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We define a filtration of the smooth concordance group based on the genus of representative knots. We use the Heegaard Floer ε- and Υ-invariants to prove the quotient groups with respect to this filtration are infinitely generated. Results are applied to three infinite families of topologically slice knots.

1. Introduction

Let $\mathcal{C}$ be the smooth concordance group. Let $\mathcal{G}_k$ denote the subgroup of $\mathcal{C}$ generated by knots of genus not greater than k. Clearly $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots \subseteq \mathcal{G}_k \subseteq \cdots \subseteq \mathcal{C}$ and $\bigcup_{k=1}^{\infty} \mathcal{G}_k = \mathcal{C}$. This gives a filtration of $\mathcal{C}$. We call it the *genus filtration*.

There is another way to understand $\mathcal{G}_k$. Recall that the concordance genus g_c of a knot K is defined to be the minimal genus of a knot K' concordant to K. It is obvious that $g_c(K) = \min\{k \mid K \text{ is concordant to } K' \text{ and } g(K') \le k\}$. This motivates the following definition.

Definition 1.1. The *splitting concordance genus* of a knot K is

$$g_{\mathrm{sp}}(K) := \min\{k \mid K \text{ is concordant to } K_1 \# \cdots \# K_m \text{ for some } m \text{ and } g(K_1), \ldots, g(K_m) \le k\}.$$

That is to say, $g_{\mathrm{sp}}(K)$ is the filtration level of K in $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_k \subseteq \cdots \subseteq \mathcal{C}$. By [Endo 1995], $\mathcal{G}_1$ contains a $\mathbb{Z}^\infty$ subgroup whose elements are topologically slice.

Let $\mathcal{C}_{TS} \subseteq \mathcal{C}$ be the subgroup of topologically slice knots. Recently several results have appeared which reveal that the group $\mathcal{C}_{TS}$ is quite large. For example, in [Hom 2015a; Ozsvath et al. 2014] it is shown that $\mathcal{C}_{TS}$ contains $\mathbb{Z}^\infty$ as a direct summand. In [Hedden et al. 2012] it is shown that $\mathcal{C}_{TS}$ contains $\mathbb{Z}^\infty$ as a subgroup whose nonzero elements are not concordant to knots of Alexander polynomial one. In [Hedden et al. 2016] it is shown that $\mathcal{C}_{TS}$ contains $\mathbb{Z}_2^\infty$ as a subgroup whose nonzero elements are not concordant to knots of Alexander polynomial one.

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We will show that $\mathcal{C}_{TS}$ is large in another sense. We will prove $\mathcal{C}_{TS} \not\subseteq \mathcal{G}_k$ for any k. Moreover, the difference between $\mathcal{C}_{TS}$ and $\mathcal{G}_k$ is large. Corollary 1.4 states that $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_k)$ contains a direct summand isomorphic to $\mathbb{Z}^\infty$.

Our examples will be built from those of [Hom 2014a; Ozsvath et al. 2014]. Let $\mathrm{Wh}(K)$ denote the untwisted Whitehead double of a knot K. Additionally let $K_{p,q}$ denote the (p,q)-cable of K, let $J_n = (\mathrm{Wh}(T_{2,3}))_{n,n+1} \# -T_{n,n+1}$, and let $J'_n = (\mathrm{Wh}(T_{2,3}))_{n,2n-1} \# -T_{n,2n-1}$. These knots are topologically slice and used to prove the following theorems.

Theorem 1 [Hom 2015a, Theorem 1]. *The group $\mathcal{C}_{TS}$ contains a summand which is isomorphic to $\mathbb{Z}^\infty$ and generated by $\{J_n\}_{n=2}^\infty$.*

Theorem 2 [Ozsvath et al. 2014, Theorem 1.20]. *The topologically slice knots $\{J'_n\}_{n=2}^\infty$ form a basis for a free direct summand of $\mathcal{C}_{TS}$.*

We will prove the following results.

