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DASHAN FAN, HUOXIONG WU AND FAYOU ZHAO

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Using transference techniques, we show that $L^p(\mathbb{R}^n)$ estimates for many operators may be transferred to the $L^p(\mathbb{T}^n)$ estimates on the n-torus $\mathbb{T}^n$ via measure-preserving actions of $\mathbb{R}^n$. These operators include the maximal bilinear Hilbert transform, the oscillation, and the variation and short variation operators of the Hilbert transform on the torus $\mathbb{T}$. As an extension, we study the (maximal) bilinear Riesz transforms on the n-torus $\mathbb{T}^n$.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{R}^2_+$ the upper half plane

$$\mathbb{R}^2_+ = \{(x, y) = x + iy \in \mathbb{C} : y > 0\}.$$

The boundary of $\mathbb{R}^2_+$ is the real line $\mathbb{R}$. Consider the boundary condition $f \in L^p(\mathbb{R})$, where f is real-valued and $1 \le p < \infty$. It is well known that the Poisson integral

$$u(x, y) = P_y(f)(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\frac{y}{(x-t)^2 + y^2}\,dt$$

is the solution of the Dirichlet problem on $\mathbb{R}^2_+$. Precisely, u is a harmonic function on $\mathbb{R}^2_+$ and $u(x, y)$ tends to $f(x)$ nontangentially for almost all $x \in \mathbb{R}$ as $y \to 0^+$. There is a (unique) harmonic function

$$v(x, y) = Q_y(f)(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\frac{x-t}{(x-t)^2+y^2}\,dt$$

such that

$$F(z) = u(x, y) + iv(x, y)$$

is an analytic function on $\mathbb{R}^2_+$. This function $Q_y(f)(x)$ is called the conjugate Poisson integral of f. From [Stein and Weiss 1971, p. 186], we know that the

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function $F(z)$, with $z = x + iy$, has the nontangential limit $f(x) + \frac{i}{\pi} Hf(x)$ for almost all $x \in \mathbb{R}$. Here H is the Hilbert transform defined by

$$Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x) \tag{1}$$

and

$$H_\varepsilon f(x) = \int_{|t|>\varepsilon} \frac{f(x-t)}{t}\, dt.$$

To study the pointwise convergence of $Hf(x)$, one then needs to study the truncated maximal Hilbert transform

$$H^* f(x) = \sup_{\varepsilon > 0} |H_\varepsilon f(x)|.$$

Let

$$D = \{z = x + iy \in \mathbb{C} : |z| < 1\}$$

be the unit disc. Its boundary $\mathbb{T} = \partial D$ is the one-dimensional torus. Without loss of generality, we may identify the torus $\mathbb{T}$ with its fundamental interval $\left[-\frac{1}{2}, \frac{1}{2}\right)$. The Dirichlet problem on D with the boundary condition $\tilde{f} \in L^p(\mathbb{T})$ similarly raises an analytic function $\widetilde{F}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y)$. The function $\widetilde{F}(z)$, for $z = x + iy \in D$, has the nontangential limit $\tilde{f}(x) + \frac{i}{\pi} \widetilde{H}\tilde{f}(x)$ for almost all points $x \in \mathbb{T}$. Here $\widetilde{H}$ is the periodic version of the Hilbert transform defined by

$$\widetilde{H}\tilde{f}(x) = \lim_{\varepsilon \to 0} \widetilde{H}_\varepsilon \tilde{f}(x)$$

and

$$\widetilde{H}_\varepsilon \tilde{f}(x) = \int_{\varepsilon < |t| < \frac{1}{2}} \tilde{f}(x-t) \cot(\pi t)\, dt. \tag{2}$$

By computing the Fourier coefficients, one can see that $\widetilde{H}\tilde{f}(x)$ has the Fourier series

$$\widetilde{H}\tilde{f}(x) = \sum_{k=-\infty}^{\infty} i \operatorname{sgn}(k) a_k e^{2\pi i k x}$$

for any

$$\tilde{f}(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k x}$$

(see also [Edwards and Gaudry 1977]). It is known that $\sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) a_k e^{2\pi i k x}$ (up to a constant multiplier) is the conjugate Fourier series of $\tilde{f}$.

The bilinear Hilbert transform $\mathcal{H}(f, g)$ and the maximal bilinear Hilbert transform $\mathcal{H}^*(f, g)$ are defined respectively as

$$\mathcal{H}(f, g)(x) = \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} \frac{f(x-t)g(x+t)}{t}\, dt \tag{3}$$

and

$$\mathcal{H}^*(f, g)(x) = \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} \frac{f(x-t)g(x+t)}{t}\, dt \right|.$$

The operator $\mathcal{H}(f, g)$ is not merely a formal extension from the Hilbert transform. It has deep roots in the study of certain harmonic analysis and PDE problems. The study of the bilinear Hilbert transform $\mathcal{H}(f, g)$ was initiated by Calderón when he studied certain Cauchy integrals $C_\gamma(f)$ along the Lipschitz curves. In order to obtain the L^2 boundedness of $C_\gamma(f)$, Calderón introduced a commutator (now known as the first Calderón commutator) and raised a famous conjecture, which says that $\mathcal{H}$ is a bounded operator from $L^\infty \times L^2 \to L^2$; see [Jones 1994]. The conjecture was solved in a more general setting by Lacey and Thiele in their celebrated theorem:

Theorem A1 [Lacey and Thiele 1997]. *Let* $1 < q, r \le \infty$, *and* $\frac{2}{3} < p < \infty$. *Then*

$$\|\mathcal{H}(f, g)\|_{L^p(\mathbb{R})} \preceq \|f\|_{L^q(\mathbb{R})} \|g\|_{L^r(\mathbb{R})},$$

provided $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

The notation $A \preceq B$ for $A, B > 0$ means that there exists a constant $c > 0$ independent of all essential variables such that $A \le cB$. We also use the notation $A \simeq B$ when $A \preceq B$ and $B \preceq A$.

The proof of the theorem by Lacey and Thiele involves a very elegant method of time-frequency analysis. The essence of the matter lies in their formulation and proof of certain almost orthogonal results on the phase space. Maximal forms of these results must be proved. These maximal inequalities rely in an essential way on a novel maximal inequality of Bourgain [1989; 1990]. By refining these maximal bilinear estimates and Bourgain's lemma, Lacey further obtained the following remarkable theorem:

Theorem A2 [Lacey 2000]. *Let* $1 < q, r \le \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *If* $\frac{2}{3} < p < \infty$, *then*

$$\|\mathcal{H}^*(f, g)\|_{L^p(\mathbb{R})} \preceq \|f\|_{L^q(\mathbb{R})} \|g\|_{L^r(\mathbb{R})}.$$

Based on Theorems A1 and A2, it is natural to expect to establish analogous theorems for the periodic bilinear Hilbert transform on the torus. Here, the bilinear Hilbert transform and its maximal operator on the torus are defined, initially on $C^\infty(\mathbb{T})$, by

$$\widetilde{\mathcal{H}}(\tilde{f}, \tilde{g})(x) = \text{p.v.} \int_{|t|<\frac{1}{2}} \tilde{f}(x-t)\tilde{g}(x+t)\cot(\pi t)\, dt$$

and

$$\widetilde{\mathcal{H}}^*(\tilde{f}, \tilde{g})(x) = \sup_{\varepsilon>0} \left| \int_{\varepsilon<|t|<\frac{1}{2}} \tilde{f}(x-t)\tilde{g}(x+t)\cot(\pi t)\, dt \right|.$$

However, it seems quite difficult to adopt the time-frequency method used in [Lacey and Thiele 1997]. Thus, in [Fan and Sato 2001], the authors used a "transference" method to reduce the boundedness of $\widetilde{\mathcal{H}}(\tilde{f}, \tilde{g})$ to the boundedness of $\mathcal{H}(f, g)$ by estimating an error term. The method of transference is a useful tool for obtaining norm estimates independent of the dimension for classical operators acting on $L^p(\mathbb{R}^n)$ (see [Auscher and Carro 1994; Blasco and Gillespie 2009; Coifman and Weiss 1977; Gillespie and Torrea 2004; Rubio de Francia 1989]). Fan and Sato [2001] proved the de Leeuw-type theorems (see [de Leeuw 1965]) for the transference of multilinear operators on Lebesgue spaces from $\mathbb{R}^n$ to the n-torus. In particular, they proved:

