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ALMOST EVERYWHERE CONVERGENCE FOR MODIFIED BOCHNER–RIESZ MEANS AT THE CRITICAL INDEX FOR $p \geq 2$

MARCO ANNONI

Boundedness for a maximal modified Bochner–Riesz operator between weighted L^2 spaces is proved. As a consequence, we have sufficient conditions for a.e. convergence of the modified Bochner–Riesz means at the critical exponent $p_\lambda = 2n/(n-2\lambda-1)$.

1. Introduction

This paper contains the results proved in the author's doctoral dissertation [Annoni 2010] and referenced by S. Lee and A. Seeger [2015], but yet unpublished in a mathematical journal. For λ, $R > 0$, let B_R^λ denote the Bochner–Riesz operators and m_λ the Fourier multipliers introduced in [Bochner 1936]:

$$B_R^\lambda(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) m_\lambda\left(\frac{|\xi|}{R}\right) e^{2\pi i \xi x}\, d\xi, \quad m_\lambda(t) = (1-t^2)_+^\lambda.$$

For $p < 2$, results related to almost everywhere convergence and maximal operators have been proved by Tao [1998; 2002], Ashurov [1983], and Ahmedov, Ashurov, and Mahmud [Ashurov et al. 2010]. For $p \geq 2$, partial results on almost everywhere convergence of $B_R^\lambda(f)$ to f as $R \to \infty$ have been achieved in [Carbery 1983; Christ 1985]. Carbery, Rubio de Francia, and Vega [Carbery et al. 1988] obtained a.e. convergence in the range $2 \leq p < p_\lambda$ and $\lambda > 0$.

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In this paper, the situation at the critical exponent $p_\lambda = 2n/(n-2\lambda-1)$ is studied by considering the modified Bochner–Riesz multipliers $m_{\lambda,\gamma}$

$$m_{\lambda,\gamma}(t) = \frac{(1-t^2)_+^\lambda}{(1-\log(1-t^2))^\gamma},$$

which were introduced by Seeger [1987]. Seeger [1996] showed that $m_{\lambda,\gamma}$ is an $L^{p_\lambda}(\mathbb{R}^2)$ multiplier for $\gamma > 1/p'_\lambda$ (where $1/p_\lambda + 1/p'_\lambda = 1$). His results easily extend to dimensions $n \geq 3$ when $\lambda \geq (n-1)/(2(n+1))$ and had already been proven to be sharp in [Seeger 1987] when $n = 2$.

In order to investigate for which values of γ the means $B_R^{\lambda,\gamma}$ defined via $m_{\lambda,\gamma}$ converge a.e. for functions in L^{p_λ}, we study the maximal operator $B_*^{\lambda,\gamma}$. The following theorem is my main result.

Theorem 1.1. *Let $1 < 1+2\lambda < n$ and $0 \leq \mu < 2\gamma - 2$. Then there is a constant $C = C(n,\lambda,\gamma,\mu)$ such that*

$$\int_{\mathbb{R}^n} |B_*^{\lambda,\gamma}(f)(x)|^2\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^2\,dx \tag{1}$$

for all $f \in L^2(\mathbb{R}^n, dx)$ and

$$\int_{\mathbb{R}^n} |B_*^{\lambda,\gamma}(f)(x)|^2 w_{\lambda,\mu}(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^2\, w_{\lambda,\mu}(x)\,dx \tag{2}$$

for all $f \in L^2(\mathbb{R}^n, w_{\lambda,\mu}(x)\,dx)$, where $w_{\lambda,\mu} = \omega_{\lambda,\mu}(|x|)$ and

$$\omega_{\lambda,\mu}(t) = \begin{cases} \dfrac{1}{t^{2\lambda+1}} & \text{if } 0 < t \leq 1, \\ \dfrac{1}{t^{2\lambda+1}(\log(et))^\mu} & \text{if } t > 1. \end{cases} \tag{3}$$

For $(2\lambda+1)/n < \mu$, we also have $L^{p_\lambda} \subseteq L^2 + L^2(w_{\lambda,\mu})$. Hence:

Corollary 1.2. *If $1 < 1+2\lambda < n$, $f \in L^{p_\lambda}(\mathbb{R}^n)$, and $\gamma > 1/p'_\lambda + 1/2$, we have*

$$\lim_{R\to\infty} B_R^{\lambda,\gamma}(f)(x) = f(x) \tag{4}$$

for almost every $x \in \mathbb{R}^n$. If $f \in L^p(\mathbb{R}^n)$ for $2 \leq p < p_\lambda$, then the condition $\gamma \geq 0$ suffices for (4) *to hold.*

When I first proved this result, it was natural to wonder whether the condition $\gamma > 1/p'_\lambda + 1/2$ was sharp. Lee, Rogers, and Seeger [Lee et al. 2014] have since proved among other things that, if

$$\frac{2(n+1)}{n-1} < p < \infty, \quad n \geq 2,$$

and $m \in B^2_{\alpha,q}$, then the maximal operator

$$M_m(f) := \sup_{t>0}\left|\left(\hat{f}\, m(t|\cdot|)\right)^\vee\right|$$

is bounded from $L^{p,q'}$ to L^p. This can be applied to $m = m_{\lambda,\gamma}$ to conclude that the condition $\gamma > 1/p'_\lambda + 1/2$ in Corollary 1.2 can be replaced by $\gamma > 1/p'_\lambda$, if we further assume $(n-1)/(2(n+1)) < \lambda$.

Lee and Seeger [2015] have gone much further, proving that a.e. convergence of

$$S_t(f) := \left(\hat{f}\, m_{\lambda,\gamma} \circ \rho(t(\cdot))\right)^\vee$$

to f (where ρ is an arbitrary homogeneous "distance" function, that is a homogeneous function that satisfies $\rho(\xi) > 0$ if $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\rho(0) = 0$) holds for every $f \in L^{p_\lambda,q}$ when $q \geq 1$ if and only if $\gamma > 1/q'$, for all $0 < \lambda < (n-1)/2$. For $q = p_\lambda$ and $\rho(\xi) = |\xi|$, this implies Corollary 1.2. In particular, they proved that the condition $\gamma > 1/q'$ is sharp.

The sufficiency of the condition $\gamma > 1/q'$ in [Lee and Seeger 2015] is presented as a consequence of a boundedness estimate between appropriate homogeneous Herz spaces — see [Baernstein and Sawyer 1985; Gilbert 1972] — of a maximal operator defined via an arbitrary quasiradial multiplier $h \circ \rho$, provided that h lies in an appropriate Besov space. A particular case of the same theorem also implies a characterization of boundedness for certain convolution operators on L^2 spaces that are weighted with power weights. In order to prove the sufficiency of the condition on γ, both of our papers use the approach of [Carbery et al. 1988], to some extent. However, much of my work is necessary to deal with the weight $w_{\lambda,\mu}$, that isn't homogenous. The first choice of Lee and Seeger was to keep working with a homogeneous weight, but to use the observation that, for $p > 2$, the space $L^{p,2}$ is embedded in $L^2(|x|^{-n(1-2/p)}\, dx)$. By sharpening the analysis in [loc. cit.], this idea would only have yielded their result for $q = 2$. They solved the problem for all q by using Herz spaces, embedding theorems, and innovations that were needed to work with a more general "distance" function ρ and multiplier h.

The necessity of the condition $\gamma > 1/q'$ starts with the reminder that the operators S_t $(t > 0)$ are naturally defined on the Schwartz class $\mathcal{S}$ and extended on bigger spaces by using density. So, they proved that each operator S_t is continuous from $\mathcal{S}$ — equipped with the $L^{p_\lambda,q}$ norm and topology — to $\mathcal{S}'$ only if $\gamma > 1/q'$.

This paper. The proof of Theorem 1.1 follows closely the idea developed in [Carbery et al. 1988], but accounts for the necessity to work with nonhomogeneous weights.

In Section 2, Theorem 1.1 is reduced to Lemma 2.1, which is in turn reduced to Lemma 5.2 in Section 5. Lemma 5.2 is proved in Section 6.

In Section 3, an upper bound is given for the Fourier transform of $w^{(1)}_{\lambda,\mu}$, which is $w_{\lambda,\mu}$ smoothened in a neighborhood of the spherical surface $\|x\| = 1$. An analytic continuation argument is needed to prove that the upper bound holds for all $0 < \lambda < (n-1)/2$. This upper bound will be used to prove Lemma 5.2.

In Section 4, a new weight $\widetilde{w}_{N,\lambda,\mu}$ is exhibited that is comparable to $1/w_{\lambda,\mu}$ and that has an algebraic form needed in the computations of Section 5.

In Section 5, Lemma 2.1 is reduced to Lemma 5.2. Lemma 5.2 contains weighted Fourier inequalities for the special weight used in this paper. It is crucial that the "constants" appearing in both such inequalities have a certain functional form with respect to the parameter t. So, general results such as those in [Benedetto and Heinig 2003] were not sufficient.

Section 6 contains the proofs of Lemma 5.2 and Corollary 1.2.

We shall refer to [Carbery et al. 1988] for every piece of the proof that doesn't differ significantly. Yet, the reader can find more details of the proof contained in that reference in [Grafakos 2014, Subsection 10.5.2].

2. Reduction of Theorem 1.1 to Lemma 2.1

We will only need to show (2), as the proof of (1) is contained in [Grafakos 2014] for the case $\gamma = 0$ (which implies it for all $\gamma \geq 0$). Let φ, ψ be smooth functions, supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\left[\frac{1}{8}, \frac{5}{8}\right]$ respectively, with values in [0, 1], that satisfy

$$\varphi(t) + \sum_{k=0}^{\infty} \psi\left(\frac{1-t}{2^{-k}}\right) = 1$$

for all $t \in [0, 1)$. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. We define $m_{\lambda,\gamma,00}(t) = m_{\lambda,\gamma}(te_1)\,\varphi(t)$ and

$$m_{\lambda,\gamma,k}(t) = 2^{k\lambda} m_{\lambda,\gamma}(te_1)\,\psi\left(\frac{1-t}{2^{-k}}\right), \quad k = 0, 1, 2, \dots \tag{5}$$

We define $\widetilde{m}_{\lambda,\gamma,k}$, $(S_{\lambda,\gamma,k})_t$, $(S_{\lambda,\gamma,k})_*$, and $G_{\lambda,\gamma,k}$ from $m_{\lambda,\gamma,k}$, analogous to how $\widetilde{m}^\delta$, S_t^δ, S_*^δ, and G^δ were defined from m^δ in [Carbery et al. 1988]. Similarly, we also define $(\widetilde{S}_{\lambda,\gamma,k})_t$, $(\widetilde{S}_{\lambda,\gamma,k})_*$, and $\widetilde{G}_{\lambda,\gamma,k}$ by using $\widetilde{m}_{\lambda,\gamma,k}$ instead of $m_{\lambda,\gamma,k}$. For $m_{\lambda,\gamma,k}$ we have the estimate

$$\sup_{0 \leq t \leq 1} \left| \frac{d^\ell}{dt^\ell} m_{\lambda,\gamma,k}(t) \right| \leq C_{\lambda,\gamma,\ell} \frac{2^{k\ell}}{k^\gamma} \tag{6}$$

for all $\ell \in \mathbb{Z}^+ \cup \{0\}$. As in [loc. cit.], these inequalities follow:

$$\|B_*^{\lambda,\gamma}\| \leq \|(S_{\lambda,\gamma,00})_*\| + \sum_{k=0}^{\infty} 2^{-k\lambda} \|(S_{\lambda,\gamma,k})_*\|, \tag{7}$$

$$\left\|(S_{\lambda,\gamma,k})_*(f)\right\|_{L^2(w_{\lambda,\mu})}^2 \leq 2^{k+1} \left\|G_{\lambda,\gamma,k}(f)\right\|_{L^2(w_{\lambda,\mu})} \left\|\widetilde{G}_{\lambda,\gamma,k}(f)\right\|_{L^2(w_{\lambda,\mu})}. \tag{8}$$

By reasoning as in [loc. cit.], one then shows without difficulty that the right-hand side in (8) can be controlled by the left-hand side of the inequality in the result we are about to state:

Lemma 2.1. *For $k > 4$ we have*

$$\int_{\mathbb{R}^n} \int_1^2 |(S_{\lambda,\gamma,k})_{a\,t}(f)(x)|^2 \,\frac{dt}{t}\, w_{\lambda,\mu}(x)\,dx \le C_{n,\lambda,\mu,\gamma,k} \int_{\mathbb{R}^n} |f(x)|^2 w_{\lambda,\mu}(x)\,dx$$

for all $a > 0$ and for all functions f in $L^2(w_{\lambda,\mu})$, with

$$C_{n,\lambda,\mu,\gamma,k} = C_{n,\lambda,\mu,\gamma} \frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}.$$

We need not to worry about $k \le 4$ because it is easily verified that $w_{\lambda,\mu}$ is an A_2 weight under the conditions of Theorem 1.1 and therefore

$$\|(S_{\lambda,\gamma,k})_*(f)\|^2_{L^2(w_{\lambda,\mu})} < \infty$$

for every k. Inequality (8) and Lemma 2.1 then imply:

$$\big\|(S_{\lambda,\gamma,k})_*\big\|_{L^2(w_{\lambda,\mu})\to L^2(w_{\lambda,\mu})} \le C'(n,\lambda,\gamma)\Big(\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\Big)^{1/2}. \tag{9}$$

So, the right-hand side of (7) is finite if $\mu < 2\gamma - 2$. Theorem 1.1 is now proved modulo Lemma 2.1.

3. An upper bound for $|\widehat{w}_{\lambda,\mu}|$

The main result of this section will be used in Section 6. Let $\theta \in C^\infty(\mathbb{R})$ satisfy $0 \le \theta \le 1$, $\operatorname{supp}(\theta) \subset \big[\frac{9}{10}, \frac{11}{10}\big]$, $\theta \equiv 1$ on $\big[\frac{19}{20}, \frac{21}{20}\big]$. Now define

$$\omega^{(1)}_{\lambda,\mu}(t) = \omega_{\lambda,\mu}(t)\big(1-\theta(t)\big) + \theta(t). \tag{10}$$

and $w^{(1)}_{\lambda,\mu}(x) = \omega^{(1)}_{\lambda,\mu}(|x|)$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Then $w^{(1)}_{\lambda,\mu}$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and $w^{(1)}_{\lambda,\mu} \approx_{\lambda,\mu} w_{\lambda,\mu}$; that is, $w^{(1)}_{\lambda,\mu}(x)$ and $w_{\lambda,\mu}(x)$ are comparable with comparability constant depending on λ and μ only. The goal of this section is to prove this result:

Theorem 3.1. *Let $w_{\lambda,\mu}$ and $w^{(1)}_{\lambda,\mu}$ be defined as above. Then for every λ satisfying $\frac{n-1}{4} < \lambda < \frac{n-1}{2}$ and every $\mu \ge 0$ there exists a constant $C_{n,\lambda,\mu}$ such that*

$$|\widehat{w}_{\lambda,\mu}(\xi)| \le \Omega_{\lambda,\mu}(\xi) := \begin{cases} C_{n,\lambda,\mu} \dfrac{1}{|\xi|^{n-2\lambda-1}\big(\log \frac{e}{|\xi|}\big)^{\mu}} & \text{if } |\xi| \le 1, \\ C_{n,\lambda,\mu} \dfrac{1}{|\xi|^{n-2\lambda-1}} & \text{if } |\xi| \ge 1, \end{cases} \tag{11}$$

and, for all λ satisfying $0 < \lambda < \frac{n-1}{2}$ and μ as above, there exists a constant $C'_{n,\lambda,\mu}$ such that

$$|\widehat{w}^{(1)}_{\lambda,\mu}(\xi)| \le C'_{n,\lambda,\mu}\,\Omega_{\lambda,\mu}(\xi) \tag{12}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof. We begin with the proof of (11) As $w_{\lambda,\mu}$ is radial, its Fourier transform is given by

$$\widehat{w}_{\lambda,\mu}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \omega_{\lambda,\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr,$$

where J_k denotes the k-th Bessel function. It is well known — see [Watson 1944] — that $|J_k(r)| \le C_k\, r^k$ when $r \le 2\pi$ and $|J_k(r)| \le C_k\, r^{-\frac{1}{2}}$ when $r \ge 2\pi$. We control $|\widehat{w}_{\lambda,\mu}(\xi)|$ in two cases:

Case 1: $\frac{1}{|\xi|} \le 1$. Then

$$|\widehat{w}_{\lambda,\mu}(\xi)| \le C_n\left(\int_0^{\frac{1}{|\xi|}} r^{-2\lambda-1+\frac{n-2}{2}+\frac{n}{2}}\, dr\right) + \frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_{\frac{1}{|\xi|}}^1 r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}\, dr\right) + \frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_1^\infty \frac{r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}}{(\log(er))^\mu}\, dr\right).$$

Case 2: $\frac{1}{|\xi|} \ge 1$. Then

$$|\widehat{w}_{\lambda,\mu}(\xi)| \le C_n\left(\int_0^1 r^{\frac{n-2}{2}+\frac{n}{2}-2\lambda-1}\, dr\right) + C_n\left(\int_1^{\frac{1}{|\xi|}} \frac{1}{(\log(er))^\mu} r^{\frac{n-2}{2}+\frac{n}{2}-2\lambda-1}\, dr\right) + \frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_{\frac{1}{|\xi|}}^\infty \frac{1}{(\log(er))^\mu} r^{-\frac{1}{2}+\frac{n}{2}-2\lambda-1}\, dr\right).$$

If $\lambda > \frac{n-1}{4}$ and $\lambda < \frac{n-1}{2}$, all integrals converge and (11) easily follows by using calculus.

The same holds with $w_{\lambda,\mu}$ replaced by $w^{(1)}_{\lambda,\mu}$ and the proof is almost identical. Then, an analytic continuation argument and the smoothness of $w^{(1)}_{\lambda,\mu}$ can be used to prove that (12) holds in the bigger range $0 < \lambda < \frac{n-1}{2}$. The argument involves many details that we omit but that may be split in two pieces.

In the first one, given any $\lambda' \in \big(0, \frac{n-1}{4}\big]$, we use more asymptotic estimates of the Bessel functions — see [Watson 1944] — and iterated integration by parts to rewrite the right-hand side of

$$\widehat{w}^{(1)}_{\lambda,\mu}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \omega^{(1)}_{\lambda,\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr \tag{13}$$

in a way that also is well defined when λ ranges in a complex neighborhood $\mathcal{O}_{\lambda'}$ of the real interval $\big(\lambda', \frac{n-1}{2}\big)$. We can call such extension $\tilde{u}_{\lambda,\mu}(\xi)$, and show that

$$|\tilde{u}_{\lambda,\mu}(\xi)| \le C'_{n,\lambda,\mu}\, \Omega_{\lambda,\mu}(\xi),$$

as in (12).

In the second one, for the same value of λ' and the same neighborhood $\mathcal{O}_{\lambda'}$, we use the dominated convergence theorem to prove that, for a given test function φ defined on $\mathbb{R}^n$, the right-hand side of

$$\int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\, w^{(1)}_{\lambda,\mu}(\xi)\, d\xi = \int_{\mathbb{R}^n} \varphi(\xi)\, \tilde{u}_{\lambda,\mu}(\xi)\, d\xi, \tag{14}$$

rewritten after the first piece of the argument, is holomorphic, hence analytic, on $\mathcal{O}_{\lambda'}$. It can be proved easily that the left-hand side of (14) is also analytic on $\mathcal{O}_{\lambda'}$. Since (14) holds when $\lambda \in \left(\frac{n-1}{4}, \frac{n-1}{2}\right)$, we conclude from the analytic continuation theorem that (14) also holds when $\lambda \in \left(\lambda', \frac{n-1}{2}\right)$. Then $\widehat{w}^{(1)}_{\lambda,\mu} = \tilde{u}_{\lambda,\mu}$, since φ is arbitrary. The arbitrariness of λ' concludes the proof. □

4. A useful weight comparable to $1/w_{\lambda,\mu}$

In this section we show that $1/w_{\lambda,\mu}$ is comparable to another weight which can be written in a more useful way for our purposes, a fact that will be used in the next section. More precisely, let $u_{\lambda,\mu}$ and $\widetilde{w}_{N,\lambda,\mu}$ be defined by:

$$u_{\lambda,\mu}(y) = \begin{cases} |y|^{-n-2\lambda-1}\left(\log \frac{e}{|y|}\right)^{\mu} & \text{if } |y| < 1, \\ |y|^{-n-2\lambda-1} & \text{if } |y| \geq 1. \end{cases} \tag{15}$$

$$\widetilde{w}_{N,\lambda,\mu}(x) = \int_{\mathbb{R}^n} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy, \tag{16}$$

where N is a large enough integer independent of x.

The goal of this section is to prove that there exist constants $C_{1,n,\lambda,\mu,N}$ and $C_{2,n,\lambda,\mu,N}$ such that

$$\frac{C_{1,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \leq \widetilde{w}_{N,\lambda,\mu}(x) \leq \frac{C_{2,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \tag{17}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Let us write $\widetilde{w}_{N,\lambda,\mu} = \widetilde{w}_{N,\lambda,\mu,1} + \widetilde{w}_{N,\lambda,\mu,2}$, where

$$\widetilde{w}_{N,\lambda,\mu,1}(x) = \int_{|y|\leq \frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy, \tag{18}$$

$$\widetilde{w}_{N,\lambda,\mu,2}(x) = \int_{|y|> \frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy. \tag{19}$$

Observe that in (18),

$$C_1|\langle x, y\rangle| \leq |e^{i\langle x,y\rangle} - 1| \leq |x|\,|y| \tag{20}$$

for an absolute constant $0 < C_1 < 1$. Now, we estimate $\widetilde{w}_{N,\lambda,\mu,1}$.

Case 1: $\frac{1}{|x|} \leq 1$. Given a positive constant $C > 0$, in view of (20),

$$\begin{aligned}\widetilde{w}_{N,\lambda,\mu,1}(x) &= \int_{|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N\, |y|^{-n-2\lambda-1}\Big(\log\frac{e}{|y|}\Big)^{\mu}\, dy\\ &\geq C_{C,N}\int_{\Omega(x)} |x|^N\, |y|^N\, |y|^{-n-2\lambda-1}\Big(\log\frac{e}{|y|}\Big)^{\mu}\, dy\\ &= C_{n,C,N}\, |x|^N\, \frac{|S^{n-1}|}{e^{2\lambda+1-N}}\int_{e|x|}^{\infty} s^{2\lambda-N}(\log s)^{\mu}\, ds,\end{aligned}$$

where $\Omega(x) = \big\{y : |y| \leq \frac{1}{|x|} \text{ and } C \leq \big|\big\langle \frac{x}{|x|}, \frac{y}{|y|}\big\rangle\big|\big\}$. In order for this integral to converge, we need $N > 2\lambda + 1$. Later we will also need N to be even. So, we set $N = N_\lambda := 2\lceil 2\lambda+1\rceil$. It easily follows that there exist constants $C_{n,\lambda,N}$ and $C_{\lambda,\mu,N}$ such that $\widetilde{w}_{N,\lambda,\mu,1}(x) \geq C_{n,\lambda,N}/w_{\lambda,\mu}(x)$ for all $x \in \mathbb{R}^n$ satisfying $|x| \geq C_{\lambda,\mu,N}$. An easier computation and (20) yield $\widetilde{w}_{N,\lambda,\mu,1}(x) \leq C'_{n,\lambda,N}/w_{\lambda,\mu}(x)$ in Case 1 for all $x \in \mathbb{R}^n$ satisfying $|x| \geq C_{\lambda,\mu,N}$. So, on $\{x \in \mathbb{R}^n : |x| \geq \max\{1, C_{\lambda,\mu,N}\}\}$ we have

$$\widetilde{w}_{N,\lambda,\mu,1} \approx_{n,\lambda,N} 1/w_{\lambda,\mu} \tag{21}$$

Case 2: $\frac{1}{|x|} > 1$. Let us use the decomposition $\widetilde{w}_{N,\lambda,\mu,1}(x) = I + II$, where

$$\begin{aligned} I &= \int_{|y|\leq 1} |e^{i\langle x,y\rangle} - 1|^N\, |y|^{-n-2\lambda-1}\Big(\log\frac{e}{|y|}\Big)^{\mu}\, dy \approx_{n,\lambda,\mu,N} |x|^N,\\ II &= \int_{1<|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N\, |y|^{-n-2\lambda-1}\, dy \approx_{n,\lambda,N} |x|^{2\lambda+1},\end{aligned}$$

This proves that

$$\widetilde{w}_{N,\lambda,\mu,1}(x) \approx_{n,\lambda,\mu,N} |x|^N + |x|^{2\lambda+1} \approx_{n,\lambda,N} |x|^{2\lambda+1} = \frac{1}{w_{\lambda,\mu}(x)} \tag{22}$$

on $\{x \in \mathbb{R}^n : |x| \leq 1\}$. If $C_{\lambda,\mu,N} \leq 1$, then relations (21) and (22) immediately imply that $\widetilde{w}_{N,\lambda,\mu,1} \approx_{n,\lambda,\mu,N} 1/w_{\lambda,\mu}$ on $\mathbb{R}^n$. Otherwise, just observe that both functions $\widetilde{w}_{N,\lambda,\mu,1}$ and $1/w_{\lambda,\mu}$ are positive and continuous on the compact annulus $1 \leq |x| \leq C_{\lambda,\mu,N}$. We still have to show that $\widetilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} 1/w_{\lambda,\mu}$. Let us define

$$\widetilde{\widetilde{w}}_{\lambda,\mu,2}(x) = \int_{|y|>\frac{1}{|x|}} u_{\lambda,\mu}(y)\, dy. \tag{23}$$

Then

$$\widetilde{w}_{N,\lambda,\mu,2}(x) \leq 2^N\, \widetilde{\widetilde{w}}_{\lambda,\mu,2}(x). \tag{24}$$

We will prove that the inverse inequality also holds (with a constant different from 2^N), so that we have $\tilde{\tilde{w}}_{\lambda,\mu,2} \approx_{N,\lambda,\mu,n} \tilde{w}_{N,\lambda,\mu,2}$. Now, let us prove that

$$\tilde{\tilde{w}}_{\lambda,\mu,2} \approx_{N,\lambda,\mu,n} 1/w_{\lambda,\mu}.$$

Case 1: $\frac{1}{|x|} > 1$. Then:

$$\tilde{\tilde{w}}_{\lambda,\mu,2}(x) = \frac{|S^{n-1}|}{2\lambda+1}\,|x|^{2\lambda+1} \approx_{\lambda,n} |x|^{2\lambda+1} = \frac{1}{w_{\lambda,\mu}(x)}.$$

Case 2: $\frac{1}{|x|} \leq 1$. Then:

$$\begin{aligned}\tilde{\tilde{w}}_{\lambda,\mu,2}(x) &= C_{n,\lambda} + \frac{|S^{n-1}|}{e^{2\lambda+1}} \int_e^{e|x|} t^{2\lambda} (\log t)^{\mu}\, dt \\ &\approx_{\lambda,\mu,n} |x|^{2\lambda+1} (\log(e|x|))^{\mu} = \frac{1}{w_{\lambda,\mu}(x)}.\end{aligned}$$

This concludes the proof that $\tilde{\tilde{w}}_{\lambda,\mu,2} \approx_{\lambda,\mu,n} 1/w_{\lambda,\mu}$ on $\mathbb{R}^n \setminus \{0\}$. Now we need to prove that there exists a constant $C_{N,\lambda,\mu,n}$ such that the inequality

$$\tilde{\tilde{w}}_{\lambda,\mu,2} \leq C_{N,\lambda,\mu,n}\, \tilde{w}_{N,\lambda,\mu,2}$$

holds on $\mathbb{R}^n \setminus \{0\}$. Since both $\tilde{\tilde{w}}_{\lambda,\mu,2}$ and $\tilde{w}_{N,\lambda,\mu,2}$ are radial, it will be enough to prove that the functions $t \mapsto \tilde{w}_{N,\lambda,\mu,2}(te_1)$ and $t \mapsto \tilde{\tilde{w}}_{\lambda,\mu,2}(te_1)$ are comparable on $\mathbb{R}^+$, where $e_1 = (1, 0, \dots, 0)$. Observe that $|e^{i\langle te_1, y\rangle} - 1| > \sqrt{2}$ on $G^t := \bigcup_{k\in\mathbb{Z}} G_k^t$, where

$$G_k^t := \left\{ y \in \mathbb{R}^n : \langle e_1, y\rangle \in \left[\frac{(4k+1)\pi}{2t}, \frac{(4k+3)\pi}{2t}\right) \right\}$$

for all $t > 0$ and $k \in \mathbb{Z}$. Therefore $u_{\lambda,\mu}(y) \approx_N |e^{i\langle te_1, y\rangle} - 1|^N u_{\lambda,\mu}(y)$ on G^t. In particular, there exists a constant C_N such that

$$\int_{G^t} u_{\lambda,\mu}(y)\, dy \leq C_N \int_{G^t} |e^{i\langle te_1, y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy.$$

If $t > 0$ and $k \in \mathbb{Z} \setminus \{0\}$ we define

$$R_k^t := \left\{ y \in \mathbb{R}^n : \langle e_1, y\rangle \in \left[\frac{(4k-1)\pi}{2t}, \frac{(4k+1)\pi}{2t}\right) \right\}$$

and

$$R_0^t := \left\{ y \in \mathbb{R}^n : \langle e_1, y\rangle \in \left[\frac{-\pi}{2t}, \frac{\pi}{2t}\right) \text{ and } |y| > \frac{1}{t} \right\}.$$

As

$$\int_{R_k^t} u_{\lambda,\mu}(y)\, dy \leq \int_{G_{k-1}^t} u_{\lambda,\mu}(y)\, dy$$

for all $k \in \mathbb{Z}^+$, and

$$\int_{R_k^t} u_{\lambda,\mu}(y)\, dy \leq \int_{G_k^t} u_{\lambda,\mu}(y)\, dy$$

for all $k \in \mathbb{Z}^-$, we also have

$$\int_{\bigcup_{k\in\mathbb{Z}\setminus\{0\}} R_k^t} u_{\lambda,\mu}(y)\,dy \le \int_{G^t} u_{\lambda,\mu}(y)\,dy \le C_N \int_{G^t} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)\,dy.$$

Since

$$\left\{ y : |\langle e_1, y\rangle| > \frac{\pi}{2t} \right\} = G^t \cup \bigcup_{k\in\mathbb{Z}\setminus\{0\}} R_k^t,$$

we have

$$\int_{|\langle e_1,y\rangle|>\frac{\pi}{2t}} u_{\lambda,\mu}(y)\,dy \le 2C_N \int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)\,dy.$$

Since $u_{\lambda,\mu}$ is radial, we can replace e_1 by e_j in the inequality above for $j = 2, \dots, n$. Let $|y|_\infty := \sup_{1\le j\le n} |\langle e_j, y\rangle|$. Then

$$\int_{\{y\in\mathbb{R}^n : |y|_\infty > \frac{\pi}{2t}\}} u_{\lambda,\mu}(y)\,dy \le 2nC_N \int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)\,dy. \tag{25}$$

Inequality (25) and the Lemma 4.1 easily imply — see (19) and (23) for details — that $\widetilde{\widetilde{w}}_{\lambda,\mu,2}(te_1) \le C_{n,\lambda,\mu,N} \cdot \widetilde{w}_{N,\lambda,\mu,2}(te_1)$.

Lemma 4.1. *Let $u_{\lambda,\mu}$ be as in* (15). *Then, for all $n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$, and $C > 1$ there exists a constant $D = D(n, \lambda, C) \in \mathbb{R}$ such that $u_{\lambda,\mu}\big(\frac{y}{C}\big) \le D\, u_{\lambda,\mu}(y)$ for all $y \in \mathbb{R}^n \setminus \{0\}$. We can choose $D = C^{n+2\lambda+1}(\log(eC))^\mu$.*

The proof of Lemma 4.1 is left to the reader. This completes the proof that $\widetilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} \frac{1}{w_{\lambda,\mu}}$ on $\mathbb{R}^n \setminus \{0\}$ and therefore the proof of (17), that is the claim of this section.

5. Reduction of Lemma 2.1 to Lemma 5.2

By duality, the inequality in Lemma 2.1 can be expressed as

$$\left\| \int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\cdot))(x)\,\frac{dt}{t} \right\|_{L^2(\frac{dx}{w_{\lambda,\mu}(x)})} \le C\,\|h(t,x)\|_{L^2(\frac{dt}{t}\frac{dx}{w_{\lambda,\mu}(x)})} \tag{26}$$

for all functions $h(t,x)$ in the appropriate space, where

$$C = C_{n,\lambda,\mu,\gamma,k} = \sqrt{C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}}.$$

In view of the result of Section 4, for every $f \in L^2\big(\mathbb{R}^n, \frac{1}{w_{\lambda,\mu}}\big)$,

$$\|f\|^2_{L^2(\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \approx \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_{\lambda,\mu}(y) \left| \left(\sum_{j=0}^{N_\lambda/2} \widehat{f}(g_{j,y}(\xi))\, b_j \right) \right|^2 dy\,d\xi, \tag{27}$$

where

$$g_{j,y}(\xi) = \left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi}\right) y\right), \quad b_j = (-1)^j \binom{N_\lambda/2}{j},$$

Plancherel's identity was used, and the implicit comparability constants depend on λ, μ, n only. We can substitute the left-hand side of (26) by using (27) on the function

$$f(x) = \int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\cdot))(x)\,\frac{dt}{t}.$$

For such a function we have

$$\left|\left(\sum_{j=0}^{N_\lambda/2} \widehat{f}(g_{j,y}(\xi))\, b_j\right)\right|^2 \tag{28}$$

$$= \left|\int_1^2 \left(\sum_{j=0}^{N_\lambda/2} \widehat{h}(t, g_{j,y}(\xi))\, m_{\lambda,\gamma,k}(at|g_{j,y}(\xi)|)\, b_j\right) \frac{dt}{t}\right|^2.$$

Since $m_{\lambda,\gamma,k}$ is supported in $\left[1 - \frac{5}{8\cdot 2^k}, 1 - \frac{1}{8\cdot 2^k}\right]$, the Cauchy–Schwarz inequality in the t variable allows us to control the right-hand side of (28) by

$$\frac{C_\lambda}{2^k} \int_1^2 \left|\sum_{j=0}^{N_\lambda/2} \widehat{h}(t, g_{j,y}(\xi)) \cdot m_{\lambda,\gamma,k}(a\,t|g_{j,y}(\xi)|)\, b_j\right|^2 \frac{dt}{t} =: H_{k,\lambda,\gamma}(y,\xi). \tag{29}$$

So, if we can show

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_{\lambda,\mu}(y)\, H_{k,\lambda,\gamma}(y,\xi)\, dy\, d\xi \le C\,\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}} \|h(t,x)\|^2_{L^2(\frac{dt}{t}\,\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \tag{30}$$

for a constant $C := C_{n,\lambda,\mu,\gamma}$, then (26) is proved. But (30) follows from the following pointwise (with respect to t) estimate:

$$\|(S_{\lambda,\gamma,k})_t(h)(x)\|^2_{L^2(\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \le C_{n,\lambda,\mu,\gamma}\,\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\,\|h\|^2_{L^2(\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \tag{31}$$

if (31) holds for all $t > 0$ rather than just $t \in [1,2]$ (which allowed us to drop the parameter a), and for all $h \in L^2\big(\mathbb{R}^n, dx/w_{\lambda,\mu}(x)\big)$. In order to see that (31) implies (30), just use (27) with $f(x) = \big(\widehat{h}(\cdot)\, m_{\lambda,\gamma,k}(t|\cdot|)\big)^{\vee}(x) = (S_{\lambda,\gamma,k})_t(h)(x)$, to rewrite the left-hand side of (31).

By duality, (31) is equivalent to

$$\|(S_{\lambda,\gamma,k})_t(h)\|^2_{L^2(w_{\lambda,\mu})} \le C_{n,\lambda,\mu,\gamma}\,\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\,\|h\|^2_{L^2(w_{\lambda,\mu})} \tag{32}$$

for all $h \in L^2(w_{\lambda,\mu}(x)\,dx)$, $t > 0$. So, the latter also yields the inequality in Lemma 2.1 for every f in the appropriate space and every $a > 0$. We now need to prove (32).

We denote by $(K_{\lambda,\gamma,k})_t(x)$ the kernel of the operator $(S_{\lambda,\gamma,k})_t$, i.e., the inverse Fourier transform of the multiplier $m_{\lambda,\gamma,k}(t\,|\cdot|)$. $(K_{\lambda,\gamma,k})_t$ is radial on $\mathbb{R}^n$, and it is convenient to decompose it radially as

$$(K_{\lambda,\gamma,k})_t = (K_{\lambda,\gamma,k})_1^{(0)} + \sum_{j=1}^{\infty} (K_{\lambda,\gamma,k})_t^{(j)},$$

where

$$(K_{\lambda,\gamma,k})_1^{(0)}(x) = (K_{\lambda,\gamma,k})_t(x)\,\theta(2^{-(k+3)}x/t),$$
$$(K_{\lambda,\gamma,k})_t^{(j)}(x) = (K_{\lambda,\gamma,k})_t(x)\,\big(\theta(2^{-(j+k+3)}\,x/t) - \theta(2^{-(k+2+j)}\,x/t)\big),$$

for some radial smooth function θ supported in the ball $B(0,2)$ and equal to one on $B(0,1)$.

To prove estimate (32) we make use of the subsequent lemmas.

Lemma 5.1. *For all $M \geq 2n$ there is a constant $C_{\lambda,\gamma,k,M} = C_{\lambda,\gamma,k,M}(n,\theta)$ such that for all $j = 0,1,2,\dots,$*

$$\sup_{\xi\in\mathbb{R}^n} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-jM}}{k^\gamma} \tag{33}$$

and also

$$|\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-(j+l)M}}{k^\gamma} \tag{34}$$

whenever $\big|t\,|\xi| - 1\big| \geq 2^{l-k-3}$ and $l \geq 4$. Also,

$$|\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-(j+k+3)M}}{k^\gamma}(1+t\,|\xi|)^{-M} \tag{35}$$

whenever $|t\,\xi| \leq \frac{1}{8}$ or $|t\,\xi| \geq \frac{15}{8}$.

Proof. The proof for $t = 1$ follows the lines of the proof of Lemma 10.5.5 in [Grafakos 2014, p. 413]. Just observe that estimate (10.5.9) in p. 409 of that reference is now replaced by (6), which explains why the factor $1/k^\gamma$ appears. The general case (any $t > 0$) is straightforward in view of the fact that

$$\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi) = \widehat{(K_{\lambda,\gamma,k})_1^{(j)}}(t\,\xi). \qquad \square$$

Lemma 5.2. *The inequalities*

$$\int_{||t\xi|-1|\,<\,\varepsilon} |\hat f(\xi)|^2\,d\xi \leq C_{n,\lambda,\mu}\,\omega_{\lambda,\mu}(t)\,\varepsilon \int_{\mathbb{R}^n} |f(x)|^2\,\frac{dx}{w_{\lambda,\mu}(x)} \tag{36}$$

and

$$\int_{\mathbb{R}^n} |\hat f(\xi)|^2\,\frac{d\xi}{(1+|t\xi|)^M} \leq C_{n,\lambda,\mu,M}\,\omega_{\lambda,\mu}(t)\int_{\mathbb{R}^n} |f(x)|^2\,\frac{dx}{w_{\lambda,\mu}(x)}, \tag{37}$$

hold for all Schwartz functions f, $t > 0$, $M \geq 2n$, all $0 < \varepsilon < 2$, λ, and μ as in Theorem 1.1.

The proof of Lemma 5.2 is postponed to Section 6.

By reasoning as in [Grafakos 2014, p. 414] and using the estimates in Lemmas 5.1 and 5.2 instead of those in Lemma 10.5.5 in [op. cit., p. 413] and Lemma 10.5.6 in [op. cit., p. 414], we can prove

$$\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2\, dx \le C\, \frac{2^{-2jM}}{2^k k^{2\gamma}}\, \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \tag{38}$$

for another constant $C = C_{n,\lambda,\mu,\gamma,M}$. By duality, this is equivalent to

$$\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2\, w_{\lambda,\mu}(x)\, dx \le C\, \frac{2^{-2jM}}{2^k k^{2\gamma}}\, \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2\, dx. \tag{39}$$

Given a Schwartz function f, we write

$$f_0 = f \chi_{Q_0^{(n,k,j,t)}},$$

where $Q_0^{(n,k,j,t)}$ is a cube centered at the origin of side length $C_n\, 2^{j+k+4} t$ (note that $\operatorname{supp} (K_{\lambda,\gamma,k})_t^{(j)} \subseteq B(0, 2^{j+k+4} t)$). Then for $x \in Q_0^{(n,k,j,t)}$ we have the inequality

$$|x| \le \sqrt{n}\, C_n 2^{j+k+4} t;$$

hence, (39) implies

$$\begin{aligned} &\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_0)(x)|^2\, w_{\lambda,\mu}(x)\, dx \\ &\quad \le C_{n,\lambda,\mu,\gamma,M}\, \frac{2^{-2jM}}{2^k k^{2\gamma}}\, \frac{\omega_{\lambda,\mu}(t)}{\omega_{\lambda,\mu}(\sqrt{n}\, C_n 2^{j+k+4} t)} \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2\, w_{\lambda,\mu}(x)\, dx \end{aligned} \tag{40}$$

because the function $1/\omega_{\lambda,\mu}$ is increasing. A simple computation shows that

$$\sup_{t>0} \frac{\omega_{\lambda,\mu}(at)}{\omega_{\lambda,\mu}(t)} = \frac{1}{a^{2\lambda+1}} \quad \text{and} \quad \sup_{t>0} \frac{\omega_{\lambda,\mu}(at)}{\omega_{\lambda,\mu}(t)} = \frac{(\log(e/a))^{\mu}}{a^{2\lambda+1}} \tag{41}$$

if $a > 1$ and if $a \le 1$, respectively. Therefore, for all j and k such that $j + k \ge C_n'$ for a suitable purely dimensional constant C_n',

$$\sup_{t>0} \frac{\omega_{\lambda,\mu}(t)}{\omega_{\lambda,\mu}(\sqrt{n}\, C_n\, 2^{j+k+4} t)} \le C''_{n,\lambda,\mu}\, 2^{(j+k)(2\lambda+1)} (j^{\mu} + k^{\mu}), \tag{42}$$

where we used the hypothesis on j and k and the fact that

$$(j+k)^{\mu} \le C_{\mu} (j^{\mu} + k^{\mu}).$$

It follows from (42) and (40) that $\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_0)(x)|^2\, w_{\lambda,\mu}(x)\, dx$ is bounded by

$$C\, 2^{j(2\lambda+1-2M)}\, \frac{2^{2k\lambda}}{k^{2\gamma}}\, (j^{\mu} + k^{\mu}) \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2\, w_{\lambda,\mu}(x)\, dx,$$

for $C = C_{n,\lambda,\mu,\gamma,M}$, provided that

$$j + k \geq C_n'. \tag{43}$$

Now write $\mathbb{R}^n \setminus Q_0^{(n,k,j,t)}$ as a mesh of cubes $Q_i^{(n,k,j,t)}$, indexed by $i \in \mathbb{Z} \setminus \{0\}$, of side lengths $C_n 2^{j+k+4} t$ (the same side length of $Q_0^{(n,k,j,t)}$) and centers c_{Q_i}. By using (33), reasoning as in [Grafakos 2014, p. 415] as well as simply noting that $2^{2k\lambda}(j^\mu + k^\mu) \geq 1$, we can find that the pieces

$$\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_i)(x')|^2 \, w_{\lambda,\mu}(x') \, dx'$$

are bounded by

$$C_{\lambda,\mu,\gamma,M} \, 2^{-2jM} \, \frac{2^{2k\lambda}}{k^{2\gamma}} (j^\mu + k^\mu) \int_{Q_i^{(n,k,j,t)}} |f_i(x)|^2 \, w_{\lambda,\mu}(x) \, dx$$

whenever f_i is supported in $Q_i^{(n,k,j,t)}$ and, in turn, that

$$\begin{aligned} \|(K_{\lambda,\gamma,k})_t^{(j)} * f\|_{L^2(w_{\lambda,\mu})} \qquad & \\ \leq C''_{n,\lambda,\mu,\gamma,M} \, 2^{j(\lambda+\frac{1}{2}-M)} \, \frac{2^{k\lambda}}{k^\gamma} \, (j^{\frac{\mu}{2}} + k^{\frac{\mu}{2}}) \, \|f\|_{L^2(w_{\lambda,\mu})} & \end{aligned} \tag{44}$$

(in view of the argument in [Grafakos 2014]). Observe that condition (43) is satisfied if we assume $k \geq C_n'$, which we can as the convergence of (7) only depends on the estimates we have for k big enough. So, for $k \geq C_n'$, by using (44) and summing over $j = 0, 1, 2, \ldots$, we deduce (32) if we just choose $M > n/2$ (remember that $n > 2\lambda + 1$). In turn, (32) is equivalent to (31), which is equivalent to (26), which is equivalent to the inequality in Lemma 2.1. Therefore, this completes the proof of the lemma, modulo Lemma 5.2

6. Proof of Lemma 5.2

6.1. ***Proof of inequality* (36).** We reduce estimate (36) by duality to

$$\int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 \, w_{\lambda,\mu}(\xi) \, d\xi \leq C_{n,\lambda,\mu} \, \omega_{\lambda,\mu}(t) \, \varepsilon \int_{||tx|-1| \leq \varepsilon} |g(x)|^2 \, dx \tag{45}$$

for functions g supported in the annulus $\big||tx| - 1\big| \leq \varepsilon$. In Section 3 we observed that

$$w_{\lambda,\mu} \approx_{\lambda,\mu} w_{\lambda,\mu}^{(1)}$$

and proved in Theorem 3.1 that the function $|\widehat{w}_{\lambda,\mu}^{(1)}|$ is bounded by a scalar multiple of $\Omega_{\lambda,\mu}$ (see (12)) in the whole range $\lambda \in (0, (n-1)/2)$. Therefore, we can start to prove (45) as follows:

$$\begin{aligned}(46)\quad \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 w_{\lambda,\mu}(\xi)\,d\xi &\approx_{\lambda,\mu} \int_{\mathbb{R}^n} \big(\widehat{g}\,\overline{\widehat{g}}\big)^{\vee}(x)\,\widehat{w}^{(1)}_{\lambda,\mu}(x)\,dx \\ &\le C_{n,\lambda,\mu} \int_{\mathbb{R}^n} (|g| * |\tilde{\bar{g}}|)(x)\,\Omega_{\lambda,\mu}(x)\,dx \\ &= C_{n,\lambda,\mu} \iint_{\substack{\left||t\,y|-1\right|\le\varepsilon \\ \left||t\,x|-1\right|\le\varepsilon}} |g(x)|\,|\tilde{\bar{g}}(y)|\,\Omega_{\lambda,\mu}(x-y)\,dx\,dy \\ &\le C_{n,\lambda,\mu} B(n,\lambda,\mu,\varepsilon,t)\|g\|_{L^2}^2\end{aligned}$$

where $\widetilde{g}(x) = g(-x)$ and

$$(47)\qquad B(n,\lambda,\mu,\varepsilon,t) = \frac{1}{t^n} \sup_{\{x:\left||x|-1\right|\le\varepsilon\}} \int_{\left||y|-1\right|\le\varepsilon} \Omega^t_{\lambda,\mu}(y-x)\,dy,$$

where $\Omega^t_{\lambda,\mu}(x) := \Omega_{\lambda,\mu}(x/t)$. The last inequality of (46) is proved by interpolation between the $L^1(S) \to L^1(S)$ and $L^\infty(S) \to L^\infty(S)$ estimates for the linear operator

$$L_{\lambda,\mu,t,\varepsilon}(g)(x) = \int_S g(y)\,\Omega_{\lambda,\mu}(y-x)\,dy,$$

where

$$S = \big\{y \in \mathbb{R}^n : \left||t\,y|-1\right| \le \varepsilon\big\},$$

using the Cauchy–Schwarz inequality. It remains to establish that

$$(48)\qquad B(n,\lambda,\mu,\varepsilon,t) \le C_{n,\lambda,\mu}\,\omega_{\lambda,\mu}(t)\,\varepsilon.$$

Then we reason as in [Grafakos 2014, pp. 417, 418]: we apply a rotation and a change of variable to the integrals in (47) to push the dependence on x to the domain of integration, then control the supremum in (47) by integrating $\Omega_{\lambda,\mu}$ over the bigger set

$$\big\{y : \left||y-e_1|-1\right| \le 2\varepsilon\big\},$$

finally we split this latter integral over the sets S_0, S_ℓ, S_* defined in [op. cit.] to be

$$\begin{aligned} S_0 &= \big\{y \in \mathbb{R}^n : \left||y-e_1|-1\right| \le 2\varepsilon,\ |y| \le \varepsilon\big\}, \\ S_\ell &= \big\{y \in \mathbb{R}^n : \left||y-e_1|-1\right| \le 2\varepsilon,\ \ell\varepsilon \le |y| \le (\ell+1)\varepsilon\big\}, \\ S_* &= \big\{y \in \mathbb{R}^n : \left||y-e_1|-1\right| \le 2\varepsilon,\ |y| \ge 1\big\}. \end{aligned}$$

In the end, matters reduce to proving the estimates

$$\int_{S_0} \Omega^t_{\lambda,\mu}(y)\,dy \le C'_{n,\lambda,\mu}\, t^n\, \omega_{\lambda,\mu}(t)\,\varepsilon^{2\lambda+1}, \tag{49}$$

$$\sum_{\ell=1}^{[\frac{1}{\varepsilon}]+1} \int_{S_\ell} \Omega^t_{\lambda,\mu}(y)\,dy \le C_{\lambda,\mu}\, t^n\, \varepsilon\omega_{\lambda,\mu}(t), \tag{50}$$

$$\int_{S_*} \Omega^t_{\lambda,\mu}(y)\,dy \le C_n\,\varepsilon\omega_{\lambda,\mu}(t)\,t^n. \tag{51}$$

In proving the inequalities above, we can assume without loss of generality that $t \ge 2$, because when $t < 2$ the proof of Lemma 5.2 is an immediate consequence of Lemma 10.5.6 in [op. cit., p. 414]. We can also assume that $t \ge C_{n,\lambda,\mu}$, due to the compactness of $[2, C_{n,\lambda,\mu}]$ and the continuity and positivity of the functions involved. For a suitable constant $C_{n,\lambda,\mu}$ and $t \ge \max\{2, C_{n,\lambda,\mu}\}$, (49) is proved by using calculus (note that the integrand in (49) is radial and the domain of integration is a sphere); (50) is proved by using the maximum of the integrand over each set S_ℓ, then by comparing the sum with an integral, finally by using calculus to estimate the integral; (51) is proved by using the maximum of the integrand over S_*. The condition that $t \ge 2 > \varepsilon$ was used in both (49) and (50) and (41) was used in (51).

By combining estimates (49), (50), and (51), we obtain (48). This concludes the proof of (45) and, therefore, of (36). □

6.2. ***Proof of inequality* (37).** Inequality (37) is already known for $t \le 1$; see equation (10.5.22) in [Grafakos 2014, p. 414]. Indeed, if $0 < t \le 1$ then $\omega_{\lambda,\mu}(t) = 1/t^{2\lambda+1}$, and (37) follows by dilation from the case $t = 1$, the one shown in [op. cit.]. For $t > 1$ define:

$$A_1^t = \Big\{\xi \in \mathbb{R}^n : |\xi| \le \frac{1}{t}\Big\}, \qquad A_2^t = \Big\{\xi \in \mathbb{R}^n : \frac{1}{t} < |\xi| \le \frac{2+\sqrt{t}}{t}\Big\},$$
$$A_3^t = \Big\{\xi \in \mathbb{R}^n : \frac{2+\sqrt{t}}{t} < |\xi| \le \frac{2+t}{t}\Big\}, \quad A_4^t = \Big\{\xi \in \mathbb{R}^n : \frac{2+t}{t} < |\xi|\Big\}.$$

We will prove (37) by proving that

$$I_j := \int_{A_j^t} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\,d\xi \le C_{n,\lambda,\mu,M}\,\omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \tag{52}$$

for each $j = 1, 2, 3, 4$. For $j = 1$, first observe that $1/(1+|t\,\xi|)^M \approx_M 1$ on A_1^t and then argue as in the proof of (36), at the beginning of this section. By duality, we reduce (52) with $j = 1$ to

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\, w_{\lambda,\mu}(\xi)\,d\xi \le C_{n,\lambda,\mu}\,\omega_{\lambda,\mu}(t) \int_{A_1^t} |f(x)|^2\,dx \tag{53}$$

for all functions f supported in the ball A_1^t. By proceeding as in (46), we can prove that

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 w_{\lambda,\mu}(\xi)\, d\xi \le B'(n,\lambda,\mu,t)\|f\|_{L^2}^2$$

for every f supported in A_1^t, where $B'(n,\lambda,\mu,t)$, now, is defined by

$$\begin{aligned} B'(n,\lambda,\mu,t) &= \sup_{\{x:|x|\le\frac{1}{t}\}} \int_{|y|\le\frac{1}{t}} \Omega_{\lambda,\mu}(y-x)\, dy \\ &= \frac{1}{t^n} \sup_{\{x:|x|\le 1\}} \int_{|y+x|\le 1} \Omega_{\lambda,\mu}\Big(\frac{y}{t}\Big)\, dy \end{aligned} \tag{54}$$

and all we still need to show is that

$$B'(n,\lambda,\mu,t) \le C_{n,\lambda,\mu}\, \omega_{\lambda,\mu}(t). \tag{55}$$

Since $|x| \le 1$ and $|x+y| \le 1$ we have $|y| \le 2$. So, (55) is a consequence of

$$\frac{1}{t^n}\int_{|y|\le 2} \Omega_{\lambda,\mu}\Big(\frac{y}{t}\Big)\, dy \le C_{n,\lambda,\mu}\, \omega_{\lambda,\mu}(t), \tag{56}$$

which can be proved similarly to (49).