Theorem 1.2. *$\{J_n\}_{n=k}^\infty$ forms a basis for a $\mathbb{Z}^\infty$ summand of $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_{\lfloor k/2 \rfloor})$ for any $k \geq 2$.*

Theorem 1.3. *$\{J'_n\}_{n=k}^\infty$ forms a basis for a $\mathbb{Z}^\infty$ summand of $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_{k-1})$ for any $k \geq 2$.*

Hence we have the following consequence.

Corollary 1.4. *For any $k \in \mathbb{N}$ we have $\mathcal{C}_{TS} \not\subseteq \mathcal{G}_k$. Moreover, the quotient group $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_k)$ contains a direct summand isomorphic to $\mathbb{Z}^\infty$.*

One can define another subgroup $\mathcal{H}_k$ of $\mathcal{C}$ generated by knots of 4-genus not greater than k. Clearly $\mathcal{G}_k \subseteq \mathcal{H}_k$. It is natural to ask whether $\mathcal{H}_k/\mathcal{G}_k$ is infinitely generated. We show the answer is affirmative by proving the following:

Theorem 1.5. *The quotient group $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_k)$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$ whose basis elements have slice genus* 1 *for any $k \geq 2$.*

Conjecture 1.6. (1) For any $k \in \mathbb{N}$, the quotient $(\mathcal{C}_{TS} \cap \mathcal{G}_{k+1})/(\mathcal{C}_{TS} \cap \mathcal{G}_k)$ contains a direct summand isomorphic to $\mathbb{Z}^\infty$ whose basis elements have slice genus 1.

(2) For any $k \in \mathbb{N}$, the group $\mathcal{C}/\mathcal{H}_n$ is nontrivial.

This paper is organized as follows. In Section 2 we use Alexander polynomials to prove the splitting concordance genus can be arbitrarily large. In Section 3 we review Hom's ε-invariant and develop an obstruction, which is used to prove Theorems 1.2 and 1.5. In Section 4 we use the Υ-invariant to develop an obstruction and prove Theorem 1.3.

2. A first glance at the genus filtration

Given a knot K, let $\Delta_K(t)$ be its Alexander polynomial, and $\operatorname{breadth}(\Delta_K(t))$ be the maximal exponent of $\Delta_K(t)$ minus the minimal exponent of $\Delta_K(t)$. Recall that for any K, $\operatorname{breadth}(\Delta_K(t)) \le 2g(K)$. Moreover, if K is slice, recall that it must satisfy the Fox–Milnor condition, factoring as $t^{\pm n} f(t) f(t^{-1})$. Based on these facts, we can prove the following theorem, generalizing [Livingston 2004, Theorem 2.2].

Proposition 2.1. *For any knot K, if $p(t)$ appears an odd number of times in the irreducible factorization of $\Delta_K(t)$ in $\mathbb{Z}[t, t^{-1}]$, then*

$$g_{\mathrm{sp}}(K) \ge \tfrac{1}{2} \operatorname{breadth}(p(t)).$$

Proof. By definition, we can choose knots $K_1, \dots, K_m$ such that K is concordant to $K_1 \# \cdots \# K_m$ and $g(K_i) \le g_{\mathrm{sp}}(K)$ for each $1 \le i \le m$. Thus $K \# -K_1 \# \cdots \# -K_m$ is a slice knot and its Alexander polynomial $\Delta_K(t)\Delta_{K_1}(t) \cdots \Delta_{K_m}(t)$ must factor as $t^{\pm n} f(t) f(t^{-1})$ for some $f \in \mathbb{Z}[t, t^{-1}]$. If some $p(t)$ appears an odd number of times in the irreducible factorization of $\Delta_K(t)$, it must appear in the irreducible factorization of one of $\Delta_{K_1}(t), \dots, \Delta_{K_m}(t)$. Since $2g_{\mathrm{sp}}(K) \ge \operatorname{breadth}(\Delta_{K_i}(t))$ for each $1 \le i \le m$, we conclude that $2g_{\mathrm{sp}}(K) \ge \operatorname{breadth}(p(t))$. □

