Pacific Journal of Mathematics

NATURAL COMMUTING OF VANISHING CYCLES AND THE VERDIER DUAL

DAVID B. MASSEY

Volume 284 No. 2 October 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.284.431

NATURAL COMMUTING OF VANISHING CYCLES AND THE VERDIER DUAL

DAVID B. MASSEY

We prove that the shifted vanishing cycles and nearby cycles commute with Verdier dualizing up to a *natural* isomorphism, even when the coefficients are not in a field.

1. Introduction

In this short, technical, paper, we prove a result whose full statement is missing from the literature, and which may be surprising even to some experts in the field. To state this result, we need to use technical notions and notations; references are [Kashiwara and Schapira 1990; Dimca 2004; Schürmann 2003; Massey 2003, Appendix B]. We should remark immediately that the definition that we use (see below) for the vanishing cycles is the standard one, which is shifted by one from the definition in [Kashiwara and Schapira 1990].

We fix a base ring, R, which is a commutative, regular, Noetherian ring, with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$). Throughout this paper, by a topological space, we will mean a locally compact space. When we write that $\mathbf{A}^\bullet$ is complex of sheaves on a topological space, X, we mean that $\mathbf{A}^\bullet$ is an object in $D^b(X)$, the derived category of bounded complexes of sheaves of R-modules on X. When X is complex analytic, we may also require that $\mathbf{A}^\bullet$ is (complex) constructible, and write $\mathbf{A}^\bullet \in D^b_c(X)$. We remind the reader that constructibility includes the assumption that the stalks of all cohomology sheaves are finitely generated R-modules (so that, by our assumption on R, each stalk complex $\mathbf{A}^\bullet_x$, for $x \in X$, is perfect, i.e., is quasi-isomorphic to a bounded complex of finitely generated projective R-modules).

We let $\mathcal{D} = \mathcal{D}_X$ denote the Verdier dualizing operator on $D^b_c(X)$. We will always write simply $\mathcal{D}$, since the relevant topological space will always be clear.

Suppose that $f : X \to \mathbb{C}$ is a complex analytic function, where X is an arbitrary complex analytic space, and suppose that we have a complex of sheaves $\mathbf{A}^\bullet$ on X. We let ψ_f and ϕ_f denote the nearby and vanishing cycle functors, respectively. Henceforth, we shall always write these functors composed with a shift by -1, that is, we shall write $\psi_f[-1] := \psi_f \circ [-1]$ and $\phi_f[-1] := \phi_f \circ [-1]$. In order

MSC2010: 32B10, 32B15, 32C18, 32C35.
Keywords: vanishing cycles, nearby cycles, Verdier dual, constructible complexes.

to eliminate any possible confusion over indexing/shifting: with the definitions that we are using, if $F_{f,p}$ denotes the Milnor fiber of f at $p \in f^{-1}(0)$ inside the open ball $\mathring{B}_\epsilon(p)$ (using a local embedding into affine space), then, for all $k \in \mathbb{Z}$, $H^k(\psi_f[-1]\mathbf{A}^\bullet)_p \cong \mathbb{H}^{k-1}(F_{f,p}; \mathbf{A}^\bullet)$ and $H^k(\phi_f[-1]\mathbf{A}^\bullet)_p \cong \mathbb{H}^k(\mathring{B}_\epsilon(p), F_{f,p}; \mathbf{A}^\bullet)$.

The questions in which we are interested are:

(i) Do isomorphisms $\mathcal{D} \circ \psi_f[-1] \cong \psi_f[-1] \circ \mathcal{D}$ and $\mathcal{D} \circ \phi_f[-1] \cong \phi_f[-1] \circ \mathcal{D}$ exist even if R is *not* a field?

(ii) Do there exist such isomorphisms which are *natural*?

We show that the answer to both is *yes*.

Is this result known and/or surprising? Some references, such as [Brylinski and Monteiro Fernandes 1986; Dimca 2004; Massey 2003, Appendix B], state that there exist nonnatural isomorphisms, and require that the base ring is a field. Schürmann [2003, Corollary 5.4.4] proves the natural isomorphism exists on the stalk level, even when R is not a field.

In the l-adic algebraic context, Illusie [1994] proves that the Verdier dual and nearby cycles commute, up to natural isomorphism. M. Saito [1988; 1989] proves the analogous result in the complex analytic setting, with field coefficients. One can obtain our full result by combining Proposition 8.4.13, Proposition 8.6.3, and Exercise VIII.15 of [Kashiwara and Schapira 1990], though our proof here is completely different. In fact, our proof is similar to the discussion on duality of local Morse data following Remark 5.1.7 in [Schürmann 2003].

