Pacific
Journal of
Mathematics

REMARKS ON QUANTUM UNIPOTENT SUBGROUPS AND THE DUAL CANONICAL BASIS

YOSHIYUKI KIMURA

Volume 286 No. 1 January 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 286, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.286.125

REMARKS ON QUANTUM UNIPOTENT SUBGROUPS AND THE DUAL CANONICAL BASIS

YOSHIYUKI KIMURA

We prove the tensor product decomposition of the half of the quantized universal enveloping algebra associated with a Weyl group element which was conjectured by Berenstein and Greenstein (preprint, 2014, arXiv 1411.1391; see Conjecture 5.5) using the theory of the dual canonical basis. In fact, based on the compatibility between the decomposition and the dual canonical basis, a weak multiplicity-free property between the factors is established.

1. Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody Lie algebra and w be a Weyl group element. In [Kimura 2012], we studied the compatibility of the dual canonical basis and the quantum coordinate ring of the unipotent subgroup associated with a finite subset $\Delta_+ \cap w\Delta_-$, where Δ_+ (resp. Δ_-) is the set of positive (resp. negative) roots of $\mathfrak{g}$. The purpose of this paper is to study the compatibility of the dual canonical basis and the "quantum coordinate ring" of the pro-unipotent subgroup associated with a cofinite subset $\Delta_+ \cap w\Delta_+$.

Let $\boldsymbol{U}_q(\mathfrak{g})$ be the quantized enveloping algebra and

$$\boldsymbol{U}_q(\mathfrak{g}) \simeq \boldsymbol{U}_q^+(\mathfrak{g}) \otimes \boldsymbol{U}_q^0(\mathfrak{g}) \otimes \boldsymbol{U}_q^-(\mathfrak{g})$$

be its triangular decomposition. Let $\boldsymbol{U}_q^{\geq 0}(\mathfrak{g})$ be the subalgebra generated by $\boldsymbol{U}_q^+(\mathfrak{g})$ and $\boldsymbol{U}_q^0(\mathfrak{g})$. Let $T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell} : \boldsymbol{U}_q \to \boldsymbol{U}_q$ be Lusztig's symmetry associated with a Weyl group element w, where $\boldsymbol{i} = (i_1, \dots, i_\ell)$ is a reduced word of w. It is known that $T_w \in \operatorname{Aut}(\boldsymbol{U}_q(\mathfrak{g}))$ does not depend on the choice of reduced word.

Berenstein and Greenstein [2014, Conjecture 5.5] conjectured the following tensor product decomposition of the half $\boldsymbol{U}_q^-$ in general. We show the multiplicity-free property of the multiplications of the dual canonical basis elements between the

This work was supported by the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers "Mathematical Science of Symmetry, Topology and Moduli, Evolution of International Research Network based on OCAMI" and JSPS Grant-in-Aid for Scientific Research (S) 24224001.
MSC2010: 17B37, 13F60.
Keywords: quantum groups, dual canonical basis.

finite part and the cofinite part. We also prove the decomposition in the dual integral form $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ of the Lusztig integral form $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ with respect to Kashiwara's nondegenerate bilinear form.

Theorem 1.1. (1) *For a Weyl group element $w \in W$, multiplication in $\boldsymbol{U}_q^-$ defines an isomorphism of vector spaces over $\mathbb{Q}(q)$:*

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}) \otimes (\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-) \xrightarrow{\sim} \boldsymbol{U}_q^-.$$

(2) *For a Weyl group element $w \in W$, we set*

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0})_{\mathcal{A}}^{\mathrm{up}} := \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_w \boldsymbol{U}_q^{\geq 0}$$

and

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)_{\mathcal{A}}^{\mathrm{up}} := \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_w \boldsymbol{U}_q^-.$$

Then multiplication in $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ defines an isomorphism of free $\mathcal{A}$-modules:

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0})_{\mathcal{A}}^{\mathrm{up}} \otimes_{\mathcal{A}} (\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}.$$

Remark 1.2. (1) Theorem 1.1(1) can be shown directly in finite-type cases using the Poincaré–Birkhoff–Witt bases of $\boldsymbol{U}_q^-$ (see [Berenstein and Greenstein 2014, Proposition 5.3]). Hence it is a new result only in infinite-type cases.

(2) For the proof of Theorem 1.1(1), we use the dual canonical bases and the multiplication formula for them; in particular we will prove Theorem 1.1(2). After finishing this work, the author was informed of a proof which does not involve the theory of the dual canonical basis by Toshiyuki Tanisaki [2015, Proposition 2.10], who also proved the tensor product decomposition in Lusztig form, De Concini–Kac form and De Concini–Procesi form.

We note that the De Concini–Kac form (resp. De Concini–Procesi form) is related to the dual integral form of Lusztig's integral form with respect to the Kashiwara (resp. Lusztig) nondegenerate bilinear form on $\boldsymbol{U}_q^-$. Since the multiplicative structure of the dual canonical basis does not depend on the choice of nondegenerate bilinear form (and hence the definition of the dual canonical basis), our argument yields results for the tensor product decompositions of the De Concini–Kac form and the De Concini–Procesi form.

Remark 1.3. We note that the fact that $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}$ has a Poincaré–Birkhoff–Witt basis was shown by Beck, Chari and Pressley [Beck et al. 1999, Proposition 2.3] in general. (Throughout that paper, it is assumed that the generalized Cartan matrix is of symmetric affine type, but it should be noted that the assumption is not used in the proof of [Beck et al. 1999, Proposition 2.3]. For more details, see Theorem 2.18). The injectivity in Theorem 1.1 can be easily proved by the linear independence of the Poincaré–Birkhoff–Witt monomials (see [Lusztig 1993, Theorem 40.2.1(a)]) and the triangular decomposition of the quantized enveloping algebra (see [Lusztig 1993, Section 3.2]). Hence the nontrivial assertion is the surjectivity in Theorem 1.1.

Theorem 1.4. (1) *For a Weyl group element $w \in W$ and for a reduced word $\boldsymbol{i} = (i_1, \dots, i_\ell)$ of w, we have*

$$\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^- = \boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$

(2) *We have that $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$ is compatible with the dual canonical basis; that is, $\boldsymbol{B}^{\mathrm{up}} \cap \boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$ is a $\mathbb{Q}(q)$-basis of $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$. In fact, there exists a subset $\mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-) \subset \mathscr{B}(\infty)$ such that*

$$\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^- = \bigoplus_{b \in \mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

Using the theory of crystal bases, we can obtain the characterization of the subset $\mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)$. For $w \in W$, we have the decomposition theorem of the crystal basis $\mathscr{B}(\infty)$ of $\boldsymbol{U}_q^-$ associated with a Weyl group element (and a reduced word) and the corresponding multiplication formula. We consider the map Ω_w associated with a Weyl group element which was introduced by Saito [1994] (and Baumann, Kamnitzer and Tingley [Baumann et al. 2014]):

$$\Omega_w := (\tau_{\leq w}, \tau_{>w}) : \mathscr{B}(\infty) \to \mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}) \times \mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-),$$

where $\tau_{\leq w}(b)$ and $\tau_{>w}(b)$ are defined by crystal bases as follows:

$$\begin{aligned}
&L(b, \boldsymbol{i}) := \big(\varepsilon_{i_1}(b), \varepsilon_{i_2}(\hat{\sigma}_{i_1}^* b), \dots, \varepsilon_{i_\ell}(\hat{\sigma}_{i_{\ell-1}}^* \cdots \hat{\sigma}_{i_1}^* b)\big) \in \mathbb{Z}_{\geq 0}^\ell,\\
&b(\boldsymbol{c}, \boldsymbol{i}) := f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots T_{i_1} \cdots T_{i_{\ell-1}}(f_{i_\ell}^{(c_\ell)}) \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty),\\
&\tau_{\leq \boldsymbol{i}}(b) := b(L(b, \boldsymbol{i}), \boldsymbol{i}) \in \mathscr{B}(\infty),\\
&\tau_{> \boldsymbol{i}}(b) := \sigma_{i_1} \cdots \sigma_{i_\ell} \hat{\sigma}_{i_\ell}^* \cdots \hat{\sigma}_{i_1}^* b \in \mathscr{B}(\infty).
\end{aligned}$$

The following is the multiplicity-free result of the multiplication of the dual canonical basis elements in the finite part and the cofinite part.

Theorem 1.5. *Let w be a Weyl group element and $\boldsymbol{i} = (i_1, \dots, i_\ell)$ be a reduced word of w. For a crystal basis element $b \in \mathscr{B}(\infty)$, we have*

$$G^{\mathrm{up}}(\tau_{\leq \boldsymbol{i}}(b)) G^{\mathrm{up}}(\tau_{> \boldsymbol{i}}(b)) \in G^{\mathrm{up}}(b) + \sum_{L(b', \boldsymbol{i}) < L(b, \boldsymbol{i})} q\mathbb{Z}[q] G^{\mathrm{up}}(b'),$$

where $L(b', \boldsymbol{i}) < L(b, \boldsymbol{i})$ in the left lexicographic order on $\mathbb{Z}_{\geq 0}^\ell$ associated with a reduced word $\boldsymbol{i}$.

Using induction on the lexicographic order on each root space, we obtain the surjectivity in Theorem 1.1(2). In particular, Theorem 1.1(1) can be shown.

Since the subalgebras $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}$ and $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$ are compatible with the dual canonical basis and since the dual bar-involution σ which characterizes the dual canonical basis is a (twisted) anti-involution, we obtain the tensor product factorization in the opposite order.

Corollary 1.6. *For a Weyl group element $w \in W$, multiplication in $\boldsymbol{U}_q^-$ defines an isomorphism of vector spaces*:

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-) \otimes (\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}) \xrightarrow{\sim} \boldsymbol{U}_q^-.$$

2. Review of quantum unipotent subgroups and the dual canonical basis

2A. ***Quantum universal enveloping algebra.*** In this subsection, we give a brief review of the definition of quantum universal enveloping algebra. The reader is referred to [Kashiwara 1991; 1993a; 1993b] for more details.

2A1. Let I be a finite index set.

Definition 2.1. A *root datum* is a quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ which consists of

(1) a square matrix $(a_{ij})_{i,j \in I}$, called the symmetrizable generalized Cartan matrix, that is, an I-indexed $\mathbb{Z}$-valued matrix which satisfies
  (a) $a_{ii} = 2$ for $i \in I$,
  (b) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
  (c) there exists a diagonal matrix $\mathrm{diag}(d_i)_{i \in I}$ such that $(d_i a_{ij})_{i,j \in I}$ is symmetric and d_i are positive integers;

(2) P : a free abelian group (the weight lattice);

(3) $\Pi = \{\alpha_i \mid i \in I\} \subset P$: the set of simple roots such that $\Pi \subset P \otimes_{\mathbb{Z}} \mathbb{Q}$ is linearly independent;

(4) $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$: the dual lattice (the coweight lattice) of P with perfect pairing $\langle \cdot, \cdot \rangle : P^\vee \otimes_{\mathbb{Z}} P \to \mathbb{Z}$;

(5) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$: the set of simple coroots, satisfying
  (a) $a_{ij} = \langle h_i, \alpha_j \rangle$ for all $i, j \in I$,
  (b) there exists $\{\Lambda_i\}_{i \in I} \subset P$, called the set of fundamental weights, satisfying $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ for $i, j \in I$.

We say $\Lambda \in P$ is *dominant* if $\langle h_i, \Lambda \rangle \geq 0$ for any $i \in I$ and denote by P_+ the set of dominant integral weights. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ be the root lattice. Let $Q_\pm = \pm \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we set $|\xi| = \sum_{i \in I} \xi_i$.