Example 2.2. The Alexander polynomial of the torus knot $T_{p,q}$ is

$$\Delta_{T_{p,q}}(t) = ((t^{pq} - 1)(t - 1))/((t^p - 1)(t^q - 1)),$$

in whose irreducible factorization the cyclotomic polynomial Φ_{pq} appears exactly once. Hence $g_{\mathrm{sp}}(T_{p,q}) \ge \varphi(pq)/2$, where φ is Euler's totient function. If p and q are prime, we have $g_{\mathrm{sp}}(T_{p,q}) \ge ((p-1)(q-1))/2$. This is actually an equality, because $g(T_{p,q}) = ((p-1)(q-1))/2$.

Corollary 2.3. *$\mathcal{C}/\mathcal{G}_k$ is nontrivial for any $k \in \mathbb{N}$.*

Working a little harder, we can show the following.

Proposition 2.4. *$\mathcal{C}/\mathcal{G}_k$ contains an infinitely generated free subgroup for any $k \in \mathbb{N}$.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of strictly increasing prime numbers with $p_1 > k$. We will prove that the torus knots $\{T_{p_{2n-1}, p_{2n}}\}_{n=1}^{\infty}$ are linearly independent in $\mathcal{C}/\mathcal{G}_k$.

Suppose towards a contradiction that $\#_{i=1}^{l} c_i T_{p_{2n_i-1}, p_{2n_i}}$, where $0 < n_1 < \cdots < n_l$ and $c_1, \dots, c_l$ are nonzero integers, is concordant to $K_1 \# \cdots \# K_m$ with $g(K_j) \le k$ for $1 \le j \le m$. Notice that $\Delta_{T_{p_{2n_i-1}, p_{2n_i}}}(t) = \Phi_{p_{2n_i-1} p_{2n_i}}$, where $\Phi_{p_{2n_i-1} p_{2n_i}}$ is the cyclotomic polynomial, which is irreducible of degree $(p_{2n_i-1} - 1)(p_{2n_i} - 1)$. By a combinatorial formula [Litherland 1979, Proposition 1] for the Tristram–Levine signature functions of torus knots, $\sigma_\omega(T_{p_{2n_i-1}, p_{2n_i}})$ jumps by ± 2 at the primitive $(p_{2n_i-1} p_{2n_i})$-th roots of unity. Since the products $p_{2n_i-1} p_{2n_i}$ are distinct for $i = 1, \dots, l$, we know $\sigma_\omega(\#_{i=1}^{l} c_i T_{p_{2n_i-1}, p_{2n_i}})$ has a jump discontinuity at a primitive $(p_{2n_1-1}p_{2n_1})$-th root of unity. Hence $\sigma_\omega(K_1\#\cdots\#K_m)$ also has a jump discontinuity at a primitive $(p_{2n_1-1}p_{2n_1})$-th root of unity, and so does one of $\sigma_\omega(K_1),\ldots,\sigma_\omega(K_m)$. Without loss of generality, assume that $\sigma_\omega(K_1)$ has a jump discontinuity at a primitive $(p_{2n_1-1}p_{2n_1})$-th root of unity. Since jump discontinuities of the Tristram–Levine signature function can only appear at roots of the Alexander polynomial, it follows that $\Delta_{K_1}(t)$ has a root at a primitive $(p_{2n_1-1}p_{2n_1})$-th root of unity and thus is divisible by $\Phi_{p_{2n_i-1}p_{2n_i}}$, but this is impossible because

$$\deg\Delta_{K_1}(t)\le 2g(K_1)\le 2k<(p_{2n_i-1}-1)(p_{2n_i}-1). \qquad \square$$

3. Obstruction by ε-invariant

We assume the reader is familiar with knot Floer homology defined by Ozsváth and Szabó [2004b] and independently Rasmussen [2003] and the ε-invariant defined by Hom [2014a]. We briefly recall some of their properties for later use.