Theorem B. *Let* $1 < q, r \leq \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *If* $\frac{2}{3} < p < \infty$, *then*

$$\|\widetilde{\mathcal{H}}(\tilde{f}, \tilde{g})\|_{L^p(\mathbb{T})} \preceq \|\tilde{f}\|_{L^q(\mathbb{T})} \|\tilde{g}\|_{L^r(\mathbb{T})}.$$

Note that

$$\widetilde{\mathcal{H}}(\tilde{f}, \tilde{g})(x) \simeq \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \operatorname{sgn}(k_1 - k_2) a_{k_1} a_{k_2} e^{2\pi i (k_1 + k_2) x},$$

where

$$\tilde{f}(x) = \sum_{k_1 \in \mathbb{Z}} a_{k_1} e^{2\pi i k_1 x} \quad \textit{and} \quad \tilde{g}(x) = \sum_{k_2 \in \mathbb{Z}} a_{k_2} e^{2\pi i k_2 x}.$$

In [Fan and Sato 2001], the authors also studied the boundedness of the maximal multiplier operator

$$\widetilde{T}^{**}(\tilde{f}, \tilde{g})(x) = \sup_{\varepsilon > 0} \left| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} m(\varepsilon k_1, \varepsilon k_2) a_{k_1} a_{k_2} e^{2\pi i (k_1 + k_2) x} \right|,$$

where m is a bounded and continuous function (see also [Berkson et al. 2006; 2007; Blasco et al. 2005; Grafakos and Honzík 2006] for transference methods on maximal bilinear operators). For the bilinear Hilbert transform, clearly we have

$$\widetilde{\mathcal{H}}(\tilde{f}, \tilde{g})(x) = \widetilde{\mathcal{H}}^{**}(\tilde{f}, \tilde{g})(x),$$

since

$$\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \operatorname{sgn}(\varepsilon k_1 - \varepsilon k_2) a_{k_1} a_{k_2} e^{2\pi i (k_1 + k_2) x} = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \operatorname{sgn}(k_1 - k_2) a_{k_1} a_{k_2} e^{2\pi i (k_1 + k_2) x}.$$

This observation indicates

$$\widetilde{\mathcal{H}}^{*}(\tilde{f}, \tilde{g})(x) \neq \widetilde{\mathcal{H}}^{**}(\tilde{f}, \tilde{g})(x).$$

Since the boundedness of $\widetilde{\mathcal{H}}^{*}(\tilde{f}, \tilde{g})$ still remains open, the first aim of this paper is to solve this problem by establishing the following analog of Lacey's theorem.

Theorem 1.1. *Let* $1 < q, r \le \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *If* $\frac{2}{3} < p < \infty$, *then*

$$\|\widetilde{\mathcal{H}}^*(\tilde{f}, \tilde{g})\|_{L^p(\mathbb{T})} \preceq \|\tilde{f}\|_{L^q(\mathbb{T})}\|\tilde{g}\|_{L^r(\mathbb{T})}.$$

We adopt the method in [Fan and Sato 2001] to prove the theorem, in which the main issue is to estimate error terms in order to reduce the boundedness of $L^q(\mathbb{T}) \times L^r(\mathbb{T}) \to L^p(\mathbb{T})$ to the known result for $L^q(\mathbb{R}) \times L^r(\mathbb{R}) \to L^p(\mathbb{R})$. This method additionally allows us to treat other operators related to the maximal Hilbert transform. Recall that the limits (1) and (3) mentioned above exist almost everywhere. Motivated by probability and ergodic theory [Bourgain 1989; Jones 1997; 1998], in order to obtain extra information on their convergence rate, as well as an estimate on the number of λ-jumps they can have, Campbell, Jones, Reinhold and Wierdl [Campbell et al. 2000] studied the oscillation and variation of the family (H_ε) as ε approaches 0 as follows.

For each fixed sequence $(t_k) \searrow 0$, define the oscillation and variation operators by

$$\mathscr{O}(H_* f)(x) = \Bigl(\sum_{k=1}^{\infty} \sup_{t_{k+1} \le \varepsilon_{k+1} < \varepsilon_k \le t_k} |H_{\varepsilon_k} f(x) - H_{\varepsilon_{k+1}} f(x)|^2\Bigr)^{\frac{1}{2}}, \tag{4}$$

$$\mathscr{V}_\varrho(H_* f)(x) = \sup_{(\varepsilon_k)\searrow 0} \Bigl(\sum_{k=1}^{\infty} |H_{\varepsilon_k} f(x) - H_{\varepsilon_{k+1}} f(x)|^\varrho\Bigr)^{\frac{1}{\varrho}}, \tag{5}$$

respectively. Also, define

$$V_k(H_* f)(x) = \sup_{(\varepsilon_j)\searrow 0} \Bigl(\sum_{\frac{1}{2^k} < \varepsilon_{j+1} < \varepsilon_j \le \frac{1}{2^{k-1}}} |H_{\varepsilon_j} f(x) - H_{\varepsilon_{j+1}} f(x)|^2\Bigr)^{\frac{1}{2}},$$

where the supremum is taken over all decreasing sequences (ε_j). Then the "short variation operator" is defined by

$$S_V(H_* f)(x) = \Bigl(\sum_{k=-\infty}^{\infty} V_k(H_* f(x))^2\Bigr)^{\frac{1}{2}}. \tag{6}$$

For convenience, all the integrals are defined on the Schwartz class.

We recall the following results from [Campbell et al. 2000].

Theorem C. *The oscillation operator* $\mathscr{O}(H_*)$ *satisfies*

$$\|\mathscr{O}(H_* f)\|_{L^p(\mathbb{R})} \le c_p \|f\|_{L^p(\mathbb{R})}$$

for $1 < p < \infty$ *and* $|\{x \in \mathbb{R} : \mathscr{O}(H_* f)(x) > \lambda\}| \le (c/\lambda)\|f\|_{L^1(\mathbb{R})}$.

Theorem D. *If* $\varrho > 2$, *then the variation operator* $\mathscr{V}_\varrho(H_*)$ *satisfies*

$$\|\mathscr{V}_\varrho(H_* f)\|_{L^p(\mathbb{R})} \le c(p, \varrho)\|f\|_{L^p(\mathbb{R})}$$

for $1 < p < \infty$ *and* $|\{x \in \mathbb{R} : \mathscr{V}_\varrho(H_* f)(x) > \lambda\}| \le (c(\varrho)/\lambda)\|f\|_{L^1(\mathbb{R})}$.

Theorem E. *The short variation operator* $S_V(H_*)$ *satisfies*

$$\|S_V(H_*f)\|_{L^p(\mathbb{R})} \le c_p \|f\|_{L^p(\mathbb{R})}$$

for $1 < p < \infty$ *and* $|\{x \in \mathbb{R} : S_V(H_*f)(x) > \lambda\}| \le (c/\lambda)\|f\|_{L^1(\mathbb{R})}$.

Our second aim is to use the transference method to study the analogs of these operators on the torus. For each fixed sequence $(t_k) \searrow 0$, define the oscillation and variation operators on the torus by

$$\mathscr{O}(\widetilde{H}_*\tilde{f})(x) = \left(\sum_{k=1}^{\infty} \sup_{t_{k+1}\le\varepsilon_{k+1}<\varepsilon_k\le t_k} |\widetilde{H}_{\varepsilon_k}\tilde{f}(x) - \widetilde{H}_{\varepsilon_{k+1}}\tilde{f}(x)|^2\right)^{\frac{1}{2}},$$

$$\mathscr{V}_\varrho(\widetilde{H}_*\tilde{f})(x) = \sup_{(\varepsilon_k)\searrow 0}\left(\sum_{k=1}^{\infty} |\widetilde{H}_{\varepsilon_k}\tilde{f}(x) - \widetilde{H}_{\varepsilon_{k+1}}\tilde{f}(x)|^\varrho\right)^{\frac{1}{\varrho}},$$

respectively. Also, define the operator $V_k(\widetilde{H}_*)$ on the torus by

$$V_k(\widetilde{H}_*\tilde{f})(x) = \sup_{(\varepsilon_j)\searrow 0}\left(\sum_{\frac{1}{2^k}<\varepsilon_{j+1}<\varepsilon_j\le\frac{1}{2^{k-1}}} |\widetilde{H}_{\varepsilon_j}\tilde{f}(x) - \widetilde{H}_{\varepsilon_{j+1}}\tilde{f}(x)|^2\right)^{\frac{1}{2}},$$

where the supremum is taken over all decreasing sequences (ε_j). Then define the "short variation operator" on the torus by

$$S_V(\widetilde{H}_*\tilde{f})(x) = \left(\sum_{k=-\infty}^{\infty} (V_k(\widetilde{H}_*\tilde{f}(x))^2\right)^{\frac{1}{2}}.$$

For simplicity, we define these operators on the space $C^\infty(\mathbb{T})$.