When $j=2$, we use

$$I_2 \le \sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \frac{1}{(2+\ell)^M} \int_{\frac{1+\ell}{t}<|\xi|\le\frac{2+\ell}{t}} |\widehat{f}(\xi)|^2\, d\xi. \tag{57}$$

Next, we apply estimate (36) on each of the latter integrals. We are already assuming that $t>1$. Since $\omega_{\lambda,\mu}(t) \approx_{\lambda,\mu,J} 1$ on any compact subinterval J of $(0,\infty)$, we can in fact assume $t \ge 3$. Now we control the right-hand side of (57) with

$$\begin{aligned} C_{n,\lambda,\mu} \sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \frac{1}{(2+\ell)^M}\, \omega_{\lambda,\mu}\Big(\frac{2t}{3+2\ell}\Big) \frac{1}{3+2\ell} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \\ \le C_{n,\lambda,\mu,M}\, \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)}, \end{aligned} \tag{58}$$

provided $M > 2\lambda+1$. This proves that (52) holds for $j=2$.

If $j=3$ then $(2+\sqrt{t})/t < |\xi|$, which implies that

$$\frac{1}{(1+|t\xi|)^M} \le \frac{1}{(3+\sqrt{t})^M}.$$

Then apply (36). Observe that, as long as $t>1$, we have that the quantity $\tilde{t}$ that now plays the role of t in (36) is bounded above and below by absolute constants, so $\omega_{\lambda,\mu}(\tilde{t}) \approx_{\lambda,\mu} 1$. In addition, for t in the same range, we have $\tilde{\varepsilon} \le 1$ ($\tilde{\varepsilon}$ being the quantity that now plays the role of ε in (36)). These considerations imply that

$$I_3 \le \frac{1}{(3+\sqrt{t})^M} \int_{\frac{2+\sqrt{t}}{t} < |\xi| \le \frac{2+t}{t}} |\widehat{f}(\xi)|^2 \, d\xi \tag{59}$$
$$\le C'_{n,\lambda,\mu} \frac{1}{(3+\sqrt{t})^M} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)}$$
$$\le C''_{n,\lambda,\mu,M}\, \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)},$$

last inequality holding for a suitable constant $C''_{n,\lambda,\mu,M}$, provided that $M > 4\lambda + 2$.

It only remains to prove (52) with $j = 4$. We have

$$I_4 \le \sum_{\ell=\lfloor t \rfloor+1}^{\infty} \int_{\frac{1+\ell}{t} < |\xi| \le \frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\xi|)^M} \, d\xi. \tag{60}$$

Again, we apply (36) to the integral in the last term of (60), which is therefore controlled by

$$C_{n,\lambda,\mu} \sum_{\ell=\lfloor t \rfloor+1}^{\infty} \frac{1}{(2+\ell)^M} \frac{1}{\left(\frac{2t}{3+2\ell}\right)^{2\lambda+1}} \frac{1}{(3+2\ell)} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)}$$
$$\le C_{n,\lambda,\mu,M} \frac{1}{t^{2\lambda+1}} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \frac{1}{(1+t)^{M-2\lambda-1}},$$

which yields the desired inequality, provided that $M > 2\lambda + 1$. By choosing any $M > 4\lambda + 2$ (as required after (59)), we conclude the proof of (37) and of the claimed statement.

Proof of Corollary 1.2. The proof in [Carbery et al. 1988] can be used with $m_{\lambda,\gamma}$ instead of m_1^λ to account for the case where $\gamma \ge 0$ and $2 \le p < p_\lambda$. When $p = p_\lambda$ and $\gamma > 1/p'_\lambda + 1/2$, values of μ satisfying $(2\lambda+1)/n < \mu < 2\gamma - 2$ exist. For such μ, since $1 < 1 + 2\lambda < n$, we can use Theorem 1.1. Since (4) trivially holds for all $f \in \mathcal{S}$, the boundedness of $B_*^{\lambda,\gamma}$ implies that it also holds for every $f \in L^2(\mathbb{R}^n, dx)$ and every $f \in L^2(\mathbb{R}^n, w_{\lambda,\mu})$. But then it must hold for every $f \in L^2 + L^2(w_{\lambda,\mu})$. Since $(2\lambda+1)/n < \mu$ we have $L^{p_\lambda} \subseteq L^2 + L^2(w_{\lambda,\mu})$, concluding the proof. □

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MARCO ANNONI
1513 INDIANA ST.
APT. A
ST CHARLES, IL 60174
UNITED STATES
marcoannoni.81@gmail.com

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UNIQUENESS OF CONFORMAL RICCI FLOW USING ENERGY METHODS

THOMAS BELL

We analyze an energy functional associated to conformal Ricci flow along closed manifolds with constant negative scalar curvature. Given initial conditions we use this functional to demonstrate the uniqueness of both the metric and the pressure function along conformal Ricci flow.

1. Introduction

Uniqueness of Ricci flow on closed manifolds was originally proved by Hamilton [1982]. Chen and Zhu [2006] subsequently proved uniqueness on complete noncompact manifolds with bounded curvature. The method employed in [Chen and Zhu 2006] utilizes DeTurck Ricci flow. Recently Kotschwar [2014] used energy techniques to give another proof of the uniqueness on complete manifolds. Kotschwar's proof does not rely on DeTurck Ricci flow. A natural question is whether similar techniques can be applied to demonstrate uniqueness of other geometric flows. One such flow is conformal Ricci flow, introduced by Fischer [2004]. Ricci flow preserves many important properties of metrics, but it generally does not preserve the property of constant scalar curvature. Conformal Ricci flow is a modification of Ricci flow which is intended for this purpose, and for this reason it is restricted to the class of metrics of constant scalar curvature. Conformal Ricci flow is, like Ricci flow, a weakly parabolic flow of the metric on manifolds, except that conformal Ricci flow is coupled with an elliptic equation.

Let (M^n, g_0) be a smooth n-dimensional Riemannian manifold with a metric g_0 of constant scalar curvature s_0. Conformal Ricci flow on M is defined as follows:

$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\operatorname{Ric}_{g(t)} + 2\dfrac{s_0}{n} g(t) - 2p(t)g(t), \\ s(g(t)) = s_0 \end{cases} \qquad \text{on } M \times [0, T]. \tag{1}$$

Here $g(t)$, $t \in [0, T]$, is a family of metrics on M with $g(0) = g_0$, $s(g(t))$ is the scalar curvature of $g(t)$, and $p(t)$, $t \in [0, T]$, is a family of functions on M.

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In [Fischer 2004; Lu et al. 2014] we see that (1) is equivalent to the following system:

$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\,\mathrm{Ric}_{g(t)} + 2\dfrac{s_0}{n}g(t) - 2p(t)g(t) \\ ((n-1)\Delta_{g(t)} + s_0)p(t) = -\Big\langle \mathrm{Ric}_{g(t)} - \dfrac{s_0}{n}g(t),\, \mathrm{Ric}_{g(t)} - \dfrac{s_0}{n}g(t)\Big\rangle. \end{cases} \tag{2}$$

Throughout we use V to denote the following symmetric 2-tensor:

$$V(t) = \mathrm{Ric}_{g(t)} - \frac{s_0}{n}g(t) + p(t)g(t). \tag{3}$$

In this paper we use Kotschwar's energy techniques to give a proof of the uniqueness of conformal Ricci flow for closed manifolds with metrics of constant negative scalar curvature. It is worth noting similarities to the study of certain elliptic-hyperbolic systems done by Andersson and Moncrief [2011]. The existence of solutions to conformal Ricci flow has been shown by Fischer [2004] and by Lu, Qing, and Zheng [2014], the latter paper using DeTurck conformal Ricci flow. More precisely we prove the following uniqueness theorem of conformal Ricci flow:

Theorem 1. *Let (M^n, g_0) be a closed manifold with constant negative scalar curvature s_0. Suppose $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ are two solutions of* (1) *on $M \times [0, T]$ with $\tilde{g}(0) = g(0)$. Then $(\tilde{g}(t), \tilde{p}(t)) = (g(t), p(t))$ for $0 \le t \le T$.*

2. The differences between $g(t)$ and $\tilde{g}(t)$

Let $g(t)$ and $\tilde{g}(t)$ be as in Theorem 1. We treat g as our background metric and $\tilde{g}$ as our alternative metric. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of g and $\tilde{g}$ respectively. Similarly, let R and $\tilde{R}$ represent the full Riemannian curvature tensors of g and $\tilde{g}$ respectively.

Let $h = g - \tilde{g}$, and $A = \nabla - \tilde{\nabla}$. Explicitly, $A^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$ where Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of ∇ and $\tilde{\nabla}$ respectively. Also let $S = R - \tilde{R}$ and $q = p - \tilde{p}$.

In this section we find bounds on h, A, S, q, ∇q, and $\nabla\nabla q$ (see Propositions 3 and 5). Throughout this chapter we use the convention $X * Y$ to denote any finite sum of tensors of the form $X \cdot Y$. We use $C(X)$ to denote a finite sum of tensors of the form X.

2.1. ***Preliminary calculations.*** First we calculate some useful expressions for quantities which arise in the proofs of Propositions 3 and 5. We calculate

$$g^{ij} - \tilde{g}^{ij} = g^{ik}(\tilde{g}^{j\ell}\tilde{g}_{k\ell}) - \tilde{g}^{j\ell}(g^{ik}g_{k\ell}) = -g^{ik}\tilde{g}^{j\ell}h_{k\ell},$$

i.e.,

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h.$$

If X is any tensor which is not a function we have

$$(\nabla - \tilde{\nabla})X = A * X.$$

We check this when X is a (1, 1)-tensor. Calculating in local coordinates we see

$$\begin{aligned}(\nabla_i - \tilde{\nabla}_i)X_j^k &= \partial_i X_j^k - \Gamma_{ij}^\ell X_\ell^k + \Gamma_{i\ell}^k X_j^\ell - \partial_i X_j^k + \tilde{\Gamma}_{ij}^\ell X_\ell^k - \tilde{\Gamma}_{i\ell}^k X_j^\ell \\ &= A_{i\ell}^k X_j^\ell - A_{ij}^\ell X_\ell^k = A * X.\end{aligned}$$

If f is a function however, then we have the following:

$$(\nabla^i - \tilde{\nabla}^i)f = (g^{ij} - \tilde{g}^{ij})\partial_j f = -g^{ik}\tilde{g}^{j\ell}h_{k\ell}\partial_j f = -g^{ik}h_{k\ell}\tilde{\nabla}^\ell f,$$

or in other words

$$(\nabla - \tilde{\nabla})f = h * \tilde{\nabla} f.$$

We now calculate

$$\nabla \tilde{g}^{-1} = (\nabla - \tilde{\nabla})\tilde{g}^{-1} = \tilde{g}^{-1} * A.$$

The following calculation is also important.

$$\nabla_i h_{jk} = \nabla_i g_{jk} - \nabla_i \tilde{g}_{jk} = -(\nabla_i - \tilde{\nabla}_i)\tilde{g}_{jk}.$$

Thus we have

$$\nabla h = \tilde{g} * A.$$

Now we are able to calculate the following for a function f.

$$\begin{aligned}\nabla(\nabla - \tilde{\nabla})f &= \nabla(h * \tilde{\nabla} f) \\ &= \nabla h * \tilde{\nabla} f + h * (\nabla - \tilde{\nabla})\tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f \\ &= \tilde{g} * A * \tilde{\nabla} f + h * A * \tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f.\end{aligned}$$

Now let

$$\begin{aligned}U_{ijk\ell}^a &= g^{ab}\nabla_b \tilde{R}_{ijk\ell} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}_{ijk\ell} \\ &= g^{ab}(\nabla_b - \tilde{\nabla}_b)\tilde{R}_{ijk\ell} + (g^{ab} - \tilde{g}^{ab})\tilde{\nabla}_b \tilde{R}_{ijk\ell} \\ &= A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R},\end{aligned} \tag{4}$$

and we may calculate

$$\begin{aligned}\nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) &= \nabla_a(g^{ab}\nabla_b \tilde{R} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) + g^{ab}\nabla_a\nabla_b(R - \tilde{R}) \\ &= \operatorname{div} U + \Delta S.\end{aligned}$$

We summarize the above calculations in the following lemma:

Lemma 2. *Using the notation defined at the beginning of this section,*

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h, \tag{5}$$

$$(\nabla - \tilde{\nabla})X = A * X, \tag{6}$$

$$(\nabla - \tilde{\nabla})f = h * \tilde{\nabla} f, \tag{7}$$

$$\nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A, \tag{8}$$

$$\nabla h = \tilde{g} * A, \tag{9}$$

$$\nabla(\nabla - \tilde{\nabla})f = \tilde{g} * A * \tilde{\nabla} f + h * A * \tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f, \tag{10}$$

$$U = A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R}, \tag{11}$$

$$\nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) = \operatorname{div} U + \Delta S, \tag{12}$$

where U is defined in (4).

2.2. *Bounds on time derivatives of h, A and S.* In this subsection we derive bounds on the time derivatives of h, A and S. In particular we prove the following proposition. Here, as well as throughout this chapter, we let C denote a constant dependent only upon n while N denotes a constant with further dependencies.

Proposition 3. *Let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ be two solutions of* (1) *on $M \times [0, T]$. Using the notation defined at the beginning of this section, there exist constants N_h, N_A and N_S such that*

$$\left|\frac{\partial}{\partial t} h\right| \le N_h |h| + C(|S| + |q|), \tag{13}$$

$$\left|\frac{\partial}{\partial t} A\right| \le N_A(|h| + |A|) + C(|\nabla S| + |\nabla q|), \tag{14}$$

$$\left|\frac{\partial}{\partial t} S - \Delta S - \operatorname{div} U\right| \le N_S(|h| + |A| + |S| + |q|) + C|\nabla\nabla q|, \tag{15}$$

where U is defined in (4).

Proof. We start with the time derivative of h. By (1) we have

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= -2(R_{ij} - \tilde{R}_{ij}) + 2\frac{s_0}{n}(g_{ij} - \tilde{g}_{ij}) - 2(p\, g_{ij} - \tilde{p}\, \tilde{g}_{ij}) \\
&= -2S^k_{kij} + 2\frac{s_0}{n} h_{ij} - 2[(p - \tilde{p})g_{ij} + \tilde{p}(g_{ij} - \tilde{g}_{ij})] \\
&= -2S^k_{kij} + 2\frac{s_0}{n} h_{ij} - 2q\, g_{ij} - 2\tilde{p}\, h_{ij}.
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t} h = C(S) + C(s_0 h) + C(q) + \tilde{p} * h$$

and

$$\left|\frac{\partial}{\partial t}h\right| \le C((|s_0|+|\tilde{p}|)|h|+|S|+|q|). \tag{16}$$

This proves (13).

Recall the definition of V from (3):

$$V(t) = \mathrm{Ric}_{g(t)} - \frac{s_0}{n}g(t) + p(t)g(t). \tag{17}$$

We may define $\tilde{V}$ similarly using our alternate metric $\tilde{g}$. Since V and $\tilde{V}$ are symmetric 2-tensors, then by [Chow et al. 2006, p. 108] we may calculate

$$\frac{\partial}{\partial t}A^k_{ij} = \tilde{g}^{k\ell}(\tilde{\nabla}_i\tilde{V}_{j\ell}+\tilde{\nabla}_j\tilde{V}_{i\ell}-\tilde{\nabla}_\ell\tilde{V}_{ij}) - g^{k\ell}(\nabla_i V_{j\ell}+\nabla_j V_{i\ell}-\nabla_\ell V_{ij}). \tag{18}$$

We proceed to calculate

$$\begin{aligned}
&\tilde{g}^{k\ell}\tilde{\nabla}_i\tilde{V}_{j\ell} - g^{k\ell}\nabla_i V_{j\ell} \\
&\quad= \tilde{g}^{k\ell}(\tilde{\nabla}_i\tilde{R}_{j\ell}) - g^{k\ell}(\nabla_i R_{j\ell}) + \tilde{g}^{k\ell}\tilde{\nabla}_i(\tilde{p}\tilde{g}_{j\ell}) - g^{k\ell}\nabla_i(pg_{j\ell}) \\
&\quad= (\tilde{g}^{k\ell}-g^{k\ell})\tilde{\nabla}_i\tilde{R}_{j\ell} + g^{k\ell}(\tilde{\nabla}_i-\nabla_i)\tilde{R}_{j\ell} - g^{k\ell}\nabla_i(S^m_{mj\ell}) + \delta^k_j\tilde{\nabla}_i\tilde{p} - \delta^k_j\nabla_i p \\
&\quad= \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + h*\tilde{\nabla}\tilde{p} + C(\nabla q),
\end{aligned} \tag{19}$$

where we have used (7) to get the last equality. Similarly we find

$$\tilde{g}^{k\ell}\tilde{\nabla}_j\tilde{V}_{i\ell} - g^{k\ell}\nabla_j V_{i\ell} = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + h*\tilde{\nabla}\tilde{p} + C(\nabla q). \tag{20}$$

Now we consider

$$\begin{aligned}
&-\tilde{g}^{k\ell}\tilde{\nabla}_\ell\tilde{V}_{ij} + g^{k\ell}\nabla_\ell V_{ij} \\
&\qquad= \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + \tilde{g}^{k\ell}\tilde{g}_{ij}\tilde{\nabla}_\ell\tilde{p} - g^{k\ell}g_{ij}\nabla_\ell p \\
&\qquad= \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + (\tilde{g}^{k\ell}-g^{k\ell})\tilde{g}_{ij}\tilde{\nabla}_\ell\tilde{p} \\
&\qquad\qquad + g^{k\ell}(\tilde{g}_{ij}-g_{ij})\tilde{\nabla}_\ell\tilde{p} + g^{k\ell}g_{ij}(\tilde{\nabla}_\ell-\nabla_\ell)\tilde{p} \\
&\qquad\qquad + g^{k\ell}g_{ij}\nabla_\ell(\tilde{p}-p) \\
&\qquad= \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) \\
&\qquad\qquad + \tilde{g}^{-1}*h*\tilde{g}*\tilde{\nabla}\tilde{p} + h*\tilde{\nabla}\tilde{p} + C(\nabla q).
\end{aligned} \tag{21}$$

Hence by (18), (19), (20) and (21),

$$\frac{\partial}{\partial t}A = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + h*\tilde{\nabla}\tilde{p} + C(\nabla q) + \tilde{g}^{-1}*h*\tilde{g}*\tilde{\nabla}\tilde{p}$$

and

$$\left|\frac{\partial}{\partial t}A\right| \le C\big((|\tilde{g}^{-1}||\tilde{\nabla}\tilde{R}|+|\tilde{\nabla}\tilde{p}|+|\tilde{g}^{-1}||\tilde{g}||\tilde{\nabla}\tilde{p}|)|h|+|\tilde{R}||A|+|\nabla S|+|\nabla q|\big). \tag{22}$$

This proves (14).

By [Chow et al. 2006, equation (2.67)] we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}^{\ell} &= g^{\ell m}(\nabla_i\nabla_k V_{jm}-\nabla_i\nabla_m V_{jk}-\nabla_j\nabla_k V_{im}+\nabla_j\nabla_m V_{ik})\\ &\qquad -g^{\ell m}(R_{ijk}^{r}V_{rm}+R_{ijm}^{q}V_{kq})\\ &= g^{\ell m}(-\nabla_i\nabla_k R_{jm}+\nabla_i\nabla_m R_{jk}+\nabla_j\nabla_k R_{im}-\nabla_j\nabla_m R_{ik})\\ &\qquad + g^{\ell m}(-g_{jm}\nabla_i\nabla_k p+g_{jk}\nabla_i\nabla_m p+g_{im}\nabla_j\nabla_k p-g_{ik}\nabla_j\nabla_m p)\\ &\qquad + g^{\ell m}(R_{ijk}^{r}R_{rm}+R_{ijm}^{r}R_{kr})-\frac{s_0}{n}g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})p\\ &\qquad + g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})p.\end{aligned} \tag{23}$$

Following the calculations in [Chow et al. 2006, pp. 119–120] we have

$$\begin{aligned}\Delta R_{ijk}^{\ell} &= g^{ab}\nabla_a\nabla_b R_{ijk}^{\ell} = g^{ab}(-\nabla_a\nabla_i R_{jbk}^{\ell}-\nabla_a\nabla_j R_{bik}^{\ell})\\ &= g^{ab}\big(-\nabla_i\nabla_a R_{jbk}^{\ell}+R_{aij}^{m}R_{mbk}^{\ell}+R_{aib}^{m}R_{jmk}^{\ell}+R_{aik}^{m}R_{jbm}^{\ell}-R_{aim}^{\ell}R_{jbk}^{m}\\ &\qquad -\nabla_j\nabla_a R_{bik}^{\ell}+R_{ajb}^{m}R_{mik}^{\ell}+R_{aji}^{m}R_{bmk}^{\ell}+R_{ajk}^{m}R_{bim}^{\ell}-R_{ajm}^{\ell}R_{bik}^{m}\big)\\ &= g^{\ell m}(-\nabla_i\nabla_k R_{jm}+\nabla_i\nabla_m R_{jk}+\nabla_j\nabla_k R_{im}-\nabla_j\nabla_m R_{ik})\\ &\qquad + g^{mr}(-R_{ir}R_{jmk}^{\ell}-R_{jr}R_{mik})\\ &\qquad + g^{ab}\big(R_{aij}^{m}R_{mbk}^{\ell}+R_{aik}^{m}R_{jbm}^{\ell}-R_{aim}^{\ell}R_{jbk}^{m}+R_{aji}^{m}R_{bmk}^{\ell}\\ &\qquad\qquad +R_{ajk}^{m}R_{bim}^{\ell}-R_{ajm}^{\ell}R_{bik}^{m}\big).\end{aligned} \tag{24}$$

Combining (23) and (24) we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}^{\ell} &= \Delta R_{ijk}^{\ell}+g^{mr}(R_{ir}R_{jmk}^{\ell}+R_{jr}R_{mik}^{\ell})\\ &\qquad + g^{ab}\big(-R_{aij}^{m}R_{mbk}^{\ell}-R_{aik}^{m}R_{jbm}^{\ell}+R_{aim}^{\ell}R_{jbk}^{m}\\ &\qquad\qquad -R_{aji}^{m}R_{bmk}^{\ell}-R_{ajk}^{m}R_{bim}^{\ell}+R_{ajm}^{\ell}R_{bik}^{m}\big)\\ &\qquad + g^{\ell m}(-g_{jm}\nabla_i\nabla_k p+g_{jk}\nabla_i\nabla_m p+g_{im}\nabla_j\nabla_k p-g_{ik}\nabla_j\nabla_m p)\\ &\qquad + g^{\ell m}(R_{ijk}^{r}R_{rm}+R_{ijm}^{r}R_{kr})-\frac{s_0}{n}g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})\\ &\qquad + g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})p.\end{aligned} \tag{25}$$

Hence the evolution of S is

$$
\begin{aligned}
\frac{\partial}{\partial t} S^{\ell}_{ijk} &= \Delta R^{\ell}_{ijk} - \tilde{\Delta} \tilde{R}^{\ell}_{ijk} \\
&\quad + g^{mr}(R_{ir} R^{\ell}_{jmk} + R_{jr} R^{\ell}_{jmk}) - \tilde{g}^{mr}(\tilde{R}_{ir} \tilde{R}^{\ell}_{jmk} + \tilde{R}_{jr} \tilde{R}^{\ell}_{mik}) \\
&\quad + g^{ab}\big(-R^{m}_{aij} R^{\ell}_{mbk} - R^{m}_{aik} R^{\ell}_{jbm} + R^{\ell}_{aim} R^{m}_{jbk} - R^{m}_{aji} R^{\ell}_{bmk} \\
&\qquad\qquad - R^{m}_{ajk} R^{\ell}_{bim} + R^{\ell}_{ajm} R^{m}_{bik}\big) \\
&\quad - \tilde{g}^{ab}\big(-\tilde{R}^{m}_{aij} \tilde{R}^{\ell}_{mbk} - \tilde{R}^{m}_{aik} \tilde{R}^{\ell}_{jbm} + \tilde{R}^{\ell}_{aim} \tilde{R}^{m}_{jbk} - \tilde{R}^{m}_{aji} \tilde{R}^{\ell}_{bmk} \\
&\qquad\qquad - \tilde{R}^{m}_{ajk} \tilde{R}^{\ell}_{bim} + \tilde{R}^{\ell}_{ajm} \tilde{R}^{m}_{bik}\big) \\
&\quad + g^{\ell m}(-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p) \\
&\quad - \tilde{g}^{\ell m}(-\tilde{g}_{jm} \tilde{\nabla}_i \tilde{\nabla}_k \tilde{p} + \tilde{g}_{jk} \tilde{\nabla}_i \tilde{\nabla}_m \tilde{p} + \tilde{g}_{im} \tilde{\nabla}_j \tilde{\nabla}_k \tilde{p} - \tilde{g}_{ik} \tilde{\nabla}_j \tilde{\nabla}_m \tilde{p}) \\
&\quad + g^{\ell m}(R^{r}_{ijk} R_{rm} + R^{r}_{ijm} R_{kr}) - \tilde{g}^{\ell m}(\tilde{R}^{r}_{ijk} \tilde{R}_{rm} + \tilde{R}^{r}_{ijm} \tilde{R}_{kr}) \\
&\quad - \frac{s_0}{n} g^{\ell m}(R^{r}_{ijk} g_{rm} + R^{r}_{ijm} g_{kr}) + \frac{s_0}{n} \tilde{g}^{\ell m}(\tilde{R}^{r}_{ijk} \tilde{g}_{rm} + \tilde{R}^{r}_{ijm} \tilde{g}_{kr}) \\
&\quad + g^{\ell m}(R^{r}_{ijk} g_{rm} + R^{r}_{ijm} g_{kr}) p - \tilde{g}^{\ell m}(\tilde{R}^{r}_{ijk} \tilde{g}_{rm} + \tilde{R}^{r}_{ijm} \tilde{g}_{kr}) \tilde{p}.
\end{aligned} \tag{26}
$$

Looking at the individual components, we see

$$
\begin{aligned}
\Delta R - \tilde{\Delta} \tilde{R} &= g^{ab} \nabla_a \nabla_b R - \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{R} \\
&= \nabla_a (g^{ab} \nabla_b R) - \nabla_a (\tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + (\nabla_a - \tilde{\nabla}_a)(\tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) \\
&= \nabla_a (g^{ab} \nabla_b R - \tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + \tilde{g}^{-1} * A * \tilde{\nabla} \tilde{R},
\end{aligned} \tag{27}
$$

while

$$
\begin{aligned}
g^{-1} R R - \tilde{g}^{-1} \tilde{R} \tilde{R} &= (g^{-1} - \tilde{g}^{-1})(\tilde{R} \tilde{R}) + g^{-1}(R R - \tilde{R} \tilde{R}) \\
&= \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + g^{-1}(R - \tilde{R}) \tilde{R} + g^{-1}(R R - R \tilde{R}) \\
&= \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R,
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
g^{-1} g \nabla \nabla p - \tilde{g}^{-1} \tilde{g} \tilde{\nabla} \tilde{\nabla} \tilde{p} &= (g^{-1} - \tilde{g}^{-1}) \tilde{g} \tilde{\nabla} \tilde{\nabla} \tilde{p} + g^{-1}(g - \tilde{g}) \tilde{\nabla} \tilde{\nabla} \tilde{p} \\
&\qquad + g^{-1} g (\nabla \nabla p - \tilde{\nabla} \tilde{\nabla} \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} \\
&\qquad + g^{-1} g (\nabla - \tilde{\nabla})(\tilde{\nabla} \tilde{p}) + g^{-1} g (\nabla \nabla p - \nabla \tilde{\nabla} \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\
&\qquad + g^{-1} g \nabla (\nabla - \tilde{\nabla}) \tilde{p} + g^{-1} g \nabla \nabla (p - \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\
&\qquad + h * A * \tilde{\nabla} \tilde{p} + C(\nabla \nabla q),
\end{aligned} \tag{29}
$$

where in the last equality we used (10). We also have

$$
\begin{aligned}
g^{-1}gR-\tilde g^{-1}\tilde g\tilde R &= (g^{-1}-\tilde g^{-1})\tilde g\tilde R+g^{-1}(g-\tilde g)\tilde R+g^{-1}g(R-\tilde R)\\
&=\tilde g^{-1}*h*\tilde g*\tilde R+h*\tilde R+C(S),
\end{aligned}\tag{30}
$$

and lastly

$$
\begin{aligned}
g^{-1}gRp-\tilde g^{-1}\tilde g\tilde R\tilde p &= (g^{-1}-\tilde g^{-1})\tilde g\tilde R\tilde p+g^{-1}(g-\tilde g)\tilde R\tilde p\\
&\qquad +g^{-1}g(R-\tilde R)\tilde p+g^{-1}gR(p-\tilde p)\\
&=\tilde g^{-1}*h*\tilde g*\tilde R*\tilde p+h*\tilde R*\tilde p+S*\tilde p+R*q.
\end{aligned}\tag{31}
$$

Now by (26), (27), (28), (29), (30) and (31) we see

$$
\begin{aligned}
\frac{\partial}{\partial t}S=\nabla_a(g^{ab}\nabla_bR-\tilde g^{ab}\tilde\nabla_b\tilde R)&+\tilde g^{-1}*A*\tilde\nabla\tilde R+\tilde g^{-1}*h*\tilde R*\tilde R\\
&+S*\tilde R+S*R+\tilde g^{-1}*h*\tilde g*\tilde\nabla\tilde\nabla\tilde p+h*\tilde\nabla\tilde\nabla\tilde p+A*\tilde\nabla\tilde p\\
&+h*A*\tilde\nabla\tilde p+C(\nabla\nabla q)+\tilde g^{-1}*h*\tilde g*\tilde R+h*\tilde R+C(S)\\
&+\tilde g^{-1}*h*\tilde g*\tilde R*\tilde p+h*\tilde R*\tilde p+S*\tilde p+R*q.
\end{aligned}
$$

Hence by (12) we have

$$
\begin{aligned}
\Big|\frac{\partial}{\partial t}S-\Delta S-\operatorname{div}U\Big|\le C\Big(&\big(|\tilde g^{-1}||\tilde R|^2+|\tilde g^{-1}||\tilde g||\tilde\nabla\tilde\nabla\tilde p|+|\tilde\nabla\tilde\nabla\tilde p|\\
&+|\tilde g^{-1}||\tilde g||\tilde R|+|\tilde R|+|\tilde g^{-1}||\tilde g||\tilde R||\tilde p|+|\tilde R||\tilde p|\big)|h|\\
&+(|\tilde g^{-1}||\tilde\nabla\tilde R|+|\tilde\nabla\tilde p|+|h||\tilde\nabla\tilde p|)|A|\\
&+(|\tilde R|+|R|+1+|\tilde p|)|S|+|R||q|+|\nabla\nabla q|\Big).
\end{aligned}\tag{32}
$$

This proves (15). □

Remark 4. Upon closer observation we notice the following dependencies:

$$N_h=N_h(n,s_0,|\tilde p|).$$
$$N_A=N_A(n,s_0,|\tilde g|,|\tilde g^{-1}|,|\tilde R|,|\tilde\nabla\tilde R|,|\tilde\nabla\tilde p|).$$
$$N_S=N_S(n,s_0,|\tilde g|,|\tilde g^{-1}|,|h|,|R|,|\tilde R|,|\tilde\nabla\tilde R|,|\tilde p|,|\tilde\nabla\tilde p|,|\tilde\nabla\tilde\nabla\tilde p|).$$

M is closed, so $M\times[0,T]$ is compact. Thus, given two metrics g and $\tilde g$, all of these quantities are bounded.

2.3. ***Bounds on q and its spatial derivatives.*** We turn our attention now to finding bounds on the differences between our pressure functions p and $\tilde p$. We have the following proposition:

Proposition 5. *Let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ be two solutions of* (1) *on $M \times [0, T]$. Then there exist constants N_q and $\hat{N}_q$ such that*

$$\int_M |q|^2\, d\mu \le N_q \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu, \tag{33}$$

$$\int_M |\nabla q|^2\, d\mu \le N_q \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu, \tag{34}$$

$$\int_M |\nabla\nabla q|^2\, d\mu \le \hat{N}_q \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu. \tag{35}$$

Proof. We let f represent any smooth function or tensor on M. This is general, but in this paper we represent f by the function q, the difference of the pressure functions. Since M is compact we have

$$\int_M ((n-1)\Delta + s_0)(f) \cdot f\, d\mu = s_0 \int_M |f|^2\, d\mu - (n-1) \int_M \langle \nabla f, \nabla f \rangle\, d\mu.$$

Since $s_0 < 0$, taking the absolute value gives

$$\left| \int_M ((n-1)\Delta + s_0)(f) \cdot f\, d\mu \right| = |s_0| \int_M |f|^2\, d\mu + (n-1) \int_M |\nabla f|^2\, d\mu. \tag{36}$$

Now we deal specifically with p, $\tilde{p}$ and q. By (2) we have the following equations for the pressure functions p and $\tilde{p}$:

$$((n-1)\Delta + s_0)p = -\left\langle \mathrm{Ric} - \frac{s_0}{n} g, \mathrm{Ric} - \frac{s_0}{n} g \right\rangle. \tag{37}$$

$$((n-1)\tilde{\Delta} + s_0)\tilde{p} = -\left\langle \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g}, \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g} \right\rangle. \tag{38}$$

Now we calculate

$$\begin{aligned} \Delta p - \tilde{\Delta}\tilde{p} &= g^{ab}\nabla_a\nabla_b p - \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{p} \\ &= (g^{-1} - \tilde{g}^{-1})\tilde{\nabla}\tilde{\nabla}\tilde{p} + g^{-1}(\nabla - \tilde{\nabla})\tilde{\nabla}\tilde{p} + g^{-1}\nabla(\nabla - \tilde{\nabla})\tilde{p} + \Delta(p - \tilde{p}) \\ &= \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + \Delta q. \end{aligned} \tag{39}$$

We also compute

$$\begin{aligned} &-\left\langle \mathrm{Ric} - \frac{s_0}{n} g, \mathrm{Ric} - \frac{s_0}{n} g \right\rangle + \left\langle \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g}, \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g} \right\rangle \\ &\quad = -(g^{ik} g^{j\ell} R_{ij} R_{k\ell} - \tilde{g}^{ik} \tilde{g}^{j\ell} \tilde{R}_{ij} \tilde{R}_{k\ell}) + 2\frac{s_0}{n}(g^{ij} R_{ij} - \tilde{g}^{ij} \tilde{R}_{ij}) \\ &\quad = -(g^{-1} - \tilde{g}^{-1})\tilde{g}^{-1}\tilde{R}\tilde{R} - g^{-1}(g^{-1} - \tilde{g}^{-1})\tilde{R}\tilde{R} - g^{-1}g^{-1}(R - \tilde{R})\tilde{R} \\ &\qquad\qquad - g^{-1}g^{-1}R(R - \tilde{R}) + 2\frac{s_0}{n}(g^{-1} - \tilde{g}^{-1})\tilde{R} + 2\frac{s_0}{n} g^{-1}(R - \tilde{R}) \\ &\quad = \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\ &\qquad\qquad + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S). \end{aligned} \tag{40}$$

Combining (37), (38), (39) and (40), we see that q satisfies the following elliptic equation at each time $t \in [0, T]$:

$$
\begin{aligned}
(41)\quad Lq &= ((n-1)\Delta + s_0)(q) \\
&= \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\
&\qquad + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S).
\end{aligned}
$$

Hence

$$
(42)\qquad |Lq| = \left|((n-1)\Delta + s_0)(q)\right| \leq N(|h| + |A| + |S|).
$$

To find estimates for q and ∇q, we combine (36) and (42):

$$
\begin{aligned}
|s_0|\int_M |q|^2\,d\mu + (n-1)\int_M |\nabla q|^2\,d\mu &= \left|\int_M ((n-1)\Delta + s_0)(q)\cdot q\,d\mu\right| \\
&\leq \int_M N(|h|+|A|+|S|)|q|\,d\mu \\
&\leq \frac{|s_0|}{2}\int_M |q|^2\,d\mu + N\int_M (|h|^2+|A|^2+|S|^2)\,d\mu.
\end{aligned}
$$

Thus

$$
\frac{|s_0|}{2}\int_M |q|^2\,d\mu + (n-1)\int_M |\nabla q|^2\,d\mu \leq N\int_M (|h|^2 + |A|^2 + |S|^2)\,d\mu,
$$

and we proved (33) and (34).

To find an appropriate bound for $|\nabla\nabla q|$ we use interior regularity theory for elliptic PDEs. From (41) we see that $Lq = f$ is an elliptic equation. We then have the following estimate from [Rauch 1991, p. 229]:

$$
|q|_{H^2(W)} \leq K(|Lq|_{L^2(M)} + |q|_{H^1(M)}),
$$

where W is any compactly supported open subset of M and K depends only upon the coefficients of the operator L, the subset W and the manifold M. Since M is a closed manifold we may in fact choose $W = M$. Thus we have

$$
(43)\qquad |q|_{H^2(M)} \leq K(|Lq|_{L^2(M)} + |q|_{H^1(M)}).
$$

Upon squaring both sides we observe

$$
(44)\qquad \int_M |\nabla\nabla q|^2\,d\mu \leq |q|^2_{H^2(M)} \leq K^2\biggl(\int_M |Lq|^2\,d\mu + |q|^2_{H^1(M)}\biggr).
$$

Now (33) and (34) imply that

$$|q|^2_{H^1(M)} \le N \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu. \tag{45}$$

Combining (42), (44) and (45) we have

$$\int_M |\nabla\nabla q|^2\, d\mu \le N \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu,$$

and we have proved (35). □

Remark 6. We observe the following dependencies:

$$N_q = N_q(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|).$$
$$\hat{N}_q = \hat{N}_q(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|, K),$$

where K is from (43).

3. Energy estimates

We now define the energy functional

$$\mathcal{E}(t) = \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu, \tag{46}$$

as well as the following:

$$\mathcal{H}(t) = \int_M |h|^2\, d\mu. \tag{47}$$

$$\mathcal{A}(t) = \int_M |A|^2\, d\mu. \tag{48}$$

$$\mathcal{S}(t) = \int_M |S|^2\, d\mu. \tag{49}$$

$$\mathcal{D}(t) = \int_M |\nabla S|^2\, d\mu. \tag{50}$$

Note that $\mathcal{E}(t) = \mathcal{H}(t) + \mathcal{A}(t) + \mathcal{S}(t)$. We now estimate the evolution of the energy functional under conformal Ricci flow, $\mathcal{E}'(t)$, by first estimating the evolutions of $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{S}$.

3.1. ***Evolution of $\mathcal{H}(t)$.*** Lu, Qing and Zheng [2014] give the evolution of the volume element under conformal Ricci flow

$$\frac{\partial}{\partial t}\, d\mu_{g(t)} = -np(t)\, d\mu_{g(t)}. \tag{51}$$

Hence by (13) and (47) we have

$$\begin{aligned}\mathcal{H}'(t) &\le N\int_M |h|^2\,d\mu + \int_M 2\Big\langle \frac{\partial h}{\partial t}, h\Big\rangle\,d\mu \\ &\le N\mathcal{H}(t) + \int_M 2|h|\Big|\frac{\partial h}{\partial t}\Big|\,d\mu \\ &\le N\mathcal{H}(t) + N\int_M (|S|\,|h| + |h|^2 + |q|\,|h|)\,d\mu.\end{aligned}$$

Now we know that $N(|S|\,|h| + |q|\,|h|) \le N(|h|^2 + |S|^2 + |q|^2)$. Hence

$$\begin{aligned}\mathcal{H}'(t) &\le N\mathcal{H}(t) + N\int_M (|S|^2 + |q|^2)\,d\mu \\ &\le N\mathcal{H}(t) + N\int_M (|S|^2 + |h|^2 + |A|^2)\,d\mu \\ &\le N\mathcal{H}(t) + N\mathcal{S}(t) + N\mathcal{A}(t) = N\mathcal{E}(t).\end{aligned} \tag{52}$$

3.2. *Evolution of $\mathcal{A}(t)$.* By (14), (48) and (51) we have

$$\begin{aligned}\mathcal{A}'(t) &\le N\mathcal{A}(t) + \int_M 2|A|\Big|\frac{\partial A}{\partial t}\Big|\,d\mu \\ &\le N\mathcal{A}(t) + \int_M (N|h|\,|A| + N|A|^2 + C|\nabla S|\,|A| + C|\nabla q|\,|A|)\,d\mu.\end{aligned}$$

Now

$$N|h|\,|A| + C|\nabla S|\,|A| + C|\nabla q|\,|A| \le N|h|^2 + N|A|^2 + |\nabla S|^2 + |\nabla q|^2.$$

Hence we have that

$$\begin{aligned}\mathcal{A}'(t) &\le N\mathcal{A}(t) + \int_M (N|h|^2 + N|A|^2 + |\nabla S|^2 + |\nabla q|^2)\,d\mu \\ &\le N\mathcal{A}(t) + N\mathcal{H}(t) + \mathcal{D}(t) + N\int_M (|h|^2 + |A|^2 + |S|^2)\,d\mu \\ &\le N\mathcal{A}(t) + N\mathcal{H}(t) + N\mathcal{S}(t) + \mathcal{D}(t) = N\mathcal{E}(t) + \mathcal{D}(t).\end{aligned} \tag{53}$$

3.3. *Evolution of $\mathcal{S}(t)$.* By (15), (49) and (51) we have

$$\begin{aligned}\mathcal{S}'(t) &\le N\int_M |S|^2\,d\mu + \int_M 2\Big\langle \frac{\partial S}{\partial t}, S\Big\rangle\,d\mu \\ &\le N\mathcal{S}(t) + \int_M \big(2\langle \Delta S + \operatorname{div} V, S\rangle + N(|h| + |A| + |S| + |q|)|S| + C|\nabla\nabla q|\,|S|\big)\,d\mu \\ &\le N\mathcal{S}(t) + \int_M \big(2\langle \Delta S + \operatorname{div} V, S\rangle + N(|h|^2 + |A|^2 + |S|^2 + |q|^2 + |\nabla\nabla q|^2)\big)\,d\mu.\end{aligned}$$

Now by (33) and (35) we have

$$\begin{aligned}\mathcal{S}'(t) &\le N\mathcal{S}(t) + N\mathcal{H}(t) + N\mathcal{A}(t) \\ &\qquad + \int_M \big(2\langle \Delta S + \operatorname{div} V, S\rangle + N(|A|^2 + |S|^2 + |h|^2)\big)\, d\mu \\ &\le N\mathcal{S}(t) + N\mathcal{H}(t) + N\mathcal{A}(t) + \int_M 2\langle \Delta S + \operatorname{div} V, S\rangle\, d\mu.\end{aligned}$$

Upon integrating by parts we get

$$\begin{aligned}\mathcal{S}'(t) &\le N\mathcal{E}(t) - 2\int_M \langle \nabla S + V, \nabla S\rangle\, d\mu \\ &\le N\mathcal{E}(t) - 2\int_M |\nabla S|^2\, d\mu + \int_M 2|V|\,|\nabla S|\, d\mu.\end{aligned}$$

Now we know that

$$2|V|\,|\nabla S| \le |\nabla S|^2 + |V|^2 \le |\nabla S|^2 + N(|h|^2 + |A|^2),$$

hence

$$\mathcal{S}'(t) \le N\mathcal{E}(t) + N\int_M (|h|^2 + |A|^2)\, d\mu - \int_M |\nabla S|^2\, d\mu \le N\mathcal{E}(t) - \mathcal{D}(t). \tag{54}$$

3.4. *Proof of main theorem.* We are now ready to prove Theorem 1.

Proof. By (52), (53) and (54) we know that

$$\mathcal{H}'(t) \le N\mathcal{E}(t), \quad \mathcal{A}'(t) \le N\mathcal{E}(t) + \mathcal{D}(t) \quad \text{and} \quad \mathcal{S}'(t) \le N\mathcal{E}(t) - \mathcal{D}(t),$$

so

$$\mathcal{E}'(t) \le N\mathcal{E}(t).$$

Our initial condition $\tilde{g}(0) = g(0)$ tells us that at $t = 0$ we have $|h| = |A| = |S| = 0$. Therefore by the smoothness and integrability of our solutions we know

$$\lim_{t\to 0^+} \mathcal{E}(t) = 0,$$

so by Gronwall's inequality we know that $\mathcal{E} \equiv 0$ on $[0, T]$. Thus for $t \in [0, T]$ we have that $h \equiv 0$ and $g(t) \equiv \tilde{g}(t)$. Also, $\mathcal{E} \equiv 0$ implies $A \equiv 0$ and $S \equiv 0$, so (33) forces $q \equiv 0$. Thus $p(t) \equiv \tilde{p}(t)$. Therefore $(\tilde{g}(t), \tilde{p}(t)) = (g(t), p(t))$, $t \in [0, T]$. □

4. Further research

The arguments in this paper are only valid for conformal Ricci flow on a compact manifold with constant positive scalar curvature. In particular, if $s_0 \ge 0$ we do not have the equality (36). It is worth discovering whether or not there is some other way to compute the bounds on q and its derivatives, namely equations (33), (34) and (35).

It is also interesting to consider complete noncompact manifolds of constant scalar curvature. Previous results in Ricci flow and parabolic PDE suggest that in this case we will not achieve uniqueness of conformal Ricci flow without some sort of bound on the curvature of the manifold.

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THOMAS BELL
DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
275 TMCB
PROVO, UT 84602
UNITED STATES
thomas@mathematics.byu.edu

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A FUNCTIONAL CALCULUS AND RESTRICTION THEOREM ON H-TYPE GROUPS

HEPING LIU AND MANLI SONG

Let L be the sublaplacian and T the partial laplacian with respect to central variables on H-type groups. We investigate a class of invariant differential operators by the joint functional calculus of L and T. We establish Stein–Tomas type restriction theorems for these operators. In particular, the asymptotic behaviors of restriction estimates are given.

1. Introduction

The restriction theorem for the Fourier transform plays an important role in harmonic analysis as well as in the theory of partial differential equations. The original version is credited to E. M. Stein and P. A. Tomas, and states that the transform of an L^p-function on $\mathbb{R}^n$ has a well-defined restriction to the unit sphere S^{n-1} which is square integrable on S^{n-1}. The result is listed as follows:

Theorem 1.1 [Stein 1993; Tomas 1975]. *Let* $1 \le p \le \dfrac{2n+2}{n+3}$. *Then the estimate*

$$\|\hat{f}\|_{L^2(S^{n-1})} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1-1}$$

holds for all functions $f \in L^p(\mathbb{R}^n)$.

A simple duality argument shows that the estimate (1-1) is equivalent to the following estimate:

$$\|f * \widehat{d\sigma_r}\|_{p'} \le C_r\|f\|_p \tag{1-2}$$

for all Schwartz functions f on $\mathbb{R}^n$, where $1/p + 1/p' = 1$ and $d\sigma_r$ is the surface measure on the sphere with radius r.

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Moreover, according to the Knapp example [Stein 1993], the estimates (1-1) and (1-2) fail if $(2n+2)/(n+3) < p \leq 2$.

Many authors have worked on the topic and various new restriction theorems have been proved. The study of restriction theorems has recently obtained more and more attention. A survey of recent progress on restriction theorems can be found in [Tao 2004]. To generalize the restriction theorem on the Heisenberg group, D. Müller [1990] established the boundedness of the restriction operator with respect to the mixed L^p-norm and also gave a counterexample to show that the estimate between Lebesgue spaces for the restriction operator was necessarily trivial, due to the fact that the center of the Heisenberg group was of dimension one. Some extensions have been treated by S. Thangavelu [1991a; 1991b]. Restriction theorems have been also studied in the case of the Heisenberg motion group by P. K. Ratnakumar, R. Rawat and S. Thangavelu [Ratnakumar et al. 1997], where groups with center with dimension higher than one were first considered.

On an H-type group, let T be the laplacian on the center and L the sublaplacian. It is well known that L is positive and essentially self-adjoint. Let $L = \int_0^\infty \lambda \, dE(\lambda)$ be the spectral decomposition of L. Then the restriction operator can be formally written $\mathcal{P}_\lambda f = \delta_\lambda(L) f = \lim_{\epsilon \to \infty} \chi_{(\lambda-\epsilon,\lambda+\epsilon)}(L) f$ which is well defined for a Schwartz function f, where $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}$ is the characteristic function of the interval $(\lambda-\epsilon, \lambda+\epsilon)$. Liu and Wang [2011] investigated the restriction theorem for the sublaplacian L on H-type groups with center whose dimension was greater than one. They gave the following result:

Theorem 1.2. *Let G be an H-type group with the underlying manifold $\mathbb{R}^{2n+m}$, where $m > 1$ is the dimension of the center. Suppose $1 \leq p \leq (2m+2)/(m+3)$. Then the following estimate*

$$\|\mathcal{P}_\lambda f\|_{p'} \leq C \lambda^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \|f\|_p, \lambda > 0$$

holds for all Schwartz functions f on G.

V. Casarino and P. Ciatti [2013a; 2013b] extended the results of Müller, Liu and Wang to Métivier groups. They proved the restriction theorem for the sublaplacian and the full laplacian on Métivier groups. In fact, they also investigated the joint functional calculus of L and T. The invariant differential operators related to the joint functional calculus of L and T on H-type groups do not have the homogeneous properties in general. Thus the asymptotic behaviors of restriction estimates for these operators are also interesting. Casarino and Ciatti [2013a; 2013b] did not discuss the asymptotic behavior of the full laplacian. In this article we will show restriction estimates for these operators on H-type groups. In particular, the asymptotic behaviors of restriction estimates are given.

The outline of the paper is as follows. In the second section, we provide the necessary background for the H-type group. In the third section, by introducing the joint functional calculus of L and T, the restriction operator can be computed explicitly. In the fourth section, we prove the restriction theorem on H-type groups. In the fifth section, we describe the restriction theorems for other operators with the form of the joint functional calculus of L and T. Finally, in the last section, we show that the range of p in the restriction theorem is sharp.

2. Preliminaries

Definition 2.1 (H-type group). Let $\mathfrak{g}$ be a two step nilpotent Lie algebra endowed with an inner product $\langle \cdot, \cdot \rangle$. Its center is denoted by $\mathfrak{z}$. The algebra $\mathfrak{g}$ is said to be of H-type if $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$ and for every $t \in \mathfrak{z}$, the map $J_t : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_t u, w \rangle := \langle t, [u, w] \rangle \text{ for all } u, w \in \mathfrak{z}^\perp$$

is an orthogonal map whenever $|t| = 1$.

An H-type group is a connected and simply connected Lie group G whose Lie algebra is of H-type.

For a given $0 \neq a \in \mathfrak{z}^*$, the dual of $\mathfrak{z}$, we can define a skew-symmetric mapping $B(a)$ on $\mathfrak{z}^\perp$ by

$$\langle B(a)u, w \rangle = a([u, w]) \text{ for all } u, w \in \mathfrak{z}^\perp.$$

We denote by z_a the element of $\mathfrak{z}$ determined by

$$\langle B(a)u, w \rangle = a([u, w]) = \langle J_{z_a} u, w \rangle.$$

Since $B(a)$ is skew-symmetric and nondegenerate, the dimension of $\mathfrak{z}^\perp$ is even, i.e., $\dim \mathfrak{z}^\perp = 2n$.

