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JINGSONG CHAI AND QING ZHANG

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It is known that the multiplicity one property holds for SL_2 while the strong multiplicity one property fails. However, in this paper we show that if we require further that a pair of cuspidal representations π and π' of SL_2 have the same local components at the archimedean places and the places above 2, and they are generic with respect to the same additive character, then they also satisfy the strong multiplicity one property. The proof is based on a local converse theorem for SL_2.

1. Introduction

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ be its ring of adeles. Let G be a linear reductive algebraic group defined over F. The study of the space of automorphic forms $L^2(G(F)\backslash G(\mathbb{A}))$ has been a central topic in the Langlands program and representation theory. Let $L_0^2(G(F)\backslash G(\mathbb{A}))$ be the subspace of cuspidal representations. Suppose π is an irreducible automorphic representation of $G(\mathbb{A})$. It is known that π occurs discretely with finite multiplicity m_π in $L_0^2(G(F)\backslash G(\mathbb{A}))$.

The multiplicities m_π are important in the study of automorphic forms and number theory. By [Jacquet and Shalika 1981; Badulescu 2008] and the work of Piatetski-Shapiro, the group $G = \mathrm{GL}_n$ and its inner forms have the property of multiplicity one, that is, $m_\pi \le 1$ for any π. This is also true for SL_2 by the famous work of D. Ramakrishnan [2000]. But in general the multiplicity one property fails, for example [Blasius 1994; Gan et al. 2002; Li 1997; Labesse and Langlands 1979] to list a few.

In the case of GL_n a stronger theorem, called the strong multiplicity one, holds. It says that for two cuspidal representations π_1 and π_2, if they have isomorphic local components almost everywhere, then they coincide in the space of cusp forms (not only isomorphic). It follows from the results in [Labesse and Langlands 1979] that SL_2 does not have this strong multiplicity one property. The multiplicity one property is already rare and the strong multiplicity one is even rarer. To the authors' knowledge the examples other than GL_n in this direction are the strong multiplicity one theorems for $\mathrm{U}(2, 1)$ [Gelbart and Piatetski-Shapiro 1984; Baruch 1997] and

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GSp_4 [Soudry 1987] and the rigidity theorem for $\mathrm{SO}(2n+1)$ [Jiang and Soudry 2003, Theorem 5.3].

The main purpose of this paper is to prove a weaker version of the strong multiplicity one result for $\mathrm{Sp}_2 = \mathrm{SL}_2$. Although we know strong multiplicity one does not hold in general for a pair of cuspidal representations π_1 and π_2 of $\mathrm{SL}_2(\mathbb{A})$, if we require that both π_1 and π_2 are generic with respect to the same additive character ψ of $\mathbb{A}$, then we can show that they also satisfy the strong multiplicity one property.

The reason for the failure of the strong multiplicity one for SL_2 is the existence of L-packets. According to the local conjecture of Gan–Gross–Prasad [2012, Conjecture 17.3] there is at most one ψ-generic representation in each L-packet. For SL_2, the result is known by the local discussion in [Labesse and Langlands 1979]. In this paper, we prove a local converse theorem for $\mathrm{SL}_2(F)$ when F is a p-adic field such that its residue characteristic is not 2, which will reprove the result of Labesse and Langlands [1979] and confirm a local converse conjecture of Jiang, see [Jiang 2006, Conjecture 3.7] and [Jiang and Nien 2013, Conjecture 6.3]. This also implies our version of strong multiplicity one easily.

We now give some details of our results. Gelbart and Piatetski-Shapiro [1987] constructed some Rankin–Selberg integrals to study L-functions on the group $G_n \times \mathrm{GL}(n)$, for $G_n = \mathrm{Sp}(n)$ and $\mathrm{U}(n,n)$. In particular, in Method C in that paper, if π is a globally generic cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{A})$, then τ is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. Consider the global Shimura type zeta-integral

$$I(s,\phi,E) = \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \phi(g)\theta(g)E(g,s)dg,$$

where ϕ belongs to the space of π, $E(g,s)$ is a genuine Eisenstein series on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ built from the representation induced from the representation τ of $\mathrm{GL}_n(\mathbb{A})$ twisted by $|\det|^s$ and $\theta(g)$ is some theta series on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Note that the product $\theta(g)E(g,s)$ is well-defined on Sp_{2n}. The global integral is shown to be Eulerian. The local functional equations and unramified calculations were also carried out by Gelbart and Piatetski-Shapiro [1987]. Although we will only consider the easiest case when $n=1$ of Gelbart and Piatetski-Shapiro's construction, we remark here that Ginzburg, Rallis and Soudry [1997; 1998] generalized the above construction to $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$, for any k.

We study more details of Gelbart and Piatetski-Shapiro's local integral

$$\Psi(W_v,\phi_v,f_{s,v}) = \int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(h)(\omega_{\psi_v^{-1}}(h)\phi_v)(1)f_{s,v}(h)dh$$

(for the unexplained notations, see sections below) when v is finite. These local zeta-integrals satisfy certain functional equations, which come from the intertwining operators on induced representations and certain uniqueness statements. These functional equations can then be used to define local gamma factors $\gamma(s, \pi_v, \eta_v, \psi_v)$, where π_v is a generic representation of $\mathrm{SL}_2(F_v)$, η_v is a character of $F_v^\times$ and ψ_v is a nontrivial additive character. The main local result of this paper can be formulated as follows.

Theorem 3.10 (Local converse theorem and stability of γ). *Suppose that the residue characteristic of the p-adic field F is not 2 and ψ is a nontrivial additive character of F. Let (π, V_π) and $(\pi', V_{\pi'})$ be two ψ-generic representations of $\mathrm{SL}_2(F)$ with the same central character.*

(1) *If $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasicharacters η of $F^\times$, then $\pi \cong \pi'$.*

(2) *There is an integer $l = l(\pi, \pi')$ such that if η is a quasicharacter of $F^\times$ with conductor $\mathrm{cond}(\eta) > l$, then*

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

The proof of this result follows closely [Baruch 1995; 1997; Zhang 2015] and Howe vectors play an important role. With the help of this result, combined with a nonvanishing result on archimedean local integrals proved in Lemma 4.9, we follow the argument in [Baruch 1997, Theorem 7.2.13], or in [Casselman 1973, Theorem 2], to prove the main global result of this paper.

Theorem 4.8 (Strong multiplicity one for SL_2). *Let ψ be a nontrivial additive character of $F \setminus \mathbb{A}$ and let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be two irreducible cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A})$ with the same central character. Suppose that π and π' are both ψ-generic. Let S be a finite set of* ***finite*** *places such that no place in S is above 2. If $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.*

Remark. The restriction on residue characteristic comes from Lemma 3.3. It is expected that this restriction can be removed.

Besides the above, we also in this paper include a discussion of relations between global genericity and local genericity. An irreducible cuspidal automorphic representation (π, V_π) is called globally ψ-generic if for some $\phi \in V_\pi$, the integral

$$\int_{N(F)\setminus N(\mathbb{A})} \phi(ug)\psi^{-1}(u)du \neq 0$$

for some $g \in \mathrm{SL}_2(\mathbb{A})$. The representation π is called locally ψ-generic if each of its local component is generic for the corresponding local components of ψ. It is easy to see that if π is globally ψ-generic, then π is also locally ψ-generic. It is a conjecture that on a reductive algebraic group G, the converse is also true. This conjecture is closely related to the Ramanujan conjecture. See [Shahidi 2011] for more detailed discussions. We confirm this conjecture for SL_2.

Theorem 4.3. *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ and $\psi = \otimes \psi_v$ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then π is ψ-generic if and only if each π_v is ψ_v-generic.*

Gelbart, Rogawski and Soudry [1997, Proposition 2.5] proved similar results for $\mathrm{U}(1, 1)$ and for endoscopic cuspidal automorphic representations of $\mathrm{U}(2, 1)$. From the discussions given in [Gelbart et al. 1997] Theorem 4.3 follows directly from the results of Labesse and Langlands [1979]. Here, we include this result because we adopt a local argument (see Proposition 2.1) which is different from that given in [Labesse and Langlands 1979]. Hopefully, this local argument can be extended to more general groups.

As explained above, there is essentially nothing new in this paper. All the results and proofs should be known to the experts. Our task here is simply to try to write down the details and to check everything works out as expected.

This paper is organized as follows. In Section 2 we collect basic results about the local zeta-integrals which will be needed. In Section 3 we study the Howe vectors and use them to prove the local converse theorem and stability of local gamma factors. In Section 4 we prove the main global results.

1A. ***Notations.*** Let F be a field. In $\mathrm{SL}_2(F)$, we consider the following subgroups. Let B be the upper triangular subgroup. Let $B = TN$ be the Levi-decomposition, where T is the diagonal torus and N is the upper triangular unipotent. Denote

$$t(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in T, \ \text{for } a \in F^\times, \quad \text{and} \quad n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N, \ \text{for } b \in F.$$

Let $\bar{N}$ be the lower triangular unipotent and denote

$$\bar{n}(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Denote by St the natural inclusion $\mathrm{SO}_3(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$ and view it as the "standard" representation of ${}^L\mathrm{SL}_2 = \mathrm{SO}_3(\mathbb{C})$.

2. The local zeta-integral

2A. ***The genericity of representations of*** $\mathbf{SL_2(F)}$**.** In this section let F be a local field and ψ be a nontrivial additive character of F, which is also viewed as a character of $N(F)$. For $\kappa \in F^\times$ and $g \in \mathrm{SL}_2(F)$ we define

$$g^\kappa = \begin{pmatrix} \kappa & \\ & 1 \end{pmatrix} g \begin{pmatrix} \kappa^{-1} & \\ & 1 \end{pmatrix}.$$

Explicitly

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^{\kappa} = \begin{pmatrix} x & \kappa y \\ \kappa^{-1} z & w \end{pmatrix}.$$

Note that if $\kappa \in F^{\times,2}$, say $\kappa = a^2$, then $g^{\kappa} = t(a) g t(a)^{-1}$, i.e., $g \mapsto g^{\kappa}$ is an inner automorphism on $\mathrm{SL}_2(F)$. Let (π, V_π) be an infinite dimensional irreducible smooth representation of $\mathrm{SL}_2(F)$. We consider the representation $(\pi^{\kappa}, V_{\pi^{\kappa}})$ defined by

$$V_{\pi^{\kappa}} = V_\pi \quad \text{and} \quad \pi^{\kappa}(g) = \pi(g^{\kappa}).$$

Let ψ_κ be the character of F defined by $\psi_\kappa(b) = \psi(\kappa b)$. If (π, V_π) is ψ-generic with a nonzero ψ Whittaker functional $\Lambda : V_\pi \to \mathbb{C}$, one verifies that

$$\Lambda(\pi^{\kappa}(n) v) = \Lambda(\pi(n^{\kappa}) v) = \psi_\kappa(n) \Lambda(v)$$

for all $n \in N(F)$ and all $v \in V_{\pi^{\kappa}} = V_\pi$. Hence $(\pi^{\kappa}, V_{\pi^{\kappa}})$ is ψ_κ-generic.