The knot Floer complex and ε-invariant. The knot Floer complex associates to a knot $K\subset S^3$ a doubly filtered, free, finitely generated chain complex over $\mathbb{F}[U,U^{-1}]$, denoted by $CFK^\infty(K)$, where $\mathbb{F}$ is the field with two elements. The two filtrations are called the algebraic and Alexander filtrations and the grading of the chain complex is called the homological or Maslov grading. Multiplication by U shifts each filtration down by one and lowers the homological grading by two. $CFK^\infty(K)$ is an invariant of K up to filtered chain homotopy equivalence. Furthermore, up to filtered chain homotopy equivalence, one can assume the differential strictly lowers at least one of the filtrations [Rasmussen 2003].

A quick corollary from [Ozsváth and Szabó 2004a, Theorem 1.2] is the following.

Proposition 3.1. *If K has genus g, then there exists a representative of the filtered chain homotopy equivalence class of $CFK^\infty(K)$ all of whose elements have filtration levels (i,j) such that $-g\le i-j\le g$.*

For a subset $S\subseteq\mathbb{Z}\oplus\mathbb{Z}$ that is downward closed under the standard product partial order on $\mathbb{Z}\oplus\mathbb{Z}$, let $C\{S\}$ denote the subcomplex of $CFK^\infty(K)$ generated by elements with filtration levels in S. If S is the difference of two such subsets, let $C\{S\}$ denote the corresponding subquotient complex of $CFK^\infty(K)$. For example, $C\{i=0\}=C\{i\le 0\}/C\{i<0\}=CFK^\infty(K)\{i\le 0\}/CFK^\infty(K)\{i<0\}$. The invariant

$$\tau(K)=\min\{s\mid \text{the inclusion map } C\{i=0,\ j\le s\}\to C\{i=0\} \text{ induces a nontrivial map on homology}\}$$

is proven to be a smooth concordance invariant in [Ozsváth and Szabó 2003].

For any knot K, Hom [2014a] defines an invariant called ε taking on values -1, 0 or 1, which has the following properties.

Proposition 3.2 [Hom 2014a, Proposition 3.6]. *The invariant ε satisfies the following properties*:

(1) *If K is smoothly slice*, *then $\varepsilon(K) = 0$.*

(2) $\varepsilon(-K) = -\varepsilon(K)$.

(3) *If $\varepsilon(K) = \varepsilon(K')$*, *then $\varepsilon(K \# K') = \varepsilon(K) = \varepsilon(K')$.*

(4) *If $\varepsilon(K) = 0$*, *then $\varepsilon(K \# K') = \varepsilon(K')$.*

Thus the relation $\sim$, defined by $K \sim K' \Leftrightarrow \varepsilon(K \# -K') = 0$, is an equivalence relation coarser than smooth concordance. It gives an equivalence relation on $\mathcal{C}$ called ε-equivalence. The ε-equivalence class of K is denoted by $[\![K]\!]$. The set of all ε-equivalence classes forms a group $\mathcal{F}$ (also denoted by $\mathcal{CFK}$ in [Hom 2015a]), which is a quotient group of $\mathcal{C}$. The kernel of the natural homomorphism from $\mathcal{C}$ to $\mathcal{F}$ is $\{[K] \in \mathcal{C} \mid \varepsilon(K) = 0\}$, where $[K]$ denotes the concordance class of K.

According to [Hom 2014b, Proposition 4.1], ε induces a total order on $\mathcal{F}$. The proof uses Proposition 3.2. The total order is defined by

$$[\![K]\!] > [\![K']\!] \Leftrightarrow \varepsilon(K \# -K') = 1.$$

Moreover, this order respects the addition operation on $\mathcal{F}$. Therefore there is a quotient homomorphism from $\mathcal{C}$ to the totally ordered abelian group $\mathcal{F}$, which can be used to show linear independence in $\mathcal{C}$.

Some facts about totally ordered abelian groups. Let G be a totally ordered abelian group, that is an abelian group with a total order respecting the addition operation. Denote its identity element by 0.