We establish the following theorems.

Theorem 1.2. *The oscillation operator* $\mathscr{O}(\widetilde{H}_*)$ *satisfies*

$$\|\mathscr{O}(\widetilde{H}_*\tilde{f})\|_{L^p(\mathbb{T})} \le c_p \|\tilde{f}\|_{L^p(\mathbb{T})}$$

for $1 < p < \infty$ *and* $|\{x \in \mathbb{T} : \mathscr{O}(\widetilde{H}_*\tilde{f})(x) > \lambda\}| \le (c/\lambda)\|\tilde{f}\|_{L^1(\mathbb{T})}$.

Theorem 1.3. *If* $\varrho > 2$, *then the variation operator* $\mathscr{V}_\varrho(\widetilde{H}_*)$ *satisfies*

$$\|\mathscr{V}_\varrho(\widetilde{H}_*\tilde{f})\|_{L^p(\mathbb{T})} \le c(p,\varrho) \|\tilde{f}\|_{L^p(\mathbb{T})}$$

for $1 < p < \infty$ *and* $|\{x \in \mathbb{T} : \mathscr{V}_\varrho(\widetilde{H}_*\tilde{f})(x) > \lambda\}| \le (c(\varrho)/\lambda)\|\tilde{f}\|_{L^1(\mathbb{T})}$.

Theorem 1.4. *The short variation operator* $S_V(\widetilde{H}_*)$ *satisfies*

$$\|S_V(\widetilde{H}_*\tilde{f})\|_{L^p(\mathbb{T})} \le c_p \|\tilde{f}\|_{L^p(\mathbb{T})}$$

for $1 < p < \infty$ *and* $|\{x \in \mathbb{T} : S_V(\widetilde{H}_*\tilde{f})(x) > \lambda\}| \le (c/\lambda)\|\tilde{f}\|_{L^1(\mathbb{T})}$.

As we mentioned, the method in [Fan and Sato 2001] shows that $L^p(\mathbb{R}^n)$ estimates for many linear operators may be transferred to their corresponding L^p estimates on the torus via measure-preserving actions of $\mathbb{R}^n$. As a further application and extension, we consider the bilinear Riesz transform on the n-torus. Recall that the bilinear singular integral with rough kernel on $\mathbb{R}^n$ is defined by

$$T_\Omega(f, g)(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} f(x-y)g(x+y)\frac{\Omega(y')}{|y|^n}\, dy, \tag{7}$$

where $\Omega(y')$ is a function defined on the unit sphere S^{n-1} in Euclidean space $\mathbb{R}^n$ and whose integral over S^{n-1} is zero. One then obtains the bilinear Riesz transform by taking $\Omega(x) = x_j/|x|$, where x_j is the j-th component of x. Using the same transference method, we also can transfer the L^p-boundedness of the maximal bilinear Riesz transform from $\mathbb{R}^n$ to $\mathbb{T}^n$. This fact is discussed in the last two sections.

Throughout this article, we use the letter c to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

2. Proof of Theorems 1.1–1.4

In this section we give the proof of Theorems 1.1–1.4. As is well known, Euler discovered two remarkable expressions for circular functions, one as an infinite product and the other as an infinite series. For the sine function he established the formula

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty}\Big(1 - \frac{x^2}{k}\Big) = \pi x \prod_{k\in\mathbb{Z}\setminus\{0\}}\Big(1 + \frac{x}{k}\Big)$$

(see [Varadarajan 2007]). By logarithmic differentiation one obtains

$$\cot(\pi x) = \text{p.v.}\ \frac{1}{\pi}\sum_{k\in\mathbb{Z}}\Big(\frac{1}{x+k}\Big), \tag{8}$$

where p.v. means the Cauchy principal value, that is, that the sum has to be interpreted as the limit

$$\lim_{N\to+\infty}\sum_{k=-N}^{N}\frac{1}{x+k} = \frac{1}{x} + \sum_{k=1}^{\infty}\Big(\frac{1}{x+k} - \frac{1}{x-k}\Big), \quad N\in\mathbb{Z}^+. \tag{9}$$

Let $\chi_A(t)$ be the characteristic function of the set $A = \{t\in\mathbb{R} : |t| > 1\}$. To prove Theorem 1.1, we need the following lemma.

Lemma 2.1. *For $\varepsilon < |t| \le \frac{1}{2}$, we have*

$$\cot(\pi t)\chi_A\Big(\frac{t}{\varepsilon}\Big) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}}\frac{1}{t+k}\chi_A\Big(\frac{t+k}{\varepsilon}\Big) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big)$$

and the estimate

$$\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big)\right| \preceq |t|\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big),$$

where A^C is the complement of the set A.

Proof. Using (8), we write the term $\cot(\pi t)\chi_A(t/\varepsilon)$ as

$$\begin{aligned}\cot(\pi t)\chi_A\Big(\frac{t}{\varepsilon}\Big) &= \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{1}{t+k}\chi_A\Big(\frac{t}{\varepsilon}\Big)\\ &= \frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_A\Big(\frac{t}{\varepsilon}\Big) + \frac{1}{\pi t}\chi_A\Big(\frac{t}{\varepsilon}\Big).\end{aligned}$$

Since χ_A is the characteristic function of the set $\{t \in \mathbb{R} : |t| > 1\}$, it is easy to see that for $\varepsilon < |t| \le \frac{1}{2}$ and $k \in \mathbb{Z}\setminus\{0\}$, we have

$$\chi_A\Big(\frac{t+k}{\varepsilon}\Big) = 1.$$

The fact above leads to

$$\frac{1}{\pi t}\chi_A\Big(\frac{t}{\varepsilon}\Big) = \frac{1}{\pi t}\chi_A\Big(\frac{t}{\varepsilon}\Big) + \frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_A\Big(\frac{t+k}{\varepsilon}\Big) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}.$$

Hence we have

$$\cot(\pi t)\chi_A\Big(\frac{t}{\varepsilon}\Big) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{1}{t+k}\chi_A\Big(\frac{t+k}{\varepsilon}\Big) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big).$$

It now remains to estimate

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big).$$

From (9), we know

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big) = \sum_{k=1}^{\infty}\Big(\frac{1}{t+k}+\frac{1}{t-k}\Big)\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big) = 2t\sum_{k=1}^{\infty}\frac{1}{t^2-k^2}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big).$$

Using this yields

$$\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big)\right| \preceq |t|\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big),$$

which completes the proof. □

Proof of Theorem 1.1. For simplicity, we introduce some notation. Denote by

$$\mathfrak{R}(t)=\frac{\chi_A(t)}{t},\qquad \mathfrak{R}_\varepsilon(t)=\frac{1}{\varepsilon}\mathfrak{R}\Big(\frac{t}{\varepsilon}\Big), \tag{10}$$

$$\widetilde{\mathfrak{R}}_\varepsilon(t)=\frac{1}{\varepsilon}\sum_{k\in\mathbb{Z}}\mathfrak{R}\Big(\frac{t+k}{\varepsilon}\Big),\qquad r_\varepsilon(t)=\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{t+k}\chi_{A^C}\Big(\frac{t}{\varepsilon}\Big). \tag{11}$$

Then we have

$$\mathcal{H}_\varepsilon(f,g)(x)=\int_{\mathbb{R}} f(x-t)g(x+t)\mathfrak{R}_\varepsilon(t)\,dt.$$

Let

$$\widetilde{\mathbb{H}}_\varepsilon(\tilde f,\tilde g)(x)=\int_{|t|<\frac12}\tilde f(x-t)\tilde g(x+t)\widetilde{\mathfrak{R}}_\varepsilon(t)\,dt.$$