For a given $0 \neq a \in \mathfrak{z}^*$, we can choose an orthonormal basis

$$\{E_1(a), E_2(a), \dots, E_n(a), \overline{E}_1(a), \overline{E}_2(a), \dots, \overline{E}_n(a)\}$$

of $\mathfrak{z}^\perp$ such that

$$B(a)E_i(a) = |z_a| J_{\frac{z_a}{|z_a|}} E_i(a) = |a| \overline{E}_i(a)$$

and

$$B(a)\overline{E}_i(a) = -|a| E_i(a).$$

We set $m = \dim \mathfrak{z}$. Throughout this paper we assume that $m > 1$. We can choose an orthonormal basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_m\}$ of $\mathfrak{z}$ such that $a(\epsilon_1) = |a|$ and $a(\epsilon_j) = 0$, $j = 2, 3, \dots, m$. Then we can denote the elements of $\mathfrak{g}$ by

$$(z, t) = (x, y, t) = \sum_{i=1}^{n} (x_i E_i + y_i \overline{E}_i) + \sum_{j=1}^{m} t_j \epsilon_j.$$

We identify G with its Lie algebra $\mathfrak{g}$ via the exponential map. The group law on H-type group G has the form

$$(z,t)(z',t') = (z+z', t+t'+\tfrac{1}{2}[z,z']), \tag{2-1}$$

where $[z,z']_j = \langle z, U^j z'\rangle$ for a suitable skew-symmetric matrix U^j, $j = 1,2,\dots,m$.

Theorem 2.2. *G is an H-type group with underlying manifold $\mathbb{R}^{2n+m}$, with the group law* (2-1) *and the matrix U^j, $j = 1,2,\dots,m$ satisfies the following conditions*:

(i) *U^j is a $2n \times 2n$ skew-symmetric and orthogonal matrix, $j = 1,2,\dots,m$.*

(ii) *$U^i U^j + U^j U^i = 0$, where $i, j = 1,2,\dots,m$ with $i \neq j$.*

Proof. See [Bonfiglioli and Uguzzoni 2004]. □

Remark 2.3. In particular, $\langle z, U^1 z'\rangle = \sum_{j=1}^{n}(x'_j y_j - y'_j x_j)$.

Remark 2.4. All the above expressions depend on a given $0 \neq a \in \mathfrak{z}^*$, but we will suppress a from them for simplification.

Remark 2.5. It is well know that H-type algebras are closely related to Clifford modules [Reimann 2001]. H-type algebras can be classified by the standard theory of Clifford algebras. Especially, on the H-type group G, there is a relation between the dimension of the center and its orthogonal complement space. That is $m+1 \leq 2n$ (see [Kaplan and Ricci 1983]).

The left invariant vector fields which agree respectively with $\partial/\partial x_j$, $\partial/\partial y_j$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{m}\left(\sum_{l=1}^{2n} z_l U^k_{l,j}\right)\frac{\partial}{\partial t_k},$$

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{k=1}^{m}\left(\sum_{l=1}^{2n} z_l U^k_{l,j+n}\right)\frac{\partial}{\partial t_k},$$

where $z_l = x_l,\ z_{l+n} = y_l,\ l = 1,2,\dots,n$.

The vector fields $T_k = \partial/\partial t_k,\ k = 1,2,\dots,m$ correspond to the center of G. In terms of these vector fields, we introduce the sublaplacian L and full laplacian Δ respectively

$$L = -\sum_{j=1}^{n}(X_j^2 + Y_j^2) = -\Delta_z + \tfrac{1}{4}|z|^2 T - \sum_{k=1}^{m}\langle z, U^k \nabla_z\rangle T_k, \tag{2-2}$$

$$\Delta = L + T, \tag{2-3}$$

where

$$\Delta_z = \sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j^2}, \quad T = -\sum_{k=1}^{m} \frac{\partial^2}{\partial t_k^2}, \quad \nabla_z = \Big(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{2n}}\Big)^t.$$

3. The restriction operator

First we recall some results about the scaled special Hermite expansion. We refer the reader to [Thangavelu 1993, 2004] for details. Letting $\lambda > 0$, the twisted laplacian (or the scaled special Hermite expansion) L_λ is defined by

$$L_\lambda = -\Delta_z + \frac{\lambda^2 |z|^2}{4} - i\lambda \sum_{j=1}^{n} \Big(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \Big),$$

where we identify $z = x + iy \in \mathbb{C}^n$ with $z = (x, y) \in \mathbb{R}^{2n}$.

For $f, g \in L^1(\mathbb{C}^n)$, we define the λ-twisted convolution by

$$f \times_\lambda g = \int_{\mathbb{C}^n} f(z-w) g(w) e^{\frac{1}{2} i \lambda \operatorname{Im} z \cdot \overline{w}} \, dw.$$

Set Laguerre function $\varphi_k^\lambda(z) = L_k^{n-1}\big(\frac{1}{2}\lambda |z|^2\big) e^{-\frac{1}{4}\lambda |z|^2}$, $k = 0, 1, 2, \dots$, where L_k^{n-1} is the Laguerre polynomial of type $(n-1)$ and degree k.

For any Schwartz function f on $\mathbb{C}^n$, we have the scaled special Hermite expansion

$$f(z) = \Big(\frac{\lambda}{2\pi}\Big)^n \sum_{k=0}^{\infty} f \times_\lambda \varphi_k^\lambda(z), \tag{3-1}$$

which is an orthogonal form. We also have

$$\|f\|^2 = \Big(\frac{\lambda}{2\pi}\Big)^n \sum_{k=0}^{\infty} \|f \times_\lambda \varphi_k^\lambda\|^2. \tag{3-2}$$

Moreover, $f \times_\lambda \varphi_k^\lambda$ is an eigenfunction of L_λ with the eigenvalue $(2k+n)\lambda$ and

$$\|f \times_\lambda \varphi_k^\lambda\|_2 \le (2k+n)^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \lambda^{n\left(\frac{1}{p}-\frac{3}{2}\right)} \|f\|_p \quad \text{for } 1 \le p < \frac{6n+2}{3n+4} \tag{3-3}$$

(see [Thangavelu 1991b]).

Now we turn to the expression for the restriction operator. We may identify $\mathfrak{z}^*$ with $\mathfrak{z}$. Therefore, we will write $\langle a, t\rangle$ instead of $a(t)$ for $a \in \mathfrak{z}^*$ and $t \in \mathfrak{z}$.

Lemma 3.1. *Let $0 \neq a \in \mathfrak{z}^*$. If $f(z,t) = e^{-i\langle a,t\rangle}\varphi(z)$, then*

$$Lf(z,t) = e^{-i\langle a,t\rangle} L_{|a|}\varphi(z).$$

Proof. Because $\langle a, t\rangle = |a| t_1$ and $\langle z, U^1 \nabla_z\rangle = \sum_{j=1}^n \big(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}\big)$, Lemma 3.1 is easily deduced from the expression (2-2). □

Set $e_k^a(z,t) = e^{-i\langle a,t\rangle}\varphi_k^{|a|}(z)$. For $f \in \mathscr{S}(G)$, let

$$f^a(z) = \int_{\mathbb{R}^m} f(z,t)e^{i\langle a,t\rangle}\,dt$$

be the Fourier transform of f with respect to the central variable t. It is easy to obtain

$$f * e_k^a(z,t) = e^{-i\langle a,t\rangle} f^a \times_{|a|} \varphi_k^{|a|}(z). \tag{3-4}$$

Note that $f * e_k^a$ is an eigenfunction of T with the eigenvalue $|a|^2$. Furthermore, it follows from Lemma 3.1 that $f * e_k^a$ is an eigenfunction of L with the eigenvalue $(2k+n)|a|$. Thus $f * e_k^a$ is a joint eigenfunction of the operators L and T.

For a Schwartz function f on an H-type group, using the inversion formula for the Fourier transform together with (3-1) and (3-4), we have

$$\begin{aligned}
f(z,t) &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} f^a(z)e^{-i\langle a,t\rangle}\,da\\
&= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\biggl(\frac{|a|^n}{(2\pi)^n}\sum_{k=0}^{\infty} f^a \times_{|a|} \varphi_k^{|a|}(z)\biggr)e^{-i\langle a,t\rangle}\,da\\
&= \frac{1}{(2\pi)^{n+m}}\int_{\mathbb{R}^m}\sum_{k=0}^{\infty} f * e_k^a(z,t)|a|^n\,da\\
&= \int_0^\infty\biggl(\frac{1}{(2\pi)^{n+m}}\sum_{k=0}^{\infty}\lambda^{n+m-1}\int_{S^{m-1}} f * e_k^{\lambda\tilde{a}}(z,t)\,d\sigma(\tilde{a})\biggr)\,d\lambda.
\end{aligned}$$

The operators L and T extend to a pair of strongly commuting self-adjoint operators. Therefore, they admit a joint spectral decomposition. By the spectral theorem, we can define the joint functional calculus of L and T. The joint functional calculus of L and T was investigated in [Casarino and Ciatti 2013a]. As in that paper, we define the operator $\delta_\mu(h(L,T))$ for a suitable function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as

$$h(L,T)f(z,t) = \int_0^\infty\biggl(\frac{1}{(2\pi)^{n+m}}\sum_{k=0}^{\infty} h\bigl((2k+n)\lambda,\lambda^2\bigr)\lambda^{n+m-1}\int_{S^{m-1}} f * e_k^{\lambda\tilde{a}}(z,t)\,d\sigma(\tilde{a})\biggr)\,d\lambda,$$

where we make the assumption on h that the expression on the right-hand side is a well-defined distribution for all Schwartz functions f. We also suppose $h((2k+n)\lambda,\lambda^2)$ is a strictly monotonic differentiable positive function of λ on $\mathbb{R}_+$, with the domain (A,B) where $0 \le A < B \le \infty$. Then for each $\mu \in (A,B)$, the equation

$$h((2k+n)\lambda,\lambda^2) = \mu$$

may be solved for each k. We denote the solution by $\lambda = \lambda_k(\mu)$ and λ_k' denotes the derivative of λ_k. Replacing λ with μ in the integral, we obtain

$$h(L,T)f(z,t) = \int_A^B \mu\left(\frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)|\lambda_k'(\mu)| \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a})\right) d\mu,$$

which is the spectral decomposition of $h(L,T)$.

Thus, given a Schwartz function f, the spectral decomposition with respect to $h(L,T)$ is

$$f(z,t) = \int_A^B \left(\frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)|\lambda_k'(\mu)| \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a})\right) d\mu.$$

We can also use this equation to introduce the spectral resolution of $h(L,T)$, which is defined by

$$\mathcal{P}_\mu^h f(z,t) = \delta_\mu(h(L,T))f(z,t) = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon} \chi_{(\mu-\epsilon,\mu+\epsilon)}(h(L,T))f, \tag{3-5}$$

where f is a Schwartz function and $\chi_{(\mu-\epsilon,\mu+\epsilon)}$ is the characteristic function of the interval $(\mu-\epsilon,\mu+\epsilon)$. We easily find

$$\mathcal{P}_\mu^h f(z,t) = \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)|\lambda_k'(\mu)| \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a}).$$

Specifically, for the full laplacian Δ, $h(\xi,\eta) = \xi+\eta$, so we have $\mu = (2k+n)\lambda+\lambda^2$, which yields

$$\lambda_k(\mu) = \frac{1}{2}\sqrt{4\mu + (2k+n)^2} - \frac{2k+n}{2} \quad \text{and} \quad \lambda_k'(\mu) = \frac{1}{\sqrt{4\mu+(2k+n)^2}}. \tag{3-6}$$

Therefore,

$$\mathcal{P}_\mu^\Delta f(z,t) = \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)\lambda_k'(\mu) \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a}).$$

4. The restriction theorem

Our main result is the following theorem.

Theorem 4.1. *Let G be an H-type group with the underlying manifold $\mathbb{R}^{2n+m}$, where $m > 1$ is the dimension of the center. Let $h(\xi,\eta) = \xi^\alpha + \eta^\beta$, $\alpha, \beta > 0$. Then* *for* $1 \le p \le (2m+2)/(m+3)$, *we have for all Schwartz functions* f:

$$\text{if } \alpha < 2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1,\\ \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1,\end{cases}$$

$$\text{if } \alpha > 2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1,\\ \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1,\end{cases}$$

$$\text{if } \alpha = 2\beta \left\{ \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, \quad 0 < \mu < \infty. \right.$$

First, we have the following abstract statement.

Proposition 4.2. *The function* $h((2k+n)\lambda, \lambda^2)$ *is a strictly monotonic differentiable positive function of* λ *on* $\mathbb{R}_+$, *with the domain* (A, B) *where* $0 \le A < B \le \infty$. *Then for* $1 \le p \le (2m+2)/(m+3)$, *the estimate*

$$\|\mathcal{P}^h_\mu f\|_{p'} \le C_\mu \|f\|_p$$

holds, *where*

$$C_\mu \le C \sum_{k=0}^{\infty} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda'_k(\mu)| \tag{4-1}$$

for all Schwartz functions f *and all positive* $\mu \in (A, B)$.

The proof of Proposition 4.2 coincides essentially with Theorem 4.1 in [Casarino and Ciatti 2013a] and we omit it. To obtain our Theorem 4.1, it suffices to show the convergence of the series in (4-1). Next we will exploit the following estimates, which can be easily proved by comparing the sums with the corresponding integrals:

Lemma 4.3. *Fix* $\nu \in \mathbb{R}$. *There exists* $C_\nu > 0$ *such that for* $A > 0$ *and* $n \in \mathbb{Z}_+$, *we have*

$$\sum_{\substack{m\in\mathbb{N}\\ 2m+n\ge A}} (2m+n)^\nu \le C_\nu A^{\nu+1}, \quad \nu < -1; \tag{4-2}$$

$$\sum_{\substack{m\in\mathbb{N}\\ 2m+n\le A}} (2m+n)^\nu \le C_\nu A^{\nu+1}, \quad \nu > -1. \tag{4-3}$$

Now Theorem 4.1 follows from the result in the following lemma.

Lemma 4.4. *Let* $h(\xi, \eta) = \xi^\alpha + \eta^\beta$, $\alpha, \beta > 0$. *The series in* (4-1) *has the estimate*

$$
\begin{aligned}
&\textit{if}\ \alpha<2\beta \begin{cases} C_\mu \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & \mu>1,\\ C_\mu \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & 0<\mu\le 1,\end{cases}\\
&\textit{if}\ \alpha>2\beta \begin{cases} C_\mu \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & \mu>1,\\ C_\mu \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & 0<\mu\le 1,\end{cases}\\
&\textit{if}\ \alpha=2\beta \begin{cases} C_\mu \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & 0<\mu<\infty.\end{cases}
\end{aligned}
$$

Proof. The function $h(\xi,\eta)=\xi^\alpha+\eta^\beta$, $\alpha,\beta>0$, so $\mu=(2k+n)^\alpha\lambda_k^\alpha(\mu)+\lambda_k^{2\beta}(\mu)$, which yields

$$\lambda_k'(\mu)=\frac{1}{\alpha(2k+n)^\alpha\lambda_k^{\alpha-1}(\mu)+2\beta\lambda_k^{2\beta-1}(\mu)}.$$

To study the convergence of this series, we need to distinguish three cases according to the relation of α and 2β: $\alpha<2\beta$, $\alpha>2\beta$ and $\alpha=2\beta$. In order not to burden the exposition, we only prove the case $\alpha<2\beta$, and the other cases are analogous.

If $\alpha<2\beta$, then when $\mu\le 1$, it is easy to see that $\lambda_k(\mu)\sim\mu^{\frac{1}{\alpha}}/(2k+n)$ and $\lambda_k'(\mu)\sim\mu^{\frac{1}{\alpha}-1}/(2k+n)$, so that the series

$$
\begin{aligned}
C_\mu &\le C\sum_{k=0}^{\infty}(2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1}\lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)|\\
&\le C\sum_{k=0}^{\infty}(2k+n)^{2n(\frac{1}{p}-\frac{1}{2})-1}\left(\frac{\mu^{\frac{1}{\alpha}}}{2k+n}\right)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\frac{\mu^{\frac{1}{\alpha}-1}}{2k+n}\\
&\le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\sum_{k=0}^{\infty}\frac{1}{(2k+n)^{2m\left(\frac{1}{p}-\frac{1}{2}\right)+1}}\\
&\le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}
\end{aligned}
\tag{4-4}
$$

converges.

When $\mu>1$, we split the sum into two parts, the sum over those k such that $(2k+n)^\alpha\lambda_k^\alpha(\mu)\ge\lambda^{2\beta}(\mu)$ and those such that $(2k+n)^\alpha\lambda_k^\alpha(\mu)<\lambda^{2\beta}(\mu)$. They are denoted by I and II respectively.

For the first part, $(2k+n)^\alpha\lambda_k^\alpha(\mu)\ge\lambda^{2\beta}(\mu)$ implies

$$\lambda_k(\mu)\sim\frac{\mu^{\frac{1}{\alpha}}}{2k+n},\quad \lambda_k'(\mu)\sim\frac{\mu^{\frac{1}{\alpha}-1}}{2k+n},\quad\text{and}\quad 2k+n\ge\mu^{\frac{2\beta-\alpha}{2\alpha\beta}}.$$

Then we control the first part I by

$$
\begin{aligned}
\mathrm{I} &\le C \sum_{2k+n \ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)| \\
&\le C \sum_{2k+n \ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \left(\frac{\mu^{\frac{1}{\alpha}}}{2k+n}\right)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \frac{\mu^{\frac{1}{\alpha}-1}}{2k+n} \\
&\le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \sum_{2k+n \ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} \frac{1}{(2k+n)^{2m\left(\frac{1}{p}-\frac{1}{2}\right)+1}}.
\end{aligned}
$$

By (4-2), we have

$$
\mathrm{I} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \frac{1}{\mu^{\frac{2\beta-\alpha}{2\alpha\beta}\left(2m\left(\frac{1}{p}-\frac{1}{2}\right)\right)}} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}. \tag{4-5}
$$

For the second part, $(2k+n)^\alpha \lambda_k^\alpha(\mu) < \lambda^{2\beta}(\mu)$ implies

$$
\lambda_k(\mu) \sim \mu^{\frac{1}{2\beta}}, \quad \lambda_k'(\mu) \sim \mu^{\frac{1}{2\beta}-1}, \quad \text{and} \quad 2k+n < \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}.
$$

Then we control the second part II by

$$
\begin{aligned}
\mathrm{II} &\le C \sum_{2k+n < \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)| \\
&\le C \sum_{2k+n < \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \left(\mu^{\frac{1}{2\beta}}\right)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \mu^{\frac{1}{2\beta}-1} \\
&\le C\mu^{\frac{1}{\beta}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \sum_{2k+n < \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1}.
\end{aligned}
$$

Because $1 \le p \le (2m+2)/(m+3)$, we obtain $2n\left(\frac{1}{p}-\frac{1}{2}\right)-1 \ge -1$. Hence, by (4-3) we get

$$
\sum_{2k+n < \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lesssim \mu^{\frac{2\beta-\alpha}{2\alpha\beta}\left(2n\left(\frac{1}{p}-\frac{1}{2}\right)\right)} = \mu^{\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)n\left(\frac{1}{p}-\frac{1}{2}\right)}.
$$

Thus, for the second part we also have

$$
\mathrm{II} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}. \tag{4-6}
$$

Finally, the estimate for the case $\alpha < 2\beta$ follows from (4-4), (4-5) and (4-6). This completes the proof of the first case.

Combining Proposition 4.2 and Lemma 4.4, Theorem 4.1 comes out easily. □

Especially, in the case $\Delta = L + T,\ h(\xi, \eta) = \xi + \eta$, we obtain the restriction theorem associated with the full laplacian on H-type groups.

Corollary 4.5. *For $1 \le p \le (2m+2)/(m+3)$, the estimates*

$$\|\mathcal{P}^{\Delta}_{\mu} f\|_{p'} \le C\mu^{(2n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ \mu > 1$$

and

$$\|\mathcal{P}^{\Delta}_{\mu} f\|_{p'} \le C\mu^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ 0 < \mu \le 1$$

hold for all Schwartz functions f.

5. Examples

Similarly to what we have done so far in Theorem 4.1, we now discuss other operators with the form of the joint functional calculus of L and T. We obtain the following results. We omit the arguments which are really similar to that of Theorem 4.1.

Example 5.1. Let $h(\xi, \eta) = (\xi^{\alpha} + \eta^{\beta})^{-1},\ \alpha, \beta > 0$. For $1 \le p \le (2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha < 2\beta \begin{cases} \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1, \\ \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha > 2\beta \begin{cases} \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1, \\ \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha = 2\beta \left\{ \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, \quad 0 < \mu < \infty. \right.$$

Example 5.2. Let $h(\xi, \eta) = (1+\xi)^{-1}$. For $1 \le p \le (2m+2)/(m+3)$, the estimates

$$\|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ \text{ when } \mu \to 0^+,$$

and

$$\|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C(1-\mu)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ \text{ when } \mu \to 1^-,$$

hold for all Schwartz functions f.

More generally, we have:

Example 5.3. Let $h(\xi,\eta)=(\xi^{\alpha}+\eta^{\beta})^{\gamma}$, $\alpha,\beta,\gamma>0$. For $1\le p\le(2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha<2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac1p-\frac12\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha>2\beta, \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac1p-\frac12\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha=2\beta \left\{ \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p, \quad 0<\mu<\infty. \right.$$

Example 5.4. Letting $h(\xi,\eta)=(\xi^{\alpha}+\eta^{\beta})^{-\gamma}$, $\alpha,\beta,\gamma>0$, then for $1\le p\le(2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha<2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac1p-\frac12\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha>2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac1p-\frac12\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha=2\beta \left\{ \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p, \quad 0<\mu<\infty. \right.$$

Example 5.5. Let $h(\xi,\eta)=(1+\xi^{\alpha}+\eta^{\beta})^{-\gamma}$, $\alpha,\beta,\gamma>0$. Then for $1\le p\le(2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha\le 2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac1p-\frac12\right)-1}\|f\|_p & \text{when } \mu\to 0^+,\\ \|\mathcal{P}^h_\mu f\|_{p'}\le C(1-\mu^{\frac1\gamma})^{\frac{2}{\alpha}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p & \text{when } \mu\to 1^-,\end{cases}$$

$$\text{if } \alpha>2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac1p-\frac12\right)-1}\|f\|_p & \text{when } \mu\to 0^+,\\ \|\mathcal{P}^h_\mu f\|_{p'}\le C(1-\mu^{\frac1\gamma})^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac1p-\frac12\right)-1}\|f\|_p & \text{when } \mu\to 1^-.\end{cases}$$

6. Sharpness of the range p

In this section we only give an example to show that the range of p in the restriction theorem associated with the full laplacian Δ is sharp. The example is constructed similarly to the counterexample of Müller [1990], which shows that the estimates between Lebesgue spaces for the operators $\mathcal{P}^{\Delta}_{\mu}$ are necessarily trivial.

Let $\varphi\in C_c^{\infty}(\mathbb{R}^m)$ be a radial function such that $\varphi(a)=\psi(|a|)$, where $\psi\in C_c^{\infty}(\mathbb{R})$, with $\psi=1$ on a neighborhood of the point n and $\psi=0$ near 0. Let h be a Schwartz function on $\mathbb{R}^m$ and define

$$f(z,t) = \int_{\mathbb{R}^m} \varphi(a)\hat{h}(a)e^{-\frac{|a|}{4}|z|^2}e^{-i\langle a,t\rangle}|a|^n\,da.$$

Denote

$$\begin{aligned} g(z,t) &= \int_{\mathbb{R}^m} \varphi(a)e^{-\frac{|a|}{4}|z|^2}e^{-i\langle a,t\rangle}|a|^n\,da \\ &= \int_{\mathbb{R}^{m+2n}} \varphi(a)e^{-\frac{|\xi|^2}{|a|}}e^{-i(\langle a,t\rangle+\langle \xi,z\rangle)}\,d\xi\,da. \end{aligned}$$

Hence $\widehat{g(\xi,a)} = \varphi(a)e^{-\frac{|\xi|^2}{|a|}}$, which shows that $\hat{g}$ and consequently g are Schwartz functions. On the other hand, we have $f = h *_t g$, where $*_t$ denotes the involution about the central variable. By Lemma 3.1, we have $\Delta\big(e^{-i\langle a,t\rangle}e^{-\frac{1}{4}|a||z|^2}\big) = (n\lambda+\lambda^2)e^{-i\langle a,t\rangle}e^{-\frac{1}{4}|a||z|^2}$. Therefore, we write f by the integration with polar coordinates as

$$\begin{aligned} f(z,t) &= \int_0^\infty \left(\lambda^{n+m-1}\psi(\lambda)e^{-\frac{\lambda}{4}|z|^2}\int_{S^{m-1}} \hat{h}(\lambda w)e^{-i\lambda\langle w,t\rangle}\,d\sigma(w)\right)d\lambda \\ &= \int_0^\infty \Bigg(\lambda_\Delta(\mu)^{n+m-1}\lambda'_\Delta(\mu)\psi(\lambda_\Delta(\mu))e^{-\frac{\lambda_\Delta(\mu)}{4}|z|^2} \\ &\qquad\qquad \int_{S^{m-1}} \hat{h}(\lambda_\Delta(\mu)w)e^{-i\lambda_\Delta(\mu)\langle w,t\rangle}\,d\sigma(w)\Bigg)d\mu \\ &= \int_0^\infty \mathcal{P}^\Delta_\mu f(z,t)\,d\mu, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}^\Delta_\mu f(z,t) &= \lambda_\Delta(\mu)^{n+m-1}\lambda'_\Delta(\mu)\psi(\lambda_\Delta(\mu))e^{-\frac{\lambda_\Delta(\mu)}{4}|z|^2} \\ &\qquad \times \int_{S^{m-1}} \hat{h}(\lambda_\Delta(\mu)w)e^{-i\lambda_\Delta(\mu)\langle w,t\rangle}\,d\sigma(w), \\ \lambda_\Delta(\mu) &= \frac{\sqrt{n^2+4\mu}-n}{2}. \end{aligned}$$

Therefore, letting $\mu = 2n^2$, we have $\lambda_\Delta(2n^2) = n,\ \lambda'_\Delta(2n^2) = 1/(3n)$ and

$$\begin{aligned} \mathcal{P}^\Delta_{2n^2} f(z,t) &= \tfrac{1}{3}n^{n+m-2}e^{-\frac{n|z|^2}{4}}\int_{S^{m-1}} \hat{h}(nw)e^{-in\langle w,t\rangle}\,d\sigma(w) \\ &= \tfrac{1}{3}n^{n-1}e^{-\frac{n|z|^2}{4}}\,h * \widehat{d\sigma_n}(t). \end{aligned}$$

From the restriction theorem associated the full laplacian on H-type groups, we have the estimate $\|\mathcal{P}^\Delta_{2n^2} f\|_{L^{p'}(G)} \le C\|f\|_{L^p(G)}$.

Because of

$$\|\mathcal{P}^{\Delta}_{2n^2} f\|_{L^{p'}(G)} = C\|h * \widehat{d\sigma_n}\|_{L^{p'}(\mathbb{R}^m)} \tag{6-1}$$

and

$$\|f\|_{L^p(G)} \le \|h\|_{L^p(\mathbb{R}^m)}\|g\|_{L^1_t L^p_z} \lesssim \|h\|_{L^p(\mathbb{R}^m)}, \tag{6-2}$$

where the mixed Lebesgue norm is defined by

$$\|g\|_{L^1_t L^p_z} = \left(\int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^m}|f(z,t)|\,dt\right)^p dz\right)^{\frac{1}{p}},$$

we have $\|h * \widehat{d\sigma_n}\|_{L^{p'}(\mathbb{R}^m)} \le C\|h\|_{L^p(\mathbb{R}^m)}$.

From the sharpness of the Stein–Tomas theorem which is guaranteed by the Knapp counterexample, this would imply $p \le (2m+2)/(m+3)$. Hence the range of p can not be extended. With the same tricks we can prove the range of p for the restriction theorem associated with the functional calculus is also sharp.

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HEPING LIU
LMAM, SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
CHINA

hpliu@pku.edu.cn

MANLI SONG
SCHOOL OF NATURAL AND APPLIED SCIENCES
NORTHWESTERN POLYTECHNICAL UNIVERSITY
XI'AN, 710129
CHINA

mlsong@nwpu.edu.cn

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IDENTITIES INVOLVING CYCLIC AND SYMMETRIC SUMS OF REGULARIZED MULTIPLE ZETA VALUES

TOMOYA MACHIDE

There are two types of regularized multiple zeta values: harmonic and shuffle types. The first purpose of the present paper is to give identities involving cyclic sums of regularized multiple zeta values of both types for depth less than 5. Michael Hoffman, in "Quasi-symmetric functions and mod *p* multiple harmonic sums" (*Kyushu Journal of Mathematics* 69 (2015), 345–366) proved an identity involving symmetric sums of regularized multiple zeta values of harmonic type for arbitrary depth. The second purpose is to prove Hoffman's identity for shuffle type. We also give a connection between the identities involving cyclic sums and symmetric sums, for depth less than 5.

1. Introduction and statement of results

Multiple zeta values (MZVs) are real numbers that are variations of special values of the Riemann zeta function $\zeta_1(s) = \sum_{m=1}^{\infty} 1/m^s$ with integer arguments. Regularized multiple zeta values (RMZVs) are generalizations of MZVs, which are defined in [Ihara et al. 2006] as constant terms of certain polynomials. There are two types of RMZVs: harmonic and shuffle types. It is known that these values satisfy a great many relations over $\mathbb{Q}$, including, for example, extended harmonic and shuffle relations, Drinfeld associator relations, and Kawashima's relations (e.g., see [Drinfeld 1990; Ihara et al. 2006; Kawashima 2009]). New classes of relations are being studied, but their exact structure is not yet fully understood.

The first purpose (Theorem 1.1) of the present paper is to give identities involving cyclic sums of RMZVs of both types for depth less than 5. Hoffman [1992, Theorem 2.2] proved an identity involving symmetric sums of MZVs for arbitrary depth, and then, he extended it to RMZVs of harmonic type [Hoffman 2015, Theorem 2.3]. The second purpose (Theorem 1.2) is to prove Hoffman's identity for shuffle type. We also show that Theorem 1.1 yields Theorem 1.2, for depth less than 5 (see Corollary 1.3).

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We will begin by introducing the notation and terminology that will be used to state our results. An MZV is a convergent series defined by

$$\zeta_n(\mathbf{l}_n) := \sum_{m_1>\cdots>m_n>0} \frac{1}{m_1^{l_1}\cdots m_n^{l_n}},$$

where $\mathbf{l}_n = (l_1, \ldots, l_n)$ is an (ordered) index set of positive integers with $l_1 \geq 2$. In other words, MZVs are images under the real-valued function ζ_n with the domain $\{(l_1, \ldots, l_n) \in \mathbb{N}^n \mid l_1 \geq 2\}$, where $\mathbb{N}$ denotes the set of positive integers. We call $w_n(\mathbf{l}_n) = l_1 + \cdots + l_n$ the weight, and $d_n(\mathbf{l}_n) = n$ the depth. Ihara, Kaneko, and Zagier [Ihara et al. 2006] extended MZVs to two types of RMZV (harmonic and shuffle) with two different renormalization procedures for divergent series $\zeta_n(\mathbf{l}_n)$ of $l_1 = 1$. The former and latter types are denoted by $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, and they inherit the harmonic and shuffle relation structures, respectively. The following are a few examples of these values: $\zeta_1^*(1) = \zeta_1^{ш}(1) = \zeta_2^{ш}(1, 1) = 0$ and $\zeta_2^*(1, 1) = -\zeta_1(2)/2 \neq 0$. In other words, RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$ are images under two different extension functions of ζ_n to the domain $\mathbb{N}^n$.

Let $\mathfrak{S}_n$ denote the symmetric group of degree n, and let $e = e_n$ denote its unit element. Let $\mathfrak{C}_3$ and $\mathfrak{C}_4$ be the cyclic subgroups in $\mathfrak{S}_3$ and $\mathfrak{S}_4$ given by $\mathfrak{C}_3 = \langle(123)\rangle = \{e, (123), (132)\}$ and $\mathfrak{C}_4 = \langle(1234)\rangle = \{e, (1234), (13)(24), (1432)\}$, respectively. We set $\mathfrak{C}_2 = \langle(12)\rangle$ (or $\mathfrak{C}_2 = \mathfrak{S}_2$) for convenience. The group ring $\mathbb{Z}[\mathfrak{S}_n]$ of $\mathfrak{S}_n$ over $\mathbb{Z}$ acts on a function f of n variables in a natural way by

$$(f \mid \Gamma)(x_1, \ldots, x_n) := \sum a_i f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}),$$

where $\Gamma = \sum a_i\sigma_i \in \mathbb{Z}[\mathfrak{S}_n]$. This is a right action, that is, $f \mid (\Gamma_1\Gamma_2) = (f|\Gamma_1) \mid \Gamma_2$. For a subset H in $\mathfrak{S}_n$, we define the sum of all elements in H by

$$\Sigma_H := \sum_{\sigma\in H} \sigma \in \mathbb{Z}[\mathfrak{S}_n].$$

That is, $(f \mid \Sigma_H)(x_1, \ldots, x_n)$ is $\sum_{\sigma\in H} f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. In particular, if H is a group, it is $\sum_{\sigma\in H} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ because $H = H^{-1}$. For positive integers $n_1, \ldots, n_j, n$ with $n_1 + \cdots + n_j = n$, we define real-valued functions with the domain $\mathbb{N}^n$ by

$$\zeta^\dagger_{(n_1,\ldots,n_j)}(\mathbf{l}_n) := \zeta^\dagger_{n_1}(l_1, \ldots, l_{n_1})\zeta^\dagger_{n_2}(l_{n_1+1}, \ldots, l_{n_1+n_2}) \cdots \zeta^\dagger_{n_j}(l_{n_1+n_2+\cdots+n_{j-1}+1}, \ldots, l_n),$$

where $\dagger \in \{*, ш\}$. For example,

$$\zeta^\dagger_{(1,1)}(\mathbf{l}_2) = \zeta^\dagger_1(l_1)\zeta^\dagger_1(l_2) \quad \text{and} \quad \zeta^\dagger_{(2,1)}(\mathbf{l}_3) = \zeta^\dagger_2(l_1, l_2)\zeta^\dagger_1(l_3).$$

We define the characteristic functions χ_n^* and $\chi_n^{\sqcup\!\sqcup}$ of the set $\mathbb{N}^n$ by

$$\chi_n^*(\mathbf{l}_n) = 1 \quad \text{and} \quad \chi_n^{\sqcup\!\sqcup}(\mathbf{l}_n) = \begin{cases} 0 & \text{if } n > 1,\ l_1 = \cdots = l_n = 1, \\ 1 & \text{otherwise,} \end{cases} \tag{1-1}$$

respectively. We have defined $\chi_1^{\sqcup\!\sqcup}(1)$ to be not 0 but 1, though this definition will not be used in Theorem 1.1. We will need it to hold consistency between Definitions (1-1) and (1-7); to prove Theorem 1.2, (1-7) is required.

Theorem 1.1 is stated as follows.

Theorem 1.1. *Let $\mathbf{l}_n = (l_1, \ldots, l_n)$ be an index set in $\mathbb{N}^n$, and let $L_n = w_n(\mathbf{l}_n)$ be its weight. Then we have the following identities for RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{\sqcup\!\sqcup}(\mathbf{l}_n)$ of $n = 2, 3,$ and 4:*

$$(\zeta_2^\dagger \mid \Sigma_{\mathfrak{C}_2})(\mathbf{l}_2) = \zeta_{(1,1)}^\dagger(\mathbf{l}_2) - \chi_2^\dagger(\mathbf{l}_2)\zeta_1(L_2), \tag{1-2}$$

$$(\zeta_3^\dagger \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3) = -\zeta_{(1,1,1)}^\dagger(\mathbf{l}_3) + (\zeta_{(2,1)}^\dagger \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3) + \chi_3^\dagger(\mathbf{l}_3)\zeta_1(L_3), \tag{1-3}$$

$$\begin{aligned}(\zeta_4^\dagger \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4) = \zeta_{(1,1,1,1)}^\dagger(\mathbf{l}_4) &- (\zeta_{(2,1,1)}^\dagger \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4) + (\zeta_{(2,2)}^\dagger \mid \Sigma_{\mathfrak{C}_4^0})(\mathbf{l}_4), \\ &+ (\zeta_{(3,1)}^\dagger \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4) - \chi_4^\dagger(\mathbf{l}_4)\zeta_1(L_4),\end{aligned} \tag{1-4}$$

where $\dagger \in \{*, \sqcup\!\sqcup\}$, and $\mathfrak{C}_4^0$ in (1-4) is the subset $\{e, (1234)\}$ of $\mathfrak{C}_4$.

We note that (1-2) can be easily obtained from the harmonic relations

$$\zeta_1^*(l_1)\zeta_1^*(l_2) = \zeta_2^*(l_1, l_2) + \zeta_2^*(l_2, l_1) + \zeta_1^*(l_1 + l_2)$$

for RMZVs of harmonic type of depth 2; thus our main results are (1-3) and (1-4) (see Section 5 for their straightforward expressions).

We now recall Hoffman's identity. Let $|P|$ be the number of elements of a set P. For any partition $\Pi = \{P_1, \ldots, P_m\}$ of the set $\{1, \ldots, n\}$, we define an integer $\tilde{c}_n(\Pi)$ by

$$\tilde{c}_n(\Pi) := (-1)^{n-m} \prod_{i=1}^{m} (|P_i| - 1)!. \tag{1-5}$$

For $\dagger \in \{*, \sqcup\!\sqcup\}$, we define a real number $\zeta^\dagger(\mathbf{l}_n; \Pi)$ by

$$\zeta^\dagger(\mathbf{l}_n; \Pi) := \prod_{i=1}^{m} \chi^\dagger(\mathbf{l}_n; P_i)\Big(\sum_{p \in P_i} l_p\Big), \tag{1-6}$$

where

$$\chi^\dagger(\mathbf{l}_n; P_i) := \begin{cases} 0 & \text{if } \dagger = \sqcup\!\sqcup,\ |P_i| > 1, \text{ and } l_p = 1 \text{ for all } p \in P_i, \\ 1 & \text{otherwise.} \end{cases} \tag{1-7}$$

For example,

$$\chi^{ш}((2,1,1);\{2,3\})=0 \quad \text{and} \quad \chi^{ш}((2,1,1);\{1,3\})=\chi^{ш}((2,1,1);\{3\})=1.$$

We note that $\chi^{\dagger}(\mathbf{l}_n; P_i) = \chi_j^{\dagger}(l_{p_1}, \dots, l_{p_j})$ if $P_i = \{p_1, \dots, p_j\}$. For any index $\mathbf{l}_n = (l_1, \dots, l_n)$ in $\mathbb{N}^n$, Hoffman [2015, Theorem 2.3] proved the following identity involving symmetric sums of RMZVs of harmonic type:

$$(\zeta_n^{\dagger} \mid \Sigma_{\mathfrak{S}_n})(\mathbf{l}_n) = \sum_{\text{partitions } \Pi \text{ of } \{1,\dots,n\}} \tilde{c}_n(\Pi)\zeta^{\dagger}(\mathbf{l}_n; \Pi), \tag{1-8}$$

where $\dagger = *$. In the case that $l_i > 1$ for all i, he proved (1-8) in [Hoffman 1992, Theorem 2.2]. (In this case, $\zeta_n(\mathbf{l}_n) = \zeta_n^*(\mathbf{l}_n) = \zeta_n^{ш}(\mathbf{l}_n)$.)

Theorem 1.2 is stated as follows.

Theorem 1.2. *Identity* (1-8) *for* $\dagger = ш$ *holds.*

Corollary 1.3 gives a connection between identities involving cyclic sums and symmetric sums of RMZVs, for depth less than 5.

Corollary 1.3. *Let* $\dagger \in \{*, ш\}$. *Identity* (1-2) *yields* (1-8) *for* $n = 2$, *identities* (1-2) *and* (1-3) *yield* (1-8) *for* $n = 3$, *and identities* (1-2), (1-3), *and* (1-4) *yield* (1-8) *for* $n = 4$.

Remark 1.4. Hoffman proved (1-8) for $\dagger = *$ under a general algebraic setup, i.e., the harmonic algebra $\mathfrak{H}_*^1$ that will be introduced in Section 2. (To be more precise, he used the algebra of quasisymmetric functions that is isomorphic to $\mathfrak{H}_*^1$.) The constant terms of the polynomials $Z_{\mathbf{l}_n}^*(T)$ defined in [Ihara et al. 2006] are RMZVs $\zeta_n^*(\mathbf{l}_n)$, and the polynomials $Z_{\mathbf{l}_n}^*(T)$ have the same harmonic relation structure as RMZVs $\zeta_n^*(\mathbf{l}_n)$ (see Section 2 for details). Thus, (1-8) for $\dagger = *$ also holds in the case of $Z_{\mathbf{l}_n}^*(T)$. This fact will be necessary to prove Theorem 1.2.

We now briefly explain how Theorem 1.1, Theorem 1.2, and Corollary 1.3 can be proved. We first prove the identities in Theorem 1.1 for $\dagger = *$ from harmonic relations of RMZVs $\zeta_n^*(\mathbf{l}_n)$. Ihara et al. [2006, Theorem 1] gave a class of relations over $\mathbb{Q}$ between RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, which we call renormalization relations. Using renormalization relations, we derive the identities in Theorem 1.1 for $\dagger = ш$ from those for $\dagger = *$, and we complete the proof of Theorem 1.1. Similarly, we prove Theorem 1.2 by combining the renormalization relations and (1-8) for $\dagger = *$ in which $\zeta_n^*(\mathbf{l}_n)$ are replaced by $Z_{\mathbf{l}_n}^*(T)$. We show Corollary 1.3 by focusing on the fact that $\mathfrak{C}_n$ is a subgroup of $\mathfrak{S}_n$, i.e., $(\zeta_n^{\dagger} \mid \Sigma_{\mathfrak{C}_n})(\mathbf{l}_n)$ is a partial sum of $(\zeta_n^{\dagger} \mid \Sigma_{\mathfrak{S}_n})(\mathbf{l}_n)$.

It is worth noting that Theorem 1.1 gives the following property, which is an analog of the parity property [Borwein and Girgensohn 1996; Euler 1776; Ihara et al. 2006; Tsumura 2004]; any cyclic sum of RMZVs of depth less than 5, or

$$(\zeta_n^\dagger \mid \Sigma_{\mathfrak{C}_n})(\mathbf{l}_n) = \sum_{j=1}^{n} \zeta_n^\dagger(l_j, \ldots, l_n, l_1, \ldots, l_{j-1}) \tag{1-9}$$

for $n = 2, 3, 4$ and $\dagger \in \{*, ш\}$, is a rational linear combination of the Riemann zeta value $\zeta_1(l_1 + \cdots + l_n)$ and products of RMZVs of smaller depth and weight. It appears that the existence of such a property for depth greater than 4 is an open problem. (The case of symmetric sums of general depth easily follows from (1-8); there is a stronger property from (1-8), such that any symmetric sum can be written in terms of only Riemann zeta values.) It is also worth noting that Hoffman and Ohno [2003] studied a class of relations involving

$$\sum_{j=1}^{n} \zeta_n(l_j + 1, l_{j+1}, \ldots, l_n, l_1, \ldots, l_{j-1}),$$

whose form is quite similar to (1-9), but the first indices differ.

The paper is organized as follows. In Section 2, we review some facts of RMZVs by referring to [Hoffman 1997; Ihara et al. 2006]. Sections 3 and 4 have two and three subsections, respectively. Sections 3.1 and 3.2 are devoted to calculating harmonic relations for RMZVs $\zeta_n^*(\mathbf{l}_n)$ and renormalization relations between RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, respectively, for depth less than 5. We then prove Theorem 1.1 in Section 4.1, Theorem 1.2 in Section 4.2, and Corollary 1.3 in Section 4.3. We give some examples of Theorems 1.1 and 1.2 in Section 5.

Remark 1.5. (i) Although the ideas of the proofs are the same, the computational complexity of proving (1-4) is much greater than that required to prove (1-2) and (1-3). We recommend that, on first reading, those readers who are interested only in the ideas skip over the statements relating to the proof of (1-4) (or statements in the case of depth 4).

(ii) This paper is an expansion of Section 2.1 in [Machide 2012]. The remainder of the results of that article has been amplified in [Machide 2015].

2. Preparation

Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ be the noncommutative polynomial algebra over $\mathbb{Q}$ in two indeterminates x and y, and let $\mathfrak{H}^0$ and $\mathfrak{H}^1$ be its subalgebras $\mathbb{Q} + x\mathfrak{H}y$ and $\mathbb{Q} + \mathfrak{H}y$, respectively. These algebras satisfy the inclusion relations $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$. Let z_l denote $x^{l-1}y$ for any integer $l \geq 1$. Every word $w = w_0 y$ in the set $\{x, y\}$ with terminal letter y is expressed as $w = z_{l_1} \cdots z_{l_n}$ uniquely, and so $\mathfrak{H}^1$ is the free algebra generated by z_l $(l = 1, 2, 3, \ldots)$. We define the harmonic product $*$ on $\mathfrak{H}^1$

inductively by

$$1 * w = w * 1 = w, \tag{2-1}$$

$$z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_{k+l}(w_1 * w_2), \tag{2-2}$$

for any integers $k, l \geq 1$ and words $w, w_1, w_2 \in \mathfrak{H}^1$, and then extend it by $\mathbb{Q}$-bilinearity. This product gives the subalgebras $\mathfrak{H}^0$ and $\mathfrak{H}^1$ structures of commutative $\mathbb{Q}$-algebras [Hoffman 1997], which we denote by $\mathfrak{H}^0_*$ and $\mathfrak{H}^1_*$, respectively; note that $\mathfrak{H}^0_*$ is a subalgebra of $\mathfrak{H}^1_*$. In a similar way, we can define the shuffle product ш on $\mathfrak{H}^1$ and the commutative $\mathbb{Q}$-algebras $\mathfrak{H}^0_ш$ and $\mathfrak{H}^1_ш$ (see [Ihara et al. 2006; Reutenauer 1993] for details).

Let $Z : \mathfrak{H}^0 \to \mathbb{R}$ be the $\mathbb{Q}$-linear map (evaluation map) given by

$$Z(z_{l_1} \cdots z_{l_n}) = \zeta_n(\mathbf{l}_n) \quad (z_{l_1} \cdots z_{l_n} \in \mathfrak{H}^0). \tag{2-3}$$

We know from [Hoffman 1997] that Z is homomorphic on both products $*$ and ш, that is,

$$Z(w_1 * w_2) = Z(w_1 \,ш\, w_2) = Z(w_1)Z(w_2)$$

for $w_1, w_2 \in \mathfrak{H}^0$. Let $\mathbb{R}[T]$ be the polynomial ring in a single indeterminate with real coefficients. Through the isomorphisms $\mathfrak{H}^1_* \simeq \mathfrak{H}^0_*[y]$ and $\mathfrak{H}^1_ш \simeq \mathfrak{H}^0_ш[y]$, which were proved in [Hoffman 1997] and [Reutenauer 1993], respectively, Ihara et al. [2006, Proposition 1] considered the algebra homomorphisms

$$Z^* : \mathfrak{H}^1_* \to \mathbb{R}[T] \quad \text{and} \quad Z^ш : \mathfrak{H}^1_ш \to \mathbb{R}[T],$$

respectively, which are uniquely characterized by the property that they extend the evaluation map Z and send y to T. For any word $w = z_{l_1} \cdots z_{l_n} \in \mathfrak{H}^1$, we denote by $Z^*_{\mathbf{l}_n}(T)$ and $Z^ш_{\mathbf{l}_n}(T)$ the images under the maps Z^* and $Z^ш$, respectively, of the word w, that is,

$$Z^*_{l_1,\dots,l_n}(T) = Z^*(z_{l_1} \cdots z_{l_n}) \quad \text{and} \quad Z^ш_{l_1,\dots,l_n}(T) = Z^ш(z_{l_1} \cdots z_{l_n}). \tag{2-4}$$

(The notation $Z^*_{\mathbf{l}_n}(T)$ and $Z^ш_{\mathbf{l}_n}(T)$ will be used when we focus on the variable T and the corresponding index set $\mathbf{l}_n$ of the word $z_{l_1} \cdots z_{l_n}$.) Then the RMZVs $\zeta^*(\mathbf{l}_n)$ and $\zeta^ш(\mathbf{l}_n)$ of the harmonic and shuffle types are defined as

$$\zeta^*(l_1, \dots, l_n) := Z^*_{l_1,\dots,l_n}(0) \quad \text{and} \quad \zeta^ш(l_1, \dots, l_n) := Z^ш_{l_1,\dots,l_n}(0), \tag{2-5}$$

respectively. Obviously, $\zeta^*_n(\mathbf{l}_n) = \zeta^ш_n(\mathbf{l}_n) = \zeta_n(\mathbf{l}_n)$ if $l_1 > 1$. We have

$$Z^*(z_{k_1} \cdots z_{k_m} * z_{l_1} \cdots z_{l_n}) = Z^*(z_{k_1} \cdots z_{k_m}) Z^*(z_{l_1} \cdots z_{l_n})$$

for index sets $(k_1, \dots, k_m)$ and $(l_1, \dots, l_n)$, since Z^* is homomorphic, and so we see from the first equations of (2-4) and (2-5) that the RMZVs $\zeta^*_n(\mathbf{l}_n)$ satisfy the harmonic relations. In Section 3.1, we will calculate these relations in detail for depth less than 5. (We can also see that the RMZVs $\zeta_n^{ш}(\mathbf{l}_n)$ satisfy the shuffle relations since $Z^{ш}$ is homomorphic, but we will not discuss this in the present paper.)

Let $A(u)=\sum_{k=0}^{\infty}\gamma_k u^k$ be the Taylor expansion of $e^{\gamma u}\Gamma(1+u)$ near $u=0$, where γ is Euler's constant and $\Gamma(x)$ is the gamma function. The renormalization map $\rho:\mathbb{R}[T]\to\mathbb{R}[T]$ is an $\mathbb{R}$-linear map defined by

$$\rho(e^{Tu})=A(u)e^{Tu}. \tag{2-6}$$

That is, images $\rho(T^m)$ are determined by comparing the coefficients of u^m on both sides of (2-6), and expressed as

$$\rho(T^m)=m!\sum_{i=0}^{m}\gamma_i\frac{T^{m-i}}{(m-i)!}\quad (m=0,1,2,\dots). \tag{2-7}$$

Then the renormalization formula proved by Ihara et al. [2006, Theorem 1] is

$$\rho(Z^*_{\mathbf{l}_n}(T))=Z^{ш}_{\mathbf{l}_n}(T). \tag{2-8}$$

Combining (2-5) and (2-8) with $T=0$, we can obtain relations between RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, or renormalization relations. In Section 3.2, we will calculate these relations in detail for depth less than 5.

3. Relations

3.1. *Harmonic relations.* We begin by defining the notation that we will use to state the harmonic relations of RMZVs $\zeta_n^*(\mathbf{l}_n)$ of depth less than 5 in terms of real-valued functions.

We first define analogs of the weight map $w_n:\mathbb{N}^n\to\mathbb{N}$ of depth n. For positive integers $n_1,\dots,n_j,n$ with $n_1+\dots+n_j=n$, we define the map $w_{(n_1,\dots,n_j)}$ from $\mathbb{N}^n$ to $\mathbb{N}^j$ by

$$w_{(n_1,\dots,n_j)}(\mathbf{l}_n):=(w_{n_1}(l_1,\dots,l_{n_1}),\dots,w_{n_j}(l_{n_1+n_2+\dots+n_{j-1}+1},\dots,l_n)). \tag{3-1}$$

For example, $w_{(2,1)}(l_1,l_2,l_3)=(l_1+l_2,l_3)$ and $w_{(1,2,1)}(l_1,l_2,l_3,l_4)=(l_1,l_2+l_3,l_4)$. We define a subset U_3 in $\mathfrak{S}_3$ as

$$U_3=\{e_3,(23),(123)\}, \tag{3-2}$$

and subsets U_4, V_4^0, V_4, W_4^0, W_4^1, W_4, and X_4 in $\mathfrak{S}_4$ as

$$U_4=\{e_4,(34),(234),(1234)\}, \tag{3-3}$$

$$\begin{aligned} V_4^0&=\{(23),(1243)\},\\ V_4&=\{e_4,(13)(24),(123),(243)\}\cup V_4^0, \end{aligned} \tag{3-4}$$

$$W_4^0 = \{(23), (24)\},$$
$$W_4^1 = \{(34), (1234), (1243), (1324)\} \cup W_4^0, \tag{3-5}$$
$$W_4 = \{e_4, (13)(24), (123), (124), (234), (243)\} \cup W_4^1,$$

$$X_4 = \{(14), (23)\} \cup \mathfrak{C}_4. \tag{3-6}$$

We have the inclusion relations $V_4^0 \subset V_4$ and $W_4^0 \subset W_4^1 \subset W_4$. We denote by $\mathfrak{A}_3$ and $\mathfrak{A}_4$ the alternating groups of degree 3 and 4, respectively. Note that $\mathfrak{A}_3 = \mathfrak{C}_3$. Functional composition $\circ$ satisfies the distributive law, i.e.,

$$\Big(\sum_i f_i\Big) \circ \Big(\sum_j g_j\Big) = \sum_{i,j} f_i \circ g_j,$$

where f_i are real-valued functions with the domain $\mathbb{N}^n$, and g_j are vector-valued functions with a same domain whose images are included in $\mathbb{N}^n$. The notation $f \circ g \mid \sigma$ is unambiguous since $(f \circ g) \mid \sigma = f \circ (g \mid \sigma)$.

