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ON NONRADIAL SINGULAR SOLUTIONS OF SUPERCRITICAL BIHARMONIC EQUATIONS

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We develop a gluing method for fourth-order ODEs and construct infinitely many nonradial singular solutions for a biharmonic equation with supercritical exponent.

1. Introduction

In this paper we are concerned with positive singular solutions of the biharmonic equation

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n,\ n \geq 6, \tag{1-1}$$

where $p > (n+4)/(n-4)$.

Equation (1-1) arises in both physics and geometry. In recent decades there has been much research into classifying solutions to (1-1). When $1 < p \leq (n+4)/(n-4)$, all nonnegative solutions to (1-1) have been completely classified [Lin 1998; Wei and Xu 1999]: if $p < (n+4)/(n-4)$, then (1-1) admits no nontrivial nonnegative regular solution, while for $p = (n+4)/(n-4)$, i.e., the critical case, any positive regular solution of (1-1) can be written in the form

$$u_{\lambda,\xi} = \big(n(n-4)(n-2)(n+2)\big)^{-\frac{1}{8}(n-4)} \left(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\right)^{\frac{1}{2}(n-4)}, \quad \xi \in \mathbb{R}^n.$$

However, the question of the complete classification of positive regular solutions of (1-1) in the supercritical case, i.e., $p > (n+4)/(n-4)$, remains largely open.

The structure of positive radial solutions of (1-1) with $p > (n+4)/(n-4)$ has been studied by Gazzola and Grunau [2006] and Guo and Wei [2010]. For the fourth-order ODE

$$\begin{cases} \Delta^2 u(r) = u^p(r), & r \in [0,\infty), \\ u(0) = a,\ u''(0) = b,\ u'(0) = u'''(0) = 0, \end{cases} \tag{1-2}$$

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it is known from [Gazzola and Grunau 2006] that for any $a > 0$ there is a unique $b_0 := b_0(a) < 0$ such that the unique solution u_{a,b_0} of (1-2) satisfies $u_{a,b_0} \in C^4(0, \infty)$, $u'_{a,b_0}(r) < 0$ and

$$\lim_{r\to\infty} r^\alpha u_{a,b_0}(r) = K_0^{1/(p-1)},$$

where $\alpha = 4/(p-1)$ and

$$K_0 = \frac{8\big((n-2)(n-4)(p-1)^3+2(n^2-10n+20)(p-1)^2-16(n-4)(p-1)+32\big)}{(p-1)^4}.$$

This implies that $u_{a,b_0}(r) > 0$ for all $r > 0$ and $u_{a,b_0}(r) \to 0$ as $r \to \infty$. Moreover, it is known from [Guo and Wei 2010] that if $5 \le n \le 12$ or if $n \ge 13$ and $(n+4)/(n-4) < p < p_c(n)$, then $u_{a,b_0} - K_0^{1/(p-1)} r^{-\alpha}$ changes sign infinitely many times in $(0, \infty)$, and if $n \ge 13$ and $p \ge p_c(n)$, then $u(r) < K_0^{1/(p-1)} r^{-\alpha}$ for all $r > 0$ and the solutions are strictly ordered with respect to the initial value $a = u_{a,b_0}(0)$. Here $p_c(n)$ refers to the unique value of $p > (n+4)/(n-4)$ such that

$$p_c(n) = \begin{cases} +\infty & \text{if } 4 \le n \le 12, \\ \dfrac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \ge 13. \end{cases}$$

Very recently, Dávila, Dupaigne, Wang and Wei [Dávila et al. 2014] proved that all stable or finite Morse index solutions of (1-1) are trivial provided $1 < p < p_c(n)$. According to a result in [Guo and Wei 2010] and [Karageorgis 2009] all radial solutions are stable when $p \ge p_c(n)$. Thus the result in [Dávila et al. 2014] is sharp.

We now turn to the singular solutions of (1-1). It is easily seen that

$$u_s(x) := K_0^{1/(p-1)} |x|^{-4/(p-1)} \tag{1-3}$$

is a singular solution of (1-1). In other words, u_s satisfies the equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{1-4}$$

As far as we know, the radial singular solution in (1-3) is the only singular solution to (1-4) known so far. The question we shall address in this paper is whether or not there are nonradial singular solutions to (1-4). To this end, we first discuss the corresponding second-order Lane–Emden equation

$$\Delta u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^n, \tag{1-5}$$

which has been widely studied. We refer to [Budd and Norbury 1987; Bidaut-Véron and Véron 1991; Dancer et al. 2011; Farina 2007; Guo 2002; Gidas and Spruck 1981; Gui et al. 1992; Johnson et al. 1993; Joseph and Lundgren 1972/73; Korevaar et al. 1999; Zou 1995] and the references therein. Farina [2007] proved that if

$(n+2)/(n-2) < p < p^c(n)$, the Morse index of any regular solution u of (1-5) is ∞. Here $p^c(n)$ is the Joseph–Lundgren exponent [Joseph and Lundgren 1972/73]:

$$p^c(n) = \begin{cases} +\infty & \text{if } 2 \le n \le 10, \\ \dfrac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \ge 11. \end{cases}$$

In [Dancer et al. 2011], Dancer, Du and Guo showed that if Ω_0 is a bounded domain containing 0, then u is a solution of (1-5) in $\Omega_0 \setminus \{0\}$; if u has finite Morse index and $(n+2)/(n-2) < p < p^c(n)$, then $x = 0$ must be a removable singularity of u. They also showed that if Ω_0 is a bounded domain containing 0, u is a solution of (1-5) in $\mathbb{R}^n \setminus \Omega_0$ that has finite Morse index, and $(n+2)/(n-2) < p < p^c(n)$, then u must be a fast decay solution. It is easily seen that (1-5) has a radial singular solution

$$u^s(x) := u^s(r) = \left(\frac{2}{p-1} \left(n - 2 - \frac{2}{p-1} \right) \right)^{1/(p-1)} |x|^{-2/(p-1)}.$$

Recently, Dancer, Guo and Wei [Dancer et al. 2012] obtained infinitely many positive nonradial singular solutions of (1-5) provided $p \in ((n+1)/(n-3), p^c(n-1))$. The proof of that result is via a gluing of outer and inner solutions.

The main result in this paper is the following theorem.

Theorem 1.1. *Let $n \ge 6$. Assume that*

$$\frac{n+3}{n-5} < p < p_c(n-1).$$

Then (1-1) *admits infinitely many nonradial singular solutions.*

The proof of Theorem 1.1 is via a gluing of inner and outer solutions, as in [Dancer et al. 2012]. In the second-order case, one glues $(u(r), u'(r))$ at some intermediate point. However, since (1-1) is of fourth order, we have to match the inner solution and outer solution up to the third derivative $(u(r), u'(r), u''(r), u'''(r))$. Some essential obstructions appear when matching the inner and outer solutions. As far as we know this is the first paper on gluing inner and outer solutions for fourth-order ODE problems.

In the following, we sketch the proof of Theorem 1.1. After performing a separation of variables for a solution u of (1-1), $u(x) = r^{-\alpha} w(\theta)$, finding a nonradial singular solution of (1-1) is equivalent to finding a nonconstant solution of the equation

$$\Delta^2_{S^{n-1}} w + k_1(n) \Delta_{S^{n-1}} w + k_0(n) w = w^p, \tag{1-6}$$

where

$$k_0(n) = (n-4-\alpha)(n-2-\alpha)(2+\alpha)\alpha,$$
$$k_1(n) = -\big((n-4-\alpha)(2+\alpha)+(n-2-\alpha)\alpha\big).$$

It is clear that $w(\theta) = (k_0(n))^{1/(p-1)}$ is the constant solution of (1-6), which provides the radial singular solution of (1-1) that is given in (1-3).

In order to construct positive nonradial singular solutions of (1-1), we need to find positive nonconstant solutions of (1-6), which is a fourth-order inhomogeneous nonlinear ODE; therefore, we shall construct infinitely many positive nonconstant radially symmetric solutions of (1-6), i.e., solutions that only depend on the geodesic distance $\theta \in [0, \pi)$. We only consider the simple case $w(\theta) = w(\pi-\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. In this case, (1-6) can be written in the form

$$\begin{cases} T_1 w(\theta) + k_1(n) T_2 w(\theta) + k_0(n) w = w^p, & w(\theta) > 0,\ 0 < \theta < \frac{\pi}{2}, \\ w'(0),\, w'''(0) \text{ exist},\ w'\big(\frac{\pi}{2}\big) = w'''\big(\frac{\pi}{2}\big) = 0, \end{cases} \tag{1-7}$$

where T_1, T_2 are the differential operators defined by

$$T_1 w(\theta) = \frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\left(\sin^{n-2}\theta \frac{d}{d\theta}\left(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\left(\sin^{n-2}\theta \frac{dw(\theta)}{d\theta}\right)\right)\right)$$

and

$$T_2 w(\theta) = \frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\left(\sin^{n-2}\theta \frac{dw(\theta)}{d\theta}\right).$$

A key observation is that

$$w_*(\theta) = A_p(\sin\theta)^{-\alpha}, \quad \theta \in \big(0, \tfrac{\pi}{2}\big], \tag{1-8}$$

with

$$A_p^{p-1} = (n-5-\alpha)(n-3-\alpha)(2+\alpha)\alpha \ (:= k_0(n-1)),$$

is a singular solution of (1-7) with a singular point at $\theta = 0$. (Note that this is a singular solution in one dimension less.) We will construct the inner and outer solutions of (1-7) and glue them at some point close to 0, which gives solutions of (1-7). The main difficulty is the matching of four parameters, which correspond to matching u and its derivatives up to the third order.

This paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we construct inner solutions of (1-7) by studying an initial value problem of (1-7) with large initial values at $\theta = 0$. In Section 4, we construct outer solutions of (1-7). We first study an initial value problem of (1-7) with the initial values at $\theta = \frac{\pi}{2}$, then we analyze the asymptotic behaviors of the solutions of this initial value problem near $\theta = 0$. Finally, in Section 5, we match the inner and outer solutions constructed in Sections 3 and 4 to obtain solutions of (1-1). This completes the proof of Theorem 1.1. We leave some computational results to the Appendix.

2. Preliminaries

In this section, we present some known results which will be used subsequently.

Let $u = u(r)$ be a positive radial solution of (1-1). Using the Emden–Fowler transformation

$$u(r) = r^{-\alpha} v(t), \quad t = \ln r, \tag{2-1}$$

we see that $v(t)$ satisfies the equation

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t), \quad t \in (-\infty, \infty), \tag{2-2}$$

where the coefficients K_0, K_1, K_2, K_3 are given in [Gazzola and Grunau 2006]:

$$K_0 = \frac{8}{(p-1)^4}\big((n-2)(n-4)(p-1)^3 + 2(n^2-10n+20)(p-1)^2 - 16(n-4)(p-1) + 32\big),$$

$$K_1 = -\frac{2}{(p-1)^3}\big((n-2)(n-4)(p-1)^3 + 4(n^2-10n+20)(p-1)^2 - 48(n-4)(p-1) + 128\big),$$

$$K_2 = \frac{1}{(p-1)^2}\big((n^2-10n+20)(p-1)^2 - 24(n-4)(p-1) + 96\big),$$

$$K_3 = \frac{2}{p-1}\big((n-4)(p-1) - 8\big).$$

By direct calculation it is easy to see that $K_0 = k_0$. The characteristic polynomial (linearized at $K_0^{1/(p-1)}$) of (2-2) is

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + (1-p)K_0$$

and the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$

$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)},$$

where

$$N_1 := -(n-4)(p-1) + 8,$$

$$N_2 := (n^2 - 4n + 8)(p-1)^2,$$

$$N_3 := (9n-34)(n-2)(p-1)^4 + 8(3n-8)(n-6)(p-1)^3 + (16n^2 - 288n + 832)(p-1)^2 - 128(n-6)(p-1) + 256.$$

Let $\tilde{\nu}_j = \nu_j - \alpha$ for $j = 1, 2, 3, 4$.

Proposition 2.1 [Guo and Wei 2010]. *For any $n \geq 5$ and $p > (n+4)/(n-4)$,*

$$\tilde{\nu}_2 < 2-n < 0 < \tilde{\nu}_1. \tag{2-3}$$

(1) *For any $5 \leq n \leq 12$ or $n \geq 13$ and $(n+4)/(n-4) < p < p_c(n)$, we have $\tilde{\nu}_3, \tilde{\nu}_4 \notin \mathbb{R}$ and $\Re(\tilde{\nu}_3) = \Re(\tilde{\nu}_4) = \frac{1}{2}(4-n) < 0$.*

(2) *For any $n \geq 13$ and $p = p_c(n)$, we have $\tilde{\nu}_3 = \tilde{\nu}_4 = \frac{1}{2}(4-n)$.*

(3) *For any $n \geq 13$ and $p > p_c(n)$, we have*

$$\tilde{\nu}_2 < 4-n < \tilde{\nu}_4 < \tfrac{1}{2}(4-n) < \tilde{\nu}_3 < 0 < \tilde{\nu}_1, \quad \tilde{\nu}_3 + \tilde{\nu}_4 = 4-n. \tag{2-4}$$

Theorem 2.2 [Gazzola and Grunau 2006]. *For any $k \geq 1$,*

$$\lim_{t\to\infty} v(t) = K_0^{1/(p-1)}, \quad \lim_{t\to\infty} v^{(k)}(t) = 0 \tag{2-5}$$

Remark. We see that K_i $(i = 0, 1, 2, 3)$ and $\nu_j, \tilde{\nu}_j$ $(j = 1, 2, 3, 4)$ above depend on n and p. In the following, by abuse of notation, we use $K_i, \nu_j, \tilde{\nu}_j$ with the dimension n replaced by $n-1$ and write $k_0 = k_0(n)$ and $k_1 = k_1(n)$.

3. Inner solutions

In this section, we construct inner solutions of (1-7).

Let $Q \gg 1$ be a large constant and $\tilde{b}$ be a constant which will be given below. We consider the initial value problem

$$\begin{cases} T_1 w(\theta) + k_1 T_2 w(\theta) + k_0 w = w^p, \\ w(0) = Q,\ w'(0) = 0,\ w''(0) = (\tilde{b}+\mu) Q^{1+2/\alpha},\ w'''(0) = 0, \end{cases} \tag{3-1}$$

where $\mu > 0$ is a small constant. Since $Q \gg 1$, we set $Q = \epsilon^{-4/(p-1)}$ $(:= \epsilon^{-\alpha})$ with $\epsilon > 0$ sufficiently small.

