Pacific Journal of Mathematics

A KIRCHBERG-TYPE TENSOR THEOREM FOR OPERATOR SYSTEMS

KYUNG HOON HAN

Volume 286 No. 1 January 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 286, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.286.91

A KIRCHBERG-TYPE TENSOR THEOREM FOR OPERATOR SYSTEMS

KYUNG HOON HAN

We construct operator systems $\mathfrak{C}_I$ that are universal in the sense that all operator systems can be realized as their quotients. They satisfy the operator system lifting property. Without relying on the theorem by Kirchberg, we prove the Kirchberg-type tensor theorem

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H).$$

Combining this with a result of Kavruk, we give a new operator system theoretic proof of Kirchberg's theorem and show that Kirchberg's conjecture is equivalent to its operator system analogue

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\mathrm{c}} \mathfrak{C}_I.$$

It is natural to ask whether the universal operator systems $\mathfrak{C}_I$ are projective objects in the category of operator systems. We show that an operator system from which all unital completely positive maps into operator system quotients can be lifted is necessarily one-dimensional. Moreover, a finite-dimensional operator system satisfying a perturbed lifting property can be represented as the direct sum of matrix algebras. We give an operator system theoretic approach to the Effros–Haagerup lifting theorem.

1. Introduction

Every Banach space can be realized as a quotient of $\ell_1(I)$ for a suitable choice of index set I. Moreover, every linear map $\varphi : \ell_1(I) \to E/F$ lifts to $\tilde{\varphi} : \ell_1(I) \to E$ with $\|\tilde{\varphi}\| < (1+\varepsilon)\|\varphi\|$. On noncommutative sides, $\bigoplus_1 T_{n_i}$ (respectively $C^*(\mathbb{F})$) plays such a role in the category of operator spaces (respectively C^*-algebras). The purpose of this paper is to find operator systems that play such a role in the category of operator systems.

We construct operator systems $\mathfrak{C}_I$ that are universal in the sense that all operator systems can be realized as their quotients. The method of construction is motivated

This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2012R1A1A1012190).
MSC2010: 46L06, 46L07, 47L07.
Keywords: operator system, tensor product, quotient, Kirchberg's theorem.

by [Blecher 1992, Proposition 3.1] and the coproduct of operator systems [Fritz 2014; Kerr and Li 2009]. The index set I is chosen to be sufficiently large that we can index the set $\mathcal{S}^+_{\|\cdot\|\le 1}$ of positive contractive elements in an operator system $\mathcal{S}$. The operator system $\mathfrak{C}_I$ is realized as the infinite coproduct of $\{M_k \oplus M_k\}_{k\in\mathbb{N}}$ admitting copies of $M_k \oplus M_k$ up to the cardinality of I.

We prove that the operator systems $\mathfrak{C}_I$ satisfy the operator system lifting property: for any unital C^*-algebra $\mathcal{A}$ with its closed ideal $\mathcal{I}$ and the quotient map $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$, every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{A}$. It is helpful to picture the situation using the commutative diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "\pi"] \\ \mathfrak{C}_I \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & \mathcal{A}/\mathcal{I} \end{tikzcd}$$

For a free group $\mathbb{F}$ and a Hilbert space H, Kirchberg [1994, Corollary 1.2] proved that

$$C^*(\mathbb{F}) \otimes_{\min} B(H) = C^*(\mathbb{F}) \otimes_{\max} B(H).$$

The proof was later simplified in [Pisier 1996] and [Farenick and Paulsen 2012] using operator space theory and operator system theory, respectively. Kirchberg's theorem is striking if we recall that $C^*(\mathbb{F})$ and $B(H)$ are universal objects in the C^*-algebra category: every C^*-algebra is a C^*-quotient of $C^*(\mathbb{F})$ and a C^*-subalgebra of $B(H)$ for suitable choices of $\mathbb{F}$ and H.

For suitable choices of I and H, every operator system is a subsystem of $B(H)$ by the Choi–Effros theorem [Choi and Effros 1977] and is a quotient of $\mathfrak{C}_I$ which is proved in Section 3. We will prove a Kirchberg-type tensor theorem

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$$

in Section 4. The proof is independent of Kirchberg's theorem. Combining this with Kavruk's idea [2012] we give a new operator system theoretic proof of Kirchberg's theorem.

We also prove that the operator system analogue

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$$

of Kirchberg's conjecture

$$C^*(\mathbb{F}) \otimes_{\min} C^*(\mathbb{F}) = C^*(\mathbb{F}) \otimes_{\max} C^*(\mathbb{F})$$

is equivalent to Kirchberg's conjecture itself.

In the final section, we consider several lifting problems of completely positive maps. It is natural to ask whether the universal operator system $\mathfrak{C}_I$ is a projective object in the category of operator systems. In other words, for any operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, does every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{S}/\mathcal{J}$ lift to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{S}$? The answer is negative in an extreme manner. An operator system satisfying such a lifting property is necessarily one-dimensional. This is essentially due to Archimedeanization of quotients [Paulsen and Tomforde 2009]. Even though some perturbation is allowed, there is also rigidity: for a finite-dimensional operator system E and a faithful state ω, the following are equivalent:

(i) If $\varepsilon > 0$ and $\varphi : E \to \mathcal{S}/\mathcal{J}$ is a completely positive map for an operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, then there exists a self-adjoint lifting $\tilde{\varphi} : E \to \mathcal{S}$ of φ such that $\tilde{\varphi} + \varepsilon\omega 1_{\mathcal{S}}$ is completely positive.

(ii) E is unitally completely order isomorphic to the direct sum of matrix algebras.

In order to prove it, we give a characterization of nuclearity via the projectivity and the minimal tensor product: an operator system $\mathcal{S}$ is nuclear if and only if

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 \to \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

is a quotient map for any quotient map $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$.

Finally, we present an operator system theoretic approach to the Effros–Haagerup lifting theorem [Effros and Haagerup 1985].

2. Preliminaries

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Following [Kavruk et al. 2011], henceforth abbreviated [KPTT1], an *operator system structure* on $\mathcal{S} \otimes \mathcal{T}$ is defined as a family of cones $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ satisfying

(T1) $(\mathcal{S}\otimes\mathcal{T}, \{M_n(\mathcal{S}\otimes_\tau\mathcal{T})^+\}_{n=1}^\infty, 1_{\mathcal{S}}\otimes 1_{\mathcal{T}})$ is an operator system denoted by $\mathcal{S}\otimes_\tau\mathcal{T}$,

(T2) $M_m(\mathcal{S})^+ \otimes M_n(\mathcal{T})^+ \subset M_{mn}(\mathcal{S} \otimes_\tau \mathcal{T})^+$ for all $m, n \in \mathbb{N}$, and

(T3) if $\varphi : \mathcal{S} \to M_m$ and $\psi : \mathcal{T} \to M_n$ are unital completely positive maps, then $\varphi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is a unital completely positive map.

By an *operator system tensor product,* we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, we have that $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, and denote it by $\mathcal{S} \otimes_\tau \mathcal{T}$. We call an operator system tensor product τ *functorial,* if the following property is satisfied:

(T4) For any operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1, \mathcal{T}_2$ and unital completely positive maps $\varphi : \mathcal{S}_1 \to \mathcal{T}_1$, $\psi : \mathcal{S}_2 \to \mathcal{T}_2$, the map $\varphi \otimes \psi : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2$ is unital completely positive.

Given a linear mapping $\varphi : V \to W$ between vector spaces, its n-th amplification $\varphi_n : M_n(V) \to M_n(W)$ is defined as $\varphi_n([x_{i,j}]) = [\varphi(x_{i,j})]$. For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = \left\{ \begin{array}{l} X \in M_n(\mathcal{S} \otimes \mathcal{T}) : (\varphi \otimes \psi)_n(X) \in M_{nkl}^+ \text{ for all unital} \\ \text{completely positive maps } \varphi : \mathcal{S} \to M_k,\, \psi : \mathcal{T} \to M_l \end{array} \right\}.$$

Then the family $\{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+\}_{n=1}^\infty$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$. Moreover, if we let $\iota_\mathcal{S} : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ and $\iota_\mathcal{T} : \mathcal{T} \to \mathcal{B}(\mathcal{K})$ be any unital complete order embeddings, then this is the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the embedding $\iota_\mathcal{S} \otimes \iota_\mathcal{T} : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ [KPTT1, Theorem 4.4]. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})\}_{n=1}^\infty, 1_\mathcal{S} \otimes 1_\mathcal{T})$ the *minimal* tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\min} \mathcal{T}$.

The mapping $\min : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\min} \mathcal{T}$ is an injective, associative, symmetric and functorial operator system tensor product. The positive cone of the minimal tensor product is the largest among all possible positive cones of operator system tensor products at each matrix level [KPTT1, Theorem 4.6]. The operator system minimal tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$ of unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a dense subsystem of C^*-minimal tensor product $\mathcal{A} \otimes_{C^*\min} \mathcal{B}$ [KPTT1, Corollary 4.10].

For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$D_n^{\max}(\mathcal{S}, \mathcal{T}) = \left\{ \alpha(P \otimes Q)\alpha^* : P \in M_k(\mathcal{S})^+,\ Q \in M_l(\mathcal{T})^+,\ \alpha \in M_{n,kl}, k, l \in \mathbb{N} \right\}.$$

This is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with order unit $1_\mathcal{S} \otimes 1_\mathcal{T}$. Let $\{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty$ be the Archimedeanization of the matrix ordering $\{D_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n=1}^\infty$. Then it can be written as

$$M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+ = \{X \in M_n(\mathcal{S} \otimes \mathcal{T}) : \forall \varepsilon > 0,\ X + \varepsilon I_n \otimes 1_\mathcal{S} \otimes 1_\mathcal{T} \in D_n^{\max}(\mathcal{S}, \mathcal{T})\}.$$

We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty, 1_\mathcal{S} \otimes 1_\mathcal{T})$ the *maximal* operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\max} \mathcal{T}$.

The mapping $\max : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\max} \mathcal{T}$ is an associative, symmetric and functorial operator system tensor product. The positive cone of the maximal tensor product is the smallest among all possible positive cones of operator system tensor products at each matrix level [KPTT1, Theorem 5.5]. The operator system maximal tensor product $\mathcal{A} \otimes_{\max} \mathcal{B}$ of unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a dense subsystem of C^*-maximal tensor product $\mathcal{A} \otimes_{C^*\max} \mathcal{B}$ [KPTT1, Theorem 5.12].

For completely positive maps $\varphi : \mathcal{S} \to B(H)$ and $\psi : \mathcal{T} \to B(H)$, let $\varphi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to B(H)$ be the map given on simple tensors by $(\varphi \cdot \psi)(x \otimes y) = \varphi(x)\psi(y)$.

We put

$$M_n(\mathcal{S}\otimes_{\mathrm{c}}\mathcal{T})^+ = \left\{ \begin{array}{l} X\in M_n(\mathcal{S}\otimes\mathcal{T}) : (\varphi\cdot\psi)_n(X)\geq 0 \text{ for all completely positive} \\ \text{maps } \varphi:\mathcal{S}\to B(H),\ \psi:\mathcal{T}\to B(H) \text{ with commuting ranges} \end{array} \right\}.$$

Then the family $\{M_n(\mathcal{S}\otimes_{\mathrm{c}}\mathcal{T})^+\}_{n=1}^{\infty}$ is an operator system structure on $\mathcal{S}\otimes\mathcal{T}$. We call the operator system $(\mathcal{S}\otimes\mathcal{T}, \{M_n(\mathcal{S}\otimes_{\mathrm{c}}\mathcal{T})\}_{n=1}^{\infty}, 1_{\mathcal{S}}\otimes 1_{\mathcal{T}})$ the *commuting* tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S}\otimes_{\mathrm{c}}\mathcal{T}$. If $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{S}$ is an operator system, then we have

$$\mathcal{A}\otimes_{\mathrm{c}}\mathcal{S} = \mathcal{A}\otimes_{\max}\mathcal{S}$$

[KPTT1, Theorem 6.7]. Hence, the maximal tensor product and the commuting tensor product are two different means of extending the C^*-maximal tensor product from the category of C^*-algebras to operator systems.

For an inclusion $\mathcal{S}\subset B(H)$, we let $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T}$ be the operator system with underlying space $\mathcal{S}\otimes\mathcal{T}$ whose matrix ordering is induced by the inclusion $\mathcal{S}\otimes\mathcal{T}\subset B(H)\otimes_{\max}\mathcal{T}$. We call the operator system $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T}$ the *enveloping left* operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$. The mapping $\mathrm{el}:\mathcal{O}\times\mathcal{O}\to\mathcal{O}$ sending $(\mathcal{S},\mathcal{T})$ to $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T}$ is a left injective functorial operator system tensor product. Here, $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T}$ is independent of the choice of any injective operator system containing $\mathcal{S}$ instead of $B(H)$.

Given an operator system $\mathcal{S}$, we call $\mathcal{J}\subset\mathcal{S}$ the kernel, provided that it is the kernel of a unital completely positive map from $\mathcal{S}$ to another operator system. If we define a family of positive cones $M_n(\mathcal{S}/\mathcal{J})^+$ on $M_n(\mathcal{S}/\mathcal{J})$ as

$$M_n(\mathcal{S}/\mathcal{J})^+ := \big\{[x_{i,j}+\mathcal{J}]_{i,j} : \forall\varepsilon>0,\ \exists\, k_{i,j}\in\mathcal{J},\ \varepsilon I_n\otimes 1_{\mathcal{S}}+[x_{i,j}+k_{i,j}]_{i,j}\in M_n(\mathcal{S})^+\big\},$$

then $(\mathcal{S}/\mathcal{J}, \{M_n(\mathcal{S}/\mathcal{J})^+\}_{n=1}^{\infty}, 1_{\mathcal{S}/\mathcal{J}})$ satisfies all the conditions of an operator system, from Proposition 3.4 in [Kavruk et al. 2013], henceforth abbreviated [KPTT2]. We call this the quotient operator system. With this definition, the first isomorphism theorem can be proved: if $\varphi:\mathcal{S}\to\mathcal{T}$ is a unital completely positive map with $\mathcal{J}\subset\ker\varphi$, then the map $\widetilde{\varphi}:\mathcal{S}/\mathcal{J}\to\mathcal{T}$ given by $\widetilde{\varphi}(x+\mathcal{J})=\varphi(x)$ is a unital completely positive map from Proposition 3.6 in the same paper. In particular, when

$$M_n(\mathcal{S}/\mathcal{J})^+ = \big\{[x_{i,j}+\mathcal{J}]_{i,j} : \exists\, k_{i,j}\in\mathcal{J},\ [x_{i,j}+k_{i,j}]_{i,j}\in M_n(\mathcal{S})^+\big\}$$

for all $n\in\mathbb{N}$, we call the kernel $\mathcal{J}$ completely order proximinal.