Remark 3.1. For integers j, n with $1 \le j \le n-1$, let $\mathrm{sh}_j^{(n)}$ be the shuffle elements given in [Ihara et al. 2006], which are elements in $\mathbb{Z}[\mathfrak{S}_n]$ and defined as

$$\mathrm{sh}_j^{(n)} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \left(\substack{\sigma(1)<\cdots<\sigma(j) \\ \sigma(j+1)<\cdots<\sigma(n)}\right)}} \sigma.$$

The elements Σ_{U_3}, Σ_{U_4}, and Σ_{V_4} are equal to $\mathrm{sh}_2^{(3)}$, $\mathrm{sh}_3^{(4)}$, and $\mathrm{sh}_2^{(4)}$, respectively. The element Σ_{W_4} cannot be written in terms of only a shuffle element, but it is equal to $\mathrm{sh}_2^{(4)}\, \Sigma_{\langle (34)\rangle} = \Sigma_{V_4} \Sigma_{\langle (34)\rangle}$ as we will see in (3-36), below.

The harmonic relations we desire are listed below.

Proposition 3.2 (case of depth 2). *We have*

$$\zeta^*_{(1,1)} = \zeta^*_2 \mid \Sigma_{\mathfrak{C}_2} + \zeta_1 \circ w_2. \tag{3-7}$$

Proposition 3.3 (case of depth 3). *We have*

$$\zeta^*_{(2,1)} = \zeta^*_3 \mid \Sigma_{U_3} + \zeta^*_2 \circ (w_{(2,1)} \mid (123) + w_{(1,2)}), \tag{3-8}$$

$$\zeta^*_{(1,1,1)} = \zeta^*_3 \mid \Sigma_{\mathfrak{S}_3} + \zeta^*_2 \circ (w_{(2,1)} + w_{(1,2)}) \mid \Sigma_{\mathfrak{C}_3} + \zeta_1 \circ w_3. \tag{3-9}$$

Proposition 3.4 (case of depth 4). *We have*

$$\zeta^*_{(3,1)} = \zeta^*_4 \mid \Sigma_{U_4} + \zeta^*_3 \circ ((w_{(2,1,1)} + w_{(1,2,1)}) \mid (234) + w_{(1,1,2)}), \tag{3-10}$$

$$\zeta^*_{(2,2)} = \zeta^*_4 \mid \Sigma_{V_4} + \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{V_4^0} + \zeta^*_2 \circ w_{(2,2)} \mid (23), \tag{3-11}$$

$$\zeta^*_{(2,1,1)} = \zeta^*_4 \mid \Sigma_{W_4} + \zeta^*_3 \circ \big(w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}} + w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}} \tag{3-12}$$

$$+\zeta_2^* \circ (w_{(2,2)} \mid \Sigma_{W_4^0} + w_{(3,1)} \mid (24) + w_{(1,3)})\big),$$

$$\zeta^*_{(1,1,1,1)} = \zeta_4^* \mid \Sigma_{\mathfrak{S}_4} + \zeta_3^* \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{\mathfrak{A}_4} + \zeta_2^* \circ (w_{(2,2)} \mid \Sigma_{X_4} + (w_{(3,1)} + w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}) + \zeta_1 \circ w_4, \tag{3-13}$$

where $W^1_{4,\sigma}$ $(\sigma \in \{(34), (1234), (1324)\})$ *in* (3-12) *mean the subsets* $W_4^1 \setminus \{\sigma\}$.

We will show Lemmas 3.5, 3.6, and 3.7 to prove Propositions 3.2, 3.3, and 3.4, respectively. These lemmas calculate the harmonic products of the generators z_l of $\mathfrak{H}^1_*$ for the corresponding depths.

Lemma 3.5 (case of depth 2). *For positive integers* l_1, l_2, *we have*

$$z_{l_1} * z_{l_2} = z_{l_1}z_{l_2} + z_{l_2}z_{l_1} + z_{l_1+l_2}. \tag{3-14}$$

Lemma 3.6 (case of depth 3). *For positive integers* l_1, l_2, l_3, *we have*

$$z_{l_1}z_{l_2} * z_{l_3} = z_{l_1}z_{l_2}z_{l_3} + z_{l_1}z_{l_3}z_{l_2} + z_{l_3}z_{l_1}z_{l_2} + z_{l_1+l_3}z_{l_2} + z_{l_1}z_{l_2+l_3}, \tag{3-15}$$

$$\begin{aligned} z_{l_1} * z_{l_2} * z_{l_3} = {} & z_{l_1}z_{l_2}z_{l_3} + z_{l_1}z_{l_3}z_{l_2} + z_{l_2}z_{l_1}z_{l_3} + z_{l_2}z_{l_3}z_{l_1} + z_{l_3}z_{l_1}z_{l_2} \\ & + z_{l_3}z_{l_2}z_{l_1} + z_{l_1+l_2}z_{l_3} + z_{l_1+l_3}z_{l_2} + z_{l_2+l_3}z_{l_1} \\ & + z_{l_1}z_{l_2+l_3} + z_{l_2}z_{l_1+l_3} + z_{l_3}z_{l_1+l_2} + z_{l_1+l_2+l_3}. \end{aligned} \tag{3-16}$$

Lemma 3.7 (case of depth 4). *For positive integers* l_1, l_2, l_3, l_4, *we have*

$$\begin{aligned} z_{l_1}z_{l_2}z_{l_3} * z_{l_4} = {} & z_{l_1}z_{l_2}z_{l_3}z_{l_4} + z_{l_1}z_{l_2}z_{l_4}z_{l_3} + z_{l_1}z_{l_4}z_{l_2}z_{l_3} + z_{l_4}z_{l_1}z_{l_2}z_{l_3} \\ & + z_{l_1+l_4}z_{l_2}z_{l_3} + z_{l_1}z_{l_2+l_4}z_{l_3} + z_{l_1}z_{l_2}z_{l_3+l_4}, \end{aligned} \tag{3-17}$$

$$\begin{aligned} z_{l_1}z_{l_2} * z_{l_3}z_{l_4} = {} & z_{l_1}z_{l_2}z_{l_3}z_{l_4} + z_{l_1}z_{l_3}z_{l_2}z_{l_4} + z_{l_1}z_{l_3}z_{l_4}z_{l_2} \\ & + z_{l_3}z_{l_1}z_{l_2}z_{l_4} + z_{l_3}z_{l_1}z_{l_4}z_{l_2} + z_{l_3}z_{l_4}z_{l_1}z_{l_2} \\ & + z_{l_1+l_3}z_{l_2}z_{l_4} + z_{l_1+l_3}z_{l_4}z_{l_2} + z_{l_1}z_{l_2+l_3}z_{l_4} \\ & + z_{l_3}z_{l_1+l_4}z_{l_2} + z_{l_1}z_{l_3}z_{l_2+l_4} + z_{l_3}z_{l_1}z_{l_2+l_4} + z_{l_1+l_3}z_{l_2+l_4}, \end{aligned} \tag{3-18}$$

$$\begin{aligned} z_{l_1}z_{l_2} * z_{l_3} * z_{l_4} = {} & z_{l_1}z_{l_2} * z_{l_3}z_{l_4} + z_{l_1}z_{l_2} * z_{l_4}z_{l_3} \\ & + z_{l_3+l_4}z_{l_1}z_{l_2} + z_{l_1}z_{l_3+l_4}z_{l_2} + z_{l_1}z_{l_2}z_{l_3+l_4} \\ & + z_{l_1+l_3+l_4}z_{l_2} + z_{l_1}z_{l_2+l_3+l_4}, \end{aligned} \tag{3-19}$$

$$\begin{aligned} z_{l_1} * z_{l_2} * z_{l_3} * z_{l_4} = {} & z_{l_1}z_{l_2} * z_{l_3} * z_{l_4} + z_{l_2}z_{l_1} * z_{l_3} * z_{l_4} \\ & + z_{l_1+l_2}z_{l_3}z_{l_4} + z_{l_1+l_2}z_{l_4}z_{l_3} + z_{l_3}z_{l_1+l_2}z_{l_4} + z_{l_4}z_{l_1+l_2}z_{l_3} \\ & + z_{l_3}z_{l_4}z_{l_1+l_2} + z_{l_4}z_{l_3}z_{l_1+l_2} + z_{l_1+l_2}z_{l_3+l_4} + z_{l_3+l_4}z_{l_1+l_2} \\ & + z_{l_1+l_2+l_3}z_{l_4} + z_{l_1+l_2+l_4}z_{l_3} + z_{l_3}z_{l_1+l_2+l_4} + z_{l_4}z_{l_1+l_2+l_3} \\ & + z_{l_1+l_2+l_3+l_4}. \end{aligned} \tag{3-20}$$

Proof of Lemma 3.5. Identity (3-14) follows from Equations (2-1) and (2-2) with $w_1 = w_2 = 1$. □

Proof of Lemma 3.6. We see from (2-1), (2-2), and (3-14) that

$$\begin{aligned}
z_{l_1} z_{l_2} * z_{l_3} &= z_{l_1}(z_{l_2} * z_{l_3}) + z_{l_3}(z_{l_1} z_{l_2} * 1) + z_{l_1+l_3}(z_{l_2} * 1)\\
&= z_{l_1}(z_{l_2} z_{l_3} + z_{l_3} z_{l_2} + z_{l_2+l_3}) + z_{l_3} z_{l_1} z_{l_2} + z_{l_1+l_3} z_{l_2}\\
&= z_{l_1} z_{l_2} z_{l_3} + z_{l_1} z_{l_3} z_{l_2} + z_{l_3} z_{l_1} z_{l_2} + z_{l_1+l_3} z_{l_2} + z_{l_1} z_{l_2+l_3},
\end{aligned}$$

which proves (3-15). We see from (3-14) and (3-15) that

$$\begin{aligned}
z_{l_1} * z_{l_2} * z_{l_3} &= (z_{l_1} z_{l_2} + z_{l_2} z_{l_1} + z_{l_1+l_2}) * z_{l_3}\\
&= z_{l_1} z_{l_2} * z_{l_3} + z_{l_2} z_{l_1} * z_{l_3} + z_{l_1+l_2} * z_{l_3}\\
&= z_{l_1} z_{l_2} z_{l_3} + z_{l_1} z_{l_3} z_{l_2} + z_{l_3} z_{l_1} z_{l_2} + z_{l_1+l_3} z_{l_2} + z_{l_1} z_{l_2+l_3}\\
&\quad + z_{l_2} z_{l_1} z_{l_3} + z_{l_2} z_{l_3} z_{l_1} + z_{l_3} z_{l_2} z_{l_1} + z_{l_2+l_3} z_{l_1} +_{l_2} z_{l_1+l_3}\\
&\quad + z_{l_1+l_2} z_{l_3} + z_{l_3} z_{l_1+l_2} + z_{l_1+l_2+l_3},
\end{aligned}$$

which proves (3-16), and completes the proof. □

Proof of Lemma 3.7. We see from (2-1), (2-2), and (3-15) that

$$\begin{aligned}
z_{l_1} z_{l_2} z_{l_3} * z_{l_4} &= z_{l_1}(z_{l_2} z_{l_3} * z_{l_4}) + z_{l_4}(z_{l_1} z_{l_2} z_{l_3} * 1) + z_{l_1+l_4}(z_{l_2} z_{l_3} * 1)\\
&= z_1(z_{l_2} z_{l_3} z_{l_4} + z_{l_2} z_{l_4} z_{l_3} + z_{l_4} z_{l_2} z_{l_3} + z_{l_2+l_4} z_{l_3} + z_{l_2} z_{l_3+l_4})\\
&\quad + z_{l_4} z_{l_1} z_{l_2} z_{l_3} + z_{l_1+l_4} z_{l_2} z_{l_3}\\
&= z_{l_1} z_{l_2} z_{l_3} z_{l_4} + z_{l_1} z_{l_2} z_{l_4} z_{l_3} + z_{l_1} z_{l_4} z_{l_2} z_{l_3} + z_{l_4} z_{l_1} z_{l_2} z_{l_3}\\
&\quad + z_{l_1+l_4} z_{l_2} z_{l_3} + z_{l_1} z_{l_2+l_4} z_{l_3} + z_{l_1} z_{l_2} z_{l_3+l_4},
\end{aligned}$$

which proves (3-17). We see from (2-2), (3-14), and (3-15) that

$$\begin{aligned}
z_{l_1} z_{l_2} * z_{l_3} z_{l_4} &= z_{l_1}(z_{l_2} * z_{l_3} z_{l_4}) + z_{l_3}(z_{l_1} z_{l_2} * z_{l_4}) + z_{l_1+l_3}(z_{l_2} * z_{l_4})\\
&= z_{l_1}(z_{l_3} z_{l_4} z_{l_2} + z_{l_3} z_{l_2} z_{l_4} + z_{l_2} z_{l_3} z_{l_4} + z_{l_3+l_2} z_{l_4} + z_{l_3} z_{l_4+l_2})\\
&\quad + z_{l_3}(z_{l_1} z_{l_2} z_{l_4} + z_{l_1} z_{l_4} z_{l_2} + z_{l_4} z_{l_1} z_{l_2} + z_{l_1+l_4} z_{l_2} + z_{l_1} z_{l_2+l_4})\\
&\quad + z_{l_1+l_3}(z_{l_2} z_{l_4} + z_{l_4} z_{l_2} + z_{l_2+l_4})\\
&= z_{l_1} z_{l_2} z_{l_3} z_{l_4} + z_{l_1} z_{l_3} z_{l_2} z_{l_4} + z_{l_1} z_{l_3} z_{l_4} z_{l_2} + z_{l_3} z_{l_1} z_{l_2} z_{l_4}\\
&\quad + z_{l_3} z_{l_1} z_{l_4} z_{l_2} + z_{l_3} z_{l_4} z_{l_1} z_{l_2} + z_{l_1+l_3} z_{l_2} z_{l_4} + z_{l_1+l_3} z_{l_4} z_{l_2}\\
&\quad + z_{l_1} z_{l_2+l_3} z_{l_4} + z_{l_3} z_{l_1+l_4} z_{l_2} + z_{l_1} z_{l_3} z_{l_2+l_4} + z_{l_3} z_{l_1} z_{l_2+l_4} + z_{l_1+l_3} z_{l_2+l_4},
\end{aligned}$$

which proves (3-18). We see from (3-14) and (3-15) that

$$\begin{aligned}
z_{l_1} z_{l_2} * z_{l_3} * z_{l_4} &= z_{l_1} z_{l_2} * (z_{l_3} z_{l_4} + z_{l_4} z_{l_3} + z_{l_3+l_4})\\
&= z_{l_1} z_{l_2} * z_{l_3} z_{l_4} + z_{l_1} z_{l_2} * z_{l_4} z_{l_3} + z_{l_1} z_{l_2} * z_{l_3+l_4}\\
&= z_{l_1} z_{l_2} * z_{l_3} z_{l_4} + z_{l_1} z_{l_2} * z_{l_4} z_{l_3}\\
&\quad + z_{l_1} z_{l_2} z_{l_3+l_4} + z_{l_1} z_{l_3+l_4} z_{l_2} + z_{l_3+l_4} z_{l_1} z_{l_2} + z_{l_1+l_3+l_4} z_{l_2} + z_{l_1} z_{l_2+l_3+l_4},
\end{aligned}$$

which proves (3-19). We see from (3-14) and (3-16) that

$$\begin{aligned}
z_{l_1} * z_{l_2} * z_{l_3} * z_{l_4} &= (z_{l_1} z_{l_2} + z_{l_2} z_{l_1} + z_{l_1+l_2}) * z_{l_3} * z_{l_4} \\
&= z_{l_1} z_{l_2} * z_{l_3} * z_{l_4} + z_{l_2} z_{l_1} * z_{l_3} * z_{l_4} + z_{l_1+l_2} * z_{l_3} * z_{l_4} \\
&= z_{l_1} z_{l_2} * z_{l_3} * z_{l_4} + z_{l_2} z_{l_1} * z_{l_3} * z_{l_4} \\
&\quad + z_{l_1+l_2} z_{l_3} z_{l_4} + z_{l_1+l_2} z_{l_4} z_{l_3} + z_{l_3} z_{l_1+l_2} z_{l_4} + z_{l_4} z_{l_1+l_2} z_{l_3} \\
&\quad + z_{l_3} z_{l_4} z_{l_1+l_2} + z_{l_4} z_{l_3} z_{l_1+l_2} + z_{l_1+l_2+l_3} z_{l_4} + z_{l_1+l_2+l_4} z_{l_3} + z_{l_3+l_4} z_{l_1+l_2} \\
&\quad + z_{l_1+l_2} z_{l_3+l_4} + z_{l_3} z_{l_1+l_2+l_4} + z_{l_4} z_{l_1+l_2+l_3} + z_{l_1+l_2+l_3+l_4},
\end{aligned}$$

which proves (3-20), and completes the proof. □

We are now in a position to prove Propositions 3.2 and 3.3.

Proof of Proposition 3.2. Let $\mathbf{l}_2 = (l_1, l_2)$ be an index set in $\mathbb{N}^2$. Applying the map Z^* to both sides of (3-14) and substituting $T = 0$, we obtain

$$\begin{aligned}
\zeta_1^*(l_1)\zeta_1^*(l_2) &= \zeta_2^*(l_1, l_2) + \zeta_2^*(l_2, l_1) + \zeta_1^*(l_1 + l_2) \\
&= \sum_{\sigma \in \mathfrak{C}_2} \zeta_2^*(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}) + \zeta_1^*(l_1 + l_2).
\end{aligned}$$

We thus have

$$\zeta_{(1,1)}^*(\mathbf{l}_2) = (\zeta_2^* \mid \Sigma_{\mathfrak{C}_2})(\mathbf{l}_2) + \zeta_1^* \circ w_2(\mathbf{l}_2),$$

which proves (3-7) because $\mathbf{l}_2$ is arbitrary and $\zeta_1^* \circ w_2(\mathbf{l}_2) = \zeta_1 \circ w_2(\mathbf{l}_2)$ by virtue of $w_2(\mathbf{l}_2) = l_1 + l_2 \geq 2$. □

Proof of Proposition 3.3. Let $\mathbf{l}_3 = (l_1, l_2, l_3)$ be an index set in $\mathbb{N}^3$. Applying the map Z^* to both sides of (3-15) and substituting $T = 0$, we obtain

$$\begin{aligned}
&\zeta_2^*(l_1, l_2)\zeta_1^*(l_3) \\
&\quad = \zeta_3^*(l_1, l_2, l_3) + \zeta_3^*(l_1, l_3, l_2) + \zeta_3^*(l_3, l_1, l_2) + \zeta_2^*(l_1 + l_3, l_2) + \zeta_2^*(l_1, l_2 + l_3) \\
&\quad = \sum_{\sigma \in U_3} \zeta_3^*(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}, l_{\sigma^{-1}(3)}) + \zeta_2^*(l_{\tau^{-1}(1)} + l_{\tau^{-1}(2)}, l_{\tau^{-1}(3)}) + \zeta_2^*(l_1, l_2 + l_3),
\end{aligned}$$

where $\tau = (123)$. We thus have

$$\zeta_{(2,1)}^*(\mathbf{l}_3) = (\zeta_3^* \mid \Sigma_{U_3})(\mathbf{l}_3) + (\zeta_2^* \circ w_{(2,1)} \mid (123))(\mathbf{l}_3) + \zeta_2^* \circ w_{(1,2)}(\mathbf{l}_3),$$

which proves (3-8). In a similar way, we obtain from (3-16) that

$$\zeta_{(1,1,1)}^*(\mathbf{l}_3) = (\zeta_3^* \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_3) + (\zeta_2^* \circ (w_{(2,1)} + w_{(1,2)}) \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3) + \zeta_1^* \circ w_3(\mathbf{l}_3),$$

which proves (3-9). □

We require another lemma for the proof of Proposition 3.4, since the proof is more complicated than those of Propositions 3.2 and 3.3.

Lemma 3.8. *Let* $\mathrm{id} = \mathrm{id}_4$ *mean the identity map on* $\mathbb{N}^4$. *We have the following equations in maps with the domain* $\mathbb{N}^4$:

(i)

$$w_{(2,2)} \mid (23)\Sigma_{\langle(34)\rangle} = w_{(2,2)} \mid \Sigma_{W_4^0}, \tag{3-21}$$

$$w_{(i,j,k)} \mid \Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} = \begin{cases} w_{(i,j,k)} \mid (\Sigma_{W_4^1} - (34) - (1324)) & ((i,j,k) \in I), \\ w_{(i,j,k)} \mid (\Sigma_{W_4^1} - (24) - (1234)) & ((i,j,k) = (1,2,1)), \end{cases} \tag{3-22}$$

$$\mathrm{id} \mid \Sigma_{V_4}\Sigma_{\langle(34)\rangle} = \mathrm{id} \mid \Sigma_{W_4}, \tag{3-23}$$

where I *in* (3-22) *means the set* $\{(2,1,1), (1,1,2)\}$.

(ii)

$$w_{(3,1)} \mid (24)\Sigma_{\langle(12)\rangle} = w_{(3,1)} \mid (\Sigma_{\mathfrak{C}_4} - e - (1234)), \tag{3-24}$$

$$w_{(1,3)} \mid \Sigma_{\langle(12)\rangle} = w_{(1,3)} \mid (\Sigma_{\mathfrak{C}_4} - (13)(24) - (1234)), \tag{3-25}$$

$$w_{(2,2)} \mid \Sigma_{W_4^0}\Sigma_{\langle(12)\rangle} = w_{(2,2)} \mid (\Sigma_{X_4} - e - (13)(24)), \tag{3-26}$$

$$w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}}\Sigma_{\langle(12)\rangle} = w_{(2,1,1)} \mid (\Sigma_{\mathfrak{A}_4} - e - (12)(34)), \tag{3-27}$$

$$w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}}\Sigma_{\langle(12)\rangle} = w_{(1,2,1)} \mid (\Sigma_{\mathfrak{A}_4} - (123) - (134)), \tag{3-28}$$

$$w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}}\Sigma_{\langle(12)\rangle} = w_{(1,1,2)} \mid (\Sigma_{\mathfrak{A}_4} - (13)(24) - (14)(23)), \tag{3-29}$$

$$\mathrm{id} \mid \Sigma_{W_4}\Sigma_{\langle(12)\rangle} = \mathrm{id} \mid \Sigma_{\mathfrak{S}_4}. \tag{3-30}$$

We now prove Proposition 3.4. We will then discuss a proof of Lemma 3.8.

Proof of Proposition 3.4. Let $\mathbf{l}_4 = (l_1, l_2, l_3, l_4)$ be an index set in $\mathbb{N}^4$. Applying the map Z^* to both sides of (3-17) and substituting $T = 0$, we obtain

$$\begin{aligned}
&\zeta_3^*(l_1,l_2,l_3)\zeta_1^*(l_4) \\
&\quad = \zeta_4^*(l_1,l_2,l_3,l_4) + \zeta_4^*(l_1,l_2,l_4,l_3) + \zeta_4^*(l_1,l_4,l_2,l_3) + \zeta_4^*(l_4,l_1,l_2,l_3) \\
&\qquad\quad + \zeta_3^*(l_1+l_4,l_2,l_3) + \zeta_3^*(l_1,l_2+l_4,l_3) + \zeta_3^*(l_1,l_2,l_3+l_4) \\
&\quad = \sum_{\sigma\in U_4} \zeta_4^*(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}, l_{\sigma^{-1}(3)}, l_{\sigma^{-1}(4)}) \\
&\qquad\quad + \zeta_3^*(l_{\tau^{-1}(1)} + l_{\tau^{-1}(2)}, l_{\tau^{-1}(3)}, l_{\tau^{-1}(4)}) \\
&\qquad\quad + \zeta_3^*(l_{\tau^{-1}(1)}, l_{\tau^{-1}(2)} + l_{\tau^{-1}(3)}, l_{\tau^{-1}(4)}) + \zeta_3^*(l_1,l_2,l_3+l_4),
\end{aligned}$$

where $\tau = (234)$. We thus have

$$\zeta^*_{(3,1)}(\mathbf{l}_4) = (\zeta_4^* \mid \Sigma_{U_4})(\mathbf{l}_4) + \zeta_3^* \circ ((w_{(2,1,1)} + w_{(1,2,1)}) \mid (234) + w_{(1,1,2)})(\mathbf{l}_4),$$

which proves (3-10). Similarly, we have by (3-18),

$$\zeta^*_{(2,2)}(\mathbf{l}_4) = (\zeta^*_4 \mid \Sigma_{V_4})(\mathbf{l}_4) \\ +(\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{V_4^0})(\mathbf{l}_4) + (\zeta^*_2 \circ w_{(2,2)} \mid (23))(\mathbf{l}_4),$$

which proves (3-11).

As we calculated above by using (3-17) and (3-18), we can deduce from (3-19) and (3-20) that

$$\zeta^*_{(2,1,1)} = \zeta^*_{(2,2)} \mid \Sigma_{\langle(34)\rangle} + \zeta^*_3 \circ (w_{(2,1,1)} \mid (1324) + w_{(1,2,1)} \mid (24) \\ +w_{(1,1,2)} \mid (34)) + \zeta^*_2 \circ (w_{(3,1)} \mid (24) + w_{(1,3)}) \tag{3-31}$$

and

$$\begin{aligned}\zeta^*_{(1,1,1,1)} &= \zeta^*_{(2,1,1)} \mid \Sigma_{\langle(12)\rangle} \\ &+\zeta^*_3 \circ \big(w_{(2,1,1)} \mid (e + (12)(34)) \\ &\qquad +w_{(1,2,1)} \mid ((123)+(134))+w_{(1,1,2)} \mid ((13)(24)+(14)(23))\big) \\ &+\zeta^*_2 \circ \big(w_{(2,2)} \mid (e + (13)(24)) + w_{(3,1)} \mid (e + (1234)) \\ &\qquad +w_{(1,3)} \mid ((13)(24) + (1234))\big) \\ &+\zeta_1 \circ w_4,\end{aligned} \tag{3-32}$$

respectively. Combining (3-11) and the equations of Lemma 3.8(i), we obtain

$$\begin{aligned}\zeta^*_{(2,2)} \mid \Sigma_{\langle(34)\rangle} &= \zeta^*_4 \mid \Sigma_{V_4}\Sigma_{\langle(34)\rangle} \\ &\quad +\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} \\ &\quad +\zeta^*_2 \circ w_{(2,2)} \mid (23)\Sigma_{\langle(34)\rangle} \\ &= \zeta^*_4 \mid \Sigma_{W_4} \\ &\quad +\zeta^*_3 \circ \big((w_{(2,1,1)}+w_{(1,1,2)}) \mid \Sigma_{W^1_{4,\{(34),(1324)\}}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,\{(24),(1234)\}}}\big) \\ &\quad +\zeta^*_2 \circ w_{(2,2)} \mid \Sigma_{W_4^0},\end{aligned} \tag{3-33}$$

where $W^1_{4,\{\sigma,\tau\}}$ denotes $W^1_4 \setminus \{\sigma, \tau\}$. A straightforward calculation shows that

$$\begin{aligned}&\zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_{4,\{(34),(1324)\}}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,\{(24),(1234)\}}}\big) \\ &\qquad + \zeta^*_3 \circ (w_{(2,1,1)} \mid (1324) + w_{(1,2,1)} \mid (24) + w_{(1,1,2)} \mid (34)) \\ &\quad = \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_4} \\ &\qquad - \zeta^*_3 \circ (w_{(2,1,1)} \mid (34) + w_{(1,2,1)} \mid (1234) + w_{(1,1,2)} \mid (1324)) \\ &\quad = \zeta^*_3 \circ \big(w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}} + w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}}\big)\end{aligned}$$

and so, substituting (3-33) into the right-hand side of (3-31) gives

$$\begin{aligned}
\zeta^*_{(2,1,1)} &= \zeta^*_4 \mid \Sigma_{W_4} \\
&\quad +\zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_{4,\{(34),(1324)\}}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,\{(24),(1234)\}}}\big) \\
&\quad +\zeta^*_2 \circ w_{(2,2)} \mid \Sigma_{W^0_4} \\
&\quad +\zeta^*_3 \circ (w_{(2,1,1)} \mid (1324) + w_{(1,2,1)} \mid (24) + w_{(1,1,2)} \mid (34)) \\
&\quad +\zeta^*_2 \circ (w_{(3,1)} \mid (24) + w_{(1,3)}) \\
&= \zeta^*_4 \mid \Sigma_{W_4} \\
&\quad +\zeta^*_3 \circ \big(w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}} + w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}}\big) \\
&\quad +\zeta^*_2 \circ (w_{(2,2)} \mid \Sigma_{W^0_4} + w_{(3,1)} \mid (24) + w_{(1,3)}),
\end{aligned}$$

which proves (3-12). Similarly, combining (3-12) and the equations of Lemma 3.8(ii), we obtain

$$\begin{aligned}
&\zeta^*_{(2,1,1)} \mid \Sigma_{\langle(12)\rangle} \\
&\quad = \zeta^*_4 \mid \Sigma_{\mathfrak{S}_4} \\
&\qquad +\zeta^*_3 \circ \big(w_{(2,1,1)} \mid (\Sigma_{\mathfrak{A}_4} - e - (12)(34)) + w_{(1,2,1)} \mid (\Sigma_{\mathfrak{A}_4} - (123) - (134)) \\
&\qquad\qquad + w_{(1,1,2)} \mid (\Sigma_{\mathfrak{A}_4} - (13)(24) - (14)(23))\big) \\
&\qquad +\zeta^*_2 \circ \big(w_{(2,2)} \mid (\Sigma_{X_4} - e - (13)(24)) + w_{(3,1)} \mid (\Sigma_{\mathfrak{C}_4} - e - (1234)) \\
&\qquad\qquad + w_{(1,3)} \mid (\Sigma_{\mathfrak{C}_4} - (13)(24) - (1234))\big).
\end{aligned}$$

Substituting this into the right-hand side of (3-32) proves (3-13). □

We will show Lemma 3.8 for the completeness of the proof of Proposition 3.4.

For a subgroup H in $\mathfrak{S}_4$, we define an equivalence relation $\equiv$ on $\mathfrak{S}_4$ such that $\sigma \equiv \tau \bmod H$ if and only if $\sigma\tau^{-1} \in H$, and we denote by $[\sigma]_H$ the equivalence class of σ. Note that $[\sigma]_H$ is the right coset $H\sigma$ of $\mathfrak{S}_4$. Table 1 below gives all the equivalence classes in $\mathfrak{S}_4$ modulo certain subgroups, where we denote by $\langle\sigma_1, \dots, \sigma_i\rangle$ the subgroup generated by permutations $\sigma_1, \dots, \sigma_i$. (We have already used $\langle\sigma\rangle$ to denote a cyclic subgroup.) We extend the congruence relation $\equiv$ on $\mathfrak{S}_4$ to that on its group ring $\mathbb{Z}[\mathfrak{S}_4]$, as follows. Let $\sum_{i=1}^m a_i\sigma_i$ and $\sum_{j=1}^n b_j\tau_j$ be elements in $\mathbb{Z}[\mathfrak{S}_4]$. Without loss of generality, we may assume that $\sigma_a \neq \sigma_b$ and $\tau_a \neq \tau_b$ if $a \neq b$. We then say that

$$\sum_{i=1}^m a_i\sigma_i \equiv \sum_{j=1}^n b_j\tau_j \mod H$$

if and only if $m = n$ and there is a permutation $\rho \in \mathfrak{S}_m$ such that $a_i = b_{\rho(i)}$ and $\sigma_i \equiv \tau_{\rho(i)} \bmod H$ $(i = 1, \dots, m)$. The equivalence classes in Table 1 will be necessary when we prove some congruence equations in $\mathbb{Z}[\mathfrak{S}_4]$.

The following congruence equations in $\mathbb{Z}[\mathfrak{S}_4]$ are useful for proving Lemma 3.8.

Lemma 3.9. *The following congruence equations hold*:

(i)

$$(23)\Sigma_{\langle(34)\rangle} \equiv \Sigma_{W_4^0} \mod \langle(12), (34)\rangle, \tag{3-34}$$

$$\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} \equiv \begin{cases} \Sigma_{W_4^1} - (34) - (1324) \mod \langle(12)\rangle \textit{ or } \mod \langle(34)\rangle, \\ \Sigma_{W_4^1} - (24) - (1234) \mod \langle(23)\rangle, \end{cases} \tag{3-35}$$

$$\Sigma_{V_4}\Sigma_{\langle(34)\rangle} \equiv \Sigma_{W_4} \mod \langle e\rangle. \tag{3-36}$$

(ii)

$$(24)\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{C}_4} - e - (1234) \mod \langle(12), (123)\rangle, \tag{3-37}$$

$$\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{C}_4} - (13)(24) - (1234) \mod \langle(23), (234)\rangle, \tag{3-38}$$

$$\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{X_4} - e - (13)(24) \mod \langle(12), (34)\rangle, \tag{3-39}$$

$$\Sigma_{W^1_{4,(34)}}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{A}_4} - e - (12)(34) \mod \langle(12)\rangle, \tag{3-40}$$

$$\Sigma_{W^1_{4,(1234)}}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{A}_4} - (123) - (134) \mod \langle(23)\rangle, \tag{3-41}$$

$$\Sigma_{W^1_{4,(1324)}}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{A}_4} - (13)(24) - (14)(23) \mod \langle(34)\rangle, \tag{3-42}$$

$$\Sigma_{W_4}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{S}_4} \mod \langle e\rangle. \tag{3-43}$$

Proof. Before proving the congruence equations, we introduce an identity in $\mathbb{Z}[\mathfrak{S}_n]$, which immediately follows from the definition:

$$\Sigma_H\Sigma_K = \Sigma_{H_1}\Sigma_K + \cdots + \Sigma_{H_n}\Sigma_K, \tag{3-44}$$

where H and K are subsets in $\mathfrak{S}_n$ such that $H_1, \dots, H_n$ are a partition of H (i.e., a set of subsets of H satisfying $\bigcup_{i=1}^n H_i = H$ and $H_i \cap H_j = \phi$ for $i \neq j$).

We first prove the congruence equations stated in (i). We obtain from $\Sigma_{\langle(34)\rangle} = e + (34)$ that

$$(23)\Sigma_{\langle(34)\rangle} = (23) + (234). \tag{3-45}$$

Since $\{(24), (124), (234), (1234)\}$ is an equivalence class modulo $\langle(12), (34)\rangle$ as we see in Table 1,

$$(234) \equiv (24) \mod \langle(12), (34)\rangle.$$

Thus, noting the definition of W_4^0 in (3-5), we have

$$(23)\Sigma_{\langle(34)\rangle} \equiv (23) + (24) = \Sigma_{W_4^0} \mod \langle(12), (34)\rangle,$$

mod	All equivalence classes
$\langle(12), (123)\rangle$	$\{e, (12), (13), (23), (123), (132)\}$, $\{(14), (14)(23), (142), (143), (1423), (1432)\}$, $\{(24), (13)(24), (124), (243), (1243), (1324)\}$, $\{(34), (12)(34), (134), (234), (1234), (1342)\}$.
$\langle(23), (234)\rangle$	$\{e, (23), (24), (34), (234), (243)\}$, $\{(12), (12)(34), (132), (142), (1342), (1432)\}$, $\{(13), (13)(24), (123), (143), (1243), (1423)\}$, $\{(14), (14)(23), (124), (134), (1234), (1324)\}$.
$\langle(12), (34)\rangle$	$\{e, (12), (34), (12)(34)\}$, $\{(13), (132), (143), (1432)\}$, $\{(14), (134), (142), (1342)\}$, $\{(23), (123), (243), (1243)\}$, $\{(24), (124), (234), (1234)\}$, $\{(13)(24), (14)(23), (1324), (1423)\}$.
$\langle(12)\rangle$	$\{e, (12)\}$, $\{(13), (132)\}$, $\{(14), (142)\}$, $\{(23), (123)\}$, $\{(24), (124)\}$, $\{(34), (12)(34)\}$, $\{(13)(24), (1324)\}$, $\{(14)(23), (1423)\}$, $\{(134), (1342)\}$, $\{(143), (1432)\}$, $\{(234), (1234)\}$, $\{(243), (1243)\}$.
$\langle(23)\rangle$	$\{e, (23)\}$, $\{(12), (132)\}$, $\{(13), (123)\}$, $\{(14), (14)(23)\}$, $\{(24), (243)\}$, $\{(34), (234)\}$, $\{(12)(34), (1342)\}$, $\{(13)(24), (1243)\}$, $\{(124), (1324)\}$, $\{(134), (1234)\}$, $\{(142), (1432)\}$, $\{(143), (1423)\}$.
$\langle(34)\rangle$	$\{e, (34)\}$, $\{(12), (12)(34)\}$, $\{(13), (143)\}$, $\{(14), (134)\}$, $\{(23), (243)\}$, $\{(24), (234)\}$, $\{(13)(24), (1423)\}$, $\{(14)(23), (1324)\}$, $\{(123), (1243)\}$, $\{(124), (1234)\}$, $\{(132), (1432)\}$, $\{(142), (1342)\}$.
$\langle(13)(24)\rangle$	$\{e, (13)(24)\}$, $\{(12), (1423)\}$, $\{(13), (24)\}$, $\{(14), (1243)\}$, $\{(23), (1342)\}$, $\{(34), (1324)\}$, $\{(12)(34), (14)(23)\}$, $\{(123), (142)\}$, $\{(124), (143)\}$, $\{(132), (234)\}$, $\{(134), (243)\}$, $\{(1234), (1432)\}$.

Table 1. All equivalence classes (or all right cosets $H\sigma$) in $\mathfrak{S}_4$ modulo subgroups H.

which proves (3-34). A calculation shows that

$$(1243)\Sigma_{\langle(34)\rangle} = (1243) + (124), \tag{3-46}$$

and so we see from (3-44), (3-45), and (3-46) that

$$\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} = (23)\Sigma_{\langle(34)\rangle} + (1243)\Sigma_{\langle(34)\rangle} = (23) + (124) + (234) + (1243). \tag{3-47}$$

Using (3-47) and the equivalence classes modulo $\langle(12)\rangle$, $\langle(23)\rangle$, and $\langle(34)\rangle$ in Table 1, we obtain

$$\begin{aligned}\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} &\equiv \begin{cases}(23) + (24) + (1234) + (1243) & \text{mod } \langle(12)\rangle \quad \text{or} \quad \text{mod } \langle(34)\rangle,\\ (23) + (34) + (1243) + (1324) & \text{mod } \langle(23)\rangle,\end{cases}\\ &= \begin{cases}\Sigma_{W_4^1} - (34) - (1324) & \text{mod } \langle(12)\rangle \quad \text{or} \quad \text{mod } \langle(34)\rangle,\\ \Sigma_{W_4^1} - (24) - (1234) & \text{mod } \langle(23)\rangle,\end{cases}\end{aligned}$$

which proves (3-35). Direct calculations show that

$$\begin{aligned}(13)(24)\,\Sigma_{\langle(34)\rangle} &= (13)(24)+(1324),\\ (123)\,\Sigma_{\langle(34)\rangle} &= (123)+(1234),\\ (243)\,\Sigma_{\langle(34)\rangle} &= (243)+(24),\end{aligned}$$

which together with (3-47) yields

$$\begin{aligned}\Sigma_{V_4}\Sigma_{\langle(34)\rangle} &= \Sigma_{\{e,(13)(24),(123),(243)\}}\Sigma_{\langle(34)\rangle}+\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle}\\ &= e+(34)+(13)(24)+(1324)+(123)+(1234)+(243)+(24)\\ &\qquad +(23)+(124)+(234)+(1243).\end{aligned}$$

We obtain (3-36) because the right-hand side of this equation is Σ_{W_4}, by definition.

We next prove the congruence equations stated in (ii). We easily see that

$$(24)\,\Sigma_{\langle(12)\rangle}=(24)+(142) \quad\text{and}\quad \Sigma_{\langle(12)\rangle}=e+(12). \tag{3-48}$$

Using (3-48) and the equivalence classes modulo $\langle(12),(123)\rangle$ and $\langle(23),(234)\rangle$ in Table 1, we obtain

$$\begin{aligned}(24)\,\Sigma_{\langle(12)\rangle} &\equiv (13)(24)+(1432) \;=\; \Sigma_{\mathfrak{C}_4}-e-(1234) \mod \langle(12),(123)\rangle,\\ \Sigma_{\langle(12)\rangle} &\equiv e+(1432) \;=\; \Sigma_{\mathfrak{C}_4}-(13)(24)-(1234) \mod \langle(23),(234)\rangle,\end{aligned}$$

which prove (3-37) and (3-38), respectively. A direct calculation shows that

$$(23)\,\Sigma_{\langle(12)\rangle}=(23)+(132), \tag{3-49}$$

and so we see from (3-48) and (3-49) that

$$\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle}=(23)\,\Sigma_{\langle(12)\rangle}+(24)\,\Sigma_{\langle(12)\rangle}=(23)+(24)+(132)+(142). \tag{3-50}$$

Using (3-50) and the equivalence classes modulo $\langle(12),(34)\rangle$ in Table 1, we obtain

$$\begin{aligned}\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle} &\equiv (23)+(1234)+(1432)+(14)\\ &= \Sigma_{X_4}-e-(13)(24) \mod \langle(12),(34)\rangle,\end{aligned}$$

which proves (3-39). Direct calculations show that

$$\begin{aligned}(34)\,\Sigma_{\langle(12)\rangle} &= (34)+(12)(34),\\ (1234)\,\Sigma_{\langle(12)\rangle} &= (1234)+(134),\\ (1243)\,\Sigma_{\langle(12)\rangle} &= (1243)+(143),\\ (1324)\,\Sigma_{\langle(12)\rangle} &= (1324)+(14)(23),\end{aligned} \tag{3-51}$$

and so we see from (3-50) and (3-51) that

$$\begin{aligned}\Sigma_{W_4^1}\Sigma_{\langle(12)\rangle} &= \Sigma_{\{(34),(1234),(1243),(1324)\}}\Sigma_{\langle(12)\rangle}+\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle}\\ &=(34)+(12)(34)+(1234)+(134)+(1243)+(143)+(1324)+(14)(23)\\ &\qquad+(23)+(24)+(132)+(142),\end{aligned}$$

which can be restated as

$$\begin{aligned}\Sigma_{W_4^1}\Sigma_{\langle(12)\rangle} &= (23)+(24)+(34)+(12)(34)+(14)(23)+(132)+(134)\\ &\qquad+(142)+(143)+(1234)+(1243)+(1324).\end{aligned}\tag{3-52}$$

Equation (3-52) together with the first equation of (3-51) gives

$$\begin{aligned}\Sigma_{W^1_{4,(34)}}\Sigma_{\langle(12)\rangle} &= \Sigma_{W_4^1\setminus\{(34)\}}\Sigma_{\langle(12)\rangle}\\ &= \Sigma_{W_4^1}\Sigma_{\langle(12)\rangle}-(34)\Sigma_{\langle(12)\rangle}\\ &=(23)+(24)+(14)(23)\\ &\quad+(132)+(134)+(142)+(143)+(1234)+(1243)+(1324).\end{aligned}$$

Using this equation and the equivalence classes modulo $\langle(12)\rangle$ in Table 1, we obtain

$$\begin{aligned}\Sigma_{W^1_{4,(34)}}\Sigma_{\langle(12)\rangle} &\equiv (123)+(124)+(14)(23)+(132)+(134)+(142)+(143)\\ &\qquad+(234)+(243)+(13)(24)\mod\langle(12)\rangle.\end{aligned}\tag{3-53}$$

Since $(ijk)=(ik)(ij)$ is an even permutation for a tuple (i,j,k) of distinct integers i,j,k, and since $\mathfrak{A}_4$ consists of even permutations in $\mathfrak{S}_4$ and $|\mathfrak{A}_4|=12$, we can express $\Sigma_{\mathfrak{A}_4}$ as

$$\begin{aligned}\Sigma_{\mathfrak{A}_4} &= e+(12)(34)+(13)(24)+(14)(23)\\ &\qquad+(123)+(124)+(132)+(134)+(142)+(143)+(234)+(243).\end{aligned}\tag{3-54}$$

Combining (3-53) and (3-54) proves (3-40). Similarly, (3-52) together with the second equation of (3-51) and the equivalence classes modulo $\langle(23)\rangle$ in Table 1 yields

$$\begin{aligned}\Sigma_{W^1_{4,(1234)}}\Sigma_{\langle(12)\rangle} &= (23)+(24)+(34)+(12)(34)+(14)(23)\\ &\quad+(132)+(142)+(143)+(1243)+(1324)\\ &\equiv e+(243)+(234)+(12)(34)+(14)(23)\\ &\quad+(132)+(142)+(143)+(13)(24)+(124)\\ &=\Sigma_{\mathfrak{A}_4}-(123)-(134)\mod\langle(23)\rangle,\end{aligned}\tag{3-55}$$

and (3-52) together with the fourth equation of (3-51) and the equivalence classes modulo $\langle(34)\rangle$ in Table 1 yields

$$\begin{aligned}\Sigma_{W^1_{4,(1324)}}\Sigma_{\langle(12)\rangle} &= (23)+(24)+(34)+(12)(34)\\ &\quad +(132)+(134)+(142)+(143)+(1234)+(1243)\\ &\equiv (243)+(234)+e+(12)(34)\\ &\quad +(132)+(134)+(142)+(143)+(124)+(123)\\ &= \Sigma_{\mathfrak{A}_4}-(13)(24)-(14)(23) \mod \langle(34)\rangle.\end{aligned} \tag{3-56}$$

Equations (3-55) and (3-56) prove (3-41) and (3-42), respectively. Direct calculations show that

$$\begin{aligned}(13)(24)\Sigma_{\langle(12)\rangle} &= (13)(24)+(1423),\\ (123)\Sigma_{\langle(12)\rangle} &= (123)+(13),\\ (124)\Sigma_{\langle(12)\rangle} &= (124)+(14),\\ (234)\Sigma_{\langle(12)\rangle} &= (234)+(1342),\\ (243)\Sigma_{\langle(12)\rangle} &= (243)+(1432),\end{aligned} \tag{3-57}$$

and so we can see from (3-52) and (3-57) that

$$\begin{aligned}\Sigma_{W_4}\Sigma_{\langle(12)\rangle} &= \Sigma_{\{e,(13)(24),(123),(124),(234),(243)\}}\Sigma_{\langle(12)\rangle}+\Sigma_{W^1_4}\Sigma_{\langle(12)\rangle}\\ &= \Sigma_{\mathfrak{S}_4},\end{aligned} \tag{3-58}$$

which proves (3-43), and completes the proof. □

The following statement holds: the maps $w_{(3,1)}$, $w_{(1,3)}$, $w_{(2,2)}$, $w_{(2,1,1)}$, $w_{(1,2,1)}$, and $w_{(1,1,2)}$ are invariant under the subgroups

$$\langle(12),(123)\rangle,\ \langle(23),(234)\rangle,\ \langle(12),(34)\rangle,\ \langle(12)\rangle,\ \langle(23)\rangle,\ \langle(34)\rangle,$$

respectively. In fact, this statement immediately follows from (3-1) and the fact that w_n is invariant under $\mathfrak{S}_n$, i.e., $(w_n\mid\sigma)(\mathbf{l}_n)=w_n(\mathbf{l}_n)$ for any $\sigma\in\mathfrak{S}_n$. Note that $\langle(12),(123)\rangle$ and $\langle(23),(234)\rangle$ are equivalent to the symmetric groups on $\{1,2,3\}$ and $\{2,3,4\}$, respectively.

We are now able to prove Lemma 3.8.

Proof of Lemma 3.8. We can obtain (3-21) by using (3-34) because of the invariance of $w_{(2,2)}$ under $\langle(12),(34)\rangle$. Similarly, we can obtain the equations from (3-22) through (3-30) by using the congruence equations from (3-35) through (3-43), respectively. □

3.2. ***Renormalization relations.*** For any real-valued functions $f_1, \dots, f_j$ of n variables, we define the product $f_1 \cdots f_j$ of the functions by using the multiplication in the real number field such that

$$(f_1 \cdots f_j)(x_1, \dots, x_n) := f_1(x_1, \dots, x_n) \times \cdots \times f_j(x_1, \dots, x_n).$$

For real-valued functions $g_{n_1}, \dots, g_{n_j}$ such that each g_{n_i} has n_i variables, we define the function $g_{n_1} \otimes \cdots \otimes g_{n_j}$ of $n = n_1 + \cdots + n_j$ variables by

$$g_{n_1} \otimes g_{n_2} \otimes \cdots \otimes g_{n_j}(x_1, \dots, x_n) :=$$
$$g_{n_1}(x_1, \dots, x_{n_1}) \times g_{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}) \times \cdots \times g_{n_j}(x_{n_1+n_2+\cdots+n_{j-1}+1}, \dots, x_n).$$

Note that $\zeta^{\dagger}_{(n_1,n_2,\dots,n_j)} = \zeta^{\dagger}_{n_1} \otimes \zeta^{\dagger}_{n_2} \otimes \cdots \otimes \zeta^{\dagger}_{n_j}$. We define a characteristic function $\check{\chi}^{ш}_n$ of the set $\mathbb{N}^n$ by

$$\check{\chi}^{ш}_n(\mathbf{l}_n) := \begin{cases} 1 & \text{if } l_1 = \cdots = l_n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3-59}$$

For example, $\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2$ is the two-variable function such that

$$\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(\mathbf{l}_2) = \check{\chi}^{ш}_2(l_1, l_2) \times \zeta_1 \circ w_2(l_1, l_2) = \check{\chi}^{ш}_2(l_1, l_2)\zeta_1(l_1 + l_2),$$

and $(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1$ is the three-variable function such that

$$(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1(\mathbf{l}_3) = \check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(\mathbf{l}_2) \times \zeta^{ш}_1(l_3) = \check{\chi}^{ш}_2(l_1, l_2)\zeta_1(l_1 + l_2)\zeta^{ш}_1(l_3).$$

The renormalization relations for depth less than 5 are written in terms of real-valued functions, as follows.

Proposition 3.10. *We have*

$$\zeta^*_1 = \zeta^{ш}_1, \tag{3-60}$$
$$\zeta^*_2 = \zeta^{ш}_2 - \tfrac{1}{2}\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2, \tag{3-61}$$
$$\zeta^*_3 = \zeta^{ш}_3 - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1 + \tfrac{1}{3}\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3, \tag{3-62}$$
$$\zeta^*_4 = \zeta^{ш}_4 - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 + \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 + \tfrac{1}{16}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4. \tag{3-63}$$

We require two lemmas to prove Proposition 3.10.