Let $w(\theta) = \epsilon^{-\alpha} v(\theta/\epsilon)$. Then we have $v(0) = 1$, $v'(0) = 0$, $v''(0) = \tilde{b} + \mu$, $v'''(0) = 0$ and $v(r)$ (for $r = \theta/\epsilon$) satisfies the equation

$$\begin{aligned} v^{(4)}(r) &+ 2(n-2)\epsilon \cot(\epsilon r) v'''(r) \\ &+ \left((n-2)(n-4)\frac{\epsilon^2}{\sin^2(\epsilon r)} - (n-2)^2\epsilon^2 + k_1\epsilon^2\right) v'' \\ &+ \left((n-2)k_1\epsilon^3 \cot(\epsilon r) - (n-2)(n-4)\epsilon^3 \frac{\cot(\epsilon r)}{\sin^2(\epsilon r)}\right) v'(r) + k_0\epsilon^4 v(r) = v^p(r) \end{aligned} \tag{3-2}$$

with initial conditions

$$v(0) = 1, \quad v'(0) = 0, \quad v''(0) = \tilde{b} + \mu, \quad v'''(0) = 0.$$

For $\epsilon > 0$ sufficiently small, we have

$$\epsilon \cot(\epsilon r) = \frac{1}{r} - \tfrac{1}{3}\epsilon^2 r + \sum_{k=1}^{\infty} l_k \epsilon^{2k+2} r^{2k+1},$$

$$\epsilon^2 \sin^{-2}(\epsilon r) = \frac{1}{r^2} + \tfrac{1}{3}\epsilon^2 + \sum_{k=1}^{\infty} m_k \epsilon^{2k+2} r^{2k},$$

$$\epsilon^3 \cot(\epsilon r) \sin^{-2}(\epsilon r) = \frac{1}{r^3} + \sum_{k=1}^{\infty} n_k \epsilon^{2k+2} r^{2k-1}.$$

So (3-2) can be written in the form

(3-3)

$$v^{(4)}(r) + \left(\frac{2(n-2)}{r} - \tfrac{2}{3}(n-2)\epsilon^2 r + \sum_{k=1}^{\infty} l'_k \epsilon^{2k+2} r^{2k+1} \right) v'''(r)$$
$$+ \left(\frac{(n-2)(n-4)}{r^2} + \left(\tfrac{1}{3}(n-2)(n-4) - (n-2)^2 + k_1\right)\epsilon^2 + \sum_{k=1}^{\infty} m'_k \epsilon^{2k+2} r^{2k} \right) v''(r)$$
$$- \left(\frac{(n-2)(n-4)}{r^3} - (n-2)k_1 r^{-1}\epsilon^2 + \sum_{k=1}^{\infty} n'_k \epsilon^{2k+2} r^{2k-1} \right) v'(r) + k_0 \epsilon^4 v(r) = v^p(r)$$

with initial conditions

$$v(0) = 1, \quad v''(0) = \tilde{b} + \mu, \quad v'(0) = v'''(0) = 0.$$

The first approximation to the solution of (3-3) is the radial solution $v_0(r)$ of the problem

(3-4) $\quad \Delta^2 v = v^p \text{ in } \mathbb{R}^{n-1}, \quad v(0) = 1,\ v'(0) = 0,\ v''(0) = \tilde{b} + \mu,\ v'''(0) = 0.$

We write $v_0 = v_{01} + v_{02}$, where v_{01} satisfies

(3-5) $\quad \Delta^2 v = v^p, \quad v(0) = 1,\ v'(0) = 0,\ v''(0) = \tilde{b},\ v'''(0) = 0,$

and v_{02} satisfies

(3-6) $\quad \Delta^2 v = v_0^p - v_{01}^p, \quad v(0) = 0,\ v'(0) = 0,\ v''(0) = \mu,\ v'''(0) = 0.$

We now choose $\tilde{b} < 0$ to be the unique value such that the solution v_{01} is the unique positive radial ground state of (3-5).

Lemma 3.1. *Assume that $v_{01}(r)$ and $v_{02}(r)$ are the solutions to* (3-5) *and* (3-6), *respectively. For $(n+3)/(n-5) < p < p_c(n-1)$, there exists $R_0 \gg 1$ such that for $r \geq R_0$, the solution $v_{01}(r)$ satisfies*

(3-7) $$v_{01}(r) = A_p r^{-\alpha} + \frac{a_0 \cos(\beta \ln r) + b_0 \sin(\beta \ln r)}{r^{(n-5)/2}} + O(r^{2\sigma - \alpha}),$$

where $\beta = \sqrt{4\sqrt{N_3} - N_2}/(2(p-1))$ (*with* n *being replaced by* $n-1$ *in* N_2 *and* N_3) *and* $\sqrt{a_0^2 + b_0^2} \neq 0$.

The solution $v_{02}(r)$ *satisfies*

$$v_{02}(r) = \mu B_p r^{\tilde{\nu}_1} + O(\mu^2 r^{\nu_1 + \tilde{\nu}_1} + \mu r^{\tilde{\nu}_1 + \alpha - (n-5)/2}), \tag{3-8}$$

with $B_p \neq 0$ *when* $\mu = O(1/(r^{\nu_1 - \sigma}))$ *for* r *in any interval* $[e^T, e^{10T}]$ *with* $T \gg 1$ *and* $\sigma = \alpha - \frac{1}{2}(n-5)$.

Proof. The proof of this lemma is divided into two steps. We consider $v_{01}(r)$ in the first step. The main arguments in the proof are similar to those in the proof of Theorem 3.1 of [Guo 2014].

Using the Emden–Fowler transformation

$$v_{01}(r) = r^{-\alpha} v(t), \quad t = \ln r \quad (r > 0), \tag{3-9}$$

and letting $v(t) = A_p - h(t)$, we see that $h(t)$ satisfies

$$h^{(4)}(t) + K_3 h'''(t) + K_2 h''(t) + K_1 h'(t) + (1-p) K_0 h(t) + O(h^2) = 0 \tag{3-10}$$

for $t > 1$. Note that $r^\alpha v_{01}(r) \to A_p$ as $r \to \infty$ and hence $h(t) \to 0$ as $t \to \infty$. It follows from Proposition 2.1 that $\tilde{\nu}_3, \tilde{\nu}_4 \notin \mathbb{R}$ and $\Re(\tilde{\nu}_3) = \Re(\tilde{\nu}_4) = \frac{1}{2}(5-n) < 0$ and $\tilde{\nu}_2 < 3-n < 0 < \tilde{\nu}_1$ provided $(n+3)/(n-5) < p < p_c(n-1)$. Let $\nu_3 = \sigma + i\beta$, where $\beta = \sqrt{4\sqrt{N_3} - N_2}/(2(p-1))$ and $\sigma = -\frac{1}{2}(n-5) + \alpha < 0$ for $p > (n+3)/(n-5)$.

We can write (3-10) as

$$(\partial_t - \nu_4)(\partial_t - \nu_3)(\partial_t - \nu_2)(\partial_t - \nu_1) h(t) = H(h(t)), \tag{3-11}$$

where $H(h(t)) = O(h^2)$. We claim that for any $T \gg 1$, there exist constants A_i and B_i $(i = 1, 2, 3, 4)$ such that

$$\begin{aligned} h(t) = {} & A_1 e^{\sigma t} \cos \beta t + A_2 e^{\sigma t} \sin \beta t + A_3 e^{\nu_2 t} + A_4 e^{\nu_1 t} \\ & + B_1 \int_T^t e^{\sigma(t-s)} \sin \beta(t-s) H(h(s)) \, \mathrm{d}s + B_2 \int_T^t e^{\sigma(t-s)} \cos \beta(t-s) H(h(s)) \, \mathrm{d}s \\ & + B_3 \int_T^t e^{\nu_2(t-s)} H(h(s)) \, \mathrm{d}s + B_4 \int_T^t e^{\nu_1(t-s)} H(h(s)) \, \mathrm{d}s. \end{aligned}$$

Moreover, each A_i depends on T and ν_i $(i = 1, 2, 3, 4)$, while each B_i depends only on ν_i $(i = 1, 2, 3, 4)$. In fact, it follows from (3-11) and the theory of second-order ODEs (see [Hartman 1982]) that

$$\begin{aligned} & (\partial_t - \nu_2)(\partial_t - \nu_1) h(t) \\ & \qquad = A_1' e^{\sigma t} \cos \beta t + A_2' e^{\sigma t} \sin \beta t + \frac{1}{\beta} \int_T^t e^{\sigma(t-s)} \sin \beta(t-s) H(h(s)) \, \mathrm{d}s, \end{aligned} \tag{3-12}$$

where A_1' and A_2' are constants depending on T, ν_3 and ν_4. Multiplying both sides of (3-12) by $e^{-\nu_2 t}$ and integrating it from T to t, we obtain

$$(\partial_t - \nu_1)h(t) = A_3' e^{\nu_2 t} + \int_T^t e^{\nu_2(t-s)}(A_1' e^{\sigma s}\cos\beta s + A_2' e^{\sigma s}\sin\beta s)\,\mathrm{d}s \\ + \frac{1}{\beta}\int_T^t e^{\nu_2(t-s)}\int_T^s e^{\sigma(s-\xi)}\sin\beta(s-\xi)H(h(\xi))\,\mathrm{d}\xi\,\mathrm{d}s.$$

We now switch the order of integration and find that

$$(\partial_t - \nu_1)h(t) \\ = A_1'' e^{\sigma t}\cos\beta t + A_2'' e^{\sigma t}\sin\beta t + A_3'' e^{\nu_2 t} + B_1'\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\,\mathrm{d}s \\ + B_2'\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\,\mathrm{d}s + B_3'\int_T^t e^{\nu_2(t-s)}H(h(s))\,\mathrm{d}s,$$

where A_1'', A_2'' and A_3'' depend on T and ν_i $(i=2,3,4)$, and where the B_i' $(i=1,2,3)$ depend only on ν_i $(i=2,3,4)$. Repeating the same argument once again, we obtain our claim. Using the fact that $\int_T^t = \int_T^\infty - \int_t^\infty$, we have

$$B_4\int_T^t e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s = B_4\int_T^\infty e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s - B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s \\ = B_4 e^{\nu_1 t}\int_T^\infty e^{-\nu_1 s}H(h(s))\,\mathrm{d}s - B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s.$$

By combining $B_4 e^{\nu_1 t}\int_T^\infty e^{-\nu_1 s}H(h(s))\,\mathrm{d}s$ and $A_4 e^{\nu_1 t}$, we can also write $h(t)$ as

$$h(t) = A_1 e^{\sigma t}\cos\beta t + A_2 e^{\sigma t}\sin\beta t + A_3 e^{\nu_2 t} + M_4 e^{\nu_1 t} \\ + B_1\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\,\mathrm{d}s \\ + B_2\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\,\mathrm{d}s \\ + B_3\int_T^t e^{\nu_2(t-s)}H(h(s))\,\mathrm{d}s - B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s.$$

Since $h(t)\to 0$ as $t\to\infty$, we have $M_4 = 0$ (note $\nu_1 > 0$). Setting

$$h_1(t) = A_1 e^{\sigma t}\cos\beta t + A_2 e^{\sigma t}\sin\beta t + A_3 e^{\nu_2 t}$$

and

$$h_2(t) = B_1\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\,\mathrm{d}s + B_2\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\,\mathrm{d}s \\ + B_3\int_T^t e^{\nu_2(t-s)}H(h(s))\,\mathrm{d}s - B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s$$

and noting that $H(h(t)) = O(h^2(t))$, we see that

$$|h_2(t)| \le C(\tilde{h}_1(t) + \tilde{h}_2(t)), \tag{3-13}$$

where $C > 0$ is independent of T and

$$\tilde{h}_1(t) = \max\left\{\int_T^t e^{\sigma(t-s)}|h_1(s)|^2\,\mathrm{d}s,\ \int_T^t e^{\nu_2(t-s)}|h_1(s)|^2\,\mathrm{d}s,\ \int_t^\infty e^{\nu_1(t-s)}|h_1(s)|^2\,\mathrm{d}s\right\},$$

$$\tilde{h}_2(t) = \max\left\{\int_T^t e^{\sigma(t-s)}|h_2(s)|^2\,\mathrm{d}s,\ \int_T^t e^{\nu_2(t-s)}|h_2(s)|^2\,\mathrm{d}s,\ \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|^2\,\mathrm{d}s\right\}.$$

We now show

$$|h_2(t)| = o(e^{\sigma t}). \tag{3-14}$$

There are three cases to be considered:

(1) $|h_2(t)| \le \left(\tilde{h}_1(t) + \int_T^t e^{\sigma(t-s)}|h_2(s)|^2\,\mathrm{d}s\right)$,

(2) $|h_2(t)| \le C\left(\tilde{h}_1(t) + \int_T^t e^{\nu_2(t-s)}|h_2(s)|^2\,\mathrm{d}s\right)$,

(3) $|h_2(t)| \le C\left(\tilde{h}_1(t) + \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|^2\,\mathrm{d}s\right)$.

We only consider cases (1) and (3); case (2) is similar. For case (1), we have

$$|h_2(t)| \le C\left(\tilde{h}_1(t) + \int_T^t e^{\sigma(t-s)}|h_2(s)|^2\,\mathrm{d}s\right). \tag{3-15}$$

Thus,

$$|h_2(t)| \le C\left(\tilde{h}_1(t) + \max_{t\ge T}|h_2(t)|\int_T^t e^{\sigma(t-s)}|h_2(s)|\,\mathrm{d}s\right). \tag{3-16}$$

Let $m(t) = \int_T^t e^{-\sigma s}|h_2(s)|\,\mathrm{d}s$. Then it can be seen from (3-16) that

$$m'(t) \le C\tilde{h}_1(t)e^{-\sigma t} + C\max_{t\ge T}|h_2(t)|m(t). \tag{3-17}$$

For any $\epsilon > 0$ sufficiently small, we can choose T sufficiently large so that $0 < d_T := C\max_{t\ge T}|h_2(t)| < \epsilon$. It follows from (3-17) that

$$m(t) \le Ce^{d_T t}\int_T^t \tilde{h}_1(s)e^{-\sigma s}e^{-d_T s}\,\mathrm{d}s. \tag{3-18}$$

Substituting $m(t)$ in (3-18) into (3-16), we see that

$$|h_2(t)| \le C\tilde{h}_1(t) + Cd_T e^{(\sigma+d_T)t}\int_T^t \tilde{h}_1(s)e^{-\sigma s}e^{-d_T s}\,\mathrm{d}s. \tag{3-19}$$

Note that $\sigma + d_T < 0$ for T sufficiently large. We can combine $\nu_2 < \sigma$ with $h_1(t) = O(e^{\sigma t})$ to get $\tilde{h}_1(t) = o(e^{\sigma t})$. On the other hand, from (3-19) we can obtain that $|h_2(t)| = o(e^{(\sigma+d_T)t})$. Substituting these into (3-15), we eventually have

$$|h_2(t)| = o(e^{\sigma t}). \tag{3-20}$$

For case (3), we have

$$|h_2(t)| \le C\left(\tilde{h}_1(t) + \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|^2\,\mathrm{d}s\right). \tag{3-21}$$

Thus,

$$|h_2(t)| \le C\tilde{h}_1(t) + C\max_{t\ge T}|h_2(t)| \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|\,\mathrm{d}s. \tag{3-22}$$

Letting $l(t) = \int_t^\infty e^{-\nu_1 s}|h_2(s)|\,\mathrm{d}s$, we see from (3-22) that

$$-l'(t) \le C\tilde{h}_1(t)e^{-\nu_1 t} + d_T l(t). \tag{3-23}$$

It follows from (3-23) that

$$l(s) \le Ce^{-d_T t}\int_t^\infty \tilde{h}_1(s)e^{-\nu_1 s}e^{d_T s}\,\mathrm{d}s. \tag{3-24}$$

Since $\tilde{h}_1(t) = o(e^{\sigma t})$, we obtain from (3-24) that

$$l(s) = o(e^{(\sigma-\nu_1)t}).$$

Substituting this into (3-22), we also have

$$|h_2(t)| = o(e^{\sigma t}).$$

We now write $h(t)$ as

$$\begin{aligned} h(t) = M_1 e^{\sigma t}\cos\beta t + M_2 e^{\sigma t}\sin\beta t + A_3 e^{\nu_2 t} \\ - B_1\int_t^\infty e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\,\mathrm{d}s \\ - B_2\int_t^\infty e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\,\mathrm{d}s \\ + B_3\int_T^t e^{\nu_2(t-s)}H(h(s))\,\mathrm{d}s - B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\,\mathrm{d}s. \end{aligned}$$

Then, it follows from $H(h(t)) = O(h^2(t))$, $h_1(t) = O(e^{\sigma t})$, $h_2(t) = o(e^{\sigma t})$ and $\nu_2 < 2\sigma$ that

$$h(t) = M_1 e^{\sigma t}\cos(\beta t) + M_2 e^{\sigma t}\sin(\beta t) + A_3 e^{\nu_2 t} + O(e^{2\sigma t}). \tag{3-25}$$

This implies that (3-7) holds for some a_0 and b_0. By an argument similar to the one used in the proof of [Guo and Wei 2010, Theorem 3.3], we can show $a_0^2+b_0^2\neq 0$. This completes the proof of the first step.