Recently, J. Schürmann proved in Proposition A.1 of the Appendix in [Brav et al. 2015], that the duality isomorphism constructed here fits with a corresponding (more complicated) duality isomorphism in Saito's theory of mixed Hodge modules.

Furthermore, this duality result and the half-space description used for the vanishing cycles have recently become very promising in the study of Donaldson–Thomas invariants of suitable moduli spaces (as in [Brav et al. 2015] and [Kontsevich and Soibelman 2011, Section 7]).

Our proof is relatively simple, and consists of three main steps: proving a small lemma about pairs of closed sets which cover a space, using a convenient characterization/definition of the vanishing cycles, and using that the stratified critical values of f are locally isolated. The nearby cycle result follows as a quick corollary of the result for vanishing cycles.

2. Two lemmas

We shall use $\simeq$ to denote natural isomorphisms of functors. If $\mathbf{A}^\bullet$ is a bounded complex, then, by $\operatorname{supp} \mathbf{A}^\bullet$, we mean, by definition, the closure of the union of the support of all (nontrivial) cohomology sheaves.

The following is an easy generalization of the fact that, if i is the inclusion of an open set, then $i^* \simeq i^!$ (see, for instance, [Dimca 2004, Corollary 3.2.12]).

Lemma 2.1. *Suppose Z is a locally compact subset of X, and let $j : Z \hookrightarrow X$ denote the inclusion. Let $D^b_Z(X)$ denote the full subcategory of $D^b(X)$ of complexes $\mathbf{A}^\bullet$ such that $Z \cap \operatorname{supp} \mathbf{A}^\bullet$ is an open subset of $\operatorname{supp} \mathbf{A}^\bullet$. Then, there is a natural isomorphism of functors $j^! \simeq j^*$ from $D^b_Z(X)$ to $D^b(Z)$.*

Proof. Let $\mathbf{A}^\bullet \in D^b_Z(X)$. We will show that the natural map $j^! \to j^*$ of functors from $D^b(X)$ to $D^b(Z)$ yields an isomorphism $j^!\mathbf{A}^\bullet \to j^*\mathbf{A}^\bullet$.

Let $Y := \operatorname{supp} \mathbf{A}^\bullet$. Let $m : Y \hookrightarrow X$, $\hat{m} : Z \cap Y \hookrightarrow Z$ and $\hat{j} : Z \cap Y \hookrightarrow Y$ denote the inclusions. Then,

$$j^!\mathbf{A}^\bullet \cong j^! m_* m^* \mathbf{A}^\bullet \cong \hat{m}_* \hat{j}^! m^* \mathbf{A}^\bullet \cong \hat{m}_! \hat{j}^* m^* \mathbf{A}^\bullet \cong j^* m_! m^* \mathbf{A}^\bullet \cong j^* \mathbf{A}^\bullet,$$

where we used, in order, that $\mathbf{A}^\bullet \cong m_* m^* \mathbf{A}^\bullet$, since Y is the support of $\mathbf{A}^\bullet$, Proposition 2.6.7 of [Kashiwara and Schapira 1990] on Cartesian squares, that $\hat{m}_* \simeq \hat{m}_!$ and $\hat{j}^! \simeq \hat{j}^*$, since $\hat{m}$ is a closed inclusion and $\hat{j}$ is an open inclusion, Proposition 2.6.7 of [Kashiwara and Schapira 1990] again, and that $\mathbf{A}^\bullet \cong m_! m^* \mathbf{A}^\bullet$. □

The lemma that we shall now prove certainly looks related to many propositions we have seen before, and may be known, but we cannot find a reference. The lemma tells us that, in our special case, the morphism of functors described in Proposition 3.1.9(iii) of [Kashiwara and Schapira 1990] is an isomorphism.