Since the kernel $\mathcal{J}$ in an operator system $\mathcal{S}$ is a closed subspace, the operator space structure of $\mathcal{S}/\mathcal{J}$ can be interpreted in two ways: first, as the operator space quotient and second, as the operator space structure induced by the operator system quotient. The two matrix norms can be different. For a specific example, see [KPTT2, Example 4.4].

For a unital completely positive surjection $\varphi : \mathcal{S} \to \mathcal{T}$, we call $\varphi : \mathcal{S} \to \mathcal{T}$ *a complete order quotient map* [Han 2011, Definition 3.1] if for any Q in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take an element P in $M_n(\mathcal{S})$ so that it satisfies

$$P + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \quad \text{and} \quad \varphi_n(P) = Q,$$

or equivalently, if for any Q in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take a positive element P in $M_n(\mathcal{S})$ satisfying

$$\varphi_n(P) = Q + \varepsilon I_n \otimes 1_{\mathcal{S}}.$$

This definition is compatible with [Farenick et al. 2013, Proposition 3.2]: every strictly positive element lifts to a strictly positive element. An element $x \in \mathcal{S}$ is called *strictly positive* if there exists $\delta > 0$ such that $x \geq \delta 1_{\mathcal{S}}$. The map $\varphi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map if and only if the induced map $\widetilde{\varphi} : \mathcal{S}/\ker\varphi \to \mathcal{T}$ is a unital complete order isomorphism. In other operator system references, this is termed a complete quotient map. To avoid confusion with complete quotient maps in operator space theory, we use the terminology of *a complete order quotient map* throughout this paper. In this paper, we say that a linear map $\Phi : V \to W$ for operator spaces V and W is a *complete quotient map* if Φ_n maps the open unit ball of $M_n(V)$ onto the open unit ball of $M_n(W)$. When $\varphi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map (respectively a complete order embedding), we will use the special type arrow as $\varphi : \mathcal{S} \twoheadrightarrow \mathcal{T}$ (respectively $\varphi : \mathcal{S} \hookrightarrow \mathcal{T}$) throughout the paper.

The normed space dual $\mathcal{S}^*$ of an operator system $\mathcal{S}$ is matrix ordered by the cones

$$M_n(\mathcal{S}^*)^+ = \{\text{completely positive maps from } \mathcal{S} \text{ to } M_n\},$$

where we identify $[\varphi_{i,j}] \in M_n(\mathcal{S}^*)$ with the mapping $x \in \mathcal{S} \mapsto [\varphi_{i,j}(x)] \in M_n$. Unfortunately, duals of operator systems fail to be operator systems in general due to the lack of matrix order unit. When $\mathcal{S}$ is finite-dimensional, there exists a state ω_0 on $\mathcal{S}$ such that $(\mathcal{S}^*, \{M_n(\mathcal{S}^*)^+\}_{n\in\mathbb{N}}, \omega_0)$ is an operator system [Choi and Effros 1977, Corollary 4.5]. In fact, we can show that every faithful state on $\mathcal{S}$ plays such a role by the compactness of $\mathcal{S}^+_{\|\cdot\|=1}$.

Let f (respectively g) be a state on $M_n(\mathcal{S})$ (respectively $M_n(\mathcal{T})$). We identify a subsystem $M_n \otimes \mathbb{C}1_{\mathcal{S}}$ of $M_n(\mathcal{S})$ (respectively $M_n \otimes \mathbb{C}1_{\mathcal{T}}$ of $M_n(\mathcal{T})$) with M_n. We call (f, g) a compatible pair whenever $f|_{M_n} = g|_{M_n}$. An operator system structure is defined on the amalgamated direct sum $\mathcal{S} \oplus \mathcal{T}/\langle(1_{\mathcal{S}}, -1_{\mathcal{T}})\rangle$ identifying each order unit. For $s \in M_n(\mathcal{S})$ and $t \in M_n(\mathcal{T})$, we define

(1) $(s+t)^* = s^* + t^*$,

(2) $s + t \geq 0$ if and only if $f(s) + g(t) \geq 0$ for all compatible pairs (f, g).

This operator system is denoted by $\mathcal{S} \oplus_1 \mathcal{T}$ and called the coproduct of operator systems $\mathcal{S}$ and $\mathcal{T}$. The canonical inclusion from $\mathcal{S}$ (respectively $\mathcal{T}$) into $\mathcal{S} \oplus_1 \mathcal{T}$

is a complete order embedding. The coproducts of operator systems satisfy the universal property: for unital completely positive maps $\varphi : \mathcal{S} \to \mathcal{R}$ and $\psi : \mathcal{T} \to \mathcal{R}$, there is a unique unital completely positive map $\Phi : \mathcal{S} \oplus_1 \mathcal{T} \to \mathcal{R}$ that extends both φ and ψ, i.e., such that the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[d, hook] \arrow[dr, "\varphi"] & \\ \mathcal{S} \oplus_1 \mathcal{T} \arrow[r, dashed, "\Phi"] & \mathcal{R} \\ \mathcal{T} \arrow[u, hook'] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes [Fritz 2014, Proposition 3.3]. The coproduct $\mathcal{S} \oplus_1 \mathcal{T}$ can be realized as a quotient operator system. The map

$$s + t \in \mathcal{S} \oplus_1 \mathcal{T} \mapsto 2(s, t) + \langle 1_{\mathcal{S}}, -1_{\mathcal{T}} \rangle \in \mathcal{S} \oplus \mathcal{T} / \langle 1_{\mathcal{S}}, -1_{\mathcal{T}} \rangle$$

is a unital complete order isomorphism [Kavruk 2014].

We refer to [KPTT1; KPTT2; Kavruk 2014; Fritz 2014] for general information on tensor products, quotients, duals and coproducts of operator systems.

3. Universal operator systems $\mathfrak{C}_I$

The coproduct of two operator systems can be generalized to any family of operator systems in a way parallel to [Fritz 2014]. Suppose that $\{\mathcal{S}_\iota\}_{\iota \in I}$ is a family of operator systems. We consider their algebraic direct sum $\bigoplus_{\iota \in I} \mathcal{S}_\iota$ consisting of finitely supported elements and its subspace

$$N = \operatorname{span}\Big\{ n_{\iota_1} - n_{\iota_2} \in \bigoplus_{\iota \in I} \mathcal{S}_\iota : \iota_1, \iota_2 \in I \Big\},$$

where

$$n_{\iota_0}(\iota) = \begin{cases} 1_{\mathcal{S}_{\iota_0}} & \text{if } \iota = \iota_0, \\ 0 & \text{otherwise.} \end{cases}$$

The algebraic quotient

$$\Big(\bigoplus_{\iota \in I} \mathcal{S}_\iota \Big) / N$$

can be regarded as an amalgamated direct sum of $\{\mathcal{S}_\iota\}_{\iota \in I}$ identifying all order units $1_{\mathcal{S}_\iota}$ over $\iota \in I$. We denote general elements in $M_n\big((\bigoplus_{\iota \in I} \mathcal{S}_\iota)/N \big)$ in brief by

$$\sum_{\iota \in F} x_\iota, \quad \text{where } x_\iota \in M_n(\mathcal{S}_\iota),\ F \text{ is a finite subset of } I.$$

Let ω_ι be a state on $M_n(\mathcal{S}_\iota)$ for each $\iota \in F$. We identify each subsystem $M_n \otimes \mathbb{C}1_{\mathcal{S}_\iota}$ of $M_n(\mathcal{S}_\iota)$ with M_n. Whenever $\omega_{\iota_1}|_{M_n} = \omega_{\iota_2}|_{M_n}$ for each $\iota_1, \iota_2 \in F$, we call the family $\{\omega_\iota\}_{\iota\in F}$ compatible. On $M_n\big((\bigoplus_{\iota\in I}\mathcal{S}_\iota)/N\big)$, we define the involution by

$$\Big(\sum_{\iota\in F} x_\iota\Big)^* = \sum_{\iota\in F} x_\iota^*$$

and the positive cone as

$$\sum_{\iota\in F} x_\iota \in M_n\big((\bigoplus_{\iota\in I}\mathcal{S}_\iota)/N\big)^+ \quad \Longleftrightarrow \quad \sum_{\iota\in F}\omega_\iota(x_\iota)\ge 0$$

for any compatible family $\{\omega_\iota\}_{\iota\in F}$ of states. The triple

$$\big((\bigoplus_{\iota\in I}\mathcal{S}_\iota)/N,\ \big\{M_n\big((\bigoplus_{\iota\in I}\mathcal{S}_\iota)/N\big)^+\big\}_{n\in\mathbb{N}},\ n_\iota+N\big)$$

is denoted by $\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}$ and called the coproduct of operator systems $\{\mathcal{S}_\iota\}_{\iota\in I}$. The following is an immediate generalization of the results on the coproduct of two operator systems studied in [Fritz 2014] to any family of operator systems.

Proposition 3.1. *Suppose that $\{\mathcal{S}_\iota\}_{\iota\in I}$ (respectively $\{\mathcal{A}_\iota\}_{\iota\in I}$) is a family of operator systems (respectively unital C^*-algebras). Then:*

(i) *$\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}$ is an operator system.*

(ii) *For any subset $J\subset I$, the inclusion*

$$\bigoplus_1\{\mathcal{S}_\iota:\iota\in J\}\subset\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}$$

is completely order isomorphic.

(iii) *For unital completely positive maps $\varphi_\iota:\mathcal{S}_\iota\to\mathcal{R}$, there exists a unique unital completely positive map $\Phi:\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}\to\mathcal{R}$ which extends all φ_ι, i.e., such that the diagram*

$$\begin{tikzcd} \mathcal{S}_\iota \arrow[d, hook] \arrow[dr, "\varphi_\iota"] & \\ \bigoplus_1\{\mathcal{S}_\iota:\iota\in I\} \arrow[r, dashed, "\Phi"] & \mathcal{R} \end{tikzcd}$$

commutes.

(iv) *We have $\sum_{\iota\in F}x_\iota\in M_n\big(\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}\big)^+$ if and only if there exist $\alpha_\iota\in M_n$ for $\iota\in F$ such that*

$$\sum_{\iota\in F}\alpha_\iota=0 \quad \textit{and} \quad x_\iota+\alpha_\iota\otimes 1_{\mathcal{S}_\iota}\in M_n(\mathcal{S}_\iota)^+.$$

(v) *The coproduct $\bigoplus_1\{\mathcal{A}_\iota:\iota\in I\}$ is an operator subsystem of the unital C^*-algebra free product $*_{\iota\in I}\mathcal{A}_\iota$.*

The following is an immediate generalization of [Kavruk 2012, Proposition 4.7] to any finite family of operator systems.

Proposition 3.2. *Suppose that $\mathcal{S}_1, \dots, \mathcal{S}_n$ are operator systems and*

$$N = \operatorname{span}\{n_i - n_j \in \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n : 1 \le i, j \le n\} \quad (n_i(j) = \delta_{i,j} 1_{\mathcal{S}_i}).$$

Then, N is a kernel in $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n$ and the map

$$\sum_{i=1}^{n} x_i \in \mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_n \mapsto n(x_i) + N \in \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n / N$$

is a unital complete order isomorphism.

Suppose that I is an index set and $\{I_k\}_{k \in \mathbb{N}}$ is a sequence of index sets having the same cardinality as I. Let $M_k(C([0, 1]))_{\iota_k}$ denote the copy of $M_k(C([0, 1]))$ for each index $\iota_k \in I_k$. We denote the copy of $1 \in C([0, 1])$ (respectively $t \in C([0, 1])$) in $M_k(C([0, 1])_{\iota_k}$ by 1_{ι_k} (respectively t_{ι_k}). For each $\iota_k \in I_k$, we let $\mathfrak{C}_{\iota_k}$ be an operator subsystem of $M_k(C([0, 1])_{\iota_k}$ generated by

$$\{e_{ij} \otimes 1_{\iota_k} : 1 \le i, j \le k\} \quad \text{and} \quad \{e_{ij} \otimes t_{\iota_k} : 1 \le i, j \le k\}.$$

We define the operator system $\mathfrak{C}_I$ as the coproduct

$$\bigoplus\nolimits_1 \{\mathfrak{C}_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

The operator system $\mathfrak{C}_I$ depends only on the cardinality of the index set I.

Proposition 3.3. *The operator system $\mathfrak{C}_I$ is unitally completely order isomorphic to the coproduct*

$$\bigoplus\nolimits_1 \{(M_k \oplus M_k)_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

Proof. It is sufficient to show that each $\mathfrak{C}_{\iota_k}$ is unitally completely order isomorphic to the direct sum $M_k \oplus M_k$. For $\alpha, \beta \in M_{nk}$, we have

$\alpha \otimes 1_{\iota_k} + \beta \otimes t_{\iota_k}$ is positive in $M_n(\mathfrak{C}_{\iota_k})$
$\Leftrightarrow \alpha \otimes 1 + \beta \otimes t$ is positive in $M_n(M_k(C([0, 1])))$
$\Leftrightarrow \forall t \in [0, 1],\ f(t) = \alpha + t\beta \in M_{nk}^{+}$ (since $M_n(M_k(C([0, 1]))) \simeq C([0, 1], M_{nk})$)
$\Leftrightarrow \alpha, \alpha + \beta \in M_{nk}^{+}$ (because f is affine).