We now proceed to the second step. Setting $v_{02}=\mu\tilde{v}_{02}$, we see that $\tilde{v}_{02}(r)$ satisfies

$$\Delta^2\tilde{v}_{02}-pv_{01}^{p-1}\tilde{v}_{02}=\mu^{-1}\big((v_{01}+\mu\tilde{v}_{02})^p-v_{01}^p-p\mu v_{01}^{p-1}\tilde{v}_{02}\big) \tag{3-26}$$

with initial conditions

$$\tilde{v}_{02}(0)=0,\quad \tilde{v}_{02}'(0)=0,\quad \tilde{v}_{02}''(0)=1,\quad \tilde{v}_{02}'''(0)=0.$$

Using the Emden–Fowler transformation

$$\tilde{v}_{02}(r)=r^{-\alpha}\hat{v}(t),\quad t=\ln r\quad (r>0),$$

and the expression obtained for $v_{01}(r)$, we see that $\hat{v}(t)$ satisfies

$$\hat{v}^{(4)}+K_3\hat{v}'''+K_2\hat{v}''+K_1\hat{v}'+(1-p)K_0\hat{v}=f(r,\mu,\hat{v}), \tag{3-27}$$

where

$$f(r,\mu,\hat{v})=O(\mu\hat{v}+r^{\alpha-(n-5)/2})\hat{v}$$

provided that $\mu\hat{v}=o(1)$ for t sufficiently large. It follows from (3-27) that

$$\begin{aligned}\hat{v}(t)=\hat{A}_1e^{\sigma t}\cos\beta t+\hat{A}_2e^{\sigma t}\sin\beta t+\hat{A}_3e^{\nu_2 t}+\hat{A}_4e^{\nu_1 t}\\ +\hat{B}_1\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)f(r,\mu,\hat{v}(s))\,\mathrm{d}s\\ +\hat{B}_2\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)f(r,\mu,\hat{v}(s))\,\mathrm{d}s\\ +\hat{B}_3\int_T^t e^{\nu_2(t-s)}f(r,\mu,\hat{v}(s))\,\mathrm{d}s+\hat{B}_4\int_T^t e^{\nu_1(t-s)}f(r,\mu,\hat{v}(s))\,\mathrm{d}s,\end{aligned}$$

where $\hat{A}_i=\hat{A}_i(T,\nu_1,\nu_2,\nu_3,\nu_4)$ $(i=1,2,3,4)$ and $\hat{B}_i=\hat{B}_i(\nu_1,\nu_2,\nu_3,\nu_4)$. We first show that $\tilde{v}_{02}$ is strictly increasing in $(0,\infty)$. Using the initial values, we can find $R\in(0,\infty)$ such that $\tilde{v}_{02}(r)>0$ for $r\in(0,R)$. Writing (3-26) as

$$\mu\Delta^2\tilde{v}_{02}=(v_{01}+\mu\tilde{v}_{02})^p-v_{01}^p,$$

we obtain that $(\Delta\tilde{v}_{02})'>0$, and hence $\Delta\tilde{v}_{02}>\Delta\tilde{v}_{02}(0)=n-1$ for $r\in(0,R)$, which implies that $(\tilde{v}_{02})'(r)>0$ for $r\in(0,R)$. Moreover, we can deduce that $R=\infty$ and $\tilde{v}_{02}'(r)>0$ for $r\in(0,\infty)$. Therefore, $\hat{v}$ is increasing in $(0,\infty)$. Next, we claim that $\hat{A}_4 \neq 0$ for any $T \gg 1$ sufficiently large. Indeed, for $t \in [T, 10T]$,

$$\begin{aligned} e^{-\nu_1 t}\hat{v}(t) &= \hat{A}_4 + \tilde{g}(t) + \hat{B}_1 e^{(\sigma-\nu_1)t}\int_T^t e^{-\sigma s}\sin\beta(t-s) f(r,\mu,\hat{v}(s))\,\mathrm{d}s \\ &\qquad + \hat{B}_2 e^{(\sigma-\nu_1)t}\int_T^t e^{-\sigma s}\cos\beta(t-s) f(r,\mu,\hat{v}(s))\,\mathrm{d}s \\ &\qquad + \hat{B}_3 e^{(\nu_2-\nu_1)t}\int_T^t e^{-\nu_2 s} f(r,\mu,\hat{v}(s))\,\mathrm{d}s + \hat{B}_4\int_T^t e^{-\nu_1 s} f(r,\mu,\hat{v}(s))\,\mathrm{d}s \\ &\le |\hat{A}_4| + |\tilde{g}(t)| + \Bigl(\sum_{j=1}^4 |\hat{B}_j|\Bigr)\max_{t\in[T,10T]}(\mu\hat{v} + e^{(\alpha-(n-5)/2)t})\int_T^t e^{-\nu_1 s}\hat{v}(s)\,\mathrm{d}s, \end{aligned}$$

where

$$\tilde{g}(t) = \hat{A}_1 e^{(\sigma-\nu_1)t}\cos\beta t + \hat{A}_2 e^{(\sigma-\nu_1)t}\sin\beta t + \hat{A}_3 e^{(\nu_2-\nu_1)t}.$$

Since

$$\Bigl(\sum_{j=1}^4 |\hat{B}_j|\Bigr)\max_{t\in[T,10T]}(\mu\hat{v} + e^{(\alpha-(n-5)/2)t}) = \tau = o(1),$$

we have

$$e^{-\nu_1 t}\hat{v}(t) \le |\hat{A}_4| + |\tilde{g}(t)| + \tau\int_T^t e^{-\nu_1 s}\hat{v}(s)\,\mathrm{d}s. \tag{3-28}$$

Let $\ell(t) = \int_T^t e^{-\nu_1 s}\hat{v}(s)\,\mathrm{d}s$. We see that

$$(e^{-\tau t}\ell(t))' \le (|\hat{A}_4| + |\tilde{g}(t)|)e^{-\tau t}. \tag{3-29}$$

Integrating (3-29) in $[T, t]$, we obtain

$$\ell(t) \le \frac{|\hat{A}_4| + \max_{t\in[T,10T]}|\tilde{g}(t)|}{\tau} e^{\tau(t-T)}.$$

If we choose $\tau(t-T) \le C$ for $t \in [T, 10T]$, i.e., $\tau = O(1/T)$, we see that

$$\ell(t) \le \frac{(|\hat{A}_4| + \max_{t\in[T,10T]}|\tilde{g}(t)|)C}{\tau}. \tag{3-30}$$

Substituting this into (3-28), we have

$$e^{-\nu_1 t}\hat{v}(t) \le |\hat{A}_4|(1+C) + |\tilde{g}(t)| + C\max_{t\in[T,10T]}|\tilde{g}(t)|. \tag{3-31}$$

Suppose $\hat{A}_4 = 0$. We see from (3-31) and the expression of $|\tilde{g}(t)|$ that

$$\hat{v}(t) = o(1) \quad \text{for all } t \in [T, 10T].$$

This contradicts the fact that $\hat{v}$ is increasing in $(0, \infty)$. Therefore, $\hat{A}_4 \neq 0$ and our claim holds. Moreover, it is known from (3-31) and the expression of $\hat{v}(t)$ that

$$\hat{v}(t) = B_p e^{\nu_1 t} + O(\mu e^{2\nu_1 t} + e^{(\sigma+\nu_1)t}) \tag{3-32}$$

with $B_p \neq 0$ and $\mu = O(e^{(-\nu_1+\sigma)t})$. Therefore,

$$v_{02}(r) = \mu B_p r^{\tilde{\nu}_1} + O(\mu^2 r^{\nu_1+\tilde{\nu}_1} + \mu r^{\tilde{\nu}_1+\sigma})$$

with $B_p \neq 0$ and $\mu = O(1/r^{\nu_1-\sigma})$. □

Lemma 3.2. *Let p satisfy the conditions of Lemma 3.1 and $v_1(r)$ be the unique solution of the equation*

$$\begin{cases} v_1^{(4)}(r) + \dfrac{2(n-2)}{r} v_1'''(r) + \dfrac{(n-2)(n-4)}{r^2} v_1''(r) - \dfrac{(n-2)(n-4)}{r^3} v_1'(r) \\ \qquad - \frac{2}{3}(n-2) r v_0'''(r) + \big(\frac{1}{3}(n-2)(n-4) - (n-2)^2 + k_1\big) v_0''(r) \\ \qquad\qquad + \dfrac{(n-2)k_1}{r} v_0'(r) = p v_0^{p-1}(r) v_1(r), \\ v_1(0) = 0,\ v_1'(0) = 0,\ v_1''(0) = 0,\ v_1'''(0) = 0. \end{cases} \tag{3-33}$$

Then for $r \in [e^T, e^{10T}]$ with $T \gg 1$ and $\mu = O(1/r^{\nu_1-\sigma})$,

$$\begin{aligned} v_1(r) = C_p r^{2-\alpha} &+ r^{2-(n-5)/2}(a_1 \cos(\beta \ln r) + b_1 \sin(\beta \ln r)) \\ &+ \mu D_p r^{2+\tilde{\nu}_1} + O(\mu^2 r^{\tilde{\nu}_1+\nu_1+2} + \mu r^{\tilde{\nu}_1+\sigma+2}) + o(r^{2-(n-5)/2}), \end{aligned} \tag{3-34}$$

where C_p satisfies

$$E_1 C_p - p A_p^{p-1} C_p = F_1 A_p, \tag{3-35}$$

with

$$\begin{aligned} E_1 &= (1+\alpha)(1-\alpha)(2-\alpha)\alpha - 2(n-2)(2-\alpha)(1-\alpha)\alpha \\ &\qquad - (n-2)(n-4)(2-\alpha) + (n-2)(n-4)(2-\alpha)(1-\alpha), \\ F_1 &= \big((n-2)^2 - k_1 - \tfrac{1}{3}(n-2)(n-4)\big)\alpha(\alpha+1) \\ &\qquad - \tfrac{2}{3}(n-2)\alpha(\alpha+1)(\alpha+2) + k_1(n-2)\alpha, \end{aligned}$$

and where D_p satisfies

$$E_2 D_p = F_2 B_p, \tag{3-36}$$

with

$$E_2 = (2+\tilde{\nu}_1)(\tilde{\nu}_1+n-1)(\tilde{\nu}_1+n-3)\tilde{\nu}_1 - pA_p^{p-1},$$
$$F_2 = \tfrac{2}{3}(n-2)(\tilde{\nu}_1-1)(\tilde{\nu}_1-2)\tilde{\nu}_1 + \big((n-2)^2 - k_1 - \tfrac{1}{3}(n-2)(n-4)\big)(\tilde{\nu}_1-1)\tilde{\nu}_1 - k_1(n-2)\tilde{\nu}_1 + p(p-1)A_p^{p-2}C_p,$$

and where (a_1, b_1) *is the solution of*

$$\begin{cases} Aa_1 - Bb_1 = G, \\ Ba_1 + Ab_1 = H, \end{cases}$$

with

$$A = \tfrac{1}{16}(n^4 - 12n^3 + 14n^2 + 132n - 135) - pA_p^{p-1} + \tfrac{1}{2}(n^2 - 6n - 35)\beta^2 + \beta^4,$$
$$B = (2n^2 - 12n - 6)\beta + 8\beta^3,$$
$$G = p(p-1)A_p^{p-2}C_p a_0 + \tfrac{1}{12}(n^4 - 11n^3 + 41n^2 - 61n + 30)a_0 + \tfrac{1}{4}(n^2 - 6n + 5)k_1 a_0 + \tfrac{1}{6}(4n^2 + 3n - n^3 - 14)b_0\beta - 2k_1 b_0\beta + \tfrac{1}{3}(n^2 - 9n + 14)a_0\beta^2 + a_0 k_1\beta^2 - \tfrac{2}{3}(n-2)b_0\beta^3,$$
$$H = p(p-1)A_p^{p-2}C_p b_0 + \tfrac{1}{12}(n^4 - 11n^3 + 41n^2 - 61n + 30)b_0 + \tfrac{1}{4}(n^2 - 6n + 5)k_1 b_0 - \tfrac{1}{6}(4n^2 + 3n - n^3 - 14)a_0\beta + 2k_1 a_0\beta + \tfrac{1}{3}(n^2 - 9n + 14)b_0\beta^2 + b_0 k_1\beta^2 + \tfrac{2}{3}(n-2)a_0\beta^3.$$

Remark. We need to show that $E_2 \neq 0$ and that the 2×2 matrix $K = \left[\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right]$ is invertible. This will be proved in the Appendix.

Proof. The uniqueness of solutions to (3-33) follows from standard ODE theory since all the initial conditions are zero and the inhomogeneous term is locally Lipschitz. Analyzing the terms which contain v_0 in (3-33) and using the Taylor expansion for v_0^{p-1} for $r \in [e^T, e^{10T}]$, after direct computation we can find the leading terms which are of the orders

$$r^{-2-\alpha}, \quad r^{(1-n)/2}\cos(\beta\ln r), \quad r^{(1-n)/2}\sin(\beta\ln r), \quad \mu r^{\tilde{\nu}_1-2}.$$

By the above observation, we can assume

$$v_1(r) = C_p r^{2-\alpha} + \tilde{f}(r) r^{2-(n-5)/2} + \mu D_p r^{2+\tilde{\nu}_1} + o(r^{2-(n-5)/2}) + O(\mu^2 r^{\tilde{\nu}_1+\nu_1+2} + \mu r^{\tilde{\nu}_1+\sigma+2}),$$

where

$$\tilde{f}(r) = a_1\cos(\beta\ln r) + b_1\sin(\beta\ln r).$$

Using (3-7) and (3-8), we can get C_p, D_p, a_1 and b_1 by direct calculation. □

Furthermore, we can obtain the following proposition.