2A2. Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a root datum. We set $\mathfrak{h} := P^\vee \otimes_{\mathbb{Z}} \mathbb{C}$. A triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a Cartan datum or a realization of a generalized Cartan matrix A.

It is known that there exists a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ satisfying

(1) $(\alpha_i, \alpha_i) = d_i a_{ij}$,

(2) $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in \mathfrak{h}^*$.

Definition 2.2. Let $\mathfrak{g}$ be the *symmetrizable Kac–Moody Lie algebra* associated with a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, that is, a Lie algebra which is generated by $\{e_i\}_{i\in I} \cup \{f_i\}_{i\in I} \cup \mathfrak{h}$ with the following relations:

(1) $[h_1, h_2] = 0$ for $h_1, h_2 \in \mathfrak{h}$,

(2) $[h, e_i] = \langle h, \alpha_i\rangle e_i$ and $[h, f_i] = -\langle h, \alpha_i\rangle f_i$ for $h \in \mathfrak{h}$ and $i \in I$,

(3) $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$ for $i, j \in I$,

(4) $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ and $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$, where $\mathrm{ad}(x)(y) = [x, y]$.

Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the Lie subalgebra which is generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$). We have the triangular decomposition and the root space decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*\setminus\{0\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \langle h, \alpha\rangle x \ \ \forall h \in \mathfrak{h}\}$. The set $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is called the root system of $\mathfrak{g}$.

2A3. We fix a root datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. We introduce an indeterminate q. For $i \in I$, we set $q_i = q^{d_i}$. For $\xi = \sum \xi_i\alpha_i \in Q$, we set $q_\xi := \prod_{i\in I} q_i^{\xi_i}$.

For $n \in \mathbb{Z}$ and $i \in I$, we set

$$[n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

and $[n]_i! = [n]_i[n-1]_i \cdots [1]_i$ for $n > 0$ and $[0]! = 1$.

Definition 2.3. The *quantized enveloping algebra* $\boldsymbol{U}_q(\mathfrak{g})$ associated with a root datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the $\mathbb{Q}(q)$-algebra which is generated by $\{e_i\}_{i\in I}$, $\{f_i\}_{i\in I}$ and $\{q^h \mid h \in P^\vee\}$ with the following relations:

(1) $q^0 = 1$ and $q^{h+h'} = q^h q^{h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i$ and $q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$ for $i \in I$ and $h \in P^\vee$,

(3) $e_i f_j - f_j e_i = \delta_{ij}(k_i - k_i^{-1})/(q_i - q_i^{-1})$, where $k_i = q^{d_i h_i}$,

(4) $$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$$ (q-Serre relations),

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$ for $i \in I$ and $k \in \mathbb{Z}_{>0}$.

2A4. Let $\boldsymbol{U}_q^0$ be the subalgebra of $\boldsymbol{U}_q(\mathfrak{g})$ which is generated by $\{q^h \mid h \in P^\vee\}$; it is isomorphic to the group algebra

$$\mathbb{Q}(q)[P^\vee] := \bigoplus_{h \in P^\vee} \mathbb{Q}(q) q^h$$

over $\mathbb{Q}(q)$. For $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we set

$$k_\xi := \prod_{i \in I} k_i^{\xi_i} = \prod_{i \in I} q^{d_i \xi_i h_i}.$$

Let $\boldsymbol{U}_q^+$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{e_i\}_{i \in I}$, let $\boldsymbol{U}_q^-$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{f_i\}_{i \in I}$, let $\boldsymbol{U}_q^{\geq 0}$ be the $\mathbb{Q}(q)$-subalgebra generated by $\boldsymbol{U}_q^0$ and $\boldsymbol{U}_q^+$, and let $\boldsymbol{U}_q^{\leq 0}$ be the $\mathbb{Q}(q)$-subalgebra generated by $\boldsymbol{U}_q^0$ and $\boldsymbol{U}_q^-$.

Theorem 2.4 [Lusztig 1993, Corollary 3.2.5]. *The multiplication of $\boldsymbol{U}_q$ induces the triangular decomposition of $\boldsymbol{U}_q(\mathfrak{g})$ as vector spaces over $\mathbb{Q}(q)$:*

$$\boldsymbol{U}_q(\mathfrak{g}) \cong \boldsymbol{U}_q^+ \otimes \boldsymbol{U}_q^0 \otimes \boldsymbol{U}_q^- \cong \boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^0 \otimes \boldsymbol{U}_q^+. \tag{2-1}$$

2A5. For $\xi \in \pm Q$, we define $\boldsymbol{U}_q^\pm(\mathfrak{g})_\xi$ by

$$\boldsymbol{U}_q^\pm(\mathfrak{g})_\xi := \{x \in \boldsymbol{U}_q^\pm(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h, \xi \rangle} x \text{ for } h \in P^\vee\}. \tag{2-2}$$

Then we have a root space decomposition

$$\boldsymbol{U}_q^\pm(\mathfrak{g}) = \bigoplus_{\xi \in Q_\pm} \boldsymbol{U}_q^\pm(\mathfrak{g})_\xi.$$

An element $x \in \boldsymbol{U}_q^\pm(\mathfrak{g})$ is called homogeneous if $x \in \boldsymbol{U}_q^\pm(\mathfrak{g})_\xi$ for some $\xi \in Q_\pm$.

2A6. We define a $\mathbb{Q}(q)$-algebra anti-involution $* : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$ by

$$*(e_i) = e_i, \quad *(f_i) = f_i, \quad *(q^h) = q^{-h}. \tag{2-3}$$

We call this the *star involution*.

We define a $\mathbb{Q}$-algebra automorphism $^- : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$ by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q} = q^{-1}, \quad \overline{q^h} = q^{-h}. \tag{2-4}$$

We call this the *bar involution*.

These two involutions preserve $\boldsymbol{U}_q^+(\mathfrak{g})$ and $\boldsymbol{U}_q^-(\mathfrak{g})$, and we have $^- \circ * = * \circ {}^-$.

2A7. In this article, we choose the following comultiplication $\Delta = \Delta_-$ on $\boldsymbol{U}_q(\mathfrak{g})$:

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i. \tag{2-5}$$

2A8. We define a $\mathbb{Q}(q)$-algebra structure on $\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-$ by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = q^{-(\mathrm{wt}(x_2),\mathrm{wt}(y_1))} x_1 x_2 \otimes y_1 y_2, \tag{2-6}$$

where x_i, y_i $(i = 1, 2)$ are homogeneous elements. Let $r = r_- : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-$ be the $\mathbb{Q}(q)$-algebra homomorphism defined by

$$r(f_i) = f_i \otimes 1 + 1 \otimes f_i \quad (i \in I).$$

We call this the twisted comultiplication. Then it is known that there exists a unique $\mathbb{Q}(q)$-valued nondegenerate symmetric bilinear form $(\cdot,\cdot) : \boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^- \to \mathbb{Q}(q)$ with the following properties:

$$(1,1)=1,\ (f_i, f_j)=\delta_{ij},\ (r(x), y_1 \otimes y_2)=(x, y_1 y_2),\ (x_1 \otimes x_2, r(y))=(x_1 x_2, y)$$

for homogeneous $x, y_1, y_2 \in \boldsymbol{U}_q^-$, where the form

$$(\cdot \otimes \cdot, \cdot \otimes \cdot) : (\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-) \otimes (\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-) \to \mathbb{Q}(q)$$

is defined by $(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2 \otimes y_2)$ for $x_1, x_2, y_1, y_2 \in \boldsymbol{U}_q^-$.

2A9. For $i \in I$, we define the unique $\mathbb{Q}(q)$-linear map ${}_ir : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$ (resp. $r_i : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$) by

$$({}_ir(x), y) = (x, f_i y),$$
$$(r_i(x), y) = (x, y f_i).$$

Lemma 2.5 [Lusztig 1993, Section 1.2.13]. *For $x, y \in \boldsymbol{U}_q^-$, we have q-boson relations*:

$${}_ir(xy) = {}_ir(x)y + q^{(\mathrm{wt}\,x,\alpha_i)} x\, {}_ir(y),$$
$$r_i(xy) = q^{(\mathrm{wt}\,y,\alpha_i)} r_i(x)y + x\, r_i(y).$$

Lemma 2.6 [Lusztig 1993, Proposition 3.1.6]. *We have*

$$[e_i, x] = \frac{r_i(x)k_i - k_i^{-1}\,{}_ir(x)}{q_i - q_i^{-1}} \quad \text{for } x \in \boldsymbol{U}_q^-. \tag{2-7}$$

Using the q-boson relation, we obtain the following result.

Lemma 2.7 [Lusztig 1993, Lemma 38.1.2, Proposition 38.1.6]. *For each $i \in I$, any element $x \in \boldsymbol{U}_q^-$ can be written uniquely as*

$$x = \sum_{c \geq 0} f_i^{(c)} x_c \quad \text{with } x_c \in \mathrm{Ker}({}_ir).$$

2B. ***Canonical basis and dual canonical basis.*** We give a brief review of the theory of the canonical basis and the dual canonical basis following Kashiwara. Note that Kashiwara called them the lower global basis and the upper global basis.

2B1. We define $\mathbb{Q}$-subalgebras $\mathcal{A}_0$, $\mathcal{A}_\infty$ and $\mathcal{A}$ of $\mathbb{Q}(q)$ by

$$\begin{aligned}\mathcal{A}_0 &:= \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 0\},\\ \mathcal{A}_\infty &:= \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = \infty\},\\ \mathcal{A} &:= \mathbb{Q}[q^{\pm 1}].\end{aligned}$$

2B2. We introduce the crystal basis of $\boldsymbol{U}_q^-$. For more details, see [Kashiwara 1991, Section 3]. We define the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\boldsymbol{U}_q^-$ by

$$\begin{aligned}\tilde{e}_i x &= \sum_{c\geq 1} f_i^{(c-1)} x_c,\\ \tilde{f}_i x &= \sum_{c\geq 0} f_i^{(c+1)} x_c,\end{aligned}$$

and we set

$$\mathscr{L}(\infty) := \sum_{\substack{\ell\geq 0\\ i_1,\dots,i_\ell\in I}} \mathcal{A}_0 \tilde{f}_{i_1}\cdots\tilde{f}_{i_\ell} 1 \subset \boldsymbol{U}_q^-,$$

$$\mathscr{B}(\infty) := \{ \tilde{f}_{i_1}\cdots\tilde{f}_{i_\ell} 1 \bmod q\mathscr{L}(\infty) \mid l \geq 0, i_1,\dots,i_\ell \in I\} \subset \mathscr{L}(\infty)/q\mathscr{L}(\infty).$$

Then $\mathscr{L}(\infty)$ is an $\mathcal{A}_0$-lattice with $\mathbb{Q}(q)\otimes_{\mathcal{A}_0}\mathscr{L}(\infty) \simeq \boldsymbol{U}_q^-$ that is stable under $\tilde{e}_i$ and $\tilde{f}_i$, and $\mathscr{B}(\infty)$ is a $\mathbb{Q}$-basis of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$. We also have induced maps $\tilde{f}_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ and $\tilde{e}_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty)\sqcup\{0\}$ with the property that $\tilde{f}_i\,\tilde{e}_i\, b = b$ for $b \in \mathscr{B}(\infty)$ with $\tilde{e}_i b \neq 0$. We call $(\mathscr{B}(\infty), \mathscr{L}(\infty))$ the (lower) crystal basis of $\boldsymbol{U}_q^-$ and call $\mathscr{L}(\infty)$ the (lower) crystal lattice. We denote $1 \bmod q\mathscr{L}(\infty)$ by u_∞.

