Pacific Journal of Mathematics

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Volume 284 No. 2 October 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.284.439

THE NEF CONES OF AND MINIMAL-DEGREE CURVES IN THE HILBERT SCHEMES OF POINTS ON CERTAIN SURFACES

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We determine the nef cones of the Hilbert schemes of points on certain surfaces X with $h^1(X, \mathcal{O}_X) = 0$. Then we apply the results to Hirzebruch surfaces, and study the minimal-degree curves in the Hilbert schemes of points on Hirzebruch surfaces. Our results generalize those in Li, Qin, and Zhang (2003).

1. Introduction

Hilbert schemes are classical objects in algebraic geometry, and have been studied extensively since their constructions by Grothendieck. Hilbert schemes of points on smooth surfaces are known to be smooth and irreducible, and have deep connections with combinatorics, representation theory and string theory. Ample divisors on these Hilbert schemes were considered in [Beltrametti and Sommese 1991; 1993; Catanese and Gœttsche 1990]. The nef cones of the Hilbert schemes of points on smooth surfaces were first investigated in [Li et al. 2003] when the surface is the projective plane. Recently, these nef cones were further understood in [Arcara et al. 2013; Bertram and Coskun 2013; Bolognese et al. 2015] via Bridgeland stability.

In this paper, we generalize the methods and results in [Li et al. 2003], and prove a structure theorem for the nef cones of the Hilbert schemes of points on certain surfaces. To state our result, let X be a smooth projective complex surface. The nef cone of X is the span of the nef divisors on X. We use $\mathrm{NE}(X)$ to denote the cone spanned by all the effective curves in X. It is well-known that $\mathrm{NE}(X)$ is dual to the nef cone of X. Let $X^{[n]}$ be the Hilbert scheme of points in X. By [Fogarty 1968; Iarrobino 1977], $X^{[n]}$ is a smooth irreducible variety of dimension $2n$.

Theorem 1.1. *Let $n \geq 2$, and let the surface X satisfy $h^1(X, \mathcal{O}_X) = 0$. Assume that the nef cone of X is the span of the divisors $F_1, \dots, F_t$, and the cone* $\mathrm{NE}(X)$ *is the*

Qin partially supported by a grant from the Simons Foundation. Tu partially supported by NSFC grant 11171258.
MSC2010: 14C05.
Keywords: nef cones, Hilbert schemes.

span of the curves $C_1, \dots, C_t$ *with* $F_i \cdot C_j = \delta_{i,j}$ *for all* i *and* j. *Assume further that*

$$\mathcal{O}_X\left((n-1)\sum_{i=1}^{t} F_i\right)$$

is $(n-1)$*-very ample. Then*

(i) *the nef cone of the Hilbert scheme* $X^{[n]}$ *is spanned by*

$$D_{F_1}, \dots, D_{F_t}, (n-1)\sum_{i=1}^{t} D_{F_i} - B_n/2; \tag{1-1}$$

(ii) *the cone* $\mathrm{NE}(X^{[n]})$ *is spanned by the classes*

$$\beta_{C_1} - (n-1)\beta_n, \dots, \beta_{C_t} - (n-1)\beta_n, \beta_n. \tag{1-2}$$

In the above theorem, B_n denotes the boundary divisor of the Hilbert scheme $X^{[n]}$ consisting of the elements $\xi \in X^{[n]}$ which are not smooth as subschemes of $X^{[n]}$, and β_n is the minimal curve class contracted by the Hilbert–Chow morphism $X^{[n]} \to X^{(n)}$ sending an element $\xi \in X^{[n]}$ to its support (with multiplicities) in the n-th symmetric product $X^{(n)}$ of X. We refer to (2-4), (2-3) and Definition 2.1 for the definitions of D_F, β_C and $(n-1)$-very ampleness, respectively. Theorem 1.1 is proved in Section 2. Our main idea in the proof of Theorem 1.1 is to construct curves in $X^{[n]}$ which provide us with information about the nef divisors in $X^{[n]}$.

In Section 3, we apply Theorem 1.1 to the case when X is a Hirzebruch surface, and recover a result in [Bertram and Coskun 2013]. Moreover, when X is a Hirzebruch surface, we classify all the curves in the Hilbert scheme $X^{[n]}$ whose homology classes are contained in the list (1-2). These curves have minimal degree in the sense that their intersection numbers with certain very ample divisors in $X^{[n]}$ are all equal to 1. We compute the normal bundles of these curves, and prove that their moduli spaces are unobstructed, i.e., are smooth with the expected dimensions.

Conventions. Let $0 \le k \le n$ and V be an n-dimensional vector space. We use the Grassmannian $\mathbb{G}(V, k)$ to denote the set of all k-dimensional quotients of V, or equivalently, the set of all $(n-k)$-dimensional subspaces of V. Also, we take $\mathbb{P}(V) = \mathbb{G}(V, 1)$. So the set of lines in $\mathbb{P}(V)$ is the Grassmannian $\mathbb{G}(V, 2)$.

2. The nef cones of the Hilbert schemes of points on certain surfaces

In this section, we study the nef cones of the Hilbert schemes of points on certain surfaces with $h^1(X, \mathcal{O}_X) = 0$. Our goal is to prove Theorem 1.1.

Let X be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ be the corresponding sheaf of ideals and $\mathcal{O}_\xi$ the structure sheaf. The subset

$$B_n = \{\xi \in X^{[n]} \mid |\operatorname{Supp}(\xi)| < n\} \tag{2-1}$$

is defined to be the boundary of $X^{[n]}$. Let C be a real surface in X, and fix distinct points $x_1, \dots, x_{n-1} \in X$ which are not contained in C. Define

$$\beta_n = \{\xi + x_2 + \cdots + x_{n-1} \in X^{[n]} \mid \operatorname{Supp}(\xi) = \{x_1\}\}, \tag{2-2}$$

$$\beta_C = \{x + x_1 + \cdots + x_{n-1} \in X^{[n]} \mid x \in C\}, \tag{2-3}$$

$$D_C = \{\xi \in X^{[n]} \mid \operatorname{Supp}(\xi) \cap C \neq \varnothing\}. \tag{2-4}$$

Note that β_C is a curve, and D_C a divisor, in $X^{[n]}$ when C is a smooth algebraic curve in X. We extend the notions β_C and D_C to all the divisors C in X by linearity.