Proposition 3.3. *Let*

$$\frac{n+3}{n-5} < p < p_c(n-1)$$

and $v(r)$ be a solution of (3-2). *Then for $\epsilon > 0$ sufficiently small,*

$$v(r) = v_0(r) + \sum_{k=1}^{\infty} \epsilon^{2k} v_k(r).$$

Moreover, for $r \in [e^T, e^{10T}]$ with $T \gg 1$ and $\mu = O(1/r^{\nu_1-\sigma})$,

$$v_k(r) = \sum_{j=1}^{k} d_j^k r^{2j-\alpha} + \sum_{j=1}^{k} e_j^k r^{2j-(n-5)/2} \sin(\beta \ln r + E_j^k) + \sum_{j=1}^{k} \mu f_j^k r^{2j+\tilde{\nu}_1} + O(\mu^2 r^{\tilde{\nu}_1+\nu_1+2k} + \mu r^{\tilde{\nu}_1+\sigma+2k}) + o(r^{2k-(n-5)/2}), \tag{3-37}$$

where $d_j^k, e_j^k, f_j^k, E_j^k$ $(j = 1, 2, \ldots, k)$ are constants. Moreover,

$$d_1^1 = C_p, \quad e_1^1 = \sqrt{a_1^2 + b_1^2}, \quad f_1^1 = D_p, \quad \sin E_1^1 = a_1/e_1^1, \quad \cos E_1^1 = b_1/e_1^1,$$

where C_p, a_1, b_1, D_p are given in Lemma 3.2.

Proof. Substituting

$$v(r) = v_0(r) + \sum_{i=1}^{\infty} \epsilon^{2i} v_i(r)$$

into (3-3), we expand (3-3) according to the order of ϵ. Considering the constant order and the ϵ^2 order, we get (3-4) and (3-33), respectively. We note that only the terms $v_0, v_1, \ldots, v_k$ carry ϵ^{2k}. Suppose we have found v_{k-1}. Then we can determine v_k by studying the equation of order ϵ^{2k} in (3-3), i.e.,

$$\begin{cases} v_k^{(4)}(r) + \dfrac{2(n-2)}{r} v_k'''(r) + \dfrac{(n-2)(n-4)}{r^2} v_k''(r) - \dfrac{(n-2)(n-4)}{r^3} v_k'(r) \\ \qquad - \frac{2}{3}(n-2) r v_{k-1}'''(r) + \left(\frac{1}{3}(n-2)(n-4) - (n-2)^2 + k_1\right) v_{k-1}''(r) \\ \qquad\qquad + \dfrac{(n-2)k_1}{r} v_{k-1}'(r) + \displaystyle\sum_{i=1}^{k-1} \bigl(l_i' r^{2i+1} v_{k-i-1}'''(r) + m_i' r^{2i} v_{k-i-1}''(r) \\ \qquad\qquad\qquad + n_i' r^{2i-1} v_{k-i-1}'(r) \bigr) + k_0 v_{k-1}(r) = \dfrac{d^k}{dt^k} \Bigl(\displaystyle\sum_{i=0}^{k} t^i v_i \Bigr)^p \Big|_{t=0}, \\ v_k(0) = 0, \ v_k'(0) = 0, \ v_k''(0) = 0, \ v_k'''(0) = 0, \end{cases}$$

where l_i', m_i', n_i' are given in (3-3). Following our arguments in Lemma 3.2, we find the leading order of the terms involving $v_0, v_1, \ldots, v_{k-1}$ in the above equation, and then we assume v_k has the expansion in (3-37). By substituting (3-37) into the equation of order ϵ^{2k} and comparing each order, we can compute the terms $d_j^k, e_j^k, f_j^k, E_j^k$ $(j = 1, 2, \ldots, k)$. □

Theorem 3.4. *Let*

$$\frac{n+3}{n-5} < p < p_c(n-1)$$

and $w^{\mathrm{inn}}_{\epsilon,\mu}(\theta)$ *be the solution of* (1-7) *with*

$$w(0) = \epsilon^{-\alpha}, \quad w_\theta(0) = 0, \quad w_{\theta\theta}(0) = (\tilde{b} + \mu)\epsilon^{-\alpha-2}, \quad w_{\theta\theta\theta}(0) = 0.$$

Then for any sufficiently small $\epsilon > 0$, $\theta/\epsilon \in [e^T, e^{10T}]$ *with* $T \gg 1$, *and* $\mu = O((\epsilon/\theta)^{\nu_1-\sigma})$, *there holds*

$$\begin{aligned}
&w^{\mathrm{inn}}_{\epsilon,\mu}(\theta)\\
&= \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + B_p\mu\epsilon^{-\nu_1}\theta^{\tilde{\nu}_1} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d_j^k \epsilon^{2(k-j)}\theta^{2j-\alpha}\\
&\quad + \epsilon^{(n-5)/2-\alpha}\Biggl(\frac{a_0\cos\bigl(\beta\ln\frac{\theta}{\epsilon}\bigr) + b_0\sin\bigl(\beta\ln\frac{\theta}{\epsilon}\bigr)}{\theta^{(n-5)/2}} + \frac{a_1\cos\bigl(\beta\ln\frac{\theta}{\epsilon}\bigr) + b_1\sin\bigl(\beta\ln\frac{\theta}{\epsilon}\bigr)}{\theta^{(n-5)/2-2}}\\
&\qquad + \sum_{k=2}^{\infty}\Bigl(\sum_{j=1}^{k} e_j^k\epsilon^{2(k-j)}\theta^{2j-(n-5)/2}\sin\bigl(\beta\ln\tfrac{\theta}{\epsilon} + E_j^k\bigr) + o(\theta^{2k-(n-5)/2})\Bigr)\\
&\qquad + O(\theta^{2-(n-5)/2})\Biggr)\\
&\quad + \epsilon^{-\alpha}\sum_{k=1}^{\infty}\Bigl(\sum_{j=1}^{k}(\mu f_j^k\epsilon^{2k-2j-\tilde{\nu}_1}\theta^{2j+\tilde{\nu}_1})\\
&\qquad + O(\mu^2\theta^{\tilde{\nu}_1+\nu_1+2k}\epsilon^{-\tilde{\nu}_1-\nu_1} + \mu\theta^{\tilde{\nu}_1+\sigma+2k}\epsilon^{-\tilde{\nu}_1-\sigma})\\
&\qquad + O\Bigl(\mu^2\Bigl(\frac{\theta}{\epsilon}\Bigr)^{\tilde{\nu}_1+\nu_1} + \mu\Bigl(\frac{\theta}{\epsilon}\Bigr)^{\tilde{\nu}_1+\sigma}\Bigr)\Bigr).
\end{aligned}$$

Proof. This is a direct consequence of Proposition 3.3 by setting $r = \theta/\epsilon$. □

We now obtain some useful lemmas.

Lemma 3.5. *Let* $(n+3)/(n-5) < p < p_c(n-1)$ *and*

$$v(Q, \mu, \theta) = Qv_0(Q^{(p-1)/4}\theta).$$

Then for $Q^{(p-1)/4}\theta \in [e^T, e^{10T}]$ *with* $T \gg 1$,

$$\mu = O\biggl(\frac{1}{(Q^{(p-1)/4}\theta)^{\nu_1-\sigma}}\biggr)$$

and $n = 0, 1, 2$, *we have that* $v(Q, \mu, \theta)$ *satisfies*

$$\begin{aligned}
&\frac{\partial^n}{\partial Q^n}(v(Q,\mu,\theta)) \\
&= \frac{\partial^n}{\partial Q^n}\left(\frac{A_p}{\theta^\alpha}\right) + \frac{\partial^n}{\partial Q^n}\big(C\theta^{-(n-5)/2} Q^{-((p-1)(n-5)/8-1)} \sin\big(\beta \ln(Q^{(p-1)/4}\theta) + \kappa\big)\big) \\
&\qquad + Q^{\tilde{\nu}_2/\alpha+1-n} O(\theta^{\tilde{\nu}_2}) + \mu B_p Q^{\tilde{\nu}_1/\alpha+1-n}\theta^{\tilde{\nu}_1} \\
&\qquad\qquad + O\big(\mu^2 Q^{(\tilde{\nu}_1+\nu_1)/\alpha+1-n}\theta^{\tilde{\nu}_1+\nu_1} + \mu Q^{(\tilde{\nu}_1+\sigma)/\alpha+1-n}\theta^{\sigma+\tilde{\nu}_1}\big),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^n}{\partial Q^n}(v'_\theta(Q,\mu,\theta)) \\
&= \frac{\partial^n}{\partial Q^n}\left(-\alpha\frac{A_p}{\theta^{\alpha+1}}\right) \\
&\quad + \frac{\partial^{n+1}}{\partial Q^n\partial\theta}\big(C\theta^{-(n-5)/2} Q^{-((p-1)(n-5)/8-1)} \sin\big(\beta \ln(Q^{(p-1)/4}\theta) + \kappa\big)\big) \\
&\qquad + Q^{\tilde{\nu}_2/\alpha+1-n} O(\theta^{\tilde{\nu}_2-1}) + \mu\tilde{\nu}_1 B_p Q^{\tilde{\nu}_1/\alpha+1-n}\theta^{\tilde{\nu}_1-1} \\
&\qquad\qquad + O\big(\mu^2 Q^{(\tilde{\nu}_1+\nu_1)/\alpha+1-n}\theta^{\tilde{\nu}_1+\nu_1-1} + \mu Q^{(\tilde{\nu}_1+\sigma)/\alpha+1-n}\theta^{\sigma+\tilde{\nu}_1-1}\big),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^n}{\partial Q^n}\left(\frac{\partial^2}{\partial\theta^2}v(Q,\mu,\theta)\right) \\
&= \frac{\partial^n}{\partial Q^n}\left(\alpha(\alpha+1)\frac{A_p}{\theta^{\alpha+2}}\right) \\
&\quad + \frac{\partial^{n+2}}{\partial Q^n\partial\theta^2}\big(C\theta^{-(n-5)/2} Q^{-((p-1)(n-5)/8-1)} \sin\big(\beta \ln(Q^{(p-1)/4}\theta) + \kappa\big)\big) \\
&\qquad + Q^{\tilde{\nu}_2/\alpha+1-n} O(\theta^{\tilde{\nu}_2-2}) + \mu\tilde{\nu}_1(\tilde{\nu}_1-1) B_p Q^{\tilde{\nu}_1/\alpha+1-n}\theta^{\tilde{\nu}_1-2} \\
&\qquad\qquad + O\big(\mu^2 Q^{(\tilde{\nu}_1+\nu_1)/\alpha+1-n}\theta^{\tilde{\nu}_1+\nu_1-2} + \mu Q^{(\tilde{\nu}_1+\sigma)/\alpha+1-n}\theta^{\sigma+\tilde{\nu}_1-2}\big),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^n}{\partial Q^n}\left(\frac{\partial^3}{\partial\theta^3}v(Q,\mu,\theta)\right) \\
&= \frac{\partial^n}{\partial Q^n}\left(-\alpha(\alpha+1)(\alpha+2)\frac{A_p}{\theta^{\alpha+3}}\right) \\
&\quad + \frac{\partial^{n+3}}{\partial Q^n\partial\theta^3}\big(C\theta^{-(n-5)/2} Q^{-((p-1)(n-5)/8-1)} \sin\big(\beta \ln(Q^{(p-1)/4}\theta) + \kappa\big)\big) \\
&\qquad + Q^{\tilde{\nu}_2/\alpha+1-n} O(\theta^{\tilde{\nu}_2-3}) + \mu\tilde{\nu}_1(\tilde{\nu}_1-1)(\tilde{\nu}_1-2) B_p Q^{\tilde{\nu}_1/\alpha+1-n}\theta^{\tilde{\nu}_1-3} \\
&\qquad\qquad + O\big(\mu^2 Q^{(\tilde{\nu}_1+\nu_1)/\alpha+1-n}\theta^{\tilde{\nu}_1+\nu_1-3} + \mu Q^{(\tilde{\nu}_1+\sigma)/\alpha+1-n}\theta^{\sigma+\tilde{\nu}_1-3}\big),
\end{aligned}$$

where $\kappa = \tan^{-1}(b_0/a_0)$ *and* $C = \sqrt{a_0^2 + b_0^2}$.