The absolute value of an element $a \in G$ is defined to be

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Definition 3.3. Two nonzero elements a and b of G are *Archimedean equivalent*, denoted by $a \sim_A b$, if there exists a natural number N such that $N \cdot |a| > |b|$ and $N \cdot |b| > |a|$. If a and b are not Archimedean equivalent and $|a| < |b|$, we say that b dominates a. We write $a \ll b$ if $a > 0$, $b > 0$ and b dominates a.

Property A. An element $a \in G$ *satisfies Property A* if for every $b \in G$ such that $b \sim_A a$, we have that $b = ka + c$, where k is an integer and c is dominated by a.

We have the following two facts:

Lemma 3.4 [Hom 2014b, Lemma 4.7]. *If $0 < a_1 \ll a_2 \ll a_3 \ll \cdots$ in G*, *then $a_1, a_2, a_3, \ldots$ are linearly independent in G.*

Lemma 3.5 [Hom 2015a, Proposition 1.3]. *If $0 < a_1 \ll a_2 \ll a_3 \ll \cdots$ in G and each a_i satisfies Property A, then $a_1, a_2, a_3, \ldots$ generate (as a basis) a direct summand isomorphic to $\mathbb{Z}^\infty$ in G.*

The following lemmas are proven in [Hom 2014b] and [Hom 2015a] respectively.

Lemma 3.6 [Hom 2014b, Remark 4.9]. *We have $0 < [\![J_n]\!] \ll [\![J_{n+1}]\!]$ for any $n \geq 2$.*

Lemma 3.7 [Hom 2015a, Proposition 4.1, Lemmas 5.2 and 5.3]. *The class $[\![J_n]\!]$ satisfies Property A for any $n \geq 2$.*

It is straightforward to check that $\{a : |a| \ll x\}$ is a subgroup of G for any $x > 0$ in G. Denote this subgroup by G_x. Let φ_x be the quotient homomorphism. Define a relation $<$ in G/G_x by $\varphi_x(a) < \varphi_x(b)$ if and only if $a < b$ and $b - a \notin G_x$.

Proposition 3.8. *The relation $<$ makes G/G_x into a totally ordered abelian group with the following properties: If $0 < a \ll b$ in G and $b \notin G_x$, then $0 \leq \varphi_x(a) \ll \varphi_x(b)$ in G/G_x. If a satisfies Property A in G, then $\varphi_x(a)$ satisfies Property A in G/G_x.*

Proof. First we check that the relation $<$ in G/G_x is well defined. Suppose $\varphi_x(a) < \varphi_x(b)$. Let $c \in G_x$. We must show $\varphi_x(a+c) < \varphi_x(b)$ and $\varphi_x(a) < \varphi_x(b+c)$. Since $b - a > 0$ and $b - a \notin G_x$ it is easy to verify that $b - a \gg |y|$ for any $y \in G_x$. Thus $b - a \pm c > 0$. Additionally $b - a \notin G_x$ implies $b - a \pm c \notin G_x$. Hence $\varphi_x(a + c) < \varphi_x(b)$ and $\varphi_x(a) < \varphi_x(b + c)$, which means the definition does not depend on the choices of a and b.

Next we verify $<$ is a strict total order on G/G_x that respects the addition operation. For trichotomy, let $\varphi_x(a)$ and $\varphi_x(b)$ be two distinct elements in G/G_x. Then $b - a \notin G_x$. Thus $b - a \neq 0$ and exactly one of $a < b$ and $b < a$ is true. Hence exactly one of $\varphi_x(a) < \varphi_x(b)$ and $\varphi_x(b) < \varphi_x(a)$ is true by definition. For transitivity, let $\varphi_x(a), \varphi_x(b), \varphi_x(c) \in G/G_x$ satisfy $\varphi_x(a) < \varphi_x(b)$ and $\varphi_x(b) < \varphi_x(c)$. Then $a < b$, $b < c$ and $b - a, c - b \notin G_x$. Immediately $a < c$. Suppose towards a contradiction that $c - a \in G_x$. Then the fact that $b - a \gg |y|$ for any $y \in G_x$ implies $b - a - (c - a) > 0$, which contradicts $b < c$. Hence $c - a \notin G_x$ and $\varphi_x(a) < \varphi_x(c)$ by definition. For consistency with the addition operation, let $\varphi_x(a)$, $\varphi_x(b)$, $\varphi_x(c) \in G/G_x$ and $\varphi_x(a) < \varphi_x(b)$. Then $a < b$ and $b - a \notin G_x$. Thus $a + c < b + c$ and $(b + c) - (a + c) \notin G_x$. Hence $\varphi_x(a) + \varphi_x(c) = \varphi_x(a + c) < \varphi_x(b + c) = \varphi_x(b) + \varphi_x(c)$ by definition.