Hence, the mapping

$$\alpha \otimes 1_{\iota_k} + \beta \otimes t_{\iota_k} \in \mathfrak{C}_{\iota_k} \mapsto (\alpha, \alpha + \beta) \in M_k \oplus M_k, \quad \alpha, \beta \in M_k$$

is a unital complete order isomorphism. □

A C^*-cover $(\mathcal{A}, \iota)$ of an operator system $\mathcal{S}$ is a unital C^*-algebra $\mathcal{A}$ with a unital complete order embedding $\iota : \mathcal{S} \hookrightarrow \mathcal{A}$ such that $\iota(\mathcal{S})$ generates $\mathcal{A}$ as a C^*-algebra. The enveloping C^*-algebra $C_e^*(\mathcal{S})$ is a C^*-cover of $\mathcal{S}$ satisfying the universal minimal property: for any C^*-cover $\iota : \mathcal{S} \hookrightarrow \mathcal{A}$, there is a unique unital $*$-homomorphism

$$\pi : \mathcal{A} \to C_e^*(\mathcal{S})$$

such that $\pi(\iota(x)) = x$ for all $x \in \mathcal{S}$ [Hamana 1979].

Let $\mathcal{S}$ be an operator subsystem of $\mathcal{T}$. We say that $\mathcal{S}$ is relatively weakly injective in $\mathcal{T}$ if

$$\mathcal{S} \otimes_{\mathrm{c}} \mathcal{R} \hookrightarrow \mathcal{T} \otimes_{\mathrm{c}} \mathcal{R}$$

for any operator system $\mathcal{R}$. The following are equivalent [Bhattacharya 2014, Theorem 4.1]:

(i) $\mathcal{S}$ is relatively weakly injective in $\mathcal{T}$.

(ii) $\mathcal{S} \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty) \hookrightarrow \mathcal{T} \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty)$.

(iii) For any unital completely positive map $\varphi : \mathcal{S} \to B(H)$, there exists a unital completely positive map $\Phi : \mathcal{T} \to \varphi(\mathcal{S})''$ such that $\Phi|_{\mathcal{S}} = \varphi$.

Theorem 3.4. *Suppose that I is an index set and $\{I_k\}_{k\in\mathbb{N}}$ is a sequence of index sets having the same cardinality as I. Then,*

(i) *the unital C^*-algebra free product*

$$*_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k}$$

is a C^*-cover of $\mathfrak{C}_I$;

(ii) *the unital C^*-algebra free product*

$$*_{k\in\mathbb{N},\iota_k\in I_k} (M_k \oplus M_k)_{\iota_k}$$

is a C^*-envelope of $\mathfrak{C}_I$;

(iii) *for a unital C^*-algebra $\mathcal{A}$, every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{A}$ has completely positive extensions*

$$\Phi : *_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k} \to \mathcal{A} \quad \textit{and} \quad \Psi : *_{k\in\mathbb{N},\iota_k\in I_k} (M_k \oplus M_k)_{\iota_k} \to \mathcal{A};$$

(iv) *$\mathfrak{C}_I$ is relatively weakly injective in both*

$$*_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k} \quad \textit{and} \quad *_{k\in\mathbb{N},\iota_k\in I_k} (M_k \oplus M_k)_{\iota_k}.$$

Proof. (i) By Proposition 3.1(iv), (v), we have

$$
\begin{aligned}
&\sum_{\iota_k\in F} x_{\iota_k} \in M_n(\mathfrak{C}_I)^+\\
&\qquad\iff \exists \alpha_{\iota_k}\in M_n,\ \sum_{\iota_k\in F}\alpha_{\iota_k}=0 \text{ and } x_{\iota_k}+\alpha_{\iota_k}\otimes 1_{\mathfrak{C}_{\iota_k}}\in M_n(\mathfrak{C}_{\iota_k})^+\\
&\qquad\iff \exists \alpha_{\iota_k}\in M_n,\ \sum_{\iota_k\in F}\alpha_{\iota_k}=0 \text{ and } x_{\iota_k}+\alpha_{\iota_k}\otimes 1_{\mathfrak{C}_{\iota_k}}\in M_n(M_k(C([0,1]))_{\iota_k})^+\\
&\qquad\iff \sum_{\iota_k\in F} x_{\iota_k}\in M_n\Big(\bigoplus\nolimits_1\{M_k(C([0,1]))_{\iota_k}: k\in\mathbb{N},\ \iota_k\in I_k\}\Big)^+\\
&\qquad\iff \sum_{\iota_k\in F} x_{\iota_k}\in M_n(*_{k\in\mathbb{N},\iota_k\in I_k}M_k(C([0,1]))_{\iota_k})^+.
\end{aligned}
$$

Hence, $\mathfrak{C}_I$ is an operator subsystem of $*_{k\in\mathbb{N},\iota_k\in I_k}M_k(C([0,1]))_{\iota_k}$. By the Weierstrass approximation theorem, each $\mathfrak{C}_{\iota_k}$ generates $M_k(C([0,1]))_{\iota_k}$ as a C^*-algebra. Hence, $*_{k\in\mathbb{N},\iota_k\in I_k}M_k(C([0,1]))_{\iota_k}$ is a C^*-cover of $\mathfrak{C}_I$.

(ii) The proof is motivated by [Farenick and Paulsen 2012, Theorem 2.6]. Suppose that

$$\mathfrak{C}_I\subset B(H) \quad\text{and}\quad *_{k\in\mathbb{N},\iota_k\in I_k}(M_k\oplus M_k)_{\iota_k}\subset B(K).$$

Let $\mathcal{A}$ be a C^*-algebra generated by $\mathfrak{C}_I$ in $B(H)$. By the Arveson extension theorem, the canonical inclusion from $\mathfrak{C}_I=\bigoplus_1\{(M_k\oplus M_k)_{\iota_k}: k\in\mathbb{N},\ \iota_k\in I_k\}$ into $*_{k\in\mathbb{N},\iota_k\in I_k}(M_k\oplus M_k)_{\iota_k}$ extends to a unital completely positive map $\rho:\mathcal{A}\to B(K)$. Then letting $\rho=V^*\pi(\cdot)V$ be a minimal Stinespring decomposition of ρ for a $*$-representation $\pi:\mathcal{A}\to B(\widehat{K})$ and an isometry $V:K\to\widehat{K}$, we have the commutative diagram

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "\pi"] \arrow[drr, "\rho"] & & B(\widehat{K}) \arrow[d, "V^*\cdot V"] \\
\mathfrak{C}_I \arrow[u, hook] \subset *_{k\in\mathbb{N},\iota_k\in I_k}(M_k\oplus M_k)_{\iota_k} & \subset & B(K)
\end{tikzcd}
$$

For a unitary matrix U in $(M_k\oplus M_k)_{\iota_k}$ (U need not be unitary in $\mathcal{A}$), we can write $\pi(U)$ in the operator matrix form

$$\pi(U)=\begin{pmatrix} U & B\\ C & D\end{pmatrix}.$$

Since U is unitary in $B(K)$ and

$$1=\|U\|\le\left\|\begin{pmatrix} U & B\\ C & D\end{pmatrix}\right\|=\|\pi(U)\|\le 1,$$

we have $B = 0 = C$ by the C^*-axiom. It follows that ρ is multiplicative on

$$\mathcal{U} := \{U \in (U(k) \oplus U(k))_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

By the spectral theorem, every matrix can be written as a linear combination of unitary matrices. It follows that the set $\mathcal{U}$ generates $\mathcal{A}$ as a C^*-algebra. We can regard ρ as a surjective $*$-homomorphism from $\mathcal{A}$ onto $*_{k\in\mathbb{N},\iota_k\in I_k}(M_k \oplus M_k)_{\iota_k}$. Hence, $*_{k\in\mathbb{N},\iota_k\in I_k}(M_k \oplus M_k)_{\iota_k}$ is the universal quotient of all C^*-algebras generated by $\mathfrak{C}_I$.

(iii) Since each $\mathfrak{C}_{\iota_k}$ is unitally completely order isomorphic to $M_k \oplus M_k$ which is injective, there exists a unital completely positive projection

$$P_{\iota_k} : M_k(C([0,1]))_{\iota_k} \to \mathfrak{C}_{\iota_k}.$$

By [Boca 1991, Theorem 3.1], the unital free products

$$*_{k\in\mathbb{N},\iota_k\in I_k}(\varphi|_{\mathfrak{C}_{\iota_k}} \circ P_{\iota_k}) : *_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k} \to \mathcal{A}$$

and

$$*_{k\in\mathbb{N},\iota_k\in I_k}\varphi|_{\mathfrak{C}_{\iota_k}} : *_{k\in\mathbb{N},\iota_k\in I_k}(M_k \oplus M_k)_{\iota_k} \to \mathcal{A}$$

are completely positive extensions of φ.

(iv) Let $\varphi : \mathfrak{C}_I \to B(H)$ be a unital completely positive map. The double commutant $\varphi(\mathcal{S})''$ of its range is a C^*-algebra. The relative weak injectivity follows from (iii) and [Bhattacharya 2014, Theorem 4.1]. □

Theorem 3.5. *Suppose that $\mathcal{S}$ is an operator system and $\mathcal{S}^+_{\|\cdot\|\le 1}$ is indexed by a set I. Then, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_I$. Furthermore, the kernel is completely order proximinal and every positive element $x \in M_k(\mathcal{S})$ can be lifted to a positive element $\tilde{x} \in M_k(\mathfrak{C}_I)$ with $\|\tilde{x}\| \le k^2\|x\|$.*

Proof. Let $\{I_k\}_{k\in\mathbb{N}}$ be a sequence of index sets with the same cardinality as I. Then each element in $M_k(\mathcal{S})^+_{\|\cdot\|\le 1}$ can be indexed by I_k. Suppose that $\mathcal{S} \subset B(H)$. Then for each index $\iota_k \in I_k$, we define a unital completely positive map $\Phi_{\iota_k} : M_k(C([0,1]))_{\iota_k} \to B(H)$ as

$$\Phi_{\iota_k}(\alpha \otimes f) = \frac{1}{k}\begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \alpha \otimes f(x_{\iota_k}) \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} = \frac{1}{k}\sum_{i,j} \alpha_{i,j} f(x_{\iota_k})_{i,j},$$

where $x_{\iota_k} \in M_k(\mathcal{S})^+_{\|\cdot\|\le 1}$ and each e_i is a column vector. Let $\varphi_{\iota_k} : \mathfrak{C}_{\iota_k} \to \mathcal{S}$ be its restriction on $\mathfrak{C}_{\iota_k}$. By Proposition 3.1(iii), there exists a unital completely positive map $\Phi : \mathfrak{C}_I \to \mathcal{S}$ which extends all φ_{ι_k} over $\iota_k \in I_k$, $k \in \mathbb{N}$. Since $\mathcal{S}^+_{\|\cdot\|\le 1}$ is contained in the range of Φ, Φ is surjective.

Choose an element $x_{\iota_k} \in M_k(\mathcal{S})^+_{\|\cdot\|=1}$. From

$$\Phi_k(k[E_{ij} \otimes t_{\iota_k}]_{i,j}) = [k\Phi(E_{ij} \otimes t_{\iota_k})]_{i,j} = [x_{\iota_k}(i, j)]_{i,j} = x_{\iota_k}$$

and

$$k[E_{i,j} \otimes t_{\iota_k}]_{i,j} = k[E_{ij}]_{i,j} \otimes t_{\iota_k} = k \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \begin{pmatrix} e_1^t & \cdots & e_n^t \end{pmatrix} \otimes t_{\iota_k} \in M_{k^2}(C([0, 1]))^+,$$

we see that $\Phi : \mathfrak{C}_I \to \mathcal{S}$ is a complete order quotient map whose kernel is completely order proximinal. Moreover, we have

$$\begin{aligned} \|k[E_{i,j} \otimes t_{\iota_k}]_{i,j}\|_{M_k(C([0,1]))} &= \|k[E_{i,j}]_{i,j}\| \\ &= k \left\| \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} \begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \right\| \\ &= k \left\| \begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} \right\| = k^2. \end{aligned}$$ □

We define the operator system $\mathfrak{C}_1$ as the coproduct

$$\bigoplus\nolimits_1 \{M_k \oplus M_k : k \in \mathbb{N}\}.$$

Note that $\mathfrak{C}_1 = \mathfrak{C}_I$ when $|I| = 1$.

Theorem 3.6. *Suppose that an operator system $\mathcal{S}$ is a countable union of its finite-dimensional subsystems. Then, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1$.*

Proof. First, we show that every finite-dimensional operator system is an operator system quotient of $\mathfrak{C}_{\mathbb{N}}$. Let E be a finite-dimensional operator system. We index a countable dense subset D_k of $M_k(E)^+_{\|\cdot\|\le 1}$ by $\mathbb{N}$. Define a unital completely positive map $\Phi : \mathfrak{C}_{\mathbb{N}} \to E$ as in Theorem 3.5. Since the range of Φ is a dense subspace of a finite-dimensional space E, Φ is surjective.

Choose $\varepsilon > 0$ and an element x in $M_k(E)^+_{\|\cdot\|\le 1}$. Since E is finite-dimensional, the inverse of $\widetilde{\Phi} : \mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi \to E$ is completely bounded. Let

$$\|\widetilde{\Phi}^{-1} : E \to \mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi\|_{cb} \le M.$$

Take $y \in D_k$ so that $\|x - y\| \le \varepsilon/(2M)$. Since $\|\widetilde{\Phi}_k^{-1}(x - y)\| \le \varepsilon/2$, we have

$$\widetilde{\Phi}_k^{-1}(x - y) + \frac{\varepsilon}{2} I_k \otimes 1_{\mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi} \in M_k(\mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi)^+.$$

There exists a positive element z in $M_k(\mathfrak{C}_{\mathbb{N}})$ satisfying

$$z + \operatorname{Ker}\Phi_k = \widetilde{\Phi}_k^{-1}(x - y) + \varepsilon I_k \otimes 1_{\mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi},$$

which implies

$$\Phi_k(z) = \widetilde{\Phi}_k(z + \operatorname{Ker}\Phi) = x - y + \varepsilon I_k \otimes 1_E.$$

As in the proof of Theorem 3.5, we can take a positive element $\tilde{y}$ in $M_k(\mathfrak{C}_{\mathbb{N}})$ such that $\Phi_k(\tilde{y}) = y$. It follows that

$$\Phi_k(z + \tilde{y}) = (x - y + \varepsilon I_k \otimes 1_E) + y = x + \varepsilon I_k \otimes 1_E.$$

Hence, E is an operator system quotient of $\mathfrak{C}_{\mathbb{N}}$.