Lemma 2.2. *Let X be a locally compact space, and let Z_1 and Z_2 be closed subsets of X such that $X = Z_1 \cup Z_2$. Denote the inclusion maps by*

$$j_1 : Z_1 \hookrightarrow X, \quad j_2 : Z_2 \hookrightarrow X, \quad \hat{j}_1 : Z_1 \cap Z_2 \hookrightarrow Z_2,$$
$$\hat{j}_2 : Z_1 \cap Z_2 \hookrightarrow Z_1, \quad m = j_1 \hat{j}_2 = j_2 \hat{j}_1 : Z_1 \cap Z_2 \hookrightarrow X.$$

Then, we have the following natural isomorphisms

$$m^* j_{2!} j_2^! \simeq \hat{j}_1^* j_2^! \simeq \hat{j}_2^! j_1^* \simeq m^* j_{1*} \hat{j}_{2!} \hat{j}_2^! j_1^*. \tag{1}$$

Proof. Let $i_1 : X - Z_1 \hookrightarrow X$ and $i_2 : X - Z_2 \hookrightarrow X$ denote the open inclusions. We make use of Proposition 2.6.7 of [Kashiwara and Schapira 1990] on Cartesian squares repeatedly. We also use repeatedly that, if j is a closed inclusion, then $j_* \simeq j_!$ and $j^* j_* \simeq j^* j_! \simeq \mathrm{id}$.

We find

$$m^* j_{2!} j_2^! = (j_2 \hat{j}_1)^* j_{2!} j_2^! \simeq \hat{j}_1^* j_2^* j_{2!} j_2^! \simeq \hat{j}_1^* j_2^!,$$

which proves the first isomorphism in (1).

We also find

$$m^* j_{1*} \hat{j}_{2!} \hat{j}_2^! j_1^* = (j_1 \hat{j}_2)^* j_{1*} \hat{j}_{2!} \hat{j}_2^! j_1^* \simeq \hat{j}_2^* j_1^* j_{1*} \hat{j}_{2!} \hat{j}_2^! j_1^* \simeq \hat{j}_2^* \hat{j}_{2!} \hat{j}_2^! j_1^* \simeq \hat{j}_2^! j_1^*,$$

which proves the last isomorphism in (1).

It remains for us to demonstrate the middle isomorphism.

Let $l_2 : X - Z_2 = Z_1 - (Z_1 \cap Z_2) \to X$ denote the (open) inclusion. Then, we have the natural distinguished triangle

$$j_{2!}j_2{}^! \to \mathrm{id} \to l_{2*}l_2{}^* \xrightarrow{[1]},$$

which yields the distinguished triangle

$$\hat{j}_2^! j_1^* j_{2!} j_2{}^! \to \hat{j}_2^! j_1^* \to \hat{j}_2^! j_1^* l_{2*} l_2{}^* \xrightarrow{[1]} .$$

Now, $\hat{j}_2^! j_1^* j_{2!} j_2{}^! \simeq \hat{j}_2^! \hat{j}_{2!} \hat{j}_1^* j_2^! \simeq \hat{j}_1^* j_2^!$ and so, if we can show that $\hat{j}_2^! j_1^* l_{2*} l_2{}^* = 0$, then we will be finished.

This is easy. The support of $l_{2*}l_2{}^*$ is contained in Z_1; hence $j_1^* l_{2*} l_2{}^* \simeq j_1^! l_{2*} l_2{}^*$. Therefore,

$$\hat{j}_2^! j_1^* l_{2*} l_2{}^* \simeq \hat{j}_2^! j_1^! l_{2*} l_2{}^* \simeq \hat{j}_1^! j_2^! l_{2*} l_2{}^*,$$

and $j_2^! l_{2*} = 0$. □

3. The main theorem

Let $f : X \to \mathbb{C}$ be complex analytic, and let $\mathbf{A}^\bullet \in D_c^b(X)$. For any real number θ, let

$$Z_\theta := f^{-1}(e^{i\theta}\{v \in \mathbb{C} \mid \operatorname{Re} v \le 0\})$$

and let

$$L_\theta := f^{-1}(e^{i\theta}\{v \in \mathbb{C} \mid \operatorname{Re} v = 0\}).$$

Let $j_\theta : Z_\theta \hookrightarrow X$ and $p : f^{-1}(0) \hookrightarrow X$ denote the inclusions.

By Lemma 1.3.2 of [Schürmann 2003], or following Exercise VIII.13 of [Kashiwara and Schapira 1990] (but reversing the inequality, and using a different shift), we define (or characterize up to natural isomorphism) the shifted vanishing cycles of $\mathbf{A}^\bullet$ along f to be

$$\phi_f[-1]\mathbf{A}^\bullet := p^* R\Gamma_{Z_0}(\mathbf{A}^\bullet) \simeq \; p^* j_{0!} j_0^! \mathbf{A}^\bullet.$$

In fact, for each θ, we define the shifted vanishing cycles of $\mathbf{A}^\bullet$ along f at the angle θ to be

$$\phi_f^\theta[-1]\mathbf{A}^\bullet := p^* j_{\theta!} j_\theta^! \mathbf{A}^\bullet.$$

There are the well-known natural isomorphisms $\widetilde{T}_f^\theta : \phi_f^0[-1] \to \phi_f^\theta[-1]$, induced by rotating $\mathbb{C}$ counterclockwise by an angle θ around the origin. The natural isomorphism $\widetilde{T}_f^{2\pi} : \phi_f[-1] \to \phi_f[-1]$ is the usual monodromy automorphism on the vanishing cycles.