Lemma 3.11. *Let $P(T) = \sum_{j=0}^{n} a_j T^j$ be a polynomial whose degree n is less than 5. Then the constant term of $\rho(P(T)) - P(T)$ is*

$$\rho(P(T))\,|_{T=0} - P(0) = \begin{cases} 0 & (n < 2), \\ a_2\zeta_1(2) & (n = 2), \\ a_2\zeta_1(2) - 2a_3\zeta_1(3) & (n = 3), \\ a_2\zeta_1(2) - 2a_3\zeta_1(3) + \tfrac{27}{2}a_4\zeta_1(4) & (n = 4). \end{cases} \tag{3-64}$$

Lemma 3.12. *Let n be an integer with $1 \le n \le 4$, and let $\mathbf{l}_n = (l_1, \dots, l_n) \in \mathbb{N}^n$. Then*

$$Z^*_{\mathbf{l}_1}(T) \approx 0, \tag{3-65}$$

$$Z^*_{\mathbf{l}_2}(T) \approx \tfrac{1}{2}\check{\chi}^{ш}_2(\mathbf{l}_2)T^2, \tag{3-66}$$

$$Z^*_{\mathbf{l}_3}(T) \approx \tfrac{1}{2}\check{\chi}^{ш}_2 \otimes \zeta^*_1(\mathbf{l}_3)T^2 + \tfrac{1}{6}\check{\chi}^{ш}_3(\mathbf{l}_3)T^3, \tag{3-67}$$

$$Z^*_{\mathbf{l}_4}(T) \approx \tfrac{1}{2}\check{\chi}^{ш}_2 \otimes \zeta^*_2(\mathbf{l}_4)T^2 + \tfrac{1}{6}\check{\chi}^{ш}_3 \otimes \zeta^*_1(\mathbf{l}_4)T^3 + \tfrac{1}{24}\check{\chi}^{ш}_4(\mathbf{l}_4)T^4, \tag{3-68}$$

where $\approx$ means the congruence relation on $\mathbb{R}[T]$ modulo $\mathbb{R}T + \mathbb{R}$, i.e., $P(T) \approx Q(T)$ if and only if $\deg(P(T) - Q(T)) < 2$.

We will now prove Proposition 3.10. We will then discuss proofs of Lemmas 3.11 and 3.12.

Proof of Proposition 3.10. We first introduce an identity for proving (3-60), (3-61), (3-62), and (3-63): for any index set $\mathbf{l}_n = (l_1, \dots, l_n)$ in $\mathbb{N}^n$ with $n \ge 2$,

$$\check{\chi}^{ш}_n(\mathbf{l}_n)\zeta_1(n) = (\check{\chi}^{ш}_n \cdot \zeta_1 \circ w_n)(\mathbf{l}_n), \tag{3-69}$$

which can be rewritten in terms of real-valued functions with the domain $\mathbb{N}^n$ as

$$\zeta_1(n)\check{\chi}^{ш}_n = \check{\chi}^{ш}_n \cdot \zeta_1 \circ w_n.$$

Identity (3-69) is obtained by the fact that $\check{\chi}^{ш}_n(\mathbf{l}_n) = 0$ unless $l_1 = \dots = l_n = 1$, and the fact that $\zeta_1(n) = \zeta_1(l_1 + \dots + l_n) = \zeta_1(w_n(\mathbf{l}_n)) = \zeta_1 \circ w_n(\mathbf{l}_n)$ if $l_1 = \dots = l_n = 1$ and $n \ge 2$.

It follows from (3-64) and (3-65) that

$$\rho(Z^*_{\mathbf{l}_1}(T))\,|_{T=0} - Z^*_{\mathbf{l}_1}(0) = 0.$$

Using (2-5) and (2-8) with $T = 0$, we can restate this identity as

$$\zeta^{ш}_1(\mathbf{l}_1) - \zeta^*_1(\mathbf{l}_1) = 0,$$

which proves (3-60). Similarly, we obtain from (3-64) and (3-66) that

$$\zeta^{ш}_2(\mathbf{l}_2) - \zeta^*_2(\mathbf{l}_2) = \tfrac{1}{2}\check{\chi}^{ш}_2(\mathbf{l}_2)\zeta_1(2),$$

which proves (3-61) since $\check{\chi}^{ш}_2(\mathbf{l}_2)\zeta_1(2) = \check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(\mathbf{l}_2)$ by (3-69). We can obtain from (3-64) and (3-67) that

$$\zeta^{ш}_3(\mathbf{l}_3) - \zeta^*_3(\mathbf{l}_3) = \tfrac{1}{2}\check{\chi}^{ш}_2 \otimes \zeta^*_1(\mathbf{l}_3) \cdot \zeta_1(2) - \tfrac{1}{3}\check{\chi}^{ш}_3(\mathbf{l}_3)\zeta_1(3),$$

which proves (3-62) since $\check{\chi}^{ш}_3(\mathbf{l}_3)\zeta_1(3) = \check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3(\mathbf{l}_3)$ and

$$\begin{aligned}\check{\chi}^{ш}_2 \otimes \zeta^*_1(\mathbf{l}_3) \cdot \zeta_1(2) &= (\check{\chi}^{ш}_2(l_1, l_2)\zeta^*_1(l_3))\zeta_1(2) = (\check{\chi}^{ш}_2(l_1, l_2)\zeta_1(2))\zeta^*_1(l_3) \\ &\overset{\substack{(3\text{-}60)\\(3\text{-}69)}}{=} (\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(l_1, l_2)) \cdot \zeta^{ш}_1(l_3) = (\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1(\mathbf{l}_3).\end{aligned}$$

We can obtain from (3-64) and (3-68) that

$$\begin{aligned}(3\text{-}70)\quad &\zeta_4^{\sqcup\!\sqcup}(\mathbf{l}_4)-\zeta_4^{*}(\mathbf{l}_4)\\ &=\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}\otimes\zeta_2^{*}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{1}{3}\check{\chi}_3^{\sqcup\!\sqcup}\otimes\zeta_1^{*}(\mathbf{l}_4)\cdot\zeta_1(3)+\tfrac{9}{16}\check{\chi}_4^{\sqcup\!\sqcup}(\mathbf{l}_4)\zeta_1(4).\end{aligned}$$

The first term on the right-hand side of (3-70) can be calculated as

$$(3\text{-}71)\qquad \tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}\otimes\zeta_2^{*}(\mathbf{l}_4)\cdot\zeta_1(2)=\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}\otimes\zeta_2^{\sqcup\!\sqcup}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{5}{8}\check{\chi}_4^{\sqcup\!\sqcup}(\mathbf{l}_4)\zeta_1(4).$$

In fact, we see from (3-61) and (3-69) that $\zeta_2^{*}(l_3,l_4)=\zeta_2^{\sqcup\!\sqcup}(l_3,l_4)-\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}(l_3,l_4)\zeta_1(2)$, and so

$$\begin{aligned}\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}\otimes\zeta_2^{*}(\mathbf{l}_4)\cdot\zeta_1(2)&=\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}(l_1,l_2)\zeta_2^{*}(l_3,l_4)\zeta_1(2)\\ &=\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}(l_1,l_2)\zeta_2^{\sqcup\!\sqcup}(l_3,l_4)\zeta_1(2)-\tfrac{1}{4}\check{\chi}_2^{\sqcup\!\sqcup}(l_1,l_2)\check{\chi}_2^{\sqcup\!\sqcup}(l_3,l_4)\zeta_1(2)^2\\ &=\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}\otimes\zeta_2^{\sqcup\!\sqcup}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{1}{4}\check{\chi}_4^{\sqcup\!\sqcup}(\mathbf{l}_4)\zeta_1(2)^2,\end{aligned}$$

where we note that, by definition, $\check{\chi}_4^{\sqcup\!\sqcup}(\mathbf{l}_4)=\check{\chi}_2^{\sqcup\!\sqcup}(l_1,l_2)\check{\chi}_2^{\sqcup\!\sqcup}(l_3,l_4)$. This equality proves (3-71) because

$$(3\text{-}72)\qquad \zeta_1(2)^2=\tfrac{5}{2}\zeta_1(4),$$

which follows from Euler's results $\zeta_1(2)=\pi^2/6$ and $\zeta_1(4)=\pi^2/90$. Since

$$\check{\chi}_3^{\sqcup\!\sqcup}\otimes\zeta_1^{*}(\mathbf{l}_4)=\check{\chi}_3^{\sqcup\!\sqcup}\otimes\zeta_1^{\sqcup\!\sqcup}(\mathbf{l}_4)$$

by (3-60), combining (3-70) and (3-71) gives

$$\begin{aligned}(3\text{-}73)\quad &\zeta_4^{\sqcup\!\sqcup}(\mathbf{l}_4)-\zeta_4^{*}(\mathbf{l}_4)\\ &=\tfrac{1}{2}\check{\chi}_2^{\sqcup\!\sqcup}\otimes\zeta_2^{\sqcup\!\sqcup}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{1}{3}\check{\chi}_3^{\sqcup\!\sqcup}\otimes\zeta_1^{\sqcup\!\sqcup}(\mathbf{l}_4)\cdot\zeta_1(3)-\tfrac{1}{16}\check{\chi}_4^{\sqcup\!\sqcup}(\mathbf{l}_4)\zeta_1(4).\end{aligned}$$

By (3-69), the right-hand side of (3-73) can be rewritten as

$$\begin{aligned}(3\text{-}74)\quad &(\text{RHS of (3-73)})\\ &=\tfrac{1}{2}(\zeta_1(2)\check{\chi}_2^{\sqcup\!\sqcup})\otimes\zeta_2^{\sqcup\!\sqcup}(\mathbf{l}_4)-\tfrac{1}{3}(\zeta_1(3)\check{\chi}_3^{\sqcup\!\sqcup})\otimes\zeta_1^{\sqcup\!\sqcup}(\mathbf{l}_4)-\tfrac{1}{16}\check{\chi}_4^{\sqcup\!\sqcup}(\mathbf{l}_4)\zeta_1(4)\\ &=\tfrac{1}{2}(\check{\chi}_2^{\sqcup\!\sqcup}\cdot\zeta_1\circ w_2)\otimes\zeta_2^{\sqcup\!\sqcup}(\mathbf{l}_4)-\tfrac{1}{3}(\check{\chi}_3^{\sqcup\!\sqcup}\cdot\zeta_1\circ w_3)\otimes\zeta_1^{\sqcup\!\sqcup}(\mathbf{l}_4)-\tfrac{1}{16}\check{\chi}_4^{\sqcup\!\sqcup}\cdot\zeta_1\circ w_4(\mathbf{l}_4).\end{aligned}$$

Equating (3-73) and (3-74), we obtain (3-63). □

We will now show Lemmas 3.11 and 3.12 for the completeness of the proof of Proposition 3.10.

Proof of Lemma 3.11. Let O denote the Landau symbol. By definition,

$$(3\text{-}75)\qquad A(u)=\sum_{k=0}^{\infty}\gamma_k u^k\;=\;\exp\biggl(\sum_{m=2}^{\infty}\frac{(-1)^m\zeta_1(m)}{m}u^m\biggr)$$

near $u=0$. Thus,

$$\begin{aligned} A(u) &= 1+\left(\frac{\zeta_1(2)}{2}u^2-\frac{\zeta_1(3)}{3}u^3+\frac{\zeta_1(4)}{4}u^4+O(u^5)\right)+\frac{1}{2}\left(\frac{\zeta_1(2)}{2}u^2+O(u^3)\right)^2+\cdots \\ &= 1+\frac{\zeta_1(2)}{2}u^2-\frac{\zeta_1(3)}{3}u^3+\left(\frac{\zeta_1(4)}{4}+\frac{\zeta_1(2)^2}{8}\right)u^4+O(u^5), \end{aligned}$$

and so

$$\gamma_0=1, \quad \gamma_1=0, \quad \gamma_2=\frac{\zeta_1(2)}{2}, \quad \gamma_3=-\frac{\zeta_1(3)}{3},$$
$$\gamma_4=\frac{2\zeta_1(4)+\zeta_1(2)^2}{8}=\frac{9\zeta_1(4)}{16}$$

where we have used (3-72) for the last equality. Therefore, we see from (2-7) that

$$\rho(T^j)=\begin{cases} 1 & (j=0), \\ T & (j=1), \\ T^2+\zeta_1(2) & (j=2), \\ T^3+3\zeta_1(2)T-2\zeta_1(3) & (j=3), \\ T^4+6\zeta_1(2)T^2-8\zeta_1(3)T+\frac{27}{2}\zeta_1(4) & (j=4), \end{cases}$$

and so $\rho(1)\,|_{T=0}=1$, $\rho(T)\,|_{T=0}=0$, $\rho(T^2)\,|_{T=0}=\zeta_1(2)$, $\rho(T^3)\,|_{T=0}=-2\zeta_1(3)$, and $\rho(T^4)\,|_{T=0}=27\zeta_1(4)/2$. Since

$$\rho(P(T))\,|_{T=0}-P(0)=\sum_{j=0}^{n}a_j\rho(T^j)\,|_{T=0}-a_0,$$

we obtain (3-64). □

Proof of Lemma 3.12. We first recall a result in [Ihara et al. 2006] that will be required to prove Lemma 3.12. Let $\mathrm{reg}_*^T:\mathfrak{H}_*^1(\simeq\mathfrak{H}_*^0[y])\to\mathfrak{H}_*^0[T]$ be the algebra homomorphism defined in [Ihara et al. 2006, Section 3], which is characterized by the property that it is the identity on $\mathfrak{H}^0$ and sends y to T. Let $\mathrm{reg}_*:\mathfrak{H}_*^1\to\mathfrak{H}_*^0$ be the algebra homomorphism obtained by specializing reg_*^T to $T=0$. It immediately follows that

$$Z(\mathrm{reg}_*(z_{k_1}\cdots z_{k_n}))=Z^*(z_{k_1}\cdots z_{k_n})\,|_{T=0}\;=\;\zeta_n^*(k_1,\ldots,k_n)$$

for positive integers $k_1,\ldots,k_n$, since $Z^*:\mathfrak{H}_*^1\to\mathbb{R}[T]$ is the homomorphism characterized by the property that it extends the evaluation map $Z:\mathfrak{H}^0\to\mathbb{R}$ and sends y to T. Ihara et al. [2006, Corollary 5] showed that

$$w = \sum_{j=0}^{m} \frac{1}{j!} \mathrm{reg}_*(y^{m-j} w_0) * y^{*j}, \tag{3-76}$$

for $w \in \mathfrak{H}^1$, $w_0 \in \mathfrak{H}^0$, and $m \geq 0$ with $w = y^m w_0$.

We set $w = z_{l_1} \cdots z_{l_n} \in \mathfrak{H}^1$ for the given index set $\mathbf{l}_n$. The element w_0 can be written as $w_0 = z_{l_{m+1}} \cdots z_{l_n}$, where $l_{m+1} \geq 2$. (We set $w_0 = 1$ if $l_1 = \cdots = l_n = 1$.) Let $\{1\}^k$ denote k repetitions of 1. Applying Z^* to both sides of (3-76) gives

$$Z^*_{\mathbf{l}_n}(T) = \sum_{j=0}^{m} \frac{T^j}{j!} \zeta^*_{n-j}(\{1\}^{m-j}, l_{m+1}, \ldots, l_n), \tag{3-77}$$

where we define $\zeta^*_0(\phi) = 1$ for the case that $j = m = n$. For an integer j with $0 \leq j \leq m$, we see from (3-59) and $l_1 = \cdots = l_m = 1$ that $\check{\chi}^{ш}_j(l_1, \ldots, l_j) = 1$, and so

$$\begin{aligned} \zeta^*_{n-j}(\{1\}^{m-j}, l_{m+1}, \ldots, l_n) &= \zeta^*_{n-j}(l_{j+1}, \ldots, l_m, l_{m+1}, \ldots, l_n) \\ &= \check{\chi}^{ш}_j(l_1, \ldots, l_j) \zeta^*_{n-j}(l_{j+1}, \ldots, l_n) \\ &= \check{\chi}^{ш}_j \otimes \zeta^*_{n-j}(\mathbf{l}_n), \end{aligned} \tag{3-78}$$

where we define $\check{\chi}^{ш}_0(\phi) = 1$ and $\check{\chi}^{ш}_0 \otimes \zeta^*_n(\mathbf{l}_n) = \zeta^*_n(\mathbf{l}_n)$. Combining (3-77) and (3-78), we obtain

$$Z^*_{\mathbf{l}_n}(T) = \sum_{j=0}^{m} \frac{T^j}{j!} \check{\chi}^{ш}_j \otimes \zeta^*_{n-j}(\mathbf{l}_n). \tag{3-79}$$

Since $l_{m+1} \geq 2$, it follows from (3-59) that $\check{\chi}^{ш}_j(l_1, \ldots, l_j) = 0$ if $m < j \leq n$. Thus, (3-79) can be rewritten as

$$Z^*_{\mathbf{l}_n}(T) = \sum_{j=0}^{n} \frac{T^j}{j!} \check{\chi}^{ш}_j \otimes \zeta^*_{n-j}(\mathbf{l}_n). \tag{3-80}$$

Identities (3-65), (3-66), (3-67), and (3-68) are obtained from (3-80) for $n = 1, 2, 3$, and 4, respectively. □

4. Proofs

4.1. *Proof of Theorem 1.1.* Before proving Theorem 1.1 we introduce the following identity, which can be easily obtained by definitions (1-1) and (3-59): For $n \geq 2$,

$$\chi^{ш}_n + \check{\chi}^{ш}_n = l_n, \tag{4-1}$$

where I_n is the constant function whose value is 1. Identity (4-1) dose not hold when $n = 1$, but we will not need this case.

We now prove (1-2) and (1-3) in Theorem 1.1.

Proof of (1-2). By (3-7), we easily obtain

$$\zeta_2^{*} \mid \Sigma_{\mathfrak{C}_2} = \zeta_{(1,1)}^{*} - \zeta_1 \circ w_2, \tag{4-2}$$

which proves (1-2) for $\dagger = *$.

We can deduce the following identities from (3-60) and (3-61):

$$\zeta_{(1,1)}^{*} = \zeta_{(1,1)}^{ш} \qquad \text{and} \qquad \zeta_2^{*} \mid \Sigma_{\mathfrak{C}_2} = \zeta_2^{ш} \mid \Sigma_{\mathfrak{C}_2} - \check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2, \tag{4-3}$$

where we have used in the second identity the property that $\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2$ is invariant under $\mathfrak{S}_2$, or $\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2 \mid \Sigma_{\mathfrak{C}_2} = 2\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2$. Substituting (4-3) into (4-2) gives

$$\zeta_2^{ш} \mid \Sigma_{\mathfrak{C}_2} - \check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2 = \zeta_{(1,1)}^{ш} - \zeta_1 \circ w_2.$$

By (4-1) with $n = 2$, we can write this identity as

$$\zeta_2^{ш} \mid \Sigma_{\mathfrak{C}_2} = \zeta_{(1,1)}^{ш} - \chi_2^{ш} \cdot \zeta_1^{ш} \circ w_2, \tag{4-4}$$

which proves (1-2) for $\dagger = ш$. □

Proof of (1-3). Since $\mathfrak{C}_3 = \{e, (123), (132)\}$ and $U_3 = \{e, (23), (123)\}$, direct calculations give the following equations in $\mathbb{Z}[\mathfrak{S}_3]$:

$$(123)\Sigma_{\mathfrak{C}_3} = \Sigma_{\mathfrak{C}_3}, \quad \Sigma_{U_3}\Sigma_{\mathfrak{C}_3} = \Sigma_{\mathfrak{S}_3} + \Sigma_{\mathfrak{C}_3}.$$

We thus see from (3-8) that

$$\zeta_{(2,1)}^{*} \mid \Sigma_{\mathfrak{C}_3} = \zeta_3^{*} \mid (\Sigma_{\mathfrak{S}_3} + \Sigma_{\mathfrak{C}_3}) + \zeta_2^{*} \circ (w_{(2,1)} + w_{(1,2)}) \mid \Sigma_{\mathfrak{C}_3}. \tag{4-5}$$

Subtracting (4-5) from (3-9), we obtain $\zeta_{(1,1,1)}^{*} - \zeta_{(2,1)}^{*} \mid \Sigma_{\mathfrak{C}_3} = -\zeta_3^{*} \mid \Sigma_{\mathfrak{C}_3} + \zeta_1 \circ w_3$. This identity is equivalent to

$$\zeta_3^{*} \mid \Sigma_{\mathfrak{C}_3} = -\zeta_{(1,1,1)}^{*} + \zeta_{(2,1)}^{*} \mid \Sigma_{\mathfrak{C}_3} + \zeta_1 \circ w_3, \tag{4-6}$$

which proves (1-3) for $\dagger = *$.

We can deduce the following identities from (3-60), (3-61), and (3-62):

$$\zeta_{(1,1,1)}^{*} = \zeta_{(1,1,1)}^{ш}, \tag{4-7}$$

$$\zeta_{(2,1)}^{*} \mid \Sigma_{\mathfrak{C}_3} = \zeta_{(2,1)}^{ш} \mid \Sigma_{\mathfrak{C}_3} - \tfrac{1}{2}(\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{ш} \mid \Sigma_{\mathfrak{C}_3}, \tag{4-8}$$

$$\zeta_3^{*} \mid \Sigma_{\mathfrak{C}_3} = \zeta_3^{ш} \mid \Sigma_{\mathfrak{C}_3} - \tfrac{1}{2}(\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{ш} \mid \Sigma_{\mathfrak{C}_3} + \check{\chi}_3^{ш} \cdot \zeta_1 \circ w_3, \tag{4-9}$$

where we have used in the third identity the property that $\check{\chi}_3^{ш} \cdot (\zeta_1 \circ w_3)$ is invariant under $\mathfrak{S}_3$. By (4-1) with $n = 3$, substituting (4-7), (4-8), and (4-9) into (4-6) yields

$$\zeta_3^{ш} \mid \Sigma_{\mathfrak{C}_3} = -\zeta_{(1,1,1)}^{ш} + \zeta_{(2,1)}^{ш} \mid \Sigma_{\mathfrak{C}_3} + \chi_3^{ш} \cdot \zeta_1 \circ w_3. \tag{4-10}$$

Identity (4-10) proves (1-3) for $\dagger = ш$, and we complete the proof. $\square$

We now prepare two lemmas before proving (1-4), because the proof of (1-4) is more complicated than those of (1-2) and (1-3). The identities of Lemma 4.1 (resp. Lemma 4.2) correspond to (4-5) (resp. (4-7), (4-8), and (4-9)) in the proof of (1-3).

Lemma 4.1. *We have*

$$\begin{aligned}\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} = {}& \zeta^*_4 \mid (2\Sigma_{\mathfrak{C}_4}+(12)\Sigma_{\mathfrak{C}_4}+(34)\Sigma_{\mathfrak{C}_4})\\ &+\zeta^*_3\circ(w_{(2,1,1)}+w_{(1,2,1)}+w_{(1,1,2)}) \mid (\Sigma_{\mathfrak{A}_4}-(13)\Sigma_{\mathfrak{C}_4}-(23)\Sigma_{\mathfrak{C}_4}),\end{aligned} \tag{4-11}$$

$$\begin{aligned}\zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} = {}& \zeta^*_4 \mid (\Sigma_{\mathfrak{C}_4}+(14)\Sigma_{\mathfrak{C}_4}+(23)\Sigma_{\mathfrak{C}_4})\\ &+\zeta^*_3\circ(w_{(2,1,1)}+w_{(1,2,1)}+w_{(1,1,2)}) \mid (23)\Sigma_{\mathfrak{C}_4}\\ &+\zeta^*_2\circ w_{(2,2)} \mid (23)\Sigma_{\mathfrak{C}^0_4},\end{aligned} \tag{4-12}$$

$$\begin{aligned}\zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} = {}& \zeta^*_4 \mid (2\Sigma_{\mathfrak{S}_4}+\Sigma_{\mathfrak{C}_4}-(13)\Sigma_{\mathfrak{C}_4})\\ &+\zeta^*_3\circ(w_{(2,1,1)}+w_{(1,2,1)}+w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4}-(13)\Sigma_{\mathfrak{C}_4})\\ &+\zeta^*_2\circ(w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4}+2(23)\Sigma_{\mathfrak{C}^0_4})+(w_{(3,1)}+w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}).\end{aligned} \tag{4-13}$$

Lemma 4.2. *We have*

$$\zeta^*_{(1,1,1,1)} = \zeta^{ш}_{(1,1,1,1)}, \tag{4-14}$$

$$\zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} = \zeta^{ш}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_{(1,1)} \mid \Sigma_{\mathfrak{C}_4}, \tag{4-15}$$

$$\zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} = \zeta^{ш}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\mathfrak{C}_4} + \tfrac{5}{4}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4, \tag{4-16}$$

$$\begin{aligned}\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} = {}& \zeta^{ш}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_{(1,1)} \mid \Sigma_{\mathfrak{C}_4}\\ &+ \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 \mid \Sigma_{\mathfrak{C}_4},\end{aligned} \tag{4-17}$$

$$\begin{aligned}\zeta^*_4 \mid \Sigma_{\mathfrak{C}_4} = {}& \zeta^{ш}_4 \mid \Sigma_{\mathfrak{C}_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\mathfrak{C}_4}\\ &+ \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 \mid \Sigma_{\mathfrak{C}_4} + \tfrac{1}{4}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4.\end{aligned} \tag{4-18}$$

We now prove (1-4). We will then discuss proofs of Lemmas 4.1 and 4.2.

Proof of identity (1-4). Direct calculations show that

$$\begin{aligned}\Sigma_{\mathfrak{C}_4} &= e + (13)(24) + (1234) + (1432),\\ (12)\Sigma_{\mathfrak{C}_4} &= (12) + (143) + (234) + (1324),\\ (13)\Sigma_{\mathfrak{C}_4} &= (13) + (24) + (12)(34) + (14)(23),\\ (14)\Sigma_{\mathfrak{C}_4} &= (14) + (123) + (243) + (1342),\\ (23)\Sigma_{\mathfrak{C}_4} &= (23) + (134) + (142) + (1243),\\ (34)\Sigma_{\mathfrak{C}_4} &= (34) + (124) + (132) + (1423),\end{aligned} \tag{4-19}$$

from which we see that

$$\Sigma_{\mathfrak{S}_4} = (e + (12) + (13) + (14) + (23) + (34))\Sigma_{\mathfrak{C}_4}, \tag{4-20}$$

i.e., $\{\mathfrak{C}_4, (12)\mathfrak{C}_4, (13)\mathfrak{C}_4, (14)\mathfrak{C}_4, (23)\mathfrak{C}_4, (34)\mathfrak{C}_4\}$ gives a left $\mathfrak{C}_4$-coset decomposition of $\mathfrak{S}_4$. By (4-20), the sum of (4-11) and (4-12) yields

$$\begin{aligned}\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0} = \zeta^*_4 \mid (\Sigma_{\mathfrak{S}_4} + 2\Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4})&\\ +\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4})&\\ +\zeta^*_2 \circ w_{(2,2)} \mid (23)\Sigma_{\mathfrak{C}_4^0}.&\end{aligned}$$

Subtracting (4-13) from this identity, we obtain

$$\begin{aligned}&\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0} - \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4}\\ &\quad = -\zeta^*_4 \mid (\Sigma_{\mathfrak{S}_4} - \Sigma_{\mathfrak{C}_4})\\ &\qquad - \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{\mathfrak{A}_4}\\ &\qquad - \zeta^*_2 \circ (w_{(2,2)} \mid \Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4^0}) + (w_{(3,1)} + w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}).\end{aligned} \tag{4-21}$$

We see from (3-6) and the equivalence classes modulo $\langle(12), (34)\rangle$ in Table 1 that $\Sigma_{X_4} = (14) + (23) + \Sigma_{\mathfrak{C}_4} \equiv (23) + (134) + \Sigma_{\mathfrak{C}_4} = (23)\Sigma_{\mathfrak{C}_4^0} + \Sigma_{\mathfrak{C}_4} \mod \langle(12), (34)\rangle$, and so

$$w_{(2,2)} \mid \Sigma_{X_4} = w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4^0}).$$

Thus the sum of (3-13) and (4-21) yields

$$\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0} - \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(1,1,1,1)} = \zeta^*_4 \mid \Sigma_{\mathfrak{C}_4} + \zeta_1 \circ w_4,$$

which is equivalent to

$$\zeta^*_4 \mid \Sigma_{\mathfrak{C}_4} = \zeta^*_{(1,1,1,1)} - \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0} + \zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} - \zeta_1 \circ w_4. \tag{4-22}$$

Identity (4-22) proves (1-4) for $\dagger = *$.

Combining (4-14)–(4-17) (or considering (4-14) − (4-15) + (4-16) + (4-17), roughly speaking), we can restate the right-hand side of (4-22) as

$$\begin{aligned}(\text{RHS of (4-22)}) = \zeta^{ш}_{(1,1,1,1)} - \zeta^{ш}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^{ш}_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0} + \zeta^{ш}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}&\\ - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\mathfrak{C}_4}&\\ + \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 \mid \Sigma_{\mathfrak{C}_4}&\\ + (\tfrac{5}{4}\check{\chi}^{ш}_4 - I_4) \cdot (\zeta_1 \circ w_4).&\end{aligned} \tag{4-23}$$

Equating (4-18), (4-22), and (4-23), we obtain

$$\zeta^{ш}_4 \mid \Sigma_{\mathfrak{C}_4} = \zeta^{ш}_{(1,1,1,1)} - \zeta^{ш}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^{ш}_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0} + \zeta^{ш}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + (\check{\chi}^{ш}_4 - I_4) \cdot (\zeta_1 \circ w_4),$$

which, together with (4-1) of $n = 4$, proves (1-4) for $\dagger = ш$, and we complete the proof. □

We will show Lemmas 4.1 and 4.2 for the completeness of the proof of (1-4). We first prove Lemma 4.2.

Proof of Lemma 4.2. We easily see from (3-60) that $\zeta^{*}_{(\{1\}^n)} = \zeta^{ш}_{(\{1\}^n)}$, which with $n = 4$ proves (4-14). Multiplying both sides of (3-61) by $\zeta^{*}_{(1,1)} = \zeta^{ш}_{(1,1)}$ from the right, in the sense of the operator $\otimes$, gives

$$\zeta^{*}_{(2,1,1)} = \zeta^{ш}_{(2,1,1)} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_{(1,1)}.$$

Applying $\Sigma_{\mathfrak{C}_4}$ to both sides of this equation, we obtain (4-15). We can similarly obtain (4-17), by using (3-62) and $\zeta^{*}_1 = \zeta^{ш}_1$ instead of (3-61) and $\zeta^{*}_{(1,1)} = \zeta^{ш}_{(1,1)}$, respectively. We also obtain (4-18) by applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-63), since $\check{\chi}^{ш}_4 \cdot (\zeta_1 \circ w_4)$ is invariant under $\mathfrak{S}_4$.

We prove (4-16). We easily see that $f \otimes g \mid (13)(24) = g \otimes f$ for any functions f and g of two variables, and so we obtain from (3-61) and $\zeta^{*}_{(2,2)} = \zeta^{*}_2 \otimes \zeta^{*}_2 (= \zeta^{*\otimes 2}_2)$ that

$$\begin{aligned} \zeta^{*}_{(2,2)} &= (\zeta^{ш}_2 - \tfrac{1}{2}\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes (\zeta^{ш}_2 - \tfrac{1}{2}\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \\ &= \zeta^{ш}_{(2,2)} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid (e + (13)(24)) + \tfrac{1}{4}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2)^{\otimes 2}. \end{aligned} \tag{4-24}$$

We see from (3-69), (3-72), and $\check{\chi}^{ш\otimes 2}_2 = \check{\chi}^{ш}_4$ that

$$(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2)^{\otimes 2} = \zeta_1(2)^2 \check{\chi}^{ш\otimes 2}_2 = \tfrac{5}{2}\zeta_1(4)\check{\chi}^{ш}_4 = \tfrac{5}{2}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4,$$

by which we can restate (4-24) as

$$\zeta^{*}_{(2,2)} = \zeta^{ш}_{(2,2)} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\langle (13)(24)\rangle} + \tfrac{5}{8}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4. \tag{4-25}$$

Since $\mathfrak{C}^0_4 = \{e, (1234)\} \subset \mathfrak{C}_4 = \{e, (1234), (13)(24), (1432)\}$,

$$\Sigma_{\langle (13)(24)\rangle} \Sigma_{\mathfrak{C}^0_4} = \Sigma_{\mathfrak{C}_4}. \tag{4-26}$$

Applying $\Sigma_{\mathfrak{C}^0_4}$ to both sides of (4-25), we obtain (4-16), and this completes the proof. □

We now prove Lemma 4.1.

Proof of Lemma 4.1. Let $\sigma \in \{(12), (23), (34)\}$. By the equivalence classes modulo $\langle\sigma\rangle$ in Table 1 and straightforward calculations, (3-54) yields

$$2\Sigma_{\mathfrak{A}_4} \equiv \Sigma_{\mathfrak{S}_4} \mod \langle\sigma\rangle, \tag{4-27}$$

and (4-19) yields

$$
\begin{aligned}
&\Sigma_{\mathfrak{C}_4} \equiv (12)\Sigma_{\mathfrak{C}_4}, \quad (13)\Sigma_{\mathfrak{C}_4} \equiv (34)\Sigma_{\mathfrak{C}_4}, \quad (14)\Sigma_{\mathfrak{C}_4} \equiv (23)\Sigma_{\mathfrak{C}_4} \mod \langle (12)\rangle,\\
&\Sigma_{\mathfrak{C}_4} \equiv (23)\Sigma_{\mathfrak{C}_4}, \quad (12)\Sigma_{\mathfrak{C}_4} \equiv (34)\Sigma_{\mathfrak{C}_4}, \quad (13)\Sigma_{\mathfrak{C}_4} \equiv (14)\Sigma_{\mathfrak{C}_4} \mod \langle (23)\rangle,\\
&\Sigma_{\mathfrak{C}_4} \equiv (34)\Sigma_{\mathfrak{C}_4}, \quad (12)\Sigma_{\mathfrak{C}_4} \equiv (13)\Sigma_{\mathfrak{C}_4}, \quad (14)\Sigma_{\mathfrak{C}_4} \equiv (23)\Sigma_{\mathfrak{C}_4} \mod \langle (34)\rangle.
\end{aligned} \tag{4-28}
$$

Thus, we deduce from (4-20) that

$$\Sigma_{\mathfrak{A}_4} \equiv \alpha\Sigma_{\mathfrak{C}_4} + \beta\Sigma_{\mathfrak{C}_4} + \gamma\Sigma_{\mathfrak{C}_4} \mod \langle \sigma\rangle, \tag{4-29}$$

where (α, β, γ) is a 3-tuple of $\{e, (12), (13), (14), (23), (34)\}$ such that

$$
\begin{cases}
\alpha \in \{e, (12)\},\ \beta \in \{(13), (34)\},\ \gamma \in \{(14), (23)\} & (\sigma = (12)),\\
\alpha \in \{e, (23)\},\ \beta \in \{(12), (34)\},\ \gamma \in \{(13), (14)\} & (\sigma = (23)),\\
\alpha \in \{e, (34)\},\ \beta \in \{(12), (13)\},\ \gamma \in \{(14), (23)\} & (\sigma = (34)).
\end{cases} \tag{4-30}
$$

We now prove (4-11). Since either $g\mathfrak{C}_4 = h\mathfrak{C}_4$ or $g\mathfrak{C}_4 \cap h\mathfrak{C}_4 = \phi$ for any $g, h \in \mathfrak{S}_4$, we can see from the first and second equations of (4-19) that

$$(1234)\Sigma_{\mathfrak{C}_4} = \Sigma_{\mathfrak{C}_4} \quad \text{and} \quad (234)\Sigma_{\mathfrak{C}_4} = (12)\Sigma_{\mathfrak{C}_4},$$

respectively. By (3-3) and (3-44), we obtain

$$\Sigma_{U_4}\Sigma_{\mathfrak{C}_4} = \Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4} + (234)\Sigma_{\mathfrak{C}_4} + (1234)\Sigma_{\mathfrak{C}_4} = 2\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4}$$

Thus, applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-10) yields

$$
\begin{aligned}
\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} = \zeta^*_4 \mid (2\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4})&\\
+\zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,2,1)}) \mid (12)\Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid \Sigma_{\mathfrak{C}_4}\big)&.
\end{aligned} \tag{4-31}
$$

We know from (4-29) and (4-30) that

$$\Sigma_{\mathfrak{A}_4} \equiv (13)\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4} + \begin{cases} (12)\Sigma_{\mathfrak{C}_4} & \mod \langle (12)\rangle \quad \text{or} \quad \mod \langle (23)\rangle,\\ \Sigma_{\mathfrak{C}_4} & \mod \langle (34)\rangle. \end{cases}$$

Since $w_{(2,1,1)}$, $w_{(1,2,1)}$, and $w_{(1,1,2)}$ are invariant under $\langle (12)\rangle$, $\langle (23)\rangle$, and $\langle (34)\rangle$, respectively, we have

$$
\begin{aligned}
&w_{(i,j,k)} \mid (\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4} - (23)\Sigma_{\mathfrak{C}_4})\\
&\qquad\qquad \equiv \begin{cases} w_{(i,j,k)} \mid (12)\Sigma_{\mathfrak{C}_4} & ((i, j, k) = (2, 1, 1), (1, 2, 1)),\\ w_{(i,j,k)} \mid \Sigma_{\mathfrak{C}_4} & ((i, j, k) = (1, 1, 2)), \end{cases}
\end{aligned}
$$

and so

$$
\begin{aligned}
\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4} - (23)\Sigma_{\mathfrak{C}_4})&\\
= \zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,2,1)}) \mid (12)\Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid \Sigma_{\mathfrak{C}_4}\big)&.
\end{aligned} \tag{4-32}
$$

Combining (4-31) and (4-32), we obtain (4-11).

We can easily see that

$$\Sigma_{V_4^0} = (23)\,\Sigma_{\langle(13)(24)\rangle} \quad \text{and} \quad \Sigma_{V_4} = (e+(123)+(23))\,\Sigma_{\langle(13)(24)\rangle},$$

which together with (4-26) give

$$\Sigma_{V_4^0}\Sigma_{\mathfrak{C}_4^0} = (23)\,\Sigma_{\mathfrak{C}_4} \quad \text{and} \quad \Sigma_{V_4}\Sigma_{\mathfrak{C}_4^0} = \Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}_4},$$

respectively, where we note that $(123)\,\Sigma_{\mathfrak{C}_4} = (14)\,\Sigma_{\mathfrak{C}_4}$ by the fourth equation of (4-19). Thus, applying $\Sigma_{\mathfrak{C}_4^0}$ to both sides of (3-11), we obtain (4-12).

Lastly, we prove (4-13). We can obtain the following identity by applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-12):

$$\begin{aligned}\text{(4-33)}\quad \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} = \zeta^*_4 \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\,\Sigma_{\mathfrak{C}_4}) \\ +\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4}) \\ -\zeta^*_3 \circ (w_{(2,1,1)} \mid (34)\,\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (12)\,\Sigma_{\mathfrak{C}_4}) \\ +\zeta^*_2 \circ (w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\,\Sigma_{\mathfrak{C}_4^0}) + (w_{(3,1)} + w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}).\end{aligned}$$

(We will prove (4-33) in Lemma 4.3 below because the proof is not short.) We can also obtain by (4-28)

$$(23)\,\Sigma_{\mathfrak{C}_4} + (13)\,\Sigma_{\mathfrak{C}_4} \equiv (14)\,\Sigma_{\mathfrak{C}_4} + \begin{cases} (34)\,\Sigma_{\mathfrak{C}_4} & \text{mod } \langle(12)\rangle, \\ \Sigma_{\mathfrak{C}_4} & \text{mod } \langle(23)\rangle, \\ (12)\,\Sigma_{\mathfrak{C}_4} & \text{mod } \langle(34)\rangle. \end{cases}$$

Thus, $(w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (13)\,\Sigma_{\mathfrak{C}_4}$ can be expressed as

$$\begin{aligned}(w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (13)\,\Sigma_{\mathfrak{C}_4} \\ = (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (-(23)\,\Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4}) \\ +(w_{(2,1,1)} \mid (34)\,\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (12)\,\Sigma_{\mathfrak{C}_4}).\end{aligned}$$

Adding $-(w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4})$ to both sides of this equation, and then multiplying both sides by -1, we obtain

$$\begin{aligned}\text{(4-34)}\quad \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} - (13)\,\Sigma_{\mathfrak{C}_4}) \\ = \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4}) \\ -\zeta^*_3 \circ (w_{(2,1,1)} \mid (34)\,\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (12)\,\Sigma_{\mathfrak{C}_4}).\end{aligned}$$

Combining (4-33) and (4-34) proves (4-13). □

Lemma 4.3. (i) *Let* $\sigma \in \{(12), (23), (34)\}$. *The following congruence equations hold*:

$$\text{(4-35)}\qquad \Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} \equiv \Sigma_{\mathfrak{C}_4} + 2(23)\,\Sigma_{\mathfrak{C}_4^0} \quad \text{mod } \langle(12), (34)\rangle,$$

$$\Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4} \equiv 2\Sigma_{\mathfrak{A}_4} + (23)\Sigma_{\mathfrak{C}_4} - (14)\Sigma_{\mathfrak{C}_4} \mod \langle\sigma\rangle, \tag{4-36}$$

$$\Sigma_{W_4}\Sigma_{\mathfrak{C}_4} \equiv 2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4} \mod \langle e\rangle. \tag{4-37}$$

(ii) *Identity* (4-33) *holds.*

Proof. We first prove the assertion (i). We see from (3-5) and (3-44) that

$$\Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} = (23)\Sigma_{\mathfrak{C}_4} + (24)\Sigma_{\mathfrak{C}_4}$$

and from the third equation of (4-19), we see that

$$(24)\Sigma_{\mathfrak{C}_4} = (13)\Sigma_{\mathfrak{C}_4}.$$

We thus obtain

$$\Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} = (13)\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4}. \tag{4-38}$$

Equation (4-38) proves (4-35), since

$$\begin{aligned}(13)\Sigma_{\mathfrak{C}_4} &\equiv (1432) + (1234) + e + (13)(24) \;=\; \Sigma_{\mathfrak{C}_4} \mod \langle(12), (34)\rangle,\\ (23)\Sigma_{\mathfrak{C}_4} &\equiv 2((23) + (134)) \;=\; 2(23)\Sigma_{\mathfrak{C}_4^0} \mod \langle(12), (34)\rangle,\end{aligned}$$

which can be seen from the equivalence classes modulo $\langle(12), (34)\rangle$ in Table 1. By virtue of (4-20), calculations similar to (4-38) show that

$$\begin{aligned}\Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4} &= \Sigma_{\{(34),(1234),(1243),(1324)\}}\Sigma_{\mathfrak{C}_4} + \Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4}\\ &= ((34)\Sigma_{\mathfrak{C}_4} + \Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4}) + ((13)\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4})\\ &= \Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (13)\Sigma_{\mathfrak{C}_4} + 2(23)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4}\\ &= \Sigma_{\mathfrak{S}_4} + (23)\Sigma_{\mathfrak{C}_4} - (14)\Sigma_{\mathfrak{C}_4},\end{aligned} \tag{4-39}$$

and

$$\begin{aligned}\Sigma_{W_4}\Sigma_{\mathfrak{C}_4} &= \Sigma_{\{e_4,(13)(24),(123),(124),(234),(243)\}}\Sigma_{\mathfrak{C}_4} + \Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4}\\ &= (\Sigma_{\mathfrak{C}_4} + \Sigma_{\mathfrak{C}_4} + (14)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (14)\Sigma_{\mathfrak{C}_4})\\ &\qquad + (\Sigma_{\mathfrak{S}_4} + (23)\Sigma_{\mathfrak{C}_4} - (14)\Sigma_{\mathfrak{C}_4})\\ &= \Sigma_{\mathfrak{S}_4} + 2\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (14)\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4}\\ &= 2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4}.\end{aligned} \tag{4-40}$$

Then we obtain (4-36) by (4-27) and (4-39), and obtain (4-37) by (4-40).

We now prove the assertion (ii), or (4-33). We can deduce from (4-35), (4-36), and (4-37) that

$$\zeta_2^* \circ w_{(2,2)} \mid \Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} = \zeta_2^* \circ w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\Sigma_{\mathfrak{C}_4^0}), \tag{4-41}$$

$$\zeta_3^* \circ w_{(i,j,k)} \mid \Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4} = \zeta_3^* \circ w_{(i,j,k)} \mid (2\Sigma_{\mathfrak{A}_4} + (23)\Sigma_{\mathfrak{C}_4} - (14)\Sigma_{\mathfrak{C}_4}), \tag{4-42}$$

$$\zeta_4^* \mid \Sigma_{W_4}\Sigma_{\mathfrak{C}_4} = \zeta_4^* \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4}), \tag{4-43}$$

respectively, where $(i, j, k) \in \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$. Applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-12) and substituting (4-41), (4-42), and (4-43) into it, we obtain

$$\begin{aligned} \zeta_{(2,1,1)}^* \mid \Sigma_{\mathfrak{C}_4} = {} & \zeta_4^* \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4}) \\ & + \zeta_3^* \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} + (23)\Sigma_{\mathfrak{C}_4} - (14)\Sigma_{\mathfrak{C}_4}) \\ & - \zeta_3^* \circ (w_{(2,1,1)} \mid (34)\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid (1234)\Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (1324)\Sigma_{\mathfrak{C}_4}) \\ & + \zeta_2^* \circ (w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\Sigma_{\mathfrak{C}_4^0}) + w_{(3,1)} \mid (24)\Sigma_{\mathfrak{C}_4} + w_{(1,3)} \mid \Sigma_{\mathfrak{C}_4}). \end{aligned} \tag{4-44}$$

We see from the third equation of (4-19) and the equivalence classes modulo $\langle(12), (123)\rangle$ in Table 1 that

$$\begin{aligned} (24)\Sigma_{\mathfrak{C}_4} &= (13) + (24) + (12)(34) + (14)(23) \\ &\equiv e + (13)(24) + (1234) + (1432) = \Sigma_{\mathfrak{C}_4} \mod \langle(12), (123)\rangle, \end{aligned}$$

and so

$$\zeta_2^* \circ w_{(3,1)} \mid (24)\Sigma_{\mathfrak{C}_4} = \zeta_2^* \circ w_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}. \tag{4-45}$$

We also have

$$\zeta_3^* \circ w_{(1,2,1)} \mid (1234)\Sigma_{\mathfrak{C}_4} = \zeta_3^* \circ w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4}, \tag{4-46}$$

$$\zeta_3^* \circ w_{(1,1,2)} \mid (1324)\Sigma_{\mathfrak{C}_4} = \zeta_3^* \circ w_{(1,1,2)} \mid (12)\Sigma_{\mathfrak{C}_4}, \tag{4-47}$$

since $(1234)\Sigma_{\mathfrak{C}_4} = \Sigma_{\mathfrak{C}_4}$ and $(1324)\Sigma_{\mathfrak{C}_4} = (12)\Sigma_{\mathfrak{C}_4}$ by the first and second equations of (4-19), respectively. Combining (4-44), (4-45), (4-46), and (4-47), we obtain (4-33). □

4.2. ***Proof of Theorem 1.2.*** We denote by $\mathcal{P}(A)$ the set of partitions of a set A; if A is the empty set ϕ, we set $\mathcal{P}(\phi) = \{\phi\}$. We denote by $\mathbb{N}_n$ the subset $\{1, 2, \dots, n\}$ in the set $\mathbb{N}$.

For $\dagger \in \{*, ш\}$, $\mathbf{l}_n = (l_1, \dots, l_n) \in \mathbb{N}^n$, and $\Pi = \{P_1, \dots, P_m\} \in \mathcal{P}(\mathbb{N}_n)$, we define a polynomial $Z^\dagger_{\mathbf{l}_n;\Pi}(T)$ with real coefficients by

$$Z^\dagger_{\mathbf{l}_n;\Pi}(T) := \prod_{i=1}^{m} \chi^\dagger(\mathbf{l}_n;\, P_i) Z^\dagger_{l_{P_i}}(T), \tag{4-48}$$

where $l_{P_i} = \sum_{p \in P_i} l_p$. For example,

$$Z^\dagger_{(2,1,1);\{\{1,2,3\}\}}(T) = Z^\dagger_{2+1+1}(T) = \zeta_1(4)$$

and

$$Z^\dagger_{(2,1,1);\{\{1,2\},\{3\}\}}(T) = Z^\dagger_{2+1}(T) Z^\dagger_1(T) = \zeta_1(3)T,$$

where note that $\chi^\dagger((2,1,1);\{1,2,3\}) = \chi^\dagger((2,1,1);\{1,2\}) = \chi^\dagger((2,1,1);\{3\}) = 1$ by the definition (1-7). Since $Z^\dagger_k(T) = \zeta^\dagger_1(k)$ for $k \geq 2$, we can see from (1-6) and (4-48) that the difference between $Z^\dagger_{\mathbf{l}_n;\Pi}(T)$ and $\zeta^\dagger(\mathbf{l}_n;\Pi)$ depends on only the difference between $Z^\dagger_1(T) = T$ and $\zeta^\dagger_1(1) = 0$, and so

$$Z^\dagger_{\mathbf{l}_n;\Pi}(T)\,\Big|_{\,T=0} = \zeta^\dagger(\mathbf{l}_n;\Pi).$$

By the correspondence between $\mathfrak{H}^1$ and the algebra of quasisymmetric functions, which is given by

$$z_{l_1}\cdots z_{l_n} \longleftrightarrow M_{(l_1,\dots,l_n)} := \sum_{i_1<\cdots<i_n} t_{i_1}^{l_1}\cdots t_{i_n}^{l_n} \in \operatorname*{proj\,lim}_p \mathbb{Z}[t_1,\dots,t_p],$$

we can restate [Hoffman 2015, Theorem 2.3] as

$$\sum_{\sigma\in\mathfrak{S}_n} z_{l_{\sigma(1)}}\cdots z_{l_{\sigma(n)}} = \sum_{\Pi=\{P_1,\dots,P_m\}\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi)\, z_{l_{P_1}} * \cdots * z_{l_{P_m}}. \tag{4-49}$$

We see from (1-7) that $\chi^*(\mathbf{l}_n; P) = 1$ for any $\mathbf{l}_n \in \mathbb{N}^n$ and $P \subset \mathbb{N}_n$. Thus, applying Z^* to both sides of (4-49) yields the following identity (4-50). Since $\mathfrak{S}_n^{-1} = \mathfrak{S}_n$, (4-50) with $T = 0$ proves (1-8) for $\dagger = *$.

Theorem 4.4 (see [Hoffman 2015, Theorem 2.3]). *For any index set $\mathbf{l}_n = (l_1,\dots,l_n)$ in $\mathbb{N}^n$,*

$$\sum_{\sigma\in\mathfrak{S}_n} Z^*_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T) = \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi)\, Z^*_{\mathbf{l}_n;\Pi}(T). \tag{4-50}$$

We may show (4-51) in order to prove (1-8) for $\dagger = ш$, or Theorem 1.2.

Proposition 4.5. *For any index set $\mathbf{l}_n = (l_1,\dots,l_n)$ in $\mathbb{N}^n$,*

$$\rho\biggl(\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi)\, Z^*_{\mathbf{l}_n;\Pi}(T)\biggr) = \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi)\, Z^{ш}_{\mathbf{l}_n;\Pi}(T). \tag{4-51}$$

In fact, we can easily prove Theorem 1.2, as follows.

Proof of Theorem 1.2. We see from (2-8) that

$$\rho\biggl(\sum_{\sigma\in\mathfrak{S}_n} Z^*_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T)\biggr) = \sum_{\sigma\in\mathfrak{S}_n} Z^{ш}_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T). \tag{4-52}$$

By (4-51) and (4-52), applying ρ to both sides of (4-50) yields

$$\sum_{\sigma\in\mathfrak{S}_n} Z^{ш}_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T) = \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi)\, Z^{ш}_{\mathbf{l}_n;\Pi}(T) \tag{4-53}$$

which with $T = 0$ proves (1-8) for $\dagger = ш$. □

For subsets A and B in $\mathbb{N}_n$, we define a subset $\mathcal{P}_B(A)$ in $\mathcal{P}(A)$ by

$$\mathcal{P}_B(A) := \{\Pi = \{P_1, \dots, P_m\} \in \mathcal{P}(A) \mid P_i \not\subset B \text{ for all } i\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then

$$\mathcal{P}_B(A) = \big\{\{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{1, 2, 3\}\}\big\},$$

where

$$\mathcal{P}(A) = \big\{\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}, \{\{1, 2, 3\}\}\big\}.$$

We note that $\mathcal{P}_B(A) = \phi$ if $A = \phi$ (or $\mathcal{P}(A) = \{\phi\}$), because the empty set ϕ is a subset of any set, i.e., $\phi \subset B$. We denote by $A^{c_n} = A^c$ the complement of A in $\mathbb{N}_n$, and by $\sqcup$ the disjoint union.

We will show (4-51) for the completeness of the proof of Theorem 1.2. For this, we will require Lemmas 4.7, 4.8, and 4.9.

Remark 4.6. The condition $B \neq \mathbb{N}_n$ in Lemma 4.7 is necessary for taking an element in $\mathcal{P}_B(A^c)$. In fact, if $B = \mathbb{N}_n$, then $P \subset B$ for any subset P in A^c, and so $\mathcal{P}_B(A^c) = \phi$. That is, (4-54) in Lemma 4.7 does not hold in the case that $B = \mathbb{N}_n$.