Because of Lemma 2.1, one has

$$\widetilde{H}^*(\tilde f,\tilde g)(x)\le\tfrac{1}{\pi}\sup_{\varepsilon>0}\big|\widetilde{\mathbb{H}}_\varepsilon(\tilde f,\tilde g)(x)\big|+\tfrac{1}{\pi}M(\tilde f,\tilde g)(x),$$

where

$$M(\tilde f,\tilde g)(x)=\sup_{\varepsilon>0}\bigg|\int_{|t|<\frac12}\tilde f(x-t)\tilde g(x+t)r_\varepsilon(t)\,dt\bigg|.$$

By Lemma 2.1, the Minkowski integral inequality and Hölder's inequality, we see that, for $p\ge 1$,

$$\begin{aligned}\|M(\tilde f,\tilde g)\|_{L^p(\mathbb{T})}&\preceq\int_{-\frac12}^{\frac12}\|\tilde f(x-t)\tilde g(x+t)\|_{L^p(\mathbb{T},dx)}|t|\,dt\\&\preceq\|\tilde f\|_{L^q(\mathbb{T})}\|\tilde g\|_{L^r(\mathbb{T})}.\end{aligned}$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned}\bigg\|\int_{-\frac12}^{\frac12}\tilde f(x-t)\tilde g(x+t)r_\varepsilon(t)\,dt\bigg\|_{L^{1/2}(\mathbb{T})}&\preceq\bigg\|\int_{-\frac12}^{\frac12}|\tilde f(x-t)\tilde g(x+t)r_\varepsilon(t)|\,dt\bigg\|_{L^1(\mathbb{T})}\\&\preceq\|\tilde f\|_{L^1(\mathbb{T})}\|\tilde g\|_{L^1(\mathbb{T})}.\end{aligned}$$

Then an interpolation yields that, for all $\frac12\le p<\infty$,

$$\|M(\tilde f,\tilde g)\|_{L^p(\mathbb{T})}\preceq\|\tilde f\|_{L^q(\mathbb{T})}\|\tilde g\|_{L^r(\mathbb{T})}.$$

Thus, to prove the theorem, we only need to show that

$$\Big\|\sup_{\varepsilon>0}\big|\widetilde{\mathbb{H}}_\varepsilon(\tilde f,\tilde g)\big|\Big\|_{L^p(\mathbb{T})}\preceq\|\tilde f\|_{L^q(\mathbb{T})}\|\tilde g\|_{L^r(\mathbb{T})}.$$

It is easy to compute that the Fourier coefficients of $\widetilde{\mathfrak{R}}_\varepsilon(t)$ are

$$c_{m,\varepsilon} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k\in\mathbb{Z}} \frac{1}{\varepsilon}\mathfrak{R}\Big(\frac{t+k}{\varepsilon}\Big) e^{-i2\pi mt}\,dt \tag{12}$$
$$= \int_{\mathbb{R}} \mathfrak{R}(t) e^{-i2\pi mt\varepsilon}\,dt = -i\int_{|t|>1} \frac{\sin(2\pi mt\varepsilon)}{t}\,dt$$
$$= -\pi i\,\mathrm{sgn}(\varepsilon m) + i\int_{|t|\le 2\pi} \frac{\sin(\varepsilon mt)}{t}\,dt.$$

Clearly, $c_{m,\varepsilon}$ is uniformly bounded on $\varepsilon > 0$ and $m \in \mathbb{Z}$. On the other hand, it is not difficult to check that

$$\int_{|t|<\frac{1}{2}} \tilde f(x-t)\tilde g(x+t)\widetilde{\mathfrak{R}}_\varepsilon(t)\,dt = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_m b_n c_{m-n,\varepsilon} e^{2\pi i(n+m)x},$$

where

$$\tilde f(x) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi imx} \quad\text{and}\quad \tilde g(x) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi inx}.$$

Now pick $\Psi \in \mathscr{S}(\mathbb{R})$ satisfying $\Psi(x) = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$, $\mathrm{supp}(\Psi) \subset \big[-\frac{3}{4}, \frac{3}{4}\big]$ and $0 \le \Psi(x) \le 1$. For any positive N, denote the function Ψ^N by

$$\Psi^N(x) = \Psi(x/N). \tag{13}$$

Consider the error term given by

$$E_{N,\varepsilon}(\tilde f, \tilde g)(x) = \Psi(x/N)^2 \widetilde{\mathbb{H}}_\varepsilon(\tilde f, \tilde g)(x) - \mathcal{H}_\varepsilon\big(\Psi^N \tilde f, \Psi^N \tilde g\big)(x).$$

The error term $E_{N,\varepsilon}(\tilde f, \tilde g)(x)$ roughly gives the difference of $\mathcal{H}_\varepsilon$ on $\mathbb{R}$ and $\widetilde{\mathbb{H}}_\varepsilon$ on the torus. By checking the Fourier transform, we have

$$\mathcal{H}_\varepsilon\big(\Psi^N \tilde f, \Psi^N \tilde g\big)(x) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_m b_n e^{2\pi i(n+m)x} \times \int_{\mathbb{R}} \mathfrak{R}_\varepsilon(t)\Psi\Big(\frac{x+t}{N}\Big)\Psi\Big(\frac{x-t}{N}\Big)e^{2\pi i(n-m)t}\,dt.$$

The definition of the inverse Fourier transform on the space of Schwartz functions shows that

$$\int_{\mathbb{R}} \mathfrak{R}_\varepsilon(t)\Psi\Big(\frac{x+t}{N}\Big)\Psi\Big(\frac{x-t}{N}\Big)e^{2\pi i(n-m)t}\,dt$$
$$= \int_{\mathbb{R}} \mathfrak{R}_\varepsilon(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat\Psi(u)\widehat\Psi(v)e^{2\pi iu(x+t)/N}e^{2\pi iv(x-t)/N}\,du\,dv\; e^{2\pi i(n-m)t}\,dt$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat\Psi(u)\widehat\Psi(v)e^{2\pi ix(u+v)/N}\Big(\int_{\mathbb{R}} \mathfrak{R}_\varepsilon(t)e^{2\pi i(u-v)t/N}e^{2\pi i(n-m)t}\,dt\Big)\,du\,dv.$$

Therefore, we obtain that

$$E_{N,\varepsilon}(\tilde{f},\tilde{g})(x) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_m b_n e^{2\pi i(n+m)x} \times \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\Psi}(u)\widehat{\Psi}(v)(c_{m-n,\varepsilon} - c_{m-n+(v-u)/N,\varepsilon})e^{2\pi i x(u+v)/N}\,du\,dv.$$

If $m = n$, then

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\Psi}(u)\widehat{\Psi}(v)(c_{m-n,\varepsilon} - c_{m-n+(v-u)/N,\varepsilon})e^{2\pi i x(u+v)/N}\,du\,dv \\
&\quad = i\int_{\mathbb{R}}\int_{u>v} \widehat{\Psi}(u)\widehat{\Psi}(v)\biggl(\int_{|t|>1} \frac{\sin \varepsilon 2\pi t(v-u)/N}{t}\,dt\biggr)e^{2\pi i x(u+v)/N}\,du\,dv \\
&\qquad - i\int_{\mathbb{R}}\int_{v>u} \widehat{\Psi}(u)\widehat{\Psi}(v)\biggl(\int_{|t|>1} \frac{\sin \varepsilon 2\pi t(u-v)/N}{t}\,dt\biggr)e^{2\pi i x(u+v)/N}\,du\,dv \\
&\quad = 0.
\end{aligned}$$

If $m \neq n$, for any sufficiently small $\delta > 0$, we choose an $L > 0$ such that

$$\left|\int_{u^2+v^2>L} \widehat{\Psi}(u)\widehat{\Psi}(v)(c_{m-n,\varepsilon} - c_{m-n+(v-u)/N,\varepsilon})e^{2\pi i x(u+v)/N}\,du\,dv\right| < \delta.$$

Now we let N be sufficiently large so that, for $u^2 + v^2 \leq L$,

$$\operatorname{sgn}(m-n) = \operatorname{sgn}(m-n+(v-u)/N).$$

By this choice, for all $0 < \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned}
&\left|\int_{u^2+v^2\leq L} \widehat{\Psi}(u)\widehat{\Psi}(v)(c_{m-n,\varepsilon} - c_{m-n+(v-u)/N,\varepsilon})e^{2\pi i x(u+v)/N}\,du\,dv\right| \\
&\preceq \int_{u^2+v^2\leq L} \left|\widehat{\Psi}(u)\widehat{\Psi}(v)\int_{|t|<2\pi} \frac{\sin(\varepsilon t(m-n)) - \sin(\varepsilon t(m-n+(v-u)/N))}{t}\,dt\right|\,du\,dv \\
&= o(1) \quad \text{as } N \to \infty.
\end{aligned}$$