For $n = 0, 1$, we have

$$\frac{\partial^n}{\partial\mu^n}(v(Q,\mu,\theta)) = \mu^{1-n} B_p Q^{\tilde\nu_1/\alpha+1}\theta^{\tilde\nu_1} + O\big(\mu^{2-n} Q^{(\tilde\nu_1+\nu_1)/\alpha+1}\theta^{\tilde\nu_1+\nu_1} + \mu^{1-n} Q^{(\tilde\nu_1+\sigma)/\alpha+1}\theta^{\sigma+\tilde\nu_1}\big),$$

$$\frac{\partial^n}{\partial\mu^n}\Big(\frac{\partial}{\partial\theta} v(Q,\mu,\theta)\Big) = \mu^{1-n}\tilde\nu_1 B_p Q^{\tilde\nu_1/\alpha+1}\theta^{\tilde\nu_1-1} + O\big(\mu^{2-n} Q^{(\tilde\nu_1+\nu_1)/\alpha+1}\theta^{\tilde\nu_1+\nu_1-1} + \mu^{1-n} Q^{(\tilde\nu_1+\sigma)/\alpha+1}\theta^{\sigma+\tilde\nu_1-1}\big),$$

$$\frac{\partial^n}{\partial\mu^n}\Big(\frac{\partial^2}{\partial\theta^2} v(Q,\mu,\theta)\Big) = \mu^{1-n}\tilde\nu_1(\tilde\nu_1-1) B_p Q^{\tilde\nu_1/\alpha+1}\theta^{\tilde\nu_1-2} + O\big(\mu^{2-n} Q^{(\tilde\nu_1+\nu_1)/\alpha+1}\theta^{\tilde\nu_1+\nu_1-2} + \mu^{1-n} Q^{(\tilde\nu_1+\sigma)/\alpha+1}\theta^{\sigma+\tilde\nu_1-2}\big),$$

$$\frac{\partial^n}{\partial\mu^n}\Big(\frac{\partial^3}{\partial\theta^2} v(Q,\mu,\theta)\Big) = \mu^{1-n}\tilde\nu_1(\tilde\nu_1-1)(\tilde\nu_1-2) B_p Q^{\tilde\nu_1/\alpha+1}\theta^{\tilde\nu_1-3} + O\big(\mu^{2-n} Q^{(\tilde\nu_1+\nu_1)/\alpha+1}\theta^{\tilde\nu_1+\nu_1-3} + \mu^{1-n} Q^{(\tilde\nu_1+\sigma)/\alpha+1}\theta^{\sigma+\tilde\nu_1-3}\big),$$

while for $n = 2$, we have

$$\frac{\partial^2}{\partial\mu^2}\Big(\frac{\partial^m}{\partial\theta^m} v(Q,\mu,\theta)\Big) = O(Q^{(\tilde\nu_1+\nu_1)/\alpha+1}\theta^{\tilde\nu_1+\nu_1-m}), \quad m = 0, 1, 2, 3.$$

Proof. These estimates are obtained by the expansions of $v_{01}(r)$ and $v_{02}(r)$ given above and direct calculation. □

Lemma 3.6. *In the region*

$$\theta = |O(Q^{\sigma/((2-\sigma)\alpha)})|, \quad \mu = O(\theta^{2-2\nu_1/\sigma}), \quad \sigma = -\tfrac{1}{2}(n-5-2\alpha),$$

the solution $w(Q,\mu,\theta)$ of (1-7) *with*

$$w(Q,\mu,0) = Q, \qquad w_\theta(Q,\mu,0) = 0,$$
$$w_{\theta\theta}(Q,\mu,0) = (\tilde b+\mu)Q^{1+2/\alpha}, \quad w_{\theta\theta\theta}(Q,\mu,0) = 0$$

satisfies

$$(1)\quad \left|\frac{\partial^{m+n}}{\partial Q^n\partial\theta^m} w(Q,\mu,\theta) - \frac{\partial^{m+n}}{\partial Q^n\partial\theta^m} v(Q,\mu,\theta)\right| = Q^{-(n-5)(p-1)/8-(n-1)}\big|o(\theta^{-(n-5)/2-m})\big|,$$

(2) $$\left|\frac{\partial^{m+n}}{\partial\mu^n\partial\theta^m}w(Q,\mu,\theta)-\frac{\partial^{n+m}}{\partial\mu^n\partial\theta^m}v(Q,\mu,\theta)\right| = \left|O(\mu^{2-n}Q^{(\tilde\nu_1+\nu_1)/\alpha+1}\theta^{\tilde\nu_1+\nu_1-m})\right|.$$

Proof. This lemma can be obtained from Lemma 3.5 and Theorem 3.4. Note that

$$\epsilon = Q^{-1/\alpha},\quad \sigma/\alpha=\tfrac{1}{8}(p-1)(n-5)-1.$$

Moreover,

$$Q^{(p-1)/4}\theta\in[e^T,e^{10T}]$$

provided that Q is sufficiently large. □

Now we write the inner solutions obtained in Theorem 3.4 in terms of the parameters Q and μ.

Theorem 3.7. *Let* $(n+3)/(n-5)<p<p_c(n-1)$ *and let* $w^{\text{inn}}_{Q,\mu}(\theta)$ *be an inner solution of problem* (1-7) *with* $w(0)=Q,\ w_\theta(0)=0,\ w_{\theta\theta}(0)=(\tilde b+\mu)Q^{1+2/\alpha}$, $w_{\theta\theta\theta}(0)=0$. *Then for any sufficiently large* $Q>0$ *and* $\theta=|O(Q^{\sigma/((2-\sigma)\alpha)})|=|O(\mu^{\sigma/(2\sigma-2\nu_1)})|$,

$$\begin{aligned} w^{\text{inn}}_{Q,\mu}(\theta)=\frac{A_p}{\theta^\alpha}+\frac{C_p}{\theta^{\alpha-2}}+B_p\mu Q^{\nu_1/\alpha}\theta^{\tilde\nu_1}+\sum_{k=2}^{\infty}\sum_{j=1}^{k}d_j^kQ^{-(p-1)(k-j)/2}\theta^{2j-\alpha}\\ +Q^{\sigma/\alpha}\Biggl(\frac{a_0\cos\bigl(\beta\ln(Q^{(p-1)/4}\theta)\bigr)+b_0\sin\bigl(\beta\ln(Q^{(p-1)/4}\theta)\bigr)}{\theta^{(n-5)/2}}\\ +\frac{a_1\cos\bigl(\beta\ln(Q^{(p-1)/4}\theta)\bigr)+b_1\sin\bigl(\beta\ln(Q^{(p-1)/4}\theta)\bigr)}{\theta^{(n-5)/2-2}}\\ +O(\theta^{2-(n-5)/2})\\ +\sum_{k=2}^{\infty}\Biggl(\sum_{j=1}^{k}e_j^kQ^{-(p-1)(k-j)/2}\theta^{2j-(n-5)/2}\\ \times\sin\bigl(\beta\ln(Q^{(p-1)/4}\theta)+E_j^k\bigr)+o(\theta^{2k-(n-5)/2})\Biggr)\Biggr)\\ +Q\sum_{k=1}^{\infty}\Biggl(\sum_{j=1}^{k}\bigl(\mu f_j^kQ^{-(2k-2j-\tilde\nu_1)/\alpha}\theta^{2j+\tilde\nu_1}\bigr)\\ +O\bigl(\mu^2Q^{(\tilde\nu_1+\nu_1)/\alpha}\theta^{\tilde\nu_1+\nu_1+2k}+\mu Q^{(\tilde\nu_1+\sigma)/\alpha}\theta^{\tilde\nu_1+\sigma+2k}\bigr)\Biggr).\end{aligned}$$

4. Outer solutions

In this section, we construct outer solutions for (1-7). Let $w_*(\theta)$ be the singular solution given in (1-8).

Lemma 4.1. *The equation*

(4-1) $$T_1\phi(\theta)+k_1T_2\phi(\theta)+k_0\phi=pw_*^{p-1}(\theta)\phi(\theta),\quad 0<\theta<\tfrac{\pi}{2},$$

admits a solution, which can be written as

$$\phi(\theta)=\theta^{-(n-5)/2}\big(c_1\cos\big(\beta\ln\tfrac{\theta}{2}\big)+c_2\sin\big(\beta\ln\tfrac{\theta}{2}\big)\big)+O(\theta^{2-(n-5)/2})\quad \text{as } \theta\to 0, \tag{4-2}$$

where c_1, c_2 *are constants such that* $c_1^2+c_2^2\neq 0$, *and also admits another solution, which can be written as*

$$\psi(\theta)=c_0\theta^{\tilde{\nu}_2}+O(\theta^{\tilde{\nu}_2+2})\quad \text{as } \theta\to 0, \tag{4-3}$$

where c_0 *is a nonzero constant. Here* T_1 *and* T_2 *are differential operators defined in* (1-7).

Proof. For the equations

$$\begin{cases} T_1\phi_1(\theta)+k_1T_2\phi_1(\theta)+k_0\phi_1(\theta)=pw_*^{p-1}(\theta)\phi_1(\theta), & 0<\theta<\frac{\pi}{2},\\ \phi_1\big(\frac{\pi}{2}\big)=1,\ \phi_1'\big(\frac{\pi}{2}\big)=0,\ \phi_1''\big(\frac{\pi}{2}\big)=0,\ \phi_1'''\big(\frac{\pi}{2}\big)=0, \end{cases} \tag{4-4}$$

and

$$\begin{cases} T_1\phi_2(\theta)+k_1T_2\phi_2(\theta)+k_0\phi_2(\theta)=pw_*^{p-1}(\theta)\phi_2(\theta), & 0<\theta<\frac{\pi}{2},\\ \phi_2\big(\frac{\pi}{2}\big)=0,\ \phi_2'\big(\frac{\pi}{2}\big)=0,\ \phi_2''\big(\frac{\pi}{2}\big)=1,\ \phi_2'''\big(\frac{\pi}{2}\big)=0, \end{cases} \tag{4-5}$$

we claim that both $\phi_1(\theta)$ and $\phi_2(\theta)$ are strictly decreasing for $\theta\in\big(0,\frac{\pi}{2}\big)$. We only show the case of $\phi_2(\theta)$; the case of $\phi_1(\theta)$ can be treated similarly.

Let us set

$$A(\theta)=\frac{d}{d\theta}\left(\sin^{n-2}\theta\frac{d\phi_2(\theta)}{d\theta}\right).$$

Before proving that $\phi_2(\theta)$ is decreasing, we first present a useful fact. If $A(\theta)>0$ for $\theta\in\big(\theta_0,\frac{\pi}{2}\big)$, where $\theta_0\in\big(0,\frac{\pi}{2}\big)$, then for $\theta\in\big(\theta_0,\frac{\pi}{2}\big)$, we have $\phi_2'(\theta)<0$ and $\phi_2(\theta)>0$. The proof of this fact is simple; thus we omit it here. Next, we show that $\phi_2(\theta)$ is decreasing. By using the boundary condition of ϕ_2 at $\theta=\frac{\pi}{2}$, we have $A\big(\frac{\pi}{2}\big)=1$ and find $\theta_1\in\big(0,\frac{\pi}{2}\big)$ such that $A(\theta)>0$ for $\theta\in\big(\theta_1,\frac{\pi}{2}\big)$; then $\phi_2(\theta)>0$ for $\theta\in\big(\theta_1,\frac{\pi}{2}\big)$. Using the fact that $k_1(n)<0$ and the second conclusion in Lemma A.1, we have

$$T_1\phi_2(\theta)=(pw_*^{p-1}-k_0)\phi_2(\theta)-k_1\frac{A(\theta)}{\sin^{n-2}\theta}>0\quad \text{for } \theta\in\big(\theta_1,\tfrac{\pi}{2}\big).$$

Now we are going to show that $\theta_1=0$. If not, $\theta_1\in\big(0,\frac{\pi}{2}\big)$ and $A(\theta_1)=0$. For $\theta\in\big(\theta_1,\frac{\pi}{2}\big)$, we have

$$\frac{d}{d\theta}\left(\sin^{n-2}\theta\frac{d}{d\theta}\left(\frac{A(\theta)}{\sin^{n-2}\theta}\right)\right)>0.$$

Using this inequality and

$$\frac{d}{d\theta}\left(\frac{A(\theta)}{\sin^{n-2}\theta}\right)\bigg|_{\theta=\frac{\pi}{2}}=0,$$

we have

$$\frac{d}{d\theta}\left(\frac{A(\theta)}{\sin^{n-2}\theta}\right) < 0 \quad \text{for } \theta \in \left(\theta_1, \tfrac{\pi}{2}\right). \tag{4-6}$$

It follows from (4-6) that

$$\frac{A(\theta)}{\sin^{n-2}\theta} > 1 \quad \text{for } \theta \in \left(\theta_1, \tfrac{\pi}{2}\right), \tag{4-7}$$

which contradicts the fact that $A(\theta_1) = 0$. Thus, $A(\theta) > 0$ and $\phi_2'(\theta) < 0$ for $\theta \in \left(0, \frac{\pi}{2}\right)$. Hence, we have proved the claim.

We now prove that there are $D_1 \neq 0$ and $D_2 \neq 0$ such that for θ near 0,

$$\phi_1(\theta) = D_1\theta^{\tilde{\nu}_2} + O(\theta^{2+\tilde{\nu}_2}) \tag{4-8}$$

and

$$\phi_2(\theta) = D_2\theta^{\tilde{\nu}_2} + O(\theta^{2+\tilde{\nu}_2}). \tag{4-9}$$

We only show (4-9). The proof of (4-8) is similar. Using the Emden–Fowler transformation

$$\tilde{\phi}(t) = (\sin\theta)^{\alpha}\phi_2(\theta), \quad t = \ln\left(\tan\tfrac{\theta}{2}\right),$$

we obtain that $\tilde{\phi}(t)$, for $t \in (-\infty, 0)$, satisfies the homogeneous equation

$$\tilde{\phi}^{(4)}(t) + a_3(t)\tilde{\phi}'''(t) + a_2(t)\tilde{\phi}''(t) + a_1(t)\tilde{\phi}'(t) + a_0(t)\tilde{\phi}(t) = 0, \tag{4-10}$$

where

$$\begin{aligned} a_3(t) &= K_3 + O(e^{2t}), \quad a_2(t) = K_2 + O(e^{2t}), \\ a_1(t) &= K_1 + O(e^{2t}), \quad a_0(t) = (1-p)K_0. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{\phi}^{(4)}(t) + K_3\tilde{\phi}'''(t) + K_2\tilde{\phi}''(t) + K_1\tilde{\phi}'(t) + (1-p)K_0\tilde{\phi}(t) \\ = O(e^{2t}(\tilde{\phi}'''(t) + \tilde{\phi}''(t) + \tilde{\phi}'(t))). \end{aligned} \tag{4-11}$$

Following the arguments in the proof of Lemma 3.1, we can write the solutions of (4-11) as (for any $T \ll -1$):

$$\begin{aligned} \tilde{\phi}(t) = {} & A_5 e^{\sigma t}\cos\beta t + A_6 e^{\sigma t}\sin\beta t + A_7 e^{\nu_2 t} + A_8 e^{\nu_1 t} \\ & + B_5\int_{-\infty}^{t} e^{\sigma(t-s)}\sin\beta(t-s)g(s, \tilde{\phi}(s))\,\mathrm{d}s \\ & + B_6\int_{-\infty}^{t} e^{\sigma(t-s)}\cos\beta(t-s)g(s, \tilde{\phi}(s))\,\mathrm{d}s \\ & + B_7\int_{-\infty}^{t} e^{\nu_2(t-s)}g(s, \tilde{\phi}(s))\,\mathrm{d}s + B_8\int_{T}^{t} e^{\nu_1(t-s)}g(s, \tilde{\phi}(s))\,\mathrm{d}s, \end{aligned} \tag{4-12}$$

where $g(t,\tilde{\phi}(t))$ is the right-hand side of (4-11), A_8 depends on T and each B_{i+4} depends only on ν_i $(i=1,2,3,4)$. It is known from (4-12) that if $A_7=0$, then for $|t|$ sufficiently large,

$$\tilde{\phi}(t)=A_5e^{\sigma t}\cos\beta t+A_6e^{\sigma t}\sin\beta t+O(e^{(2+\sigma)t}) \tag{4-13}$$

with $A_5^2+A_6^2\neq 0$ or

$$\tilde{\phi}(t)=A_8e^{\nu_1 t}+O(e^{(2+\nu_1)t}) \tag{4-14}$$

with $A_8\neq 0$. Otherwise, if $A_5^2+A_6^2=0$ and $A_8=0$, we know that $\tilde{\phi}(t)=O(e^{(2+\nu_1)t})$. Substituting this into (4-12), we see that $\tilde{\phi}(t)=O(e^{(4+\nu_1)t})$; repeating this procedure, we eventually obtain that $\tilde{\phi}(t)\equiv 0$. This is impossible. Therefore, for θ near 0,

$$\phi_2(\theta)=A_5\theta^{-(n-5)/2}\cos\bigl(\beta\ln\tfrac{\theta}{2}\bigr)+A_6\theta^{-(n-5)/2}\sin\bigl(\beta\ln\tfrac{\theta}{2}\bigr)+O(\theta^{2-(n-5)/2})$$

or

$$\phi_2(\theta)=A_8\theta^{\tilde{\nu}_1}+O(\theta^{2+\tilde{\nu}_1}).$$

But these contradict the fact that $\phi_2(\theta)$ is strictly decreasing for $\theta\in\bigl(0,\frac{\pi}{2}\bigr)$. Thus, we prove the claim and get (4-9).

