cap K$. We form a quotient $Y_H = \bar{H} \cap \bar{K} \times L$ of every isolated \mathfrak{H} -frame $(H \cap K) \times L$.

Now we define the quotients of interfered \mathfrak{H} -frames. Let $\hat{A} = \mathfrak{g}_{N(H)}(\hat{Z})$ and let A be the image of \hat{A} under $\hat{Z} \rightarrow Z$. There is a local isometry $A \rightarrow S_\Gamma$, to which we apply canonical completion to produce a finite cover $\ddot{S}_\Gamma \rightarrow S_\Gamma$ where A embeds. By [Bou-Rabee et al. 2015, Lemma 2.8], $\deg(\ddot{S}_\Gamma \rightarrow S_\Gamma) = |\ddot{S}_\Gamma^{(0)}| = |A^{(0)}| \leq |Z^{(0)}|$. Let $\bar{H} \cap \bar{K} = \text{im}(H \cap K \rightarrow \ddot{S}_\Gamma)$, and map the interfered \mathfrak{H} -frame $(H \cap K) \times L$ to $Y_H = \bar{H} \cap \bar{K} \times L$.

Constructing Y . We now construct a compact cube complex Y' from Z and the various quotients Y_H . A hyperplane W in K separates H_1 from \hat{Z} only if $W \in \mathfrak{H}$. Each \mathfrak{H} -hyperplane frame has the form $(H_i \cap K) \times L_i = (H_i \cap K) \times [0, m_i]$, parametrized so that $(H_i \cap K) \times \{0\}$ is the closest combinatorial hyperplane in the frame to \hat{Z} . We form $Y'(1)$ by gluing Y_{H_1} to Z along the image of $\mathfrak{g}_{\hat{Z}}((H_1 \cap K) \times \{0\})$, enabled by the fact that the quotients of interfered \mathfrak{H} -frames are compatible with $\hat{Z} \rightarrow Z$. In a similar manner, form $Y'(i)$ from $Y'(i-1)$ and Y_{H_i} by identifying the image of $(H_{i-1} \cap K) \times \{m_{i-1}\} \cap (H_i \cap K) \times \{0\}$ in $Y_{H_{i-1}} \subset Y'(i-1)$ with its image in Y_{H_i} . Let $Y' = Y'(k)$.

Let

$$K' = \lfloor \hat{Z} \rfloor \cup \bigcup_{H_i \in \mathfrak{H}} (H_i \cap K) \times L_i.$$

Since $H_i \cap H_j = \emptyset$ for $i \neq j$, there exists a map $(K', \tilde{x}) \rightarrow (Y', x)$ and a map $(Y', x) \rightarrow (S_\Gamma, x)$ such that the composition is precisely the restriction to K' of the covering map $(\tilde{S}_\Gamma, \tilde{x}) \rightarrow (S_\Gamma, x)$.

If $Y' \rightarrow S_\Gamma$ fails to be a local isometry, then there exists i and nontrivial open cubes $e \subset \bar{H}_{i-1} \cap \bar{K} \times \{m_{i-1}\}$ (or Z if $i = 1$) and $c \subset \bar{H}_i \cap \bar{K} \times \{0\}$ such that S_Γ

contains an open cube $\bar{e} \times \bar{c}$, where \bar{e}, \bar{c} are the images of e, c under $\tilde{S}_\Gamma \rightarrow S_\Gamma$, respectively. Moreover, since $\mathfrak{g}_{\tilde{Z}}(H_i \cap K) \subseteq \mathfrak{g}_{\tilde{Z}}(H_{i-1} \cap K)$, we can assume that \bar{c} is disjoint from each immersed hyperplane of S_Γ crossing Z . Hence the closure $Cl(\bar{c})$ is a standard torus. Glue $Cl(\bar{e}) \times Cl(\bar{c})$ to Y' , if necessary, in the obvious way. Note that this gluing adds no new 0-cubes to Y' . Indeed, every 0-cube of $Cl(\bar{e}) \times Cl(\bar{c})$ is identified with an existing 0-cube of Y' lying in $\overline{H_{i-1} \cap K} \times \{m_{i-1}\}$. Adding $Cl(\bar{e}) \times Cl(\bar{c})$ also preserves the existence of a local injection from our cube complex to S_Γ . Either this new complex admits a local isometry to S_Γ , or there is a missing cube of the form $\bar{e} \times \bar{c}$ where $Cl(\bar{c})$ is a standard torus and \bar{e} lies in Y' . We add cubes of this type until we have no missing corners. That the process terminates in a local isometry with compact domain Y is a consequence of the following facts: at each stage, every missing cube has the form $\bar{e} \times \bar{c}$ where \bar{e} lies in Y' and $Cl(\bar{c})$ is a standard torus, so the number of 0-cubes remains unchanged; each gluing preserves the existence of a local injection to S_Γ ; each gluing increases the number of positive dimensional cubes containing some 0-cube; cubes that we add are images of cubes in K , which is compact.

There exists a combinatorial path γ in K' joining \tilde{x} to $g\tilde{x}$. It follows from the existence of γ that the convex hull of K' is precisely equal to K . Hence, there exists a based cubical map $(K, \tilde{x}) \rightarrow (Y, x) \rightarrow (S_\Gamma, x)$, so that the composition is the restriction of the covering map $(\tilde{S}_\Gamma, \tilde{x}) \rightarrow (S_\Gamma, x)$. Therefore, any closed path in S_Γ representing g lifts to a nonclosed path at x in Y . It is easily verified that the number of 0-cubes in Y is bounded by $|Z^{(0)}|(m_1 + \dots + m_k)$, where each m_i is the length of L_i , and hence $|Y^{(0)}| \leq |Z^{(0)}|(|g| + 1)$. Thus, Y is the desired cube complex. \square

Remark 3.2. When $\dim S_\Gamma = 1$, arguing as above shows that Y can be chosen so that $|Y^{(0)}| \leq |Z^{(0)}| + |g|$. Hence, if F is freely generated by \mathcal{S} , with $|\mathcal{S}| = r$, then $\text{Sep}_{F, \mathcal{S}}^{Q, K}(Q, n) \leq (2r)^K + n$.

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