Next, we show that if b dominates a in G and $b \notin G_x$, then $\varphi_x(b)$ dominates $\varphi_x(a)$. Suppose $0 < a \ll b$ in G and $b \notin G_x$. Then $0 < Na < b$ for any $N \in \mathbb{N}$. Additionally, the fact that $b \gg |y|$ for any $y \in G_x$ implies $Na + y < b$, $\forall y \in G_x$. It follows that $b - Na > 0$ and that $b - Na \notin G_x$. Hence $0 \leq \varphi_x(a) \ll \varphi_x(b)$ in G/G_x by definition.

Finally we show that if a has Property A in G, then $\varphi_x(a)$ has Property A in G/G_x. Suppose a satisfies Property A in G, that is, if $b \sim_A a$ in G then $b = ka + c$ for some integer k and some $c \in G$ dominated by a. Without loss of generality we assume $\varphi_x(a) \neq 0$. Let $\varphi_x(b) \sim_A \varphi_x(a)$, so $b \sim_A a$ in G. Otherwise either $|a| \ll |b|$ or $|b| \ll |a|$, which would imply $|\varphi_x(a)| \ll |\varphi_x(b)|$ or $|\varphi_x(b)| \ll |\varphi_x(a)|$. Thus $b = ka + c$ for some integer k and some $c \in G$ dominated by a. Thus $\varphi_x(b) = k\varphi_x(a) + \varphi_x(c)$. Since c is dominated by a, we know $\varphi_x(c)$ is dominated by $\varphi_x(a)$. Hence $\varphi_x(a)$ satisfies Property A in G/G_x. □

Restriction on the Archimedean equivalence class by genus. Given a knot K with $\varepsilon(K) = 1$, Hom [2015a, Section 3] defines a tuple of numerical invariants $\boldsymbol{a}^+(K) = (a_1(K), \dots, a_n(K))$. Here each $a_i(K)$ is a positive integer, and the number n depends on K. It is shown that $\boldsymbol{a}^+(K)$ is an invariant of the ε-equivalence class $[\![K]\!]$ (see [Hom 2015a, Proposition 3.1]).

Computations in [Hom 2014b] show the following result.

Lemma 3.9 [Hom 2014b, p.568]. *We have* $\boldsymbol{a}^+(J_p) = (1, p, \dots)$.

The integers a_1 and a_2 are useful in determining domination.

Lemma 3.10 [Hom 2014b, Lemmas 6.3 and 6.4]. *If* $\boldsymbol{a}^+(K) = (a_1(K), \dots)$ *and* $\boldsymbol{a}^+(K') = (a_1(K'), \dots)$ *with* $a_1(K) > a_1(K') > 0$, *then* $[\![K]\!] \ll [\![K']\!]$.

Additionally, if $\boldsymbol{a}^+(K) = (a_1(K), a_2(K), \dots)$ *and* $\boldsymbol{a}^+(K') = (a_1(K'), a_2(K'), \dots)$ *with* $a_1(K) = a_1(K') > 0$ *and* $a_2(K) > a_2(K') > 0$, *then* $[\![K]\!] \gg [\![K']\!]$.

Based on Proposition 3.1, the following is shown.

Lemma 3.11 ([Hom 2015b, Theorem 1.2 and Lemma 2.3]). *Suppose that* $\varepsilon(K) = 1$, *and* $a_2(K)$ *is defined, then* $|\tau(K) - a_1(K) - a_2(K)| \leq g(K)$.