2B3. It is also known that the star involution $* : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$ induces an $\mathcal{A}_0$-linear isomorphism $* : \mathscr{L}(\infty) \to \mathscr{L}(\infty)$ and a bijection $* : \mathscr{B}(\infty) \to \mathscr{B}(\infty)$; see [Kashiwara 1991, Proposition 5.2.4; 1993b, Theorem 2.1.1]. We set

$$\begin{aligned}\tilde{f}_i^* &:= * \circ \tilde{f}_i \circ * : \mathscr{B}(\infty) \to \mathscr{B}(\infty),\\ \tilde{e}_i^* &:= * \circ \tilde{e}_i \circ * : \mathscr{B}(\infty) \to \mathscr{B}(\infty)\sqcup\{0\}.\end{aligned}$$

2B4. Let $\overline{\mathscr{L}(\infty)} = \{\bar{x} \mid x \in \mathscr{L}(\infty)\}$. Then the natural map

$$\mathscr{L}(\infty)\cap\overline{\mathscr{L}(\infty)}\cap\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}} \to \mathscr{L}(\infty)/q\mathscr{L}(\infty)$$

is an isomorphism of $\mathbb{Q}$-vector spaces. Let G^{low} be the inverse of this isomorphism. The image

$$\boldsymbol{B}^{\mathrm{low}} = \{ G^{\mathrm{low}}(b) \mid b \in \mathscr{B}(\infty)\} \subset \mathscr{L}(\infty)\cap\overline{\mathscr{L}(\infty)}\cap\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}$$

is an $\mathcal{A}$-basis of $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ and is called the canonical basis or the lower global basis of $\boldsymbol{U}_q^-$.

2B5. The important property of the canonical basis is the following compatibility with the left and right ideals which are generated by Chevalley generators $\{f_i\}_{i\in I}$.

Theorem 2.8 [Lusztig 1993, Theorems 14.3.2 and 14.4.3; Kashiwara 1991, Theorem 7]. *For $i \in I$ and $n \geq 1$, $f_i^n \boldsymbol{U}_q^-$ and $\boldsymbol{U}_q^- f_i^n$ are compatible with the canonical basis; that is, $f_i^n \boldsymbol{U}_q^- \cap \boldsymbol{B}^{\mathrm{low}}$ (resp. $\boldsymbol{U}_q^- f_i^n \cap \boldsymbol{B}^{\mathrm{low}}$) is a basis of $f_i^n \boldsymbol{U}_q^-$ (resp. $\boldsymbol{U}_q^- f_i^n$). In fact, we have*

$$f_i^n \boldsymbol{U}_q^- \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i(b)\geq m}} \mathcal{A}G^{\mathrm{low}}(b),$$

$$\boldsymbol{U}_q^- f_i^n \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i^*(b)\geq m}} \mathcal{A}G^{\mathrm{low}}(b).$$

2B6. Let $\sigma : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$ be the $\mathbb{Q}$-linear map defined by

$$(\sigma(x), y) = \overline{(x, \overline{y})}$$

for arbitrary $x, y \in \boldsymbol{U}_q^-$. Let $\sigma(\mathscr{L}(\infty)) := \{\sigma(x) \mid x \in \mathscr{L}(\infty)\}$ and set the dual integral form:

$$\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} := \{x \in \boldsymbol{U}_q^- \mid (x, \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}) \subset \mathcal{A}\}.$$

$\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ has an $\mathcal{A}$-subalgebra of $\boldsymbol{U}_q^-$. The natural map

$$\mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \to \mathscr{L}(\infty)/q\mathscr{L}(\infty)$$

is also an isomorphism of $\mathbb{Q}$-vector spaces, so let G^{up} be the inverse of the above isomorphism. Then

$$\boldsymbol{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b) \mid b \in \mathscr{B}(\infty)\} \subset \mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$$

is an $\mathcal{A}$-basis of $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ and is called the dual canonical basis or the upper global basis of $\boldsymbol{U}_q^-$.

Proposition 2.9 [Kimura 2012, Proposition 4.26(1)]. *For $i \in I$ and $c \geq 1$, let $f_i^{\{c\}} = f_i^{(c)}/(f_i^{(c)}, f_i^{(c)})$. Then we have*

$$f_i^{\{c\}} = q_i^{c(c-1)/2} f_i^c.$$

2B7. For the dual canonical basis, we have the following expansion of left and right multiplication with respect to the Chevalley generators and their (shifted) powers.

Theorem 2.10 [Kashiwara 2012, Proposition 2.2; Oya 2015, Proposition 4.14 (ii)]. *For $b \in \mathscr{B}(\infty)$, $i \in I$ and $c \geq 1$, we have*

$$f_i^{\{c\}} G^{\mathrm{up}}(b) = q_i^{-c\varepsilon_i(b)} G^{\mathrm{up}}(\tilde{f}_i^c b) + \sum_{\varepsilon_i(b') < \varepsilon_i(b)+c} F_{i;b,b'}^{\{c\}}(q) G^{\mathrm{up}}(b'), \tag{2-8a}$$

$$G^{\mathrm{up}}(b) f_i^{\{c\}} = q_i^{-c\varepsilon_i^*(b)} G^{\mathrm{up}}(\tilde{f}_i^{*c} b) + \sum_{\varepsilon_i^*(b') < \varepsilon_i^*(b)+c} F_{i;b,b'}^{*\{c\}}(q) G^{\mathrm{up}}(b'), \tag{2-8b}$$

where

$$F_{i;b,b'}^{\{c\}}(q) := (f_i^{\{c\}} G^{\mathrm{up}}(b), G^{\mathrm{low}}(b')) = q_i^{c(c-1)/2} (G^{\mathrm{up}}(b), ({}_i r)^c G^{\mathrm{low}}(b')) \in q_i^{-c\varepsilon_i(b)} q\mathbb{Z}[q],$$

$$F_{i;b,b'}^{*\{c\}}(q) := (G^{\mathrm{up}}(b) f_i^{\{c\}}, G^{\mathrm{low}}(b')) = q_i^{c(c-1)/2} (G^{\mathrm{up}}(b), (r_i)^c G^{\mathrm{low}}(b')) \in q_i^{-c\varepsilon_i^*(b)} q\mathbb{Z}[q].$$

2C. *Braid group action and the (dual) canonical basis.* In this subsection, we recall the compatibility between Lusztig's braid symmetry and the (dual) canonical basis (for more details, see [Kimura 2012, Sections 4.4 and 4.6]).

2C1. *Braid group action on quantized enveloping algebra.* Let W be the Weyl group and $\{s_i\}_{i \in I}$ be the set of simple reflections, and let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function.

Following Lusztig [1993, Section 37.1.3], we define the $\mathbb{Q}(q)$-algebra automorphisms

$$T'_{i,\epsilon} : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$$

and

$$T''_{i,\epsilon} : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$$

for $i \in I$ and $\epsilon \in \{\pm 1\}$ by the following formulae:

$$T'_{i,\epsilon}(q^h) = q^{s_i(h)}; \tag{2-9a}$$

$$T'_{i,\epsilon}(e_j) = \begin{cases} -k_i^{\epsilon} e_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{\epsilon r} e_i^{(r)} e_j e_i^{(s)} & \text{for } j \neq i; \end{cases} \tag{2-9b}$$

$$T'_{i,\epsilon}(f_j) = \begin{cases} -e_i k_i^{-\epsilon} & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{-\epsilon r} f_i^{(s)} f_j f_i^{(r)} & \text{for } j \neq i; \end{cases} \tag{2-9c}$$

and

(2-10a) $$T''_{i,-\epsilon}(q^h) = q^{s_i(h)};$$

(2-10b) $$T''_{i,-\epsilon}(e_j) = \begin{cases} -f_i k_i^{-\epsilon} & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{\epsilon r} e_i^{(s)} e_j e_i^{(r)} & \text{for } j \neq i; \end{cases}$$

(2-10c) $$T''_{i,-\epsilon}(f_j) = \begin{cases} -k_i^{\epsilon} e_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-\epsilon r} f_i^{(r)} f_j f_i^{(s)} & \text{for } j \neq i. \end{cases}$$

It is known that $\{T'_{i,\epsilon}\}_{i\in I}$ and $\{T''_{i,\epsilon}\}_{i\in I}$ satisfy the braid relation.

Lemma 2.11 [Lusztig 1993, Proposition 37.1.2(d), Section 37.2.4].

(1) *We have* $T'_{i,\epsilon} \circ T''_{i,-\epsilon} = T''_{i,-\epsilon} \circ T'_{i,\epsilon} = \mathrm{id}$.

(2) *We have* $* \circ T'_{i,\epsilon} \circ * = T''_{i,-\epsilon}$ *for* $i \in I$ *and* $\epsilon \in \{\pm 1\}$.

In the following, we write $T_i = T''_{i,1}$ and $T_i^{-1} = T'_{i,-1}$ as in [Saito 1994, Proposition 1.3.1].

2C2.

Proposition 2.12 [Lusztig 1993, Proposition 38.1.6, Lemma 38.1.5].

(1) *For* $i \in I$, *we have*

$$\boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^- = \{x \in \boldsymbol{U}_q^- \mid {}_i r(x) = 0\},$$
$$\boldsymbol{U}_q^- \cap T_i^{-1} \boldsymbol{U}_q^- = \{x \in \boldsymbol{U}_q^- \mid r_i(x) = 0\}.$$

(2) *For* $i \in I$, *we have the following orthogonal decomposition with respect to* $(\cdot,\cdot)_-$:

$$\boldsymbol{U}_q^- = (\boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-) \oplus f_i \boldsymbol{U}_q^- = (\boldsymbol{U}_q^- \cap T_i^{-1} \boldsymbol{U}_q^-) \oplus \boldsymbol{U}_q^- f_i.$$

Corollary 2.13. *For* $i \in I$, *the subalgebra* $\boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$ *(resp.* $\boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$*) is compatible with the dual canonical basis*; *that is*, *we have*

$$\boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^- \cap \boldsymbol{U}_q^-(\mathfrak{g})^{\mathrm{up}}_{\mathcal{A}} = \bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i(b)=0}} \mathcal{A} G^{\mathrm{up}}(b),$$
$$\boldsymbol{U}_q^- \cap T_i^{-1} \boldsymbol{U}_q^- \cap \boldsymbol{U}_q^-(\mathfrak{g})^{\mathrm{up}}_{\mathcal{A}} = \bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i^*(b)=0}} \mathcal{A} G^{\mathrm{up}}(b).$$

2C3.

Proposition 2.14 [Saito 1994, Proposition 3.4.7, Corollary 3.4.8].