Proposition 2.1. *If π is both ψ- and ψ_κ-generic, then $\pi \cong \pi^{\kappa}$.*

Proof. If F is nonarchimedean, the proof is similar to the $U(1,1)$ case as in [Zhang 2015].

If F is archimedean the case $F = \mathbb{C}$ is easy, as every κ has a square root in $\mathbb{C}$. Now consider $F = \mathbb{R}$. We will work with the category of smooth representations of moderate growth of finite length. The Whittaker functional is an exact functor from this category to the category of vector spaces by [Casselman et al. 2000, Theorem 8.2].

We first consider the case when $I(\chi) = \mathrm{Ind}_B^G(\chi)$ for some quasicharacter χ of $F^{\times}$. For $f \in I(\chi)$, consider the function f^{κ} on $\mathrm{SL}_2(F)$ defined by $f^{\kappa}(g) = f(g^{\kappa^{-1}})$. It is clear that $f^{\kappa} \in I(\chi)^{\kappa}$ and the map $f \mapsto f^{\kappa}$ defines an isomorphism $I(\chi) \to I(\chi)^{\kappa}$.

By results in [Vogan 1981, Chapter 2], if π is not a fully induced representation then it can be embedded into a principal series $I(\chi)$. This $I(\chi)$ has two irreducible infinite dimensional subrepresentations, use π' to denote the other one. The quotient of $I(\chi)$ by the sum of π and π', denoted by π'', is finite dimensional, i.e., we have a short exact sequence

$$0 \to \pi \oplus \pi' \to I(\chi) \to \pi'' \to 0.$$

First, by [Casselman et al. 2000, Theorem 6.1], we know that the Whittaker functionals on $I(\chi)$ are one dimensional for either ψ or ψ_κ. Note that π'' cannot be generic as it is finite dimensional. Since the Whittaker functor is exact, it follows that the dimension of Whittaker functionals on $\pi \oplus \pi'$ is also one for either ψ or ψ_κ. By the assumption π is both ψ- and ψ_κ-generic, thus π' is neither ψ- nor ψ_κ-generic.

Now since π is ψ-generic, π^κ is ψ_κ-generic. Hence the image of π under the isomorphism $I(\chi) \to I(\chi)^\kappa$ given by $f \mapsto f^\kappa$ is again ψ_κ-generic and hence it has to be ψ-generic and isomorphic to π, which finishes the proof. $\square$

2B. ***Weil representations of*** $\widetilde{\mathbf{SL}}_2$**.** Let $\widetilde{\mathrm{SL}}_2$ be the metaplectic double cover of SL_2. Then we have an exact sequence

$$0 \to \mu_2 \to \widetilde{\mathrm{SL}}_2 \to \mathrm{SL}_2 \to 0,$$

where $\mu_2 = \{\pm 1\}$.

The product on $\widetilde{\mathrm{SL}}_2(F)$ is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 c(g_1, g_2)),$$

where $c : \mathrm{SL}_2(F) \times \mathrm{SL}_2(F) \to \{\pm 1\}$ is defined by Hilbert symbols as

$$c(g_1, g_2) = (\mathbf{x}(g_1), \mathbf{x}(g_2))_F (-\mathbf{x}(g_1)\mathbf{x}(g_2), \mathbf{x}(g_1 g_2))_F,$$

where

$$\mathbf{x}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & c \neq 0, \\ d & c = 0, \end{cases}$$

and $(\ ,\ )_F$ is the Hilbert symbol. For these formulas for the Kubota cocycle see [Kubota 1969, Section 3].

For a subgroup A of $\mathrm{SL}_2(F)$, we denote by $\widetilde{A}$ the preimage of A in $\widetilde{\mathrm{SL}}_2(F)$, which is a subgroup of $\widetilde{\mathrm{SL}}_2(F)$. For an element $g \in \mathrm{SL}_2(F)$, we sometimes abuse notation by writing $(g, 1) \in \widetilde{\mathrm{SL}}_2(F)$ as g.

A representation π of $\widetilde{\mathrm{SL}}_2(F)$ is called genuine if $\pi(\zeta g) = \zeta \pi(g)$ for all $g \in \widetilde{\mathrm{SL}}_2(F)$ and $\zeta \in \mu_2$. Let ψ be an additive character of F. Then there is a Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2(F)$ on the space $\mathcal{S}(F)$ of Schwartz–Bruhat functions on F. For $f \in \mathcal{S}(F)$, we have the well-known formulas:

$$\begin{gathered}
\left(\omega_\psi \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) f(x) = \gamma(\psi) \hat{f}(x), \\
\left(\omega_\psi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) f(x) = \psi(bx^2) f(x), \quad b \in F \\
\left(\omega_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right) f(x) = |a|^{1/2} \frac{\gamma(\psi)}{\gamma(\psi_a)} f(ax), \quad a \in F^\times. \\
\omega_\psi(\zeta) f(x) = \zeta f(x), \quad \zeta \in \mu_2.
\end{gathered}$$

Here $\hat{f}(x) = \int_F f(y)\psi(2xy)dy$, where dy is normalized so that $\hat{\hat{f}}(x) = f(-x)$, $\gamma(\psi)$ is the Weil index and $\psi_a(x) = \psi(ax)$.

Let $\widetilde{T}$ be the inverse image of $T = \{t(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, a \in F^\times\} \subset \mathrm{SL}_2(F)$ in $\widetilde{\mathrm{SL}}_2(F)$. The product in $\widetilde{T}$ is given by the Hilbert symbol, i.e.,

$$(t(a), \zeta_1)(t(b), \zeta_2) = (t(ab), \zeta_1\zeta_2(a, b)_F).$$

The function

$$\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$$

satisfies

$$\mu_\psi(a)\mu_\psi(b) = \mu_\psi(ab)(a, b)_F,$$

and thus extends to a genuine character of $\widetilde{T}$.

The representation ω_ψ is not irreducible and we have $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$, where ω_ψ^+ and ω_ψ^- are the subrepresentations on even and odd functions in $\mathcal{S}(F)$, respectively. All the above facts can be found in [Gelbart and Piatetski-Shapiro 1980, Section 1].

2C. ***The local zeta-integral.*** Let $\mu_\psi(a) = \gamma(\psi)/\gamma(\psi_a)$ be as above, which is viewed as a character of $\widetilde{T}$. Let η be a quasicharacter of $F^\times$ and η_s denote the character $\eta|\cdot|^s$ of $F^\times$. Consider the induced representation $I(s, \eta, \psi) = \mathrm{Ind}_{\widetilde{B}(F)}^{\widetilde{\mathrm{SL}}_2(F)}(\eta_{s-1/2}\mu_\psi)$.

Let (π, V) be a ψ-generic representation of $\mathrm{SL}_2(F)$ with its Whittaker model $\mathcal{W}(\pi, \psi)$. Choose $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in I(s, \eta, \psi^{-1})$. Note that $(\omega_{\psi^{-1}}(h)\phi)(1) f_s(h)$ is well-defined as a function on $\mathrm{SL}_2(F)$ and consider the integral

$$\Psi(W, \phi, f_s) = \int_{N(F)\backslash \mathrm{SL}_2(F)} W(h)(\omega_{\psi^{-1}}(h)\phi)(1) f_s(h)\, dh.$$

By results in [Gelbart et al. 1987, Sections 5 and 12], the above integral is absolutely convergent when $\mathrm{Re}(s)$ is large enough and has a meromorphic continuation to the whole plane.

Remark. Gelbart and Piatetski-Shapiro [1987, Method C] constructed a global zeta-integral for $\mathrm{Sp}_{2n} \times \mathrm{GL}_n$ which showed that it is Eulerian. They also sketched a proof of the local functional equation. The above integral is the simplest case of the Gelbart and Piatetski-Shapiro integral, namely when $n = 1$.

2D. ***Local functional equation.*** The trilinear form $(W, \phi, f_s) \mapsto \Psi(W, \phi, f_s)$ defines an element in

$$\mathrm{Hom}_{\mathrm{SL}_2}(\pi \otimes \omega_{\psi^{-1}} \otimes I(s, \eta, \psi^{-1}), \mathbb{C}),$$

which has dimension at most one. The proof of this fact is given in [Gelbart et al. 1987, §11] and also can be deduced by the uniqueness of the Fourier–Jacobi model for SL_2, see [Sun 2012]. Let

$$M_s : I(s, \eta, \psi^{-1}) \to I(1-s, \eta^{-1}, \psi^{-1})$$

be the standard intertwining operator, i.e.,

$$M_s(f_s)(g) = \int_N f_s(wng)dn.$$

By the one dimensionality of the above Hom space we get the following:

Proposition 2.2. *There is a meromorphic function $\gamma(s, \pi, \eta, \psi)$ such that*

$$\Psi(W, \phi, M_s(f_s)) = \gamma(s, \pi, \eta, \psi)\Psi(W, \phi, f_s),$$

for all $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in I(s, \eta, \psi^{-1})$.

2E. ***Unramified calculation.*** The unramified calculation of Method C is in fact not included in [Gelbart et al. 1987], but it can be simply done in the SL_2-case.