Next we prove our obstruction theorem.

Proposition 3.12. *Suppose J is a knot with* $\boldsymbol{a}^+(J) = (1, b, \dots)$ *with* $b \geq 2n$ *for some positive integer n. Then for any knot* $K \in \mathcal{G}_n$, *we have* $|[\![K]\!]| \ll [\![J]\!]$.

Proof. Before proving the proposition for $K \in \mathcal{G}_n$, first consider the case $g(K) \leq n$. We may further assume that $[\![K]\!] > 0$, since $[\![-K]\!] > 0$ if $[\![K]\!] < 0$ and the proposition is trivial if $[\![K]\!] = 0$. Notice that $a_1(K)$ is always defined [Hom 2014b, §6]. If $a_1(K) > 1$, then $[\![K]\!] \ll [\![J]\!]$ by Lemma 3.10. If $a_1(K) = 1$, then $a_2(K)$ is defined [Hom 2015a, Lemma 3.7]. Observe that $\tau(K) - a_1(K) - a_2(K) \geq -g(K)$ by Lemma 3.11. Combining this with $\tau(K) \leq g_4(K) \leq g(K)$, it follows that $g(K) - a_1(K) - a_2(K) \geq -g(K)$. This implies $a_2(K) \leq 2n - 1$, if $a_1(K) = 1$. Hence $|[\![K]\!]| \ll [\![J]\!]$ by Lemma 3.10.

Generally, let $K \in \mathcal{G}_n$. Then $K = K_1 + \cdots + K_m$, where $g(K_i) \leq n$ for $i = 1, \dots, m$. Since $[\![K]\!] = [\![K_1]\!] + \cdots + [\![K_m]\!]$, we know $|[\![K]\!]| \leq |[\![K_1]\!]| + \cdots + |[\![K_m]\!]|$. Then the conclusion follows from the last paragraph. □

Applying the obstruction to concrete families of knots.

Proof of Theorem 1.2. Fix an integer $k \geq 2$. Under the quotient homomorphism from $\mathcal{C}$ to $\mathcal{F}$, the image of $\mathcal{G}_{\lfloor k/2 \rfloor}$ is included in $\mathcal{F}_{[\![J_k]\!]} = \{[\![K]\!] : |[\![K]\!]| \ll [\![J_k]\!]\}$ by Proposition 3.12 and Lemma 3.9. This gives a homomorphism from $\mathcal{C}/\mathcal{G}_{\lfloor k/2 \rfloor}$ to $\mathcal{F}/\mathcal{F}_{[\![J_k]\!]}$. By Lemma 3.6 and Proposition 3.8, the family $\{J_n\}_{n=k}^{\infty}$ maps to a family of elements with Property A and each term is dominated by the next. Hence $\{J_n\}_{n=k}^{\infty}$ forms a basis of a direct summand isomorphic to $\mathbb{Z}^{\infty}$ by Lemma 3.5. Note that since the J_n are topologically slice, the above argument can be restricted to the subgroup $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_{\lfloor k/2 \rfloor})$ of $\mathcal{C}/\mathcal{G}_{\lfloor k/2 \rfloor}$ to complete the proof. □

Proof of Theorem 1.5. Instead of $\{J_n\}$, we use another family of topologically slice knots $\{L_n\}$, where $L_n = (\mathrm{Wh}(T_{2,3}))_{n,1} \# -(\mathrm{Wh}(T_{2,3}))_{n-1,1}$. These knots have slice genus 1 [Hom 2015b, Lemma 3.1]. Additionally, Hom [2015b] computes that $a_1(L_n) = 1$ and $a_2(L_n) = n$. By the same argument as the above proof, except for applying Lemma 3.4 rather than Lemma 3.5, we immediately know $\{L_n\}_{n=2k}^{\infty}$ are linearly independent in $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_k)$. □

4. Obstruction by Υ-invariant

Ozsvath et al. [2014] introduced a new family of knot invariants, $\Upsilon_K(t)$. We refer the reader to their construction, and confine ourselves to recalling the basic properties of the Υ-invariant.