Nakajima [1997] and Grojnowski [1996] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes $X^{[n]}$. Let $H^*(X)$ be the total cohomology of X with $\mathbb{C}$-coefficients. Denote the Heisenberg operators of Nakajima and Grojnowski by $\mathfrak{a}_m(\alpha)$ where $m \in \mathbb{Z}$ and $\alpha \in H^*(X)$. Set

$$\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}).$$

Then the space $\mathbb{H}_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $\mathfrak{a}_m(\alpha)$ with the highest weight vector being

$$|0\rangle = 1 \in H^*(X^{[0]}) = \mathbb{C}.$$

It follows that the n-th component $H^*(X^{[n]})$ in the Fock space $\mathbb{H}_X$ is linearly spanned by the *Heisenberg monomial classes*

$$\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle,$$

where $k \geq 0$, $n_1, \dots, n_k > 0$ and $n_1 + \cdots + n_k = n$. We have

$$\beta_n = \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \tag{2-5}$$

$$\beta_C = \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle, \tag{2-6}$$

$$B_n = \frac{1}{(n-2)!}\mathfrak{a}_{-1}(1_X)^{n-2}\mathfrak{a}_{-2}(1_X)|0\rangle, \tag{2-7}$$

$$D_C = \frac{1}{(n-1)!}\mathfrak{a}_{-1}(1_X)^{n-1}\mathfrak{a}_{-1}(C)|0\rangle, \tag{2-8}$$

where x and 1_X denote the cohomology classes corresponding to a point $x \in X$ and the surface X, respectively. Abusing notation, we also use C to denote the cohomology class corresponding to the real surface C.

The following important definition is from [Beltrametti and Sommese 1991].

Definition 2.1. Let $n \geq 1$. A line bundle L on the surface X is $(n-1)$-*very ample* if the restriction $H^0(X, L) \to H^0(X, \mathcal{O}_\xi \otimes L)$ is surjective for every $\xi \in X^{[n]}$.

The concept of $(n-1)$-very ampleness relates $X^{[n]}$ to a Grassmannian as follows. The surjective map in Definition 2.1 represents an element in $\mathbb{G}(H^0(X, L), n)$. So if L is $(n-1)$-very ample, then we obtain a morphism

$$\varphi_n(L) : X^{[n]} \to \mathbb{G}(H^0(X, L), n). \tag{2-9}$$

Let $h = h^0(X, L)$, and let $\mathfrak{P} : \mathbb{G}(\mathbb{C}^h, n) \to \mathbb{P}\big(\big(\bigwedge^{h-n} \mathbb{C}^h\big)^*\big)$ be the Plücker embedding. Then we see from the Appendix of [Beltrametti and Sommese 1991] that

$$(\mathfrak{P} \circ \varphi_n(L))^* \mathcal{H} = \mathcal{O}_{X^{[n]}}(D_{c_1(L)} - B_n/2), \tag{2-10}$$

where $\mathcal{H}$ is the hyperplane line bundle over the projective space $\mathbb{P}\big(\big(\bigwedge^{h-n} \mathbb{C}^h\big)^*\big)$.

Lemma 2.2. *If L is $(n-1)$-very ample, then the divisor $D_{c_1(L)} - B_n/2$ is nef. If L is n-very ample, then the divisor $D_{c_1(L)} - B_n/2$ is very ample.* □

The first statement in Lemma 2.2 follows immediately from (2-10), and the second statement was proved in [Catanese and Gœttsche 1990].

In $X^{[n]} \times X$, we have the universal codimension-2 subscheme

$$\mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \operatorname{Supp}(\xi)\} \subset X^{[n]} \times X. \tag{2-11}$$

Define the incidence variety $X^{[n-1,n]} = \{(\xi, \eta) \in X^{[n-1]} \times X^{[n]} \mid \xi \subset \eta\}$. It is well-known [Cheah 1998; Tikhomirov 1994] that $X^{[n-1,n]}$ is smooth and of dimension $2n$. Define

$$\begin{aligned} f_n &: X^{[n-1,n]} \to X^{[n-1]} && \text{with } f_n(\xi, \eta) = \xi, \\ g_n &: X^{[n-1,n]} \to X^{[n]} && \text{with } g_n(\xi, \eta) = \eta, \\ \rho &: X^{[n-1,n]} \to X && \text{with } \rho(\xi, \eta) = \operatorname{Supp}(I_\xi / I_\eta). \end{aligned}$$

Set $\phi_n = (f_n, \rho) : X^{[n-1,n]} \to X^{[n-1]} \times X$. By Proposition 2.2 in [Ellingsrud and Strømme 1998], ϕ_n is the blowing-up morphism of $X^{[n-1]} \times X$ along $\mathcal{Z}_{n-1}$.

Next, let C be an irreducible curve in X. Let $\xi = x_1 + \cdots + x_{n-1} \in X^{[n-1]}$, where $x_1, \ldots, x_{n-1}$ are distinct smooth points on C. Let $(C + \xi)$ be the closure of $(C - \operatorname{Supp}(\xi)) + \xi$ in $X^{[n]}$. Alternatively, consider

$$\begin{array}{ccccccc} \widetilde{C}_\xi & \subset & \widetilde{X}_\xi & \subset & X^{[n-1,n]} & \xrightarrow{g_n} & X^{[n]} \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \phi_n} & & \\ \{\xi\} \times C & \subset & \{\xi\} \times X & \subset & X^{[n-1]} \times X & & \end{array} \tag{2-12}$$

where $\widetilde{C}_\xi$ and $\widetilde{X}_\xi$ are the strict transforms of $\{\xi\} \times C$ and $\{\xi\} \times X$ in $X^{[n-1,n]}$, respectively. Since ϕ_n is the blowing-up morphism of $X^{[n-1]} \times X$ along $\mathcal{Z}_{n-1}$, it follows that $\widetilde{X}_\xi$ is isomorphic to the blowup of $\{\xi\} \times X \cong X$ at $x_1, \dots, x_{n-1}$. For $1 \le i \le (n-1)$, let E_i be the exceptional divisor in $\widetilde{X}_\xi$ over x_i. Then we obtain

$$(\phi_n|_{\widetilde{X}_\xi})^*(\{\xi\} \times C) = \widetilde{C}_\xi + \sum_{i=1}^{n-1} E_i \tag{2-13}$$

in the Chow group $A_1(\widetilde{X}_\xi)$. Notice that $g_n(\widetilde{C}_\xi) = (C+\xi)$ and

$$g_n(E_i) = M_2(x_i) + x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_{n-1}.$$