Let F be a nonarchimedian local field with odd residue characteristic. Suppose everything is unramified. Then the character μ_ψ is unramified, [Szpruch 2009, p. 2188]. Suppose the representation (π, V) has Satake parameter a, which means that π is the unramified component $\mathrm{Ind}_{B(F)}^{\mathrm{SL}_2(F)}(\nu)$ for an unramified character ν and $a = \nu(p_F)$, where p_F is some prime element of F. Let

$$b_k = t(p_F^k) = \mathrm{diag}(p_F^k, p_F^{-k}),$$

and W be the spherical Whittaker functional normalized by $W(e) = 1$. Then $W(b_k) = 0$ for $k < 0$ and

$$W(b_k) = \frac{q^{-k}}{a-1}(a^{k+1} - a^{-k}),$$

by the general Casselman–Shalika formula [1980, Theorem 5.4]. For $k \geq 0$ we have

$$(\omega_{\psi^{-1}}(b_k)\phi)(1) = \mu_{\psi^{-1}}(p_F^k)|p_F^k|^{1/2},$$

where ϕ is the characteristic function of the ring of integers $\mathcal{O}_F$. On the other hand, let f_s be the standard spherical section of $I(s, \eta, \psi^{-1})$ normalized by $f_s(1) = 1$. Then we have

$$f_s(b_k) = \eta(p_F^k)|p_F^k|^{s+1/2}\mu_{\psi^{-1}}(p_F^k).$$

Since $\mu_{\psi^{-1}}(p_F^k)\mu_{\psi^{-1}}(p_F^k) = (p_F^k, p_F^k)_F = (p_F, -1)_F^k$, we have

$$\begin{aligned}\Psi(W, \phi, f_s) &= \int_{F^\times}\int_K W(t(a)k)\omega_{\psi^{-1}}(t(ak)\phi)(1) f_s(t(a)k)|a|^{-2}dkda\\ &= \int_{F^\times} W(t(a))\omega_{\psi^{-1}}(t(a))\phi(1) f_s(t(a))|a|^{-2}da\\ &= \sum_{k\geq 0} W(b_k)(\omega_{\psi^{-1}}(b_k)\phi)(1) f_s(b_k)|p_F^k|^{-2}\\ &= \frac{1}{a-1}\sum_{k\geq 0}(a^{k+1}-a^{-k})(p_F, -1)^k\eta(p_F)^k q_F^{-ks}\\ &= \frac{1+c}{(1-ac)(1-a^{-1}c)} = \frac{1-c^2}{((1-ac)(1-c)(1-a^{-1}c)}\\ &= \frac{L(s, \pi, St\otimes\eta\chi)}{L(2s, \eta^2)},\end{aligned}$$

where $c = (p_F, -1)\eta(p_F)q_F^{-s}$, and $\chi(a) = (a, -1)_F$. Recall that St is the standard representation of ${}^L\mathrm{SL}_2 = \mathrm{SO}_3(\mathbb{C})$.

Remark. From the calculation of the μ_ψ given in [Szpruch 2009, Lemmas 1.5 and 1.10], one can check that

$$M_s(f_s) = \frac{L(2s-1, \eta^2)}{L(2s, \eta^2)} f_{1-s},$$

where f_s and f_{1-s} are the standard spherical sections in, respectively, $I(s, \eta, \psi^{-1})$ and $I(1-s, \eta^{-1}, \psi^{-1})$. Thus the factor $L(2s, \eta^2)$ appearing in the above unramified calculation will play the role of the normalizing factor of a global intertwining operator or Eisenstein series.

3. Howe vectors and the local converse theorem

In this section, we assume F is a p-adic field with odd residue characteristic. We will follow Baruch's method [1995; 1997] to give a proof of the local converse theorem for generic representations of $\mathrm{SL}_2(F)$.

3A. ***Howe vectors.*** Let ψ be an unramified character. For a positive integer m, let $K_m = (1 + M_{2\times 2}(\mathcal{P}_F^m)) \cap \mathrm{SL}_2(F)$ where $\mathcal{P}_F = (p_F)$ denotes the maximal ideal in $\mathcal{O}_F$. Define a character τ_m of K_m by

$$\tau_m(k) = \psi(p_F^{-2m}k_{12})$$

for $k = (k_{ij}) \in K_m$. It is easy to see that τ_m is indeed a character on K_m.

Let $d_m = t(p_F^{-m})$. Consider the subgroup $J_m = d_m K_m d_m^{-1}$. Then

$$J_m = \begin{pmatrix} 1+\mathcal{P}_F^m & \mathcal{P}_F^{-m} \\ \mathcal{P}_F^{3m} & 1+\mathcal{P}_F^m \end{pmatrix} \cap \mathrm{SL}_2(F).$$

Define $\psi_m(j) = \tau_m(d_m^{-1} j d_m)$ for $j \in J_m$. For a subgroup $H \subset \mathrm{SL}_2(F)$, denote $H_m = H \cap J_m$. It is easy to check that $\psi_m|_{N_m} = \psi|_{N_m}$.

Let π be an irreducible smooth ψ-generic representation of $\mathrm{SL}_2(F)$ and let $v \in V_\pi$ be a vector such that $W_v(1) = 1$. For $m \geq 1$, as in [Baruch 1995; 1997] we consider

$$v_m = \frac{1}{\mathrm{Vol}(N_m)} \int_{N_m} \psi(n)^{-1} \pi(n) v dn. \tag{3-1}$$

Let $L \geq 1$ be an integer such that v is fixed by K_L. Following E. M. Baruch, we call $v_m, m \geq L$ **Howe vectors**.

Lemma 3.1. *We have*:

(1) $W_{v_m}(1) = 1$.

(2) *If* $m \geq L$ *then* $\pi(j) v_m = \psi_m(j) v_m$ *for all* $j \in J_m$.

(3) *If* $k \leq m$ *then*

$$v_m = \frac{1}{\mathrm{Vol}(N_m)} \int_{N_m} \psi(u)^{-1} \pi(u) v_k du.$$

The proof of this lemma is the same as the proof in the U(2, 1) case, which is given in [Baruch 1997, Lemma 5.2].

Lemma 3.2. *Let* $m \geq L$ *and* $t = t(a)$ *for* $a \in F^\times$:

(1) *If* $W_{v_m}(t) \neq 0$, *we have*

$$a^2 \in 1 + \mathcal{P}_F^m.$$

(2) *If* $W_{v_m}(tw) \neq 0$, *we have*

$$a^2 \in \mathcal{P}^{-3m}.$$

Proof.

(1) Take $x \in \mathcal{P}^{-m}$. We then have $n(x) \in N_m \subset J_m$. From the relation

$$tn(x) = n(a^2 x) t$$

and (2) of Lemma 3.1 we have

$$\psi(x) W_{v_m}(t) = \psi(a^2 x) W_{v_m}(t).$$

If $W_{v_m}(t) \neq 0$ we get $\psi(x) = \psi(a^2 x)$ for all $x \in \mathcal{P}^{-m}$. Since ψ is unramified we get $a^2 \in 1 + \mathcal{P}^m$.

(2) For $x \in \mathcal{P}^{3m}$ we have $\bar{n}(x) \in \bar{N}_m$. From the relation $tw\bar{n}(x) = n(-a^2x)tw$ and Lemma 3.1 (2) we get

$$W_{v_m}(tw) = \psi(-a^2x)W_{v_m}(tw).$$

Thus if $W_{v_m}(tw) \neq 0$ we get $\psi(-a^2x) = 1$ for all $x \in \mathcal{P}^{3m}$. Thus $a^2 \in \mathcal{P}^{-3m}$. □

Lemma 3.3. *For $m \geq 1$ the squaring map from $1+\mathcal{P}^m \to 1+\mathcal{P}^m$, sending $a \mapsto a^2$, is well-defined and surjective.*

This lemma requires that the residue field of F is not of characteristic 2 which we assume throughout this section.

Proof. For $x \in \mathcal{P}^m$, it is clear that $(1+x)^2 = 1+2x+x^2 \in 1+\mathcal{P}^m$. Thus the square map is well-defined. On the other hand, we take $u \in 1+\mathcal{P}^m$ and consider the equation $f(X) := X^2 - u = 0$. We have $f'(X) = 2X$. Since $q^{-m} = |1-u| = |f(1)| < |f'(1)|^2 = |2|^2 = 1$ by Newton's Lemma, see for example [Lang 1994, Proposition 2, Chapter II], there is a root $a \in \mathcal{O}_F$ of $f(X)$ such that

$$|a-1| \leq \frac{|f(1)|}{|f'(1)|^2} = |1-u| = q^{-m}.$$

Thus we get a root $a \in 1+\mathcal{P}^m$ of $f(X)$. This completes the proof. □

Let $Z = \{\pm 1\}$ and identify Z with the center of $\mathrm{SL}_2(F)$. Denote by ω_π the central character of π.

Corollary 3.4. *Let $m \geq L$. Then we have*

$$W_{v_m}(t(a)) = \begin{cases} \omega_\pi(z) & \text{if } a = za' \text{ for some } z \in Z \text{ and } a' \in 1+\mathcal{P}^m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $W_{v_m}(t(a)) \neq 0$. Then by Lemma 3.2 we have $a^2 \in 1+\mathcal{P}^m$. By Lemma 3.3 there exists an $a' \in 1+\mathcal{P}^m$ such that $a^2 = (a')^2$. Thus $a = za'$ for some $z \in Z$. Since $a' \in 1+\mathcal{P}^m$ we get $t(a') \in J_m$. The assertion follows from Lemma 3.1. □

From now on, we fix two ψ-generic representations (π, V_π) and $(\pi', V_{\pi'})$ with the same central characters. Fix v and v' such that $W_v(1) = 1 = W_{v'}(1)$. Let L be an integer such that both v and v' are fixed by K_L. For $m \geq 1$ consider the Howe vectors v_m and v'_m.

By Corollary 3.4 and the fact that $\omega_\pi = \omega_{\pi'}$ we get the following:

Corollary 3.5. *For $m \geq L$ we have $W_{v_m}(g) = W_{v'_m}(g)$ for all $g \in B$.*

Lemma 3.6 (Baruch). *If $m \geq 4L$ and $n \in N - N_m$ we have*

$$W_{v_m}(twn) = W_{v'_m}(twn),$$

for all $t \in T$.

Proof. This is a special case of [Baruch 1995, Lemma 6.2.2]. A similar result for $\mathrm{U}(2,1)$ is given in [Baruch 1997, Proposition 5.7]. We just remark that the proof of this lemma depends on Corollary 3.5, and hence requires that the residue characteristic of F is not 2. $\square$

3B. *Induced representations.* Note that $\overline{N}(F)$ and $N(F)$ split in $\widetilde{\mathrm{SL}}_2(F)$. Moreover, for $g_1 \in N$ and $g \in \overline{N}$ we have $c(g_1, g_2) = 1$. In fact if $g_1 = n(y)$ and $g_2 = \bar{n}(x)$ with $x \neq 0$ we have $\mathbf{x}(g_1) = 1$ and $\mathbf{x}(g_2) = x$. Thus

$$c(g_1, g_2) = (1, x)_F(-x, x)_F = 1.$$

This shows that $N(F) \cdot \overline{N}(F) \subset \mathrm{SL}_2(F)$, where $\mathrm{SL}_2(F)$ denotes the subset of $\widetilde{\mathrm{SL}}_2(F)$ which consists of elements of the form $(g, 1)$ for $g \in \mathrm{SL}_2(F)$.