Next, we show that $\mathfrak{C}_{\mathbb{N}}$ is an operator system quotient of $\mathfrak{C}_1$. We enumerate the coproduct summands of $\mathfrak{C}_{\mathbb{N}}$ as

$$(M_1 \oplus M_1)_1,\ (M_1 \oplus M_1)_2,\ (M_2 \oplus M_2)_1,\ (M_1 \oplus M_1)_3,\ (M_2 \oplus M_2)_2,\ (M_3 \oplus M_3)_1, \ldots$$

and denote them by $M_{a_k} \oplus M_{a_k}$. Since $k \geq a_k$, the identity map on M_{a_k} is factorized as $Q_k \circ J_k$ for unital completely positive maps

$$J_k : A \in M_{a_k} \mapsto A \oplus \omega(A) I_{k-a_k} \in M_k \quad (\omega : \text{a state on } M_{a_k})$$

and

$$Q_k : A \in M_k \mapsto [A_{i,j}]_{1 \leq i,j \leq a_k} \in M_{a_k}.$$

By the universal property of the coproduct, there exists a unital completely positive map $J : \mathfrak{C}_{\mathbb{N}} \to \mathfrak{C}_1$ (respectively $Q : \mathfrak{C}_1 \to \mathfrak{C}_{\mathbb{N}}$) which extends all $J_k \oplus J_k : M_{a_k} \oplus M_{a_k} \to M_k \oplus M_k$ (respectively $Q_k \oplus Q_k : M_k \oplus M_k \to M_{a_k} \oplus M_{a_k}$). Then, the identity map on $\mathfrak{C}_{\mathbb{N}}$ is factorized as $Q \circ J$. Hence, $\mathfrak{C}_{\mathbb{N}}$ is an operator system quotient of $\mathfrak{C}_1$.

Suppose that $\mathcal{S} = \bigcup_{k=1}^{\infty} E_k$ for finite dimensional subsystems E_k of $\mathcal{S}$. We can find complete order quotient maps $\Psi_k : \mathfrak{C}_1 \to E_k$. By the universal property of the coproduct, there exists a unital completely positive map $\Psi : \mathfrak{C}_{\mathbb{N}} \to \mathcal{S}$ which extends all Ψ_k. It is easy to check that Ψ is a complete order quotient map. Since $\mathfrak{C}_{\mathbb{N}}$ is an operator system quotient of $\mathfrak{C}_1$, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1$. □

Theorem 3.7. *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{I}$ is a closed ideal in it. Every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{A}$, i.e., such that the diagram*

$$\begin{tikzcd} & \mathcal{A} \arrow[d, two heads] \\ \mathfrak{C}_I \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & \mathcal{A}/\mathcal{I} \end{tikzcd}$$

commutes.

Proof. Let z_{ι_k} be the direct sum of two Choi matrices associated to the restrictions of $\varphi|_{(M_k \oplus M_k)_{\iota_k}}$ on each two blocks M_k, that is,

$$z_{\iota_k} = [\varphi|_{(M_k \oplus M_k)_{\iota_k}}(E_{i,j} \oplus 0_k)]_{i,j} \oplus [\varphi|_{(M_k \oplus M_k)_{\iota_k}}(0_k \oplus E_{i,j})]_{i,j}.$$

Then z_{ι_k} belongs to the positive cone of $M_k(\mathcal{A}/I) \oplus M_k(\mathcal{A}/I)$. Then let $\tilde{z}_{\iota_k} \in M_k(\mathcal{A}) \oplus M_k(\mathcal{A})$ be a positive lifting z_{ι_k}. Its corresponding mapping

$$\tilde{\varphi}_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \to \mathcal{A}$$

is a completely positive lifting of $\varphi|_{(M_k \oplus M_k)_{\iota_k}}$. We let

$$\tilde{\varphi}_{\iota_k}(I_{2k}) = 1 + h, \quad h = h^+ - h^- \quad (h \in \mathcal{I}, \quad h^+, h^- \in \mathcal{I}^+)$$

and take a state ω on $(M_k \oplus M_k)_{\iota_k}$. Considering

$$\alpha \in (M_k \oplus M_k)_{\iota_k} \mapsto (1+h^+)^{-\frac{1}{2}}(\tilde{\varphi}_{\iota_k}(\alpha) + \omega(\alpha)h^-)(1+h^+)^{-\frac{1}{2}} \in \mathcal{A}$$

as in [KPTT2, Remark 8.3], we may assume that the lifting $\tilde{\varphi}_{\iota_k}$ is unital. By the universal property of the coproduct, there exists a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{S}$ that extends all $\tilde{\varphi}_{\iota_k}$. □

The universal C^*-algebra $C^*_u(\mathcal{S})$ is the C^*-cover of $\mathcal{S}$ satisfying the universal property: if $\varphi : \mathcal{S} \to \mathcal{A}$ is a unital completely positive map for a unital C^*-algebra $\mathcal{A}$, then there exists a $*$-homomorphism $\pi : C^*_u(\mathcal{S}) \to \mathcal{A}$ such that $\pi \circ \iota = \varphi$ [Kirchberg and Wassermann 1998]. For a unital completely positive map $\varphi : \mathcal{S} \to \mathcal{T}$ and a complete order embedding $\iota : \mathcal{T} \to C^*_u(\mathcal{T})$, we denote the unique $*$-homomorphic extension of $\iota \circ \varphi : \mathcal{S} \to C^*_u(\mathcal{T})$ by $C^*_u(\varphi)$. We can regard $C^*_u(\,\cdot\,)$ as a functor from the category of operator systems to the category of C^*-algebras.

Corollary 3.8. *Let $\mathcal{S}$ be an operator system and $Q : \mathfrak{C}_I \to \mathcal{S}$ be a complete order quotient map. The following are equivalent*:

(i) *$\mathcal{S}$ has the operator system lifting property*;

(ii) *$C^*_u(Q) : C^*_u(\mathfrak{C}_I) \to C^*_u(\mathcal{S})$ has a unital $*$-homomorphic right inverse*;

(iii) *$C^*_u(Q) : C^*_u(\mathfrak{C}_I) \to C^*_u(\mathcal{S})$ has a unital completely positive right inverse.*

Proof. (i) ⇒ (ii). The inclusion $\iota : \mathcal{S} \subset C^*_u(\mathcal{S})$ lifts to a unital completely positive map $\tilde{\iota} : \mathcal{S} \to C^*_u(\mathfrak{C}_I)$. Its $*$-homomorphic extension $\rho : C^*_u(\mathcal{S}) \to C^*_u(\mathfrak{C}_I)$ is the right inverse of $C^*_u(Q) : C^*_u(\mathfrak{C}_I) \to C^*_u(\mathcal{S})$.

(ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). Suppose that $\varphi : \mathcal{S} \to \mathcal{A}/\mathcal{I}$ is a unital completely positive map for a unital C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$. By Theorem 3.7, $\varphi \circ Q : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\psi : \mathfrak{C}_I \to \mathcal{A}$. Let $\rho : C^*_u(\mathfrak{C}_I) \to \mathcal{A}$ (respectively $\sigma : C^*_u(\mathcal{S}) \to \mathcal{A}/\mathcal{I}$) be a unique $*$-homomorphic extension of ψ (respectively φ). Suppose that r is a unital completely positive right inverse of $C^*_u(Q)$. We thus have the diagram

$$\begin{tikzcd}
C_u^*(\mathfrak{C}_I) \arrow[rr, "\rho"] \arrow[dr, shift left, "C_u^*(Q)"] & & \mathcal{A} \arrow[dd, two heads, "\pi"] \\
& C_u^*(\mathcal{S}) \arrow[ul, shift left, "r"] \arrow[dr, "\sigma"] & \\
\mathfrak{C}_I \arrow[uu, hook] \arrow[r, two heads, "Q"] & \mathcal{S} \arrow[u, hook, "\iota"] \arrow[r, "\varphi"] & \mathcal{A}/\mathcal{I}
\end{tikzcd}$$

Let us show that

$$\tilde{\varphi} := \rho \circ r \circ \iota : \mathcal{S} \to \mathcal{A}$$

is a lifting of φ. Since $\mathfrak{C}_I$ generates $C_u^*(\mathfrak{C}_I)$ as a C^*-algebra, $\pi \circ \psi = \varphi \circ Q$ implies that

$$\pi \circ \rho = \sigma \circ C_u^*(Q).$$

For $x \in \mathcal{S}$, we have

$$\pi \circ \tilde{\varphi}(x) = \pi \circ \rho \circ r(x) = \sigma \circ C_u^*(Q) \circ r(x) = \varphi(x). \qquad \square$$

4. A Kirchberg-type tensor theorem for operator systems

For a free group $\mathbb{F}$ and a Hilbert space H, Kirchberg [1994, Corollary 1.2] proved that

$$C^*(\mathbb{F}) \otimes_{\min} B(H) = C^*(\mathbb{F}) \otimes_{\max} B(H).$$

Kirchberg's theorem is striking if we recall that $C^*(\mathbb{F})$ and $B(H)$ are universal objects in the C^*-algebra category: every C^*-algebra is a C^*-quotient of $C^*(\mathbb{F})$ and a C^*-subalgebra of $B(H)$ for suitable choices of $\mathbb{F}$ and H. Every operator system is a quotient of $\mathfrak{C}_I$ and a subsystem of $B(H)$ for suitable choices of I and H. Hence we may say that

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H),$$

the proof of which will follow, is the Kirchberg-type theorem in the category of operator systems.

If $\mathcal{S}$ has the operator system local lifting property, $\mathcal{S} \otimes_{\min} B(H) = \mathcal{S} \otimes_{\max} B(H)$ [KPTT2, Theorem 8.6]. From this, Theorem 3.7 immediately yields that $\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$. The proof of [KPTT2, Theorem 8.6] depends on Kirchberg's theorem. We give a direct proof of $\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$ that is independent of Kirchberg's theorem. By combining this with [Kavruk 2012], we present a new operator system theoretic proof of Kirchberg's theorem in Corollary 4.4.

Theorem 4.1. *For an index set I and a Hilbert space H, we have*

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H).$$

Proof. Let z be a positive element in $\mathfrak{C}_I \otimes_{\min} B(H)$. We write $z = \sum_{\iota_k \in F} z_{\iota_k}$ for a finite subset F of $\bigcup_{k=1}^{\infty} I_k$ and $z_{\iota_k} \in \mathfrak{C}_{\iota_k} \otimes B(H)$. By Proposition 3.1(ii) and the injectivity of the minimal tensor product, we can regard z as a positive element in

$$\textstyle\bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H).$$

We apply Proposition 3.2 to $\bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\}$ to obtain the complete order isomorphism

$$\Phi : \sum_{\iota_k \in F} x_{\iota_k} \in \textstyle\bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \mapsto |F|(x_{\iota_k})_{\iota_k \in F} + N \in \displaystyle\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N,$$

where $|F|$ denotes the number of elements of the set F and

$$N = \operatorname{span}\Big\{ n_{\iota_l} - n_{\iota'_m} \in \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \iota_l, \iota'_m \in F \Big\} \quad (n_{\iota_k}(\iota'_j) = \delta_{\iota_k, \iota'_j} I_k \oplus I_k).$$

Let

$$Q : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \twoheadrightarrow \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N$$

be the canonical quotient map. By [Farenick and Paulsen 2012, Proposition 1.15], its dual map

$$Q^* : \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N \Big)^* \hookrightarrow \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \Big)^*$$

is a complete order embedding. The range of Q^* is the annihilator

$$N^{\perp} = \Big\{ \varphi \in \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \Big)^* : N \subset \operatorname{Ker} \varphi \Big\}.$$

The linear map $\gamma_k : M_k \to M_k^*$ defined as

$$\gamma_k(\alpha)(\beta) = \sum_{i,j=1}^{k} \alpha_{i,j} \beta_{i,j} = \operatorname{tr}(\alpha \beta^t)$$

is a complete order isomorphism [Paulsen et al. 2011, Theorem 6.2]. Define a complete order isomorphism

$$\Gamma : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \to \bigoplus_{\iota_k \in F} (M_k^* \oplus M_k^*)_{\iota_k} \simeq \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \Big)^*$$

by

$$\langle \Gamma((\alpha_{\iota_k})), (\beta_{\iota_k}) \rangle = \Big\langle \Big((\gamma_k \oplus \gamma_k) \Big(\frac{\alpha_{\iota_k}}{2k} \Big) \Big), (\beta_{\iota_k}) \Big\rangle = \sum_{\iota_k \in F} \frac{1}{2k} \operatorname{tr}(\alpha_{\iota_k} \beta_{\iota_k}^t).$$

Then, Γ^{-1} maps the annihilator $N^\perp$ onto the operator subsystem

$$K = \left\{(\alpha_{\iota_k}) \in \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \frac{\operatorname{tr}(\alpha_{\iota_l})}{l} = \frac{\operatorname{tr}(\alpha_{\iota'_m})}{m} \text{ for all } \iota_l, \iota'_m \in F\right\}$$

of $\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}$. We have obtained complete order isomorphisms

$$\left(\bigoplus\nolimits_1 \left\{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\right\}\right)^* \simeq \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N\Big)^* \simeq N^\perp \simeq K.$$

Considering the duals of the above isomorphisms, we obtain a complete order isomorphism

$$\Lambda : \bigoplus\nolimits_1 \left\{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\right\} \to K^*,$$

which maps each $\sum_{\iota_k \in F} \beta_{\iota_k} \in \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\}$ to a functional

$$(\alpha_{\iota_k}) \in K \mapsto \sum_{\iota_k \in F} \frac{|F|}{2k} \operatorname{tr}(\beta_{\iota_k} \alpha^t_{\iota_k}) \in \mathbb{C}.$$

In particular, Λ maps the order unit to the state ω on K defined as

$$\omega((\alpha_{\iota_k})) = \sum_{\iota_k \in F} \frac{1}{2k} \operatorname{tr}(\alpha_{\iota_k}).$$

It enables us to make the identification

$$\bigoplus\nolimits_1 \left\{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\right\} \otimes_{\min} B(H) \simeq K^* \otimes_{\min} B(H),$$

where K^* is an operator system with an order unit ω. The linear map $\varphi : K \to B(H)$ corresponding to z in a canonical way is completely positive [KPTT2, Lemma 8.5]. By the Arveson extension theorem, $\varphi : K \to B(H)$ extends to a completely positive map $\tilde{\varphi} : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \to B(H)$. We have the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} & \xrightarrow{\Phi^{-1} \circ Q} & \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \\ \Gamma \big\downarrow & & \big\downarrow \Lambda \\ \left(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}\right)^* & \xrightarrow{\quad R \quad} & K^* \end{array}$$

where R denotes the restriction. It follows that

$$(\Phi^{-1} \circ Q) \otimes \mathrm{id} : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \otimes_{\min} B(H) \to \bigoplus\nolimits_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H)$$

is a complete order quotient map. Maximal tensor products of complete order quotient maps are still complete order quotient maps [Han 2011, Theorem 3.4].