In fact, since $g_n|_{\widetilde{X}_\xi} : \widetilde{X}_\xi \to g_n(\widetilde{X}_\xi)$ is an isomorphism, we have

$$(g_n|_{\widetilde{X}_\xi})_*(\widetilde{C}_\xi) = (C+\xi) \quad \text{and} \quad (g_n|_{\widetilde{X}_\xi})_*(E_i) = \beta_n. \tag{2-14}$$

Lemma 2.3. *With the above notation, $(C+\xi) = \beta_C - (n-1)\beta_n$ in $A_1(X^{[n]})$.*

Proof. Choose two smooth curves C_1 and C_2 in X such that $C = C_1 - C_2$ in $A_1(X)$ and $\mathrm{Supp}(\xi) \cap (C_1 \cup C_2) = \varnothing$. Then in $A_1(X^{[n]})$, we have

$$\begin{aligned} &(g_n|_{\widetilde{X}_\xi})_*(\phi_n|_{\widetilde{X}_\xi})^*(\{\xi\} \times C) \\ &\qquad = (g_n|_{\widetilde{X}_\xi})_*(\phi_n|_{\widetilde{X}_\xi})^*(\{\xi\} \times C_1) - (g_n|_{\widetilde{X}_\xi})_*(\phi_n|_{\widetilde{X}_\xi})^*(\{\xi\} \times C_2) \\ &\qquad = (C_1+\xi) - (C_2+\xi) \\ &\qquad = \beta_{C_1} - \beta_{C_2} = \beta_C. \end{aligned} \tag{2-15}$$

On the other hand, applying (2-13) and (2-14), we conclude that

$$(g_n|_{\widetilde{X}_\xi})_*(\phi_n|_{\widetilde{X}_\xi})^*(\{\xi\} \times C) = (g_n|_{\widetilde{X}_\xi})_*\Bigl(\widetilde{C}_\xi + \sum_{i=1}^{n-1} E_i\Bigr) = (C+\xi) + (n-1)\beta_n.$$

Combining this with (2-15), we see that $(C+\xi) = \beta_C - (n-1)\beta_n$ in $A_1(X^{[n]})$. □

Lemma 2.4. *Let F be a divisor on X. If $D_F - d(B_n/2)$ is nef, then $d \ge 0$ and $F \cdot C \ge d(n-1)$ for every irreducible curve $C \subset X$. In particular, F is nef.*

Proof. Note that $D_F \cdot \beta_n = 0$ and $B_n \cdot \beta_n = -2$. Thus, we have

$$0 \le (D_F - d(B_n/2)) \cdot \beta_n = d.$$

Since $D_F \cdot \beta_C = F \cdot C$ and $B_n \cdot \beta_C = 0$, we conclude from Lemma 2.3 that

$$0 \le (D_F - d(B_n/2)) \cdot (\beta_C - (n-1)\beta_n) = F \cdot C - d(n-1). \qquad \square$$

Lemma 2.5. *Let F be a divisor in X. Let C be a smooth rational curve in X, and consider the n-th symmetric product $C^{(n)} = \mathrm{Hilb}^n(C) \subset X^{[n]}$. Then*

(i) *every line in $C^{(n)} \cong \mathbb{P}^n$ is homologous to $\beta_C - (n-1)\beta_n$;*

(ii) *$\mathcal{O}_{X^{[n]}}(D_F)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(C \cdot F)$ and $\mathcal{O}_{X^{[n]}}(B_n/2)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(n-1)$.*

Proof. (i) Let $x_1, \dots, x_{n-1} \in C$ be distinct, and put $\gamma = C + (x_1 + \cdots + x_{n-1})$. Then γ is a line in the projective space $C^{(n)} \cong \mathbb{P}^n$. By Lemma 2.3,

$$\gamma \sim \beta_C - (n-1)\beta_n,$$

where $\sim$ denotes homologous relation. So every line in $C^{(n)} \cong \mathbb{P}^n$ is homologous to the class $\beta_C - (n-1)\beta_n$.

(ii) Since $\gamma \cdot D_F|_{C^{(n)}} = \gamma \cdot D_F = (\beta_C - (n-1)\beta_n) \cdot D_F = C \cdot F$, we get

$$\mathcal{O}_{X^{[n]}}(D_F)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(C \cdot F).$$

Using a similar method, we obtain $\mathcal{O}_{X^{[n]}}(B_n/2)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(n-1)$. □

In the rest of the paper, we assume that $h^1(X, \mathcal{O}_X) = 0$. Then

$$\operatorname{Pic}(X^{[n]}) \cong \operatorname{Pic}(X) \oplus \mathbb{Z} \cdot (B_n/2) \tag{2-16}$$

by [Fogarty 1973]. Under this isomorphism, the divisor $D_C \in \operatorname{Pic}(X^{[n]})$ corresponds to $C \in \operatorname{Pic}(X)$. Let $\{\alpha_1, \dots, \alpha_s\}$ be a linear basis of $H^2(X)$. Then

$$\{D_{\alpha_1}, \dots, D_{\alpha_s}, B_n\} \tag{2-17}$$

is a linear basis of $H^2(X^{[n]})$. Represent $\alpha_1, \dots, \alpha_s$ by real surfaces $C_1, \dots, C_s \subset X$, respectively. Then a linear basis of $H_2(X^{[n]})$ is given by

$$\{\beta_{C_1}, \dots, \beta_{C_s}, \beta_n\}. \tag{2-18}$$

We are now ready to prove our main result in this paper.