Let X be an open compact subgroup of $N(F)$. For $x \in X$ and $i > 0$ consider the set $A(x, i) = \{\bar{n} \in \overline{N}(F) : \bar{n}x \in B \cdot \overline{N}_i\}$.

Lemma 3.7. (1) *For any positive integer c there exists an integer $i_1 = i_1(X, c)$ such that for all $i \geq i_1$, $x \in X$ and $\bar{n} \in A(x, i)$ we have*

$$\bar{n}x = nt(a)\bar{n}_0,$$

with $n \in N$, $\bar{n}_0 \in \overline{N}_i$ and $a \in 1 + \mathcal{P}^c$.

(2) *There exists an integer $i_0 = i_0(X)$ such that for all $i \geq i_0$ we have $A(x, i) = \overline{N}_i$.*

Proof. By abuse of notation, for $x \in X$ we write $x = n(x)$. Since X is compact there is a constant C such that $|x| < C$ for all $n(x) \in X \subset N$.

For $n(x) \in X$ and $\bar{n}(y) \in A(x, i)$ we have $\bar{n}(y)n(x) \in B \cdot \overline{N}_i$. Thus we can assume that

$$\bar{n}(y)n(x) = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \bar{n}(\bar{y})$$

for $a \in F^\times$, $b \in F$ and $\bar{y} \in \mathcal{P}^{3i}$. Rewrite the above expression as

$$\bar{n}(-y) \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} = n(x)\bar{n}(-\bar{y}),$$

or

$$\begin{pmatrix} a & b \\ -ay & a^{-1} - by \end{pmatrix} = \begin{pmatrix} 1 - x\bar{y} & x \\ -\bar{y} & 1 \end{pmatrix}.$$

Thus we get

$$a = 1 - x\bar{y} \text{ and } ay = \bar{y}.$$

Since $|x| < C$ and $\bar{y} \in \mathcal{P}^{3i}$ it is clear that for any positive integer c we can choose $i_1(X, c)$ such that $a = 1 - x\bar{y} \in 1 + \mathcal{P}^c$ for all $n(x) \in X$ and $\bar{n}(y) \in A(x, i)$. This proves (1).

If we take $i_0(X) = i_1(X, 1)$ we get $a \in 1+\mathcal{P} \subset \mathbb{O}^\times$ for $i \geq i_0$. From $ay = \bar{y}$ we get $y \in \mathcal{P}^{3i}$. Thus for $i \geq i_0(X)$ we have that $\bar{n}(y) \in \overline{N}_i$, i.e., $A(x, i) \subset \overline{N}_i$.

The other direction can be checked similarly if i is large. We omit the details. □

Given a positive integer i and a complex number $s \in \mathbb{C}$ we consider the following function f_s^i on $\widetilde{\mathrm{SL}}_2(F)$:

$$f_s^i(\tilde{g}) = \begin{cases} \zeta\,\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) & \text{if } \tilde{g} = \left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, \zeta\right)\bar{n}(x), \\ & \qquad\text{with } a \in F^\times, b \in F, \zeta \in \mu_2, x \in \mathcal{P}^{3i}, \\ 0 & \text{otherwise.}\end{cases}$$

Lemma 3.8. (1) *There exists an integer $i_2(\eta)$ such that for all $i \geq i_2(\eta)$, f_s^i defines a section in $I(s, \eta, \psi^{-1})$.*

(2) *Let X be an open compact subset of N. There exists an integer $I(X, \eta) \geq i_2(\eta)$ such that for all $i \geq I(X, \eta)$ we have*

$$\tilde{f}_s^i(wx) = \mathrm{vol}(\overline{N}_i) = q_F^{-3i}$$

for all $x \in X$, where $\tilde{f}_s^i = M_s(f_s^i)$ and $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Proof. (1) From the definition it is clear that

$$f_s^i\left(\left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, \zeta\right)\tilde{g}\right) = \zeta\,\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) f_s^i(\tilde{g}),$$

for $a \in F^\times$, $b \in F$, $\zeta \in \mu_2$ and $\tilde{g} \in \widetilde{\mathrm{SL}}_2(F)$. It suffices to show that for i large there is an open compact subgroup $\widetilde{H}_i \subset \widetilde{\mathrm{SL}}_2(F)$ such that $f_s^i(\tilde{g}\tilde{h}) = f_s^i(\tilde{g})$ for all $\tilde{g} \in \widetilde{\mathrm{SL}}_2(F)$ and $\tilde{h} \in \widetilde{H}_i$.

If ψ is unramified and the residue characteristic is not 2 as we assumed then the character $\mu_{\psi^{-1}}$ is trivial on $\mathbb{O}_F^\times$, see for example [Szpruch 2009, p. 2188].

Let c be a positive integer such that η is trivial on $1+\mathcal{P}^c$. Let $i_2(\eta) = \max\{c, i_0(N \cap K_c), i_1(N \cap K_c, c)\}$. For $i \geq i_2(\eta)$ we take $\widetilde{H}_i = K_{4i} = 1 + M_2(\mathcal{P}^{4i})$. Note that K_{4i} splits and thus can be viewed as a subgroup of $\widetilde{\mathrm{SL}}_2$. We now check that for $i \geq i_2(\eta)$ we have $f_s^i(\tilde{g}h) = f_s(\tilde{g})$ for all $\tilde{g} \in \widetilde{\mathrm{SL}}_2$ and $h \in K_{4i}$. We have the decomposition $K_{4i} = (N \cap K_{4i})(T \cap K_{4i})(\overline{N} \cap K_{4i})$. For $h \in \overline{N} \cap K_{4i} \subset \overline{N}_i$ we have $f_s^i(\tilde{g}h) = f_s^i(\tilde{g})$ by the definition of f_s^i. Now we take $h \in T \cap K_{4i}$. Write $h = t(a_0)$ with $a_0 \in 1+\mathcal{P}^{4i}$. We have $\bar{n}(x)h = h\bar{n}(a_0^{-2}x)$. It is clear that $x \in \mathcal{P}^{3i}$ if and only if $a_0^{-2}x \in \mathcal{P}^{3i}$. On the other hand, for any $a \in F^\times$ and $b \in F$ we have

$$c\left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, t(a_0)\right) = (a, a_0) = 1,$$

since $a_0 \in 1+\mathcal{P}_F^{4i} \subset F^{\times,2}$ by Lemma 3.3. Thus we get

$$\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta\right)\bar{n}(x)h = \left(\begin{pmatrix} aa_0 & ba_0^{-1} \\ & a^{-1}a_0^{-1} \end{pmatrix}, \zeta\right)\bar{n}(a_0^{-2}x).$$

By the definition of f_s^i, if $x \in \mathcal{P}^{3i}$ for $g = \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta\right)\bar{n}(x)$ then we get

$$f_s^i(gh) = \mu_{\psi^{-1}}(a_0 a)\eta_{s+1/2}(aa_0) = \mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) = f_s^i(g)$$

by the assumption on i.

Finally, we consider $h \in N \cap K_{4i} \subset N \cap K_c$. By the assumption on i we get

$$A(h,i) = A(h^{-1},i) = \overline{N}_i.$$

In particular, for $\bar{n} \in \overline{N}_i$ we have $\bar{n}h \in B \cdot \overline{N}_i$ and $\bar{n}h^{-1} \in B \cdot \overline{N}_i$. Now it is clear that $\tilde{g} \in \widetilde{B} \cdot \overline{N}_i$ if and only if $\tilde{g}h \in \widetilde{B} \cdot \overline{N}_i$. Thus $f_s^i(\tilde{g}) = 0$ if and only if $f_s^i(\tilde{g}h) = 0$. Moreover, for $\bar{n} \in \overline{N}_i$, we have

$$\bar{n}h = \begin{pmatrix} a_0 & b_0 \\ & a_0^{-1} \end{pmatrix}\bar{n}_0$$

for $a_0 \in 1+\mathcal{P}^c$, $b_0 \in F$ and $\bar{n}_0 \in \overline{N}_i$. Thus for $\tilde{g} = \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta\right)\bar{n}$ with $\bar{n} \in \overline{N}_i$ we get

$$\tilde{g}h = \left(\begin{pmatrix} aa_0 & ab_0 + a_0^{-1}b \\ & a_0^{-1}a^{-1} \end{pmatrix}, \zeta\right)\bar{n}_0.$$

Here we used the fact that $a_0 \in 1+\mathcal{P}^c$ is a square and thus

$$c\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} a_0 & b_0 \\ & a_0^{-1} \end{pmatrix}\right) = 1.$$

Since $\mu_{\psi^{-1}}(a_0) = 1$, $(a, a_0) = 1$ and $\eta_{s+1/2}(a_0) = 1$ we get

$$f_s^i(\tilde{g}h) = f_s^i(g).$$

This finishes the proof of (1).

(2) As in the proof of (1) let c be a positive integer such that η is trivial on $1+\mathcal{P}^c$. Take $I(X,\eta) = \max\{i_1(X,c), i_0(X)\}$. We have

$$\tilde{f_s^i}(wx) = \int_N f_s^i(w^{-1}nwx)\,dn.$$

By the definition of f_s^i, $f_s^i(w^{-1}nwx) \neq 0$ if and only if $w^{-1}nwx \in B\overline{N}_i$ if and only if $w^{-1}nw \in A(x,i) = \overline{N}_i$ for all $i \geq I(X)$ and $x \in X$. On the other hand, if

$w^{-1}nw \in A(x,i)$, we have

$$w^{-1}nwx = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \bar{n}_0$$

with $a \in 1+\mathcal{P}_F^c$. Thus

$$f_s^i(w^{-1}nwx) = \eta_{s+1/2}(a)\mu_{\psi^{-1}}(a) = 1.$$

Now the assertion is clear. □

3C. *The local converse theorem.*

Lemma 3.9. *Let ϕ^m be the characteristic function of $1+\mathcal{P}^m$. Then*

(1) *for $n \in N_m$ we have $\omega_{\psi^{-1}}(n)\phi^m = \psi^{-1}(n)\phi^m$, and*

(2) *for $\bar{n} \in \overline{N}_m$ we have $\omega_{\psi^{-1}}(\bar{n})\phi^m = \phi^m$.*

Proof.