Hence, we obtain

$$\begin{tikzcd} \bigoplus_{\iota_k \in F}(M_k \oplus M_k)_{\iota_k} \otimes_{\min} B(H) \arrow[r, equal] \arrow[d, two heads, "(\Phi^{-1}\circ Q)\otimes \mathrm{id}"'] & \bigoplus_{\iota_k \in F}(M_k \oplus M_k)_{\iota_k} \otimes_{\max} B(H) \arrow[d, two heads, "(\Phi^{-1}\circ Q)\otimes \mathrm{id}"] \\ \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H) & \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\max} B(H) \end{tikzcd}$$

The element z is also positive in $\mathfrak{C}_I \otimes_{\max} B(H)$. The same arguments apply to all matricial levels. □

The maximal tensor product and the commuting tensor product are two different means of extending the C^*-maximal tensor product from the category of C^*-algebras to operator systems. For this reason, the weak expectation property of C^*-algebras bifurcates into the weak expectation property and the double commutant expectation property of operator systems. We say that an operator system $\mathcal{S}$ has the *double commutant expectation property* provided that for every completely order isomorphic inclusion $\mathcal{S} \subset B(H)$, there exists a completely positive map $\varphi : B(H) \to \mathcal{S}''$ that fixes $\mathcal{S}$. For an operator system $\mathcal{S}$, the following are equivalent [KPTT2, Theorem 7.6; Kavruk 2012, Theorem 5.9]:

(i) $\mathcal{S}$ has the double commutant expectation property.

(ii) $\mathcal{S}$ is (el, c)-nuclear.

(iii) $\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_\infty)$.

(iv) $\mathcal{S} \otimes_{\min} (\ell^2_\infty \oplus_1 \ell^3_\infty) = \mathcal{S} \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^3_\infty)$.

Theorem 4.2. *An operator system $\mathcal{S}$ has the double commutant expectation property if and only if it satisfies*

$$\mathcal{S} \otimes_{\min} \mathfrak{C}_I = \mathcal{S} \otimes_{\mathrm{c}} \mathfrak{C}_I .$$

Proof. $\Rightarrow$) Every operator system with the double commutant expectation property is (el, c)-nuclear. Since the minimal tensor product is injective [KPTT1, Theorem 4.6], we have

$$\begin{tikzcd} B(H) \otimes_{\min} \mathfrak{C}_I \arrow[r, equal] & B(H) \otimes_{\max} \mathfrak{C}_I \\ \mathcal{S} \otimes_{\min} \mathfrak{C}_I \arrow[u, hook] & \mathcal{S} \otimes_{\mathrm{el=c}} \mathfrak{C}_I . \arrow[u, hook] \end{tikzcd}$$

$\Leftarrow$) Fix two indices $\iota_2' \in I_2$ and $\iota_3' \in I_3$. Define a unital completely positive map $\Phi : \ell^2_\infty \oplus_1 \ell^3_\infty \to \mathfrak{C}_I$ by

$$\begin{aligned} \Phi\big((a_1, a_2) + (b_1, b_2, b_3)\big) &= \mathrm{diag}(a_1, a_2, a_1, a_2) + \mathrm{diag}(b_1, b_2, b_3, b_1, b_2, b_3) \\ &\in (M_2 \oplus M_2)_{\iota_2'} \oplus_1 (M_3 \oplus M_3)_{\iota_3'} \subset \mathfrak{C}_I . \end{aligned}$$

For each index $\iota_k \neq \iota_2', \iota_3'$, we take a state ω_{ι_k} on $(M_k \oplus M_k)_{\iota_k}$ and define a unital completely positive map

$$\psi_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \to \ell^2_\infty \oplus_1 \ell^3_\infty$$

as $\psi_{\iota_k}(\alpha) = \omega_{\iota_k}(\alpha) 1_{\ell^2_\infty \oplus_1 \ell^3_\infty}$. For ι_2' and ι_3' we also define unital completely positive maps

$$\psi_{\iota_2'} : (M_2 \oplus M_2)_{\iota_2'} \to \ell^2_\infty \oplus_1 \ell^3_\infty \quad \text{and} \quad \psi_{\iota_3'} : (M_3 \oplus M_3)_{\iota_3'} \to \ell^2_\infty \oplus_1 \ell^3_\infty$$

by $\psi_{\iota_2'}(\alpha \oplus \beta) = (\alpha_{11}, \alpha_{22})$ and $\psi_{\iota_3'}(\alpha \oplus \beta) = (\alpha_{11}, \alpha_{22}, \alpha_{33})$. By the universal property of the coproduct, there exists a unital completely positive map $\Psi : \mathfrak{C}_I \to \ell^2_\infty \oplus_1 \ell^3_\infty$ that extends all ψ_{ι_k}. The identity map on $\ell^2_\infty \oplus_1 \ell^3_\infty$ is factorized through unital completely positive maps as

$$\mathrm{id}_{\ell^2_\infty \oplus_1 \ell^3_\infty} = \Psi \circ \Phi.$$

By the hypothesis, we have completely positive maps

$$\mathcal{S} \otimes_{\min} (\ell^2_\infty \oplus_1 \ell^3_\infty) \xrightarrow{\mathrm{id}_{\mathcal{S}} \otimes \Phi} \mathcal{S} \otimes_{\min} \mathfrak{C}_I = \mathcal{S} \otimes_c \mathfrak{C}_I \xrightarrow{\mathrm{id}_{\mathcal{S}} \otimes \Psi} \mathcal{S} \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^3_\infty).$$

Since the positive cone of the commuting tensor product is the subcone of that of the minimal tensor product at each matrix level, we have

$$\mathcal{S} \otimes_{\min} (\ell^2_\infty \oplus_1 \ell^3_\infty) = \mathcal{S} \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^3_\infty).$$

By [Kavruk 2012, Theorem 5.9], $\mathcal{S}$ has the double commutant expectation property. □

Since the maximal tensor product and the commuting tensor product are two different means of extending the C^*-maximal tensor product from the category of C^*-algebras to operator systems, we can regard

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I \quad \text{and} \quad \mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$$

as operator system analogues of Kirchberg's conjecture

$$C^*(\mathbb{F}) \otimes_{\min} C^*(\mathbb{F}) = C^*(\mathbb{F}) \otimes_{\max} C^*(\mathbb{F}).$$

The former is not true and the latter is equivalent to Kirchberg's conjecture itself.

Corollary 4.3. (i) $\mathfrak{C}_I \otimes_c \mathfrak{C}_I \neq \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$. *In particular*, $\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I \neq \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$.

(ii) *The Kirchberg's conjecture has an affirmative answer if and only if*

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I.$$

Proof. (i) Similarly to the proof of Theorem 4.2, we can show that the identity map on $\ell^2_\infty \oplus_1 \ell^2_\infty$ is factorized as

$$\mathrm{id}_{\ell^2_\infty \oplus_1 \ell^2_\infty} = \Psi \circ \Phi$$

for unital completely positive maps $\Phi : \ell^2_\infty \oplus_1 \ell^2_\infty \to \mathfrak{C}_I$ and $\Psi : \mathfrak{C}_I \to \ell^2_\infty \oplus_1 \ell^2_\infty$. Assume to the contrary that $\mathfrak{C}_I \otimes_{\mathrm{c}} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$. Then, we have completely positive maps

$$\begin{aligned}(\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_c (\ell^2_\infty \oplus_1 \ell^2_\infty) &\xrightarrow{\Phi\otimes\Phi} \mathfrak{C}_I \otimes_{\mathrm{c}} \mathfrak{C}_I \\ &= \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I \xrightarrow{\Psi\otimes\Psi} (\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_{\max} (\ell^2_\infty \oplus_1 \ell^2_\infty).\end{aligned}$$

Since the positive cone of the maximal tensor product at each matrix level is the subcone of that of the commuting tensor product, we have

$$(\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^2_\infty) = (\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_{\max} (\ell^2_\infty \oplus_1 \ell^2_\infty).$$

This contradicts

$$\mathrm{NC}(2) \otimes_{\mathrm{c}} \mathrm{NC}(2) \neq \mathrm{NC}(2) \otimes_{\max} \mathrm{NC}(2),$$

which was shown in [Farenick et al. 2014, Corollary 7.12]. Here, $\mathrm{NC}(n)$ is defined as the operator subsystem $\mathrm{span}\{1, h_1, \ldots, h_n\}$ of the universal C^*-algebra generated by self-adjoint contractions $h_1, \ldots, h_n$ as in Definition 6.1 of the same paper. It is unitally completely order isomorphic to the coproduct (involving n terms)

$$\ell^2_\infty \oplus_1 \cdots \oplus_1 \ell^2_\infty.$$

(ii) By [Kavruk 2012, Theorem 5.14], Kirchberg's conjecture has an affirmative answer if and only if $\ell^2_\infty \oplus_1 \ell^3_\infty$ has the double commutant expectation property. By Theorem 4.2 this is equivalent to $(\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_{\min} \mathfrak{C}_I = (\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_c \mathfrak{C}_I$. By [Kavruk 2012, Theorem 5.9] this is equivalent to $\mathfrak{C}_I$ having the double commutant expectation property, and another application of Theorem 4.2 gives the equivalence with $\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$. □

We say that an operator subsystem $\mathcal{S}$ of a unital C^*-algebra $\mathcal{A}$ *contains enough unitaries* if the unitaries in $\mathcal{S}$ generate $\mathcal{A}$ as a C^*-algebra. If $\mathcal{S} \subset \mathcal{A}$ contains enough unitaries and $\mathcal{S} \otimes_{\min} \mathcal{B} \hookrightarrow \mathcal{A} \otimes_{\max} \mathcal{B}$ completely order isomorphically for a unital C^*-algebra $\mathcal{B}$, then we have $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ [KPTT2, Proposition 9.5].

Corollary 4.4 (Kirchberg). *Let $\mathbb{F}_\infty$ be a free group on a countably infinite number of generators and H be a Hilbert space. We have*

$$C^*(\mathbb{F}_\infty) \otimes_{\min} B(H) = C^*(\mathbb{F}_\infty) \otimes_{\max} B(H).$$

Proof. Since the identity map on $\ell_\infty^2 \oplus_1 \ell_\infty^3$ is factorized through $\mathfrak{C}_I$ by unital completely positive maps, Theorem 4.1 immediately implies that

$$(\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\min} B(H) = (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\max} B(H).$$

Alternatively, applying the proof of Theorem 4.1 to commutative algebras instead of matrix algebras, we obtain

$$\begin{tikzcd} \ell_\infty^5 \otimes_{\min} B(H) \arrow[r, equal] \arrow[d, two heads] & \ell_\infty^5 \otimes_{\max} B(H) \arrow[d, two heads] \\ (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\min} B(H) & (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\max} B(H) \end{tikzcd}$$

For the remaining proof, we follow [Kavruk 2012]. Since

$$\ell_\infty^2 \oplus_1 \ell_\infty^3 \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$$

contains enough unitaries [Kavruk 2012, Theorem 4.8], we have

$$C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\min} B(H) = C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\max} B(H)$$

by [KPTT2, Proposition 9.5]. The free group $\mathbb{F}_\infty$ embeds into the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ [de la Harpe 2000]. By [Pisier 2003, Proposition 8.8], $C^*(\mathbb{F}_\infty)$ is a C^*-subalgebra of $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ complemented by a unital completely positive map. □

A wide class of operator systems shares the properties of $\mathfrak{C}_I$. Let $\mathcal{M} = \{\mathcal{M}_k\}_{k\in\mathbb{N}}$ be a sequence of direct sums of matrix algebras such that

$$\limsup_{k\to\infty} s(k) = \infty$$

when $\mathcal{M}_k = M_{d_1} \oplus \cdots \oplus M_{d_n}$ and $s(k) = \max\{d_1, \ldots, d_n\}$. Let $\mathcal{M}_{\iota_k}$ denote the copy of $\mathcal{M}_k$ for each index $\iota_k \in I_k$. We define the operator system $\mathfrak{C}_I(\mathcal{M})$ (respectively $\mathfrak{C}_1(\mathcal{M})$) as the coproduct

$$\bigoplus\nolimits_1 \{\mathcal{M}_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\} \quad (\text{respectively } \bigoplus\nolimits_1 \{\mathcal{M}_k : k \in \mathbb{N}\}).$$

In particular, we have $\mathfrak{C}_I = \mathfrak{C}_I(\mathcal{M})$ and $\mathfrak{C}_1 = \mathfrak{C}_1(\mathcal{M})$ when $\mathcal{M}_k = M_k \oplus M_k$.

Theorem 4.5. *Suppose that $\mathcal{S}$ is an operator system.*

(i) *If $\mathcal{S}^+_{\|\cdot\|\le 1}$ is indexed by a set I, then $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_I(\mathcal{M})$.*

(ii) *If $\mathcal{S}$ is a countable union of its finite dimensional subsystems, then $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1(\mathcal{M})$.*

(iii) *$\mathfrak{C}_I(\mathcal{M})$ satisfies the operator system lifting property.*

(iv) $\mathfrak{C}_I(\mathcal{M}) \otimes_{\min} B(H) = \mathfrak{C}_I(\mathcal{M}) \otimes_{\max} B(H)$.

(v) *$\mathcal{S}$ has the double commutant expectation property if and only if* $\mathcal{S} \otimes_{\min} \mathfrak{C}_I(\mathcal{M}) = \mathcal{S} \otimes_{\mathrm{c}} \mathfrak{C}_I(\mathcal{M})$.