Theorem 2.6. *Let $n \geq 2$, and let the surface X satisfy $h^1(X, \mathcal{O}_X) = 0$. Assume that the nef cone of X is the span of the divisors $F_1, \dots, F_t$, and the cone $\mathrm{NE}(X)$ is the span of the curves $C_1, \dots, C_t$ with $F_i \cdot C_j = \delta_{i,j}$ for all i and j. Assume further that $\mathcal{O}_X\big((n-1)\sum_{i=1}^t F_i\big)$ is $(n-1)$-very ample. Then*

(i) *the nef cone of the Hilbert scheme $X^{[n]}$ is spanned by*

$$D_{F_1}, \dots, D_{F_t}, (n-1)\sum_{i=1}^{t} D_{F_i} - B_n/2; \tag{2-19}$$

(ii) *the cone $\mathrm{NE}(X^{[n]})$ is spanned by the classes*

$$\beta_{C_1} - (n-1)\beta_n, \dots, \beta_{C_t} - (n-1)\beta_n, \beta_n. \tag{2-20}$$

Proof. (i) For $1 \leq j \leq n$, let $p_j : X^n \to X$ be the projection to the j-th factor. Let $X^{(n)}$ be the n-th symmetric product of X, and let $\upsilon_n : X^n \to X^{(n)}$ be the quotient map. Let $\rho_n : X^{[n]} \to X^{(n)}$ be the Hilbert–Chow morphism sending an element

$\xi \in X^{[n]}$ to its support (with multiplicities) in $X^{(n)}$. For each F_i, there exists a divisor H_i on $X^{(n)}$ such that

$$\rho_n^* H_i = D_{F_i}, \qquad \upsilon_n^* H_i = \sum_{j=1}^{n} p_j^* F_i.$$

It follows that since F_i is nef, the divisor D_{F_i} is nef as well. Since the line bundle $\mathcal{O}_X\big((n-1)\sum_{i=1}^t F_i\big)$ is $(n-1)$-very ample, we conclude from Lemma 2.2 that $(n-1)\sum_{i=1}^t D_{F_i} - B_n/2$ is a nef divisor. Thus, the cone $\boldsymbol{C}_1$ spanned by the divisors in (2-19) is contained in the nef cone of $X^{[n]}$.

Conversely, assume that $D_F - dB_n/2$ is a nef divisor on $X^{[n]}$. Let $F = \sum_{i=1}^t a_i F_i$. By Lemma 2.4, $d \geq 0$ and $F \cdot C \geq d(n-1)$ for every irreducible curve $C \subset X$. So

$$a_i = F \cdot C_i \geq d(n-1)$$

for every i. Now the nef divisor $D_F - dB_n/2$ can be written as

$$\sum_{i=1}^t a_i D_{F_i} - dB_n/2 = \sum_{i=1}^t (a_i - d(n-1))D_{F_i} + d\left((n-1)\sum_{i=1}^t D_{F_i} - B_n/2\right).$$

Therefore, $D_F - dB_n/2 \in \boldsymbol{C}_1$. It follows that $\boldsymbol{C}_1$ is the nef cone of $X^{[n]}$.

(ii) First of all, note that since the divisor F_i is nef and $F_i \cdot C_i = 1$, the curve C_i contains at least one reduced irreducible component. So the curve $\beta_{C_i} - (n-1)\beta_n$ is well-defined, and the cone $\boldsymbol{C}_2$ spanned by the curves in (2-20) is contained in the cone $\mathrm{NE}(X^{[n]})$. Conversely, assume that $\sum_{i=1}^t b_i \beta_{C_i} + c\beta_n$ is contained in $\mathrm{NE}(X^{[n]})$. Then $\big(\sum_{i=1}^t b_i \beta_{C_i} + c\beta_n\big) \cdot H \geq 0$ for every nef divisor H in $X^{[n]}$. Letting $H = D_{F_i}$, we get $b_i \geq 0$ for every i. Letting $H = (n-1)\sum_{i=1}^t D_{F_i} - B_n/2$, we obtain $(n-1)\sum_{i=1}^t b_i + c \geq 0$. Therefore, we have

$$\sum_{i=1}^t b_i \beta_{C_i} + c\beta_n = \sum_{i=1}^t b_i(\beta_{C_i} - (n-1)\beta_n) + \left((n-1)\sum_{i=1}^t b_i + c\right)\beta_n \in \boldsymbol{C}_2.$$

It follows that $\boldsymbol{C}_2$ coincides with the cone $\mathrm{NE}(X^{[n]})$. □

Corollary 2.7. *Under the same assumptions as in Theorem 2.6, if γ is an irreducible curve in the Hilbert scheme $X^{[n]}$, then γ is homologous to*

$$\sum_{i=1}^t b_i(\beta_{C_i} - (n-1)\beta_n) + c\beta_n$$

for some nonnegative integers $b_1, \dots, b_t, c$ not all equal to zero.

Proof. By Theorem 2.6 (ii), $b_1, \dots, b_t, c$ are nonnegative and not all equal to 0. Intersecting the curve γ with the divisors $D_{F_1}, \dots, D_{F_t}$ and $(n-1)\sum_{i=1}^t D_{F_i} - B_n/2$, respectively, we see that $b_1, \dots, b_t, c$ must be integers. □

3. Application to Hirzebruch surfaces

In this section, we apply Theorem 2.6 to the Hirzebruch surfaces and recover a result in [Bertram and Coskun 2013]. Then we study the curves in the Hilbert schemes of points on the Hirzebruch surfaces, which have the minimal degree. We compute their normal bundles, and prove that their moduli spaces are unobstructed.

Let X denote the Hirzebruch surface $\mathbb{F}_e$ with $e \geq 0$. Let f be a fiber of the ruling $\pi : X \to \mathbb{P}^1$, and $\sigma \subset X$ be a section of π such that $\sigma^2 = -e$. Then

$$\mathrm{Pic}(X) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f.$$

It is well-known that $a\sigma + bf$ is nef if and only if $a \geq 0$ and $b \geq ae$. The following lemma was proved in [Beltrametti and Sommese 1993].