(1) For $n = n(b) \in N_m$ we have $b \in \mathcal{P}^{-m}$. For $x \in 1+\mathcal{P}^m$ we have $bx^2 - b \in \mathcal{O}_F$. Thus

$$\omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = \psi^{-1}(b)\phi^m(x).$$

For $x \notin 1+\mathcal{P}^m$ we have $\omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = 0$. The first assertion follows.

(2) For $\bar{n} \in \overline{N}_m$ we can write $\bar{n} = w^{-1}n(b)w$ with $b \in \mathcal{P}^{3m}$. Let $\phi' = \omega_{\psi^{-1}}(w)\phi^m$. We have

$$\begin{aligned} \phi'(x) &= \gamma(\psi^{-1}) \int_F \phi^m(y)\psi^{-1}(2xy)dy \\ &= \gamma(\psi^{-1})\psi^{-1}(2x) \int_{\mathcal{P}^m} \psi^{-1}(2xz)dz \\ &= \gamma(\psi^{-1})\psi^{-1}(2x)\,\mathrm{vol}(\mathcal{P}^m)\,\mathrm{Char}(\mathcal{P}^{-m})(x), \end{aligned}$$

where $\mathrm{Char}(\mathcal{P}^{-m})$ denotes the characteristic function of the set $\mathcal{P}^{-m}$. It is clear that $\omega_{\psi^{-1}}(n(b))\phi' = \phi'$. Thus we have

$$\omega_{\psi^{-1}}(\bar{n})\phi^m = \omega_{\psi^{-1}}(w^{-1}n(b))\phi' = \omega_{\psi^{-1}}(w^{-1})\phi' = \omega_{\psi^{-1}}(w^{-1})\omega_{\psi^{-1}}(w)\phi^m = \phi^m.$$

This completes the proof. □

Given a quasicharacter η of $F^\times$ recall that we have defined a local gamma factor $\gamma(s,\pi,\eta,\psi)$ in Proposition 2.2.

Theorem 3.10. *Suppose that the residue characteristic of F is not 2 and ψ is a nontrivial additive character of F. Let (π, V_π) and $(\pi', V_{\pi'})$ be two ψ-generic representations of $\mathrm{SL}_2(F)$ with the same central character.*

(1) *If* $\gamma(s,\pi,\eta,\psi)=\gamma(s,\pi',\eta,\psi)$ *for all quasicharacters* η *of* $F^\times$, *then* $\pi\cong\pi'$.

(2) *There is an integer* $l=l(\pi,\pi')$ *such that if* η *is quasicharacter of* $F^\times$ *with conductor* $\mathrm{cond}(\eta)>l$, *then*

$$\gamma(s,\pi,\eta,\psi)=\gamma(s,\pi',\eta,\psi).$$

Remark. Theorem 3.10 can be viewed as one example of a general local converse conjecture for classical groups, see [Jiang 2006, Conjecture 3.7] or [Jiang and Nien 2013, Conjecture 6.3].

Proof. We will first treat the case where ψ is unramified and prove the general case at the end. We fix the notations $v\in V_\pi$, $v'\in V_{\pi'}$ and L as before.

Let η be a quasicharacter of $F^\times$. We take an integer $m\geq\max\{6L,\mathrm{cond}(\eta)\}$ and consider the Howe vectors v_m and v'_m. Additionally, we take an integer $i\geq\max\{i_2(\eta),I(N_m,\eta),m\}$. In particular we have a section $f_s^i\in I(s,\eta,\psi)$ as in Section 3C. Let $W_m=W_{v_m}$ or $W_{v'_m}$. We compute the integral of $\Psi(W_m,\phi^m,f_s^i)$ on the open dense subset $T\overline{N}(F)=N(F)\setminus N(F)T\overline{N}(F)$ of $N(F)\setminus\mathrm{SL}_2(F)$. For $g=nt(a)\bar{n}$ we can take the quotient measure as $dg=|a|^{-2}d\bar{n}da$. By the definition of f_s^i we get

$$\begin{aligned}\Psi(W_m,\phi^m,f_s^i)&=\int_{T\times\overline{N}(F)}W_m(t(a)\bar{n})(\omega_{\psi^{-1}}(t(a)\bar{n})\phi^m)(1)f_s^i(t(a)\bar{n})|a|^{-2}d\bar{n}da\\&=\int_{T\times\overline{N}_i}W_m(t(a)\bar{n})\mu_{\psi^{-1}}(a)|a|^{1/2}\omega_{\psi^{-1}}(\bar{n})\\&\qquad\qquad\cdot\phi^m(a)\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a)|a|^{-2}d\bar{n}da\\&=\int_{T\times\overline{N}_i}W_m(t(a)\bar{n})\omega_{\psi^{-1}}(\bar{n})\phi^m(a)\chi(a)\eta_{s-1}(a)d\bar{n}da,\end{aligned}$$

where $\chi(a)=\mu_{\psi^{-1}}(a)\mu_{\psi^{-1}}(a)=(a,-1)_F$. Since $i\geq m$ we get $\overline{N}_i\subset\overline{N}_m$. By Lemmas 3.1 and 3.9 we get $W_m(t(a)\bar{n})=W_m(t(a))$ and $\omega_{\psi^{-1}}(\bar{n})\phi^m=\phi^m$. Thus we get

$$\Psi(W_m,\phi^m,f_s^i)=q^{-3i}\int_{F^\times}W_m(t(a))\phi^m(a)\chi(a)\eta_{s-1}(a)da.$$

Since $\phi^m=\mathrm{Char}(1+\mathcal{P}^m)$ and, for $a\in 1+\mathcal{P}^m$, we have $W_m(t(a))=1$. By Lemma 3.1 we get

$$\Psi(W_m,\phi^m,f_s^i)=q^{-3i}\int_{1+\mathcal{P}^m}\chi(a)\eta(a)da.$$

Since $\chi(a)=1$ for $a\in 1+\mathcal{P}^m$ and $m\geq\mathrm{cond}(\eta)$ by assumption we get

$$\Psi(W_m,\phi^m,f_s^i)=q^{-3i-m}.$$

The above calculation works for both W_{v_m} and $W_{v'_m}$. Thus we have

$$\Psi(W_{v_m}, \phi^m, f_s^i) = \Psi(W_{v'_m}, \phi^m, f_s^i) = q^{-3i-m}. \tag{3-2}$$

Next we compute the other side of the local functional equation, $\Psi(W_m, \phi^m, \tilde{f}_s^i)$, on the open dense subset $N(F)\backslash N(F)TwN(F) \subset N(F)\backslash \mathrm{SL}_2(F)$, where $\tilde{f}_s^i = M_s(f_s^i)$. We have

$$\begin{aligned}
&\Psi(W_m, \phi^m, \tilde{f}_s^i)\\
&\quad = \int_{T\times N(F)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2} dnda\\
&\quad = \int_{T\times N_m} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2} dnda\\
&\qquad + \int_{T\times (N(F)-N_m)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2} dnda.
\end{aligned}$$

By Lemma 3.6 we get $W_{v_m}(t(a)wn) = W_{v'_m}(t(a)wn)$ for all $n \in N(F) - N_m$. Thus

$$\begin{aligned}
&\Psi(W_{v_m}, \phi^m, \tilde{f}_s^i) - \Psi(W_{v'_m}, \phi^m, \tilde{f}_s^i)\\
&\qquad = \int_{T\times N_m} (W_{v_m}(t(a)wn) - W_{v'_m}(t(a)wn))(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\\
&\qquad\qquad\qquad \cdot \tilde{f}_s^i(t(a)wn)|a|^{-2} dnda.
\end{aligned}$$

Since $i \geq I(N_m, \eta)$ we get

$$\tilde{f}_s^i(t(a)wn) = \mu_{\psi^{-1}}(a)\eta^{-1}_{3/2-s}(a)q_F^{-3i}$$

by Lemma 3.8. On the other hand, by Lemma 3.1 and Lemma 3.9, for $n \in N_m$ we get

$$\begin{gathered}
W_m(t(a)wn) = \psi(n)W_m(t(a)w),\\
(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1) = \psi^{-1}(n)(\omega_{\psi^{-1}}(t(a)w)\phi^m)(1).
\end{gathered}$$

Thus

$$\begin{aligned}
&\Psi(W_{v_m}, \phi^m, \tilde{f}_s^i) - \Psi(W_{v'_m}, \phi^m, \tilde{f}_s^i)\\
&\qquad = q_F^{-3i+m} \int_T (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\\
&\qquad\qquad \cdot \chi(a)\eta^{-1}(a)|a|^{-s} da.
\end{aligned} \tag{3-3}$$

By (3-2), (3-3) and the local functional equation we get

$$\begin{aligned}
&q^{-2m}(\gamma(s,\pi,\eta,\psi) - \gamma(s,\pi',\eta,\psi))\\
&\qquad = \int_{F^\times} (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s} da.
\end{aligned} \tag{3-4}$$

Let $k = 4L$. Since $m \geq 6L > k$, by Lemmas 3.1 and 3.6, we get

$$\begin{aligned} W_{v_m}(t(a)w) - W_{v'_m}(t(a)w) &= \frac{1}{\mathrm{vol}(N_m)} \int_{N_m} (W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn \\ &= \frac{1}{\mathrm{vol}(N_m)} \int_{N_k} (W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn \\ &= \frac{\mathrm{vol}(N_k)}{\mathrm{vol}(N_m)} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \\ &= q^{k-m}(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)). \end{aligned}$$

Now we can rewrite (3-4) as

$$\begin{aligned} &q^{-m-k}(\gamma(s,\pi,\eta,\psi) - \gamma(s,\pi',\eta,\psi)) \\ &\quad = \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da. \end{aligned} \tag{3-5}$$

By Lemma 3.2, if $a \notin \mathcal{P}^{-6L}$, i.e., $a^2 \notin \mathcal{P}^{-3k}$, we get $W_{v_k}(t(a)w) = 0 = W_{v'_k}(t(a)w)$. Thus the integral on the right side of formula (3-5) can be taken over $\mathcal{P}^{-6L}$. For $a \in \mathcal{P}^{-6L}$ and $m \geq 6L$ (as we assumed), by the calculation given in the proof of Lemma 3.9, we have

$$\begin{aligned} (\omega_{\psi^{-1}}(w)\phi^m)(a) &= \gamma(\psi^{-1})\psi^{-1}(2a)\,\mathrm{vol}(\mathcal{P}^m)\,\mathrm{Char}(\mathcal{P}^{-m})(a) \\ &= \gamma(\psi^{-1})\psi^{-1}(2a)q^{-m}. \end{aligned}$$

Plugging this into (3-5) we get

$$\begin{aligned} &q^{-k}\gamma(\psi^{-1})^{-1}(\gamma(s,\pi,\eta,\psi) - \gamma(s,\pi',\eta,\psi)) \\ &\quad = \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da. \end{aligned} \tag{3-6}$$

Now we can prove our theorem. We consider (1) first. Suppose $\gamma(s,\pi,\eta,\psi) = \gamma(s,\pi',\eta,\psi)$ for all quasicharacters η of $F^\times$. Then we get

$$\int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da = 0$$

for all quasicharacters η.