Let $\phi(\theta)=\phi_1(\theta)-(D_1/D_2)\phi_2(\theta)$. Then $\phi(\theta)$ satisfies the problem

$$\begin{cases}T_1\phi(\theta)+k_1T_2\phi(\theta)+k_0\phi(\theta)=pw_*^{p-1}(\theta)\phi(\theta), & 0<\theta<\frac{\pi}{2},\\ \phi\bigl(\frac{\pi}{2}\bigr)=1,\ \phi'\bigl(\frac{\pi}{2}\bigr)=0,\ \phi''\bigl(\frac{\pi}{2}\bigr)=-D_1/D_2,\ \phi'''\bigl(\frac{\pi}{2}\bigr)=0.\end{cases} \tag{4-15}$$

We claim that for θ near 0,

$$\phi(\theta)=\theta^{-(n-5)/2}\bigl(c_1\cos\bigl(\beta\ln\tfrac{\theta}{2}\bigr)+c_2\sin\bigl(\beta\ln\tfrac{\theta}{2}\bigr)\bigr)+O(\theta^{2-(n-5)/2}) \tag{4-16}$$

with $c_1^2+c_2^2\neq 0$. Using the Emden–Fowler transformation

$$\hat{\phi}(t)=(\sin\theta)^{\alpha}\phi(\theta),\quad t=\ln\bigl(\tan\tfrac{\theta}{2}\bigr), \tag{4-17}$$

(4-8) and (4-9), we obtain that for t near $-\infty$,

$$\hat{\phi}(t)=e^{\sigma t}(c_1\cos(\beta t)+c_2\sin(\beta t))+c_3e^{\nu_1 t}+O(e^{(2+\sigma)t}) \tag{4-18}$$

provided $c_1^2+c_2^2\neq 0$ or

$$\hat{\phi}(t)=c_3e^{\nu_1 t}+O(e^{(2+\nu_1)t}) \tag{4-19}$$

provided $c_1^2+c_2^2=0$ and $c_3\neq 0$. (Note that if both $c_1^2+c_2^2=0$ and $c_3=0$, we can obtain $\hat{\phi}(t)\equiv 0$. This is impossible.) We now show that (4-19) cannot occur. On the contrary, we see that for θ near 0,

$$\phi(\theta)=c_3\theta^{\tilde{\nu}_1}+O(\theta^{2+\tilde{\nu}_1}).$$

This implies that $\phi(\theta)\to 0$ as $\theta\to 0$. Since

$$\hat\phi(t)=O(e^{\nu_1 t}),\quad \hat\phi'(t)=O(e^{\nu_1 t}),\quad \hat\phi''(t)=O(e^{\nu_1 t}),\quad \hat\phi'''(t)=O(e^{\nu_1 t}),$$

we obtain from (4-17) that

$$\begin{aligned}\phi'(\theta)&=O(\theta^{\tilde\nu_1-1}),\\ \sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}&=O(\theta^{n-3+\tilde\nu_1}),\\ \frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)&=O(\theta^{n-4+\tilde\nu_1}).\end{aligned}$$

Similar arguments imply that

$$\sin^{n-2}\theta\frac{d}{d\theta}\Big(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)\Big)=O(\theta^{n-5+\tilde\nu_1}).$$

If we define

$$e(\theta)=\sin^{n-2}\theta\frac{d}{d\theta}\Big(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)\Big),$$

we see that $e(0)=0$. Then, we claim that ϕ changes sign in $\big(0,\frac{\pi}{2}\big)$. Suppose that this is not true. Without loss of generality, we assume $\phi>0$ in $\big(0,\frac{\pi}{2}\big)$. Then it follows from the equation of ϕ that for $\theta\in\big(0,\frac{\pi}{2}\big)$,

$$\frac{d}{d\theta}\Big(e(\theta)+k_1\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)\Big)=\sin^{n-2}\theta(pw_*^{p-1}-k_0)\phi(\theta)>0. \tag{4-20}$$

But integrating both sides of (4-20) in $\big(0,\frac{\pi}{2}\big)$ and using the boundary conditions $\phi'\big(\frac{\pi}{2}\big)=\phi'''\big(\frac{\pi}{2}\big)=0$, we obtain

$$\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta(pw_*^{p-1}-k_0)\phi(\theta)\,d\theta=0.$$

This is clearly impossible. Noticing that $\phi\neq 0$ for θ near 0, we see that there is a minimal zero point $\hat\theta\in\big(0,\frac{\pi}{2}\big)$ of ϕ. Without loss of generality, we assume that $\phi>0$ in $(0,\hat\theta)$. It follows from (4-20) that $E(\theta):=e(\theta)+k_1\sin^{n-2}\theta(d\phi(\theta)/d\theta)$ is increasing for $\theta\in(0,\hat\theta)$. Noticing $E(0)=0$, we then obtain that $E(\theta)>0$ for $\theta\in(0,\hat\theta)$. Therefore,

$$\frac{d}{d\theta}\Big(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)+k_1\phi(\theta)\Big)>0\quad\text{for }\theta\in(0,\hat\theta). \tag{4-21}$$

Moreover, by a similar argument, we have

$$\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)>0\quad\text{for }\theta\in(0,\hat\theta), \tag{4-22}$$

and

$$\frac{d\phi(\theta)}{d\theta} > 0 \quad \text{for } \theta \in (0, \hat{\theta}). \tag{4-23}$$

But (4-23) implies $\phi(\hat{\theta}) > 0$, which contradicts the fact that $\phi(\hat{\theta}) = 0$. This contradiction implies that (4-19) cannot occur and thus (4-18) holds. As a consequence, (4-16) holds and hence (4-2) holds.

Let $\psi(\theta) = \phi_1(\theta)$. We easily see that (4-3) can be obtained from (4-8). □

For any sufficiently small $\delta > \eta > 0$, we set $\psi_1(\theta)$ to be the solution of the problem

$$\begin{cases} T_1\psi_1(\theta) + k_1 T_2\psi_1(\theta) + k_0\psi_1(\theta) \\ \qquad\qquad = \eta^{-2}\big((w_* + \Phi + \Psi)^p - w_*^p - p w_*^{p-1}(\Phi + \eta^2\psi)\big), \\ (\psi_1 + \psi)\big(\frac{\pi}{2}\big) = 2, \qquad (\psi_1 + \psi)'\big(\frac{\pi}{2}\big) = 0, \\ (\psi_1 + \psi)''\big(\frac{\pi}{2}\big) = D_1\delta^2/(D_2\eta^2), \quad (\psi_1 + \psi)'''\big(\frac{\pi}{2}\big) = 0, \end{cases} \tag{4-24}$$

where $\psi(\theta)$ is given in Lemma 4.1, $\Phi = \delta^2\phi(\theta)$ and $\Psi = \eta^2(\psi_1(\theta) + \psi(\theta))$. We can see that Ψ satisfies the problem

$$\begin{cases} T_1\Psi(\theta) + k_1 T_2\Psi(\theta) + k_0\Psi(\theta) = (w_* + \Phi + \Psi)^p - w_*^p - p w_*^{p-1}\Phi, \\ \Psi\big(\frac{\pi}{2}\big) = 2\eta^2, \ \Psi'\big(\frac{\pi}{2}\big) = 0, \ \Psi''\big(\frac{\pi}{2}\big) = D_1\delta^2/D_2, \ \Psi'''\big(\frac{\pi}{2}\big) = 0. \end{cases} \tag{4-25}$$

This implies

$$\begin{cases} T_1(\Psi + \Phi) + k_1 T_2(\Psi + \Phi) + k_0(\Psi + \Phi) = (w_* + \Phi + \Psi)^p - w_*^p, \\ (\Psi + \Phi)\big(\frac{\pi}{2}\big) = 2\eta^2 + \delta^2, \quad (\Psi + \Phi)'\big(\frac{\pi}{2}\big) = 0, \\ (\Psi + \Phi)''\big(\frac{\pi}{2}\big) = 0, \qquad (\Psi + \Phi)'''\big(\frac{\pi}{2}\big) = 0. \end{cases} \tag{4-26}$$

Arguments similar to those in the proof of Lemma 4.1 imply that $\Psi(\theta) + \Phi(\theta)$ is strictly decreasing. Then

$$\Psi(\theta) + \Phi(\theta) > 0 \quad \text{for } \theta \in \big(0, \tfrac{\pi}{2}\big). \tag{4-27}$$

Setting $\psi_2(\theta) = \psi(\theta) + \psi_1(\theta)$, we easily see that ψ_2 satisfies the problem

$$\begin{cases} T_1\psi_2(\theta) + k_1 T_2\psi_2(\theta) + k_0\psi_2(\theta) \\ \qquad = p w_*^{p-1}\psi_2 + \eta^{-2}\big((w_* + \Phi + \eta^2\psi_2)^p - w_*^p - p w_*^{p-1}(\Phi + \eta^2\psi_2)\big), \\ \psi_2\big(\frac{\pi}{2}\big) = 2, \ \psi_2'\big(\frac{\pi}{2}\big) = 0, \ \psi_2''\big(\frac{\pi}{2}\big) = D_1\delta^2/(D_2\eta^2), \ \psi_2'''\big(\frac{\pi}{2}\big) = 0. \end{cases} \tag{4-28}$$

By the Emden–Fowler transformation

$$\tilde{\psi}_2(t) = (\sin\theta)^\alpha \psi_2(\theta), \qquad t = \ln\tan\tfrac{\theta}{2},$$

we see that $\tilde{\psi}_2(t)$ satisfies the problem

$$\text{(4-29)}\quad \begin{cases} \tilde{\psi}_2^{(4)}(t)+a_3(t)\tilde{\psi}_2'''(t)+a_2(t)\tilde{\psi}_2''(t) \\ \qquad\qquad +a_1(t)\tilde{\psi}_2'(t)+a_0(t)\tilde{\psi}_2(t)=G(\tilde{\psi}_2(t)), \quad -\infty<t<0, \\ \tilde{\psi}_2'(0)=0,\ \tilde{\psi}_2'''(0)=0, \end{cases}$$

where $a_0(t), a_1(t), a_2(t), a_3(t)$ are defined in (4-10), and

$$G(\tilde{\psi}_2(t)) \\ =(\sin\theta)^{4+\alpha}\eta^{-2}\big((w_*+\Phi+\eta^2\sin^{-\alpha}\theta\tilde{\psi}_2)^p-w_*^p-pw_*^{p-1}(\Phi+\eta^2\sin^{-\alpha}\theta\tilde{\psi}_2)\big).$$

Moreover, we can rewrite (4-29) in the following form (see the proof of Lemma 4.1):

$$\text{(4-30)}\quad \tilde{\psi}_2^{(4)}(t)+K_3\tilde{\psi}_2'''(t)+K_2\tilde{\psi}_2''(t)+K_1\tilde{\psi}_2'(t)+(1-p)K_0\tilde{\psi}_2(t) \\ =G(\tilde{\psi}_2(t))+g(t,\tilde{\psi}_2(t)),$$

where

$$g(t,\tilde{\psi}_2(t))=O\big(e^{2t}(\tilde{\psi}_2'''(t)+\tilde{\psi}_2''(t)+\tilde{\psi}_2'(t))\big)$$

for $t\ll -1$. Therefore, for $t<T$ with any $T\ll -1$,

$$\text{(4-31)}\quad \tilde{\psi}_2(t)=D_5e^{\nu_2 t}+D_6e^{\sigma t}\cos\beta t+D_7e^{\sigma t}\sin\beta t+D_8e^{\nu_1 t} \\ +B_5\int_{-\infty}^t e^{\sigma(t-s)}\sin\beta(t-s)(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\,\mathrm{d}s \\ +B_6\int_{-\infty}^t e^{\sigma(t-s)}\cos\beta(t-s)(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\,\mathrm{d}s \\ +B_7\int_{-\infty}^t e^{\nu_2(t-s)}(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\,\mathrm{d}s \\ +B_8\int_T^t e^{\nu_1(t-s)}(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\,\mathrm{d}s,$$

where B_5, B_6, B_7, B_8 depend only on ν_i $(i=1,2,3,4)$. Using the fact $\Psi(\theta)+\Phi(\theta)$ is strictly decreasing in $\big(0,\frac{\pi}{2}\big)$ and (4-2), we conclude that $D_5\neq 0$. Letting $\phi(\theta)=\sin^{-\alpha}\theta\tilde{\phi}(t)$, we see that for $t\in[10T, 2T]$ and $\delta^2=O(e^{(2-\sigma)t})$, $\eta^2=O(e^{(2-\nu_2)t})$,

$$\text{(4-32)}\qquad G(\tilde{\psi}_2(t))=\eta^{-2}O((\delta^2\tilde{\phi}(t)+\eta^2\tilde{\psi}_2(t))^2)=O(e^{(2+\nu_2)t}).$$

Note that

$$\tilde{\phi}(t)=e^{\sigma t}(c_1\cos(\beta t)+c_2\sin(\beta t))+O(e^{(2+\sigma)t})$$

and $\tilde{\psi}_2(t)=D_5e^{\nu_2 t}+O(e^{(2+\nu_2)t})$. Then

$$\delta^2\tilde{\phi}(t)+\eta^2\tilde{\psi}_2(t)=O(e^{2t}).$$

Therefore, it follows from (4-31) and (4-32) that

$$\tilde{\psi}_2(t) = D_5 e^{\nu_2 t} + D_6 e^{\sigma t}\cos\beta t + D_7 e^{\sigma t}\sin\beta t + O(e^{(2+\nu_2)t}) \tag{4-33}$$

provided $\delta^2 = O(e^{(2-\sigma)t})$ and $\eta^2 = O(e^{(2-\nu_2)t})$. Hence, for θ near 0,

$$\Psi(\theta) = \eta^2\Big(D_5\theta^{\tilde{\nu}_2} + \theta^{-(n-5)/2}\big(D_6\cos\big(\beta\ln\tfrac{\theta}{2}\big) + D_7\sin\big(\beta\ln\tfrac{\theta}{2}\big)\big) + O(\theta^{2+\tilde{\nu}_2})\Big) \tag{4-34}$$

with $D_5 \neq 0$ provided that

$$\theta = O(\delta^{2/(2-\sigma)}) = O(\eta^{2/(2-\nu_2)}).$$

Since $\tilde{\nu}_2 < 3-n$, we easily see that $\tilde{\nu}_2 + 2 < -(n-5) < -(n-5)/2$. Thus, $\theta^{-(n-5)/2} = o(\theta^{2+\tilde{\nu}_2})$.

Now we can obtain the following theorem.