(vi) *Kirchberg's conjecture has an affirmative answer if and only if* $\mathfrak{C}_I(\mathcal{M}) \otimes_{\min} \mathfrak{C}_I(\mathcal{M}) = \mathfrak{C}_I(\mathcal{M}) \otimes_c \mathfrak{C}_I(\mathcal{M})$.

Proof. (i) We take a subsequence $\{\mathcal{M}_{n_k}\}_{k\in\mathbb{N}}$ so that $s(n_k) \geq 2k$ and put $\mathcal{N}_k = \mathcal{M}_{n_k}$. By Proposition 3.1(ii), $\mathfrak{C}_I(\mathcal{N})$ is an operator subsystem of $\mathfrak{C}_I(\mathcal{M})$. Take a state ω_l on $\mathcal{M}_l$ for each $l \neq n_k$. By the universal property of the coproduct, there exists a unital completely positive map $P : \mathfrak{C}_I(\mathcal{M}) \to \mathfrak{C}_I(\mathcal{N})$ such that

$$P(x) = \begin{cases} x & \text{if } x \in \mathcal{M}_{n_k} \\ \omega_l(x)1 & \text{if } x \in \mathcal{M}_l, l \neq n_k. \end{cases}$$

Since P is a unital completely positive projection, $\mathfrak{C}_I(\mathcal{N})$ is an operator system quotient of $\mathfrak{C}_I(\mathcal{M})$.

We may assume that $s(k) \geq 2k$ for each $k \in \mathbb{N}$. We write $\mathcal{M}_k = M_{d_1} \oplus \cdots \oplus M_{d_m}$ and $d_l \geq 2k$. The identity map on $M_k \oplus M_k$ is factorized as $\mathrm{id}_{M_k \oplus M_k} = Q_k \circ J_k$ for the unital completely positive maps

$$J_k : A \in M_k \oplus M_k \mapsto (\omega(A) I_{d_1+\cdots+d_{l-1}}) \oplus A \oplus (\omega(A) I_{(d_l-2k)+d_{l+1}+\cdots+d_m}) \in \mathcal{M}_k$$

(where ω is a state on $M_k \oplus M_k$) and

$$Q_k : A_1 \oplus \cdots \oplus A_m \in \mathcal{M}_k \mapsto [(A_l)_{i,j}]_{1\leq i,j\leq k} \oplus [(A_l)_{i+k,j+k}]_{1\leq i,j\leq k} \in M_k \oplus M_k.$$

Let $J : \mathfrak{C}_I \to \mathfrak{C}_I(\mathcal{M})$ (respectively $Q : \mathfrak{C}_I(\mathcal{M}) \to \mathfrak{C}_I$) be the unital completely positive extension of $J_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \to \mathcal{M}_{\iota_k}$ (respectively $Q_{\iota_k} : \mathcal{M}_{\iota_k} \to (M_k \oplus M_k)_{\iota_k}$) over $k \in \mathbb{N}$, $\iota_k \in I_k$. Then, the identity map on $\mathfrak{C}_I$ is factorized as $\mathrm{id}_{\mathfrak{C}_I} = Q \circ J$. Hence, $\mathfrak{C}_I$ is an operator system quotient of $\mathfrak{C}_I(\mathcal{M})$.

(ii) By Theorem 3.6, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1$. The remaining proof is similar to (i).

(iii), (iv) The proofs of Theorems 3.7 and 4.1 work generally for coproducts of direct sums of matrix algebras.

(v), (vi) The proofs of Theorem 4.2 and Corollary 4.3 work generally for coproducts of direct sums of matrix algebras which the identity map on $\ell_2^\infty \oplus_1 \ell_3^\infty$ factorizes through. □

5. Liftings of completely positive maps

It is natural to ask whether the universal operator system $\mathfrak{C}_I$ is a projective object in the category of operator systems. In other words, for any operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, does every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{S}/\mathcal{J}$ lift to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{S}$? The answer is negative in an extreme manner.

Proposition 5.1. *An operator system $\mathcal{S}$ is one-dimensional if and only if for any operator system $\mathcal{T}$ and its kernel $\mathcal{J}$, every unital completely positive map $\varphi : \mathcal{S} \to \mathcal{T}/\mathcal{J}$ lifts to a completely positive map $\tilde{\varphi} : \mathcal{S} \to \mathcal{T}$.*

Proof. Let V^+ be the cone in $\mathbb{R}^3$ generated by

$$\{(x, y, 1) : (x-1)^2 + y^2 \leq 1,\ y \geq 0\}$$

and the origin. The triple $V := (\mathbb{C}^3, V^+, (1, 1, 2))$ is an Archimedean ordered $*$-vector space. The positive cones of the operator system $\mathrm{OMAX}(V)$ introduced in [Paulsen et al. 2011] are given as

$$M_n(\mathrm{OMAX}(V))^+ = \{X \in M_n(V) : \forall \varepsilon > 0,\ X + \varepsilon I_n \otimes 1_V \in M_n^+ \otimes V^+\},$$

where

$$M_n^+ \otimes V^+ = \left\{ \sum_{i=1}^m \alpha_i \otimes v_i \in M_n \otimes V : m \in \mathbb{N},\ \alpha_i \in M_n^+,\ v_i \in V^+ \right\}.$$

Let

$$P : (x, y, z) \in \mathrm{OMAX}(V) \mapsto (x, y) \in \ell^2_\infty$$

be the projection. We take $\varepsilon > 0$ and an element

$$\alpha_1 \otimes (1, 0) + \alpha_2 \otimes (0, 1) \in M_n(\ell^2_\infty)^+ = M_n^+ \oplus M_n^+$$

for nonzero α_2. Since

$$\begin{aligned}\alpha_1 \otimes (1,0) &+ \alpha_2 \otimes (0,1) + \varepsilon I_n \otimes (1,1) \\ &= \left(\alpha_1 + \frac{\varepsilon}{2}\left(I_n - \frac{\alpha_2}{\|\alpha_2\|}\right)\right) \otimes (1,0) + \alpha_2 \otimes \left(\frac{\varepsilon}{2\|\alpha_2\|}, 1\right) + \varepsilon I_n \otimes \left(\tfrac{1}{2}, 1\right)\end{aligned}$$

lifts to a positive element in $M_n(\mathrm{OMAX}(V))$, the projection $P : \mathrm{OMAX}(V) \to \ell^2_\infty$ is a complete order quotient map.

Suppose that $\dim \mathcal{S} \geq 2$. Let v be a positive element in $\mathcal{S}$ distinct from the scalar multiple of the identity. Considering $v - \lambda I$ for sufficiently large $\lambda > 0$, we may assume that the spectrum of v contains zero. Let ω_1 and ω_2 be states on $\mathcal{S}$ that extend, respectively, the Dirac measures

$$\begin{aligned}\delta_{\{0\}} &: \lambda 1 + \mu v \in \mathrm{span}\{1, v\} \mapsto \lambda \in \mathbb{C}, \\ \delta_{\{\|v\|\}} &: \lambda 1 + \mu v \in \mathrm{span}\{1, v\} \mapsto \lambda + \mu\|v\| \in \mathbb{C}.\end{aligned}$$

The unital completely positive map $\varphi : \mathcal{S} \to \ell^2_\infty$ defined by $\varphi = (\omega_1, \omega_2)$ cannot be lifted to a completely positive map, because the fiber of $\varphi(v) = (0, \|v\|)$ does not intersect V^+. $\square$

The absence of completely positive liftings in the above proof is essentially due to Archimedeanization of quotients [Paulsen and Tomforde 2009]. In Corollary 5.5, we will see that there is also rigidity, even though some perturbation is allowed.

A linear map between normed spaces is called a quotient map if it maps the open unit ball onto the open unit ball. Between Banach spaces, it suffices to show that the image of the open unit ball is dense in the open unit ball in Lemma A.2.1 of [Effros and Ruan 2000]. Suppose that $T : E \to F$ is a bounded linear surjection for normed spaces E and F. Let E_0 be a dense subspace of E, and $Q_0 : E_0 \to Q(E_0)$ be the surjective restriction of Q on E_0. Then, Q_0 is a quotient map if and only if $\overline{\ker Q_0} = \ker Q$ and Q is a quotient map [Defant and Floret 1993, 7.4]. This is called the quotient lemma.

Thanks to the quotient lemma, we can describe the 1-exactness of operator systems by incomplete tensor products. For an operator system $\mathcal{S}$ and a unital C^*-algebra $\mathcal{A}$ with its closed ideal $\mathcal{I}$, we denote the completion of $\mathcal{S} \otimes_{\min} \mathcal{A}$ by $\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}$ and the closure of $\mathcal{S} \otimes \mathcal{I}$ in it by $\mathcal{S}\bar{\otimes}\mathcal{I}$. When

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S}\hat{\otimes}_{\min}\mathcal{A} \to \mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}$$

is a complete order quotient map with its kernel $\mathcal{S}\bar{\otimes}\mathcal{I}$ for any C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$, $\mathcal{S}$ is called *1-exact*.

Proposition 5.2. *Suppose that $\mathcal{S}$ is an operator system and $\mathcal{A}$ is a unital C^*-algebra with its closed ideal $\mathcal{I}$. Then the map*

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S}\hat{\otimes}_{\min}\mathcal{A} \to \mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}$$

is a complete order quotient map with its kernel $\mathcal{S}\bar{\otimes}\mathcal{I}$ if and only if the map

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$$

is a complete order quotient map. Hence, an operator system $\mathcal{S}$ is 1-exact if and only if $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map for any unital C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$.

Proof. The operator space quotient and the operator system quotient of $\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}$ by $\mathcal{S}\bar{\otimes}\mathcal{I}$ are completely isometric [KPTT2, Theorem 5.1]. Since $\mathcal{S} \otimes \mathcal{I}$ is the kernel of $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$, we can also consider both the operator space quotient and the operator system quotient of $\mathcal{S} \otimes_{\min} \mathcal{A}$ by $\mathcal{S} \otimes \mathcal{I}$. It is easy to check that the operator space quotient (respectively operator system quotient) $(\mathcal{S} \otimes_{\min} \mathcal{A})/(\mathcal{S} \otimes \mathcal{I})$ is an operator subspace (respectively operator subsystem) of the operator space quotient (respectively operator system quotient) $(\mathcal{S}\hat{\otimes}_{\min}\mathcal{A})/(\mathcal{S}\bar{\otimes}\mathcal{I})$. If $z \in \mathcal{S} \otimes \mathcal{A}$ and $z + \mathcal{S}\bar{\otimes}\mathcal{I}$ is positive in the operator system quotient $(\mathcal{S}\hat{\otimes}_{\min}\mathcal{A})/(\mathcal{S}\bar{\otimes}\mathcal{I})$, then there exists $x \in \mathcal{S}\bar{\otimes}\mathcal{I}$ such that

$$z + \frac{\varepsilon}{2} 1_{\mathcal{S}} \otimes 1_{\mathcal{A}} + x \in (\mathcal{S}\hat{\otimes}_{\min}\mathcal{A})^+.$$

Take $x_0 \in \mathcal{S} \otimes I$ with $\|x - x_0\| < \varepsilon/2$. Considering $(x_0 + x_0^*)/2$, we may assume that x_0 is self-adjoint. We have

$$z+\varepsilon 1_{\mathcal{S}}\otimes 1_{\mathcal{A}}+x_0\in(\mathcal{S}\otimes_{\min}\mathcal{A})^+,$$

which implies that $z+\mathcal{S}\otimes\mathcal{I}$ is positive in $(\mathcal{S}\otimes_{\min}\mathcal{A})/(\mathcal{S}\otimes\mathcal{I})$. Hence, [KPTT2, Theorem 5.1] immediately implies that the operator space quotient and the operator system quotient of $\mathcal{S}\otimes_{\min}\mathcal{A}$ by $\mathcal{S}\otimes\mathcal{I}$ are completely isometric.

A unital linear map between operator systems is completely order isomorphic if and only if it is completely isometric, by [Effros and Ruan 2000, Corollary 5.1.2]. If a linear map between Banach spaces maps the open unit ball into the open unit ball densely, then it is a quotient map, from Lemma A.2.1 in the same paper. Combining them with the quotient lemma, we have equivalences:

$\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}\to\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}$ is a complete order quotient map with its kernel $\mathcal{S}\bar{\otimes}\mathcal{I}$

$\iff$ the operator system quotient $(\mathcal{S}\hat{\otimes}_{\min}\mathcal{A})/(\mathcal{S}\bar{\otimes}\mathcal{I})$ is completely order isomorphic to $\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}$

$\iff$ the operator space quotient $(\mathcal{S}\hat{\otimes}_{\min}\mathcal{A})/(\mathcal{S}\bar{\otimes}\mathcal{I})$ is completely isometric to $\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}$ [KPTT2, Theorem 5.1; Effros and Ruan 2000, Corollary 5.1.2]

$\iff$ the map $\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}\to\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}$ is a complete quotient map with its kernel $\mathcal{S}\bar{\otimes}\mathcal{I}$

$\iff$ the map $\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\otimes_{\min}\mathcal{A}\to\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}$ is a complete quotient map (quotient lemma, [Effros and Ruan 2000, Lemma A.2.1])

$\iff$ the operator space quotient $(\mathcal{S}\otimes_{\min}\mathcal{A})/(\mathcal{S}\otimes\mathcal{I})$ is completely isometric to $\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}$

$\iff$ the operator system quotient $(\mathcal{S}\otimes_{\min}\mathcal{A})/(\mathcal{S}\otimes\mathcal{I})$ is completely order isomorphic to $\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}$ [KPTT2, Theorem 5.1; Effros and Ruan 2000, Corollary 5.1.2]

$\iff$ $\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\otimes_{\min}\mathcal{A}\to\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}$ is a complete order quotient map. □

As pointed out in [KPTT2, Section 5], the framework of short exact sequences

$$0\to\mathcal{S}\bar{\otimes}\mathcal{I}\to\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}\to\mathcal{S}\hat{\otimes}_{\min}\mathcal{A}/\mathcal{I}\to 0$$

with complete tensor products is inappropriate if we replace ideals in C^*-algebras and C^*-quotients by kernels in operator systems and operator system quotients. Even a one-dimensional operator system does not satisfy such exactness. Instead of short exact sequences with complete tensor products, we make a replacement in

$$\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\otimes_{\min}\mathcal{A}\to\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}$$

with incomplete tensor products.