We rewrite the equality as

$$\begin{aligned} 0 &= \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da \\ &= \sum_{m=-\infty}^{\infty} \int_{|a|=q^m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)da\,q^{-ms}. \end{aligned}$$

It follows that all the coefficients in the above Laurent series in q^s have to be zero. So

$$\int_{|a|=q^m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)da = 0 \tag{3-7}$$

for all quasicharacters η.

Since the set $\{a \in F^\times : |a| = q^m\}$ is compact open in $F^\times$, the left side of equation (3-7) can be viewed as Mellin transform of a compactly supported function on $F^\times$. By the inverse Mellin transform we get

$$(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a) = 0,$$

or

$$W_{v_k}(t(a)w) = W_{v'_k}(t(a)w).$$

By Lemmas 3.1 and 3.6, Corollary 3.5 and the Bruhat decomposition $\mathrm{SL}_2(F) = B \cup BwB$ we get

$$W_{v_k}(g) = W_{v'_k}(g)$$

for all $g \in \mathrm{SL}_2(F)$. By the uniqueness of Whittaker model we get $\pi \cong \pi'$. This proves (1).

Next we consider (2). Let $l = l(\pi, \pi')$ be an integer such that $l \geq 6L$, then

$$W_{v_k}(t(a_0 a)w) = W_{v_k}(t(a)w) \quad \text{and} \quad W_{v'_k}(t(a_0 a)w) = W_{v'_k}(t(a)w)$$

for all $a_0 \in 1 + \mathcal{P}^l$ and all $a \in \mathcal{P}^{-6L}$. Such an l exists because the functions $a \mapsto W_{v_k}(t(a)w)$ and $a \mapsto W_{v'_k}(t(a)w)$ on $\mathcal{P}^{-6L} \subset F^\times$ are continuous. Note that $k = 4L$ and L only depends on the choices of v and v'. On the other hand, for $a \in \mathcal{P}^{-6L}$, it is easy to see that

$$\psi^{-1}(2a_0 a) = \psi^{-1}(2a) \quad \text{for all } a_0 \in 1 + \mathcal{P}^l,$$

since $l \geq 6L$. It is also clear that $\chi(a_0 a) = \chi(a)$ for all $a_0 \in 1 + \mathcal{P}^l$, since the character χ is unramified. As we noted before, the integrand of the right side integral of (3-6) has support in $\mathcal{P}^{-6L}$. Let η be a quasicharacter of $F^\times$ with $\mathrm{cond}(\eta) > l$. Then it is clear that the integral of the right side of (3-6) vanishes. Thus we get

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

This finishes the proof when ψ is unramified.

Now let us consider the general case when ψ is ramified. The proof is essentially the same as the unramified case. We will indicate the necessary changes in the above proof. If ψ has conductor c, i.e., $\psi(\mathcal{P}_F^c) = 1$ but $\psi(\mathcal{P}_F^{c-1}) \neq 1$, we define

$d_m = \mathrm{diag}(p_F^{-2m+c}, 1) \in \mathrm{GL}_2(F)$ and $J_m = d_m K_m d_m^{-1}$. Then

$$J_m = \begin{pmatrix} 1+\mathcal{P}_F^m & \mathcal{P}_F^{-m+c} \\ \mathcal{P}_F^{3m-c} & 1+\mathcal{P}_F^m \end{pmatrix} \cap \mathrm{SL}_2(F).$$

For $j = (j_{il})_{1\le i,l\le 2} \in J_m$ we define $\psi_m(j) = \psi(j_{12})$. It is clear that ψ_m defines a character of J_m. Given a ψ-generic representation (π, V) of $\mathrm{SL}_2(F)$ and a vector $v \in V$ we define v_m in the same way as before, i.e., by (3-1). In this case, we fix an integer L such that $L \ge c$ and v is fixed by K_L. We call $\{v_m\}_{m\ge L}$ the Howe vectors. We note that in the proof of Lemma 3.8, we used that ψ is unramified to make sure $\mu_{\psi^{-1}}$ is trivial on $\mathcal{O}_F^\times$. If ψ is ramified, by continuity, $\mu_{\psi^{-1}}$ is trivial on $1+\mathcal{P}_F^i$ for i large. This is all what we need in the proof of Lemma 3.8 to extend it to the ramified case. Now one can check easily that all of the above proofs go through and we get the theorem in general. □

4. A strong multiplicity one theorem

Let F be a number field and $\mathbb{A}$ be its adele ring.

4A. ***Global genericity.*** In this subsection we discuss the relation between global genericity and local genericity. Let φ be a cusp form on $\mathrm{SL}_2(F)\setminus \mathrm{SL}_2(\mathbb{A})$. Since the group $N(F)\setminus N(\mathbb{A})$ is compact and abelian we have the Fourier expansion

$$\varphi(g) = \sum_{\psi \in \widehat{N(F)\setminus N(\mathbb{A})}} W_\varphi^\psi(g),$$

where

$$W_\varphi^\psi(g) = \int_{N(F)\setminus N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)dg.$$

Since φ is a cusp form we get $W_\varphi^{\psi_0} \equiv 0$, where ψ_0 is the trivial character of $F\setminus\mathbb{A}$. Thus we get

$$\varphi(g) = \sum_{\substack{\psi \in \widehat{N(F)\setminus N(\mathbb{A})} \\ \psi\ne\psi_0}} W_\varphi^\psi(g).$$

Fix a nontrivial additive character ψ of $N(F)\setminus N(\mathbb{A})$. Then

$$(\widehat{N(F)\setminus N}(\mathbb{A}))\setminus\{\psi_0\} = \{\psi_\kappa : \kappa \in F^\times\},$$

where $\psi_\kappa(a) = \psi(\kappa a)$ and $a \in \mathbb{A}$. If $\kappa \in F^{\times,2}$, say $\kappa = a^2$, we have

$$W_\varphi^{\psi_\kappa}(g) = W_\varphi^\psi(t(a)g).$$

Thus we get

$$\varphi(g) = \sum_{\kappa\in F^\times/F^{\times,2}} \sum_{a\in F^\times} W_\varphi^{\psi_\kappa}(t(a)g).$$

Corollary 4.1. *If φ is a nonzero cusp form, there exists $\kappa \in F^\times$ such that*

$$W_\varphi^{\psi_\kappa} \neq 0.$$

Let (π, V_π) be a cuspidal automorphic representation of $\mathrm{SL}_2(F) \setminus \mathrm{SL}_2(\mathbb{A})$. We say π is ψ_κ-generic if there exists $\varphi \in V_\pi$ such that

$$W_\varphi^{\psi_\kappa} \not\equiv 0.$$

Corollary 4.2. *Let π be a cuspidal automorphic representation of $\mathrm{SL}_2(F)\setminus\mathrm{SL}_2(\mathbb{A})$ and ψ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then there exists $\kappa \in F^\times$ such that π is ψ_κ-generic.*

Theorem 4.3. *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ and $\psi = \otimes\psi_v$ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then π is ψ-generic if and only if each π_v is ψ_v-generic.*

Proof. A similar result is proved for $\mathrm{U}(1, 1)$ by Gelbart, Rogawski and Soudry [1997, Proposition 2.5].

It is clear that global genericity implies local genericity. Now we consider the other direction. We assume each π_v is ψ_v-generic.

We assume π is ψ_κ-generic for some $\kappa \in F^\times$, i.e., there exists $\varphi \in V_\pi$ such that

$$W_\varphi^{\psi_\kappa}(g) = \int_{N(F)\setminus N(\mathbb{A})} \varphi(ng)\psi_\kappa^{-1}(n)dn \neq 0.$$

Then π_v is also $\psi_{\kappa,v}$-generic, where $\psi_{\kappa,v}(a) = \psi_v(\kappa a)$. By Proposition 2.1 we get $\pi_v \cong \pi_v^\kappa$.

For $\varphi \in V_\pi$ consider the function $\varphi^\kappa(g) = \varphi(g^\kappa)$, where g^k is defined by

$$g^\kappa = \mathrm{diag}(\kappa, 1)g\,\mathrm{diag}(\kappa^{-1}, 1).$$

Then

$$\begin{aligned}\int_{N(F)\setminus N(\mathbb{A})} \varphi^\kappa(ng)dn &= \int_{N(F)\setminus N(\mathbb{A})} \varphi((ng)^\kappa)dn = \int_{N(F)\setminus N(\mathbb{A})} \varphi(n^\kappa g^\kappa)dn \\ &= \int_{N(F)\setminus N(\mathbb{A})} \varphi(ng^\kappa)dn = 0,\end{aligned}$$

hence φ^κ is also a cusp form. Let V_π^κ be the space which consists of functions of the form φ^κ for all $\varphi \in V_\pi$. Let π^κ denote the cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ on V_π^κ.

Claim. $(\pi^\kappa)_v = \pi_v^\kappa$.

Proof. Let $\Lambda : V_\pi \to \mathbb{C}$ be a nonzero ψ_κ-Whittaker functional for π and let Λ_v be a nonzero $(\psi_\kappa)_v$-Whittaker functional on V_{π_v} satisfying that if $\varphi = \otimes_v\varphi_v$ is a pure tensor, then

$$\Lambda(\pi(g)\varphi) = \prod_v \Lambda_v(\pi_v(g_v)\varphi_v).$$

Note that Λ is in fact given by

$$\Lambda(\varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(n)\psi_\kappa^{-1}(n)dn.$$

Then the ψ_{κ^2}-Whittaker functional of π^κ is given by

$$\int_{N(F)\backslash N(\mathbb{A})} \varphi^\kappa(n)\psi_{\kappa^2}^{-1}(n)dn.$$

This means that if $W_\varphi(g)$ is a ψ_κ-Whittaker function of π, then $W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa)$ is a ψ_{κ^2}-Whittaker function of π^κ.