(1) *Let* $x \in \boldsymbol{U}_q^- \in \mathscr{L}(\infty) \cap T_i^{-1}\boldsymbol{U}_q^-$ *with* $b := x \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty)$. *We have*

$$T_i(x) \in \mathscr{L}(\infty) \cap T_i \boldsymbol{U}_q^-,$$
$$T_i(x) \equiv \tilde{f}_i^{*\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)} b \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(2) *Let*

$$\sigma_i : \{b \in \mathscr{B}(\infty) \mid \varepsilon_i^*(b) = 0\} \to \{b \in \mathscr{B}(\infty) \mid \varepsilon_i(b) = 0\}$$

be the map defined by $\sigma_i(b) = \tilde{f}_i^{*\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)} b$. *Then* σ_i *is bijective and its inverse is given by*

$$\sigma_i^*(b) = (* \circ \sigma_i \circ *)(b) = \tilde{f}_i^{\varphi_i^*(b)} \tilde{e}_i^{*\varepsilon_i^*(b)} b.$$

The bijections σ_i and σ_i^* are called Saito crystal reflections. In [Saito 1994, Corollary 3.4.8], σ_i and σ_i^* are denoted by Λ_i and Λ_i^{-1}. Following Baumann, Kamnitzer and Tingley [Baumann et al. 2014, Section 5.5], for convenience, we extend σ_i and σ_i^* to $\mathscr{B}(\infty)$ by setting

$$\hat{\sigma}_i(b) := \sigma_i(\tilde{e}_i^{*\max}(b)),$$
$$\hat{\sigma}_i^*(b) := \sigma_i^*(\tilde{e}_i^{\max}(b)),$$

so we can consider $\hat{\sigma}_i$ and $\hat{\sigma}_i^*$ as maps from $\mathscr{B}(\infty)$ to itself.

2C4. Let ${}^i\pi : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$ (resp. $\pi^i : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^- \cap T_i^{-1} \boldsymbol{U}_q^-$) be the orthogonal projection whose kernel is $f_i \boldsymbol{U}_q^-$ (resp. $\boldsymbol{U}_q^- f_i$) in Proposition 2.12(2). We have the following relations among the braid group action and the (dual) canonical basis.

Theorem 2.15 [Lusztig 1996, Theorem 1.2; Kimura 2012, Theorem 4.23].

(1) *For* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i^*(b) = 0$, *we have*

$$T_i(\pi^i G^{\mathrm{low}}(b)) = {}^i\pi(G^{\mathrm{low}}(\sigma_i(b))),$$
$$(1 - q_i^2)^{\langle h_i, \mathrm{wt}\, b\rangle} T_i G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\sigma_i b).$$

(2) *For* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i(b) = 0$, *we have*

$$T_i^{-1}({}^i\pi G^{\mathrm{low}}(b)) = \pi^i(G^{\mathrm{low}}(\sigma_i^*(b))),$$
$$(1 - q_i^2)^{\langle h_i, \mathrm{wt}\, b\rangle} T_i^{-1} G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\sigma_i^* b).$$

We note that the constant term $(1 - q_i^2)^{\langle h_i, \mathrm{wt}\, b\rangle}$ depends on the choice of nondegenerate bilinear form on $\boldsymbol{U}_q^-(\mathfrak{g})$.

2D. ***Poincaré–Birkhoff–Witt bases.*** Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, where $\{s_i \mid i \in I\}$ is the set of simple reflections associated with $i \in I$, and let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function. For a Weyl group element w, let

$$I(w) := \{(i_1, i_2, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid s_{i_1} \cdots s_{i_{\ell(w)}} = w\}$$

be the set of reduced words of w.

2D1. Let $\Delta = \Delta_+ \sqcup \Delta_-$ be the root system of the Kac–Moody Lie algebra $\mathfrak{g}$ and the decomposition into positive and negative roots.

For a Weyl group element $w \in W$, we set

$$\Delta_+(\leq w) := \Delta_+ \cap w\Delta_- = \{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_-\},$$
$$\Delta_+(> w) := \Delta_+ \cap w\Delta_+ = \{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_+\}.$$

It is well known that $\Delta_+(\leq w)$ and $\Delta_+(>w)$ are bracket closed; that is, for $\alpha, \beta \in \Delta_+(\leq w)$ (resp. $\alpha, \beta \in \Delta_+(>w)$) with $\alpha+\beta \in \Delta_+$, we have $\alpha+\beta \in \Delta_+(\leq w)$ (resp. $\in \Delta_+(>w)$).

For a reduced word $\boldsymbol{i} = (i_1, i_2, \dots, i_\ell) \in I(w)$, we define positive roots $\beta_{\boldsymbol{i},k}$ $(1 \leq k \leq \ell)$ by the formula

$$\beta_{\boldsymbol{i},k} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leq k \leq \ell).$$

It is well known that $\Delta_+(\leq w) = \{\beta_{\boldsymbol{i},k}\}_{1 \leq k \leq \ell}$ and we put a total order on $\Delta_+(\leq w)$. We note that the convex total order on $\Delta_+(\leq w)$ is associated with a reduced word $\boldsymbol{i} \in I(w)$.

2D2. For a Weyl group element $w \in W$ and a reduced word $\boldsymbol{i} = (i_1, i_2, \dots, i_\ell) \in I(w)$, we define the root vector $f_\epsilon(\beta_{\boldsymbol{i},k})$ associated with $\beta_{\boldsymbol{i},k} \in \Delta_+(\leq w)$ and a sign $\epsilon \in \{\pm 1\}$ by

$$f_\epsilon(\beta_{\boldsymbol{i},k}) := T^\epsilon_{i_1} T^\epsilon_{i_2} \cdots T^\epsilon_{i_{k-1}}(f_{i_k}),$$

and its divided power by

$$f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)} := T^\epsilon_{i_1} T^\epsilon_{i_2} \cdots T^\epsilon_{i_{k-1}}(f^{(c)}_{i_k}) \quad \text{for } c \in \mathbb{Z}_{\geq 0}.$$

Theorem 2.16 [Lusztig 1993, Propositions 40.2.1 and 41.1.3].

(1) *For $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, $\epsilon \in \{\pm 1\}$ and $\boldsymbol{c} \in \mathbb{Z}^\ell_{\geq 0}$, we set*

$$f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) := \begin{cases} f_\epsilon(\beta_{\boldsymbol{i},1})^{(c_1)} f_\epsilon(\beta_{\boldsymbol{i},2})^{(c_2)} \cdots f_\epsilon(\beta_{\boldsymbol{i},\ell})^{(c_\ell)} & \text{if } \epsilon = +1, \\ f_\epsilon(\beta_{\boldsymbol{i},\ell})^{(c_\ell)} f_\epsilon(\beta_{\boldsymbol{i},\ell-1})^{(c_{\ell-1})} \cdots f_\epsilon(\beta_{\boldsymbol{i},1})^{(c_1)} & \text{if } \epsilon = -1. \end{cases}$$

Then $\{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}^\ell_{\geq 0}}$ is linearly independent.

(2) *The subspace of* $U_q^-(\mathfrak{g})$ *spanned by* $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ *does not depend on the choice of reduced word* $\boldsymbol{i}\in I(w)$. *We denote this subspace by* $U_q^-(\leq w,\epsilon)$. *The basis* $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ *of* $U_q^-(\leq w,\epsilon)$ *is called the Poincaré–Birkhoff–Witt basis or the lower Poincaré–Birkhoff–Witt basis.*

Definition 2.17. For a Weyl group element $w\in W$, a reduced word $\boldsymbol{i}=(i_1,\dots,i_\ell)\in I(w)$ and $\boldsymbol{c}=(c_1,\dots,c_\ell)\in\mathbb{Z}_{\geq 0}^{\ell}$, we set

$$\xi(\boldsymbol{c},\boldsymbol{i}):=-\sum_{1\leq k\leq \ell} c_k\beta_{\boldsymbol{i},k}\in Q_-.$$

We also have the following characterization of $U_q^-(\leq w,\epsilon)$.

Theorem 2.18 [Beck et al. 1999, Proposition 2.3]. *For* $w\in W$, $\epsilon\in\{\pm 1\}$ *and* $\boldsymbol{i}=(i_1,\dots,i_\ell)\in I(w)$, *let*

$$U_q^-\cap T_{w^\epsilon}^{\epsilon}U_q^{\geq 0} = \begin{cases} U_q^-\cap T_{i_1}\cdots T_{i_\ell}U_q^{\geq 0}=\{x\in U_q^-\mid T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(x)\in U_q^{\geq 0}\} & \text{if } \epsilon=+1,\\ U_q^-\cap T_{i_1}^{-1}\cdots T_{i_\ell}^{-1}U_q^{\geq 0}=\{x\in U_q^-\mid T_{i_\ell}\cdots T_{i_1}(x)\in U_q^{\geq 0}\} & \text{if } \epsilon=-1.\end{cases}$$

Then the Poincaré–Birkhoff–Witt basis $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ *forms a* $\mathbb{Q}(q)$*-basis of* $U_q^-\cap T_{w^\epsilon}^{\epsilon}U_q^{\geq 0}$; *that is,* $U_q^-(\leq w,\epsilon)=U_q^-\cap T_{w^\epsilon}^{\epsilon}U_q^{\geq 0}$.

For the convenience of readers checking the notation, we give a proof of the above theorem.

Proof. Since the $\epsilon=-1$ case can be proved from the $\epsilon=+1$ case by applying the $*$-involution, it suffices for us to prove the claim for the $\epsilon=+1$ case. For $1\leq k\leq \ell$, we have

$$\begin{aligned} T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}T_{i_1}\cdots T_{i_{k-1}}(f_{i_k}^{(c_k)}) &= T_{i_\ell}^{-1}\cdots T_{i_k}^{-1}(f_{i_k}^{(c_k)})\\ &=(-1)^{c_k}T_{i_\ell}^{-1}\cdots T_{i_{k+1}}^{-1}(e_{i_k}^{(c_k)}k_{i_k}^{c_k}). \end{aligned}$$

Since $(i_k,\dots,i_\ell)$ is a reduced word, we have $T_{i_\ell}^{-1}\cdots T_{i_{k+1}}^{-1}(e_{i_k}^{(c_k)})\in U_q^+$. Hence

$$T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}T_{i_1}\cdots T_{i_{k-1}}(f_{i_k}^{(c_k)})\in U_q^{\geq 0}.$$

So the inclusion $U_q^-(\leq w,\epsilon)\subset U_q^-\cap T_{w^\epsilon}^{\epsilon}U_q^{\geq 0}$ is shown, and it suffices to prove the opposite inclusion, that is, that the Poincaré–Birkhoff–Witt basis $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ spans $U_q^-\cap T_{w^\epsilon}^{\epsilon}U_q^{\geq 0}$.

Let $x\in U_q^-\cap T_wU_q^{\geq 0}$ be a homogeneous element. We write it as the sum

$$x=\sum_{c_1} f_{i_1}^{(c_1)}x_{c_1}$$

with $x_{c_1} \in \boldsymbol{U}_q^- \cap T_{i_1}\boldsymbol{U}_q^-$. Then we have $T_{i_1}^{-1}(x_{c_1}) \in \boldsymbol{U}_q^-$. So we write it as the sum

$$T_{i_1}^{-1}(x_{c_1}) = \sum_{c_2} f_{i_2}^{(c_2)} x_{c_1,c_2}.$$

Repeating this process, we obtain elements $x_{c_1,\dots,c_k} \in \boldsymbol{U}_q^- \cap T_{i_k}\boldsymbol{U}_q^-$ for $1 \le k \le \ell$ and

$$T_{i_k}^{-1}(x_{c_1,\dots,c_k}) = \sum_{c_{k+1}} f_{i_{k+1}}^{(c_{k+1})} x_{c_1,\dots,c_k,c_{k+1}}$$

for $1 \le k < \ell$. Then we obtain

$$\begin{aligned}
&T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(x)\\
&\quad = \sum_{c_1} T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(x_{c_1})\\
&\quad = \sum_{c_1,c_2} T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_\ell}^{-1}\cdots T_{i_2}^{-1}(f_{i_2}^{(c_2)}) T_{i_\ell}^{-1}\cdots T_{i_2}^{-1}(x_{c_1,c_2})\\
&\quad = \sum_{c_1,c_2,\dots,c_\ell} T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_\ell}^{-1}\cdots T_{i_2}^{-1}(f_{i_2}^{(c_2)}) \cdots T_{i_\ell}^{-1}(f_{i_\ell}^{(c_\ell)}) x_{c_1,\dots,c_\ell}.
\end{aligned}$$

By the assumption $x \in \boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\ge 0}$, the left-hand side is in $\boldsymbol{U}_q^{\ge 0}$. By the triangular decomposition and $T_{i_\ell}^{-1}\cdots T_{i_k}^{-1}(f_{i_k}^{(c_k)}) \in \boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\ge 0}$, we have that $x_{c_1,\dots,c_\ell} \in \boldsymbol{U}_q^- \cap \boldsymbol{U}_q^{\ge 0} = \mathbb{Q}(q)$. Hence we obtain

$$x = \sum_{c_1,\dots,c_\ell} x_{c_1,\dots,c_\ell} f_{+1}(\boldsymbol{c}, \boldsymbol{i}) \in \boldsymbol{U}_q^-(\le w, +1),$$

so $\boldsymbol{U}_q^- \cap T_{w^\epsilon}^{\epsilon} \boldsymbol{U}_q^{\ge 0} \subset \boldsymbol{U}_q^-(\le w, \epsilon)$. □

Remark 2.19. The stronger assertion for Lusztig's integral form is proved in [Beck et al. 1999, Proposition 2.3].