Theorem 5.3. *Let $\mathcal{S}$ be an operator system. Then, the following are equivalent:*

(i) *$\mathcal{S}$ is nuclear.*

(ii) *If $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ is a complete order quotient map for operator systems $\mathcal{T}_1$ and $\mathcal{T}_2$, then*

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 \to \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

is a complete order quotient map.

(iii) *If $\Phi : \mathfrak{C}_I \to \mathcal{T}$ is a complete order quotient map for an operator system $\mathcal{T}$, then*

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathfrak{C}_I \to \mathcal{S} \otimes_{\min} \mathcal{T}$$

is a complete order quotient map.

(iv) *If $\Phi : \mathcal{T} \to E$ is a complete order quotient map for an operator system $\mathcal{T}$ and a finite-dimensional operator system E, then*

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T} \to \mathcal{S} \otimes_{\min} E$$

is a complete order quotient map.

Proof. (i) $\Rightarrow$ (ii). Maximal tensor products of complete order quotient maps are still complete order quotient maps [Han 2011, Theorem 3.4]. Combining this with the hypothesis, we have a complete order quotient map

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 = \mathcal{S} \otimes_{\max} \mathcal{T}_1 \twoheadrightarrow \mathcal{S} \otimes_{\max} \mathcal{T}_2 = \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

(ii) $\Rightarrow$ (i). The proof is motivated by [Effros and Ruan 2000, Theorem 14.6.1]. Taking $\mathcal{T}_1$ as a unital C^*-algebra and $\mathcal{T}_2$ as its C^*-quotient, we see that $\mathcal{S}$ is a 1-exact operator system by Proposition 5.2. We take a finite-dimensional operator subsystem E of $\mathcal{S}$ and $\varepsilon > 0$. Then, E is a 1-exact operator system [KPTT2, Corollary 5.8], or equivalently, a 1-exact operator space [KPTT2, Proposition 5.5]. Let $E \subset B(\ell_2)$ and $P_n : \ell_2 \to \ell_2^n$ be the projection given by $P_n((\lambda_i)_{i=1}^\infty) = (\lambda_1, \dots, \lambda_n)$. For sufficiently large n, the truncation mapping

$$\varphi : x \in E \to P_n x P_n \in M_n$$

is injective with $\|\varphi^{-1}\|_{cb} < 1+\varepsilon'$ for $\varepsilon' = \varepsilon/(1+2\dim E)$ by [Pisier 1995], [Effros and Ruan 2000, Theorem 14.4.1]. Note that φ is unital completely positive and φ^{-1} is unital self-adjoint. By [Brown and Ozawa 2008, Corollary B.11], there exists a unital completely positive map $\psi : \varphi(E) \to \mathcal{S}$ with $\|\varphi^{-1} - \psi\|_{cb} \le 2\varepsilon' \dim E$. Though [Brown and Ozawa 2008, Corollary B.11] assumes that the range space is a C^*-algebra, its proof still works more generally when the range space is an operator system.

By choosing a faithful state ω on M_n, we can regard the dual space M_n^* as an operator system. Since ω is faithful on any operator subsystem, $(\varphi(E)^*, \omega|_{\varphi(E)})$

is also an operator system. The element z in $\varphi(E)^* \otimes_{\min} \mathcal{S}$ corresponding to $\psi : \varphi(E) \to \mathcal{S}$ canonically is positive [KPTT2, Lemma 8.5]. Since the duals of the complete order embeddings between finite-dimensional operator systems are complete order quotient maps [Farenick and Paulsen 2012, Proposition 1.15], the restriction $R : M_n^* \to \varphi(E)^*$ is a complete order quotient map. By the hypothesis,

$$R \otimes \mathrm{id}_{\mathcal{S}} : M_n^* \otimes_{\min} \mathcal{S} \to \varphi(E)^* \otimes_{\min} \mathcal{S}$$

is also a complete order quotient map. There exists a positive lifting $\tilde{z} \in M_n^* \otimes_{\min} \mathcal{S}$ of $z + \varepsilon' \omega|_{\varphi(E)} \otimes 1_{\mathcal{S}}$. The completely positive map $\tilde{\psi} : M_n \to \mathcal{S}$ corresponding to $\tilde{z}$ satisfies

$$\|\psi - \tilde{\psi}|_{\varphi(E)}\|_{cb} \leq \varepsilon'.$$

By the Arveson extension theorem, $\varphi : E \to M_n$ extends to a unital completely positive map $\tilde{\varphi} : \mathcal{S} \to M_n$. We thus obtain a diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[ddr, "\tilde{\varphi}"'] & E \arrow[l, "\iota"'] \arrow[r, "\iota"] \arrow[d, "\varphi"] & \mathcal{S} \\
& \varphi(E) \arrow[ur, "\psi"] \arrow[d, "\iota"] & \\
& M_n \arrow[uur, "\tilde{\psi}"'] &
\end{tikzcd}$$

where the ι denote inclusions.

It follows that

$$\begin{aligned}
\|\tilde{\psi} \circ \tilde{\varphi}(x) - x\| &\leq \|\tilde{\psi} \circ \varphi(x) - \psi \circ \varphi(x)\| + \|\psi \circ \varphi(x) - \varphi^{-1} \circ \varphi(x)\| \\
&\leq \varepsilon' \|x\| + 2\varepsilon' \dim E \|x\| \\
&= \varepsilon \|x\|.
\end{aligned}$$

for all $x \in E$. Considering the directed set

$$\{(E, \varepsilon) : E \text{ is a finite-dimensional operator subsystem of } \mathcal{S}, \varepsilon > 0\}$$

with the standard partial order, we can take nets of unital completely positive maps $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ and completely positive maps $\psi'_\lambda : M_{n_\lambda} \to \mathcal{S}$ such that $\psi'_\lambda \circ \varphi_\lambda$ converges to the map $\mathrm{id}_{\mathcal{S}}$ in the point-norm topology.

Since each φ_λ is unital, $\psi'_\lambda(I_{n_\lambda})$ converges to $1_{\mathcal{S}}$. Let us choose a state ω_λ on M_{n_λ} and set

$$\psi_\lambda(A) = \frac{1}{\|\psi'_\lambda\|} \psi'_\lambda(A) + \omega_\lambda(A)\Big(1_{\mathcal{S}} - \frac{1}{\|\psi'_\lambda\|} \psi'_\lambda(I_{n_\lambda})\Big).$$

Then $\psi_\lambda : M_{n_\lambda} \to \mathcal{S}$ is a unital completely positive map such that $\psi_\lambda \circ \varphi_\lambda$ converges to the map $\mathrm{id}_{\mathcal{S}}$ in the point-norm topology. By Corollary 3.2 of [Han and Paulsen 2011], $\mathcal{S}$ is nuclear.

(ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv). Trivial.

(iii) $\Rightarrow$ (ii). Choose a positive element z in $\mathcal{S} \otimes_{\min} \mathcal{T}_2$ and $\varepsilon > 0$. By Theorem 3.5, we can take a complete order quotient map $\Psi : \mathfrak{C}_I \to \mathcal{T}_1$. By the assumption, there exists a positive element $\tilde{z}$ in $\mathcal{S} \otimes_{\min} \mathfrak{C}_I$ satisfying $(\mathrm{id}_{\mathcal{S}} \otimes \Phi \circ \Psi)(\tilde{z}) = z + \varepsilon 1$. Thus, $\mathrm{id}_{\mathcal{S}} \otimes \Psi(\tilde{z})$ is a positive lifting of $z + \varepsilon 1$.

(iv) $\Rightarrow$ (ii). Choose a positive element $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{S} \otimes_{\min} \mathcal{T}_2$. Take E as a finite-dimensional operator subsystem of $\mathcal{T}_2$ generated by $\{y_i : 1 \le i \le n\}$ and $\mathcal{T}$ as $\Phi^{-1}(E)$. □

Remark 5.4. The equivalence of (i) and (ii) was already discovered by Kavruk independently. The proof depends on Kavruk's result that is not yet published.

Corollary 5.5. *Suppose that E is a finite-dimensional operator system and ω is a faithful state on E. The following are equivalent*:

(i) *If $\varepsilon > 0$ and $\varphi : E \to \mathcal{S}/\mathcal{J}$ is a completely positive map for an operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, then there exists a self-adjoint lifting $\tilde{\varphi} : E \to \mathcal{S}$ of φ such that $\tilde{\varphi} + \varepsilon\omega 1_{\mathcal{S}}$ is completely positive.*

(ii) *E is unitally completely order isomorphic to the direct sum of matrix algebras.*

Proof. (i) $\Rightarrow$ (ii). Condition (i) can be rephrased to state that

$$\mathrm{id}_{E^*} \otimes \pi : E^* \otimes_{\min} \mathcal{S} \to E^* \otimes_{\min} \mathcal{S}/\mathcal{J}$$

is a complete order quotient map for any operator system $\mathcal{S}$ and its kernel $\mathcal{J}$. Hence, E^* is a finite-dimensional nuclear operator system. Every finite-dimensional nuclear operator system is unitally completely order isomorphic to the direct sum of matrix algebras [Han and Paulsen 2011, Corollary 3.7]. Suppose that E^* is completely order isomorphic to $\bigoplus_{i=1}^n M_{k_i}$ for some $n, k_i \in \mathbb{N}$. Taking their duals, we see that E is completely order isomorphic to $\bigoplus_{i=1}^n M_{k_i}$. Suppose that the isomorphism maps the order unit of E to a matrix A in $\bigoplus_{i=1}^n M_{k_i}$. Then, A is positive definite. Let

$$A = U^* \operatorname{diag}(\lambda_1, \dots, \lambda_m) U, \quad \lambda_i > 0,\ m = \sum_{i=1}^n k_i$$

be a diagonalization of A. The mapping

$$\alpha \in \bigoplus_{i=1}^n M_{k_i} \mapsto U^* \operatorname{diag}\bigl(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_m}\bigr)\, \alpha \operatorname{diag}\bigl(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_m}\bigr) U \in \bigoplus_{i=1}^n M_{k_i}$$

is a complete order isomorphism that maps the identity matrix to A.

(ii) $\Rightarrow$ (i) We may assume that $E=\bigoplus_{i=1}^n M_{k_i}$. Let A be a density matrix of ω and $\lambda>0$ be its smallest eigenvalue. Suppose that $z\in\bigoplus_{i=1}^n M_{k_i}(\mathcal{S}/\mathcal{J})$ is the direct sum of Choi matrices corresponding to the restrictions of φ on each blocks M_{k_i}. There exists a lifting $\tilde{z}\in\bigoplus_{i=1}^n M_{k_i}(\mathcal{S})$ of z such that $\tilde{z}+\varepsilon\lambda I_m\otimes 1_{\mathcal{S}}$ ($m=\sum_{i=1}^n k_i$) is positive. Let $\tilde{\varphi}:E\to\mathcal{S}$ be a self-adjoint map corresponding to $\tilde{z}$. Then we have

$$\tilde{\varphi}+\varepsilon\omega 1_{\mathcal{S}}=\tilde{\varphi}+\varepsilon\operatorname{tr}(\,\cdot\,A)1_{\mathcal{S}}\geq_{cp}\tilde{\varphi}+\varepsilon\lambda\operatorname{tr}(\,\cdot\,)1_{\mathcal{S}}\geq_{cp}0. \qquad \square$$

In the last statement of Proposition 5.2, an operator systems $\mathcal{S}$ is fixed, and a C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$ are considered to be variables in

$$\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\otimes_{\min}\mathcal{A}\to\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}.$$

In the following, we switch their roles. As a result, we give an operator system theoretic proof of the Effros–Haagerup lifting theorem [Effros and Haagerup 1985, Theorem 3.2].

Theorem 5.6. *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{I}$ is its closed ideal. The following are equivalent*:

(i) $\mathrm{id}_{\mathcal{S}}\otimes\pi:\mathcal{S}\otimes_{\min}\mathcal{A}\to\mathcal{S}\otimes_{\min}\mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any operator system $\mathcal{S}$.*

(ii) $\mathrm{id}_{\mathcal{B}}\otimes\pi:\mathcal{B}\otimes_{\min}\mathcal{A}\to\mathcal{B}\otimes_{\min}\mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any unital C^*-algebra $\mathcal{B}$.*

(iii) $\mathrm{id}_{B(H)}\otimes\pi:B(H)\otimes_{\min}\mathcal{A}\to B(H)\otimes_{\min}\mathcal{A}/\mathcal{I}$ *is a complete order quotient map for a separable Hilbert space H.*

(iv) $\mathrm{id}_{E}\otimes\pi:E\otimes_{\min}\mathcal{A}\to E\otimes_{\min}\mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any finite-dimensional operator system E.*

(v) *The sequence*

$$0\to\mathcal{B}\otimes_{C^*\min}\mathcal{I}\to\mathcal{B}\otimes_{C^*\min}\mathcal{A}\to\mathcal{B}\otimes_{C^*\min}\mathcal{A}/\mathcal{I}\to 0$$

is exact for any C^*-algebra $\mathcal{B}$.

(vi) *For any finite-dimensional operator system E, every completely positive map $\varphi:E\to\mathcal{A}/\mathcal{I}$ lifts to a completely positive map $\tilde{\varphi}:E\to\mathcal{A}$.*

(vii) *For any finite-dimensional operator system E, every unital completely positive map $\varphi:E\to\mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi}:E\to\mathcal{A}$.*

(viii) *For any index set I, every unital completely positive finite rank map $\varphi:\mathfrak{C}_I\to\mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi}:\mathfrak{C}_I\to\mathcal{A}$ with* $\operatorname{Ker}\varphi=\operatorname{Ker}\tilde{\varphi}$.

(ix) *Every unital completely positive finite rank map $\varphi:\mathfrak{C}_1\to\mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi}:\mathfrak{C}_1\to\mathcal{A}$ with* $\operatorname{Ker}\varphi=\operatorname{Ker}\tilde{\varphi}$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (viii) $\Rightarrow$ (ix) are trivial. (vi) $\Rightarrow$ (vii) follows from [KPTT2, Remark 8.3]. (ii) $\Leftrightarrow$ (v) follows from Proposition 5.2. For (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i), it is sufficient to consider the first matrix level.