Hence, with $\varphi = \otimes_v \varphi_v$ a pure tensor, we have $W_\varphi(g) = \prod_v W_{\varphi_v}(g_v)$ and $\{W_{\varphi_v}(g_v)\}$ is the Whittaker model of π_v, while $W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa) = \prod_v W_{\varphi_v}(g_v^\kappa)$ and $\{W_{\varphi_v}(g_v^\kappa)\}$ is the Whittaker model of $(\pi^\kappa)_v$. Now $W_v(g_v) \to W_v(g_v^\kappa)$ gives an isomorphism between π_v^κ and $(\pi^\kappa)_v$, which proves the claim. $\square$

Now let us continue the proof of the theorem. By the claim we have $\pi_v \cong (\pi^\kappa)_v$ or $\pi \cong \pi^\kappa$. By the multiplicity one theorem for SL_2 of Ramakrishnan [2000] we get $\pi = \pi^\kappa$. Since π is ψ_κ-generic we get that π^κ is ψ_{κ^2}-generic and hence ψ-generic. Since $\pi = \pi^\kappa$ the theorem follows. $\square$

4B. ***Eisenstein series on*** $\widetilde{\mathbf{SL}}_2(\mathbb{A})$**.** Let $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ be the double cover of $\mathrm{SL}_2(\mathbb{A})$. It is well-known that $\mathrm{SL}_2(F)$ splits over the projection $\widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A})$. Let μ_ψ be the genuine character of $T(F) \setminus \widetilde{T}(\mathbb{A})$ whose local components are μ_{ψ_v} as given in §2.

Let η be a quasicharacter of $F^\times \setminus \mathbb{A}^\times$ and $s \in \mathbb{C}$. We consider the induced representation

$$I(s, \chi, \psi) = \mathrm{Ind}_{\widetilde{B}(\mathbb{A})}^{\widetilde{\mathrm{SL}}_2(\mathbb{A})}(\mu_\psi \eta_{s-1/2}).$$

For $f_s \in I(s, \eta, \psi)$ we consider the Eisenstein series $E(g, f_s)$ on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$:

$$E(g, f_s) = \sum_{B(F)\backslash \mathrm{SL}_2(F)} f_s(\gamma g),\ g \in \widetilde{\mathrm{SL}}_2(\mathbb{A}).$$

The above sum is absolutely convergent when $\mathrm{Re}(s) \gg 0$ and can be meromorphically continued to the whole s-plane.

There is an intertwining operator $M_s = M_s(\eta) : I(s, \eta, \psi) \to I(1-s, \eta^{-1}, \psi)$ with

$$M_s(f_s)(g) = \int_{N(F)\backslash N(\mathbb{A})} f_s(wng)dn.$$

The above integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and defines a meromorphic function of $s \in \mathbb{C}$.

Proposition 4.4. (1) *If $\eta^2 \neq 1$, then the Eisenstein series $E(g, f_s)$ is holomorphic for all s. If $\eta^2 = 1$, the only possible poles of $E(g, f_s)$ are at $s = 0$ and $s = 1$. Moreover, the order of the poles are at most* 1.

(2) *We have the functional equation*

$$E(g, f_s) = E(g, M_s(f_s)) \quad \textit{and} \quad M_s(\eta) \circ M_{1-s}(\eta^{-1}) = 1.$$

See [Gan et al. 2014, Proposition 6.1] for example.

4C. ***The global zeta-integral.*** Let ψ be a nontrivial additive character of $F \backslash \mathbb{A}$. Then there is a global Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ on $\mathcal{S}(\mathbb{A})$. For $\phi \in \mathcal{S}(\mathbb{A})$ we consider the theta series

$$\theta_\psi(\phi)(g) = \sum_{x \in F} (\omega_\psi(g)\phi)(x).$$

It is well-known that θ_ψ defines an automorphic form on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Let (π, V_π) be a ψ-generic cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$. For $\varphi \in V_\pi$, $\phi \in \mathcal{S}(\mathbb{A})$ and $f_s \in I(s, \eta, \psi^{-1})$ consider the integral

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\,\cdot\,, f_s)) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g)\theta_{\psi^{-1}}(\phi)(g)E(g, f_s)dg. \tag{4-1}$$

Proposition 4.5 [Gelbart et al. 1987, Theorem 4.C]. *For* $\mathrm{Re}(s) \gg 0$, *the integral $Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\,\cdot\,, f_s))$ is absolutely convergent and*

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\,\cdot\,, f_s)) = \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} W_\varphi^\psi(g)(\omega_{\psi^{-1}}(g))\phi(1) f_s(g)dg,$$

where $W_\varphi^\psi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)dn$ is the ψ-th Whittaker coefficient of φ.

Corollary 4.6. *We take $\varphi = \otimes \varphi_v$, $\phi = \otimes_v \phi_v$ and $f_s = \otimes f_{s,v}$ to be pure tensors. Let S be a finite set of places such that for all $v \notin S$, v is finite and π_v, ψ_v, $f_{s,v}$ are unramified. Then for* $\mathrm{Re}(s) \gg 0$ *we have*

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\,\cdot\,, f_s)) = \prod_{v \in S} \Psi(W_{\varphi_v}, \phi_v, f_{s,v}) \frac{L^S(s, \pi, St \otimes (\chi\eta))}{L^S(2s, \eta^2)},$$

where χ is the character of $F^\times \backslash \mathbb{A}^\times$ defined by

$$\chi((a_v)) = \prod_v (a_v, -1)_{F_v}, \qquad (a_v)_v \in \mathbb{A}^\times.$$

Moreover, we have the following functional equation

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\,\cdot\,, f_s)) = Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\,\cdot\,, M_s(f_s))).$$

This follows directly from Proposition 2.2, the unramified calculation, and the functional equation of Eisenstein series in Proposition 4.4.

Corollary 4.7. (1) *The partial L-function $L^S(s,\pi,St\otimes\chi\eta)$ can be extended to a meromorphic function of s.*

(2) *If $\eta^2\neq 1$, then $L^S(s,\pi,St\otimes\chi\eta)$ is holomorphic for* $\mathrm{Re}(s)>1/2$.

(3) *If $\eta^2=1$, then, on the region* $\mathrm{Re}(s)>1/2$, *the only possible pole of the function $L^S(s,\pi,St\otimes\chi\eta)$ is at $s=1$. Moreover, the order of the pole of $L^S(s,\pi,St\otimes(\chi\eta))$ at $s=1$ is at most* 1.

(4) *Let S_∞ be the set of infinity places of F, then we can find data $\varphi_v\in V_{\pi_v}$, $\phi_v\in\mathcal{S}(F_v)$ and $f_{s,v}\in I(s,\eta_v,\psi_v)$ for $v\in S_\infty$ such that*

$$\frac{L^S(s,\pi,St\otimes(\chi\eta))}{L^S(1-s,\pi,St\otimes(\chi\eta^{-1}))} = \prod_{v\in S_\infty}\frac{\Psi(W_{\varphi_v},\phi_v,M_s(f_{s,v}))}{\Psi(W_{\varphi_v},\phi_v,f_{s,v})}\cdot\prod_{v\in S-S_\infty}\gamma(s,\pi_v,\eta_v,\psi_v)\cdot\frac{L^S(2s-1,\eta^2)}{L^S(2-2s,\eta^{-2})},$$

where S is a large enough finite set of places which contains S_∞, all finite places v such that $v|2$ and all finite places such that our data is ramified. Here $\gamma(s,\pi_v,\eta_v,\psi_v)$ is the local gamma factors defined in Proposition 2.2.

Proof. By Proposition 4.4 and Corollary 4.6 to prove (1)-(3) it suffices to show that, for each place v and for any fixed point $s\in\mathbb{C}$, we can choose the data $(W_v,\phi_v,f_{s,v})$ such that $\Psi(W_{\varphi_v},\phi_v,f_{s,v})\neq 0$. If v is nonarchimedean this is shown in the proof of Theorem 3.10, see equation (3-2). We will prove the general case later, see Lemma 4.9. We now consider (4). For $s\in\mathbb{C}$ with $\mathrm{Re}(s)\gg 0$ we choose data $\varphi=\otimes\varphi_v$, $\phi=\otimes\phi_v$ and $f_s=\otimes f_{s,v}$ such that $\Psi(W_{\varphi_v},\phi_v,f_{s,v})\neq 0$ for each $v\in S$ and φ_v, ϕ_v, $f_{s,v}$ and ψ_v are unramified for $v\notin S$. By the Remark at the end of §2, for $v\notin S$, we have

$$M_s(f_{s,v})=\frac{L(2s-1,\eta_v^2)}{L(2s,\eta_v^2)}f_{1-s,v}.$$

Thus, by Corollary 4.6, for $\mathrm{Re}(s)\ll 0$ we have

$$Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\,\cdot\,,M_s(f_s))) = \prod_{v\in S}\Psi(W_{\varphi_v},\phi_v,M_s(f_{s,v}))\frac{L^S(1-s,\pi,St\otimes(\chi\eta^{-1}))}{L^S(2-2s,\eta^{-2})}\cdot\frac{L^S(2s-1,\eta^2)}{L^S(2s,\eta^2)}.$$

Note that the above equation also holds after meromorphic continuation. Now (4) follows from Corollary 4.6 and Proposition 2.2 directly. □

4D. *A strong multiplicity one theorem.* With the above preparation, we are now ready to prove the main global result of this paper.

Theorem 4.8. *Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be two irreducible cuspidal automorphic representations of* $\mathrm{SL}_2(\mathbb{A})$ *with the same central character. Suppose that π and π' are both ψ-generic. Let S be a finite set of* ***finite*** *places such that no place in S is above* 2*. If $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.*

Proof. The following argument follows from the proof of [Casselman 1973, Theorem 2, p. 307].