2D3. *Poincaré–Birkhoff–Witt basis and crystal basis.*

Theorem 2.20. *For $w \in W$, $\boldsymbol{i} \in (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$:*

(1) *We have $f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \in \mathscr{L}(\infty)$ and*

$$b_\epsilon(\boldsymbol{c}, \boldsymbol{i}) := f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(2) *The map $\mathbb{Z}_{\ge 0}^{\ell} \to \mathscr{B}(\infty)$ which is defined by $\boldsymbol{c} \mapsto b_\epsilon(\boldsymbol{c}, \boldsymbol{i})$ is injective. We denote the image by $\mathscr{B}(w, \epsilon)$, and this does not depend on the choice of reduced word $\boldsymbol{i} \in I(w)$.*

2D4.

Proposition 2.21 [Kimura 2012, Proposition 4.26(2)]. *For $c \ge 1$ and $1 \le k \le \ell$, let*

$$f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{\{c\}} = f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)} / \bigl(f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)}, f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)}\bigr).$$

Then we have $f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{\{c\}} = q_{i_k}^{c(c-1)/2} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^c \in \boldsymbol{B}^{\mathrm{up}}$.

Definition 2.22 (dual Poincaré–Birkhoff–Witt basis). For $w \in W$, $\boldsymbol{i} \in I(w)$ and $\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^{\ell}$, we set

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) := \frac{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})}{(f_\epsilon(\boldsymbol{c}, \boldsymbol{i}), f_\epsilon(\boldsymbol{c}, \boldsymbol{i}))},$$

and $\{f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^{\ell}}$ is called the dual Poincaré–Birkhoff–Witt basis or upper Poincaré–Birkhoff–Witt basis.

By the definition of the dual Poincaré–Birkhoff–Witt basis and the computation of $(f_\epsilon(\boldsymbol{c}, \boldsymbol{i}), f_\epsilon(\boldsymbol{c}, \boldsymbol{i}))$, we have

$$\begin{aligned}
&f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) \\
&= \begin{cases} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},2})^{\{c_2\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell})^{\{c_\ell\}} & \text{if } \epsilon = +1, \\ f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell})^{\{c_\ell\}} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell-1})^{\{c_{\ell-1}\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} & \text{if } \epsilon = -1 \end{cases} \\
&= \begin{cases} (1-q_{i_1}^2)^{\langle h_{i_1}, \xi(\boldsymbol{c}_{\ge 2}, \boldsymbol{i}_{\ge 2})\rangle} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} T_{i_1}^{\epsilon}\bigl(f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\ge 2},2})^{\{c_2\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\ge 2},\ell})^{\{c_\ell\}}\bigr) \\ \hfill \text{if } \epsilon = +1, \\ (1-q_{i_1}^2)^{\langle h_{i_1}, \xi(\boldsymbol{c}_{\ge 2}, \boldsymbol{i}_{\ge 2})\rangle} T_{i_1}^{\epsilon}\bigl(f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\ge 2},\ell})^{\{c_\ell\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\ge 2},2})^{\{c_2\}}\bigr) f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} \\ \hfill \text{if } \epsilon = -1, \end{cases}
\end{aligned}$$

where $\boldsymbol{c}_{\ge 2} = (c_2, \dots, c_\ell) \in \mathbb{Z}_{\ge 0}^{\ell-1}$, $w_{\ge 2} = s_{i_2} \cdots s_{i_\ell}$ and $\boldsymbol{i}_{\ge 2} = (i_2, \dots, i_\ell) \in I(w_{\ge 2})$.

Using the Levendorskii–Soibelman formula (see [Kimura 2012, Theorem 4.27]) and the definition of the dual canonical basis, we have the following result.

Theorem 2.23 [Kimura 2012, Theorems 4.25 and 4.29]. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. The Poincaré–Birkhoff–Witt basis satisfies the following properties*:

(1) *The subalgebra $\boldsymbol{U}_q^-(\le w, \epsilon)$ is compatible with the dual canonical basis*; *that is, there exists a subset $\mathscr{B}(\le w, \epsilon) := \mathscr{B}(\boldsymbol{U}_q^-(\le w, \epsilon)) \subset \mathscr{B}(\infty)$ such that*

$$\boldsymbol{U}_q^-(\le w, \epsilon) = \bigoplus_{b \in \mathscr{B}(\le w, \epsilon)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

(2) *The transition matrix between the dual Poincaré–Birkhoff–Witt basis and the dual canonical basis is triangular with* 1*'s on the diagonal with respect to the* (*left*) *lexicographic order $\le$ on $\mathbb{Z}_{\ge 0}^{\ell}$. More precisely, we have*

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) = G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) + \sum_{\boldsymbol{c}' < \boldsymbol{c}} d_{\boldsymbol{c},\boldsymbol{c}'}^{\boldsymbol{i}}(q) G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}', \boldsymbol{i}))$$

with

$$d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'}(q) := \big(f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i}), G^{\mathrm{low}}(b_{\epsilon}(\boldsymbol{c}',\boldsymbol{i}))\big) \in q\mathbb{Z}[q].$$

Remark 2.24. In the symmetric case, we note that it can be shown that

$$d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'}(q) = \big(f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i}), G^{\mathrm{low}}(b_{\epsilon}(\boldsymbol{c}',\boldsymbol{i}))\big) \in q\mathbb{Z}_{\geq 0}[q],$$

by the positivity of the (twisted) comultiplication with respect to the canonical basis and Proposition 2.21.

In particular, we obtain a proof of the positivity of the transition matrix from the canonical basis into the lower Poincaré–Birkhoff–Witt basis in simply laced type for an arbitrary reduced word of the longest element w_0 using the orthogonality of the (lower) Poincaré–Birkhoff–Witt basis.

For "adapted" reduced words, it was proved by Lusztig [1990, Corollary 10.7]. For an arbitrary reduced word, it was proved by Kato [2014, Theorm 4.17] using the categorification of the Poincaré–Birkhoff–Witt basis via the Khovanov–Lauda–Rouquier algebra. It was also proved by Oya [2015, Theorem 5.2].

3. Proof of the surjectivity

3A. ***Multiplication formula for $U_q^-(\leq w, \epsilon)$.*** For a Weyl group element w, a reduced word $\boldsymbol{i} \in I(w)$ and $0 \leq p < \ell$, we consider a subalgebra which is generated by

$$\{f^{\mathrm{up}}_{\epsilon}(\beta_{\boldsymbol{i},k})\}_{p+1\leq k\leq \ell}.$$

It can be shown that this subalgebra is also compatible with the dual canonical basis. This can be proved using the transition matrix between the dual Poincaré–Birkhoff–Witt basis and the dual canonical basis.

In this subsection, we give statements for the $\epsilon = +1$ case. We can obtain the corresponding claims for the $\epsilon = -1$ case by applying the $*$-involution. So we denote $f^{\mathrm{up}}_{\epsilon}(\beta_{\boldsymbol{i},k})$, $f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i})$, $b_{\epsilon}(\boldsymbol{c},\boldsymbol{i})$ by $f^{\mathrm{up}}(\beta_{\boldsymbol{i},k})$, $f^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i})$, $b(\boldsymbol{c},\boldsymbol{i})$, omitting ϵ.

Proposition 3.1. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. For $\boldsymbol{c} \in \mathbb{Z}^{\ell}_{\geq 0}$ and $0 \leq p < \ell$, we set*

$$\tau_{\leq p}(\boldsymbol{c}) := (c_1, \ldots, c_p, 0, \ldots, 0) \in \mathbb{Z}^{\ell}_{\geq 0},$$
$$\tau_{>p}(\boldsymbol{c}) := (0, \ldots, 0, c_{p+1}, \ldots, c_{\ell}) \in \mathbb{Z}^{\ell}_{\geq 0}.$$

Then we have

$$G^{\mathrm{up}}\big(b(\tau_{\leq p}(\boldsymbol{c})), \boldsymbol{i}\big) G^{\mathrm{up}}\big(b(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i})\big) \in G^{\mathrm{up}}\big(b(\boldsymbol{c},\boldsymbol{i})\big) + \sum_{\boldsymbol{d}<\boldsymbol{c}} q\mathbb{Z}[q] G^{\mathrm{up}}\big(b(\boldsymbol{d},\boldsymbol{i})\big).$$

Proof. By the transition from the dual canonical basis to the dual Poincaré–Birkhoff–Witt basis, we have

$$G^{\mathrm{up}}\big(b(\tau_{\leq p}(\boldsymbol{c}),\boldsymbol{i})\big) \in f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}),\boldsymbol{i}) + \sum_{\boldsymbol{d}_{\leq p}<\tau_{\leq p}(\boldsymbol{c})} q\mathbb{Z}[q] f^{\mathrm{up}}(\boldsymbol{d}_{\leq p},\boldsymbol{i}),$$

$$G^{\mathrm{up}}\big(b(\tau_{> p}(\boldsymbol{c}),\boldsymbol{i})\big) \in f^{\mathrm{up}}(\tau_{> p}(\boldsymbol{c}),\boldsymbol{i}) + \sum_{\boldsymbol{d}_{> p}<\tau_{> p}(\boldsymbol{c})} q\mathbb{Z}[q] f^{\mathrm{up}}(\boldsymbol{d}_{> p},\boldsymbol{i}),$$

and note that we have $\boldsymbol{d}_{\leq p} = \tau_{\leq p}(\boldsymbol{d}_{\leq p})$ and $\boldsymbol{d}_{>p} = \tau_{>p}(\boldsymbol{d}_{>p})$ by the Levendorskii–Soibelman formula in the right-hand sides.