Lemma 3.1. *$\mathcal{O}_X(a\sigma + bf)$ is n-very ample if and only if $a \geq n$ and $b \geq n + ae$.* □

Proposition 3.2. *Let $n \geq 2$, and let X be the Hirzebruch surface $\mathbb{F}_e$. Then*

(i) *the nef cone of the Hilbert scheme $X^{[n]}$ is spanned by*

$$D_f, \quad D_\sigma + eD_f, \quad (n-1)D_\sigma + (n-1)(1+e)D_f - B_n/2; \tag{3-1}$$

(ii) *the cone* $\mathrm{NE}(X^{[n]})$ *is spanned by the classes*

$$\beta_\sigma - (n-1)\beta_n, \quad \beta_f - (n-1)\beta_n, \quad \beta_n. \tag{3-2}$$

Proof. The nef cone of X is the span of $F_1 = f$ and $F_2 = \sigma + ef$, and the cone $\mathrm{NE}(X)$ is the span of $C_1 = \sigma$ and $C_2 = f$. Note that $F_i \cdot C_j = \delta_{i,j}$ for all i and j. In addition, by Lemma 3.1, the line bundle $\mathcal{O}_X((n-1)F_1 + (n-1)F_2)$ is $(n-1)$-very ample. Hence our proposition follows from Theorem 2.6. □

Proposition 3.2 has been proved in [Bertram and Coskun 2013]. We now study the curves in $X^{[n]}$ whose homology classes are contained in the list (3-2). Let

$$L_n = n\sigma + n(1+e)f.$$

By Lemma 3.1, the line bundle $\mathcal{O}_X(L_n)$ is n-very ample. By Lemma 2.2, the divisor

$$D_{L_n} - B_n/2 = nD_\sigma + n(1+e)D_f - B_n/2$$

in $X^{[n]}$ is very ample. Our next lemma characterizes the homology classes in (3-2).

Lemma 3.3. *Let γ be a curve in $X^{[n]}$ with $\gamma \cdot (nD_\sigma + n(1+e)D_f - B_n/2) = 1$. Then γ is a smooth rational curve. Moreover, $\gamma \sim \beta_n$, $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$.*

Proof. Since $nD_\sigma + n(1+e)D_f - B_n/2$ is very ample, γ is a smooth rational curve. By Corollary 2.7, $\gamma \sim a(\beta_\sigma - (n-1)\beta_n) + b(\beta_f - (n-1)\beta_n) + c\beta_n$ for some nonnegative integers a, b, c. Since $\gamma \cdot (nD_\sigma + n(1+e)D_f - B_n/2) = 1$, we obtain

$$a + b + c = 1.$$

Therefore, we have $\gamma \sim \beta_n$, $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. □

The curves in $X^{[n]}$ homologous to β_n have been classified in [Li et al. 2003]. In the rest of this section, we study the curves $\gamma \subset X^{[n]}$ which are homologous to $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. By Lemma 3.1, the line bundle $\mathcal{O}_X(L_{n-1})$ is $(n-1)$-very ample. So by (2-9), we have the morphism

$$\varphi := \varphi_n(\mathcal{O}_X(L_{n-1})) : X^{[n]} \to \mathbb{G}\big(H^0(X, \mathcal{O}_X(L_{n-1})), n\big).$$

Lemma 3.4. *Let γ be an irreducible curve in $X^{[n]}$ satisfying*

$$\gamma \cdot ((n-1)D_\sigma + (n-1)(1+e)D_f - B_n/2) = 0.$$

Then $\gamma \sim \beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. Moreover, γ is contracted by φ.

Proof. The first part of the lemma is proved by an argument similar to the proof of Lemma 3.3. For the second part, we notice from (2-10) that

$$\begin{aligned}(\mathfrak{P} \circ \varphi)^* \mathcal{H} &= \mathcal{O}_{X^{[n]}}(D_{L_{n-1}} - B_n/2) \\ &= \mathcal{O}_{X^{[n]}}((n-1)D_\sigma + (n-1)(1+e)D_f - B_n/2).\end{aligned}$$

Therefore, the curve γ is contracted by the morphism $\mathfrak{P} \circ \varphi$. Since $\mathfrak{P}$ is an embedding, the curve γ is contracted by the morphism φ. □

In the following, we fix a curve $\gamma \subset X^{[n]}$ homologous to $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. Let $X^{(n)}$ be the n-th symmetric product of X and $\upsilon_n : X^n \to X^{(n)}$ the quotient map. Let $\rho_n : X^{[n]} \to X^{(n)}$ be the Hilbert–Chow morphism sending an element $\xi \in X^{[n]}$ to its support (with multiplicities) in $X^{(n)}$. Let p_1 be the projection from X^n to the first factor.

Definition 3.5. Define C_γ to be the union of all the curves in $p_1(\upsilon_n^{-1}(\rho_n(\gamma)))$.

Lemma 3.6. *Let $\gamma \sim \beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. Then $C_\gamma \sim \sigma$ or f.*

Proof. First of all, we claim that $C_\gamma \neq \varnothing$. Indeed, if $C_\gamma = \varnothing$, then $p_1(\upsilon_n^{-1}(\rho_n(\gamma)))$ is a finite set of points in X. Since the divisor $\sigma + (1+e)f$ is very ample, we can choose a smooth curve $F \in |\sigma + (1+e)f|$ such that $F \cap p_1(\upsilon_n^{-1}(\rho_n(\gamma))) = \varnothing$. Since the elements of γ are supported in $p_1(\upsilon_n^{-1}(\rho_n(\gamma)))$, we must have $\gamma \cap D_F = \varnothing$. It follows that $\gamma \cdot D_F = 0$. However, this contradicts $\gamma \cdot D_F = 1$.

Next, assume that $C_\gamma \cdot (\sigma + (1+e)f) \geq 2$. Take a point $\xi \in \gamma$ and a smooth point $x \in C_\gamma$ such that $x \notin \mathrm{Supp}(\xi)$. Since $x \in C_\gamma \subset p_1(\upsilon_n^{-1}(\rho_n(\gamma)))$, there exists $\xi_x \in \gamma$ such that $\rho_n(\xi_x) = n_x x + \eta_x$, where $n_x \geq 1$, $\eta_x \in X^{(n-n_x)}$ and $x \notin \mathrm{Supp}(\eta_x)$. Choose a smooth curve $F \in |\sigma + (1+e)f|$ missing $\mathrm{Supp}(\eta_x) \cup \mathrm{Supp}(\xi)$, passing through x, and intersecting C_γ transversally. Then $F \cap C_\gamma$ is a finite set. Since $C_\gamma \cdot F \geq 2$, $F \cap C_\gamma$ contains one more point $y \neq x$. Hence there exists $\xi_y \in \gamma$ with $y \in \mathrm{Supp}(\xi_y)$. Thus $\xi_x, \xi_y \in \gamma \cap D_F$. Since $y \neq x$, $y \in F$ and F misses $\mathrm{Supp}(\eta_x)$, we get