Since (a_n) and (b_m) are rapidly decreasing sequences, it is easy to see that

$$\lim_{N\to\infty} \sup_{0<\varepsilon<1/2} |E_{N,\varepsilon}(\tilde{f},\tilde{g})(x)| = 0.$$

Applying that

$$\sup_{0<\varepsilon<1/2} \bigl|\widetilde{\mathbb{H}}_{\varepsilon}(\tilde{f},\tilde{g})(x)\bigr|$$

is a periodic function, together with Theorem A2, we now have, as $N \to \infty$,

$$\begin{aligned}\Big\|\sup_{0<\varepsilon<1/2}\big|\widetilde{\mathbb{H}}_\varepsilon(\tilde f,\tilde g)\big|\Big\|_{L^p(\mathbb{T})}&=\bigg(\int_{-\frac12}^{\frac12}\Big(\sup_{0<\varepsilon<1/2}\big|\widetilde{\mathbb{H}}_\varepsilon(\tilde f,\tilde g)(x)\big|\Big)^p\,dx\bigg)^{\frac1p}\\
&=\frac{1}{N^{1/p}}\bigg(\int_{-\frac N2}^{\frac N2}\sup_{0<\varepsilon<1/2}\Psi(x/N)^2\big|\widetilde{\mathbb{H}}_\varepsilon(\tilde f,\tilde g)(x)\big|^p\,dx\bigg)^{\frac1p}\\
&\preceq o(1)+\frac{1}{N^{1/p}}\bigg(\int_{\mathbb{R}}\sup_{0<\varepsilon<1/2}\big|\mathcal{H}_\varepsilon\big(\Psi^N\tilde f,\Psi^N\tilde g\big)(x)\big|^p\,dx\bigg)^{\frac1p}\\
&\preceq o(1)+\frac{1}{N^{1/p}}\|\Psi^N\tilde f\|_{L^q(\mathbb{R})}\|\Psi^N\tilde g\|_{L^r(\mathbb{R})}\\
&\preceq o(1)+\|\tilde f\|_{L^q(\mathbb{T})}\|\tilde g\|_{L^r(\mathbb{T})}.\end{aligned}$$

Thus we get the desired result by letting $N\to\infty$. □

We again will use the transference method to prove Theorems 1.2–1.4. To this end, we need a key lemma in order to estimate error terms.

Lemma 2.2. *Let* $\mathfrak{R}_\varepsilon$, $\widetilde{\mathfrak{R}}_\varepsilon$ *and* Ψ *be as defined in* (10), (11) *and* (13), *respectively. For* $k\in\mathbb{Z}^+$, *set*

$$\widetilde{H}_{\varepsilon_k,\varepsilon_{k+1}}(\tilde f)(x)=\big(\widetilde{\mathfrak{R}}_{\varepsilon_k}-\widetilde{\mathfrak{R}}_{\varepsilon_{k+1}}\big)*\tilde f(x),$$
$$H_{\varepsilon_k,\varepsilon_{k+1}}(f)(x)=\big(\mathfrak{R}_{\varepsilon_k}-\mathfrak{R}_{\varepsilon_{k+1}}\big)*f(x).$$

For fixed $N\in\mathbb{Z}^+$, *define the error term*

$$E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde f)(x)=\Psi^N(x)\widetilde{H}_{\varepsilon_k,\varepsilon_{k+1}}(\tilde f)(x)-H_{\varepsilon_k,\varepsilon_{k+1}}(\Psi^N\tilde f)(x).$$

Let $1\le p<\infty$. *As* $N\to\infty$, *we have*:

(i) *For each fixed sequence* $(t_k)\searrow 0$,

$$\bigg\|\bigg(\sum_{k=1}^{\infty}\sup_{t_{k+1}\le\varepsilon_{k+1}<\varepsilon_k\le t_k}\big|E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde f)(x)\big|^2\bigg)^{\frac12}\bigg\|_{L^p(\mathbb{T})}=o(1).$$

(ii) *If* $\frac{1}{2^k}\le\varepsilon_{k+1}<\varepsilon_k\le\frac{1}{2^{k-1}}$, *then for* $\varrho>2$,

$$\bigg\|\sup_{(\varepsilon_k)\searrow 0}\bigg(\sum_{k=1}^{\infty}\big|E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde f)(x)\big|^{\varrho}\bigg)^{\frac1\varrho}\bigg\|_{L^p(\mathbb{T})}=o(1).$$

Proof. By the previous calculation (12), we know that the Fourier coefficients of $\widetilde{\mathfrak{R}}_\varepsilon(t)$ are

$$c_{l,\varepsilon}=\int_{\mathbb{R}}\mathfrak{R}(t)e^{-i\varepsilon 2\pi\ell t}\,dt=-i\int_{|t|>1}\frac{\sin(\varepsilon 2\pi\ell t)}{t}\,dt,\quad \ell\in\mathbb{Z}. \tag{14}$$

Note that

$$\int_{|t|<\frac{1}{2}} \widetilde{\mathfrak{R}}_{\varepsilon_k}(t)\tilde{f}(x-t)\,dt = \sum_{m=-\infty}^{\infty} a_m c_{m,\varepsilon_k} e^{2\pi i m x}.$$

For fixed $N \in \mathbb{Z}^+$, we have

$$\Psi(x/N)\widetilde{H}_{\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi i m x} \int_{\mathbb{R}} \widehat{\Psi}(u)(c_{m,\varepsilon_k} - c_{m,\varepsilon_{k+1}}) e^{2\pi u x/N}\,du.$$

Also, by a similar estimate as in Theorem 1.1, we obtain

$$H_{\varepsilon_k,\varepsilon_{k+1}}(\Psi^N \tilde{f})(x) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi i m x} \int_{\mathbb{R}} \widehat{\Psi}(u)(c_{m+u/N,\varepsilon_k} - c_{m+u/N,\varepsilon_{k+1}}) e^{2\pi u x/N}\,du.$$

Consequently,

$$E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi i m x} \\ \times \int_{\mathbb{R}} \widehat{\Psi}(u)((c_{m,\varepsilon_k} - c_{m,\varepsilon_{k+1}}) - (c_{m+u/N,\varepsilon_k} - c_{m+u/N,\varepsilon_{k+1}})) e^{2\pi u x/N}\,du.$$

In order to simplify the notation, we denote by $C_N(u)$ the term

$$(c_{m,\varepsilon_k} - c_{m,\varepsilon_{k+1}}) - (c_{m+u/N,\varepsilon_k} - c_{m+u/N,\varepsilon_{k+1}}).$$

To evaluate the inner integral above, we first deal with the term $C_N(u)$. From the second expression of (14),

$$C_N(u) = 2i \int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{\sin(2\pi m t) - \sin(2\pi(m+u/N)t)}{t}\,dt.$$

We consider two cases: $m = 0$ and $m \neq 0$.

If $m = 0$, one has

$$|C_N(u)| = \left| -2i \int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{\sin(2\pi t u/N)}{t}\,dt \right| \leq 4\pi |u| \frac{1}{N}(\varepsilon_k - \varepsilon_{k+1}). \tag{15}$$

If $m \neq 0$, it follows from trigonometric identities that

$$|C_N(u)| \leq 2 \int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{|\sin(\pi t u/N)|}{t}\,dt \leq 2\pi |u| \frac{1}{N}(\varepsilon_k - \varepsilon_{k+1}). \tag{16}$$

Applying the above estimates, we have

$$\Big\|\Big(\sum_{k=1}^{\infty}\sup_{t_{k+1}\le\varepsilon_{k+1}<\varepsilon_k\le t_k}\big|E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{T})} \preceq \frac{1}{N}\Big\|\Big(\sum_{k=1}^{\infty}(t_k-t_{k+1})^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{T})}\preceq\frac{1}{N}.$$

Therefore we obtain that, as $N\to\infty$,

$$\Big\|\Big(\sum_{k=1}^{\infty}\sup_{t_{k+1}\le\varepsilon_{k+1}<\varepsilon_k\le t_k}\big|E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{T})}=o(1).$$