Hence in the product of the right-hand sides, we have four kinds of terms:

$$\begin{aligned} f^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i}) &= f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}),\boldsymbol{i}) f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}),\boldsymbol{i}),\\ f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c})+\boldsymbol{d}_{>p},\boldsymbol{i}) &= f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}),\boldsymbol{i}) f^{\mathrm{up}}(\boldsymbol{d}_{>p},\boldsymbol{i}),\\ f^{\mathrm{up}}(\tau_{> p}(\boldsymbol{c})+\boldsymbol{d}_{\leq p},\boldsymbol{i}) &= f^{\mathrm{up}}(\boldsymbol{d}_{\leq p},\boldsymbol{i}) f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}),\boldsymbol{i}),\\ f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}+\boldsymbol{d}_{>p},\boldsymbol{i}) &= f^{\mathrm{up}}(\boldsymbol{d}_{\leq p},\boldsymbol{i}) f^{\mathrm{up}}(\boldsymbol{d}_{>p},\boldsymbol{i}). \end{aligned}$$

We note that $\tau_{\leq p}(\boldsymbol{c})+\boldsymbol{d}_{>p}<\boldsymbol{c}$, $\tau_{>p}(\boldsymbol{c})+\boldsymbol{d}_{\leq p}<\boldsymbol{c}$ and $\boldsymbol{d}_{\leq p}+\boldsymbol{d}_{>p}<\boldsymbol{c}$ by the construction. Hence, using the transition from the dual Poincaré–Birkhoff–Witt basis to the dual canonical basis, we obtain the claim. □

3B. ***Compatibility of $U_q^-(>w,\epsilon)$.*** For a Weyl group element, we consider the cofinite subset $\Delta_+ \cap w\Delta_+$ and corresponding quantum coordinate ring $\boldsymbol{U}_q^-(>w,\epsilon)$.

Definition 3.2. For $w \in W$ and $\epsilon \in \{\pm 1\}$, we set

$$\boldsymbol{U}_q^-(>w,\epsilon) = \boldsymbol{U}_q^- \cap T_{w^\epsilon}^{\epsilon}\boldsymbol{U}_q^-.$$

The following is the main result in this subsection.

Theorem 3.3. *For $w \in W$ and $\epsilon \in \{\pm 1\}$, $\boldsymbol{U}_q^-(>w,\epsilon)$ is compatible with the dual canonical basis; namely, $\boldsymbol{B}^{\mathrm{up}}(>w,\epsilon) := \boldsymbol{B}^{\mathrm{up}} \cap \boldsymbol{U}_q^-(>w,\epsilon)$ is a $\mathbb{Q}(q)$-basis of $\boldsymbol{U}_q^-(>w,\epsilon)$.*

The proof of this theorem occupies the rest of this subsection, and we give the characterization of the subset $\boldsymbol{B}^{\mathrm{up}}(>w,\epsilon)$.

3B1. We provide an alternative description of $\boldsymbol{U}_q^-(>w,\epsilon)$ which is more convenient for proving the compatibility of the dual canonical basis.

Proposition 3.4. *For $w \in W$, $\boldsymbol{i} = (i_1,\dots,i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$, we have*

$$\boldsymbol{U}_q^-(>w,\epsilon) = \boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon}\boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon}T_{i_2}^{\epsilon}\boldsymbol{U}_q^- \cap \cdots \cap T_{i_1}^{\epsilon}\cdots T_{i_\ell}^{\epsilon}\boldsymbol{U}_q^-.$$

In fact, the right-hand side does not depend on the choice of reduced word $\boldsymbol{i} \in I(w)$. The above proposition can be shown by the following lemmas.

Lemma 3.5. *For a Weyl group element $w \in W$, a reduced word $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and a homogeneous element $x \in \boldsymbol{U}_q^-$, there exists $x_{\boldsymbol{c}} \in \boldsymbol{U}_q^- \cap T_{i_\ell}\boldsymbol{U}_q^-$ for $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell$ with*

(3-1)

$$T_w^{-1}(x) = \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) \cdots T_{i_\ell}^{-1} T_{i_{\ell-1}}^{-1}(f_{i_{\ell-1}}^{(c_{\ell-1})}) T_{i_\ell}^{-1}(f_{i_\ell}^{(c_\ell)}) T_{i_\ell}^{-1}(x_{\boldsymbol{c}})$$

$$\in \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(e_{i_1}^{(c_1)}) \cdots T_{i_\ell}^{-1}(e_{i_{\ell-1}}^{(c_{\ell-1})}) e_{i_\ell}^{(c_\ell)} \boldsymbol{U}_q^{\leq 0}.$$

Proof. Following the proof in [Beck et al. 1999, Proposition 2.3], we proceed by induction on the length $\ell(w)$. By Lemma 2.7, we have the decomposition

$$x = \sum f_{i_1}^{(c_1)} x_{c_1} \quad \text{with } x_{c_1} \in \boldsymbol{U}_q^- \cap T_{i_1}\boldsymbol{U}_q^-.$$

So we have $T_{i_1}^{-1}(x_{c_1}) \in \boldsymbol{U}_q^- \cap T_{i_1}^{-1}\boldsymbol{U}_q^{-1}$. Applying $T_{i_1}^{-1}$ to x, we obtain

$$T_{i_1}^{-1}(x) = \sum_{c_1 \geq 0} T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_1}^{-1}(x_{c_1}),$$

so we obtain the claim for $\ell(w) = 1$.

By induction on the length ℓ, we assume that

$$T_{i_{\ell-1}}^{-1} \cdots T_{i_1}^{-1}(x) = \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_{\ell-1}}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) \cdots T_{i_{\ell-1}}^{-1}(f_{i_{\ell-1}}^{(c_{\ell-1})}) T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}})$$

with $T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}}) \in \boldsymbol{U}_q^- \cap T_{i_{\ell-1}}^{-1}\boldsymbol{U}_q^-$. Since $T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}}) \in \boldsymbol{U}_q^-$, we have the decomposition

$$T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}}) = \sum_{c_\ell \geq 0} f_{i_\ell}^{(c_\ell)} x_{c_1,\dots,c_\ell} \quad \text{with } x_{c_1,\dots,c_\ell} \in \boldsymbol{U}_q^- \cap T_{i_\ell}\boldsymbol{U}_q^-.$$

So we obtain the following claim:

$$T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(x) = \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) \cdots T_{i_\ell}^{-1} T_{i_{\ell-1}}^{-1}(f_{i_{\ell-1}}^{(c_{\ell-1})}) T_{i_\ell}^{-1}(f_{i_\ell}^{(c_\ell)}) T_{i_\ell}^{-1}(x_{c_1,\dots,c_\ell}).$$

The second claim is clear from the definition of $\{T_i\}$ and the defining relations of $\boldsymbol{U}_q$. □

Lemma 3.6. *If $\ell(s_i w) > \ell(w)$, we have $\boldsymbol{U}_q^- \cap T_{s_i w}\boldsymbol{U}_q^- \subset \boldsymbol{U}_q^- \cap T_i\boldsymbol{U}_q^-$.*

Proof. Let $(i_1, \dots, i_\ell)$ be a reduced word of w such that $(i, i_1, \dots, i_\ell)$ is a reduced word of $s_i w$. For a homogeneous element $x \in \boldsymbol{U}_q^-$, we decompose $x = \sum_{c \geq 0} f_i^{(c)} x_c$

with $x_c \in \boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$. So we have

$$T_i^{-1} x = \sum_{c \geq 0} T_i^{-1}(f_i^{(c)}) T_i^{-1}(x_c) \in \sum_{c \geq 0} e_i^{(c)} \boldsymbol{U}_q^{\leq 0}$$

with $T_i^{-1}(x_c) \in \boldsymbol{U}_q^- \cap T_i^{-1} \boldsymbol{U}_q^-$. Apply T_w^{-1} to both sides, we have

$$\begin{aligned} &T_w^{-1} T_i^{-1} x \\ &= \sum_{c \geq 0} T_w^{-1}(T_i^{-1}(f_i^{(c)})) T_w^{-1} T_i^{-1}(x_c) \\ &\in \sum_{(c, d_1, \dots, d_\ell) \in \mathbb{Z}_{\geq 0}^{\ell+1}} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(e_i^{(c)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(e_{i_1}^{(d_1)}) \cdots T_{i_\ell}^{-1}(e_{i_{\ell-1}}^{(d_{\ell-1})}) e_{i_\ell}^{(d_\ell)} \boldsymbol{U}_q^{\leq 0}. \end{aligned}$$

Suppose $x \in \boldsymbol{U}_q^- \cap T_{s_i w} \boldsymbol{U}_q^-$ is a homogeneous element; that is, $T_w^{-1} T_i^{-1} x \in \boldsymbol{U}_q^- \cap T_{s_i w}^{-1} \boldsymbol{U}_q^-$. Since

$$\left\{ T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(e_i^{(c)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(e_{i_1}^{(d_1)}) \cdots T_{i_\ell}^{-1}(e_{i_{\ell-1}}^{(d_{\ell-1})}) e_{i_\ell}^{(d_\ell)} \mid (c, d_1, \dots, d_\ell) \in \mathbb{Z}_{\geq 0}^{\ell+1} \right\}$$

is linearly independent by the assumption $\ell(s_i w) > \ell(w)$, we have $x_c = 0$ for $c > 0$. So we obtain $x = x_0 \in \boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$. □

Proof of Proposition 3.4. We proceed by induction on the length $\ell(w)$ of a Weyl group element. When $\ell(w) = 1$, this is tautological, so we have the claim. By the induction hypothesis, we can assume that

$$\boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- = \boldsymbol{U}_q^- \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$

Then we have

$$\begin{aligned} &\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- \\ &\qquad = \boldsymbol{U}_q^- \cap T_{i_1}(\boldsymbol{U}_q^- \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-) = \boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-. \end{aligned}$$

By Lemma 3.6, we obtain the claim

$$\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- = \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$ □

3B2. Let w be a Weyl group element and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ be a reduced word. Following Saito [1994, Lemma 4.1.3] and Baumann, Kamnitzer and Tingley [Baumann et al. 2014, Proposition 5.24], we define the Lusztig datum of $b \in \mathscr{B}(\infty)$ in direction $\boldsymbol{i} \in I(w)$ and $\epsilon \in \{\pm 1\}$ ($(\boldsymbol{i}, \epsilon)$-Lusztig datum for short).

Definition 3.7 ($(\boldsymbol{i}, \epsilon)$-Lusztig datum). For $w \in W$, $\boldsymbol{i} \in I(w)$ and $\epsilon \in \{\pm 1\}$, define

$$L_\epsilon(b, \boldsymbol{i}) = \begin{cases} \left(\varepsilon_{i_1}(b), \varepsilon_{i_2}(\hat{\sigma}_{i_1}^* b), \dots, \varepsilon_{i_\ell}(\hat{\sigma}_{i_{\ell-1}}^* \cdots \hat{\sigma}_{i_1}^* b)\right) \in \mathbb{Z}_{\geq 0}^\ell & \text{if } \epsilon = +1, \\ \left(\varepsilon_{i_1}^*(b), \varepsilon_{i_2}^*(\hat{\sigma}_{i_1} b), \dots, \varepsilon_{i_\ell}^*(\hat{\sigma}_{i_{\ell-1}} \cdots \hat{\sigma}_{i_1} b)\right) \in \mathbb{Z}_{\geq 0}^\ell & \text{if } \epsilon = -1. \end{cases}$$

By construction in Theorem 2.20, we have

$$\boldsymbol{c} = L_\epsilon(b_\epsilon(\boldsymbol{c}, \boldsymbol{i}), \boldsymbol{i})$$

for $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}$; that is, the map $b_\epsilon(-, \boldsymbol{i}) : \mathbb{Z}_{\geq 0}^{\ell} \to \mathscr{B}(\infty)$ is a section of the $(\boldsymbol{i}, \epsilon)$-Lusztig datum $L_\epsilon(-, \boldsymbol{i}) : \mathscr{B}(\infty) \to \mathbb{Z}_{\geq 0}^{\ell}$.

3B3. The following gives a characterization of $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$ in terms of the $(\boldsymbol{i}, \epsilon)$-Lusztig datum.