$$y \notin \{x\} \cup \mathrm{Supp}(\eta_x) = \mathrm{Supp}(\xi_x).$$

So $\xi_x \neq \xi_y$. Since $\operatorname{Supp}(\xi) \cap F = \varnothing$, we have $\xi \notin D_F$. Since $\xi \in \gamma$ and γ is a smooth rational curve, γ is not contained in D_F. Therefore, $\gamma \cap D_F$ is a finite set of points. Since $\xi_x, \xi_y \in \gamma \cap D_F$ and $\xi_x \neq \xi_y$, we obtain $\gamma \cdot D_F \geq 2$, which contradicts $\gamma \cdot D_F = 1$.

It follows that $C_\gamma \cdot (\sigma + (1+e)f) = 1$. Since the cone $\mathrm{NE}(\mathbb{F}_e)$ is spanned by σ and f, we conclude that $C_\gamma \sim \sigma$ or f. □

By Lemma 3.6, $C_\gamma \sim \sigma$ or f. So C_γ is a smooth rational curve, and

$$\mathcal{O}_X(L_{n-1})|_{C_\gamma} \cong \mathcal{O}_{C_\gamma}(n-1).$$

Let $V_{C_\gamma} \subset H^0(X, \mathcal{O}_X(L_{n-1}))$ be the image of the injection

$$H^0(X, \mathcal{O}_X(L_{n-1} - C_\gamma)) \to H^0(X, \mathcal{O}_X(L_{n-1})),$$

which is induced by the exact sequence

$$0 \to \mathcal{O}_X(L_{n-1} - C_\gamma) \to \mathcal{O}_X(L_{n-1}) \to \mathcal{O}_{C_\gamma}(n-1) \to 0. \tag{3-3}$$

Similarly, for $\xi \in \gamma$, let $V_\xi \subset H^0(X, \mathcal{O}_X(L_{n-1}))$ be the image of the injection

$$H^0(X, \mathcal{O}_X(L_{n-1}) \otimes I_\xi) \to H^0(X, \mathcal{O}_X(L_{n-1})).$$

Since $\mathcal{O}_X(L_{n-1})$ is $(n-1)$-very ample, we obtain

$$\dim V_\xi = h^0(X, \mathcal{O}_X(L_{n-1})) - h^0(\mathcal{O}_\xi) = h^0(X, \mathcal{O}_X(L_{n-1})) - n. \tag{3-4}$$

Since the curve γ is contracted to a point by the morphism φ, the subspaces V_ξ of $H^0(X, \mathcal{O}_X(L_{n-1}))$ are independent of $\xi \in \gamma$. Set $V_\gamma = V_\xi$ where $\xi \in \gamma$.

Lemma 3.7. *If $n \geq e+1$, then $V_{C_\gamma} = V_\gamma$.*

Proof. Since $K_X = -2\sigma - (2+e)f$, the divisor $L_{n-1} - C_\gamma - K_X$ is ample if $C_\gamma = \sigma$. Similarly, since $n \geq e+1$, $L_{n-1} - C_\gamma - K_X$ is ample if $C_\gamma = f$. By the Kodaira vanishing theorem, $H^1(X, \mathcal{O}_X(L_{n-1} - C_\gamma)) = 0$. So we see from (3-3) that

$$\dim V_{C_\gamma} = h^0(X, \mathcal{O}_X(L_{n-1})) - h^0(C_\gamma, \mathcal{O}_{C_\gamma}(n-1)) = h^0(X, \mathcal{O}_X(L_{n-1})) - n.$$

In view of (3-4), we conclude that

$$\dim V_{C_\gamma} = \dim V_\xi = \dim V_\gamma.$$

Thus, to prove our lemma, it remains to prove that $V_\gamma \subset V_{C_\gamma}$. Indeed, let $f \in V_\gamma$ be a section. Let $x \in C_\gamma$. Since $C_\gamma \subset p_1(\upsilon_n^{-1}(\rho_n(\gamma)))$, there exists $\xi \in \gamma$ such that $x \in \operatorname{Supp}(\xi)$. Since $V_\gamma = V_\xi$, f vanishes at every point in $\operatorname{Supp}(\xi)$. In particular, f vanishes at x. Hence, f vanishes along the smooth curve C_γ. Therefore, $f \in V_{C_\gamma}$. It follows that $V_\gamma \subset V_{C_\gamma}$. □

Proposition 3.8. *Let $n \geq \max(2, e+1)$. Then a curve $\gamma \subset X^{[n]}$ is homologous to $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$ if and only if there is a curve $C \subset X$ such that $C \sim \sigma$ or f, and that γ is a line in* $\mathrm{Hilb}^n(C) \subset X^{[n]}$. *Moreover, the curve C is uniquely determined by the curve γ.*

Proof. The "if" part of the proposition follows from Lemma 2.5 (i). To prove the "only if" part, let $\gamma \sim \beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. By Lemma 3.6, $C := C_\gamma \sim \sigma$ or f. Fix a section $f_0 \in H^0(X, \mathcal{O}_X(C))$ whose zero locus is C. Let $\xi \in \gamma$. Since $\sigma + ef$ is basepoint-free, so is the divisor $L_{n-1} - C$. Thus there exists $f_1 \in H^0(X, \mathcal{O}_X(L_{n-1} - C))$ such that f_1 does not vanish at any point in $\mathrm{Supp}(\xi)$. Now, $f_0 \otimes f_1 \in V_C$. By Lemma 3.7, $f_0 \otimes f_1 \in V_\gamma = V_\xi$. Since f_1 does not vanish at any point in $\mathrm{Supp}(\xi)$, f_0 vanishes at ξ. Hence, ξ is a closed subscheme of C. It follows that $\gamma \subset \mathrm{Hilb}^n(C)$.