In the proof of the main theorem below, we shall use that $\mathcal{D} \circ (-)^* \simeq (-)^! \circ \mathcal{D}$ always holds; we shall also use that $\mathcal{D} \circ (-)^! \simeq (-)^* \circ \mathcal{D}$ holds in our context, but note that this uses biduality, i.e., that $\mathcal{D}\circ\mathcal{D}\simeq \mathrm{id}$, for subanalytically constructible complexes of sheaves (see of [Schürmann 2003, Corollary 2.2.7] or [Kashiwara and Schapira 1990, Exercise VIII.3]), which uses the assumption that the commutative base ring R is regular, Noetherian of finite Krull dimension, so that, for such a subanalytically constructible complex of sheaves, all stalk complexes are perfect.

We now prove the main theorem.

Theorem 3.1. *There is a natural isomorphism*

$$\phi_f[-1]\circ\mathcal{D} \;\simeq\; \mathcal{D}\circ\phi_f[-1]$$

of functors from $D^b_c(X)$ to $D^b_c(f^{-1}(0))$.

Proof. Let m denote the inclusion of $L_0 = L_\pi$ into X. Now, apply Lemma 2.2 to $X = Z_0 \cup Z_\pi$, and conclude that

$$m^* j_{0!}\, j_0^! \simeq \hat{j}_0^!\, j_\pi^*.$$

Dualizing, we obtain

$$\mathcal{D}(m^* j_{0!}\, j_0^!) \simeq \mathcal{D}(\hat{j}_0^!\, j_\pi^*) \simeq \hat{j}_0^*\, j_\pi^!\mathcal{D} \simeq m^*\, j_{\pi\,!}\, j_\pi^!\mathcal{D}, \tag{2}$$

where the second isomorphism uses that $\mathcal{D}$ "commutes" with the standard operations, and the last isomorphism results from using that $m = j_\pi\,\hat{j}_0$.

Let q denote the inclusion of $f^{-1}(0)$ into $L_0 = L_\pi$, so that the inclusion p equals mq. Applying q^* to (2), we obtain

$$q^*\mathcal{D}(m^* j_{0!}\, j_0^!) \simeq q^* m^*\, j_{\pi\,!}\, j_\pi^!\mathcal{D} \simeq p^*\, j_{\pi\,!}\, j_\pi^!\mathcal{D} = \phi_f^\pi[-1]\circ\mathcal{D} \simeq \phi_f[-1]\circ\mathcal{D},$$

where, in the last step, we used the natural isomorphism $\big(\widetilde{T}_f^\pi\big)^{-1}$.

As $\mathcal{D}(q^! m^* j_{0!}\, j_0^!) \simeq q^*\mathcal{D}(m^* j_{0!}\, j_0^!)$, it remains for us to show that $q^! m^* j_{0!} j_0^!$ is naturally isomorphic to $q^* m^* j_{0!}\, j_0^! \simeq p^* j_{0!}\, j_0^! \simeq \phi_f[-1]$. This will follow from Lemma 2.1, once we show that, for all $\mathbf{A}^\bullet \in D^b_c(X)$, $f^{-1}(0)\cap \operatorname{supp}(m^* j_{0!} j_0^!\mathbf{A}^\bullet)$ is an open subset of $\operatorname{supp}(m^* j_{0!} j_0^!\mathbf{A}^\bullet)$.

Suppose that $x \in f^{-1}(0)\cap\operatorname{supp}(m^* j_{0!} j_0^1\mathbf{A}^\bullet)$. We need to show that there exists an open neighborhood $\mathcal{W}$ of x in X such that $\mathcal{W}\cap\operatorname{supp}(m^* j_{0!} j_0^1\mathbf{A}^\bullet)\subseteq f^{-1}(0)$.