Theorem 3.8. *For $w \in W$ and $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$, we set*

$$\mathscr{B}(>w, \epsilon) = \{b \in \mathscr{B}(\infty) \mid L_\epsilon(b, \boldsymbol{i}) = 0\}.$$

Then we have

$$\boldsymbol{U}_q^-(>w, \epsilon) = \bigoplus_{b \in \mathscr{B}(>w, \epsilon)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

Proof. By Proposition 3.4, it suffices for us to prove the compatibility for the intersection $\boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon} \boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon} T_{i_2}^{\epsilon} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1}^{\epsilon} \cdots T_{i_\ell}^{\epsilon} \boldsymbol{U}_q^-$.

Since $\epsilon = -1$ can be obtained by applying the $*$-involution, we prove only the $\epsilon = +1$ case. We prove the claim by induction on the length $\ell(w)$. For $\ell(w) = 1$, it is the claim in Corollary 2.13. We consider the intersection

$$\boldsymbol{U}_q^- \cap T_{i_1}^{-1} \boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$

By the induction hypothesis, $\boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-$ is compatible with the dual canonical basis, and $\boldsymbol{U}_q^- \cap T_{i_1}^{-1} \boldsymbol{U}_q^-$ is also compatible with the dual canonical basis, so the intersection $\boldsymbol{U}_q^- \cap T_{i_1}^{-1} \boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-$ is compatible with the dual canonical basis. Applying Theorem 2.15, we obtain the claim for $\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \boldsymbol{U}_q^-$. Since

$$\begin{aligned}\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \boldsymbol{U}_q^- & \\ = \boldsymbol{U}_q^- \cap T_{i_1} \big(\boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-\big),&\end{aligned}$$

we obtain the description of $\boldsymbol{B}^{\mathrm{up}}(>w, +1)$. □

3C. *Multiplication formula between $\boldsymbol{B}^{\mathrm{up}}(\leq w, \epsilon)$ and $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$.*

3C1. We generalize the (special cases of the) formula in Theorem 2.10 using the dual canonical basis $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$.

Theorem 3.9. *For $b \in \mathscr{B}(>w, \epsilon)$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}$, we have*

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) G^{\mathrm{up}}(b) \in G^{\mathrm{up}}(\nabla_{\boldsymbol{i},\epsilon}^{\boldsymbol{c}}(b)) + \sum_{L_\epsilon(b', \boldsymbol{i}) < \boldsymbol{c}} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = +1,$$

$$G^{\mathrm{up}}(b) f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) \in G^{\mathrm{up}}(\nabla_{\boldsymbol{i},\epsilon}^{\boldsymbol{c}}(b)) + \sum_{L_\epsilon(b', \boldsymbol{i}) < \boldsymbol{c}} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = -1,$$

where

$$\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)=\begin{cases}\tilde{f}_{i_1}^{c_1}\sigma_{i_1}\cdots\tilde{f}_{i_{\ell-1}}^{c_{\ell-1}}\sigma_{i_{\ell-1}}\tilde{f}_{i_\ell}^{c_\ell}\sigma_{i_\ell}\sigma^*_{i_\ell}\cdots\sigma^*_{i_1}(b) & \text{if } \epsilon=+1,\\ \tilde{f}_{i_1}^{*c_1}\sigma^*_{i_1}\cdots\tilde{f}_{i_{\ell-1}}^{*c_{\ell-1}}\sigma^*_{i_{\ell-1}}\tilde{f}_{i_\ell}^{*c_\ell}\sigma^*_{i_\ell}\sigma_{i_\ell}\cdots\sigma_{i_1}(b) & \text{if } \epsilon=-1.\end{cases}$$

Proof. We proceed by induction on the length $\ell(w)$ of a Weyl group element. Since $\epsilon=-1$ can be obtained by applying the $*$-involution, it suffices for us to prove the $\epsilon=+1$ case. Let $w_{\geq 2}=s_{i_2}\cdots s_{i_\ell}\in W$ and $\boldsymbol{i}_{\geq 2}=(i_2,\dots,i_\ell)\in I(w_{\geq 2})$. Let $b\in\mathscr{B}(\infty)$ with $L_{+1}(b,\boldsymbol{i})=0$; that is, we have

$$\big(\varepsilon_{i_1}(b),\varepsilon_{i_2}(\sigma^*_{i_1}b),\dots,\varepsilon_{i_\ell}(\sigma^*_{i_\ell}\cdots\sigma^*_{i_1}b)\big)=(0,\dots,0).$$

So let $b_{\geq 2}:=\sigma^*_{i_1}b$; then we have

$$\begin{aligned}L_{+1}(b_{\geq 2},\boldsymbol{i}_{\geq 2})&=\big(\varepsilon_{i_2}(b_{\geq 2}),\dots,\varepsilon_{i_\ell}(\sigma^*_{i_\ell}\cdots\sigma^*_{i_2}b_{\geq 2})\big)\\&=\big(\varepsilon_{i_2}(\sigma^*_{i_1}b),\dots,\varepsilon_{i_\ell}(\sigma^*_{i_\ell}\cdots\sigma^*_{i_2}\sigma^*_{i_1}b)\big)=(0,\dots,0)\in\mathbb{Z}_{\geq 0}^{\ell-1}\end{aligned}$$

by definition of the Lusztig datum.

By the induction hypothesis for $w_{\geq 2}\in W$ and $\boldsymbol{i}_{\geq 2}\in I(w_{\geq 2})$, we have

$$f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})G^{\mathrm{up}}(b_{\geq 2})-G^{\mathrm{up}}\big(\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},\epsilon}(b_{\geq 2})\big)\in\sum_{L_\epsilon(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}}q\mathbb{Z}[q]G^{\mathrm{up}}(b'_{\geq 2})$$

with $\boldsymbol{c}_{\geq 2}=(c_2,\dots,c_\ell)\in\mathbb{Z}_{\geq 0}^{\ell-1}$. Since $\boldsymbol{U}_q^-\cap T_{i_1}^{-1}\boldsymbol{U}_q^-$ is spanned by the dual canonical basis $\{G^{\mathrm{up}}(b)\mid\varepsilon^*_{i_1}(b)=0\}$ and since $f^{\mathrm{up}}_\epsilon(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})\in\boldsymbol{U}_q^-\cap T_{i_1}^{-1}\boldsymbol{U}_q^-$ and $G^{\mathrm{up}}(b_{\geq 2})\in\boldsymbol{U}_q^-\cap T_{i_1}^{-1}\boldsymbol{U}_q^-$, we obtain $\varepsilon^*_{i_1}\big(\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},\epsilon}(b_{\geq 2})\big)=0$ and $\varepsilon^*_{i_1}(b'_{\geq 2})=0$.

We have

$$\begin{aligned}&f^{\mathrm{up}}_{+1}(\boldsymbol{c},\boldsymbol{i})G^{\mathrm{up}}(b)\\&\quad=(1-q_{i_1}^2)^{\langle h_{i_1},\xi(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})+\mathrm{wt}(b_{\geq 2})\rangle}f_{i_1}^{\{c_1\}}T_{i_1}\big(f^{\mathrm{up}}_\epsilon(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})G^{\mathrm{up}}(b_{\geq 2})\big)\\&\quad\in(1-q_{i_1}^2)^{\langle h_{i_1},\xi(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})+\mathrm{wt}(b_{\geq 2})\rangle}f_{i_1}^{\{c_1\}}\\&\qquad\qquad\times T_{i_1}\Bigg(G^{\mathrm{up}}\big(\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big)+\sum_{L_{+1}(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}}q\mathbb{Z}[q]G^{\mathrm{up}}(b'_{\geq 2})\Bigg)\\&\quad=f_{i_1}^{\{c_1\}}\Bigg(G^{\mathrm{up}}\big(\sigma_{i_1}\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big)+\sum_{L_{+1}(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}}q\mathbb{Z}[q]G^{\mathrm{up}}(\sigma_{i_1}b'_{\geq 2})\Bigg).\end{aligned}$$

We note that

$$\begin{aligned}\tilde{f}_{i_1}^{c_1}\sigma_{i_1}\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})&=\tilde{f}_{i_1}^{c_1}\sigma_{i_1}\tilde{f}_{i_2}^{c_1}\sigma_{i_2}\cdots\tilde{f}_{i_{\ell-1}}^{c_{\ell-1}}\sigma_{i_{\ell-1}}\tilde{f}_{i_\ell}^{c_\ell}\sigma_{i_\ell}\sigma^*_{i_\ell}\cdots\sigma^*_{i_2}(b_{\geq 2})\\&=\nabla^{\boldsymbol{c}}_{\boldsymbol{i},+1}(b)\end{aligned}$$

and

$$f_{i_1}^{\{c_1\}} G^{\mathrm{up}}\big(\sigma_{i_1} \nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big) \in G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},+1}(b)) + \sum_{\varepsilon_{i_1}(b'')<c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b''),$$

$$f_{i_1}^{\{c_1\}} G^{\mathrm{up}}(\sigma_{i_1} b'_{\geq 2}) \in G^{\mathrm{up}}(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}) + \sum_{\varepsilon_{i_1}(b'')<c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b'').$$

By Theorem 2.10,

$$f_{i_1}^{\{c_1\}} \Bigg(G^{\mathrm{up}}\big(\sigma_{i_1} \nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big) + \sum_{L_{+1}(b'_{\geq 2}, \boldsymbol{i}_{\geq 2})<c_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}(\sigma_{i_1} b'_{\geq 2}) \Bigg)$$

can be written in the form

$$G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},+1}(b)) + \sum_{L_{+1}(b'_{\geq 2}, \boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}) + \sum_{\varepsilon_{i_1}(b'')<c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b'').$$

Since we have $(c'_2, \dots, c'_\ell) = L_{+1}(b'_{\geq 2}, \boldsymbol{i}_{\geq 2}) < \boldsymbol{c}_{\geq 2}$, we obtain

$$L_{+1}(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}, \boldsymbol{i}) = (c_1, c'_2, \dots, c'_\ell) < \boldsymbol{c}$$

and we have

$$L_{+1}(b'', \boldsymbol{i}) = (\varepsilon_{i_1}(b''), \dots) < L_{+1}(b, \boldsymbol{i}) = (c_1, c_2, \dots, c_\ell)$$

because $\varepsilon_{i_1}(b'') < c_1$. We obtain the claim. □

Using the transition in Theorem 2.23(2) from the Poincaré–Birkhoff–Witt basis to the dual canonical basis, we obtain the following multiplicity-free result.

Theorem 3.10. *Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. For $\boldsymbol{c} \in \mathbb{Z}^{\ell}_{\geq 0}$ and $b \in \mathscr{B}(>w, \epsilon)$, we have*

$$G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) G^{\mathrm{up}}(b) \in G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)) + \sum_{L_\epsilon(b',\boldsymbol{i})<c} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = +1,$$

$$G^{\mathrm{up}}(b) G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) \in G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)) + \sum_{L_\epsilon(b',\boldsymbol{i})<c} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = -1.$$

3C2.