Similarly, we have for $\varrho>2$,

$$\Big\|\sup_{(\varepsilon_k)\searrow 0}\Big(\sum_{k=1}^{\infty}\big|E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x)\big|^{\varrho}\Big)^{\frac{1}{\varrho}}\Big\|_{L^p(\mathbb{T})}$$
$$\preceq\frac{1}{N}\Big\|\sup_{(\varepsilon_k)\searrow 0}\Big(\sum_{k=1}^{\infty}(\varepsilon_k-\varepsilon_{k+1})^{\varrho}\Big)^{\frac{1}{\varrho}}\Big\|_{L^p(\mathbb{T})}\preceq\frac{1}{N}\Big\|\sup_{(\varepsilon_k)\searrow 0}\Big(\sum_{k=1}^{\infty}(\varepsilon_k-\varepsilon_{k+1})\Big)\Big\|_{L^p(\mathbb{T})}\preceq\frac{1}{N}.$$

Thus we obtain that, as $N\to\infty$,

$$\Big\|\sup_{(\varepsilon_k)\searrow 0}\Big(\sum_{k=1}^{\infty}\big|E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x)\big|^{\varrho}\Big)^{\frac{1}{\varrho}}\Big\|_{L^p(\mathbb{T})}=o(1). \qquad\square$$

Proof of Theorem 1.2. With the previous notation, by the definition of $\widetilde{H}_\varepsilon(\tilde{f})$ in (2), we rewrite $\widetilde{H}_{\varepsilon_k}\tilde{f}-\widetilde{H}_{\varepsilon_{k+1}}\tilde{f}$ as

$$\widetilde{H}_{\varepsilon_k}\tilde{f}(x)-\widetilde{H}_{\varepsilon_{k+1}}\tilde{f}(x)=\frac{1}{\pi}\int_{|t|<\frac{1}{2}}\tilde{f}(x-t)\big(\widetilde{\mathfrak{R}}_{\varepsilon_k}(t)-\widetilde{\mathfrak{R}}_{\varepsilon_{k+1}}(t)\big)\,dt$$
$$+\frac{1}{\pi}\int_{|t|<\frac{1}{2}}\tilde{f}(x-t)(r_{\varepsilon_{k+1}}(t)-r_{\varepsilon_k}(t))\,dt,$$

with the help of Lemma 2.1.

Recall again that χ_A is the characteristic function of the set $|t|>1$. It is easy to see that for $\varepsilon_{k+1}<\varepsilon_k<|t|\le\frac{1}{2}$, we have that for all $j\in\mathbb{Z}\setminus\{0\}$,

$$\chi_A\Big(\frac{t+j}{\varepsilon_{k+1}}\Big)=\chi_A\Big(\frac{t+j}{\varepsilon_k}\Big)=1.$$

This leads to

$$r_{\varepsilon_{k+1}}(t)-r_{\varepsilon_k}(t)=0.$$

It now suffices to consider

$$\mathbb{O}(\widetilde{H}_* \tilde{f})(x) = \Bigg(\sum_{k=1}^{\infty} \sup_{t_{k+1} \le \varepsilon_{k+1} < \varepsilon_k \le t_k} \bigg| \int_{|t|<\frac{1}{2}} \tilde{f}(x-t)\big(\widetilde{\mathfrak{R}}_{\varepsilon_k}(t) - \widetilde{\mathfrak{R}}_{\varepsilon_{k+1}}(t)\big)\, dt \bigg|^2 \Bigg)^{\frac{1}{2}}.$$

By (i) of Lemma 2.2, the basic properties of operators on the torus and Theorem C, we conclude that as $N \to \infty$,

$$\begin{aligned} \|\mathbb{O}(\widetilde{H}_* \tilde{f})\|^p_{L^p(\mathbb{T})} &= \frac{1}{N} \int_{-\frac{N}{2}}^{\frac{N}{2}} \Bigg(\sum_{k=1}^{\infty} \sup_{t_{k+1} \le \varepsilon_{k+1} < \varepsilon_k \le t_k} \Big| \Psi\Big(\frac{x}{N}\Big) \widetilde{H}_{\varepsilon_k, \varepsilon_k+1}(\tilde{f})(x) \Big|^2 \Bigg)^{\frac{p}{2}} dx \\ &\preceq o(1) + \frac{1}{N} \int_{-\frac{N}{2}}^{\frac{N}{2}} \Bigg(\sum_{k=1}^{\infty} \sup_{t_{k+1} \le \varepsilon_{k+1} < \varepsilon_k \le t_k} \big| H_{\varepsilon_i, \varepsilon_i+1}(\Psi^N \tilde{f})(x) \big|^2 \Bigg)^{\frac{p}{2}} dx \\ &\preceq o(1) + \frac{1}{N} \int_{\mathbb{R}} \Big| \Psi\Big(\frac{x}{N}\Big) \tilde{f}(x) \Big|^p dx \preceq \|\tilde{f}\|_{L^p(\mathbb{T})} + o(1). \end{aligned}$$

We next show that the oscillation operator $\mathscr{O}(\widetilde{H}_*)$ is of weak type (1, 1), that is, for any $\lambda > 0$,

$$\big|\{x \in \mathbb{T} : |\mathscr{O}(\widetilde{H}_* \tilde{f})(x)| > \lambda\}\big| \le \frac{c}{\lambda} \|\tilde{f}\|_{L^1(\mathbb{T})}.$$

By the basic properties of operators on the torus, we find that for $N \in \mathbb{Z}^+$,

$$\begin{aligned} \Big|\Big\{x \in \big[-\tfrac{1}{2}, \tfrac{1}{2}\big) : |\mathscr{O}(\widetilde{H}_* \tilde{f})(x)| > \lambda\Big\}\Big| &= N^{-1} \Big|\Big\{x \in \big[-\tfrac{N}{2}, \tfrac{N}{2}\big) : |\mathscr{O}(\widetilde{H}_* \tilde{f})(x)| > \lambda\Big\}\Big| \\ &= N^{-1} \Big|\Big\{|x| \le \tfrac{N}{2} : \big|\Psi\big(\tfrac{x}{N}\big) \mathscr{O}(\widetilde{H}_* \tilde{f})(x)\big| > \lambda\Big\}\Big|. \end{aligned}$$

As in the proofs of (15) and (16), we know that $E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x) \to 0$ uniformly in x as $N \to \infty$. For any λ_1 such that $0 < \lambda_1 < \lambda$, choose N large enough that

$$\Big|\Big\{x \in \big[-\tfrac{1}{2}, \tfrac{1}{2}\big) : |\mathscr{O}(\widetilde{H}_* \tilde{f})(x)| > \lambda\Big\}\Big| \le N^{-1} \big|\{x \in \mathbb{R} : |\mathscr{O}(H_*(\Psi^N \tilde{f}))(x)| > \lambda - \lambda_1\}\big|.$$

Theorem C implies that the last term above can be controlled by

$$\frac{cN^{-1}}{\lambda - \lambda_1} \|\Psi^N \tilde{f}\|_{L^1(\mathbb{R})} = \frac{cN^{-1}N}{\lambda - \lambda_1} \|\tilde{f}\|_{L^1(\mathbb{T})} = \frac{c}{\lambda - \lambda_1} \|\tilde{f}\|_{L^1(\mathbb{T})}.$$

Since $\lambda_1 > 0$ is arbitrary, we get the desired result. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Using the same argument as in Theorem 1.2, it is enough to study

$$\mathbb{V}_\varrho(\widetilde{H}_* \tilde{f})(x) = \sup_{(\varepsilon_k) \searrow 0} \Bigg(\sum_{k=1}^{\infty} \bigg| \int_{|t|<\frac{1}{2}} \tilde{f}(x-t)\big(\widetilde{\mathfrak{R}}_{\varepsilon_k}(t) - \widetilde{\mathfrak{R}}_{\varepsilon_{k+1}}(t)\big)\, dt \bigg|^\varrho \Bigg)^{\frac{1}{\varrho}}.$$

Now by checking the proof for the oscillation operator $\mathscr{O}(\widetilde{H}_*)$, it suffices to show

$$\|\mathbb{V}_\varrho(\widetilde{H}_* \tilde{f})\|_{L^p(\mathbb{T})} \preceq \|\tilde{f}\|_{L^p(\mathbb{T})}.$$