For any knot K, the invariant $\Upsilon_K(t)$ is a piecewise linear function on [0, 2] whose derivative has finitely many discontinuities [Ozsvath et al. 2014, Proposition 1.4]. Thus, one can define $\Delta\Upsilon'_K(t_0) = \lim_{t \to t_0^+} \Upsilon'_K(t) - \lim_{t \to t_0^-} \Upsilon'_K(t)$ for any $t_0 \in (0, 2)$.

As an example, the authors of [Ozsvath et al. 2014] compute the family $\{J'_n\}$:

$$\Delta\Upsilon'_{J'_n}(t) = \begin{cases} 0 & \text{for } t < 2/(2n-1), \\ 2n-1 & \text{for } t = 2/(2n-1). \end{cases}$$

In [Ozsvath et al. 2014, Corollary 1.12] it is shown that Υ gives a homomorphism from $\mathcal{C}$ to the vector space of continuous functions on [0, 2]. Additionally,

$$K \mapsto \begin{cases} (1/q)\Delta\Upsilon'_K(p/q) & \text{if } p \text{ is even}, \\ (1/2q)\Delta\Upsilon'_K(p/q) & \text{if } p \text{ is odd}, \end{cases}$$

gives a homomorphism from $\mathcal{C}$ to $\mathbb{Z}$ for any $p/q \in (0, 2) \cap \mathbb{Q}$.

The location of singularities of Υ is related to the genus of the knot, as in the following proposition. The proof of this proposition, much like that of Lemma 3.11, is based on the fact in Proposition 3.1.

Proposition 4.1 [Livingston 2015, Theorem 8.2]. *Suppose that $\Delta\Upsilon'_K(t)$ is nonzero at $t = p/q$ with* $\gcd(p, q) = 1$. *Then $q \leq g(K)$ if p is odd, and $q \leq 2g(K)$ if p is even.*

With this proposition, we can easily prove our obstruction theorem.

Proposition 4.2. *Suppose $K \in \mathcal{G}_n$ for some positive integer n. Then $\Delta\Upsilon'_K(t) = 0$ for $t \in (0, 1/n) \cap \mathbb{Q}$.*

Proof. Before proving the proposition for $K \in \mathcal{G}_n$, first consider the case $g(K) \leq n$. If $\Upsilon_K(t)$ has a singularity at a rational number p/q with $\gcd(p, q) = 1$, then Proposition 4.1 implies $p/q \geq 1/n$.

Generally, let $K \in \mathcal{G}_n$. Then $K = K_1 + \cdots + K_m$, where $g(K_i) \leq n$ for $i = 1, \ldots, m$. If $\Upsilon_K(t)$ has a singularity at a rational number p/q, then so does one of $\Upsilon_{K_1}(t), \ldots, \Upsilon_{K_m}(t)$, since Υ is a homomorphism. The conclusion follows from the last paragraph. □

Proof of Theorem 1.3. Fix an integer $k \geq 2$. If $K \in \mathcal{G}_{k-1}$, then $\Upsilon_K(t)$ has no singularities on $(0, 1/(k-1)) \cap \mathbb{Q}$. Thus $\{K \mapsto 1/(2n-1)\Delta\Upsilon'_K(2/(2n-1))\}_{n=k}^{\infty}$ gives a homomorphism from $\mathcal{C}/\mathcal{G}_{k-1}$ to $\mathbb{Z}^\infty$. Hence $\{J'_n\}_{n=k}^{\infty}$ form a basis for a $\mathbb{Z}^\infty$ summand of $\mathcal{C}/\mathcal{G}_{k-1}$. Note that since the J'_n are topologically slice, the above argument can be restricted to the subgroup $\mathcal{C}_{TS}/(\mathcal{C}_{TS} \cap \mathcal{G}_{k-1})$ of $\mathcal{C}/\mathcal{G}_{k-1}$ to complete the proof. □

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