Theorem 4.2. *For any $\delta > \eta > 0$ sufficiently small, problem* (1-7) *admits outer solutions $w^{\text{out}}_{\delta,\eta} \in C^4\big(0, \frac{\pi}{2}\big)$ satisfying*

$$w^{\text{out}}_{\delta,\eta}(\theta) = w_*(\theta) + \Phi(\theta) + \Psi(\theta), \quad \theta \in \big(0, \tfrac{\pi}{2}\big), \tag{4-35}$$

with $(w^{\text{out}}_{\delta,\eta})'\big(\frac{\pi}{2}\big) = (w^{\text{out}}_{\delta,\eta})'''\big(\frac{\pi}{2}\big) = 0$. Moreover,

$$\begin{aligned} w^{\text{out}}_{\delta,\eta}(\theta) = {} & \frac{A_p}{\theta^\alpha} + \frac{2A_p}{3(p-1)}\frac{1}{\theta^{\alpha-2}} \\ & + \delta^2\left(\frac{\vartheta_1\cos\big(\beta\ln\frac{\theta}{2}\big) + \vartheta_2\sin\big(\beta\ln\frac{\theta}{2}\big)}{\theta^{(n-5)/2}} + O\left(\frac{1}{\theta^{(n-5)/2-2}}\right)\right) \\ & + \eta^2\big(\vartheta_3\theta^{\tilde{\nu}_2} + O(\theta^{\tilde{\nu}_2+2})\big) \end{aligned} \tag{4-36}$$

provided that

$$\theta = O(\delta^{2/(2-\sigma)}) = O(\eta^{2/(2-\nu_2)}),$$

where $\vartheta_1, \vartheta_2, \vartheta_3$ are constants independent of δ, η such that $\vartheta_1^2 + \vartheta_2^2 \neq 0$, $\vartheta_3 \neq 0$.

Proof. The proof can be obtained from the expressions of $w_*(\theta)$, $\Phi(\theta)$ and $\Psi(\theta)$ given in (1-8), (4-16) and (4-34). □

5. Infinitely many solutions of (1-7) and proof of Theorem 1.1

In this section, we construct infinitely many regular solutions for (1-7) by matching the inner and outer solutions.

We construct solutions of the problem

$$\begin{cases} T_1 w + k_1 T_2 w + k_0 w = w^p, & w(\theta) > 0,\ 0 < \theta < \frac{\pi}{2}, \\ w(0) = Q\ (:= \epsilon^{-\alpha}),\ w'\big(\frac{\pi}{2}\big) = 0,\ w''(0) = (\tilde{b} + \mu)\epsilon^{-\alpha-2},\ w'''\big(\frac{\pi}{2}\big) = 0 \end{cases} \tag{5-1}$$

by matching the inner and outer solutions given in Theorems 3.7 and 4.2. To do so, we will find $\Theta \in \left(0, \frac{\pi}{2}\right)$ with

$$\Theta = O(Q^{\sigma/((2-\sigma)\alpha)}) \quad (Q \gg 1)$$

such that the following identities hold:

$$\left(w^{\mathrm{inn}}_{Q,\mu}(\theta) - w^{\mathrm{out}}_{\delta,\eta}(\theta)\right)\big|_{\theta=\Theta} = 0, \tag{5-2}$$

$$\left(w^{\mathrm{inn}}_{Q,\mu}(\theta) - w^{\mathrm{out}}_{\delta,\eta}(\theta)\right)'_{\theta}\big|_{\theta=\Theta} = 0, \tag{5-3}$$

$$\left(w^{\mathrm{inn}}_{Q,\mu}(\theta) - w^{\mathrm{out}}_{\delta,\eta}(\theta)\right)''_{\theta}\big|_{\theta=\Theta} = 0, \tag{5-4}$$

$$\left(w^{\mathrm{inn}}_{Q,\mu}(\theta) - w^{\mathrm{out}}_{\delta,\eta}(\theta)\right)'''_{\theta}\big|_{\theta=\Theta} = 0. \tag{5-5}$$

These will be done by arguments similar to those in the proof of Lemma 6.1 of [Budd and Norbury 1987] and Theorem 1.1 of [Dancer et al. 2012]. Then, we obtain a C^4 function $w(\theta)$ defined by $w(\theta) = w^{\mathrm{inn}}_{Q,\mu}(\theta)$ for $\theta \le \Theta$ and $w(\theta) = w^{\mathrm{out}}_{\delta,\eta}(\theta)$ for $\theta \ge \Theta$ which is a solution to (5-1).

First, we observe that

$$\frac{2A_p}{3(p-1)} = C_p \tag{5-6}$$

by (3-35), where A_p, C_p are given in Section 3.

Define Q_*, δ_*^2, η_*^2 and μ_* by

$$\beta \ln Q_*^{(p-1)/4} + \kappa = \beta \ln 2^{-1} + \omega + 2m\pi, \tag{5-7}$$

$$\delta_*^2 = \sqrt{\frac{a_0^2 + b_0^2}{\vartheta_1^2 + \vartheta_2^2}}\, Q_*^{\sigma/\alpha}, \tag{5-8}$$

$$\eta_*^2 = O(Q_*^{(2-\nu_2)\sigma/((2-\sigma)\alpha)}), \quad \mu_* = O(Q_*^{(2\sigma-2\nu_1)/((2-\sigma)\alpha)}), \tag{5-9}$$

$$\mu_* B_p Q_*^{\nu_1/\alpha} = \vartheta_3 \eta_*^2 \Theta_*^{\tilde{\nu}_2 - \tilde{\nu}_1}, \tag{5-10}$$

where

$$\kappa = \tan^{-1}\left(\frac{a_0}{b_0}\right), \quad \omega = \tan^{-1}\left(\frac{\vartheta_1}{\vartheta_2}\right)$$

and $m \gg 1$ is an integer. The integer m is chosen such that the results in Sections 3 and 4 hold.

Note that

$$O(\delta_*^{2/(2-\sigma)}) = O(Q_*^{\sigma/(\alpha(2-\sigma))}),$$

$$a_0 \cos(\beta \ln(Q^{(p-1)/4}\theta)) + b_0 \sin(\beta \ln(Q^{(p-1)/4}\theta)) \\ = \sqrt{a_0^2+b_0^2}\sin(\beta\ln\theta + \beta \ln Q^{(p-1)/4} + \kappa),$$

$$\vartheta_1 \cos\big(\beta \ln \tfrac{\theta}{2}\big) + \vartheta_2 \sin\big(\beta \ln \tfrac{\theta}{2}\big) = \sqrt{\vartheta_1^2+\vartheta_2^2}\sin(\beta\ln\theta+\beta\ln 2^{-1}+\omega).$$

We will see that the Q, μ, δ^2 and η^2 required to satisfy the matching conditions (5-2)–(5-5) can be obtained as small perturbations of Q_*, μ_*, δ_*^2 and η_*^2 given in (5-7)–(5-10), i.e.,

$$Q = Q_*(1+O(Q_*^{2\sigma/((2-\sigma)\alpha)})), \tag{5-11}$$

$$\mu = \mu_*(1+O(Q_*^{2\sigma/((2-\sigma)\alpha)})), \tag{5-12}$$

$$\delta^2 = \delta_*^2(1+O(Q_*^{2\sigma/((2-\sigma)\alpha)})), \tag{5-13}$$

$$\eta^2 = \eta_*^2(1+O(Q_*^{2\sigma/((2-\sigma)\alpha)})). \tag{5-14}$$

To show this we define the function $\boldsymbol{F}(Q,\mu,\delta,\eta)$ by

$$\boldsymbol{F}(Q,\mu,\delta^2,\eta^2) = \begin{bmatrix} \Theta^{(n-5)/2}(w^{\text{inn}}_{Q,\mu}(\Theta) - w^{\text{out}}_{\delta,\eta}(\Theta)) \\ \Theta\big(\theta^{(n-5)/2}(w^{\text{inn}}_{Q,\mu}(\theta) - w^{\text{out}}_{\delta,\eta}(\theta))\big)'_\theta\big|_{\theta=\Theta} \\ \Theta^2\big(\theta^{(n-5)/2}(w^{\text{inn}}_{Q,\mu}(\theta) - w^{\text{out}}_{\delta,\eta}(\theta))\big)''_\theta\big|_{\theta=\Theta} \\ \Theta^3\big(\theta^{(n-5)/2}(w^{\text{inn}}_{Q,\mu}(\theta) - w^{\text{out}}_{\delta,\eta}(\theta))\big)'''_\theta\big|_{\theta=\Theta} \end{bmatrix}^T.$$

Now, we regard δ^2, η^2 as new variables. Taking Q_*, μ_*, δ_*^2 and η_*^2, we find a bound for $\boldsymbol{F}(Q_*,\mu_*,\delta_*^2,\eta_*^2)$ by using the behaviors of $w^{\text{inn}}_{Q,\mu}(\theta)$ and $w^{\text{out}}_{\delta,\eta}(\theta)$ given in Theorems 3.7 and 4.2 respectively. Accordingly we find for some $M>1$ suitably large,

$$\big|\Theta^{-(n-5)/2}\boldsymbol{F}(Q_*,\mu_*,\delta_*^2,\eta_*^2)\big| \le M\Theta^{4-\sigma-(n-5)/2} + \text{small terms}. \tag{5-15}$$

We seek values of Q, μ, δ^2, η^2 which are small perturbations of Q_*, μ_*, δ_*^2, η_*^2 and such that $\boldsymbol{F}(Q,\mu,\delta^2,\eta^2)=0$. As in [Dancer et al. 2012], we need to evaluate the Jacobian of $\boldsymbol{F}$ at $(Q_*,\mu_*,\delta_*^2,\eta_*^2)$:

$$\frac{\partial \boldsymbol{F}(Q,\mu,\delta^2,\eta^2)}{\partial(Q,\mu,\delta^2,\eta^2)} = \begin{bmatrix} I_1+I_3 & I_4 & -D\sin\tau & I_5 \\ \beta I_2+q_1I_3 & q_1I_4 & -\beta D\cos\tau & q_4I_5 \\ I_6 & q_2I_4 & I_8 & q_5I_5 \\ I_7 & q_3I_4 & I_9 & q_6I_5 \end{bmatrix} + \text{higher-order terms},$$

where

$$I_1 = C\left(\frac{\sigma}{\alpha}\sin\tau + \frac{\beta(p-1)}{4}\cos\tau\right)Q_*^{\sigma/\alpha-1},$$
$$I_2 = C\left(\frac{\sigma}{\alpha}\cos\tau - \frac{\beta(p-1)}{4}\sin\tau\right)Q_*^{\sigma/\alpha-1},$$
$$I_3 = \frac{\nu_1}{\alpha}B_p\mu_*\Theta^{\tilde{\nu}_1+(n-5)/2}Q_*^{\nu_1/\alpha-1}, \qquad I_4 = B_p Q_*^{\nu_1/\alpha}\Theta^{\tilde{\nu}_1+(n-5)/2},$$
$$I_5 = -\vartheta_3\Theta^{\tilde{\nu}_2+(n-5)/2}, \qquad I_6 = -\beta^2 I_1 - \beta I_2 + q_2 I_3,$$
$$I_7 = -\beta^3 I_2 + 3\beta^2 I_1 + 2\beta I_2 + q_3 I_3, \quad I_8 = \beta^2 D\sin\tau + \beta D\cos\tau,$$
$$I_9 = \beta^3 D\cos\tau - 3\beta^2 D\sin\tau - 2\beta D\cos\tau,$$
$$q_1 = \tilde{\nu}_1 + \tfrac{1}{2}(n-5), \quad q_2 = \big(\tilde{\nu}_1 + \tfrac{1}{2}(n-7)\big)q_1, \quad q_3 = \big(\tilde{\nu}_1 + \tfrac{1}{2}(n-9)\big)q_2,$$
$$q_4 = \tilde{\nu}_2 + \tfrac{1}{2}(n-5), \quad q_5 = \big(\tilde{\nu}_2 + \tfrac{1}{2}(n-7)\big)q_4, \quad q_6 = \big(\tilde{\nu}_2 + \tfrac{1}{2}(n-9)\big)q_5,$$
$$C = \sqrt{a_0^2 + b_0^2}, \quad D = \sqrt{\vartheta_1^2 + \vartheta_2^2},$$

and

$$\tau = \beta\ln\Theta + \beta\ln Q_*^{(p-1)/4} + \kappa = \beta\ln\Theta + \beta\ln 2^{-1} + \omega + 2m\pi.$$

We define the function $\boldsymbol{G}(x, y, z, w)$ by

$$\boldsymbol{G}(x, y, z, w) = \boldsymbol{F}(Q_* + xQ_*^{1-\sigma/\alpha}, \mu_* + \Theta^{-\tilde{\nu}_1-(n-5)/2}Q_*^{-\nu_1/\alpha}y, \delta_*^2 + z, \eta_*^2 + \Theta^{-\tilde{\nu}_2-(n-5)/2}w).$$

Using (5-15), (4-36) and the results in Lemmas 3.5 and 3.6, we express $\boldsymbol{G}(x, y, z, w)$ in the form

$$\boldsymbol{G}(x, y, z, w) = \boldsymbol{C} + \left[\begin{matrix} I_1' + I_3' & I_4' & -D\sin\tau & I_5' \\ \beta I_2' + q_1 I_3' & q_1 I_4' & -\beta D\cos\tau & q_4 I_5' \\ I_6' & q_2 I_4' & I_8' & q_5 I_5' \\ I_7' & q_3 I_4' & I_9' & q_6 I_5' \end{matrix} + \text{small terms}\right]\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$
$$+ \boldsymbol{E}(x, y, z, w, Q_*, \mu_*, \delta_*^2, \eta_*^2),$$

where

$$I_1' = C\left(\frac{\sigma}{\alpha}\sin\tau + \frac{\beta(p-1)}{4}\cos\tau\right), \quad I_2' = C\left(\frac{\sigma}{\alpha}\cos\tau - \frac{\beta(p-1)}{4}\sin\tau\right),$$
$$I_3' = \frac{\nu_1}{\alpha}B_p\mu_*\Theta^{\tilde{\nu}_1+(n-5)/2}Q_*^{(\nu_1-\sigma)/\alpha}, \quad I_4' = B_p,$$
$$I_5' = -\vartheta_3, \qquad I_6' = -\beta^2 I_1' - \beta I_2' + q_2 I_3',$$
$$I_7' = -\beta^3 I_2' + 3\beta^2 I_1' + 2\beta I_2' + q_3 I_3', \quad I_8' = \beta^2 D\sin\tau + \beta D\cos\tau,$$
$$I_9' = \beta^3 D\cos\tau - 3\beta^2 D\sin\tau - 2\beta D\cos\tau,$$

and where $\boldsymbol{C}$ is a constant vector independent of (x, y, z, w) which is bounded above by $M\Theta^{4-\sigma}$, and $|\boldsymbol{E}|$ is bounded independently of x, y, z, w, Q, μ, δ and η. Thus,

$$\boldsymbol{G}(x, y, z, w) = \boldsymbol{C} + L \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \boldsymbol{T}(x, y, z, w),$$

where L is a linear operator which is invertible; we shall prove this fact in Lemma A.1. If we define the operator $\boldsymbol{J}$ mapping $\mathbb{R}^4$ into itself by

$$\boldsymbol{J}(x, y, z, w) = -(L^{-1}\boldsymbol{C} + L^{-1}\boldsymbol{T}(x, y, z, w)),$$

then, provided that Q_* is sufficiently large, a direct calculation shows that $\boldsymbol{J}$ maps the set I into itself, where I is the ball

$$I = \{(x, y, z, w) : (x^2 + y^2 + z^2 + w^2)^{1/2} \le 4M(\det L)^{-1}\Theta^{4-\sigma}\}, \tag{5-16}$$

and $\det L$ is the determinant of L, which depends on $\sqrt{a_0^2 + b_0^2}$, β, D, α, B_p, ϑ_3 and ν_i $(i = 1, 2, 3, 4)$. We apply the Brouwer fixed point theorem to conclude that $\boldsymbol{J}$ has a fixed point in I. This point (x, y, z, w) satisfies $\boldsymbol{G}(x, y, z, w) = 0$ and

$$(x^2 + y^2 + z^2 + w^2)^{1/2} \le M'\Theta^{4-\sigma},$$

where M' is a constant defined in (5-16) and is independent of Q_*, μ_*, δ_*, η_* and Θ. By substituting for Q, μ, δ and η, then taking Θ to have the upper limiting value of $Q_*^{\sigma/((2-\sigma)\alpha)}$, we obtain (5-11)–(5-14). Therefore, we can find a solution to (5-1) such that (5-2)–(5-5) hold.