Write

$$H_{\varepsilon_k,\varepsilon_k+1}(f)(x) = H_{\varepsilon_k} f(x) - H_{\varepsilon_{k+1}} f(x),$$
$$\widetilde{H}_{\varepsilon_k,\varepsilon_k+1}(\tilde{f})(x) = \big(\widetilde{\mathfrak{R}}_{\varepsilon_k} - \widetilde{\mathfrak{R}}_{\varepsilon_{k+1}}\big) * \tilde{f}(x).$$

For any large integer N, we define the error term

$$E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x) = \Psi(x/N)\widetilde{H}_{\varepsilon_k,\varepsilon_k+1}(\tilde{f}) - H_{\varepsilon_k,\varepsilon_k+1}(\Psi^N \tilde{f})(x).$$

Using (ii) of Lemma 2.2, we obtain that as $N \to \infty$,

$$\Big\| \sup_{(\varepsilon_k)\searrow 0} \Big(\sum_{k=1}^{\infty} \big| E_{N,\varepsilon_k,\varepsilon_{k+1}}(\tilde{f})(x) \big|^{\varrho} \Big)^{\frac{1}{\varrho}} \Big\|_{L^p(\mathbb{T})} = o(1).$$

Finally, applying Theorem D, analogously to the proof of Theorem 1.2 we obtain

$$\begin{aligned}
\|\mathbb{V}_{\varrho}(\widetilde{H}_* \tilde{f})\|^p_{L^p(\mathbb{T})} &= \frac{1}{N} \int_{-\frac{N}{2}}^{\frac{N}{2}} \sup_{(\varepsilon_k)\searrow 0} \Big(\sum_{k=1}^{\infty} \Big| \Psi\Big(\frac{x}{N}\Big) \widetilde{H}_{\varepsilon_k,\varepsilon_k+1}(\tilde{f})(x) \Big|^{\varrho} \Big)^{\frac{p}{\varrho}} dx \\
&\preceq o(1) + \frac{1}{N} \int_{\mathbb{R}} \sup_{(\varepsilon_k)\searrow 0} \Big(\sum_{k=1}^{\infty} \big| H_{\varepsilon_k,\varepsilon_k+1}(\Psi^N \tilde{f})(x) \big|^{\varrho} \Big)^{\frac{p}{\varrho}} dx \\
&\preceq o(1) + \frac{1}{N} \int_{\mathbb{R}} \Big| \Psi\Big(\frac{x}{N}\Big) \tilde{f}(x) \Big|^p dx \preceq \|\tilde{f}\|_{L^p(\mathbb{T})} + o(1).
\end{aligned}$$

Letting $N \to \infty$, we conclude that the variation operator $\mathscr{V}_{\varrho}(\widetilde{H}_*)$ is of strong type (p, p) for $1 < p < \infty$.

The same argument as in the proof of Theorem 1.2 works for the weak type $(1, 1)$ for the variation operator $\mathscr{V}_{\varrho}(\widetilde{H}_*)$. We omit the details. □

Proof of Theorem 1.4. The proof of Theorem 1.4 is similar to that of Theorem 1.3. The only change is to consider two different cases: $p' > 2$ and $p' \leq 2$ in place of the symmetric differentiation operator used above. We leave the details to the interested reader. □

3. Extension to Riesz transforms

In this section we study the (maximal) bilinear Riesz transforms as n-dimensional extensions.

We start with the maximal bilinear singular integral with rough kernel

$$T^*_{\Omega}(f, g)(x) = \sup_{\varepsilon>0} |T_{\Omega,\varepsilon}(f, g)(x)|, \tag{17}$$

where $T_{\Omega,\varepsilon}$ is the truncated bilinear operator defined by

$$T_{\Omega,\varepsilon}(f, g)(x) = \int_{|y|>\varepsilon} \frac{f(x-y)g(x+y)}{|y|^n} \Omega(y')\, dy \quad \text{for } \varepsilon > 0.$$

Following the standard rotation method by Calderón and Zygmund (see also [Fan and Zhao ≥ 2016; Grafakos and Torres 2002]), we have the following result on $\mathbb{R}^n$.

Theorem 3.1. *Let* $1 < q, r \le \infty$, $1 \le p < \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *If* $\Omega \in L^\infty(S^{n-1})$ *is an odd function, then*

$$\|T^*_\Omega(f, g)\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}.$$

If $\Omega(x) = x_j/|x|$, $j = 1, 2, \ldots, n$, then (7) and (17) are reduced to the bilinear Riesz transforms and their maximal operators in Euclidean space $\mathbb{R}^n$:

$$R_j(f, g)(x) = C_n \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} f(x-y)g(x+y) \frac{y_j}{|y|^{n+1}}\, dy,$$

$$R^*_j(f, g)(x) = C_n \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} f(x-y)g(x+y) \frac{y_j}{|y|^{n+1}}\, dy \right|, \quad 1 \le j \le n,$$

where y_j is the j-th component of y and $C_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$.

Corollary 3.2. *Let* $1 < q, r \le \infty$, $1 \le p < \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *Then*

$$\|R^*_j(f, g)\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}.$$

As an application, we consider analogous operators on the n-dimensional torus $\mathbb{T}^n = \left[-\frac{1}{2}, \frac{1}{2}\right)^n$.

For $C^\infty(\mathbb{T}^n)$ functions $\tilde{f}$, $\tilde{g}$, write their Fourier series

$$\tilde{f}(x) = \sum_{k_1 \in \mathbb{Z}^n} a_{k_1} e^{2\pi i \langle k_1, x\rangle}, \quad \tilde{g}(x) = \sum_{k_2 \in \mathbb{Z}^n} b_{k_2} e^{2\pi i \langle k_2, x\rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product.

Let

$$Q = \left\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -\tfrac{1}{2} \le x_j < \tfrac{1}{2} \text{ for } j = 1, 2, \ldots, n\right\}$$

be the fundamental cube on which

$$\int_{\mathbb{T}^n} \tilde{f}(x)\, dx = \int_Q \tilde{f}(x)\, dx$$

for all functions $\tilde{f}$ on the torus $\mathbb{T}^n$. For $N \in \mathbb{Z}^+$, let NQ denote a cube with the same center as Q and side length N times the side length of Q. Denote by Q_ε the set given by

$$Q_\varepsilon = \{x \in Q : |x| > \varepsilon\} \quad \text{for } 0 < \varepsilon < \tfrac{1}{2}.$$

Let

$$E = \{x \in \mathbb{R}^n : |x| > 1\}$$

and $\chi_E(x)$ be the characteristic function of E. For $1 \le i \le n$, let x_i and m_i be the i-th components of $x = (x_1, \dots, x_n)$ and $m = (m_1, \dots, m_n)$, respectively. For any $x \neq 0$, the kernel of the j-th Riesz transform on $\mathbb{R}^n$ is

$$K_j(x) = \frac{x_j}{|x|^{n+1}}.$$

Then the kernel of the j-th Riesz transform on the torus is defined, in the sense of Cauchy principle value, by

$$\widetilde{K}_j(x) = \sum_{m \in \mathbb{Z}^n} \frac{x_j + m_j}{|x+m|^{n+1}}. \tag{18}$$

We now define the bilinear Riesz transform $\widetilde{R}_j$ and its maximal operator $\widetilde{R}_j^*$ on the torus $\mathbb{T}^n$, for $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{T}^n)$, by

$$\widetilde{R}_j(\tilde{f}, \tilde{g})(x) = \lim_{\varepsilon \to 0} \widetilde{R}_{j,\varepsilon}(\tilde{f}, \tilde{g})(x),$$
$$\widetilde{R}_j^*(\tilde{f}, \tilde{g})(x) = \sup_{0<\varepsilon<1/2} |\widetilde{R}_{j,\varepsilon}(\tilde{f}, \tilde{g})(x)|,$$

where $\widetilde{R}_{j,\varepsilon}$ is defined by

$$\begin{aligned}\widetilde{R}_{j,\varepsilon}(\tilde{f}, \tilde{g})(x) &= \int_{Q_\varepsilon} \widetilde{K}_j(y) \tilde{f}(x-y) \tilde{g}(x+y)\, dy \\ &= \int_Q \widetilde{K}_j(y) \chi_E\Big(\frac{y}{\varepsilon}\Big) \tilde{f}(x-y) \tilde{g}(x+y)\, dy.\end{aligned}$$

Our result can be stated as follows:

Theorem 3.3. *Let $1 < q, r \le \infty$, $1 \le p < \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then*

$$\|\widetilde{R}_j^*(\tilde{f}, \tilde{g})\|_{L^p(\mathbb{T}^n)} \preceq \|\tilde{f}\|_{L^q(\mathbb{T}^n)} \|\tilde{g}\|_{L^r(\mathbb{T}^n)}.$$

By checking the proof of Theorem 1.1, it suffices to show an easy lemma to obtain Theorem 3.3.