Lemma 4.7. *For any subset $B \subset \mathbb{N}_n$ with $B \neq \mathbb{N}_n$, we have*

$$\bigsqcup_{A \subset B} \{\Xi \sqcup \Delta \mid (\Xi, \Delta) \in \mathcal{P}(A) \times \mathcal{P}_B(A^c)\} = \mathcal{P}(\mathbb{N}_n), \tag{4-54}$$

where the disjoint union $\bigsqcup_{A \subset B}$ ranges over all subsets in B, which include the empty set ϕ.

We require some notation to state Lemma 4.8. Let A be a subset in $\mathbb{N}_n$, and let $\Xi = \{P_1, \dots, P_g\}$ be a partition in $\mathcal{P}(A)$. We can define a partition in $\mathcal{P}(\mathbb{N}_s)$ that is induced from A and Ξ, as follows. Let $a_1 < \cdots < a_s$ be the increasing sequence of integers such that $A = \{a_1, \dots, a_s\}$. Let σ_A be the permutation of $\mathfrak{S}_n$ that is uniquely determined by

$$\sigma_A^{-1}(i) = a_i \ (i = 1, \dots, s) \quad \text{and} \quad \sigma_A^{-1}(s+1) < \cdots < \sigma_A^{-1}(n);$$

by the definition, $\sigma(A) = \{\sigma_A(a_1), \dots, \sigma_A(a_s)\} = \{1, \dots, s\} = \mathbb{N}_s$. We then define the partition induced from A and Ξ as

$$\sigma_A(\Xi) := \{\sigma_A(P_1), \dots, \sigma_A(P_g)\} \in \mathcal{P}(\mathbb{N}_s).$$

We define $\sigma_A(\Xi) = \phi$ if $A = \Xi = \phi$.

Lemma 4.8. *Let A, B, and (Ξ, Δ) be as in Lemma 4.7, i.e., let A and B be subsets with $A \subset B \neq \mathbb{N}_n$, and let (Ξ, Δ) be an element in $\mathcal{P}(A) \times \mathcal{P}_B(A^c)$. We define $|\phi| = 0$ and $\tilde{c}_0(\phi) = Z^{\dagger}_{\phi;\phi}(T) = 1$.*

(i) *We have*

$$\tilde{c}_n(\Xi \cup \Delta) = \tilde{c}_{|A|}(\Xi)\tilde{c}_{|A^c|}(\Delta). \tag{4-55}$$

(ii) *Let* $\dagger \in \{*, ш\}$, *and let* $\mathbf{l}_n = (l_1, \ldots, l_n) \in \mathbb{N}^n$ *with* $\mathbf{l}_n \neq (\{1\}^n)$. *If* $B = \{j \in \mathbb{N}_n \mid l_j = 1\}$, *then*

$$Z^{\dagger}_{\mathbf{l}_n;\,\Xi\cup\Delta}(T) = \left(\prod_{i=1}^{h} \zeta_1(l_{Q_i})\right) Z^{\dagger}_{\{1\}^{|A|};\sigma_A(\Xi)}(T), \tag{4-56}$$

where $Q_1, \ldots, Q_h$ *mean the parts of* Δ (*i.e.*, $\Delta = \{Q_1, \ldots, Q_h\}$).

Lemma 4.9. *For a positive integer* n, *we have*

$$\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{*}_{\{1\}^n;\Pi}(T) = \rho^{-1}(T^n), \tag{4-57}$$

$$\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T) = T^n. \tag{4-58}$$

We now prove Proposition 4.5. We will then discuss proofs of Lemmas 4.7–4.9.

Proof of Proposition 4.5. Let $B = \{j \in \mathbb{N}_n \mid l_j = 1\} \subset \mathbb{N}_n$. We suppose that $B = \mathbb{N}_n$. Then, $\mathbf{l}_n = (\{1\}^n)$, and so, we can see from Lemma 4.9 that

$$\rho\left(\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{*}_{\{1\}^n;\Pi}(T)\right) \overset{\text{(4-57)}}{=} \rho(\rho^{-1}(T^n)) = T^n \overset{\text{(4-58)}}{=} \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T), \tag{4-59}$$

which proves (4-51) for $B = \mathbb{N}_n$.

We suppose that $B \neq \mathbb{N}_n$. Let A be a subset in B. Then we have

$$\{\sigma_A(\Xi) \mid \Xi \in \mathcal{P}(A)\} = \{\Xi' \mid \Xi' \in \mathcal{P}(\mathbb{N}_{|A|})\}, \tag{4-60}$$

because the restriction of the permutation σ_A to the subset A is a bijection from A to $\mathbb{N}_{|A|}$. From the definition (1-5) we easily see that $\tilde{c}_{|A|}(\Xi) = \tilde{c}_{|A|}(\sigma_A(\Xi))$. Thus,

$$\begin{aligned}
&\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{*}_{\mathbf{l}_n;\Pi}(T) \\
&\overset{\text{(Lemma 4.7)}}{=} \sum_{A\subset B} \sum_{\substack{\Xi\in\mathcal{P}(A)\\ \Delta\in\mathcal{P}_B(A^c)}} \tilde{c}_n(\Xi\cup\Delta) Z^{*}_{\mathbf{l}_n;\,\Xi\cup\Delta}(T) \\
&\overset{\text{(Lemma 4.8)}}{=} \sum_{A\subset B} \sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\left(\prod_{i=1}^{h} \zeta_1(l_{Q_i})\right) \sum_{\Xi\in\mathcal{P}(A)} \tilde{c}_{|A|}(\Xi) Z^{*}_{\{1\}^{|A|};\sigma_A(\Xi)}(T)
\end{aligned}$$

$$\overset{(4\text{-}60)}{=} \sum_{A \subset B} \sum_{\Delta \in \mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta) \biggl(\prod_{i=1}^{h} \zeta_1(l_{Q_i}) \biggr) \sum_{\Xi' \in \mathcal{P}(\mathbb{N}_{|A|})} \tilde{c}_{|A|}(\Xi') Z^*_{\{1\}^{|A|};\,\Xi'}(T)$$

$$\overset{(4\text{-}57)}{=} \sum_{A \subset B} \sum_{\Delta \in \mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta) \biggl(\prod_{i=1}^{h} \zeta_1(l_{Q_i}) \biggr) \rho^{-1}(T^{|A|}),$$

where $Q_1, \dots, Q_h$ mean the parts of Δ. Therefore,

$$\rho\biggl(\sum_{\Pi \in \mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^*_{\mathbf{l}_n;\,\Pi}(T) \biggr) = \sum_{A \subset B} \sum_{\Delta \in \mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta) \biggl(\prod_{i=1}^{h} \zeta_1(l_{Q_i}) \biggr) (\rho^{-1}(T^{|A|})) = \sum_{A \subset B} \sum_{\Delta \in \mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta) \biggl(\prod_{i=1}^{h} \zeta_1(l_{Q_i}) \biggr) T^{|A|}. \tag{4-61}$$

By using Lemma 4.7, Lemma 4.8, and (4-60), and by using (4-58) instead of (4-57), we can similarly prove

$$\sum_{\Pi \in \mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{\text{ш}}_{\mathbf{l}_n;\,\Pi}(T) = \sum_{A \subset B} \sum_{\Delta \in \mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta) \biggl(\prod_{i=1}^{h} \zeta_1(l_{Q_i}) \biggr) T^{|A|}. \tag{4-62}$$

Equating (4-61) and (4-62), we obtain (4-51) for $B \neq \mathbb{N}_n$. □

We prove Lemmas 4.7 and 4.8.

Proof of Lemma 4.7. Let A be a subset in B, and let (Ξ, Δ) be an element in $\mathcal{P}(A) \times \mathcal{P}_B(A^c)$. It follows from $\mathcal{P}_B(A^c) \subset \mathcal{P}(A^c)$ that $(\Xi, \Delta) \in \mathcal{P}(A) \times \mathcal{P}(A^c)$, which together with $A \sqcup A^c = \mathbb{N}_n$ yields $\Xi \sqcup \Delta \in \mathcal{P}(\mathbb{N}_n)$. Thus, the left-hand side of (4-54) is included in the right-hand side.

Let $\Pi = \{P_1, \dots, P_m\}$ be a partition in $\mathcal{P}(\mathbb{N}_n)$. We can reorder $P_1, \dots, P_m$ such that

$$P_j \subset B \ (j = 1, \dots, g) \quad \text{and} \quad P_j \not\subset B \ (j = g+1, \dots, m). \tag{4-63}$$

We define

$$\Xi := \{P_1, \dots, P_g\}, \quad \Delta := \{P_{g+1}, \dots, P_m\} \quad \text{and} \quad A := P_1 \cup \dots \cup P_g,$$

where A and Ξ mean the empty set ϕ if $g = 0$. By definition, it is obvious that $A \subset B$, $\Pi = \Xi \sqcup \Delta$, $\Xi \in \mathcal{P}(A)$, and $\Delta \in \mathcal{P}(A^c)$. We assume that $\Delta \notin \mathcal{P}_B(A^c)$. Then, either $\mathcal{P}_B(A^c) = \phi$, or there is an integer i such that $g < i \leq m$ and $P_i \subset B$. We can see from (4-63) that the latter case does not occur, and so $\mathcal{P}_B(A^c) = \phi$. Thus, the simplest partition $\{A^c\}$ of A^c does not belong to $\mathcal{P}_B(A^c)$, which yields that $A^c \subset B$. Since $A \subset B$, we have $\mathbb{N}_n = A \cup A^c \subset B$, i.e., $B = \mathbb{N}_n$, which is a contradiction to the condition $B \neq \mathbb{N}_n$. Therefore, $\Delta \in \mathcal{P}_B(A^c)$, and we can conclude that

$$A \subset B, \quad \Pi = \Xi \cup \Delta \quad \text{and} \quad (\Xi, \Delta) \in \mathcal{P}(A) \times \mathcal{P}_B(A^c).$$

This fact proves that the right-hand side of (4-54) is included in the left-hand side, since Π is arbitrary.

We should show the disjointness of the left-hand side of (4-54) in order to finish the proof. Assume that there are subsets $A_1, A_2 \subset B$ with $A_1 \neq A_2$ such that

$$\phi \neq \{\Xi_1 \sqcup \Delta_1 \mid (\Xi_1, \Delta_1) \in \mathcal{P}(A_1) \times \mathcal{P}_B(A_1^c)\} \\ \cap \{\Xi_2 \sqcup \Delta_2 \mid (\Xi_2, \Delta_2) \in \mathcal{P}(A_2) \times \mathcal{P}_B(A_2^c)\}. \tag{4-64}$$

We can take elements $(\Xi_j, \Delta_j) \in \mathcal{P}(A_j) \times \mathcal{P}_B(A_j^c)$ $(j = 1, 2)$ such that

$$\Xi_1 \sqcup \Delta_1 = \Xi_2 \sqcup \Delta_2. \tag{4-65}$$

Let $P_1 \in \Xi_1$. We easily see that $P_1 \subset B$, since $\Xi_1 \in \mathcal{P}(A_1)$ and $A_1 \subset B$. By (4-65), there is a subset $P_2 \in \Xi_2 \sqcup \Delta_2$ such that $P_1 = P_2$. If $P_2 \in \Delta_2$, then $P_2 \not\subset B$, which contradicts $P_1 \subset B$. We thus have $P_1 = P_2 \in \Xi_2$, and so $\Xi_1 \subset \Xi_2$ since P_1 is arbitrary. Similarly, we can prove $\Xi_2 \subset \Xi_1$, and we conclude that $\Xi_1 = \Xi_2$. Since Ξ_j is a partition of A_j for each $j = 1, 2$, we can obtain $A_1 = A_2$, which contradicts the assumption $A_1 \neq A_2$. Therefore, there are no subsets $A_1, A_2 \subset B$ with $A_1 \neq A_2$ such as (4-64), which proves that the left-hand side of (4-54) satisfies the disjointness. $\square$

Proof of Lemma 4.8. Let $P_1, \dots, P_g$ be the parts of Ξ, and let $Q_1, \dots, Q_h$ be those of Δ. Since $n = |A| + |A^c|$ and $\Xi \cup \Delta = \{P_1, \dots, P_g, Q_1, \dots, Q_h\}$, we see from (1-5) that

$$\begin{aligned} \tilde{c}_n(\Xi \cup \Delta) &= (-1)^{n-(g+h)} \left(\prod_{i=1}^{g} (|P_i| - 1)! \right) \left(\prod_{i=1}^{h} (|Q_i| - 1)! \right) \\ &= (-1)^{|A|-g} \left(\prod_{i=1}^{g} (|P_i| - 1)! \right) (-1)^{|A^c|-h} \left(\prod_{i=1}^{h} (|Q_i| - 1)! \right) \\ &= \tilde{c}_{|A|}(\Xi) \tilde{c}_{|A^c|}(\Delta), \end{aligned}$$

which proves (4-55). We next prove (4-56). By $\Delta \in \mathcal{P}_B(A^c)$, any part Q in Δ satisfies $Q \not\subset B = \{j \in \mathbb{N}_n \mid l_j = 1\}$, which yields that $\chi^\dagger(\mathbf{l}_n; Q) = 1$ and

$Z^{\dagger}_{l_Q}(T)=\zeta_1(l_Q)$. Thus, we can see from (4-48) that

$$\text{(4-66)}\qquad Z^{\dagger}_{\mathbf{l}_n;\,\Xi\cup\Delta}(T)=\biggl(\prod_{i=1}^{g}\chi^{\dagger}(\mathbf{l}_n;\,P_i)Z^{\dagger}_{l_{P_i}}(T)\biggr)\biggl(\prod_{i=1}^{h}\chi^{\dagger}(\mathbf{l}_n;\,Q_i)Z^{\dagger}_{l_{Q_i}}(T)\biggr)$$
$$=\biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)\biggl(\prod_{i=1}^{g}\chi^{\dagger}(\mathbf{l}_n;\,P_i)Z^{\dagger}_{l_{P_i}}(T)\biggr).$$

For any part $P=\{p_1,\dots,p_a\}$ in Ξ, every index l_{p_q} is 1, since $A\subset B$ and $\Xi\in\mathcal{P}(A)$. By this fact, we obtain

$$\chi^{\dagger}(\mathbf{l}_n;\,P)=\chi^{\dagger}_a(l_{p_1},\dots,l_{p_a})=\chi^{\dagger}_a(\{1\}^a)=\chi^{\dagger}(\{1\}^{|A|};\,\sigma_A(P))$$

and

$$Z^{\dagger}_{l_P}(T)=Z^{\dagger}_{\sum_{p\in P}l_p}(T)=Z^{\dagger}_{|P|}(T)=Z^{\dagger}_{|\sigma_A(P)|}(T).$$

Therefore,

$$\text{(4-67)}\qquad \prod_{i=1}^{g}\chi^{\dagger}(\mathbf{l}_n;\,P_i)Z^{\dagger}_{l_{P_i}}(T)=\prod_{i=1}^{g}\chi^{\dagger}(\{1\}^{|A|};\,\sigma_A(P_i))Z^{\dagger}_{|\sigma_A(P_i)|}(T)$$
$$=Z^{\dagger}_{\{1\}^{|A|};\{\sigma_A(P_1),\dots,\sigma_A(P_g)\}}(T).$$

Combining (4-66) and (4-67) proves (4-56). □

From Theorems 7.12 and 7.13 in [Stanley 2013], we can obtain the following identity in formal power series:

$$\text{(4-68)}\quad \exp\biggl(u_1u+u_2\frac{u^2}{2}+u_3\frac{u^3}{3}+\cdots\biggr)$$
$$=1+\sum_{n=1}^{\infty}u^n\sum_{\substack{i_1,i_2,\dots,i_n\ge 0\\(1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=n)}}\frac{u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n}}{1^{i_1}i_1!\,2^{i_2}i_2!\,\cdots n^{i_n}i_n!},$$

where $u,u_1,u_2,\dots$ are variables. (We can also prove (4-68) by a direct calculation of the Taylor expansion of the exponential function e^x.)

We require the following identity (4-69) to prove Lemma 4.9.

Lemma 4.10. *We have*

$$\text{(4-69)}\qquad \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)}\tilde{c}_n(\Pi)Z^{*}_{\{1\}^n;\Pi}(T)=n!\widetilde{\gamma}_n(T),$$

where we define

$$\text{(4-70)}\qquad \widetilde{\gamma}_n(T):=(-1)^n\sum_{\substack{i_1,i_2,\dots,i_n\ge 0\\(1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=n)}}\frac{(-1)^{i_1+i_2+\cdots+i_n}}{i_1!\,i_2!\,\cdots i_n!}T^{i_1}\prod_{a=2}^{n}\biggl(\frac{\zeta(a)}{a}\biggr)^{i_a}.$$

Proof of Lemma 4.10. For a partition $\Pi = \{P_1, \dots, P_g\}$ in $\mathcal{P}(\mathbb{N}_n)$ and a positive integer a, we denote by $N_a(\Pi)$ the number of the parts P_j whose cardinalities equal a, i.e.,

$$N_a(\Pi) := |\{j \in \{1, \dots, g\} \mid |P_j| = a\}|.$$

For example, $N_1(\Pi) = 2$, $N_2(\Pi) = 1$ and $N_3(\Pi) = N_4(\Pi) = 0$ if

$$\Pi = \{\{1\}, \{2\}, \{3, 4\}\} \in \mathcal{P}(\mathbb{N}_4).$$

We note that

$$g = N_1(\Pi) + \cdots + N_n(\Pi) \quad \text{and} \quad n = 1 \cdot N_1(\Pi) + \cdots + n \cdot N_n(\Pi)$$

and that

$$\prod_{i=1}^{g} (|P_i| - 1)! Z^*_{|P_i|}(T) = \prod_{a=1}^{n} ((a-1)! Z^*_a(T))^{N_a(\Pi)}.$$

Noting $\chi^*(\{1\}^n; P_i) = 1$, we obtain from (1-5) and (4-48) that

$$\begin{aligned}\tilde{c}_n(\Pi) Z^*_{\{1\}^n;\Pi}(T) &= (-1)^{n-g} \prod_{i=1}^{g} (|P_i| - 1)! Z^*_{|P_i|}(T) \\ &= (-1)^{n-(N_1(\Pi)+\cdots+N_n(\Pi))} \prod_{a=1}^{n} ((a-1)! Z^*_a(T))^{N_a(\Pi)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{\substack{i_1,\dots,i_n \ge 0 \\ (1\cdot i_1+\cdots+n\cdot i_n = n)}} \sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi) = i_a (\forall a))}} \tilde{c}_n(\Pi) Z^*_{\{1\}^n;\Pi}(T) \\ &\quad = \sum_{\substack{i_1,\dots,i_n \ge 0 \\ (1\cdot i_1+\cdots+n\cdot i_n = n)}} (-1)^{n-(i_1+\cdots+i_n)} \left(\prod_{a=1}^{n} ((a-1)! Z^*_a(T))^{i_a} \right) \sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi) = i_a (\forall a))}} 1.\end{aligned} \tag{4-71}$$

Let m be an integer with $a i_a < m$. We can choose i_a disjoint subsets

$$Q_1, \dots, Q_{i_a} \subset \mathbb{N}_m,$$

with $|Q_1| = \cdots = |Q_{i_a}| = a$ in

$$\binom{m}{a i_a} \cdot \binom{a i_a}{a} \binom{a i_a - a}{a} \cdots \binom{a}{a} \cdot \frac{1}{i_a!}$$

ways, as follows. First, we choose $a i_a$ integers $N = \{k_1, \dots, k_{a i_a}\}$ from $\mathbb{N}_m$ in $\binom{m}{a i_a}$ ways. Then we select a integers Q_1 from N, select a integers Q_2 from $N \setminus Q_1$, and so on: these combinations are

$$\binom{ai_a}{a}\binom{ai_a - a}{a}\cdots\binom{a}{a}.$$

Finally, we divide it by $i_a!$ to ignore the order of $Q_1, \dots, Q_{i_a}$, and we reach the desired result. Any partition in

$$\{\Pi \in \mathcal{P}(\mathbb{N}_n) \mid N_a(\Pi) = i_a \text{ (for all } a)\}$$

can be uniquely obtained by choosing i_1 disjoint subsets

$$Q_1^{(1)}, \dots, Q_{i_1}^{(1)} \quad \text{with } |Q_j^{(1)}| = 1 \text{ (for all } j)$$

from the set $\mathbb{N}_n$, choosing i_2 disjoint subsets

$$Q_1^{(2)}, \dots, Q_{i_2}^{(2)} \quad \text{with } |Q_j^{(2)}| = 2 \text{ (for all } j)$$

from the set $\mathbb{N}_n \setminus (Q_1^{(1)} \cup \cdots \cup Q_{i_1}^{(1)})$, and repeating it. Thus,

$$\begin{aligned}
&|\{\Pi \in \mathcal{P}(\mathbb{N}_n) \mid N_a(\Pi) = i_a \text{ (for all } a)\}| \\
&\quad = \prod_{a=1}^{n} \binom{n - 1\cdot i_1 - \cdots - (a-1)\cdot i_{a-1}}{ai_a} \cdot \binom{ai_a}{a}\binom{ai_a - a}{a}\cdots\binom{a}{a} \cdot \frac{1}{i_a!} \\
&\quad = \prod_{a=1}^{n} \binom{n - 1\cdot i_1 - \cdots - (a-1)\cdot i_{a-1}}{ai_a} (ai_a)! \frac{1}{(a!)^{i_a} i_a!} \\
&\quad = \prod_{a=1}^{n} \frac{(n - 1\cdot i_1 - \cdots - (a-1)\cdot i_{a-1})!}{(n - 1\cdot i_1 - \cdots - a\cdot i_a)!} \frac{1}{(a!)^{i_a} i_a!} \\
&\quad = n! \prod_{a=1}^{n} \frac{1}{(a!)^{i_a} i_a!},
\end{aligned}$$

which is equivalent to

$$\sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi) = i_a (\forall a))}} 1 = n! \prod_{a=1}^{n} \frac{1}{(a!)^{i_a} i_a!}.$$

Therefore, (4-71) can be rewritten as

$$\begin{aligned}
\text{(4-72)} \quad & \sum_{\substack{i_1,\dots,i_n \ge 0 \\ (1\cdot i_1 + \cdots + n\cdot i_n = n)}} \sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi) = i_a (\forall a))}} \tilde{c}_n(\Pi) Z^*_{\{1\}^n;\Pi}(T) \\
&\qquad = n!(-1)^n \sum_{\substack{i_1,\dots,i_n \ge 0 \\ (1\cdot i_1 + \cdots + n\cdot i_n = n)}} (-1)^{i_1 + \cdots + i_n} \prod_{a=1}^{n} \frac{1}{i_a!} \left(\frac{Z^*_a(T)}{a}\right)^{i_a}.
\end{aligned}$$

By (3-80), $Z_a^*(T)$ equals T if $a=1$ and $\zeta_1(a)$ if $a>1$, and so combining (4-70) and (4-72) yields

$$\sum_{\substack{i_1,\dots,i_n\geq 0\\(1\cdot i_1+\cdots+n\cdot i_n=n)}}\ \sum_{\substack{\Pi\in\mathcal{P}(\mathbb{N}_n)\\(N_a(\Pi)=i_a(\forall a))}} \tilde{c}_n(\Pi)Z^*_{\{1\}^n;\Pi}(T)=n!\widetilde{\gamma}_n(T). \tag{4-73}$$

It is obvious that $\mathcal{P}(\mathbb{N}_n)$ can be divided into disjoint subsets as follows:

$$\mathcal{P}(\mathbb{N}_n)=\bigsqcup_{\substack{i_1,\dots,i_n\geq 0\\(1\cdot i_1+\cdots+n\cdot i_n=n)}}\{\Pi\in\mathcal{P}(\mathbb{N}_n)\mid N_a(\Pi)=i_a\ (a=1,\dots,n)\}. \tag{4-74}$$

Thus, the left-hand sides of (4-69) and (4-73) are equal, and we obtain (4-69). □

We now prove Lemma 4.9.

Proof of Lemma 4.9. By (3-75) we have

$$\begin{aligned}A(u)^{-1}e^{Tu}&=\exp\biggl(-\sum_{m=2}^{\infty}\frac{(-1)^m\zeta_1(m)}{m}u^m\biggr)e^{Tu}\\&=\exp\biggl((-1)^2Tu+\sum_{m=2}^{\infty}\frac{(-1)^{m+1}\zeta_1(m)}{m}u^m\biggr),\end{aligned}$$

which, together with (4-68) for $u_1=(-1)^2T$ and $u_m=(-1)^{m+1}\zeta_1(m)$ $(m\geq 2)$, yields

$$A(u)^{-1}e^{Tu}=1+\sum_{n=1}^{\infty}u^n(-1)^n\sum_{\substack{i_1,i_2,\dots,i_n\geq 0\\(1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=n)}}\frac{(-1)^{i_1+i_2+\cdots+i_n}}{i_1!\,i_2!\,\cdots i_n!}\,T^{i_1}\prod_{a=2}^{n}\biggl(\frac{\zeta(a)}{a}\biggr)^{i_a}.$$

Thus, by Lemma 4.10,

$$A(u)^{-1}e^{Tu}=1+\sum_{n=1}^{\infty}\frac{u^n}{n!}\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)}\tilde{c}_n(\Pi)Z^*_{\{1\}^n;\Pi}(T) \tag{4-75}$$

Since the renormalization map ρ is an $\mathbb{R}$-linear map from $\mathbb{R}[T]$ to $\mathbb{R}[T]$, we can see from (2-6) that the inverse ρ^{-1} is determined by

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}\rho^{-1}(T^n)=\rho^{-1}\left(e^{Tu}\right)\ =\ \rho^{-1}\left(A(u)^{-1}\rho(e^{Tu})\right)\ =\ A(u)^{-1}e^{Tu}. \tag{4-76}$$

Equating (4-75) and (4-76), and comparing the coefficients of u^n $(n\geq 1)$, we obtain (4-57).

Let $i_1,\dots,i_n$ be nonnegative integers with $1\cdot i_1+\cdots+n\cdot i_n=n$, and let $\Pi=\{P_1,\dots,P_g\}$ be a partition in $\mathcal{P}(\mathbb{N}_n)$ with $g=i_1+\cdots+i_n$ and $N_a(\Pi)=i_a$

$(a \in \mathbb{N}_n)$. Noting (1-7) for $\dagger = ш$, we can obtain

$$\tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T) = (-1)^{n-g} \prod_{i=1}^{g} (|P_i| - 1)!\chi^{ш}(\{1\}^n;\, P_i) Z^{ш}_{|P_i|}(T)$$

$$= \begin{cases} (-1)^{n-g} \prod_{i=1}^{g} T & (|P_i| = 1 \text{ for all } i\ ), \\ 0 & (\exists\, i \ \text{ such that } \ |P_i| > 1\ ), \end{cases}$$

and so

$$\tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T) = \begin{cases} T^n & \text{if } \Pi = \big\{\underbrace{\{1\}, \ldots, \{1\}}_{n}\big\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by (4-74),

$$\sum_{\Pi \in \mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T) = \sum_{\substack{i_1,\ldots,i_n \geq 0 \\ (1\cdot i_1 + \cdots + n\cdot i_n = n)}} \ \sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi) = i_a (\forall a))}} \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T)$$

$$= \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T)\,\Big|_{\,\Pi = \big\{\underbrace{\{1\}, \ldots, \{1\}}_{n}\big\}}$$

$$= T^n,$$

which proves (4-58). □

4.3. ***Proof of Corollary 1.3.*** Let $\mathcal{P}_n$ be the set of partitions of $\{1, \ldots, n\}$, i.e., $\mathcal{P}_n = \mathcal{P}(\mathbb{N}_n)$. Let $\mathcal{P}_{n;m}$ be the subset of $\mathcal{P}_n$ which consists of partitions $\Pi = \{P_1, \ldots, P_m\}$ such that the number of the parts is m. Note that $\mathcal{P}_n = \bigsqcup_{j=1}^{n} \mathcal{P}_{n;\,j}$, In what follows, we identify a partition

$$\Pi = \{\{n_1^{(1)}, \ldots, n_{a_1}^{(1)}\}, \ldots, \{n_1^{(m)}, \ldots, n_{a_m}^{(m)}\}\}$$

with

$$n_1^{(1)} \ldots n_{a_1}^{(1)} | \ldots | n_1^{(m)} \ldots n_{a_m}^{(m)}.$$

For example,

$$\{\{1, 2, 3\}\} = 123, \quad \{\{1, 2\}, \{3\}\} = 12|3, \quad \text{and } \{\{1\}, \{2\}, \{3\}\} = 1|2|3.$$

Let n and n' be positive integers with $n < n'$. For convenience, we embed $\mathfrak{S}_n$ into $\mathfrak{S}_{n'}$ in the following way: a permutation

$$\begin{pmatrix} 1 & \ldots & n \\ j_1 & \ldots & j_n \end{pmatrix} \quad \text{of } \mathfrak{S}_n$$

is identified with the permutation

$$\begin{pmatrix} 1 & \ldots & n & n+1 & \ldots & n' \\ j_1 & \ldots & j_n & n+1 & \ldots & n' \end{pmatrix} \quad \text{of } \mathfrak{S}_{n'},$$

which fixes integers between $n+1$ and n'.

To prove Corollary 1.3, we require the following three lemmas, which state that certain sums of values $\zeta^{\dagger}_{(n_1,\ldots,n_j)}(\mathbf{l}_n)$ can be written in terms of values $\zeta^{\dagger}(\mathbf{l}_n;\Pi)$, for depths 2, 3, and 4. We assume that $\mathbf{l}_n=(l_1,\ldots,l_n)\in\mathbb{N}^n$ and $\dagger\in\{*,ш\}$ in the lemmas.

Lemma 4.11 (case of depth 2).

$$\zeta^{\dagger}_{(1,1)}(\mathbf{l}_2)=\sum_{\Pi\in\mathcal{P}_{2;2}}\zeta^{\dagger}(\mathbf{l}_2;\Pi), \tag{4-77}$$

$$(\chi^{\dagger}_2\cdot\zeta_1\circ w_2)(\mathbf{l}_2)=\sum_{\Pi\in\mathcal{P}_{2;1}}\zeta^{\dagger}(\mathbf{l}_2;\Pi). \tag{4-78}$$

Lemma 4.12 (case of depth 3).

$$(\zeta^{\dagger}_{(1,1,1)}\mid\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3)=2\sum_{\Pi\in\mathcal{P}_{3;3}}\zeta^{\dagger}(\mathbf{l}_3;\Pi), \tag{4-79}$$

$$(\zeta^{\dagger}_{(2,1)}\mid\Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3)=3\sum_{\Pi\in\mathcal{P}_{3;3}}\zeta^{\dagger}(\mathbf{l}_3;\Pi)-\sum_{\Pi\in\mathcal{P}_{3;2}}\zeta^{\dagger}(\mathbf{l}_3;\Pi), \tag{4-80}$$

$$(\chi^{\dagger}_3\cdot\zeta_1\circ w_3\mid\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3)=2\sum_{\Pi\in\mathcal{P}_{3;1}}\zeta^{\dagger}(\mathbf{l}_3;\Pi). \tag{4-81}$$

Lemma 4.13 (case of depth 4).

$$(\zeta^{\dagger}_{(1,1,1,1)}\mid\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=6\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi), \tag{4-82}$$

$$(\zeta^{\dagger}_{(2,1,1)}\mid\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=12\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)-2\sum_{\Pi\in\mathcal{P}_{4;3}}\zeta^{\dagger}(\mathbf{l}_4;\Pi), \tag{4-83}$$

$$(\zeta^{\dagger}_{(2,2)}\mid\Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=3\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)-\sum_{\Pi\in\mathcal{P}_{4;3}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)+\sum_{\Pi\in\mathcal{P}^{(2,2)}_{4;2}}\zeta^{\dagger}(\mathbf{l}_4;\Pi), \tag{4-84}$$

$$(\zeta^{\dagger}_{(3,1)}\mid\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=4\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)-2\sum_{\Pi\in\mathcal{P}_{4;3}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)+2\sum_{\Pi\in\mathcal{P}^{(3,1)}_{4;2}}\zeta^{\dagger}(\mathbf{l}_4;\Pi), \tag{4-85}$$

$$(\chi^{\dagger}_4\cdot\zeta_1\circ w_4\mid\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=6\sum_{\Pi\in\mathcal{P}_{4;1}}\zeta^{\dagger}(\mathbf{l}_4;\Pi), \tag{4-86}$$

where $\mathcal{P}^{(2,2)}_{4;2}$ *and* $\mathcal{P}^{(3,1)}_{4;2}$ *in* (4-84) *and* (4-85) *are subsets in* $\mathcal{P}_{4;2}$ *defined by*

$$\mathcal{P}^{(2,2)}_{4;2}:=\{12|34,\ 13|24,\ 14|23\}\quad\textit{and}\quad\mathcal{P}^{(3,1)}_{4;2}:=\{123|4,\ 124|3,\ 134|2,\ 234|1\},$$

respectively. Note that $\mathcal{P}_{4;2}=\mathcal{P}^{(2,2)}_{4;2}\cup\mathcal{P}^{(3,1)}_{4;2}$.

We will prove Corollary 1.3 before discussing proofs of Lemmas 4.11, 4.12, and 4.13. We will divide the proof of Corollary 1.3 into three for the cases of $n = 2$, 3, and 4.

Proof of Corollary 1.3 for $n = 2$. Substituting (4-77) and (4-78) into the right-hand side of (1-2) yields

$$(\zeta_2^\dagger \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_2) = \sum_{\Pi \in \mathcal{P}_{2;2}} \zeta^\dagger(\mathbf{l}_2;\, \Pi) - \sum_{\Pi \in \mathcal{P}_{2;1}} \zeta^\dagger(\mathbf{l}_2;\, \Pi),$$

since $\mathfrak{C}_2 = \mathfrak{S}_2$ and

$$\chi_2^\dagger(\mathbf{l}_2)\zeta_1(L_2) = \chi_2^\dagger(\mathbf{l}_2)\zeta_1(l_1 + l_2) = \chi_2^\dagger(\mathbf{l}_2) \cdot \zeta_1 \circ w_2(\mathbf{l}_2) = (\chi_2^\dagger \cdot \zeta_1 \circ w_2)(\mathbf{l}_2).$$

We have by definition (see (1-5))

$$\tilde{c}_2(\Pi) = \begin{cases} 1 & (\Pi \in \mathcal{P}_{2;2}), \\ -1 & (\Pi \in \mathcal{P}_{2;1}), \end{cases}$$

and thus we obtain by $\mathcal{P}_2 = \bigcup_{m=1}^{2} \mathcal{P}_{2;m}$

$$(\zeta_2^\dagger \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_2) = \sum_{\Pi \in \mathcal{P}_2} \tilde{c}_2(\Pi)\zeta^\dagger(\mathbf{l}_2;\, \Pi),$$

which proves (1-8) for $n = 2$. □

Proof of Corollary 1.3 for $n = 3$. Applying $\Sigma_{\mathfrak{S}_2}$ to both sides of (1-3), we obtain

$$\begin{aligned} &(\zeta_3^\dagger \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_3) \\ &\quad = -(\zeta_{(1,1,1)}^\dagger \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) + (\zeta_{(2,1)}^\dagger \mid \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) + (\chi_3^\dagger \cdot \zeta_1 \circ w_3 \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3), \end{aligned} \tag{4-87}$$

where we have used $\Sigma_{\mathfrak{S}_3} = \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2}$ on the left-hand side of (4-87). Substituting (4-79), (4-80), and (4-81) into the right-hand side of (4-87) yields

$$(\zeta_3^\dagger \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_3) = \sum_{\Pi \in \mathcal{P}_{3;3}} \zeta^\dagger(\mathbf{l}_3;\, \Pi) - \sum_{\Pi \in \mathcal{P}_{3;2}} \zeta^\dagger(\mathbf{l}_3;\, \Pi) + 2 \sum_{\Pi \in \mathcal{P}_{3;1}} \zeta^\dagger(\mathbf{l}_3;\, \Pi),$$

which proves (1-8) for $n = 3$, since

$$\tilde{c}_3(\Pi) = \begin{cases} 1 & (\Pi \in \mathcal{P}_{3;3}), \\ -1 & (\Pi \in \mathcal{P}_{3;2}), \\ 2 & (\Pi \in \mathcal{P}_{3;1}), \end{cases}$$

and $\mathcal{P}_3 = \bigcup_{m=1}^{3} \mathcal{P}_{3;m}$. □

Proof of Corollary 1.3 for $n = 4$. We can see from the fourth and sixth equations in (4-19) that $(14)\Sigma_{\mathfrak{C}_4} = (123)\Sigma_{\mathfrak{C}_4}$ and $(34)\Sigma_{\mathfrak{C}_4} = (132)\Sigma_{\mathfrak{C}_4}$, respectively, and so it follows from (4-20) that $\Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{S}_3}\Sigma_{\mathfrak{C}_4}$. Taking the inverses of both sides of this equation gives $\Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}$. Thus, applying $\Sigma_{\mathfrak{S}_3}$ to both sides of (1-4) and combining the identities in Lemma 4.13 (or considering (4-82) − (4-83) + (4-84) + (4-85) − (4-86)), we can obtain

$$\begin{aligned}
&(\zeta_4^\dagger \mid \Sigma_{\mathfrak{S}_4})(\mathbf{l}_4) \\
&= \sum_{\Pi\in\mathcal{P}_{4;4}} \zeta^\dagger(\mathbf{l}_4;\Pi) - \sum_{\Pi\in\mathcal{P}_{4;3}} \zeta^\dagger(\mathbf{l}_4;\Pi) + \sum_{\Pi\in\mathcal{P}_{4;2}^{(2,2)}} \zeta^\dagger(\mathbf{l}_4;\Pi) + 2\sum_{\Pi\in\mathcal{P}_{4;2}^{(3,1)}} \zeta^\dagger(\mathbf{l}_4;\Pi) \\
&\qquad -6\sum_{\Pi\in\mathcal{P}_{4;1}} \zeta^\dagger(\mathbf{l}_4;\Pi),
\end{aligned}$$

which proves (1-8) for $n = 4$, since

$$\tilde{c}_4(\Pi) = \begin{cases} 1 & (\Pi\in\mathcal{P}_{4;4}), \\ -1 & (\Pi\in\mathcal{P}_{4;3}), \\ 1 & (\Pi\in\mathcal{P}_{4;2}^{(2,2)}), \\ 2 & (\Pi\in\mathcal{P}_{4;2}^{(3,1)}), \\ -6 & (\Pi\in\mathcal{P}_{4;1}), \end{cases}$$

$\mathcal{P}_{4;2} = \mathcal{P}_{4;2}^{(2,2)} \cup \mathcal{P}_{4;2}^{(3,1)}$ and $\mathcal{P}_4 = \bigcup_{m=1}^4 \mathcal{P}_{4;m}$. □

We see from (1-1) that $\chi_1^\dagger(k) = 1$ for any positive integer k, and so

$$\chi_1^\dagger(k)\zeta_1^\dagger(k) = \zeta_1^\dagger(k).$$

Note that $\zeta_1(k) = \zeta_1^\dagger(k)$ for $k \geq 2$. These facts will be used repeatedly below.

We now give proofs of Lemmas 4.11, 4.12, and 4.13.

Proof of Lemma 4.11. We have $\mathcal{P}_{2;2} = \{1|2\}$ and $\mathcal{P}_{2;1} = \{12\}$ by definition. Thus,

$$\sum_{\Pi\in\mathcal{P}_{2;2}} \zeta^\dagger(\mathbf{l}_2;\Pi) = \chi_1^\dagger(l_1)\zeta_1^\dagger(l_1)\chi_1^\dagger(l_2)\zeta_1^\dagger(l_2) = \zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2) = \zeta_{(1,1)}^\dagger(\mathbf{l}_2),$$

$$\sum_{\Pi\in\mathcal{P}_{2;1}} \zeta^\dagger(\mathbf{l}_2;\Pi) = \chi_2^\dagger(l_1,l_2)\zeta_1^\dagger(l_1+l_2) = \chi_2^\dagger(l_1,l_2)\zeta_1(l_1+l_2) = (\chi_2^\dagger\cdot\zeta_1\circ w_2)(\mathbf{l}_2),$$

which prove (4-77) and (4-78), respectively. □

Proof of Lemma 4.12. We have

$$\mathcal{P}_{3;3} = \{1|2|3\}, \quad \mathcal{P}_{3;2} = \{12|3,\ 13|2,\ 23|1\}, \quad \mathcal{P}_{3;1} = \{123\},$$

by definition. In particular, $\mathcal{P}_{3;2}$ is expressed as $\bigcup_{\sigma\in\mathfrak{C}_3}\{\sigma^{-1}(1)\sigma^{-1}(2)|\sigma^{-1}(3)\}$, and so

$$\sum_{\Pi\in\mathcal{P}_{3;2}} \zeta^\dagger(\mathbf{l}_3;\Pi) = \sum_{\sigma\in\mathfrak{C}_3} \chi_2^\dagger(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)})\zeta_1^\dagger(l_{\sigma^{-1}(1)}+l_{\sigma^{-1}(2)})\chi_1^\dagger(l_{\sigma^{-1}(3)})\zeta_1^\dagger(l_{\sigma^{-1}(3)})$$

$$= \sum_{\sigma \in \mathfrak{C}_3} \chi_2^{\dagger}(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}) \zeta_1(l_{\sigma^{-1}(1)} + l_{\sigma^{-1}(2)}) \zeta_1^{\dagger}(l_{\sigma^{-1}(3)}).$$

Thus, we can obtain

$$\sum_{\Pi \in \mathcal{P}_{3;3}} \zeta^{\dagger}(\mathbf{l}_3; \Pi) = \zeta^{\dagger}_{(1,1,1)}(\mathbf{l}_3), \tag{4-88}$$

$$\sum_{\Pi \in \mathcal{P}_{3;2}} \zeta^{\dagger}(\mathbf{l}_3; \Pi) = ((\chi_2^{\dagger} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{\dagger} \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3), \tag{4-89}$$

$$\sum_{\Pi \in \mathcal{P}_{3;1}} \zeta^{\dagger}(\mathbf{l}_3; \Pi) = (\chi_3^{\dagger} \cdot \zeta_1 \circ w_3)(\mathbf{l}_3). \tag{4-90}$$

Since $\zeta^{\dagger}_{(1,1,1)}$ is invariant under $\mathfrak{S}_3$, we have $(\zeta^{\dagger}_{(1,1,1)} \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) = 2\zeta^{\dagger}_{(1,1,1)}(\mathbf{l}_3)$, which together with (4-88) proves (4-79). Similarly, we have $(\chi_3^{\dagger} \cdot \zeta_1 \circ w_3 \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) = 2\chi_3^{\dagger} \cdot \zeta_1 \circ w_3(\mathbf{l}_3)$, which together with (4-90) proves (4-81). We know from (1-2) that

$$\zeta_2^{\dagger} \mid \Sigma_{\mathfrak{S}_2} = \zeta^{\dagger}_{(1,1)} - \chi_2^{\dagger} \cdot \zeta_1 \circ w_2, \tag{4-91}$$

and so

$$\begin{aligned} \zeta^{\dagger}_{(2,1)} \mid \Sigma_{\mathfrak{S}_2} &= \zeta_2^{\dagger} \otimes \zeta_1^{\dagger} \mid \Sigma_{\mathfrak{S}_2} \\ &= (\zeta_2^{\dagger} \mid \Sigma_{\mathfrak{S}_2}) \otimes \zeta_1^{\dagger} \\ &= (\zeta^{\dagger}_{(1,1)} - \chi_2^{\dagger} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{\dagger} \\ &= \zeta^{\dagger}_{(1,1,1)} - (\chi_2^{\dagger} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{\dagger}. \end{aligned}$$

Since $\mathfrak{C}_3\mathfrak{S}_2 = \mathfrak{S}_2\mathfrak{C}_3$, we have

$$\begin{aligned} \zeta^{\dagger}_{(2,1)} \mid \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2} &= (\zeta^{\dagger}_{(2,1)} \mid \Sigma_{\mathfrak{S}_2}) | \Sigma_{\mathfrak{C}_3} \\ &= (\zeta^{\dagger}_{(1,1,1)} - (\chi_2^{\dagger} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{\dagger}) \mid \Sigma_{\mathfrak{C}_3} \\ &= 3\zeta^{\dagger}_{(1,1,1)} - (\chi_2^{\dagger} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{\dagger} \mid \Sigma_{\mathfrak{C}_3}, \end{aligned}$$

which together with (4-88) and (4-89) proves (4-80). This completes the proof. □

Proof of Lemma 4.13. Let $\mathfrak{A}_4^0$ be the subset of $\mathfrak{A}_4$ given by

$$\mathfrak{A}_4^0 = \{e, (13)(24), (123), (132), (142), (234)\} = \langle (13)(24) \rangle \mathfrak{C}_3.$$

Note that $\Sigma_{\mathfrak{A}_4} = \Sigma_{\langle (12)(34) \rangle} \Sigma_{\mathfrak{A}_4^0}$. From the definitions of $\mathcal{P}_{4;m}$ and $\mathcal{P}_{4;2}^{(i,j)}$ and some straightforward calculations, we can see that

$$\mathcal{P}_{4;4} = \{1|2|3|4\},$$

$$\mathcal{P}_{4;3} = \bigcup_{\sigma\in\mathfrak{A}_4^0} \{\sigma^{-1}(1)\sigma^{-1}(2)|\sigma^{-1}(3)|\sigma^{-1}(4)\},$$

$$\mathcal{P}_{4;2}^{(2,2)} = \bigcup_{\sigma\in\mathfrak{C}_3} \{\sigma^{-1}(1)\sigma^{-1}(2)|\sigma^{-1}(3)\sigma^{-1}(4)\},$$

$$\mathcal{P}_{4;2}^{(3,1)} = \bigcup_{\sigma\in\mathfrak{C}_4} \{\sigma^{-1}(1)\sigma^{-1}(2)\sigma^{-1}(3)|\sigma^{-1}(4)\},$$

$$\mathcal{P}_{4;1} = \{1234\}.$$

Thus, we can obtain

$$\sum_{\Pi\in\mathcal{P}_{4;4}} \zeta^\dagger(\mathbf{l}_4;\Pi) = \zeta^\dagger_{(1,1,1,1)}(\mathbf{l}_4), \tag{4-92}$$

$$\sum_{\Pi\in\mathcal{P}_{4;3}} \zeta^\dagger(\mathbf{l}_4;\Pi) = ((\chi_2^\dagger\cdot\zeta_1\circ w_2)\otimes\zeta^\dagger_{(1,1)}\mid\Sigma_{\mathfrak{A}_4^0})(\mathbf{l}_4), \tag{4-93}$$

$$\sum_{\Pi\in\mathcal{P}_{4;2}^{(2,2)}} \zeta^\dagger(\mathbf{l}_4;\Pi) = ((\chi_2^\dagger\cdot\zeta_1\circ w_2)^{\otimes 2}\mid\Sigma_{\mathfrak{C}_3})(\mathbf{l}_4), \tag{4-94}$$

$$\sum_{\Pi\in\mathcal{P}_{4;2}^{(3,1)}} \zeta^\dagger(\mathbf{l}_4;\Pi) = ((\chi_3^\dagger\cdot\zeta_1\circ w_3)\otimes\zeta^\dagger_1\mid\Sigma_{\mathfrak{C}_4})(\mathbf{l}_4), \tag{4-95}$$

$$\sum_{\Pi\in\mathcal{P}_{4;1}} \zeta^\dagger(\mathbf{l}_4;\Pi) = (\chi_4^\dagger\cdot\zeta_1\circ w_4)(\mathbf{l}_4). \tag{4-96}$$

Since $\zeta^\dagger_{(1,1,1,1)}$ and $\chi_4^\dagger\cdot\zeta_1\circ w_4$ are invariant under $\mathfrak{S}_4$, we have

$$(\zeta^\dagger_{(1,1,1,1)}\mid\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=6\zeta^\dagger_{(1,1,1,1)}(\mathbf{l}_4) \quad\text{and}\quad (\chi_4^\dagger\cdot\zeta_1\circ w_4\mid\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=6\chi_4^\dagger\cdot\zeta_1\circ w_4(\mathbf{l}_4),$$

which together with (4-92) and (4-96) prove (4-82) and (4-86), respectively.

We now prove (4-83). A direct calculation shows that $\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} = \Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{S}_2}\Sigma_{\mathfrak{A}_4}$, and so, by (4-91),

$$\begin{aligned}\zeta^\dagger_{(2,1,1)}\mid\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} &= (\zeta^\dagger_{(2,1,1)}\mid\Sigma_{\mathfrak{S}_2})|\Sigma_{\mathfrak{A}_4}\\ &= (\zeta^\dagger_2\mid\Sigma_{\mathfrak{S}_2})\otimes\zeta^\dagger_{(1,1)}|\Sigma_{\mathfrak{A}_4}\\ &= (\zeta^\dagger_{(1,1,1,1)} - (\chi_2^\dagger\cdot\zeta_1\circ w_2)\otimes\zeta^\dagger_{(1,1)})\mid\Sigma_{\mathfrak{A}_4}\\ &= 12\zeta^\dagger_{(1,1,1,1)} - (\chi_2^\dagger\cdot\zeta_1\circ w_2)\otimes\zeta^\dagger_{(1,1)}\mid\Sigma_{\mathfrak{A}_4}.\end{aligned} \tag{4-97}$$

Since $\Sigma_{\mathfrak{A}_4} = \Sigma_{\langle(12)(34)\rangle}\Sigma_{\mathfrak{A}_4^0}$ and $(\chi_2^\dagger\cdot\zeta_1\circ w_2)\otimes\zeta^\dagger_{(1,1)}$ is invariant under $\langle(12)(34)\rangle$,

$$(\chi_2^\dagger\cdot\zeta_1\circ w_2)\otimes\zeta^\dagger_{(1,1)}\mid\Sigma_{\mathfrak{A}_4} = 2(\chi_2^\dagger\cdot\zeta_1\circ w_2)\otimes\zeta^\dagger_{(1,1)}\mid\Sigma_{\mathfrak{A}_4^0}. \tag{4-98}$$

Combining (4-97) and (4-98), we obtain

$$\zeta^{\dagger}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} = 12\zeta^{\dagger}_{(1,1,1,1)} - 2(\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4^0},$$

which, together with (4-92) and (4-93), proves (4-83).

We now prove (4-84). For this, we require the identity

$$\zeta^{\dagger}_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0}\Sigma_{\mathfrak{S}_3} = (\zeta_2^{\dagger} \mid \Sigma_{\langle(12)\rangle}\otimes\zeta_2^{\dagger} \mid \Sigma_{\langle(34)\rangle})|\Sigma_{\mathfrak{C}_3}, \tag{4-99}$$

which can be verified as follows. A direct calculation shows that

$$\begin{aligned}\Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} &= (e+(12)+(34)+(12)(34))(e+(123)+(132))\\ &= e+(12)+(13)+(23)+(34)+(12)(34)\\ &\qquad +(123)+(132)+(143)+(243)+(1243)+(1432).\end{aligned}$$

From this equation and the equivalence classes modulo $\langle(13)(24)\rangle$ in Table 1, we see that

$$\Sigma_{\mathfrak{S}_4} \equiv 2\Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} \mod \langle(13)(24)\rangle.$$

We also see from $\Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}$ and (4-26) that $\Sigma_{\mathfrak{S}_4} \equiv 2\Sigma_{\mathfrak{C}_4^0}\Sigma_{\mathfrak{S}_3} \mod \langle(13)(24)\rangle$, and so

$$\Sigma_{\mathfrak{C}_4^0}\Sigma_{\mathfrak{S}_3} \equiv \Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} \mod \langle(13)(24)\rangle.$$

Since $\zeta^{\dagger}_{(2,2)}$ is invariant under $\langle(13)(24)\rangle$ and since $\Sigma_{\langle(12),(34)\rangle} = \Sigma_{\langle(12)\rangle}\Sigma_{\langle(34)\rangle}$,

$$\zeta^{\dagger}_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0}\Sigma_{\mathfrak{S}_3} = \zeta^{\dagger}_{(2,2)} \mid \Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} = (\zeta_2^{\dagger} \mid \Sigma_{\langle(12)\rangle}\otimes\zeta_2^{\dagger} \mid \Sigma_{\langle(34)\rangle})|\Sigma_{\mathfrak{C}_3},$$

which verifies (4-99). Then, by (4-91), the right-hand side of (4-99) can be calculated as

$$\begin{aligned}&(\text{RHS of (4-99)})\\ &\ = (\zeta^{\dagger}_{(1,1)} - \chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes(\zeta^{\dagger}_{(1,1)} - \chi_2^{\dagger}\cdot\zeta_1\circ w_2)|\Sigma_{\mathfrak{C}_3}\\ &\ = \{\zeta^{\dagger}_{(1,1,1,1)} - (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} - \zeta^{\dagger}_{(1,1)}\otimes(\chi_2^{\dagger}\cdot\zeta_1\circ w_2) + (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)^{\otimes 2}\} \mid \Sigma_{\mathfrak{C}_3}\\ &\ = 3\zeta^{\dagger}_{(1,1,1,1)} - (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\langle(13)(24)\rangle}\Sigma_{\mathfrak{C}_3} + (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)^{\otimes 2} \mid \Sigma_{\mathfrak{C}_3}.\end{aligned}$$

It holds that $\Sigma_{\langle(13)(24)\rangle}\Sigma_{\mathfrak{C}_3} = \Sigma_{\mathfrak{A}_4^0}$, and so (4-99) can be restated as

$$\zeta^{\dagger}_{(2,2)} \mid \Sigma_{\mathfrak{C}_4^0}\Sigma_{\mathfrak{S}_3} = 3\zeta^{\dagger}_{(1,1,1,1)} - (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4^0} + (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)^{\otimes 2} \mid \Sigma_{\mathfrak{C}_3},$$

which, together with (4-92), (4-93), and (4-94), proves (4-84).