To show that γ is a line in $\mathrm{Hilb}^n(C) \subset X^{[n]}$, let $F = \sigma + (e+1)f$. By Lemma 2.5 (ii), $\mathcal{O}_{X^{[n]}}(D_F)|_{\mathrm{Hilb}^n(C)} = \mathcal{O}_{\mathrm{Hilb}^n(C)}(1)$. So viewing γ as a curve in $\mathrm{Hilb}^n(C)$, we obtain

$$\gamma \cdot c_1(\mathcal{O}_{\mathrm{Hilb}^n(C)}(1)) = \gamma \cdot D_F = 1.$$

Therefore, γ is a line in $\mathrm{Hilb}^n(C) \subset X^{[n]}$.

Finally, the uniqueness of C follows from the observation that if $\xi \in X^{[n]}$ and $n \geq 2$, then ξ is contained in at most one curve $C \subset X$ with $C \sim \sigma$ or f. $\square$

Next, we determine the normal bundle of a curve γ in $X^{[n]}$ homologous to $\beta_f - (n-1)\beta_n$. By Proposition 3.8, there exists a unique fiber f_γ in X such that γ is a line in the n-th symmetric product $f_\gamma^{(n)} = \mathrm{Hilb}^n(f_\gamma) \subset X^{[n]}$. In particular,

$$N_{\gamma \subset f_\gamma^{(n)}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)}.$$

So we have the following exact sequence of normal bundles:

$$0 \to \mathcal{O}_\gamma(1)^{\oplus(n-1)} \to N_{\gamma \subset X^{[n]}} \to N_{f_\gamma^{(n)} \subset X^{[n]}}|_\gamma \to 0. \tag{3-5}$$

Lemma 3.9. *Let $n \geq \max(2, e+1)$. Let $\gamma \subset X^{[n]}$ be a curve homologous to $\beta_f - (n-1)\beta_n$, and f_γ be the unique fiber in X such that γ is a line in $f_\gamma^{(n)}$. Then*

(i) $N_{f_\gamma^{(n)} \subset X^{[n]}} \cong \mathcal{O}_{f_\gamma^{(n)}} \oplus \mathcal{O}_{f_\gamma^{(n)}}(-1)^{\oplus(n-1)}$;

(ii) $N_{\gamma \subset X^{[n]}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)} \oplus \mathcal{O}_\gamma \oplus \mathcal{O}_\gamma(-1)^{\oplus(n-1)}$.

Proof. (i) First of all, let $C \subset X$ be a smooth irreducible curve. Let π_n and q_n be the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X respectively. Recall the universal codimension-2 subscheme $\mathcal{Z}_n \subset X^{[n]} \times X$ from (2-11). By the results in [Altman et al. 1977], we have the isomorphism

$$N_{C^{(n)} \subset X^{[n]}} \cong \pi_{n*}(q_n^* \mathcal{O}_X(C)|_{\mathcal{Z}_n})|_{C^{(n)}}.$$

Let $\widetilde{\mathcal{Z}}_n$ be the universal subscheme in $C^{(n)} \times C$. Then we obtain

$$\pi_{n*}(q_n^*\mathcal{O}_X(C)|_{\mathcal{Z}_n})|_{C^{(n)}} \cong \tilde{\pi}_{n*}(\tilde{q}_n^*(\mathcal{O}_X(C)|_C)|_{\widetilde{\mathcal{Z}}_n}),$$

where $\tilde{\pi}_n$ and $\tilde{q}_n$ are the projections from $C^{(n)} \times C$ to $C^{(n)}$ and C, respectively. So

$$N_{C^{(n)} \subset X^{[n]}} \cong \tilde{\pi}_{n*}(\tilde{q}_n^*(\mathcal{O}_X(C)|_C)|_{\widetilde{\mathcal{Z}}_n}). \tag{3-6}$$

Replacing C in (3-6) by f_γ, we get $N_{f_\gamma^{(n)} \subset X^{[n]}} \cong \tilde{\pi}_{n*}\mathcal{O}_{\widetilde{\mathcal{Z}}_n}$. It is known that $\widetilde{\mathcal{Z}}_n \subset f_\gamma^{(n)} \times f_\gamma \cong \mathbb{P}^n \times \mathbb{P}^1$ is defined by the equation

$$a_0U^n + a_1U^{n-1}V + \cdots + a_nV^n = 0,$$

where $a_0, a_1, \ldots, a_n$ and U, V are the homogeneous coordinates on $\mathbb{P}^n$ and $\mathbb{P}^1$, respectively. So the line bundle $\mathcal{O}_{f_\gamma^{(n)} \times f_\gamma}(\widetilde{\mathcal{Z}}_n) \cong \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(\widetilde{\mathcal{Z}}_n)$ is of type $(1, n)$ in

$$\mathrm{Pic}(f_\gamma^{(n)} \times f_\gamma) \cong \mathrm{Pic}(\mathbb{P}^n \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Applying $\tilde{\pi}_{n*}$ to the exact sequence

$$0 \to \mathcal{O}_{f_\gamma^{(n)} \times f_\gamma}(-\widetilde{\mathcal{Z}}_n) \to \mathcal{O}_{f_\gamma^{(n)} \times f_\gamma} \to \mathcal{O}_{\widetilde{\mathcal{Z}}_n} \to 0,$$

we obtain the exact sequence

$$0 \to \mathcal{O}_{f_\gamma^{(n)}} \to \tilde{\pi}_{n*}\mathcal{O}_{\widetilde{\mathcal{Z}}_n} \to \mathcal{O}_{f_\gamma^{(n)}}(-1)^{\oplus(n-1)} \to 0.$$

This exact sequence splits. Thus, $\tilde{\pi}_{n*}\mathcal{O}_{\widetilde{\mathcal{Z}}_n} \cong \mathcal{O}_{f_\gamma^{(n)}} \oplus \mathcal{O}_{f_\gamma^{(n)}}(-1)^{\oplus(n-1)}$. Hence

$$N_{f_\gamma^{(n)} \subset X^{[n]}} \cong \tilde{\pi}_{n*}\mathcal{O}_{\widetilde{\mathcal{Z}}_n} \cong \mathcal{O}_{f_\gamma^{(n)}} \oplus \mathcal{O}_{f_\gamma^{(n)}}(-1)^{\oplus(n-1)}.$$