Definition 3.11. Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. We define maps $\tau_{\leq w,\epsilon} : \mathscr{B}(\infty) \to \mathscr{B}(\leq w, \epsilon)$ and $\tau_{>w,\epsilon} : \mathscr{B}(\infty) \to \mathscr{B}(>w, \epsilon)$ by

$$\tau_{\leq w,\epsilon}(b) = b_\epsilon(L_\epsilon(b, \boldsymbol{i}), \boldsymbol{i}),$$

$$\tau_{>w,\epsilon}(b) = \begin{cases} \sigma_{i_1} \cdots \sigma_{i_\ell} \hat{\sigma}^*_{i_\ell} \cdots \hat{\sigma}^*_{i_1}(b) & \text{if } \epsilon = +1, \\ \sigma^*_{i_1} \cdots \sigma^*_{i_\ell} \hat{\sigma}_{i_\ell} \cdots \hat{\sigma}_{i_1}(b) & \text{if } \epsilon = -1. \end{cases}$$

Proposition 3.12. *We have a bijection as sets*:

$$\Omega_w := (\tau_{\leq w,\epsilon}, \tau_{>w,\epsilon}) : \mathscr{B}(\infty) \to \mathscr{B}(\leq w, \epsilon) \times \mathscr{B}(>w, \epsilon).$$

We prove the multiplication property of the dual canonical basis elements between $\boldsymbol{U}_q^-(\leq w, \epsilon)$ and $\boldsymbol{U}_q^-(>w, \epsilon)$.

Theorem 3.13. *Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. For $b \in \mathscr{B}(\infty)$,*

$$G^{\mathrm{up}}(\tau_{\leq w,\epsilon}(b)) G^{\mathrm{up}}(\tau_{>w,\epsilon}(b)) \in G^{\mathrm{up}}(b) + \sum_{L_\epsilon(b',\boldsymbol{i}) < L_\epsilon(b,\boldsymbol{i})} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \ \text{if } \epsilon = +1,$$

$$G^{\mathrm{up}}(\tau_{>w,\epsilon}(b)) G^{\mathrm{up}}(\tau_{\leq w,\epsilon}(b)) \in G^{\mathrm{up}}(b) + \sum_{L_\epsilon(b',\boldsymbol{i}) < L_\epsilon(b,\boldsymbol{i})} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \ \text{if } \epsilon = -1.$$

Proof. Since $\epsilon = -1$ can be obtained by applying the $*$-involution, it suffices for us to prove the $\epsilon = +1$ case. We proceed by induction on the length $\ell(w)$.

First we have

$$\begin{aligned} &G^{\mathrm{up}}(b) - f_{i_1}^{\{\varepsilon_{i_1}(b)\}} G^{\mathrm{up}}\big(\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b\big) \\ &= G^{\mathrm{up}}(b) - (1-q_{i_1}^2)^{\langle h_{i_1}, \mathrm{wt}(\hat{\sigma}_{i_1}^* b)\rangle} f_{i_1}^{\{\varepsilon_{i_1}(b)\}} T_{i_1} G^{\mathrm{up}}(\hat{\sigma}_{i_1}^* b) \in \sum_{\varepsilon_{i_1}(b') < \varepsilon_{i_1}(b)} q\mathbb{Z}[q] G^{\mathrm{up}}(b'). \end{aligned}$$

By Theorem 3.10, we only have to compute the product

$$(1-q_i^2)^{\langle h_i, \mathrm{wt}(\hat{\sigma}_i^* b)\rangle} f_i^{\{\varepsilon_i(b)\}} T_i G^{\mathrm{up}}(\hat{\sigma}_i^* b) \times G^{\mathrm{up}}(\tau_{>w,+1}(b)).$$

We note that

$$G^{\mathrm{up}}(\tau_{>w,+1}(b)) = (1-q_i^2)^{\langle h_{i_1}, \mathrm{wt}(\tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b))\rangle} T_{i_1} G^{\mathrm{up}}(\tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)),$$

where $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$.

By the induction hypothesis for $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$ and $\boldsymbol{i}_{\geq 2} = (i_2, \dots, i_\ell) \in I(w_{\geq 2})$,

$$\begin{aligned} &G^{\mathrm{up}}(\hat{\sigma}_{i_1}^* b) - G^{\mathrm{up}}\big(\tau_{\leq w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) \\ &\qquad \in \sum_{L_{+1}(b'',\boldsymbol{i}_{\geq 2}) < L_{+1}(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\geq 2})} q\mathbb{Z}[q] G^{\mathrm{up}}(b''). \end{aligned}$$

Applying $(1-q_{i_1}^2)^{\langle h_{i_1}, \mathrm{wt}(\hat{\sigma}_{i_1}^* b)\rangle} T_{i_1}$, we obtain

$$\begin{aligned} &G^{\mathrm{up}}(\sigma_{i_1} \hat{\sigma}_{i_1}^* b) - G^{\mathrm{up}}\big(\sigma_{i_1} \tau_{\leq w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1} \tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) \\ &\qquad \in \sum_{L_{+1}(b'',\boldsymbol{i}_{\geq 2}) < L_{+1}(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\geq 2})} q\mathbb{Z}[q] G^{\mathrm{up}}(\sigma_{i_1} b''). \end{aligned}$$

We note that $\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b = \sigma_{i_1}\hat{\sigma}_{i_1}^* b$. Multiplying the second term on the left by $f_{i_1}^{\{\varepsilon_{i_1}(b)\}}$, we have

$$\begin{aligned}&f_{i_1}^{\{\varepsilon_{i_1}(b)\}} G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big)\\ &\in G^{\mathrm{up}}\big(\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) + \sum_{\varepsilon_{i_1}(b') < \varepsilon_{i_1}(b)} q\mathbb{Z}[q] G^{\mathrm{up}}(b').\end{aligned}$$

Then we obtain

$$\begin{aligned}&f_{i_1}^{\{\varepsilon_{i_1}(b)\}}\Big(G^{\mathrm{up}}\big(\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b\big) - G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big)\Big)\\ &\in \sum_{\varepsilon_{i_1}(b') < \varepsilon_{i_1}(b)} q\mathbb{Z}[q] G^{\mathrm{up}}(b') + \sum_{L_{+1}(b'', \boldsymbol{i}_{\ge 2}) < L_{+1}(\hat{\sigma}_{i_1}^* b, \boldsymbol{i}_{\ge 2})} q\mathbb{Z}[q] G^{\mathrm{up}}\big(\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1} b''\big).\end{aligned}$$

By the construction, $\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b) = \tau_{\le w}(b)$ and $\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b) = \tau_{> w}(b)$; hence we obtain the claim. $\square$

3D. *Application.* We give a slight refinement of Lusztig's result [1996, Proposition 8.3] in the dual canonical basis. The following can be shown in a similar manner using the multiplicity-free property of the multiplications of a triple of the dual canonical basis elements, so we only state the claims.

Theorem 3.14. *Let w be a Weyl group element, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $p \in [0, \ell]$ be an integer. We consider the intersection*

$$\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^- = (\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^-) \cap (\boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-).$$

(1) *The subalgebra*

$$\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-$$

is compatible with the dual canonical basis; *that is*, *there exists a subset*

$$\mathscr{B}(\boldsymbol{U}_q^- \cap T_{i_{p+1}} \cdots T_{i_\ell} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-) \subset \mathscr{B}(\infty)$$

such that

$$\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^- = \bigoplus_{b \in \mathscr{B}\left(\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-\right)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

(2) *Multiplication in $\boldsymbol{U}_q^-(\mathfrak{g})^{\mathrm{up}}_{\mathcal{A}}$ defines an isomorphism of free $\mathcal{A}$-modules*:

$$\begin{aligned}&\big(\boldsymbol{U}_q^-(s_{i_{p+1}} \cdots s_{i_\ell}, +1)\big)^{\mathrm{up}}_{\mathcal{A}} \otimes_{\mathcal{A}} \big(\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_\ell}} \boldsymbol{U}_q^-\big)^{\mathrm{up}}_{\mathcal{A}}\\ &\qquad \otimes_{\mathcal{A}} \big(\boldsymbol{U}_q^-(s_{i_p} \cdots s_{i_1}, -1)\big)^{\mathrm{up}}_{\mathcal{A}} \to \boldsymbol{U}_q^-,\end{aligned}$$

where

$$(U_q^- \cap T_{i_{p+1}} \cdots T_{i_\ell} U_q^- \cap T_{i_p}^{-1} \cdots T_{i_1}^{-1} U_q^-)_{\mathcal{A}}^{\mathrm{up}} = U_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_{i_{p+1}} \cdots T_{i_\ell} U_q^- \cap T_{i_p}^{-1} \cdots T_{i_1}^{-1} U_q^-.$$

Acknowledgements

This work was done while the author was visiting the Institut de Recherche Mathématique Avancée (Strasbourg) with support from the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers "Mathematical Science of Symmetry, Topology and Moduli, Evolution of International Research Network based on OCAMI". The author thanks Toshiyuki Tanisaki for fruitful discussions on the topic. The author is also grateful to Pierre Baumann and Fan Qin for discussions during the stay in Strasbourg. The author thanks Pierre Baumann and Hironori Oya for reading the manuscript and giving comments.

References

[Baumann et al. 2014] P. Baumann, J. Kamnitzer, and P. Tingley, "Affine Mirković–Vilonen polytopes", *Publ. Math. Inst. Hautes Études Sci.* **120** (2014), 113–205. MR Zbl

[Beck et al. 1999] J. Beck, V. Chari, and A. Pressley, "An algebraic characterization of the affine canonical basis", *Duke Math. J.* **99**:3 (1999), 455–487. MR Zbl

[Berenstein and Greenstein 2014] A. Berenstein and J. Greenstein, "Double canonical bases", preprint, 2014. Some later versions have different numbering. arXiv

[Kashiwara 1991] M. Kashiwara, "On crystal bases of the Q-analogue of universal enveloping algebras", *Duke Math. J.* **63**:2 (1991), 465–516. MR Zbl

[Kashiwara 1993a] M. Kashiwara, "Global crystal bases of quantum groups", *Duke Math. J.* **69**:2 (1993), 455–485. MR Zbl

[Kashiwara 1993b] M. Kashiwara, "The crystal base and Littelmann's refined Demazure character formula", *Duke Math. J.* **71**:3 (1993), 839–858. MR Zbl

[Kashiwara 2012] M. Kashiwara, "Notes on parameters of quiver Hecke algebras", *Proc. Japan Acad. Ser. A Math. Sci.* **88**:7 (2012), 97–102. MR Zbl

[Kato 2014] S. Kato, "Poincaré–Birkhoff–Witt bases and Khovanov–Lauda–Rouquier algebras", *Duke Math. J.* **163**:3 (2014), 619–663. MR Zbl

[Kimura 2012] Y. Kimura, "Quantum unipotent subgroup and dual canonical basis", *Kyoto J. Math.* **52**:2 (2012), 277–331. MR Zbl

[Lusztig 1990] G. Lusztig, "Canonical bases arising from quantized enveloping algebras", *J. Amer. Math. Soc.* **3**:2 (1990), 447–498. MR Zbl

[Lusztig 1993] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics **110**, Birkhäuser, Boston, 1993. MR Zbl

[Lusztig 1996] G. Lusztig, "Braid group action and canonical bases", *Adv. Math.* **122**:2 (1996), 237–261. MR Zbl

[Oya 2015] H. Oya, "Representations of quantized function algebras and the transition matrices from canonical bases to PBW bases", preprint, 2015. arXiv

[Saito 1994] Y. Saito, "PBW basis of quantized universal enveloping algebras", *Publ. Res. Inst. Math. Sci.* **30**:2 (1994), 209–232. MR Zbl

[Tanisaki 2015] T. Tanisaki, "Modules over quantized coordinate algebras and PBW-bases", preprint, 2015. arXiv

Received July 7, 2015. Revised May 28, 2016.

YOSHIYUKI KIMURA
DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE
KOBE UNIVERSITY
1-1 ROKKODAI, NADA-KU
KOBE 657-8501
JAPAN
ykimura@math.kobe-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 1 January 2017