We lastly prove (4-85) in a similar way to (4-84). We require the identity

$$\begin{aligned}&\zeta^{\dagger}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}\\ &\quad = 4\zeta^{\dagger}_{(1,1,1,1)}\zeta_1^{\dagger\otimes 4} - 2(\chi_2^{\dagger}\cdot\zeta_2^{\dagger}\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4^0} + 2(\chi_3^{\dagger}\cdot\zeta_1\circ w_3)\otimes\zeta_1^{\dagger} \mid \Sigma_{\mathfrak{C}_4},\end{aligned} \tag{4-100}$$

which can be verified as follows. Identity (1-8) for $n=3$ can be restated as

$$\zeta_3^\dagger \mid \Sigma_{\mathfrak{S}_3} = \zeta^\dagger_{(1,1,1)} - (\chi_2^\dagger \cdot \zeta_1^\dagger \circ w_2) \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_3} + 2\chi_3^\dagger \cdot \zeta_1 \circ w_3, \tag{4-101}$$

because of (4-88), (4-89), and (4-90). A direct calculation shows that

$$\begin{aligned}\Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{C}_4} = e + (14) + (34) + (13)(24) + (123) + (124) + (132) + (243)\\ + (1234) + (1342) + (1423) + (1432),\end{aligned}$$

and so we see from the equivalence classes modulo $\langle (12), (34)\rangle$ in Table 1 that

$$\Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{C}_4} \equiv 2\Sigma_{\mathfrak{A}_4^0} \mod \langle (12), (34)\rangle.$$

Since $\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} = \Sigma_{\mathfrak{S}_3}\Sigma_{\mathfrak{C}_4}$, we thus have

$$\begin{aligned}&\zeta^\dagger_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}\\ &\quad= \zeta_3^\dagger \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{S}_3}\Sigma_{\mathfrak{C}_4} = (\zeta_3^\dagger | \Sigma_{\mathfrak{S}_3}) \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_4}\\ &\quad\overset{(4\text{-}101)}{=} \zeta^\dagger_{(1,1,1,1)} \mid \Sigma_{\mathfrak{C}_4} - (\chi_2^\dagger \cdot \zeta_1^\dagger \circ w_2) \otimes \zeta^\dagger_{(1,1)} \mid \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{C}_4} + 2(\chi_3^\dagger \cdot \zeta_1 \circ w_3) \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_4}\\ &\quad= 4\zeta^\dagger_{(1,1,1,1)} - 2(\chi_2^\dagger \cdot \zeta_2^\dagger \circ w_2) \otimes \zeta^\dagger_{(1,1)} \mid \Sigma_{\mathfrak{A}_4^0} + 2(\chi_3^\dagger \cdot \zeta_1 \circ w_3) \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_4},\end{aligned}$$

which verifies (4-100). Then, combining (4-92), (4-93), (4-95), and (4-100), we obtain (4-85), which completes the proof. □

Remark 4.14. We can find that (1-2) and (1-3) are used to show Lemma 4.12; this lemma is required for the proof of (1-8) for $n=3$. Thus, not only (1-3) but also (1-2) are necessary to prove (1-8) for $n=3$. Similarly, we can find that (1-2), (1-3), and (1-4) are used to show Lemma 4.13, and thus not only (1-4) but also (1-2) and (1-3) are necessary to prove (1-8) for $n=4$.

5. Examples

We list examples of (1-3) and (1-4) in Table 2 and Table 4, respectively. We also list examples of (1-8) for $n=3$ and $n=4$ in Table 3 and Table 5, respectively, for comparison. The examples treat the case of weight less than 7. We omit examples of (1-2) and (1-8) for $n=2$ because they are essentially the harmonic relations. The following straightforward expressions of (1-3) and (1-4) are convenient for calculating the examples in Table 2 and Table 4:

$$\begin{aligned}&\zeta_3^\dagger(l_1,l_2,l_3) + \zeta_3^\dagger(l_2,l_3,l_1) + \zeta_3^\dagger(l_3,l_1,l_2)\\ &\quad= -\zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2)\zeta_1^\dagger(l_3)\\ &\qquad+ \zeta_2^\dagger(l_1,l_2)\zeta_1^\dagger(l_3) + \zeta_2^\dagger(l_2,l_3)\zeta_1^\dagger(l_1) + \zeta_2^\dagger(l_3,l_1)\zeta_1^\dagger(l_2)\\ &\qquad+ \chi_3^\dagger(l_1,l_2,l_3)\zeta_1(l_1+l_2+l_3),\end{aligned} \tag{5-1}$$

Index set	Linear relation	
(1,1,1)	$3\zeta_3^\dagger(1,1,1)=\chi_3^\dagger(1,1,1)\zeta_1(3)$	(d3-1)
(1,1,2)	$\zeta_3^\dagger(1,1,2)+\zeta_3^\dagger(1,2,1)+\zeta_3(2,1,1)=\zeta_2^\dagger(1,1)\zeta_1(2)+\zeta_1(4)$	(d3-2)
(1,1,3)	$\zeta_3^\dagger(1,1,3)+\zeta_3^\dagger(1,3,1)+\zeta_3(3,1,1)=\zeta_2^\dagger(1,1)\zeta_1(3)+\zeta_1(5)$	(d3-3)
(1,2,2)	$\zeta_3^\dagger(1,2,2)+\zeta_3(2,2,1)+\zeta_3(2,1,2)=-\zeta_1(2)\zeta_1(3)+\zeta_1(5)$	(d3-4)
(1,1,4)	$\zeta_3^\dagger(1,1,4)+\zeta_3^\dagger(1,4,1)+\zeta_3(4,1,1)=\zeta_2^\dagger(1,1)\zeta_1(4)+\zeta_1(6)$	(d3-5)
(1,2,3)	$\zeta_3^\dagger(1,2,3)+\zeta_3(2,3,1)+\zeta_3(3,1,2)$ $=\zeta_2^\dagger(1,2)\zeta_1(3)+\zeta_2(3,1)\zeta_1(2)+\zeta_1(6)$	(d3-6)
(1,3,2)	$\zeta_3^\dagger(1,3,2)+\zeta_3(3,2,1)+\zeta_3(2,1,3)$ $=\zeta_2^\dagger(1,3)\zeta_1(2)+\zeta_2(2,1)\zeta_1(3)+\zeta_1(6)$	(d3-7)
(2,2,2)	$3\zeta_3(2,2,2)=-\zeta_1(2)^3+3\zeta_2(2,2)\zeta_1(2)+\zeta_1(6)$	(d3-8)

Table 2. Examples of (1-3) (or (5-1)).

$$
\begin{aligned}
\text{(5-2)}\quad \zeta_4^\dagger(l_1,l_2,l_3,l_4)&+\zeta_4^\dagger(l_2,l_3,l_4,l_1)+\zeta_4^\dagger(l_3,l_4,l_1,l_2)+\zeta_4^\dagger(l_4,l_1,l_2,l_3)\\
&=\zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2)\zeta_1^\dagger(l_3)\zeta_1^\dagger(l_4)\\
&\qquad-\zeta_2^\dagger(l_1,l_2)\zeta_1^\dagger(l_3)\zeta_1^\dagger(l_4)-\zeta_2^\dagger(l_2,l_3)\zeta_1^\dagger(l_4)\zeta_1^\dagger(l_1)\\
&\qquad-\zeta_2^\dagger(l_3,l_4)\zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2)-\zeta_2^\dagger(l_4,l_1)\zeta_1^\dagger(l_2)\zeta_1^\dagger(l_3)\\
&\qquad+\zeta_2^\dagger(l_1,l_2)\zeta_2^\dagger(l_3,l_4)+\zeta_2^\dagger(l_2,l_3)\zeta_2^\dagger(l_4,l_1)\\
&\qquad+\zeta_3^\dagger(l_1,l_2,l_3)\zeta_1^\dagger(l_4)+\zeta_3^\dagger(l_2,l_3,l_4)\zeta_1^\dagger(l_1)\\
&\qquad+\zeta_3^\dagger(l_3,l_4,l_1)\zeta_1^\dagger(l_2)+\zeta_3^\dagger(l_4,l_1,l_2)\zeta_1^\dagger(l_3)\\
&\qquad-\chi_4^\dagger(l_1,l_2,l_3,l_4)\zeta_1(l_1+l_2+l_3+l_4).
\end{aligned}
$$

We have used $\zeta_1^\dagger(1)=0$ for all equations in the tables, and $\zeta_2^\dagger(1,k)+\zeta_2^\dagger(k,1)=-\zeta_1(k+1)$ $(k>1)$ for (d3-4), (d4-2), and (d4-3).

As was mentioned in Section 1, it holds that

$$\text{(5-3)}\qquad \zeta_2^{ш}(1,1)=0 \qquad\text{and}\qquad \zeta_2^{*}(1,1)=-\tfrac{1}{2}\zeta_1(2),$$

which follows from (3-14) with $l_1=l_2=1$, or $2z_1z_1=z_1*z_1-z_2$. In fact, applying Z^* to both sides of this equation, we obtain $2Z^*_{1,1}(T)=T^2-\zeta_1(2)$, which together with (2-8) and (3-64) gives (5-3). Lastly, we derive the following equations from (d3-1) and (d4-1) as applications of examples:

$$\text{(5-4)}\qquad \zeta_3^{ш}(1,1,1)=\zeta_4^{ш}(1,1,1,1)=0,$$

$$\text{(5-5)}\qquad \zeta_3^{*}(1,1,1)=\tfrac{1}{3}\zeta_1(3),$$

$$\text{(5-6)}\qquad \zeta_4^{*}(1,1,1,1)=\tfrac{1}{16}\zeta_1(4).$$

Index set	Linear relation	
(1,1,1)	$6\zeta_3^\dagger(1,1,1)=2\chi_3^\dagger(1,1,1)\zeta_1(3)$	(d3′-1)
(1,1,2)	$2(\zeta_3^\dagger(1,1,2)+\zeta_3^\dagger(1,2,1)+\zeta_3(2,1,1))$ $=-\chi_2^\dagger(1,1)\zeta_1(2)^2+2\zeta_1(4)$	(d3′-2)
(1,1,3)	$2(\zeta_3^\dagger(1,1,3)+\zeta_3^\dagger(1,3,1)+\zeta_3(3,1,1))$ $=-\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(3)+2\zeta_1(5)$	(d3′-3)
(1,2,2)	$2(\zeta_3^\dagger(1,2,2)+\zeta_3(2,2,1)+\zeta_3(2,1,2))$ $=-2\zeta_1(2)\zeta_1(3)+2\zeta_1(5)$	(d3′-4)
(1,1,4)	$2(\zeta_3^\dagger(1,1,4)+\zeta_3^\dagger(1,4,1)+\zeta_3(4,1,1))$ $=-\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(4)+2\zeta_1(6)$	(d3′-5)
(1,2,3)	$\zeta_3^\dagger(1,2,3)+\zeta_3^\dagger(1,3,2)+\zeta_3(2,1,3)+\zeta_3(2,3,1)$ $+\zeta_3(3,1,2)+\zeta_3(3,2,1)$ $=-(\zeta_1(2)\zeta_1(4)+\zeta_1(3)^2)+2\zeta_1(6)$	(d3′-6)
(2,2,2)	$6\zeta_3(2,2,2)=\zeta_1(2)^3-3\zeta_1(2)\zeta_1(4)+2\zeta_1(6)$	(d3′-7)

Table 3. Examples of (1-8) for $n=3$.

Index set	Linear relation	
(1,1,1,1)	$4\zeta_4^\dagger(1,1,1,1)=2\zeta_2^\dagger(1,1)^2-\chi_4^\dagger(1,1,1,1)\zeta_1(4)$	(d4-1)
(1,1,1,2)	$\zeta_4^\dagger(1,1,1,2)+\zeta_4^\dagger(1,1,2,1)+\zeta_4^\dagger(1,2,1,1)+\zeta_4(2,1,1,1)$ $=-\zeta_2^\dagger(1,1)\zeta_1(3)+\zeta_3^\dagger(1,1,1)\zeta_1(2)-\zeta_1(5)$	(d4-2)
(1,1,1,3)	$\zeta_4^\dagger(1,1,1,3)+\zeta_4^\dagger(1,1,3,1)+\zeta_4^\dagger(1,3,1,1)+\zeta_4(3,1,1,1)$ $=-\zeta_2^\dagger(1,1)\zeta_1(4)+\zeta_3^\dagger(1,1,1)\zeta_1(3)-\zeta_1(6)$	(d4-3)
(1,1,2,2)	$\zeta_4^\dagger(1,1,2,2)+\zeta_4(1,2,2,1)+\zeta_4^\dagger(2,2,1,1)+\zeta_4^\dagger(2,1,1,2)$ $=-\zeta_2^\dagger(1,1)\zeta_1(2)^2+\zeta_2^\dagger(1,1)\zeta_2(2,2)+\zeta_2^\dagger(1,2)\zeta_2(2,1)$ $+(\zeta_3^\dagger(1,1,2)+\zeta_3(2,1,1))\zeta_1(2)-\zeta_1(6)$	(d4-4)
(1,2,1,2)	$2(\zeta_4^\dagger(1,2,1,2)+\zeta_4(2,1,2,1))$ $=\zeta_2^\dagger(1,2)^2+\zeta_2(2,1)^2+2\zeta_3^\dagger(1,2,1)\zeta_1(2)-\zeta_1(6)$	(d4-5)

Table 4. Examples of (1-4) (or (5-2)).

We can easily obtain (5-4) from (d3-1) and (d4-1) for $\dagger=ш$, since

$$\chi_3^ш(1,1,1)=\chi_4^ш(1,1,1,1)=0 \quad\text{and}\quad \zeta_2^ш(1,1)=0.$$

We can also obtain (5-5) from (d3-1) for $\dagger=*$, since $\chi_3^*(1,1,1)=1$. Since $\chi_4^*(1,1,1,1)=1$ and $\zeta_2^*(1,1)=-\zeta_1(2)/2$, we obtain from (d4-1) for $\dagger=*$ that

$$4\zeta_4^*(1,1,1,1)=2\zeta_2^*(1,1)^2-\zeta_1(4)\;=\;\tfrac{1}{2}\zeta_1(2)^2-\zeta_1(4),$$

which together with (3-72) proves (5-6).

Index set	Linear relation	
(1,1,1,1)	$24\zeta_4^\dagger(1,1,1,1)=3\chi_2^\dagger(1,1)\zeta_1(2)^2-6\chi_4^\dagger(1,1,1,1)\zeta_1(4)$	(d4′-1)
(1,1,1,2)	$6(\zeta_4^\dagger(1,1,1,2)+\zeta_4^\dagger(1,1,2,1)+\zeta_4^\dagger(1,2,1,1)+\zeta_4(2,1,1,1))$ $=3\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(3)+2\chi_3^\dagger(1,1,1)\zeta_1(2)\zeta_1(3)-6\zeta_1(5)$	(d4′-2)
(1,1,1,3)	$6(\zeta_4^\dagger(1,1,1,3)+\zeta_4^\dagger(1,1,3,1)+\zeta_4^\dagger(1,3,1,1)+\zeta_4(3,1,1,1))$ $=3\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(4)+2\chi_3^\dagger(1,1,1)\zeta_1(3)^2-6\zeta_1(6)$	(d4′-3)
(1,1,2,2)	$4(\zeta_4^\dagger(1,1,2,2)+\zeta_4(1,2,1,2)+\zeta_4(1,2,2,1)+\zeta_4^\dagger(2,1,1,2)$ $+\zeta_4^\dagger(2,1,2,1)+\zeta_4^\dagger(2,2,1,1))$ $=-\chi_2^\dagger(1,1)\zeta_1(2)^3+(\chi_2^\dagger(1,1)+4)\zeta_1(2)\zeta_1(4)$ $+2\zeta_1(3)^2-6\zeta_1(6)$	(d4′-4)

Table 5. Examples of (1-8) for $n=4$.

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TOMOYA MACHIDE
JST, ERATO, KAWARABAYASHI LARGE GRAPH PROJECT
GLOBAL RESEARCH CENTER FOR BIG DATA MATHEMATICS
NATIONAL INSTITUTE OF INFORMATICS
2-1-2 HITOTSUBASHI
CHIYODA-KU
TOKYO 101-8430
JAPAN
machide@nii.ac.jp

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CONFORMALLY KÄHLER RICCI SOLITONS AND BASE METRICS FOR WARPED PRODUCT RICCI SOLITONS

GIDEON MASCHLER

We investigate Kähler metrics conformal to gradient Ricci solitons, and base metrics of warped product gradient Ricci solitons. A slight generalization of the latter we name quasi-solitons. A main assumption that is employed is functional dependence of the soliton potential, with the conformal factor in the first case, and with the warping function in the second. The main result in the first case is a partial classification in dimension $n \geq 4$. In the second case, Kähler quasi-soliton metrics satisfying the above main assumption are shown to be, under an additional genericity hypothesis, necessarily Riemannian products. Another theorem concerns quasi-soliton metrics satisfying the above main assumption, which are also conformally Kähler. With some additional assumptions it is shown that such metrics are necessarily base metrics of Einstein warped products, that is, quasi-Einstein.

1. Introduction

The study of the Ricci flow [Hamilton 1982] has inspired the introduction of a metric type generalizing the Einstein condition. A gradient Ricci soliton is a Riemannian metric satisfying

$$\mathrm{Ric} + \nabla df = \lambda g, \quad \lambda \text{ constant.}$$

The function f is called the soliton potential. Such solitons are further referred to as shrinking, steady or expanding, depending on the sign of λ.

We consider Ricci solitons in two settings: the case where they are conformal to Kähler metrics, and the case where they are warped products. Conformal classes of Ricci solitons have been studied recently in [Jauregui and Wylie 2015; Catino et al. 2016; Maschler 2015]. Kähler metrics in such a conformal class, with nontrivial conformal factor, have been examined in [Maschler 2008; Derdziński 2012]. Warped product Ricci solitons, on the other hand, have been studied extensively when the base of the warped product is one-dimensional; see for instance [Chow et al. 2007]. The cigar soliton and the Bryant soliton are examples in this category.

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In each case we focus on an auxiliary metric which at least partially determines the soliton. In the first case that would be the associated Kähler metric in the conformal class, and in the second case it is the induced metric on the base of the warped product. The latter metric is a special case of what we call a (gradient Ricci) quasi-soliton, in analogy with how base metrics of Einstein warped products are often called quasi-Einstein metrics. We consider only quasi-soliton metrics which are Kähler, or conformally Kähler.

A common thread for these two cases of auxiliary metrics is the appearance of two Hessians in their defining equations. One is the Hessian of the soliton potential f, while the other Hessian depends on the case: it is that of the conformal factor τ in the first case, and that of the warping function ℓ in the second.

These equations are, of course, more complex than the original Ricci soliton equation, and handling them in full generality still appears beyond reach. Our strategy is thus to consider mainly the case where functional dependence of the above two functions holds, in either setting. In other words, we require

(1-1i) $d\tau \wedge df = 0$ for the associated Kähler metric,

(1-1ii) $d\ell \wedge df = 0$ for the induced metric on the base of the warped product.

In the latter case we call the metric a *special* quasi-soliton.

An example where condition (1-1i) occurs in the Kähler conformally soliton case is when the conformal factor τ is additionally a potential for a Killing vector field of the Kähler metric (a Killing potential). The latter condition was studied in [Maschler 2008] and plays a role in Theorem 7.4. It turns out that the condition (1-1i) also implies, generically, the existence of a Killing potential which, however, is of a more general kind, being only functionally dependent on τ, rather than being τ itself. An instance of this more general setting was first considered in [Derdziński 2012].

Another metric type that plays an important role in all our main theorems is the SKR metric, i.e., a metric that admits a so-called special Kähler–Ricci potential. This notion was introduced by Derdzinski and Maschler [2003; 2006] for the purpose of classifying conformally Kähler Einstein metrics. In all our main theorems the proofs involve a Ricci–Hessian equation of the form

$$\alpha \nabla d\tau + \mathrm{Ric} = \gamma g,$$

for functions α and γ. The theory of SKR metrics which is then applied is closely tied to such equations.

The main results in this article are Theorems 6.2, 7.3 and 7.4. The first of these gives a partial classification of Kähler metrics conformal to gradient Ricci solitons in dimension $n \geq 4$ satisfying condition (1-1i). Theorem 7.3 presents a reducibility result for special quasi-soliton metrics which are Kähler. The conclusion of this theorem, that the metric is locally a Riemannian product, is analogous to a similar result for quasi-Einstein metrics [Case et al. 2011]. Theorem 7.4 mixes the two main themes of this paper, as it involves special quasi-soliton metrics that are conformal to an irreducible Kähler metric. With some additional assumptions, the conclusion of the theorem is that the metric must in fact be quasi-Einstein. This is in contrast with the existence of conformally Kähler quasi-Einstein metrics [Maschler 2011; Batat et al. 2015], and it remains to be seen whether this difference holds in general, or else is the result of the added assumptions.

Examples of metrics satisfying the conditions of Theorem 6.2 appear in [Maschler 2008; Derdziński 2012]. In one of the possible outcomes of the theorem, occurring in dimension four, the Ricci soliton must be non-Einstein and steady ($\lambda = 0$). There are at this time many known examples of non-Einstein steady Ricci solitons in all dimensions. Recent examples were given by Buzano, Dancer and Wang [Buzano et al. 2015] and Stolarski [2015]. A discussion of their potential relevance to this theorem is given at the end of Section 6.

The structure of the paper is as follows. After some preliminaries in Section 2, we give several forms for the conformally soliton equation in Section 3. We then determine in Section 4, in the context of the first metric type considered, certain implications of the assumption that vector fields that occur in the conformally soliton equation are of one of several well-known classical types. One such assumption which does not occur in nontrivial cases has, nonetheless, an interesting classification, which we give in the Appendix. In Section 5 we recall the salient features of SKR metric theory. The main theorem in the conformally Kähler case is given in Section 6, and the two main theorems for special quasi-soliton metrics appear in Section 7.

2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension n, and $\tau : M \to \mathbb{R}$ a C^∞ function. We write metrics conformally related to g in the form $\hat{g} = \tau^{-2} g$.

We recall a few conformal change formulas. The covariant derivative is

$$\widehat{\nabla}_w u = \nabla_w u - (d_w \log \tau) u - (d_u \log \tau) w + \langle w, u \rangle \nabla \log \tau, \tag{2-1}$$

where d_u denotes the directional derivative of a vector field u and the angle brackets stand for g. It follows that the $\hat{g}$-Hessian and $\hat{g}$-Laplacian of a C^2 function f are given by

$$\widehat{\nabla} df = \nabla df + \tau^{-1}[2\, d\tau \odot df - g(\nabla\tau, \nabla f) g], \tag{2-2i}$$

$$\widehat{\Delta} f = \tau^2 \Delta f - (n-2)\tau g(\nabla\tau, \nabla f), \tag{2-2ii}$$

where $d\tau \odot df = \frac{1}{2}(d\tau \otimes df + df \otimes d\tau)$. Finally, the well-known formula for the

Ricci tensor of $\hat{g}$ is given by

$$\widehat{\mathrm{Ric}} = \mathrm{Ric} + (n-2)\tau^{-1}\nabla d\tau + \big[\tau^{-1}\Delta\tau - (n-1)\tau^{-2}|\nabla\tau|^2\big]g, \tag{2-3}$$

where Δ denotes the Laplacian and the norm $|\cdot|$ is with respect to g.

Recall that a (real) vector field w on a complex manifold (M, J) is holomorphic if the Lie derivative $\mathcal{L}_w J$ vanishes.

Proposition 2.1. *Let ∇ be a torsion-free connection on a complex manifold (M, J). For any vector field w,*

$$\mathcal{L}_w J = \nabla_w J + [J, \nabla w],$$

where the square brackets denote the commutator.

In fact, write $(\mathcal{L}_w J)u = \mathcal{L}_w(Ju) - J\mathcal{L}_w u$ and replace each Lie derivative by the Lie brackets, and each of these by the torsion-free condition for ∇, giving $\nabla_w Ju - \nabla_{Ju} w - J\nabla_w u + J\nabla_u w$. The first and third terms together give $(\nabla_w J)(u)$, while the second and fourth terms give $[J, \nabla w](u)$.

Proposition 2.2. *Let (M, J) be a complex manifold with a Hermitian metric g. Given a C^2 function q on M, set $w = \nabla q$. Then ∇dq is J-invariant if and only if $[J, \nabla w] = 0$.*

In fact, $\nabla dq(Ja, b) = g(Ja, \nabla_b w) = -g(a, J\nabla_b w) = -g(a, J(\nabla w)(b))$, while $-\nabla dq(a, Jb) = -g(a, \nabla_{Jb} w) = -g(a, (\nabla w)(Jb))$.

In the following well-known proposition $\imath_v$ denotes interior multiplication by a vector field, while δ denotes the divergence operator.

Proposition 2.3. *Let σ be a smooth function on a Kähler manifold such that $v = \nabla\sigma$ is a holomorphic gradient vector field. Then $2\imath_v\mathrm{Ric} = -dY$ and $2\delta\nabla d\sigma = dY$ for $Y = \Delta\sigma$.*

For a proof, see [Derdzinski and Maschler 2003, (5.4) and (2.9)(c)].

3. Various forms of the conformally soliton equation

Let g be a Riemannian metric and τ a smooth function on a given manifold, for which $\hat{g} = g/\tau^2$ is a gradient Ricci soliton with soliton potential f. The soliton equation for $\hat{g}$, together with its associated scalar equation, are

$$\widehat{\mathrm{Ric}} + \widehat{\nabla} df = \lambda\hat{g}, \qquad \text{with } \lambda \text{ constant}, \tag{3-1i}$$

$$\widehat{\Delta} f - \hat{g}(\widehat{\nabla} f, \widehat{\nabla} f) + 2\lambda f = a, \qquad \text{for a constant } a. \tag{3-1ii}$$

To obtain this in terms of g, we apply equations (2-3) and (2-2i) to (3-1i). The result is

$$\mathrm{Ric} + (n-2)\tau^{-1}\nabla d\tau + \nabla df + 2\tau^{-1} d\tau \odot df = \gamma g. \tag{3-2}$$

for

$$\gamma = \tau^{-2}[\lambda + (n-1)|\nabla\tau|^2] - \tau^{-1}[\Delta\tau - g(\nabla\tau, \nabla f)], \tag{3-3}$$

with $|\nabla\tau|^2 = g(\nabla\tau, \nabla\tau)$.

We will now rewrite (3-2) in a different form. Specifically, for the vector fields $v = \nabla\tau$ and $w = \tau^2\nabla f$, equation (3-2) is equivalent to

$$\mathrm{Ric} + \alpha\mathcal{L}_v g + \beta\mathcal{L}_w g = \gamma g, \tag{3-4}$$

with $\alpha = \frac{1}{2}(n-2)\tau^{-1}$, $\beta = (2\tau^2)^{-1}$, and $\mathcal{L}$ denoting the Lie derivative. To show this, recall that for any vector fields a, b,

$$(\mathcal{L}_w g)(a, b) = g(\nabla_a w, b) + g(a, \nabla_b w), \tag{3-5}$$

or $\mathcal{L}_w g = [\nabla w + (\nabla w)^*]_\flat$, where $*$ denotes the adjoint and $\flat$ is the isomorphism associated with lowering an index. Now clearly $\mathcal{L}_v g = \mathcal{L}_{\nabla\tau} g = 2\nabla d\tau$. To compute the Lie derivative term for w, write $w = h\nabla f$. Then

$$\mathcal{L}_w g = 2h\nabla df + 2\, dh \odot df.$$

Setting $h = \tau^2$ and dividing by $2\tau^2$ gives

$$\nabla df + 2\tau^{-1} d\tau \odot df = (2\tau^2)^{-1}\mathcal{L}_{\tau^2\nabla f}\, g = \beta\mathcal{L}_w g.$$

Another form for equation (3-2) is obtained as follows. It is natural to combine the two Hessian terms into one. For this, set

$$\mu = \log\tau, \quad \theta = f + (n-2)\log\tau, \quad \psi = 2\theta - (n-2)\mu.$$

Then (3-2), (3-3) and (3-1ii) read

$$\mathrm{Ric} + \nabla d\theta + d\mu \odot d\psi = \gamma g, \quad \gamma = \lambda e^{-2\mu} - \Delta\mu + g(\nabla\theta, \nabla\mu), \tag{3-6i}$$

$$e^{2\mu}[\Delta f - g(\nabla\theta, \nabla f)] + 2\lambda f = a. \tag{3-6ii}$$

To derive (3-6ii) one uses (2-2ii), which, in terms of μ, reads

$$e^{-2\mu}\widehat{\Delta} f = \Delta f - (n-2)g(\nabla\mu, \nabla f).$$

4. The Kähler condition and distinguished vector fields

Let g be a metric which is Kähler with respect to a complex structure J on a manifold M, and conformal to a gradient Ricci soliton. Equation (3-4) then holds, and the J-invariance of g and its Ricci curvature implies that

$$\alpha\mathcal{L}_v g + \beta\mathcal{L}_w g \text{ is } J\text{-invariant}. \tag{4-1}$$

Applying (3-5) to the relation $\mathcal{L}_x g(J\cdot\,,\cdot) = -\mathcal{L}_x g(\,\cdot\,, J\cdot)$, for both $x = v$ and $x = w$, and recalling that $J^* = -J$, we see that (4-1) is equivalent to the vanishing of a commutator: $[\alpha(\nabla v + (\nabla v)^*) + \beta(\nabla w + (\nabla w)^*), J] = 0$, or

$$[\alpha(\mathcal{L}_v g)^\sharp + \beta(\mathcal{L}_w g)^\sharp, J] = 0, \tag{4-2}$$

where $\sharp$ denotes the isomorphism acting by raising an index.

The most obvious case where (4-2) holds is when both summands vanish separately, so that, w, for example, satisfies

$$[(\mathcal{L}_w g)^\sharp, J] = 0. \tag{4-3}$$

We wish to study relations between these two vanishing conditions for v and w. We first note that (4-3) includes as special cases the following three classical types of vector fields (the first being, of course, a special case of the second):

- a Killing vector field ($\mathcal{L}_w g = 0$),
- a conformal vector field ($(\mathcal{L}_w g)^\sharp = hI$, for a function h and I the identity),
- a holomorphic vector field ($[\nabla w, J] = 0$ on a Kähler manifold).

This last type is holomorphic by Proposition 2.1 in the Kähler case, and it is indeed a special case since $[\nabla w, J]^* = [(\nabla w)^*, J]$ and $[\nabla w + (\nabla w)^*, J] = 0$, the latter equality being equivalent to (4-3).

We will see in the next theorem that the Killing case does not lead to important Kähler conformally soliton metrics. However, Kähler metrics with a Killing field of the form $w = \tau^2 \nabla f$ can be classified, as we show in the Appendix.

To state the next result, we continue to assume g is Kähler and conformal to a gradient Ricci soliton $\hat{g}$, but now on a manifold of dimension $n > 2$. With notation as above for τ, f, v and w we have:

Theorem 4.1. *The following conclusions hold for the vector fields v and w*:

1. *If w is a conformal vector field, then $\hat{g}$ is Einstein.*
2. *If w is a holomorphic vector field and either v is holomorphic as well, or $\widehat{\nabla} df$ is J-invariant, then* $\mathrm{span}_{\mathbb{C}}\{v\} = \mathrm{span}_{\mathbb{C}}\{w\}$ *away from the zero sets of v and w.*

Proof. The key to both parts is that $w = \tau^2 \nabla f$ is also the $\hat{g}$-gradient of f, i.e., $w = \widehat{\nabla} f$. Therefore $\mathcal{L}_w \hat{g} = \mathcal{L}_{\widehat{\nabla} f}\hat{g} = 2\widehat{\nabla} df$. As the condition that a vector field be conformal is conformally invariant, it follows that when w is conformal, the Ricci soliton equation (3-1i) reduces, using Schur's lemma, to the Einstein equation. This proves (1).

To prove (2), note first that the combination of Propositions 2.1 and 2.2 for a Kähler metric yields the result that the vector field $v = \nabla\tau$ is holomorphic exactly when $\nabla d\tau$ is J-invariant. This in turn is equivalent, by (2-3) and the fact that the metric g and its Ricci curvature are J-invariant, to $\widehat{\mathrm{Ric}}$ being J-invariant. Finally, the latter condition is equivalent to $\widehat{\nabla} df$ being J-invariant, by the soliton equation

(3-1i). The combination, again, of Propositions 2.1 and 2.2, but this time for a Hermitian metric, yields equivalence of the latter condition with $\mathcal{L}_{\widehat{\nabla} f} J = \widehat{\nabla}_{\widehat{\nabla} f} J$, or

$$\mathcal{L}_w J = \widehat{\nabla}_w J. \tag{4-4}$$

Now from (2-1), for any vector field u,

$$\begin{aligned}(\widehat{\nabla}_w J)u &= \widehat{\nabla}_w(Ju) - J\widehat{\nabla}_w u\\ &= \nabla_w(Ju) - \tau^{-1}(d_w\tau)Ju - \tau^{-1}(d_{Ju}\tau)w + \langle w, Ju\rangle\tau^{-1}v\\ &\qquad - [J\nabla_w(u) - \tau^{-1}(d_w\tau)Ju - \tau^{-1}(d_u\tau)Jw + \langle w, u\rangle\tau^{-1}Jv]\\ &= \tau^{-1}(-\langle v, Ju\rangle w + \langle w, Ju\rangle v + \langle v, u\rangle Jw - \langle w, u\rangle Jv)\\ &= \tau^{-1}(\langle Jv, u\rangle w - \langle Jw, u\rangle v + \langle v, u\rangle Jw - \langle w, u\rangle Jv),\end{aligned}$$

where we used the fact that $\nabla_w J = 0$, and the angle brackets denote g. Combining this with (4-4) we see that as w is holomorphic, the last expression vanishes for *every* vector field u. Substituting first $u = v$ and then $u = Jv$ shows that away from the zeros of v, the vector fields w and Jw are pointwise in $\operatorname{span}\{v, Jv\}$. As this reasoning is symmetric for v and w, the result follows. □

In the examples of [Maschler 2008] the manifolds on which g and $\hat{g}$ reside are locally total spaces of holomorphic line bundles over manifolds admitting a Kähler–Einstein metric, and g is an SKR metric (see Section 5), while the conformal factor τ is a Killing potential. For these examples f is an affine function in τ^{-1} (see [Maschler 2008, Proposition 3.1]), so that, in that case, v and w are holomorphic and in fact $\operatorname{span}_{\mathbb{R}} v = \operatorname{span}_{\mathbb{R}} w$, away from the zeros of these vector fields.

5. SKR metrics

We recall here some facts from [Derdzinski and Maschler 2003] and [Maschler 2008] on the notion of an SKR metric, i.e., a Kähler metric g admitting a special Kähler–Ricci potential σ. For the definition, recall that a smooth function σ on a Kähler manifold (M, J, g) is called a Killing potential if $J\nabla\sigma$ is a Killing vector field. The definition of a special Kähler–Ricci potential consists then of the requirement that σ is a Killing potential and, at each noncritical point of it, all nonzero tangent vectors orthogonal to the complex span of $\nabla\sigma$ are eigenvectors of both the Ricci tensor and the Hessian of σ, considered as operators. This rather technical definition implies that a Ricci–Hessian equation holds for σ on a suitable open set (see [Derdzinski and Maschler 2003, Remark 7.4]), namely

$$\mathrm{Ric} + \alpha\nabla d\sigma = \gamma g, \tag{5-1}$$

for some functions α, γ which are functionally dependent on σ.

We say that equation (5-1) is a *standard* Ricci–Hessian equation if $\alpha d\alpha \neq 0$ whenever $d\sigma \neq 0$. This condition will appear in all our main theorems. However, even if it does not hold over the entire set where $d\sigma \neq 0$, these theorems will hold, with the same proofs, on any open subset of $\{d\sigma \neq 0\}$ where $\alpha d\alpha \neq 0$. We have:

Proposition 5.1. *A Kähler metric on a manifold of dimension at least four is an SKR metric, provided it satisfies a standard Ricci–Hessian equation of the form* (5-1) *with* $d\alpha \wedge d\sigma = d\gamma \wedge d\sigma = 0$.

This result appears in [Maschler 2008, Proposition 3.5] with proof referenced from [Derdzinski and Maschler 2003], a proof that has to be interpreted with the aid of [Maschler 2008, Remark 3.6]. Note also that in dimension greater than four, if the Ricci–Hessian equation of a Kähler metric satisfies $d\alpha \wedge d\sigma = 0$ then it automatically also satisfies $d\gamma \wedge d\sigma = 0$ (see [Maschler 2008, Proposition 3.3]).

If an SKR metric is locally irreducible, the theory of such metrics (see §4 of [Maschler 2008]) implies that a pair of equations holds on the open set where the Ricci–Hessian equation (5-1) holds:

$$
\begin{aligned}
(\sigma - c)^2\phi'' + (\sigma - c)[m - (\sigma - c)\alpha]\phi' - m\phi &= K,\\
-(\sigma - c)\phi'' + [\alpha(\sigma - c) - (m+1)]\phi' + \alpha\phi &= \gamma.
\end{aligned}
\tag{5-2}
$$

Here ϕ is defined pointwise as the eigenvalue of the Hessian of σ, considered as an operator, corresponding to the eigendistribution $[\operatorname{span}_{\mathbb{C}} \nabla\sigma]^{\perp}$, and c is a constant. This eigenvalue and σ are functionally dependent, so that the primes represent differentiations with respect to σ. Furthermore, K is a constant whose exact expression in terms of SKR data will not concern us, while $m = \frac{1}{2}\dim(M)$. We further have the following relations between ϕ, $\Delta\sigma$ and $Q := g(\nabla\sigma, \nabla\sigma)$:

$$
\Delta\sigma = 2m\phi + 2(\tau - c)\phi', \qquad Q = 2(\tau - c)\phi.
\tag{5-3}
$$

Note that for an irreducible SKR metric, the function ϕ is nowhere zero on the open dense set where $d\sigma \neq 0$.

In analyzing equations such as (5-2) we will repeatedly use in Section 7 the following elementary lemma, taken from [Maschler 2008].

Lemma 5.2. *For a system*

$$
\begin{aligned}
A\phi'' + B\phi' + C\phi &= D,\\
\phi' + p\phi &= q,
\end{aligned}
\tag{5-4}
$$

with rational coefficients, either $A(p^2 - p') - Bp + C = 0$ *holds identically, or else the solution is given by* $\phi = (D - A(q' - pq) - Bq)/(A(p^2 - p') - Bp + C)$.

We now state the local classification of SKR metrics (Theorem 18.1 in [Derdzinski and Maschler 2003]).

Theorem 5.3. *Let (M, g, σ) be a manifold with an SKR metric and a special Kähler–Ricci potential. Then every point for which $d\sigma \neq 0$ has a neighborhood where g is, up to a biholomorphic isometry, given explicitly on an open set in the following local model.*

Here is the model metric, which is obtained as a special case of the Calabi ansatz. For simplicity we only give it in the irreducible case. Let $\pi : (L, \langle\,\cdot\,,\cdot\,\rangle) \to (N, h)$ be a Hermitian holomorphic line bundle over a Kähler manifold which is also Einstein if $n > 4$, where $n-2$ is the (real) dimension of N. Assume that the curvature of the Chern connection associated to $\langle\,\cdot\,,\cdot\,\rangle$ is a multiple of the Kähler form of h. (Note that, if $n > 4$, N is compact and h is not Ricci flat, this implies that L is smoothly isomorphic to a rational power of the anticanonical bundle of N.) Consider, on the total space of L excluding the zero section, the metric g given by

$$g|_{\mathcal{H}} = 2|\sigma - c|\pi^* h, \quad g|_{\mathcal{V}} = \frac{Q(\sigma)}{(ar)^2}\,\mathrm{Re}\langle\,\cdot\,,\cdot\,\rangle, \tag{5-5}$$

where:

- $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal distributions of L, assumed to be g-orthogonal to each other and the latter being determined via the Chern connection of $\langle\,\cdot\,,\cdot\,\rangle$.
- c, $a \neq 0$ are constants.
- r is the norm induced by $\langle\,\cdot\,,\cdot\,\rangle$.
- σ is a function on $L \setminus 0$, obtained by composing with the norm r another function, denoted via abuse of notation by $\sigma(r)$, and obtained as follows: one fixes an open interval I and a positive C^∞ function $Q(\sigma)$ on I, solves the differential equation $(a/Q)d\sigma = d(\log r)$ to obtain a diffeomorphism $r(\sigma) : I \to (0, \infty)$, and defines $\sigma(r)$ as the inverse of this diffeomorphism.

The metric g is the model SKR metric, with special Kähler–Ricci potential $\sigma = \sigma(r)$, and $|\nabla\sigma|_g^2 = Q(\sigma(r))$.

SKR metrics on compact manifolds also admit a global classification (Theorem 16.3 of [Derdzinski and Maschler 2006]), which shows they reside only on $\mathbb{CP}^1$-bundles $\mathbb{P}(L \oplus \mathbb{C})$ over manifolds N as above, or on complex projective spaces.

6. Functional dependence

Recall equation (3-6i):

$$\mathrm{Ric} + \nabla d\theta + d\mu \odot d\psi = \gamma g, \quad \gamma = \lambda e^{-2\mu} - \Delta\mu + g(\nabla\theta, \nabla\mu), \tag{6-1}$$

with $\mu = \log\tau,\ \theta = f + (n-2)\log\tau$ and $\psi = 2\theta - (n-2)\mu$. This was one of the forms of equation (3-2) characterizing a metric g conformal to a gradient Ricci soliton. If g is also Kähler on a manifold (M, J) of real dimension at least four, constancy of θ implies that g is in fact Kähler–Einstein. This follows since, in this case, the above relation defining ψ shows that the term $d\mu \odot d\psi$ is just a constant multiple of $d\mu \otimes d\mu$, and the latter vanishes, as it is the only term in (6-1) that is not J-invariant.

Note that f cannot be constant on a nonempty open subset of M without being constant everywhere in M, by a real-analyticity argument stemming from [Ivey 1996]. Hence the same holds for θ, because we see from the previous paragraph that constancy of θ on a nonempty open set implies the same for f.

Proposition 6.1. *Assume g is Kähler and conformal to a gradient Ricci soliton in dimension $n \geq 4$ with θ nonconstant. If*

$$df \wedge d\tau = 0$$

(*equivalently*, $d\mu \wedge d\theta = 0$), *then g satisfies on an open dense set a Ricci–Hessian equation of the form*

$$\alpha \nabla d\sigma + \mathrm{Ric} = \gamma g, \tag{6-2}$$

for appropriate functionally dependent functions α and σ.

In fact, in the set where $d\theta \neq 0$, choose any function t of θ with $dt \neq 0$, so that θ and μ become functions of t, on some interval of the variable t. For the moment, t is not further specified. Denoting the derivative with respect to t by $(\dot{\ })$, we have

$$\nabla d\theta + d\mu \odot d\psi = \dot{\theta} \nabla dt + [\ddot{\theta} + 2\dot{\mu}\dot{\theta} - (n-2)\dot{\mu}^2]\, dt \odot dt. \tag{6-3}$$

Next, we choose a function σ of t such that $\dot{\sigma} > 0$ and

$$\ddot{\sigma}/\dot{\sigma} = [\ddot{\theta} + 2\dot{\mu}\dot{\theta} - (n-2)\dot{\mu}^2]/\dot{\theta} \tag{6-4}$$

on the open dense set where $\dot{\theta} \neq 0$. The right-hand side of this equation is given, so that this stipulation amounts to the requirement that an (easily solvable) ODE holds for σ.

We now fix $t = \sigma$. For this choice, (6-4) becomes

$$\ddot{\theta} + 2\dot{\mu}\dot{\theta} - (n-2)\dot{\mu}^2 = 0, \tag{6-5}$$

which holds on the image under σ of an open dense set, namely the intersection of the noncritical set of σ, with points where $\dot{\theta} \neq 0$. It follows from (6-5) and (6-3) that $\nabla d\theta + d\mu \odot d\psi = \alpha \nabla d\sigma$, with $\alpha = \dot{\theta}$. This translates the first of equations (6-1) into a Ricci–Hessian equation.

We now record some relations that will be used in the next theorem, with assumptions as in Proposition 6.1. Let $Q = g(\nabla\sigma, \nabla\sigma)$, $Y = \Delta\sigma$ and s be the scalar curvature of g. First, from (6-1),

$$\gamma = \lambda e^{-2\mu} - \dot{\mu} Y + (\alpha\dot{\mu} - \ddot{\mu}) Q, \tag{6-6}$$

as $\Delta\mu = \dot{\mu} Y + \ddot{\mu} Q$ and $g(\nabla\theta, \nabla\mu) = \alpha\dot{\mu} Q$. Next, we have

$$\alpha Y + s = n\gamma, \tag{6-7i}$$

$$\alpha dY + Y\dot{\alpha} d\sigma + ds = n d\gamma, \tag{6-7ii}$$

$$\alpha dY + \dot{\alpha} dQ + ds = 2d\gamma, \tag{6-7iii}$$

$$\alpha dQ - dY = 2\gamma d\sigma. \tag{6-7iv}$$

These equations are obtained in succession by taking the g-trace of (6-2); forming the d-image of (6-7i); finally, applying twice the divergence operator 2δ and, separately, interior multiplication by $\nabla\sigma$, i.e., $2\imath_{\nabla\sigma}$, to (6-2) and using Proposition 2.3 and the Bianchi relation $2\delta\,\mathrm{Ric} = ds$.

Further relations are obtained by subtracting (6-7iii) from (6-7ii), then applying $\cdots \wedge d\sigma$ to (6-8i), d to (6-7iv) and d followed by $\cdots \wedge d\sigma$ to (6-6), which yield

$$Y\dot{\alpha}\, d\sigma - \dot{\alpha}\, dQ = (n-2)\, d\gamma, \tag{6-8i}$$

$$\dot{\alpha}\, d\sigma \wedge dQ = (n-2)\, d\gamma \wedge d\sigma, \tag{6-8ii}$$

$$\dot{\alpha}\, d\sigma \wedge dQ = 2\, d\gamma \wedge d\sigma, \tag{6-8iii}$$

$$d\gamma \wedge d\sigma = (\alpha\dot{\mu} - \ddot{\mu})\, dQ \wedge d\sigma - \dot{\mu}\, dY \wedge d\sigma. \tag{6-8iv}$$

We can now state the following partial classification theorem.

Theorem 6.2. *Let g be a Kähler metric conformal to a gradient Ricci soliton $\hat{g}$ on a manifold M of dimension $n \geq 4$, so that equations* (3-2) *and* (6-1) *hold. If $df \wedge d\tau = 0$* (*equivalently*, $d\mu \wedge d\theta = 0$), *then one of the following must occur*:

(i) *g is a Kähler–Ricci soliton.*

(ii) *g satisfies a Ricci–Hessian equation*, *and if it is standard*, *g is an SKR metric.*

(iii) *$n = 4$ and $\hat{g}$ is an Einstein metric.*

(iv) *$n = 4$ and $\hat{g}$ is a non-Einstein steady gradient Ricci soliton* ($\lambda = 0$).

The Ricci–Hessian equation in (ii) *holds on an open dense set.*

After proving this theorem, we address its relation to various known examples.

Proof. If θ is constant, we have seen g is Kähler–Einstein, a special case of (i). Assume from now on that θ is nonconstant. Then by Proposition 6.1, g satisfies the Ricci–Hessian equation (6-2) on an open dense set.

When α is constant, so is γ, by (6-8i) and thus (6-2) gives (i). Next, we assume in the rest of this proof that α is nonconstant.

If $n > 4$ (or, $dQ \wedge d\sigma = 0$ everywhere), then $d\gamma \wedge d\sigma = 0$, as verified by subtracting (6-8iii) from (6-8ii) (or, using (6-8iii)). If the Ricci–Hessian equation is standard, taking into consideration that $d\alpha \wedge d\sigma = 0$ because $\alpha = \dot{\theta}$, Proposition 5.1 implies that (ii) holds.

So assume $n = 4$ and $dQ \wedge d\sigma \neq 0$ somewhere in M (and, consequently, almost everywhere, by an argument involving real-analyticity, valid in dimension four). By (6-7iv), (6-8iii) and (6-8iv),

$$(\dot{\alpha} + 2\alpha\dot{\mu} - 2\ddot{\mu})\, dQ - 2\dot{\mu} dY \quad \text{and} \quad 2\dot{\mu}(dY - \alpha dQ) \tag{6-9}$$

are both functional multiples of $d\sigma$. Adding these two relations, we conclude that $(\dot{\alpha} - 2\ddot{\mu})\, dQ \wedge d\sigma = 0$, so that (6-5) with $n = 4$ gives $\dot{\alpha} = 2\ddot{\mu}$ and

$$\alpha = 2(\dot{\mu} + p), \tag{6-10i}$$

$$2\dot{\alpha} + \alpha^2 = 4p^2, \tag{6-10ii}$$

$$4(\alpha\dot{\mu} - \ddot{\mu}) = (3\alpha + 2p)(\alpha - 2p), \tag{6-10iii}$$

for a constant p, where (6-10i) is obtained by integration, (6-10ii) using (6-10i) and (6-5) with $n = 4$, while (6-10iii) follows from (6-10i) and (6-10ii) by algebraic manipulations that use again $\dot{\alpha} = 2\ddot{\mu}$. Also, as $\dot{\theta} = \alpha$,

$$\dot{f} = 2p, \tag{6-11i}$$

$$p[e^{2\mu}(Y - \alpha Q) + 2\lambda\sigma] = \text{constant}. \tag{6-11ii}$$

In fact, differentiating the relation $\theta = f + (n-2)\mu$ with $n = 4$ and (6-10i) give (6-11i). Thus, f equals $2p\sigma$ plus a constant. Hence $\Delta f = 2pY$, and (6-11ii) follows from (3-6ii) and (6-10i). If $p = 0$ then f is constant, and this, by the soliton equation (3-1i), implies (iii).

Suppose, finally, that $p \neq 0$ while $n = 4$ and $dQ \wedge d\sigma \neq 0$ somewhere. As a consequence of (6-8i) and (6-10ii),

$$4\, d\gamma = (4p^2 - \alpha^2)(Y d\sigma - dQ). \tag{6-12}$$

On the other hand, (6-6), (6-10i) and (6-10iii) give

$$4\gamma = 4\lambda e^{-2\mu} + (\alpha - 2p)[(\alpha + 2p)Q + 2(\alpha Q - Y)]. \tag{6-13}$$

Since $p \neq 0$, (6-11ii) yields $\alpha Q - Y = e^{-2\mu}(2\lambda\sigma - b)$ for some constant b, so that (6-13) and (6-12) become

$$4\gamma = e^{-2\mu}[4\lambda + (2\lambda\sigma - b)(2\alpha - 4p)] + (\alpha^2 - 4p^2)Q, \tag{6-14i}$$

$$4\, d\gamma = (4p^2 - \alpha^2)[\alpha Q d\sigma - e^{-2\mu}(2\lambda\sigma - b)d\sigma - dQ]. \tag{6-14ii}$$

Thus $(4p^2-\alpha^2)[\alpha Q d\sigma - e^{-2\mu}(2\lambda\sigma - b)d\sigma]$ equals the sum of $Qd(\alpha^2-4p^2)$ and $d[e^{-2\mu}(4\lambda+(2\lambda\sigma-b)(2\alpha-4p))]$, since both expressions coincide with $4d\gamma+(4p^2-\alpha^2)\,dQ$, which for the former is clear from (6-14ii), and for the latter follows if one applies d to (6-14i). This equation yields $4e^{-2\mu}(2\lambda\sigma-b)(2p-\alpha)\alpha=0$, as seen by evaluating these expressions via the first two parts of (6-10), and subtracting the former expression from the latter. As we are assuming α is not constant, it follows necessarily that λ (and b) must be zero. This gives (iv), completing the proof. □

We remark on the relation of the four possible outcomes in this theorem to known examples. Many examples of Ricci solitons that are Kähler have been described in the literature (see for instance [Koiso 1990; Cao 1996; Pedersen et al. 1999; Wang and Zhu 2004; Dancer and Wang 2011]), and they are all examples of outcome (i), with constant conformal factor τ. A glance at the proof of Theorem 6.2 shows that the door is left open for another possibility. Namely, when θ is nonconstant but $\alpha=\dot\theta$ is constant, equation (6-5) yields that either $\mu=\log\tau$ is constant or $\dot\mu$ is constant, so that τ is an exponential in an expression affine in σ. We do not know if there exists a corresponding example of a gradient Kähler–Ricci soliton nontrivially conformal to a gradient Ricci soliton. The case of Einstein metrics conformal to other Einstein metrics is classical. On non-Einstein gradient Ricci solitons conformal to other such solitons, see [Jauregui and Wylie 2015; Maschler 2015].