(ii) By (i) and (3-5), we obtain the exact sequence

$$0 \to \mathcal{O}_\gamma(1)^{\oplus(n-1)} \to N_{\gamma \subset X^{[n]}} \to \mathcal{O}_\gamma \oplus \mathcal{O}_\gamma(-1)^{\oplus(n-1)} \to 0.$$

Since this exact sequence splits, the proof of (ii) is complete. □

Now we determine the normal bundle of a curve γ in $X^{[n]}$ homologous to $\beta_\sigma - (n-1)\beta_n$. Recall that the Hirzebruch surfaces $\mathbb{F}_e$ are deformation equivalent to either $\mathbb{F}_0$ or $\mathbb{F}_1$. If $X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then σ is a fiber of one of the two rulings on X, so the normal bundle of a curve γ in $X^{[n]}$ homologous to $\beta_\sigma - (n-1)\beta_n$ has been computed by Lemma 3.9 (ii). In the following, we concentrate on $X = \mathbb{F}_1$, which is the blowup of the projective plane at a point.

Lemma 3.10. *Let $n \geq 2$ and $X = \mathbb{F}_1$. Let $\gamma \subset X^{[n]}$ be a curve homologous to $\beta_\sigma - (n-1)\beta_n$. Then, $N_{\sigma^{(n)} \subset X^{[n]}} \cong \mathcal{O}_{\sigma^{(n)}}(-1)^{\oplus n}$ and*

$$N_{\gamma \subset X^{[n]}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)} \oplus \mathcal{O}_\gamma(-1)^{\oplus n}. \tag{3-7}$$

Proof. The proof is similar to that of Lemma 3.9, so we adopt the notation in the proof of Lemma 3.9. Since $\sigma^2 = -1$, we have $\mathcal{O}_X(\sigma)|_\sigma \cong \mathcal{O}_\sigma(-1)$. Replacing the curve C in (3-6) by σ, we conclude that

$$N_{\sigma^{(n)} \subset X^{[n]}} \cong \tilde{\pi}_{n*}(\tilde{q}_n^* \mathcal{O}_\sigma(-1)|_{\widetilde{\mathcal{Z}}_n}).$$

Applying $\tilde{\pi}_{n*}$ to the exact sequence

$$0 \to \tilde{q}_n^* \mathcal{O}_\sigma(-1) \otimes \mathcal{O}_{\sigma^{(n)} \times \sigma}(-\widetilde{\mathcal{Z}}_n) \to \tilde{q}_n^* \mathcal{O}_\sigma(-1) \to \tilde{q}_n^* \mathcal{O}_\sigma(-1)|_{\widetilde{\mathcal{Z}}_n} \to 0,$$

we obtain $\tilde{\pi}_{n*}(\tilde{q}_n^* \mathcal{O}_\sigma(-1)|_{\widetilde{\mathcal{Z}}_n}) \cong \mathcal{O}_{\sigma^{(n)}}(-1)^{\oplus n}$. Therefore, we get

$$N_{\sigma^{(n)} \subset X^{[n]}} \cong \mathcal{O}_{\sigma^{(n)}}(-1)^{\oplus n}. \tag{3-8}$$

By Proposition 3.8, γ is a line in $\sigma^{(n)} \cong \mathbb{P}^n$. Using the exact sequence

$$0 \to \mathcal{O}_\gamma(1)^{\oplus(n-1)} \to N_{\gamma \subset X^{[n]}} \to N_{\sigma^{(n)} \subset X^{[n]}}|_\gamma \to 0$$

and (3-8), we see that $N_{\gamma \subset X^{[n]}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)} \oplus \mathcal{O}_\gamma(-1)^{\oplus n}$. □

Theorem 3.11. *Let X be the Hirzebruch surface $\mathbb{F}_e$ with $e \geq 0$, let f be a fiber of the ruling on X, and let σ be a section to the ruling with $\sigma^2 = -e$.*

(i) *If $n \geq \max(2, e+1)$, then the moduli space $\mathfrak{M}(\beta_f - (n-1)\beta_n)$ of all the curves in $X^{[n]}$ homologous to $\beta_f - (n-1)\beta_n$ is irreducible and unobstructed, i.e., is smooth with the expected dimension.*

(ii) *If $e = 1$ and $n \geq 2$, then the moduli space $\mathfrak{M}(\beta_\sigma - (n-1)\beta_n)$ of all the curves in $X^{[n]}$ homologous to $\beta_\sigma - (n-1)\beta_n$ is irreducible and unobstructed.*

Proof. Under the assumptions of (i) and (ii), we see from Lemma 3.9 (ii) and (3-7) that $H^1(\gamma, N_{\gamma \subset X^{[n]}}) = 0$ if γ is a curve homologous to either $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. This implies that the moduli spaces $\mathfrak{M}(\beta_f - (n-1)\beta_n)$ and $\mathfrak{M}(\beta_\sigma - (n-1)\beta_n)$ are unobstructed. By Proposition 3.8, $\mathfrak{M}(\beta_f - (n-1)\beta_n)$ is irreducible with dimension $2n-1$, and $\mathfrak{M}(\beta_\sigma - (n-1)\beta_n)$ is irreducible with dimension $2n-2$. □

By (2-10), the composition $\mathfrak{P} \circ \varphi_n(L_n) : X^{[n]} \to \mathbb{P}^N$ (for a suitable positive integer N) is the embedding associated to the very ample divisor

$$D_{L_n} - B_n/2 = nD_\sigma + n(1+e)D_f - B_n/2.$$

By Lemma 3.3, a curve $\gamma \subset X^{[n]}$ is mapped to a line in $\mathbb{P}^N$ if and only if γ is homologous to β_n, $\beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. Therefore, regarding $X^{[n]}$ as a closed subvariety of $\mathbb{P}^N$, then the Hilbert scheme of lines in $X^{[n]}$ is the disjoint union of $\mathfrak{M}(\beta_n)$, $\mathfrak{M}(\beta_f - (n-1)\beta_n)$ and $\mathfrak{M}(\beta_\sigma - (n-1)\beta_n)$.

Acknowledgment

The authors thank the referees for carefully reading the manuscript and for providing valuable suggestions which have greatly improved the exposition of the paper.

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Received October 18, 2015. Revised January 23, 2016.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 284 No. 2 October 2016