Lemma 3.4. *For $0 < \varepsilon < \frac{1}{2}$ and $y \in Q$, we have the estimate*

$$\widetilde{K}_j(y) \chi_E\Big(\frac{y}{\varepsilon}\Big) = \frac{1}{\varepsilon^n} \sum_{m \in \mathbb{Z}^n} K_j\Big(\frac{y+m}{\varepsilon}\Big) \chi_E\Big(\frac{y+m}{\varepsilon}\Big) - \sum_{m \in \mathbb{Z}^n \setminus \{0\}} K_j(y+m) \chi_{E^C}\Big(\frac{y}{\varepsilon}\Big)$$

and

$$\Big| \sum_{m \in \mathbb{Z}^n \setminus \{0\}} K_j(y+m) \chi_{E^C}\Big(\frac{y}{\varepsilon}\Big) \Big| \preceq |y| \chi_{E^C}\Big(\frac{y}{\varepsilon}\Big),$$

where E^C is the complement of the set E.

Proof. The first equality above follows the method of Lemma 2.1. Now we estimate the second inequality. Write

$$D_j^+ = \{m \in \mathbb{Z}^n \setminus \{0\} : m_j > 0\}, \quad D_j^0 = \{m \in \mathbb{Z}^n \setminus \{0\} : m_j = 0\},$$

and

$$y^* = (y_1, y_2, \dots, y_{j-1}, -y_j, y_{j+1}, \dots, y_n).$$

Then we have

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}^n \setminus \{0\}} \frac{y_j + m_j}{|y+m|^{n+1}} \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) \\
&\quad = \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) \sum_{m \in D_j^+} \Big(\frac{y_j + m_j}{|y+m|^{n+1}} + \frac{y_j - m_j}{|y^*+m|^{n+1}}\Big) + \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) \sum_{m \in D_j^0} \frac{y_j}{|y+m|^{n+1}} \\
&\quad = \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) y_j \sum_{m \in D_j^+} \Big(\frac{1}{|y+m|^{n+1}} + \frac{1}{|y^*+m|^{n+1}}\Big) \\
&\qquad + \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) \sum_{m \in D_j^+} m_j \Big(\frac{1}{|y+m|^{n+1}} - \frac{1}{|y^*+m|^{n+1}}\Big) \\
&\qquad + \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) y_j \sum_{m \in D_j^0} \Big(\frac{1}{|y+m|^{n+1}}\Big).
\end{aligned}$$

It is trivial to get that

$$\Big|\chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) \sum_{m \in D_j^0} \frac{y_j}{|y+m|^{n+1}}\Big| \preceq \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big)|y_j|$$

and

$$\Big|\chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) y_j \sum_{m \in D_j^+} \Big(\frac{1}{|y+m|^{n+1}} + \frac{1}{|y^*+m|^{n+1}}\Big)\Big| \preceq \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big)|y_j|.$$

Using the mean value theorem,

$$|f(x) - f(y)| \le \max_{z \in I} |\nabla f(z)|\,|x-y|,$$

where I is the line segment between x and y. This leads to

$$\Big|\chi_{E^c}\Big(\frac{y}{\varepsilon}\Big) \sum_{m \in D_j^+} m_j \Big(\frac{1}{|y+m|^{n+1}} - \frac{1}{|y^*+m|^{n+1}}\Big)\Big| \preceq \chi_{E^c}\Big(\frac{y}{\varepsilon}\Big)|y_j|,$$

completing the proof. □

From the theorem, we obtain the following corollary.

Corollary 3.5. *Let* $1 < q, r \le \infty$, $1 \le p < \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *Then*

$$\|\widetilde{R}_j(\tilde{f}, \tilde{g})\|_{L^p(\mathbb{T}^n)} \preceq \|\tilde{f}\|_{L^q(\mathbb{T}^n)} \|\tilde{g}\|_{L^r(\mathbb{T}^n)}.$$

This corollary corresponds to a result by Blasco and Gillespie [2009, Theorem 1.12] which says that the bilinear Riesz transform R_j is bounded from $L^q(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, provided $1 < q, r \le \infty$, $1 \le p < \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

4. Final remarks

We want to further illustrate that our method works for many operators. In this section, we provide another example. For $\varepsilon > 0$, define

$$R_{j,\varepsilon}(f)(x) = C_n \int_{|y|>\varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy \quad \text{for } j = 1, 2, \dots, n.$$

Gillespie and Torrea [2004] introduced the oscillation, variation and short variation operators of the Riesz transform R_j in $\mathbb{R}^n$. The definitions of these three operators can be expressed in forms similar to (4), (5) and (6) with H_ε replaced by $R_{j,\varepsilon}$ in place of the symmetric differentiation operator used above. Gillespie and Torrea also established the $L^p(\mathbb{R}^n)$-boundedness of these operators for $1 < p < \infty$.

For $C^\infty(\mathbb{T}^n)$ functions $\tilde{f}$, write their Fourier series

$$\tilde{f}(x) = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i \langle k, x\rangle}.$$

We define the periodic version of $\widetilde{R}_{j,\varepsilon}$ by

$$\widetilde{R}_{j,\varepsilon}(\tilde{f})(x) = \int_{Q_\varepsilon} \widetilde{K}_j(y) \tilde{f}(x-y)\, dy,$$

where $\widetilde{K}_j$ is defined as in (18).

We now define the oscillation operator $\mathscr{O}(\widetilde{R}_j)$ on the torus by

$$\mathscr{O}(\widetilde{R}_j \tilde{f})(x) = \left(\sum_{k=1}^{\infty} \sup_{t_{k+1} \le \varepsilon_{k+1} < \varepsilon_k \le t_k} \left| \widetilde{R}_{j,\varepsilon_k} \tilde{f}(x) - \widetilde{R}_{j,\varepsilon_{k+1}} \tilde{f}(x) \right|^2 \right)^{\frac{1}{2}}$$

and the variation operator $\mathscr{V}_\varrho(\widetilde{R}_j)$ on the torus by

$$\mathscr{V}_\varrho(\widetilde{R}_j \tilde{f})(x) = \sup_{(\varepsilon_k) \searrow 0} \left(\sum_{k=1}^{\infty} \left| \widetilde{R}_{j,\varepsilon_k} \tilde{f}(x) - \widetilde{R}_{j,\varepsilon_{k+1}} \tilde{f}(x) \right|^\varrho \right)^{\frac{1}{\varrho}}.$$

Define the operator $V_k(\widetilde{R}_j)$ on the torus by

$$V_k(\widetilde{R}_j \tilde{f})(x) = \sup_{(\varepsilon_j) \searrow 0} \left(\sum_{\frac{1}{2^k} < \varepsilon_{j+1} < \varepsilon_j \le \frac{1}{2^{k-1}}} \left| \widetilde{R}_{j,\varepsilon_k} \tilde{f}(x) - \widetilde{R}_{j,\varepsilon_{k+1}} \tilde{f}(x) \right|^2 \right)^{\frac{1}{2}},$$

where the supremum is taken over all decreasing sequences (ε_j). Define the "short variation operator" on the torus by

$$S_V(\widetilde{R}_j \tilde{f})(x) = \left(\sum_{k=-\infty}^{\infty} \left(V_k(\widetilde{R}_j \tilde{f}(x)) \right)^2 \right)^{\frac{1}{2}}.$$

Applying the same techniques as in the proof of Theorem 1.2, we can easily transfer those results in [Gillespie and Torrea 2004] from $\mathbb{R}^n$ to the torus $\mathbb{T}^n$.

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DASHAN FAN
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF WISCONSIN–MILWAUKEE
MILWAUKEE, WI 53201
UNITED STATES
fan@uwm.edu

HUOXIONG WU
SCHOOL OF MATHEMATICAL SCIENCES
XIAMEN UNIVERSITY
361005 FUJIAN
CHINA
huoxwu@xmu.edu.cn

FAYOU ZHAO
DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY
200444 SHANGHAI
CHINA
fyzhao@shu.edu.cn

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