Fix a Whitney stratification of X, with respect to which $\mathbf{A}^\bullet$ is constructible. Then, select $\mathcal{W}$ so that all of the stratified critical points of f, inside $\mathcal{W}$, are contained in $f^{-1}(0)$. Suppose that there were a point $y\in\mathcal{W}$ such that $y \in f^{-1}(L_0) - f^{-1}(0)$ and the stalk cohomology of $m^* j_{0!}\, j_0^!\mathbf{A}^\bullet$ at y is nonzero. Then, by definition, y would be a point in the support $\phi_{f-f(y)}[-1]\mathbf{A}^\bullet$, which, again, is contained in the stratified critical locus of f and, hence, is contained in $f^{-1}(0)$. This contradiction concludes the proof. $\square$

We continue to let $p : f^{-1}(0) \hookrightarrow X$ denote the closed inclusion, and now let $i : X - f^{-1}(0) \hookrightarrow X$ denote the open inclusion. Consider the two fundamental distinguished triangles related to the nearby and vanishing cycles:

$$p^*[-1] \xrightarrow{\text{comp}} \psi_f[-1] \xrightarrow{\text{can}} \phi_f[-1] \xrightarrow{[1]}$$

and

$$\phi_f[-1] \xrightarrow{\text{var}} \psi_f[-1] \to p^![1] \xrightarrow{[1]} .$$

The morphisms comp, can, and var are usually referred to as the *comparison map*, *canonical map*, and *variation map*. As $p^*i_! = 0 = p^!i_*$ and as $\psi_f[-1]$ depends only on the complex outside of $f^{-1}(0)$, the top triangle, applied to $i_!i^!$ and the bottom triangle applied to i_*i^* yield natural isomorphisms

$$\alpha : \psi_f[-1] \xrightarrow{\simeq} \phi_f[-1]i_!i^! \quad \text{and} \quad \beta : \phi_f[-1]i_*i^* \xrightarrow{\simeq} \psi_f[-1].$$

Corollary 3.2. *There is a natural isomorphism*

$$\psi_f[-1] \circ \mathcal{D} \simeq \mathcal{D} \circ \psi_f[-1]$$

of functors from $D^b_c(X)$ to $D^b_c(f^{-1}(0))$.

Proof. $\psi_f[-1] \circ \mathcal{D} \simeq \phi_f[-1]i_!i^! \circ \mathcal{D} \simeq \mathcal{D} \circ \phi_f[-1]i_*i^* \simeq \mathcal{D} \circ \psi_f[-1]$. □

Acknowledgement

We thank Jörg Schürmann for providing valuable comments and references for related results.

References

[Brav et al. 2015] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendröi, "Symmetries and stabilization for sheaves of vanishing cycles", *J. Singul.* **11** (2015), 85–151. MR Zbl

[Brylinski and Monteiro Fernandes 1986] J.-L. Brylinski and T. Monteiro Fernandes, *Géométrie et analyse microlocales*, Astérisque **140–141**, Société Mathématique de France, Paris, 1986. MR Zbl

[Dimca 2004] A. Dimca, *Sheaves in topology*, Springer, Berlin, 2004. MR Zbl

[Illusie 1994] L. Illusie, "Autour du théorème de monodromie locale", pp. 9–57 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), edited by J. Fontaine, Astérisque **223**, Société Mathématique de France, Paris, 1994. MR Zbl

[Kashiwara and Schapira 1990] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften **292**, Springer, Berlin, 1990. MR Zbl

[Kontsevich and Soibelman 2011] M. Kontsevich and Y. Soibelman, "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants", *Commun. Number Theory Phys.* **5**:2 (2011), 231–352. MR Zbl

[Massey 2003] D. B. Massey, *Numerical control over complex analytic singularities*, Memoirs of the American Mathematical Society **163**:778, American Mathematical Society, Providence, RI, 2003. MR Zbl

[Saito 1988] M. Saito, "Modules de Hodge polarisables", *Publ. Res. Inst. Math. Sci.* **24**:6 (1988), 849–995. MR Zbl

[Saito 1989] M. Saito, "Duality for vanishing cycle functors", *Publ. Res. Inst. Math. Sci.* **25**:6 (1989), 889–921. MR Zbl

[Schürmann 2003] J. Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne **63**, Birkhäuser, Basel, 2003. MR Zbl

Received January 7, 2016. Revised March 9, 2016.

DAVID B. MASSEY
DEPARTMENT OF MATHEMATICS
NORTHEASTERN UNIVERSITY
360 HUNTINGTON AVENUE
BOSTON, MA 02115
UNITED STATES
d.massey@neu.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US $440/year for the electronic version, and $600/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 284 No. 2 October 2016

0030-8730(201610)284:2;1-Z