(iii) $\Rightarrow$ (iv). Let $E \subset B(H)$ for a separable Hilbert space H. Take a strictly positive element z in $E \otimes_{\min} \mathcal{A}/\mathcal{I}$ which is an operator subsystem of $B(H) \otimes_{\min} \mathcal{A}/\mathcal{I}$. By the assumption, there exists a positive lifting $\tilde{z}$ in $B(H) \otimes_{\min} \mathcal{A}$. Let $\{x_i : 1 \le i \le k\}$ be a self-adjoint basis of E and $\{\hat{x}_i : 1 \le i \le k\}$ be its dual basis. Each functional $\hat{x}_i$ on E extends to a continuous self-adjoint functional on $B(H)$ which we still denote by $\hat{x}_i$. The map $P := \sum_{i=1}^k \hat{x}_i \otimes x_i : B(H) \to B(H)$ is a self-adjoint projection onto E. Since

$$(\mathrm{id}_{B(H)} - P) \otimes \pi(\tilde{z}) = z - (P \otimes \mathrm{id}_{\mathcal{A}/\mathcal{I}})(z) = 0,$$

we have

$$(\mathrm{id}_{B(H)} - P) \otimes \mathrm{id}_{\mathcal{A}}(\tilde{z}) \in B(H) \otimes \mathcal{I}.$$

We write

$$(\mathrm{id}_{B(H)} - P) \otimes \mathrm{id}_{\mathcal{A}}(\tilde{z}) = \sum_{i=1}^{n} b_i \otimes h_i, \quad b_i \in B(H)_{sa}, h_i \in \mathcal{I}_{sa}.$$

Each h_i is decomposed into $h_i = h_i^+ - h_i^-$ for $h_i^+, h_i^- \in \mathcal{I}^+$. From

$$\begin{aligned} 0 \le \tilde{z} &= (P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^{n} b_i \otimes h_i^+ - \sum_{i=1}^{n} b_i \otimes h_i^- \\ &\le (P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^{n} \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^{n} \|b_i\| 1 \otimes h_i^- \end{aligned}$$

and

$$(\mathrm{id}_{B(H)} \otimes \pi)\left((P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^{n} \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^{n} \|b_i\| \otimes h_i^-\right) = z,$$

we see that

$$(P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^{n} \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^{n} \|b_i\| 1 \otimes h_i^- \in E \otimes_{\min} \mathcal{A}$$

is a positive lifting of z.

(iv) $\Rightarrow$ (i). Take a positive element $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$. Let E be a finite-dimensional operator system generated by $\{x_i : 1 \le i \le n\}$. Since $E \otimes_{\min} \mathcal{A}/\mathcal{I}$ is an operator subsystem of $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$, we have z also positive in $E \otimes_{\min} \mathcal{A}/\mathcal{I}$. By the hypothesis, there exists a positive element $\tilde{z}$ in $E \otimes_{\min} \mathcal{A}$ such that $(\mathrm{id}_E \otimes \pi)(\tilde{z}) = z$. This element is also positive in $\mathcal{S} \otimes_{\min} \mathcal{A}$.

(iv) $\Leftrightarrow$ (vi). Suppose that E is a finite dimensional operator system and $\varphi : E \to \mathcal{A}/\mathcal{I}$ is a completely positive map. The element z in $E^* \otimes_{\min} \mathcal{A}/\mathcal{I}$ corresponding to φ

is positive. Since E is finite-dimensional, we have $E^* \otimes_{\min} \mathcal{A} = E^* \hat{\otimes}_{\min} \mathcal{A}$. The kernel $E^* \otimes \mathcal{I}$ of $\mathrm{id}_{E^*} \otimes \pi$ is completely order proximinal in $E^* \otimes_{\min} \mathcal{A}$ [KPTT2, Corollary 5.1.5]. By the hypothesis, z lifts to a positive element $\tilde{z}$ in $E^* \otimes_{\min} \mathcal{A}$. The map $\tilde{\varphi} : E \to \mathcal{A}$ corresponding to $\tilde{z}$ is completely positive. The converse is merely the reverse of the argument.

(vii) $\Rightarrow$ (vi). The inclusion $\iota : \varphi(E) + \mathbb{C}1_{\mathcal{A}/\mathcal{I}} \subset \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\iota} : \varphi(E) + \mathbb{C}1_{\mathcal{A}/\mathcal{I}} \to \mathcal{A}$. The map $\tilde{\iota} \circ \varphi$ is the completely positive lifting of φ.

(vii) $\Rightarrow$ (viii). Let $Q : \mathfrak{C}_I \to \mathfrak{C}_I / \operatorname{Ker} \varphi$ be a quotient map. We have a factorization $\varphi = \psi \circ Q$ for $\psi : \mathfrak{C}_I / \operatorname{Ker} \varphi \to \mathcal{A}/\mathcal{I}$. By the hypothesis, ψ lifts to a unital completely positive map $\tilde{\psi} : \mathfrak{C}_I / \operatorname{Ker} \varphi \to \mathcal{A}$. Then $\tilde{\psi} \circ Q$ is a unital completely positive lifting of φ and their kernels coincide.

(ix) $\Rightarrow$ (vii). By Theorem 3.6, there exists a complete order quotient map $\Phi : \mathfrak{C}_1 \to E$. The map $\varphi \circ \Phi : \mathfrak{C}_1 \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\psi : \mathfrak{C}_1 \to \mathcal{A}$ such that their kernels coincide. Since $\operatorname{Ker} \Phi \subset \operatorname{Ker} \psi$, we get that ψ induces a map $\tilde{\varphi} : E \to \mathcal{A}/\mathcal{I}$ which is a unital completely positive lifting of φ. $\square$

The following theorem can be regarded as an operator system version of the quotient lemma.

Theorem 5.7. *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive surjection for operator systems $\mathcal{S}$ and $\mathcal{T}$. Let S_0 be an operator subsystem that is dense in $\mathcal{S}$, $\mathcal{T}_0 := \Phi(\mathcal{S}_0)$, and $\Phi_0 = \Phi|_{\mathcal{S}_0} : \mathcal{S}_0 \to \mathcal{T}_0$ be the surjective restriction. Then, the following are equivalent*:

(i) *$\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map and for any $\varepsilon > 0$, $k \in \mathbb{N}$ and a self-adjoint element $x \in \operatorname{Ker} \Phi_k$, there exists a self-adjoint element $x_0 \in \operatorname{Ker}(\Phi_0)_k$ such that $x_0 + \varepsilon 1 \geq x$.*

(ii) *$\Phi_0 : \mathcal{S}_0 \to \mathcal{T}_0$ is a complete order quotient map.*

Proof. The following arguments apply to all matricial levels.

(i) $\Rightarrow$ (ii). Choose $\varepsilon > 0$ and $\Phi_0(y_0) \in \mathcal{T}_0^+$ for a self-adjoint $y_0 \in \mathcal{S}_0$. By the hypothesis, there exist self-adjoint $x \in \operatorname{Ker} \Phi$ and $x_0 \in \operatorname{Ker} \Phi_0$ such that

$$y_0 + \tfrac{\varepsilon}{2} 1 + x \in \mathcal{S}^+ \quad \text{and} \quad x \leq x_0 + \tfrac{\varepsilon}{2} 1.$$

It follows that

$$y_0 + \varepsilon 1 + x_0 \geq y_0 + \tfrac{\varepsilon}{2} 1 + x \geq 0.$$

(ii) $\Rightarrow$ (i). Take $\varepsilon > 0$ and a self-adjoint element x in $\operatorname{Ker} \Phi$. Since $\mathcal{S}_0$ is dense in $\mathcal{S}$, there exists a self-adjoint element y_0 in $\mathcal{S}_0$ such that

$$x - \tfrac{\varepsilon}{3} 1 \leq y_0 \leq x + \tfrac{\varepsilon}{3} 1,$$

which implies that

$$\Phi_0(-y_0+\tfrac{\varepsilon}{3}1)=\Phi(-y_0+x+\tfrac{\varepsilon}{3}1)\in\mathcal{T}^+\cap\mathcal{T}_0=\mathcal{T}_0^+.$$

There exists an element x_0 in $\operatorname{Ker}\Phi_0$ such that $-y_0+\frac{2}{3}\varepsilon 1+x_0\geq 0$. From

$$x-\tfrac{\varepsilon}{3}1\leq y_0\leq\tfrac{2}{3}\varepsilon 1+x_0,$$

it follows that $x\leq\varepsilon 1+x_0$.

Now let $\Phi(y)\in\mathcal{T}^+$ for a self-adjoint $y\in\mathcal{S}$. There exists an element y_0 in $\mathcal{S}_0$ such that

$$y-\tfrac{\varepsilon}{3}1\leq y_0\leq y+\tfrac{\varepsilon}{3}1,$$

which implies that

$$\Phi_0(y_0+\tfrac{\varepsilon}{3}1)\geq\Phi(y)\geq 0.$$

There exists an element $x_0\in\operatorname{Ker}\Phi_0$ such that $y_0+x_0+\frac{2}{3}\varepsilon 1\geq 0$. It follows that

$$y+x_0+\varepsilon 1\geq y_0+x_0+\tfrac{2}{3}\varepsilon 1\geq 0. \qquad\square$$

Acknowledgements.

I would like to thank Ali S. Kavruk and Vern I. Paulsen for their careful reading of the manuscript and helpful comments.

References

[Bhattacharya 2014] A. Bhattacharya, "Relative weak injectivity of operator system pairs", *J. Math. Anal. Appl.* **420**:1 (2014), 257–267. MR Zbl

[Blecher 1992] D. P. Blecher, "The standard dual of an operator space", *Pacific J. Math.* **153**:1 (1992), 15–30. MR Zbl

[Boca 1991] F. Boca, "Free products of completely positive maps and spectral sets", *J. Funct. Anal.* **97**:2 (1991), 251–263. MR Zbl

[Brown and Ozawa 2008] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics **88**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Choi and Effros 1977] M. D. Choi and E. G. Effros, "Injectivity and operator spaces", *J. Functional Analysis* **24**:2 (1977), 156–209. MR Zbl

[Defant and Floret 1993] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies **176**, North-Holland, Amsterdam, 1993. MR Zbl

[Effros and Haagerup 1985] E. G. Effros and U. Haagerup, "Lifting problems and local reflexivity for C^*-algebras", *Duke Math. J.* **52**:1 (1985), 103–128. MR Zbl

[Effros and Ruan 2000] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs (N.S.) **23**, Oxford University Press, New York, 2000. MR Zbl

[Farenick and Paulsen 2012] D. Farenick and V. I. Paulsen, "Operator system quotients of matrix algebras and their tensor products", *Math. Scand.* **111**:2 (2012), 210–243. MR Zbl

[Farenick et al. 2013] D. Farenick, A. S. Kavruk, and V. I. Paulsen, "C^*-algebras with the weak expectation property and a multivariable analogue of Ando's theorem on the numerical radius", *J. Operator Theory* **70**:2 (2013), 573–590. MR Zbl

[Farenick et al. 2014] D. Farenick, A. S. Kavruk, V. I. Paulsen, and I. G. Todorov, "Operator systems from discrete groups", *Comm. Math. Phys.* **329**:1 (2014), 207–238. MR Zbl

[Fritz 2014] T. Fritz, "Operator system structures on the unital direct sum of C^*-algebras", *Rocky Mountain J. Math.* **44**:3 (2014), 913–936. MR Zbl

[Hamana 1979] M. Hamana, "Injective envelopes of operator systems", *Publ. Res. Inst. Math. Sci.* **15**:3 (1979), 773–785. MR Zbl

[Han 2011] K. H. Han, "On maximal tensor products and quotient maps of operator systems", *J. Math. Anal. Appl.* **384**:2 (2011), 375–386. MR Zbl

[Han and Paulsen 2011] K. H. Han and V. I. Paulsen, "An approximation theorem for nuclear operator systems", *J. Funct. Anal.* **261**:4 (2011), 999–1009. MR Zbl

[de la Harpe 2000] P. de la Harpe, *Topics in geometric group theory*, University of Chicago Press, 2000. MR Zbl

[Kavruk 2012] A. S. Kavruk, "The weak expectation property and Riesz interpolation", preprint, 2012. arXiv

[Kavruk 2014] A. S. Kavruk, "Nuclearity related properties in operator systems", *J. Operator Theory* **71**:1 (2014), 95–156. MR Zbl

[Kavruk et al. 2011] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, "Tensor products of operator systems", *J. Funct. Anal.* **261**:2 (2011), 267–299. MR Zbl

[Kavruk et al. 2013] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, "Quotients, exactness, and nuclearity in the operator system category", *Adv. Math.* **235** (2013), 321–360. MR Zbl

[Kerr and Li 2009] D. Kerr and H. Li, "On Gromov–Hausdorff convergence for operator metric spaces", *J. Operator Theory* **62**:1 (2009), 83–109. MR Zbl

[Kirchberg 1994] E. Kirchberg, "Commutants of unitaries in UHF algebras and functorial properties of exactness", *J. Reine Angew. Math.* **452** (1994), 39–77. MR Zbl

[Kirchberg and Wassermann 1998] E. Kirchberg and S. Wassermann, "C^*-algebras generated by operator systems", *J. Funct. Anal.* **155**:2 (1998), 324–351. MR Zbl

[Paulsen and Tomforde 2009] V. I. Paulsen and M. Tomforde, "Vector spaces with an order unit", *Indiana Univ. Math. J.* **58**:3 (2009), 1319–1359. MR Zbl

[Paulsen et al. 2011] V. I. Paulsen, I. G. Todorov, and M. Tomforde, "Operator system structures on ordered spaces", *Proc. Lond. Math. Soc.* (3) **102**:1 (2011), 25–49. MR Zbl

[Pisier 1995] G. Pisier, "Exact operator spaces", pp. 159–186 in *Recent advances in operator algebras* (Orléans, 1992), Astérisque **232**, Société Mathématique de France, Paris, 1995. MR Zbl

[Pisier 1996] G. Pisier, "A simple proof of a theorem of Kirchberg and related results on C^*-norms", *J. Operator Theory* **35**:2 (1996), 317–335. MR Zbl

[Pisier 2003] G. Pisier, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series **294**, Cambridge University Press, 2003. MR Zbl

Received July 17, 2015.

Kyung Hoon Han
University of Suwon
Gyeonggi-do 445-743
South Korea
kyunghoon.han@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 1 January 2017