We have shown that (5-2)–(5-5) have a solution for each large fixed m. This yields a solution of (5-1) and also gives the proof of Theorem 1.1. Hence we have:

Theorem 5.1. *For $m \gg 1$ large and Q, μ, δ and η as given in* (5-11)–(5-14), *problem* (5-1) *admits a classical solution $w_{Q,\mu,\delta,\eta}(\theta)$. Moreover, there is $\Theta = |O(Q^{\sigma/((2-\sigma)\alpha)})|$ such that* (5-2)–(5-5) *hold.*

As a consequence, problem (1-7) admits infinitely many nonconstant positive solutions. Hence, we have proved Theorem 1.1.

Appendix

We will prove a lemma which was used in the previous sections.

Lemma A.1. *For the terms E_2 and $k_0(n)$ and the matrices K and L, which were defined in previous sections, we have*

(1) $E_2 \neq 0$,

(2) $p \in \left(\dfrac{n+3}{n-5}, p_c(n-1)\right) \implies pk_0(n-1) \geq k_0(n)$,

(3) $\det K \neq 0$,

(4) $\det L \neq 0$.

Proof. First, we show that $E_2 \neq 0$. It is known that

(A-1) $E_2 = (\tilde{\nu}_1+2)\tilde{\nu}_1(\tilde{\nu}_1+n-3)(\tilde{\nu}_1+n-1) - p(n-5-\alpha)(n-3-\alpha)(2+\alpha)\alpha.$

For convenience, we use n instead of $n-1$ and $\tilde{\nu}_1(n)$ instead of $\tilde{\nu}_1(n-1)$; i.e., we study the term

(A-2) $E_2 = (\tilde{\nu}_1+2)\tilde{\nu}_1(\tilde{\nu}_1+n-2)(\tilde{\nu}_1+n) - p(n-4-\alpha)(n-2-\alpha)(2+\alpha)\alpha.$

Let $f(\alpha) = p(n-4-\alpha)(n-2-\alpha)(2+\alpha)\alpha$. Through a simple computation, we get $f(\alpha)$ and its derivative $f'(\alpha)$:

$$f(\alpha) = \alpha^4 + (12-2n)\alpha^3 + (n^2-18n+52)\alpha^2 + (6n^2-52n+96)\alpha + 8(n-2)(n-4),$$

and

$$f'(\alpha) = 4\alpha^3 + (36-6n)\alpha^2 + (2n^2-36n+104)\alpha + (6n^2-52n+96).$$

We compute the roots of $f'(\alpha)$ to find its zero points: $\frac{1}{2}(n-6\pm\sqrt{n^2+4})$ and $\frac{1}{2}(n-6)$. It is easy to see that $f(\alpha)$ is strictly increasing for $\alpha \in \big(0, \frac{1}{2}(n-6)\big)$ and decreasing for $\alpha \in \big(\frac{1}{2}(n-6), \frac{1}{2}(n-6+\sqrt{n^2+4})\big)$. We know $\alpha = 4/(p-1) < \frac{1}{2}(n-4)$ and $\frac{1}{2}(n-4) \in \big(\frac{1}{2}(n-6), \frac{1}{2}(n-6+\sqrt{n^2+4})\big)$. As a consequence, we can conclude

$$f(\alpha) \leq f\big(\tfrac{1}{2}(n-6)\big) = \tfrac{1}{16}n^4 - \tfrac{1}{2}n^2 + 1 \quad \text{for all } p \in \left(\frac{n+4}{n-4}, p_c(n)\right).$$

Let $g(x) = x(x+2)(x+n)(x+n-2) = x^4 + 2nx^3 + (n^2+2n-4)x^2 + (2n^2-4n)x$. We compute its derivative, $g'(x) = 4x^3 + 6nx^2 + (2n^2+4n-8)x + (2n^2-4n)$, and find $g'(x) > 0$ for $x > 0$ when $n \geq 5$. On the other hand, using $4\sqrt{N_3} > N_2$ for $p \in ((n+4)/(n-4), p_c(n))$, we find

$$\tilde{\nu}_1 > \tfrac{1}{2}\big(\sqrt{2(n^2-4n+8)} - (n-4)\big).$$

Therefore,

(A-3)
$$\begin{aligned} g(\tilde{\nu}_1) &\geq g\big(\tfrac{1}{2}\big(\sqrt{2(n^2-4n+8)} - (n-4)\big)\big) \\ &= 96 - 40n + 11n^2 - \tfrac{1}{2}n^3 + \tfrac{1}{16}n^4 + \sqrt{2}(24-4n+n^2)\sqrt{8-4n+n^2}. \end{aligned}$$

Comparing $\frac{1}{16}n^4 - \frac{1}{2}n^2 + 1$ and the right-hand side of (A-3), by direct computation, we can get

$$g\big(\tfrac{1}{2}\big(\sqrt{2(n^2-4n+8)} - (n-4)\big)\big) > \tfrac{1}{16}n^4 - \tfrac{1}{2}n^2 + 1 \quad \text{for } n \in (0, \infty).$$

As a result, $g(\tilde{\nu}_1) > f(\alpha)$. Hence, E_2 is nonzero.

Next, we prove $pk_0(n-1) \geq k_0(n)$ for $p \in ((n+3)/(n-5),\ p_c(n-1))$. According to the definition of $k_0(n)$, it is enough for us to show

$$p(n-5-\alpha)(n-3-\alpha) \geq (n-4-\alpha)(n-2-\alpha). \tag{A-4}$$

Using the relation $p = 4/\alpha + 1$, it is equivalent to show (after computation)

$$6\alpha^2 + (39-10n)\alpha + 4n^2 - 32n + 60 \geq 0. \tag{A-5}$$

It is known that (A-5) holds provided

$$\alpha \geq \tfrac{1}{12}\big(10n-39+\sqrt{4n^2-12n+81}\big) \quad \text{or} \quad \alpha \leq \tfrac{1}{12}\big(10n-39-\sqrt{4n^2-12n+81}\big).$$

On the other hand, since $p \in ((n+3)/(n-5),\ p_c(n-1))$, we have $\alpha < \frac{1}{2}(n-5)$. It is easy to show $\frac{1}{2}(n-5) \leq \frac{1}{12}\big(10n-39-\sqrt{4n^2-12n+81}\big)$ when $n \geq 5$. Hence, (A-5) holds. Therefore (A-4) holds.

Then, to show K is invertible, it is enough for us to show $B \neq 0$ or $A \neq 0$. Recall

$$B = (2n^2-12n-6)\beta + 8\beta^3 = (2(n-3)^2-24)\beta + 8\beta^3.$$

It is known that $2(n-3)^2 - 24 < 0$ only when $n = 6$. Since $\beta > 0$, we have $B \neq 0$ when $n \geq 7$. When $n = 6$, we find

$$A = \beta^4 - \tfrac{35}{2}\beta^2 - \tfrac{135}{16} - (1-\alpha)(3-\alpha)(2+\alpha)(4+\alpha), \qquad B = -6\beta + 8\beta^3.$$

If $B \neq 0$ for $n = 6$, we have that K is invertible, while if $B = 0$ for $n = 6$, then $A = -21-(1-\alpha)(3-\alpha)(2+\alpha)(4+\alpha) < 0$ for $\alpha \in \big(0,\ \frac{1}{2}\big)$ and K is also invertible. Therefore, we have proved the third conclusion.

Finally, we show the matrix L is invertible. Recall that L is given by

$$L := \begin{bmatrix} I'_1 + I'_3 & I'_4 & -D\sin\tau & I'_5 \\ \beta I'_2 + q_1 I'_3 & q_1 I'_4 & -\beta D\cos\tau & q_4 I'_5 \\ I'_6 & q_2 I'_4 & I'_8 & q_5 I'_5 \\ I'_7 & q_3 I'_4 & I'_9 & q_6 I'_5 \end{bmatrix}, \tag{A-6}$$

where

$$I'_1 = C\left(\frac{\sigma}{\alpha}\sin\tau + \frac{\beta(p-1)}{4}\cos\tau\right), \quad I'_2 = C\left(\frac{\sigma}{\alpha}\cos\tau - \frac{\beta(p-1)}{4}\sin\tau\right),$$

$$I'_3 = \frac{\nu_1}{\alpha} B_p \mu_* \Theta^{\tilde{\nu}_1+(n-5)/2} Q_*^{(\nu_1-\sigma)/\alpha}, \qquad I'_4 = B_p,$$

$$I'_5 = \vartheta_3, \qquad I'_6 = -\beta^2 I'_1 - \beta I'_2 + q_2 I'_3,$$

$$I'_7 = -\beta^3 I'_2 + 3\beta^2 I'_1 + 2\beta I'_2 + q_3 I'_3, \qquad I'_8 = \beta^2 D\sin\tau + \beta D\cos\tau,$$

$$I'_9 = \beta^3 D\cos\tau - 3\beta^2 D\sin\tau - 2\beta D\cos\tau.$$

Using simple linear transformations, we see that

$$\begin{bmatrix} I_1'+I_3' & I_4' & -D\sin\tau & I_5' \\ \beta I_2'+q_1I_3' & q_1I_4' & -\beta D\cos\tau & q_4I_5' \\ I_6' & q_2I_4' & I_8' & q_5I_5' \\ I_7' & q_3I_4' & I_9' & q_6I_5' \end{bmatrix} \sim \begin{bmatrix} I_1' & I_4' & -D\sin\tau & I_5' \\ \beta I_2' & q_1I_4' & -\beta D\cos\tau & q_4I_5' \\ I_6'-q_2I_3' & q_2I_4' & I_8' & q_5I_5' \\ I_7'-q_3I_3' & q_3I_4' & I_9' & q_6I_5' \end{bmatrix}$$

$$\sim \begin{bmatrix} I_1' & -D\sin\tau & I_4' & I_5' \\ \beta I_2' & -\beta D\cos\tau & q_1I_4' & q_4I_5' \\ I_6'-q_2I_3' & I_8' & q_2I_4' & q_5I_5' \\ I_7'-q_3I_3' & I_9' & q_3I_4' & q_6I_5' \end{bmatrix} \sim \begin{bmatrix} I_1' & -D\sin\tau & I_4' & -I_5' \\ \beta I_2' & -\beta D\cos\tau & q_1I_4' & -q_4I_5' \\ 0 & 0 & I_{10}' & I_{11}' \\ 0 & 0 & I_{12}' & I_{13}', \end{bmatrix},$$

where

$$I_{10}' = q_2B_p+q_1B_p+\beta^2B_p, \qquad I_{11}' = q_5\vartheta_3+q_4\vartheta_3+\beta^2\vartheta_3,$$
$$I_{12}' = q_3B_p+\beta^2q_1B_p-3\beta^2B_p-2q_1B_p, \quad I_{13}' = q_6\vartheta_3+\beta^2q_4\vartheta_3-3\beta^2\vartheta_3-2q_4\vartheta_3.$$

Here we use the first column minus I_3'/I_4' times the second column in the first step, change the places of the second and third columns in the second step, and in the end, add the second row and β times the first row to the third row and add $-3\beta^2$ times the first row and β^2-2 times the second row to the fourth row. On the other hand, since

$$\det\begin{bmatrix} I_1' & -D\sin\tau \\ \beta I_2' & -\beta D\cos\tau \end{bmatrix} \neq 0,$$

to show that L is invertible, it is enough for us to prove that the 2×2 matrix

$$\begin{bmatrix} q_2+q_1+\beta^2 & q_5+q_4+\beta^2 \\ q_3+\beta^2q_1-3\beta^2-2q_1 & q_6+\beta^2q_4-3\beta^2-2q_4 \end{bmatrix}. \tag{A-7}$$

is invertible. It follows from the definitions of q_i $(i=1,2,3,4,5,6)$ and β that $q_2+q_1+\beta^2 = q_5+q_4+\beta^2 \neq 0$. Let

$$\chi_1 = q_3+\beta^2q_1-3\beta^2-2q_1, \quad \chi_2 = q_6+\beta^2q_4-3\beta^2-2q_4.$$

Then

$$\begin{aligned}\chi_1-\chi_2 &= q_3-q_6-(q_1-q_4)(2-\beta^2)\\ &= (\tilde\nu_1-\tilde\nu_2)\big((\tilde\nu_1+\tilde\nu_2)^2-\tilde\nu_1\tilde\nu_2+\tfrac12(3n-21)(\tilde\nu_1+\tilde\nu_2)+\tfrac14(3n^2-42n+135)+\beta^2\big)\\ &= (\tilde\nu_1-\tilde\nu_2)\big(\tfrac14(n^2-10n+25)-\tilde\nu_1\tilde\nu_2+\beta^2\big),\end{aligned}$$

where we are using the fact that $\tilde\nu_1+\tilde\nu_2 = -(n-5)$. It is known (from Section 2) that

$$\tilde\nu_1\tilde\nu_2 = \frac{n^2-10n+25}{4} - \frac{N_2+4\sqrt{N_3}}{4(p-1)^2}$$

and $\beta^2 = (4\sqrt{N_3} - N_2)/(4(p-1)^2)$, where N_2 and N_3 (with the dimension n being replaced by $n-1$) are defined in Section 2. Therefore,

$$\chi_1 - \chi_2 = (\tilde{\nu}_1 - \tilde{\nu}_2)\frac{2\sqrt{N_3}}{(p-1)^2} \neq 0.$$

Hence, (A-7) is invertible. □

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