Let S_1 be a large finite set of places which contains $S_\infty \cup S$. Since $\pi_v \cong \pi'_v$ for all $v \notin S$, we have $L^{S_1}(s, \pi, St \otimes (\chi\eta)) = L^{S_1}(s, \pi', St \otimes (\chi\eta))$ and $L^{S_1}(1 - s, \pi, St \otimes (\chi\eta^{-1})) = L^{S_1}(1 - s, \pi, St \otimes (\chi\eta^{-1}))$. Thus, by Corollary 4.7 (4), for each quasicharacter η, we can find data $\varphi_v \in V_{\pi_v}$, $\phi_v \in \mathcal{S}(F_v)$ and $f_{s,v}$ for $v \in S_\infty$ such that

$$\prod_{v \in S_\infty} \frac{\Psi(W_{\varphi_v}, \phi_v, M_s(f_{s,v}))}{\Psi(W_{\varphi_v}, \phi_v, f_{s,v})} \cdot \prod_{v \in S_1 - S_\infty} \gamma(s, \pi_v, \eta_v, \psi_v) = \prod_{v \in S_\infty} \frac{\Psi(W_{\varphi'_v}, \phi_v, M_s(f_{s,v}))}{\Psi(W_{\varphi'_v}, \phi_v, f_{s,v})} \cdot \prod_{v \in S_1 - S_\infty} \gamma(s, \pi'_v, \eta_v, \psi_v),$$

where φ'_v is the image of φ_v under a fixed isomorphism $\pi_v \cong \pi'_v$ for $v \in S_\infty$. Since $\pi_v \cong \pi'_v$ for $v \in S_1 - S$, we get

$$\prod_{v \in S} \gamma(s, \pi_v, \eta_v, \psi_v) = \prod_{v \in S} \gamma(s, \pi'_v, \eta_v, \psi_v).$$

Fix $v_0 \in S$. By [Jacquet and Langlands 1970, Lemma 12.5], given an arbitrary character η_{v_0}, we can find a character η of $\mathbb{A}^\times$ which restricted to v_0 is η_{v_0} and has arbitrarily high conductor at the other places of S. By Theorem 3.10 (2) we conclude that

$$\gamma(s, \pi_{v_0}, \eta_{v_0}, \psi_{v_0}) = \gamma(s, \pi'_{v_0}, \eta_{v_0}, \psi_{v_0})$$

for all characters η_{v_0}. Thus, by Theorem 3.10 (1), we conclude that $\pi_{v_0} \cong \pi'_{v_0}$. This applies also to the other places of S. Thus we proved that $\pi_v \cong \pi'_v$ for all places v. Now the theorem follows from the multiplicity one theorem for SL_2 of [Ramakrishnan 2000]. □

Remark. We expect that the restriction about residue characteristics on the finite set S in Theorem 4.8 can be removed.

Finally, we prove a nonvanishing result about the archimedean local zeta-integrals which is used in the above proof. We formulate and prove the result both for the p-adic and the archimedean cases simultaneously.

Lemma 4.9. *Let F be a local field, ψ be a nontrivial additive character of F, η be a quasicharacter of $F^\times$ and π be a ψ-generic representation of* $\mathrm{SL}_2(F)$. *Then there exists $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in \mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}}_2(F)}(\eta_{s-1/2}\mu_\psi)$ such that*

$$\Psi(W, \phi, f_s) = \int_{N(F)\backslash \mathrm{SL}_2(F)} W(h)(\omega_{\psi^{-1}}\phi)(h) f_s(h) \neq 0.$$

Proof. We note that the Bruhat cell $\Omega = N(F)TwN(F)$ is open and dense in $\mathrm{SL}_2(F)$. Thus the above integral is reduced to

$$\Psi(W, \phi, f_s) = \int_{TN(F)} W(wt(a)n(u))(\omega_{\psi^{-1}}(wt(a)n(u))\phi)(1) f_s(wt(a)n(u))\Delta(a)\,da\,du,$$

where $\Delta(a) = |a|^{-2}$.

Using the formulas for the Weil representation $\omega_{\psi^{-1}}$ we find

$$(\omega_{\psi^{-1}}(wt(a)n(u))\phi)(x) = |a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\int_F \psi(ua^2y^2)\phi(ay)\psi(2xy)\,dy = |a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(x),$$

where $\Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ which is again a Schwartz function on F and depends continuously on a and u.

We next explain that the set $\{(g, 1) : g \in N(F)TwN(F)\}$, still denoted as Ω, is open in $\widetilde{\mathrm{SL}}_2(F)$. Note that there is a double covering map $p : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$. For any $(g, 1) \in \Omega$ its projection under p is g. As p is a covering map there exists an open neighborhood U_g of g contained in $N(F)TwN(F)$ such that $p^{-1}(U_g)$ is a disjoint union of two open subsets of $\widetilde{\mathrm{SL}}_2(F)$, each is homeomorphic to U_g by p. Then one component of $p^{-1}(U_g)$ is an open neighborhood of $(g, 1)$ in Ω, which shows that Ω is open in $\widetilde{\mathrm{SL}}_2(F)$.

Now define $f_s \in I(s, \eta, \psi^{-1})$ on the set $\{(g, 1) : g \in \mathrm{SL}_2(F)\}$ by

$$f_s(g) = \begin{cases} \delta(b)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(b) f_2(u) & \text{if } g = bwn(u) \in \Omega, \\ 0 & \text{otherwise}, \end{cases}$$

where $b \in B(F) = TN(F)$, $u \in F$ and f_2 is a compactly supported function to be determined later. Then we extend the definition of f_2 to the set $\{(g, -1) : g \in \mathrm{SL}_2(F)\}$ to make it genuine, i.e., $f_s(g, -1) = -1 f_s(g, 1)$.

Then the integral Ψ can be reduced further to

$$\Psi(W, \phi, f_s) = \int_{TN(F)} W(wau)|a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a) \cdot f_2(u)\Delta(a)\,da\,du. \tag{4-2}$$

Case 1 (F is p-adic). Consider the Howe vector W_{v_m}. By Corollary 3.4, taking m large enough, W_{v_m} can have arbitrarily small compact open support around 1 when restricted to T. Then $W_{w.v_m}(t(a^{-1})w)$ has small compact open support around $a=1$.

First choose ϕ so that $\hat{\Phi}_{a,u}(1) \neq 0$ when $a=1, u=0$. Then choose m so that $W_{w.v_m}(wt(a)) = W_{w.v_m}(t(a^{-1})w)$ has small compact support around 1 and all the other data involving a in the integral (*) are nonzero constants. For this $W_{w.v_m}$, consider $W_{w.v_m}(wt(a)u)$ with $u \in N$. When u is close to 1 enough, we have $W_{w.v_m}(wt(a)u) = W_{w.v_m}(wt(a))$ for all a in that small compact support around 1. Then take f_2 with support u close to 1 satisfying the above. With these choices of $W_{w.v_m}(g)$, f_2, ϕ, the integral (4-2) is nonzero.

Case 2 (F is archimedean). We will concentrate on the case $F=\mathbb{R}$. The case $F=\mathbb{C}$ is similar as we have the same formulas for the Weil representation by [Jacquet and Langlands 1970, Proposition 1.3]. We begin with the formulas

$$\begin{aligned}\text{(4-3)}\quad & \Psi(W,\phi,f_s) \\ &= \int_{TN(F)} W(wau)|a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot f_2(u)\Delta(a)\,da\,du,\end{aligned}$$

where $\Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ is again a Schwartz function, as is ϕ, and it depends on a and u continuously. Since the Fourier transform is an isometry of the Schwartz space we can choose ϕ so that the Fourier transform $\hat{\Phi}_{a,u}(1) > 0$ when $a=1$ and $u=0$, and it depends on a and u continuously.

Now let (π, V) be an irreducible generic smooth representation of $SL_2(\mathbb{R})$ of moderate growth. Realize π as a quotient of a smooth principal series $I(\chi,s)$, i.e.,

$$0 \to V' \to I(\chi,s) \to V \to 0.$$

Let $\lambda : V \to \mathbb{C}$ be the unique nonzero continuous Whittaker functional on V. Then the composition

$$\Lambda : I(\chi,s) \longrightarrow V \xrightarrow{\lambda} \mathbb{C}$$

gives the unique nonzero continuous Whittaker functional on $I(\chi,s)$ up to a scalar. It follows that the two spaces $\{\lambda(\pi(g)v) : g \in SL_2(F), v \in V\}$ and $\{\Lambda(R(g).f) : g \in SL_2(F), f \in I(\chi,s)\}$ are the same, although the first is the Whittaker model of π while the later may not be a Whittaker model of $I(\chi,s)$.

The Whittaker functional on $I(\chi,s)$ is given by the following

$$\Lambda(f) = \int_{N(F)} f(wu)\psi^{-1}(u)\,du,$$

when s is in some right half plane and its continuation gives Whittaker functionals for all $I(\chi,s)$. Also when f has support inside $\Omega = N(F)TwN(F)$ the above integral always converges for any s and gives the Whittaker functional.

Now for such f one computes that, for $a = t(a) \in T$,

$$\Lambda(I(a).f) = \int_{N(F)} f(wua)\psi^{-1}(u)du = \chi'(a)\int_F f(wu)\psi^{-1}(a^2u)du$$
$$= \chi'(a)\int_F f_1(u)\psi^{-1}(a^2u)du = \chi'(a)\hat{f}_1(a^2),$$

where f_1 is the restriction of f to wN which can be chosen to be a Schwartz function, $\hat{f}_1$ is its Fourier transform and χ' is a certain character. Again, as the Fourier transform gives an isometry of Schwartz functions, we can always choose f so that its Whittaker function $W_f(a)$ has arbitrarily small compact support around 1. By a right translation by w we show that one can always choose f so that $W_{w.f}(aw)$ has small compact support around 1.

In order to prove the proposition note that we have chosen Φ. Let

$$R(a,u) = |a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a)\Delta(a).$$

Then $R(a,u)$ is a continuous function of a and u, and $R(1,0) \neq 0$. This means that there exist neighborhoods U_1 of $a = 1$ and U_2 of $u = 0$, such that $R(a,u) > R(1,0)/2 > 0$ for all $a \in U_1$ and $u \in U_2$.

Now choose f so that $W_{w.f}(aw)$ has small compact support in a neighborhood V_1 of 1 with $V_1 \subset U_1$ and $W_{w.f}(w) > 0$. For this Whittaker function, since $W_{w.f}(awu)$ is continuous on u, we can choose f_2 so that it is positively supported in a neighborhood V_2 of 0 such that:

(1) $V_2 \subset U_2$.

(2) $W_{w.f}(awu) > W_{w.f}(w)/2 > 0$ for all $u \in V_2$.

Then (4-3) becomes

$$\int W_{w.f}(awu)R(a,u)f_2(u)dadu > \frac{W_{w.f}(w)}{2}\frac{R(1,0)}{2}\int_{V_1}\int_{V_2} f_2(u)dadu > 0,$$

which proves the nonvanishing. □

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JINGSONG CHAI
COLLEGE OF MATHEMATICS AND ECONOMETRICS
HUNAN UNIVERSITY
CHANGSHA, 410082
CHINA
jingsongchai@hotmail.com

QING ZHANG
DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
231 WEST 18TH AVENUE
100 MATH TOWER
COLUMBUS, OH 43210-1174
UNITED STATES
zhang.1649@osu.edu

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