Concerning the SKR metric option in possibility (ii), there are known examples of SKR metrics nontrivially conformal to Ricci solitons. Such metrics include, up to biholomorphic isometry, all Kähler conformally Einstein ones in dimension $n>4$ [Derdzinski and Maschler 2003; 2006; 2005]. Regarding SKR metrics conformal to non-Einstein gradient Ricci solitons, examples were constructed in [Maschler 2008] and [Derdziński 2012]. The former examples are special among those of the latter, as for them the conformal factor τ, rather than some function σ of it, is a Killing potential, and, more importantly, the soliton is itself Kähler with respect to another complex structure.

Note that the characteristics of the spaces that admit SKR metrics are fairly restrictive, in that they are quite specific holomorphic line bundles (if the base is not Ricci flat) over a base that is Kähler–Einstein (if $n>4$, see Section 5). Thus many of the later examples of Ricci solitons, Kähler or not, such as the cohomogeneity one metrics on vector bundles over a product of Fano Kähler–Einstein manifolds [Dancer and Wang 2011; 2009], are not SKR or conformal to SKR metrics. On the other hand, the recent examples of Stolarski [2015] live on exactly the right type of space, and it is an interesting question whether his metrics are conformal to SKR metrics.

Note that the examples in [Maschler 2008], are of a type first considered by Koiso [1990] and Cao [1996]. They, along with those in [Derdziński 2012], are also irreducible. Although the development of reducible SKR metrics runs in parallel to that of the irreducible ones, with somewhat simpler formulas, and a simpler classification of conformally Einstein such metrics, the theory of reducible SKR metrics conformal to non-Einstein Ricci solitons is currently underdeveloped, compared with the irreducible case, perhaps because the eigenfunction ϕ of the Hessian in the Ricci–Hessian equation is then identically zero, and so equations (5-2) do not hold. But see also Remark 7.2.

A Kähler conformally Einstein metric of the type given in possibility (iii) is not an SKR metric, since the latter must satisfy $dQ \wedge d\sigma = 0$, a relation that, as the proof of Theorem 6.2 shows, does not hold in this case. Instead, one has a relevant example on the two-point blowup of $\mathbb{CP}^2$, namely the Chen–LeBrun–Weber metric [2008].

Possibility (iv) is perhaps the least expected. We do not know if there are metrics of this type, and this constitutes an interesting question. The stipulation that the soliton $\hat{g}$ is steady brings to mind the four-dimensional version of the examples in [Buzano et al. 2015], which was already considered in [Ivey 1994]. However, the condition $dQ \wedge d\sigma \neq 0$ that must be satisfied points more towards a metric like that of (iii) rather than to a bundle-like metric. But unlike case (iii), such a metric cannot occur on a compact manifold, as it is well known that compact manifolds do not admit non-Einstein steady gradient Ricci solitons (see [Ivey 1993]).

7. Quasi-solitons

Many of the original examples of gradient Ricci solitons arise as warped products over a one-dimensional base (see for instance [Chow et al. 2007]). We consider here the case of an arbitrary base.

Let $\bar{g}$ be a warped product (gradient Ricci) soliton metric on a manifold $M = B \times F$, so that

$$\bar{g} = g_B + \ell^2 g_F := g + \ell^2 g_F, \quad \overline{\mathrm{Ric}} + \bar{\nabla} df = \lambda \bar{g}, \tag{7-1}$$

where ℓ is (the pullback of) a function on the base B and λ is constant. When $\bar{g}$ is Einstein, the base metric $g = g_B$ is often called quasi-Einstein. In the setting of (7-1), g_B will be a special case of what we call a *quasi-soliton* metric. The latter is defined as a metric g satisfying (7-2i) below, for some functions f and ℓ and some constant λ. There, and in what follows, we drop the subscript B in the notation for g_B-dependent quantities.

Proposition 7.1. *With notation as above, the soliton equation for* $\bar{g}$ (see (7-1)) *is equivalent to the system*

$$\mathrm{Ric} - \frac{k}{\ell} \nabla d\ell + \nabla df = \lambda g, \quad k = \dim(F), \tag{7-2i}$$

$$\mathrm{Ric}_F = \nu g_F, \tag{7-2ii}$$

(7-2iii) *where* $\nu = -\ell d_{\nabla f} \ell + \ell^2 \ell^{\#} + \lambda \ell^2$, *for* $\ell^{\#} = \ell^{-1} \Delta \ell + (k-1) \ell^{-2} |\nabla \ell|^2$.

In particular the fiber metric is Einstein if $\dim(F) > 2$, and f turns out to be a function with vanishing fiber covariant derivative (see below), so that we regard it as a function on B. Unlike the quasi-Einstein case [Kim and Kim 2003], the scalar equation on the left in (7-2iii), with ν a constant, does not follow from (7-2i).

Proof. To derive the equations, we need the well-known Ricci curvature formulas for warped products (see [O'Neill 1983]), and additionally, similar equations for the Hessian of f. For the latter we use the covariant derivative formulas for warped products, together with the known fact that for a C^1 function defined on the base, the gradient of its pullback equals the pullback of its base gradient.

Let, x, y denote lifts of vector fields on B, and u, v lifts of vector fields on F. Then

$$\begin{aligned}
&\bar\nabla_x y \text{ is the lift of } \nabla_x y \text{ on } B,\\
&\bar\nabla_x v = \bar\nabla_v x = d_x \log(\ell) v,\\
&[\bar\nabla_v w]^F \text{ is the lift of } \nabla^F_v w \text{ on } F,\\
&[\bar\nabla_v w]^B = -\bar g(v, w)\nabla \log \ell.
\end{aligned} \tag{7-3}$$

Hence,

$$\begin{aligned}
\bar\nabla df(x, y) &= \bar g\big(\bar\nabla_x(\bar\nabla f)^B, y\big) + \bar g\big(\bar\nabla_x(\bar\nabla f)^F, y\big)\\
&= g\big(\nabla_x(\bar\nabla f)^B, y\big) + \bar g\big(d_x \log \ell(\bar\nabla f)^F, y\big) = g\big(\nabla_x(\bar\nabla f)^B, y\big),\\
\bar\nabla df(x, v) &= \bar g\big(\bar\nabla_x(\bar\nabla f)^B, v\big) + \bar g\big(\bar\nabla_x(\bar\nabla f)^F, v\big) = \ell d_x \ell g_F\big((\bar\nabla f)^F, v\big),\\
\bar\nabla df(v, w) &= \bar g\big(\bar\nabla_v(\bar\nabla f)^B, w\big) + \bar g\big(\bar\nabla_v(\bar\nabla f)^F, w\big)\\
&= d_{(\bar\nabla f)^B}(\log \ell)\bar g\big(v, w\big) + \bar g\big(\nabla^F_v(\bar\nabla f)^F, w\big) - \bar g\big(v, (\bar\nabla f)^F\big)\bar g\big(\nabla \log \ell, w\big)\\
&= \ell d_{(\bar\nabla f)^B}(\ell) g_F\big(v, w\big) + \ell^2 g_F\big(\nabla^F_v(\bar\nabla f)^F, w\big).
\end{aligned}$$

We combine these with the Ricci curvature formulas

$$\begin{aligned}
&\mathrm{Ric}(x, y) = \mathrm{Ric}_B(x, y) - \Big(\frac{k}{\ell}\Big)\nabla d\ell(x, y),\\
&\mathrm{Ric}(x, v) = 0,\\
&\mathrm{Ric}(v, w) = \mathrm{Ric}_F(v, w) - \ell^\# g(v, w).
\end{aligned} \tag{7-4}$$

We now notice that the soliton equation applied to x and v implies that $(\bar\nabla f)^F = 0$ so that f can be regarded as the pullback of a function on B. This readily gives equations (7-2). □

Remark 7.2. The structure (7-1) above can at times give an example of a Ricci soliton which is conformally Kähler. Namely, $\tilde g = \ell^{-2}\bar g$ is clearly a product metric, and if, for example, $\dim B = 2$, so that the quasi-Einstein metric g_B is Kähler with respect to some complex structure on B, while g_F is chosen to be Kähler–Einstein on F, then $\tilde g$ is Kähler, reducible and conformal to a Ricci soliton.

In analogy with the previous section, we will be considering quasi-soliton metrics for which f and ℓ are functionally dependent, that is,

$$df \wedge d\ell = 0. \tag{7-5}$$

We call such metrics *special quasi-soliton* metrics.

It is known that Kähler quasi-Einstein metrics which are not Einstein do not exist on a compact manifold, and in general must be certain Riemannian product metrics [Case et al. 2011]. Similarly we show:

Theorem 7.3. *Let g be a Kähler special quasi-soliton metric on a manifold M of dimension at least four. Then g satisfies a Ricci–Hessian equation on an open set. If this equation is standard, then g is a Riemannian product there. If the dimension is greater than four, then one of the factors in this product is a Kähler–Einstein manifold of codimension two.*

Proof. As the quasi-soliton metric is special, we have $\nabla df = f'\nabla d\ell + f'' d\ell \otimes d\ell$, where the prime denotes differentiation with respect to ℓ. Then (7-2i) becomes

$$\mathrm{Ric} + \Big(f' - \frac{k}{\ell}\Big)\nabla d\ell + f'' d\ell \otimes d\ell = \lambda g. \tag{7-6}$$

In analogy with Proposition 6.1, we introduce a function σ with $d\ell \wedge d\sigma = 0$ and rewrite the special quasi-Einstein equation (7-6) as

$$\mathrm{Ric} + \tilde{\alpha}\ell'\nabla d\sigma + (\tilde{\alpha}\ell'' + f''\ell'^2)\, d\sigma \otimes d\sigma = \lambda g, \tag{7-7}$$

for $\tilde{\alpha} = f'(\ell) - k/\ell$, with the convention that primes on ℓ represent differentiations with respect to σ, while primes on f still represent differentiations with respect to ℓ. The restriction on the open set where an ODE analogous to (6-4) holds is $\alpha := \tilde{\alpha}\ell' \neq 0$ (corresponding to $\dot{\theta} \neq 0$ in Proposition 6.1). On that set, equation (7-7) becomes a Ricci–Hessian equation of the form

$$\mathrm{Ric} + \alpha\nabla d\sigma = \lambda g, \quad \alpha = \tilde{\alpha}\ell',$$

provided we choose σ so that the differential equation

$$\tilde{\alpha}\ell'' + f''\ell'^2 = 0 \tag{7-8}$$

also holds.

Assuming the Ricci–Hessian equation is standard, Proposition 5.1 now shows that g is an SKR metric on the open set described above. If g is irreducible, the theory of SKR metrics gives the two equations (5-2), which now take the form

$$(\sigma - c)^2\phi'' + (\sigma - c)[m - (\sigma - c)\alpha]\phi' - m\phi = K, \tag{7-9i}$$

$$-(\sigma - c)\phi'' + [\alpha(\sigma - c) - (m+1)]\phi' + \alpha\phi = \lambda, \tag{7-9ii}$$

where ϕ is defined pointwise as the eigenvalue of the Hessian of σ, mentioned in Section 5.

Divide (7-9i) by $\sigma - c$, add to (7-9ii) and multiply both sides of the resulting equality by -1, to obtain

$$\phi' + \Big(\frac{m}{\sigma - c} - \alpha\Big)\phi = -\lambda - \frac{K}{\sigma - c}. \tag{7-10}$$

We will apply Lemma 5.2 to the system consisting of (7-9i) (whose coefficients we now call A, B, C, D) and (7-10) (with the obvious p and q). According to the lemma, the solution ϕ is the ratio $(D - A(q' - pq) - Bq)/(A(p^2 - p') - Bp + C)$, if the denominator is nonzero. But one easily computes that $D - A(q' - pq) - Bq = 0$. However, as mentioned in Section 5, the function ϕ is nowhere zero on the set where $d\sigma \neq 0$ when g is irreducible. Hence the only possibility is that $A(p^2 - p') - Bp + C$ vanishes identically. But it is easily seen from the definitions of A, B, C, p that

$$A(p^2 - p') - Bp + C = \alpha'(\sigma - c)^2.$$

We conclude that α is constant, so g is additionally a gradient Ricci soliton. Writing this condition explicitly we get, with primes now denoting solely differentiation with respect to σ,

$$(f \circ \ell)' - k\frac{\ell'}{\ell} = b,$$

where b is constant. But equation (7-8) can also be written as

$$(f \circ \ell)'' - k\frac{\ell''}{\ell} = 0.$$

Differentiating the first of these two equations and combining it with the second shows that ℓ is constant, hence g is Einstein. But this means $\alpha \equiv 0$, contradicting that the Ricci–Hessian equation for g is standard. Hence g must be reducible. The structure of the Riemannian product constituting g follows from SKR theory. $\square$

Next we consider the problem of whether quasi-soliton metrics can be conformally Kähler. This is certainly possible for quasi-Einstein metrics (see [Maschler 2011; Batat et al. 2015]). We have the following result, analogous in form and in proof to the previous one, though it requires more assumptions and is computationally more difficult.

Theorem 7.4. *Let M be a manifold of dimension $n = 2m > 4$ and g an irreducible Kähler metric on M conformal to a special quasi-soliton $\hat{g} = g/\tau^2$ having warping function ℓ, potential f and appropriate constants k and λ. Assume τ is a Killing potential for g and $d\ell \wedge d\tau = 0$. Then g satisfies a Ricci–Hessian equation. If the latter is standard, then $\hat{g}$ is quasi-Einstein.*

Proof. Being a special quasi-soliton, $\hat{g}$ satisfies equation (7-6), i.e.,

$$\widehat{\mathrm{Ric}} + \mu\widehat{\nabla} d\ell + \chi\, d\ell \otimes d\ell = \lambda \hat{g}, \tag{7-11}$$

for $\mu = f'(\ell) - k/\ell$ and $\chi = f''(\ell)$.

Using (2-3) and (2-2i), we see that g satisfies

$$\begin{aligned}\mathrm{Ric} + (n-2)\tau^{-1}\nabla d\tau + (\tau^{-1}\Delta\tau - (n-1)\tau^{-2}Q)g \\ + \mu(\nabla d\ell + 2\tau^{-1} d\tau \odot d\ell - \tau^{-1} g(\nabla\tau, \nabla\ell)g) + \chi\, d\ell \otimes d\ell = \lambda\tau^{-2} g,\end{aligned} \tag{7-12}$$

with $Q = g(\nabla\tau, \nabla\tau)$. Since $d\ell \wedge d\tau = 0$, writing $d\ell = \ell'(\tau)d\tau$ and rearranging terms, we rewrite this equation as

$$\begin{aligned}\mathrm{Ric} + \alpha\nabla d\tau + (\mu(\ell'' + 2\tau^{-1}\ell') + (\ell')^2\chi)\, d\tau \otimes d\tau \\ = (\lambda\tau^{-2} - \tau^{-1}\Delta\tau + (\alpha + \tau^{-1})\tau^{-1}Q)g \quad \text{for } \alpha = (n-2)\tau^{-1} + \mu\ell'.\end{aligned} \tag{7-13}$$

As g is Kähler and τ is a Killing potential, the term with $d\tau \otimes d\tau$ is the only one which is not J-invariant. Hence its coefficient must vanish:

$$\mu(\ell'' + 2\tau^{-1}\ell') + (\ell')^2\chi = 0. \tag{7-14}$$

As a result, equation (7-13) is Ricci–Hessian:

$$\mathrm{Ric} + \alpha\nabla d\tau = \gamma g, \quad \text{where } \gamma = \lambda\tau^{-2} - \tau^{-1}\Delta\tau + (\alpha + \tau^{-1})\tau^{-1}Q. \tag{7-15}$$

Since clearly $d\alpha \wedge d\tau = 0$, and $n > 4$, as mentioned in Section 5, we also have $d\gamma \wedge d\tau = 0$. Under the assumption that the Ricci–Hessian equation is standard, we conclude from Proposition 5.1 that (g, τ) is an SKR metric with τ the special Kähler–Ricci potential. As in the previous theorem, irreducibility of g again implies that two ODEs hold for the horizontal Hessian eigenvalue function ϕ. They are

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K, \tag{7-16i}$$

$$\begin{aligned}-(\tau - c)\phi'' + (\alpha(\tau - c) - (m+1))\phi' + \alpha\phi = \gamma \\ = \lambda\tau^{-2} - \tau^{-1}(2m\phi + 2(\tau - c)\phi') + (\alpha + \tau^{-1})\tau^{-1} 2(\tau - c)\phi,\end{aligned} \tag{7-16ii}$$

where K, c are constants, and we have used formulas (5-3) giving $\Delta\tau$ and Q in terms of ϕ.

Simplifying the second equation, we then replace it by a first-order equation as in the previous theorem, to obtain the equivalent system

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K, \tag{7-17i}$$

$$\begin{aligned}\frac{(\tau - c)(\tau - 2c)}{\tau}\phi' - \Big(\frac{(\tau - c)(\tau - 2c)}{\tau}\alpha + \frac{2(\tau - c)^2 - m\tau(\tau - 2c)}{\tau^2}\Big)\phi \\ = \frac{K\tau^2 + \lambda\tau - \lambda c}{\tau^2}.\end{aligned} \tag{7-17ii}$$

Naming the coefficients A, B, C, D, p, q as before, we now apply Lemma 5.2 to the system (7-17). This time the computation of the two quantities used in the lemma is quite laborious, though still elementary. A symbolic computational program simplifies the result to the following.

$$\begin{aligned} &A(p^2-p')-Bp+C=\frac{(\tau-c)^2((\tau-2c)\tau\alpha'+2(\tau-c)\alpha+2-2m)}{\tau(\tau-2c)},\\ &D-A(q'-pq)-Bq=0. \end{aligned} \tag{7-18}$$

By the lemma and the fact that ϕ is nowhere zero, solutions are only possible if the first expression vanishes identically, so that α solves

$$(\tau-2c)\tau\alpha'+2(\tau-c)\alpha+2-2m=0.$$

The solutions of this equation take the form

$$\alpha=\frac{n-2}{\tau}+\frac{C}{\tau(\tau-2c)}, \tag{7-19}$$

where C is a constant. As (7-15) and (7-16ii) imply that the form of α determines that of γ, we have the following outcome. If $c=0$, the metric g is conformal to a gradient Ricci soliton [Maschler 2008, Proposition 2.4], while if $c\neq 0$ then g is conformal to a quasi-Einstein metric [Maschler 2011][1]. (The case $C=0$ is a special case of both these types, where g is conformally Einstein [Derdzinski and Maschler 2003].)

To rule out the case that $\hat{g}$ is a nontrivial gradient Ricci soliton, we note first that the expression defining α in (7-13), when compared to that in (7-19), results in

$$(f\circ\ell)'-k\frac{\ell'}{\ell}=\frac{C}{\tau(\tau-2c)}.$$

Additionally, equation (7-14) can also be written as

$$(f\circ l)''-k\frac{\ell''}{\ell}+2\Big((f\circ\ell)'-k\frac{\ell'}{\ell}\Big)\tau^{-1}=0.$$

Substituting the first of these equations in the last term of the second, and combining the result with the derivative of the first equation gives, after eliminating $(f\circ\ell)''-k\ell''/\ell$ and rearranging terms,

$$k\frac{\ell'^2}{\ell^2}=\frac{2C}{\tau^2(\tau-2c)}+\Big(\frac{C}{\tau(\tau-2c)}\Big)'=-\frac{2cC}{\tau^2(\tau-2c)^2}.$$

Hence the Ricci soliton case $c=0$ implies that ℓ is constant, so that comparing the two expressions for α again yields $C=0$, i.e., that $\hat{g}$ is Einstein, which is, of course, a special case of the quasi-Einstein condition. $\square$

[1] See (2.3) in that paper, where the quasi-Einstein case is given by $\alpha=(n-2)/\tau+a/(\tau(1+k\tau))$, where a is a constant and $k=-\frac{1}{2c}$. This corresponds to formula (7-19) with $C=-2ac$.

We comment here on the assumption in this theorem that τ is a Killing potential, which is the same assumption that singles out the examples in [Maschler 2008] among those of [Derdziński 2012]. In analogy with the previous theorems, it is possible instead to replace equation (7-13) by a similar equation involving $\nabla d\sigma$ and $d\sigma \otimes d\sigma$, for a function σ of τ that will serve, after choosing it appropriately, as the special Kähler–Ricci potential instead of τ. One can obtain then two differential equations analogous to (7-16) and (7-17) with independent variable σ. However, these equations will involve τ and its derivatives with respect to σ, and this unknown dependency hinders the determination of solutions and the corresponding α. Even if one knew this α as a function of σ, this will not easily shed light on what metric g is conformal to (with conformal factor τ). Finally, without the Killing assumption on τ, it is not clear that a similar result should be expected, as existence of more general conformally Kähler quasi-solitons may occur. This is in analogy with the fact mentioned above, that there do exist conformally Kähler quasi-Einstein metrics.

Appendix: Killing vector fields of the form $w = \tau^2\nabla f$

We consider here the classification problem for Killing fields of the form $w = \tau^2\nabla f$, a form that played an important role in Section 4. In the following τ and f will denote smooth functions on a given manifold.

Proposition A.1. *On a compact manifold, a Killing field of the form $w = \tau^2\nabla f$ must be trivial.*

Proof. First, on a compact manifold ∇f has zeros, hence so does w. Let p be a zero of $w = \tau^2\nabla f$. Since $\nabla w = 2\tau d\tau \otimes \nabla f + \tau^2\nabla d f$, and at a zero either $\tau = 0$ or $\nabla f = 0$, we see that at a zero ∇w equals either zero or $\tau^2\nabla d f$. But in the latter case ∇w is symmetric, yet it is also skew-symmetric as w is a Killing field, hence ∇w must be zero in this case as well. However, a Killing field w is uniquely determined by the values of w and ∇w at one point. As those values are zero at p, we see that w must be the zero vector field. $\square$

Without compactness, we have the following classification for such vector fields.

Theorem A.2. *A Riemannian metric g with a Killing vector field of the form $w = \tau^2\nabla f$ is, near generic points, a warped product with a one-dimensional fiber. If g is also Kähler, it is, near such points, a Riemannian product of a Kähler metric with a surface metric admitting a nontrivial Killing vector field.*

We note here that a surface with a nontrivial Killing vector field can be presented as a warped product with a one-dimensional fiber and base.

Proof. First, the orthogonal complement $\mathcal{H}$ to span(w) is generically $[\nabla f]^\perp$, which is obviously integrable. Next, $\mathcal{H}$ is totally geodesic. This follows immediately since $g(\dot{x}, w)$ is constant for any geodesic $x(t)$ and Killing field w.

By a result going back to [Hiepko 1979] and [Ponge and Reckziegel 1993] (see especially Theorem 3.1 in the survey [Zeghib 2011]), a metric is a warped product if and only if it admits two orthogonal foliations, one totally geodesic and the other spherical. In our case we have just shown the foliation orthogonal to w is totally geodesic. The fibers tangent to $\operatorname{span}(w)$, on the other hand, are certainly totally umbilic, as they are one-dimensional. This is part of the definition of spherical. The other part is that the mean curvature vector is parallel with respect to the normal connection. We now check this.

Let $w' = w/|w|$ be a unit vector parallel to w, defined away from its zeros. The mean curvature vector to the fibers is then, by definition, $n = \nabla_{w'} w'$, which takes values in $\mathcal{H}$. The requirement that $\operatorname{span}(w)$ be spherical amounts to showing that for any $x \in \mathcal{H}$, we have $g(\nabla_w n, x) = 0$. The flow of w certainly preserves itself (as $[w, w] = 0$) and also g and ∇ (as w is Killing). Therefore the flow also preserves $w' = w/\sqrt{g(w, w)}$ and thus also $n = \nabla_{w'} w'$. Hence $[w, n] = 0$, so that

$$\begin{aligned} 2g(\nabla_w n, x) &= 2g(\nabla_n w, x) = g(\nabla_n w, x) - g(n, \nabla_x w) \\ &= -g(w, \nabla_n x) + g(w, \nabla_x n) \\ &= g(w, [x, n]) = 0, \end{aligned}$$

as $\mathcal{H}$ is integrable. This concludes the first part of the proof.

What remains is to classify Kähler warped products with a one-dimensional fiber. Suppose the manifold is given by $M = B \times F$, with F the fiber (an interval). Since the base foliation corresponding to B is totally geodesic, parallel transport along one of its leaves with respect to g is the same as parallel transport with respect to the induced metric on this leaf, and therefore it preserves the tangent spaces to these leaves. It is well known that it also preserves the normal spaces to the leaves; for completeness, we show explicitly that the unit vector field w' perpendicular to the leaves is preserved. If x and y are, as usual, vector fields tangent to the leaves, then $g(w', y) = 0$, so $0 = d_x g(w', y) = g(\nabla_x w', y) + g(w', \nabla_x y) = g(\nabla_x w', y)$ because the leaves are totally geodesic, and similarly $0 = d_x g(w', w') = 2g(\nabla_x w', w')$. So $\nabla_x w'$, being orthogonal to a basis, is zero, i.e., w' is parallel in directions tangent to the leaves.

As g is Kähler, the complex structure J commutes with any ∇_x, so that Jw' is also parallel in leaf directions. But Jw' is itself tangent to leaves of the base foliation. Therefore, by the local de Rham theorem, the induced metric on any leaf splits locally into a Riemannian product so that $B = N \times I$, where the one-dimensional factor I is tangent to Jw', and N is J-invariant, hence has holomorphic (and totally geodesic) leaves in M.

Armed with this information it remains to show that, near generic points,

g is a product of a Kähler metric on N and a local metric of revolution on $I \times F$.

For this we turn to a computation that is based on the formulas (see for example [O'Neill 1983]) for the connection of the warped product metric $g = g_B + l^2 g_F$, where the function l is a (lift of) a function on B. Let t be a nontrivial vector field tangent to F which is projectable onto F. Let $s = Jt$, a vector field tangent to I. Then standard formulas for warped products give

$$\nabla_t t = (\nabla_t t)^B + (\nabla_t t)^F = -|t|^2 \nabla(\log l) + ct, \tag{A-1}$$

with c some function, and the last term takes that form because the fiber is one-dimensional. Next, as s is tangent to I, there is some function h on M such that the vector field hs is projectable onto I. Therefore, again by warped product formulas,

$$\nabla_t(hs) = hs(\log l)t. \tag{A-2}$$

But $\nabla_t(hs) = (d_t h)s + h\nabla_t s = (d_t h)s + hJ\nabla_t t = (d_t h)s - h|t|^2 J\nabla(\log l) + hcs$, by (A-1). Equating this expression with the right-hand side of (A-2) and taking components tangent to N gives $h|t|^2[J\nabla(\log l)]^N = 0$, so that, away from the zeros of h and t, $[J\nabla(\log l)]^N = 0$. Now each tangent space T_pN is J-invariant, so J commutes with the projection to N. Hence $\nabla(\log l)^N = 0$ and so $\nabla(\log l)$ is parallel to s, which means that the warping function l is constant on the leaves of N, and only changes along the fibers associated with I. Thus g is a Riemannian product of the type claimed above. □

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GIDEON MASCHLER
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
CLARK UNIVERSITY
WORCESTER, MA 01610-1477
UNITED STATES
gmaschler@clarku.edu

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CALCULATING GREENE'S FUNCTION VIA ROOT POLYTOPES AND SUBDIVISION ALGEBRAS

KAROLA MÉSZÁROS

Greene's rational function $\Psi_P(x)$ is a sum of certain rational functions in $x = (x_1, \dots, x_n)$ over the linear extensions of the poset P (which has n elements), which he introduced in his study of the Murnaghan–Nakayama formula for the characters of the symmetric group. In recent work Boussicault, Féray, Lascoux and Reiner showed that $\Psi_P(x)$ equals a valuation on a cone and calculated $\Psi_P(x)$ for several posets this way. In this paper we give an expression for $\Psi_P(x)$ for any poset P. We obtain such a formula using dissections of root polytopes. Moreover, we use the subdivision algebra of root polytopes to show that in certain instances $\Psi_P(x)$ can be expressed as a product formula, thus giving a compact alternative proof of Greene's original result and its generalizations.

1. Introduction

Given a poset P on the set $[n] = \{1, \dots, n\}$, Greene's rational function is defined by

$$\Psi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)}\right), \tag{1-1}$$

where $\mathcal{L}(P)$ denotes the set of linear extensions of P and for $w \in \mathcal{L}(P)$ and a function $f(x_1, \dots, x_n)$ we have that $w(f(x_1, \dots, x_n)) = f(x_{w(1)}, \dots, x_{w(n)})$. It was introduced by Greene [1992] in his work on the Murnaghan–Nakayama formula. Boussicault, Féray, Lascoux and Reiner [Boussicault et al. 2012] showed that

$$\Psi_P(\boldsymbol{x}) = s(K_P^{\mathrm{root}}; \boldsymbol{x}), \tag{1-2}$$

where

$$K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid i <_P j\} = \mathbb{R}_+\{e_i - e_j \mid i \lessdot_P j\} \tag{1-3}$$

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and

$$s(K;\boldsymbol{x}) := \int_K e^{-\operatorname{span}_{\mathbb{R}_+}(\boldsymbol{x},v)}\,dv, \tag{1-4}$$

for K a polyhedral cone in a Euclidean space V with inner product $\operatorname{span}_{\mathbb{R}_+}(\cdot\,,\cdot)$.

Next we explain two important results about calculating $\Psi_P(\boldsymbol{x})$. Further work on $\Psi_P(\boldsymbol{x})$ appeared in [Boussicault 2007; 2009; Boussicault and Féray 2009; Ilyuta 2009].

Greene's theorem. Let P be a *strongly planar* poset, meaning that the Hasse diagram of $P \sqcup \{\hat{0}, \hat{1}\}$ has a planar embedding with all edges directed upward in the plane. For a strongly planar poset P the edges of the Hasse diagram of P dissect the plane into bounded regions ρ such that the set of vertices of P in the boundary of ρ are two chains starting and ending at the same two elements, $\min(\rho)$ and $\max(\rho)$, respectively. Denote by $b(P)$ the set of bounded regions into which the Hasse diagram of P dissects the plane.

Greene's theorem [Greene 1992]. *For any strongly planar poset P,*

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_{\rho\in b(P)}(x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i<_P j}(x_i - x_j)}. \tag{1-5}$$

Boussicault's, Féray's, Lascoux's and Reiner's theorem. A beautiful theorem appearing in [Boussicault et al. 2012] gives an expression for $\Psi_P(\boldsymbol{x})$ for some posets P whose Hasse diagrams are bipartite graphs in terms of certain lattice paths. The setup is as follows. Let D be a skew Ferrers diagram in English notation, and let us labels its rows from top to bottom by $1, 2, \dots, r$ and its columns from right to left by $1, 2, \dots, c$. See the left of Figure 1. With this labeling the northeasternmost point of D is $(1, 1)$ and the southwesternmost is (r, c). The *bipartite poset P_D* is a poset on the set $\{x_1, \dots, x_r, y_1, \dots, y_c\}$ with order relations $x_i <_P y_j$ if and only if $(i, j) \in D$.

BFLR theorem [Boussicault et al. 2012]. *For any skew diagram D,*

$$\Psi_{P_D}(\boldsymbol{x}) = \sum_\pi \frac{1}{\prod_{(i,j)\in\pi}(x_i - y_j)}, \tag{1-6}$$

where the sum runs over all lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.

Roadmap of the paper. The objective of this paper is to (1) give a combinatorial expression of $\Psi_P(\boldsymbol{x})$ for any poset P, (2) give an alternative proof of the BFLR theorem and (3) generalize Greene's theorem. We accomplish (1) and (2) in Section 2, while we do (3) in Sections 3 and 4. In Sections 3 and 4 we also study the integer point transform of the root cone, which can be seen as a more refined invariant of the cone than Greene's function. The integer point transform of the root cone and generalizations of Greene's theorem were also investigated in [Boussicault et al. 2012]. Our tools will be root polytopes and their subdivision algebras, the latter of which were introduced in [Mészáros 2011] and put to use in [Escobar and Mészáros 2015a; 2015b; Mészáros 2015a; 2015b; 2016a; 2016b; Mészáros and Morales 2015].

2. Greene's function for an arbitrary poset

The purpose of this section is twofold. First we show how to express $\Psi_P(\boldsymbol{x})$ for any poset P in terms of $\Psi_P(\boldsymbol{x})$ for posets P whose Hasse diagrams are alternating graphs. Then we give an expression for $\Psi_P(\boldsymbol{x})$ for posets whose Hasse diagrams are alternating graphs, thereby also obtaining an expression for $\Psi_P(\boldsymbol{x})$ for any poset P. Finally, we show that for certain posets P whose Hasse diagrams are bipartite graphs we can write $\Psi_P(\boldsymbol{x})$ as a nice summation formula. The latter result originally appeared in the work of Boussicault, Féray, Lascoux and Reiner [Boussicault et al. 2012], who used triangulations of order polytopes in their proof. We phrase our proof in terms of root polytopes. The point of view of this paper is that (dissections of) root polytopes (and the root cone) are the unifying approach to the calculation of $\Psi_P(\boldsymbol{x})$.

A *root polytope* (of type A_{n-1}) is the convex hull of the origin and some of the points $e_i - e_j$ for $1 \le i < j \le n$. Given a graph G on the vertex set $[n]$ we associate to it the root polytope

$$\tilde{Q}_G = \operatorname{ConvHull}\bigl(0, e_i - e_j \mid (i, j) \in E(G),\ i < j\bigr). \tag{2-1}$$

It can be seen that $\tilde{Q}_G$ is a simplex if and only if G is acyclic and to emphasize this we sometimes denote $\tilde{Q}_G$ for acyclic graphs G by $\tilde{\Delta}_G$. In the proof of Lemma 4.2 we will also use the notation

$$\Delta_F = \operatorname{ConvHull}\bigl(e_i - e_j \mid (i, j) \in E(F),\ i < j\bigr) \tag{2-2}$$

for a forest F.

The posets P we work with in this section are on the set $[n]$ and they are labeled naturally; that is to say that if $i <_P j$ then $i < j$ in the order of natural numbers. Note that this does not pose a restriction on the results, it only makes them easier to state. Denote by $\mathcal{H}(P)$ the graph of the Hasse diagram of P. The directed transitive closure of a graph H is denoted by $\overline{H}$, and it is the graph on vertex set $V(G)$ with edges $(i, j) \in \overline{H}$ if there is an increasing path from i to j in H.

$\Psi_P(\boldsymbol{x})$ in terms of alternating posets. This subsection explains how to reduce the computation of $\Psi_P(\boldsymbol{x})$ to the computation of $\Psi_P(\boldsymbol{x})$ for posets P whose Hasse diagram is an alternating graph. A graph G on the vertex set $[n]$ is called *alternating* if there are no edges (i, j) and (j, k) in it with $i < j < k$. We call a poset on $[n]$ an *alternating poset* if its Hasse diagram is an alternating graph.

Proposition 2.1. *For any naturally labeled poset P on $[n]$ we can write*

$$\Psi_P(\boldsymbol{x}) = \sum_{L,R} \Psi_{P_{L,R}}(\boldsymbol{x}), \tag{2-3}$$

where the summation runs over all L, R such that $L \sqcup R = [n]$, and

$$G_{L,R} = \big([n], \{(i, j) \in E(G) \mid i \in L,\ j \in R,\ i < j\}\big)$$

is a connected graph, where $G = \overline{\mathcal{H}(P)}$. Furthermore, $\mathcal{H}(P_{L,R}) = G_{L,R}$ for a naturally labeled alternating poset $P_{L,R}$.

Proof. Recall that $\Psi_P(\boldsymbol{x}) = s(K_P^{\mathrm{root}}; \boldsymbol{x})$. If $K_P^{\mathrm{root}} = \bigcup_{i=1}^{l} K_i$ for interior disjoint cones K_i with $i \in [l]$ then $s(K_P^{\mathrm{root}}; \boldsymbol{x}) = \sum_{i=1}^{l} s(K_i; \boldsymbol{x})$. If $K_i = K_{P_i}^{\mathrm{root}}$ for some posets P_i with $i \in [l]$ then $\Psi_P(\boldsymbol{x}) = \sum_{i=1}^{l} \Psi_{P_i}(\boldsymbol{x})$. Therefore, to prove (2-3), it suffices to show that $K_P^{\mathrm{root}} = \bigcup_{L,R} K_{P_{L,R}}^{\mathrm{root}}$, where the union runs over all L, R such that $L \sqcup R = [n]$, $G_{L,R}$ is a connected graph ($G = \overline{\mathcal{H}(P)}$) and $\mathcal{H}(P_{L,R}) = G_{L,R}$ for a naturally labeled poset $P_{L,R}$.

Since $K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid i <_P j\}$, if $\tilde{Q}_G = \bigcup \tilde{Q}_{G_{L,R}}$ (the $\tilde{Q}_{G_{L,R}}$ are interior disjoint), where the union runs over all L, R such that $L \sqcup R = [n]$, and $G_{L,R}$ is a connected graph, then we also obtain that $K_P^{\mathrm{root}} = \bigcup_{L,R} K_{P_{L,R}}^{\mathrm{root}}$ for interior disjoint cones $K_{P_{L,R}}^{\mathrm{root}}$. The equation $\tilde{Q}_G = \bigcup \tilde{Q}_{G_{L,R}}$ follows from [Postnikov 2009, Proposition 13.3] together with the observation that $G = \overline{G}$ for our choice of G. □

We note that the cones $K_{P_{L,R}}^{\mathrm{root}}$ are generally not simplicial. One way to compute $\Psi_{P_{L,R}}(\boldsymbol{x})$ would be to triangulate $K_{P_{L,R}}^{\mathrm{root}}$ into simplicial cones with rays of the form $e_i - e_j$, since for such a cone the following simple lemma gives the value of Greene's function.

Lemma 2.2 [Boussicault et al. 2012]. *The cone K_P^{root} is simplicial if and only if the Hasse diagram of P contains no cycles. In this case it is also unimodular and*

$$\Psi_P(\boldsymbol{x}) = \frac{1}{\prod_{i \lessdot_P j}(x_i - x_j)}.$$

We remark that a proof of Lemma 2.2 different from that given in [Boussicault et al. 2012] follows immediately using the subdivision algebra of root polytopes defined in [Mészáros 2011].

Calculating $\Psi_P(\boldsymbol{x})$ for an alternating poset P. In light of Proposition 2.1, if we can calculate $\Psi_P(\boldsymbol{x})$ for an alternating poset P, then we can in turn calculate $\Psi_P(\boldsymbol{x})$ for any poset P. In this section we accomplish the former, building on the results of Li and Postnikov [2015]. The next paragraph follows the exposition of that paper.

Given an alternating graph G on the vertex set $[n]$, pick a linear order $\mathcal{O}$ on the edges of G. Let T be a spanning tree of G, and let e be an edge that does not belong to T. Let C be the unique cycle contained in the graph $([n], E(T) \cup \{e\})$. Let e^* be the maximal edge in the cycle C in the linear ordering $\mathcal{O}$ of the edges. We say that an edge e is *externally semiactive* if either $e = e^*$ or there is an odd number of edges in C between e and e^*. (Since G is alternating, all cycles in G have an even length.) Let $\mathrm{ext}_G^{\mathcal{O}}(T)$ be the number of externally semiactive edges of G with respect to a spanning tree T.

Theorem 2.3 [Li and Postnikov 2015]. *Given an alternating graph G and a linear ordering $\mathcal{O}$ of its edges, let $\mathcal{T}_G^{\mathcal{O}}$ be the set of spanning trees T with $\mathrm{ext}_G^{\mathcal{O}}(T) = 0$. Then*

$$\tilde{Q}_G = \bigcup_{T \in \mathcal{T}_G^{\mathcal{O}}} \tilde{\Delta}_T, \tag{2-4}$$

where the simplices $\tilde{\Delta}_T$ are interior disjoint.

Corollary 2.4. *For any naturally labeled poset P on $[n]$ we can write*

$$\Psi_P(\boldsymbol{x}) = \sum_{L,R} \sum_{T \in \mathcal{T}_{G_{L,R}}^{\mathcal{O}_{L,R}}} \frac{1}{\prod_{(i,j) \in E(T), i<j} (x_i - x_j)}, \tag{2-5}$$

where the summation runs over all L, R such that $L \sqcup R = [n]$, and

$$G_{L,R} = \big([n], \{(i, j) \in E(G) \mid i \in L,\ j \in R,\ i < j\}\big)$$

is a connected graph, where $G = \overline{\mathcal{H}(P)}$. Furthermore, $\mathcal{O}_{L,R}$ is an arbitrary linear order of the edges of $G_{L,R}$.

Proof. The proof follows from Proposition 2.1, Lemma 2.2 and Theorem 2.3. □

We remark that we obtained Corollary 2.4 from a particular dissection of the root polytope $\mathrm{ConvHull}(0, e_i - e_j \mid e_i <_P e_j)$ into simplices. Such a dissection then induced a dissection of $K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid e_i <_P e_j\}$ into simplicial cones. Since we know that $K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid e_i \lessdot_P e_j\}$, instead of $\mathrm{ConvHull}(0, e_i - e_j \mid e_i <_P e_j)$ one could also dissect $\mathrm{ConvHull}(0, e_i - e_j \mid e_i \lessdot_P e_j)$ into simplices and obtain an expression with fewer terms for $\Psi_P(\boldsymbol{x})$. However, since such a dissection also would not in general yield significantly fewer terms, we find the expression presented in Corollary 2.4 a fine representative of what a general formula for $\Psi_P(\boldsymbol{x})$ for an arbitrary poset P can look like. We devote the next section to particularly nice formulas for $\Psi_P(\boldsymbol{x})$ for special posets P, also demonstrating that in certain instances we can expect the formula presented in Corollary 2.4 to be far better than the formula given in (1-1), although this is not always the case.

An alternative proof of the BFLR theorem. Let P_D be the poset of a connected skew diagram D as in the BFLR theorem. Let G_D be the graph $\mathcal{H}(P_D)$ drawn on a line with vertices from left to right, $x_r, \dots, x_1, y_1, \dots, y_c$, and with edges as arcs above this line. Note that the condition that G_D comes from P_D can be translated into the conditions that G_D is bipartite on parts $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_c\}$ and for each $i \in [r]$, x_i is connected to y_j for $j \in [a_i, b_i]$, $i \in [r]$, where $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_r$ and $[1, c] = \bigcup_{i=1}^{r} [a_i, b_i]$.

Given a drawing of a graph G such that its vertices $v_1, \dots, v_n$ are arranged in this order on a horizontal line and its edges are drawn above this line, we say that G is *noncrossing* if it has no edges (v_i, v_k) and (v_j, v_l) with $i < j < k < l$. A vertex v_i of G is said to be *nonalternating* if it has both an incoming and an outgoing edge; it is called *alternating* otherwise. The graph G is alternating if all its vertices are alternating.

Lemma 2.5. *The root polytope $\tilde{Q}_{G_D}$ decomposes into $\tilde{Q}_{G_D} = \bigcup_T \tilde{\Delta}_T$, where the union runs over all noncrossing alternating trees of G_D and the simplices $\tilde{\Delta}_T$ are interior disjoint.*

Since noncrossing depends on the drawing of the graph it is essential that we remember that we drew G_D with vertices from left to right: $x_r, \dots, x_1, y_1, \dots, y_c$.

Proof of Lemma 2.5. Consider the following ordering $\mathcal{O}$ on the edges of G_D. The edges incident to y_i precede the edges incident to y_j in the ordering $\mathcal{O}$ if $1 \leq i < j \leq c$. Moreover, if edges (x_a, y_k) and (x_b, y_k) are incident to y_k for some $k \in [c]$ with $1 \leq a < b \leq r$, then (x_a, y_k) precedes (x_b, y_k) in the ordering $\mathcal{O}$. We claim that then the spanning trees T of G_D with $\mathrm{ext}_{G_D}^{\mathcal{O}} = 0$ are exactly the noncrossing alternating trees of G_D and then the lemma follows from Theorem 2.3. Indeed, note that given any noncrossing alternating tree T of G_D and an edge $e \in E(G_D) - E(T)$, in the unique cycle C of the graph T with the edge e adjoined, the edge e is always 0 edges away from the largest edge of C in the ordering $\mathcal{O}$. Thus, for any noncrossing alternating tree T of G_D we have $\mathrm{ext}_{G_D}^{\mathcal{O}} = 0$. On the other hand, given a crossing alternating spanning tree T' of G_D (note that all spanning trees of G_D are alternating) let the edges (x_i, y_j) and (x_k, y_l) cross with $k > i$ and $l < j$. Since D is a connected skew diagram, both of the edges (x_k, y_j) or (x_i, y_l) are contained in G_D. Since T' is a spanning tree of G_D, it follows that exactly one of the edges from $\{(x_k, y_j), (x_i, y_l)\}$ is in it. Adjoining the other edge as edge e we see that it is an externally semiactive edge for T, concluding the proof. □

Lemma 2.6. *The noncrossing alternating spanning trees of G_D are in bijection with the lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.*

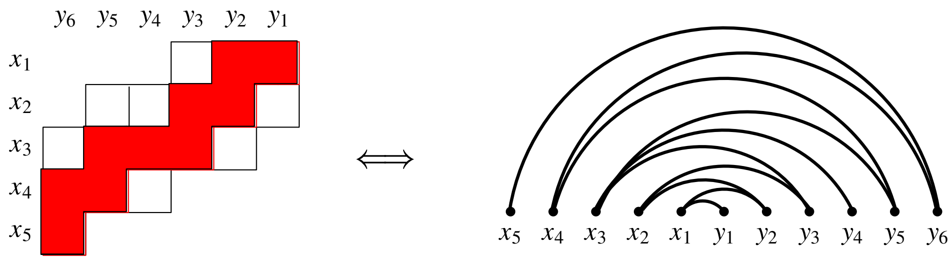

Figure 1. The correspondence between noncrossing alternating spanning trees of G_D and lattice paths from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.

Proof. The bijection is given by the map that takes a noncrossing alternating spanning tree $T = \big(\{x_r, \dots, x_1, y_1, \dots, y_c\}, \{(x_i, y_j) \mid (i, j) \in S(T)\}\big)$ of G_D to the path $\pi = S(T)$. See Figure 1. $\square$

Given a graph G on the vertex set $[n]$ such that if $(i, j) \in E(G)$ then the only increasing path from i to j in G is the edge (i, j) itself, we can define the naturally labeled poset P_G to be one on the set $[n]$ with Hasse diagram given by (the edges of) G.

Corollary 2.7 (BFLR theorem). *For any skew diagram D,*

$$\Psi_{P_D}(\boldsymbol{x}) = \sum_{\pi} \frac{1}{\prod_{(i,j)\in\pi}(x_i - y_j)}, \tag{2-6}$$

where the sum runs over all lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.

Proof. By Lemma 2.5 we have that the cone $K_{P_D}^{\mathrm{root}}$ is triangulated into simplicial cones $K_{P_T}^{\mathrm{root}}$, where the T's run over all noncrossing alternating spanning trees of G_D. By Lemma 2.6 the latter trees are in bijection with lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west, and thus by Lemma 2.2 we obtain the corollary. $\square$

Our proof for Corollary 2.7 is a special case of the proof of Corollary 2.4. We note that the formula for $\Psi_{P_D}(\boldsymbol{x})$ given in Corollary 2.7 is substantially different from the expression given in (1-1). We can see this for example by looking at the number of terms that can appear in each. When D is a diagram in the shape of an $r \times c$ rectangle, then in (1-1) we are summing over all linear extensions of the poset P_D yielding $r!c!$ terms. In comparison, in Corollary 2.7 we have $\binom{r+c-2}{r-1}$ terms corresponding to the lattice paths from $(1, 1)$ to (r, c) inside D. The latter in general can be larger than the former. However, if instead we take D to be the skew shape $D = (n, n-1, \dots, 1) \setminus (n-2, n-3, \dots, 1)$, then in Corollary 2.7 we have a single term and in (1-1) we are summing over all linear extensions of the zigzag poset P_D. In this case the number of terms in (1-1) is larger than $n!(n-1)!$, which is many more than the one term in Corollary 2.7.

3. Lifting Greene's theorem to the subdivision algebra

The objective of this section is to generalize Greene's theorem to a relation in the subdivision algebra of root polytopes. Subdivision algebras of root polytopes were introduced and studied in [Mészáros 2011], where they were used for triangulating root polytopes. Subdivision algebras were also utilized for subword complexes and flow polytopes in [Escobar and Mészáros 2015a; Mészáros 2015a; 2015b; 2016a; 2016b; Mészáros and Morales 2015]. We will see in this section that both Greene's theorem and an analogous one for the integer point transform of the root cone are special cases of a relation in the subdivision algebra.

We begin by explaining how to use subdivision algebras to subdivide root cones K_P^{root}. Since Greene's function of a poset P is a valuation on a root cone K_P^{root} and we know its expression for unimodular root cones, if we triangulate K_P^{root} into unimodular root cones, then we obtain a way to calculate Greene's function of P.

Root cones $\mathcal{C}(G)$ and their subdivisions. We establish a simpler notation for root cones here. For an arbitrary loopless graph G, define the *root cone*

$$\mathcal{C}(G) := \operatorname{span}_{\mathbb{R}_+}\big(e_i - e_j \mid (i, j) \in E(G),\, i < j\big). \tag{3-1}$$

In order for $\mathcal{C}(G)$ and $\mathcal{C}(H)$ to be distinct for distinct graphs G and H, we will mostly consider *good graphs* G, which are loopless graphs such that if there is an increasing path from vertex i to vertex j in G, which is not the edge (i, j), then the edge (i, j) is not present in G. (In particular, G contains no multiple edges.) Given a graph H let $g(H)$ be the unique good graph on the vertex set $V(H)$ such that $\mathcal{C}(H) = \mathcal{C}(g(H))$. The graph $g(H)$ can be obtained from H by repeated removal of edges (i, j) for which there is an increasing path between i and j other than the edge (i, j). In particular, all multiple edges are removed in order to obtain $g(H)$. An important property of root cones is given in the cone reduction lemma below, which can be expressed through reduction rules on graphs, as we now explain.

The *reduction rule for graphs*: given a graph G_0 on the vertex set $[n]$ and $(i, j), (j, k) \in E(G_0)$ for some $i < j < k$, let G_1, G_2, G_3 be graphs on the vertex set $[n]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0) \setminus \{(j,k)\} \cup \{(i,k)\}, \\ E(G_2) &= E(G_0) \setminus \{(i,j)\} \cup \{(i,k)\}, \\ E(G_3) &= E(G_0) \setminus \{(i,j),(j,k)\} \cup \{(i,k)\}. \end{aligned} \tag{3-2}$$

We say that G_0 *reduces* to G_1, G_2 and G_3 under the reduction rules defined by equations (3-2).

For a good graph G we define two edges $(i, j), (j, k) \in E(G)$, $i < j < k$, to be a *good pair of edges* of G if they belong to a common cycle in G, or if neither of them belongs to any cycle in G.

Lemma 3.1 (cone reduction lemma; cf. [Mészáros 2011]). *Given a good graph G_0 let $(i, j), (j, k) \in E(G_0)$ be a good pair of edges of G_0 for some $i < j < k$ and G_1, G_2 as described by equations* (3-2). *Then*

$$\mathcal{C}(G_0) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2) \tag{3-3}$$

and

$$\mathcal{C}(G_3) = \mathcal{C}(G_1) \cap \mathcal{C}(G_2), \tag{3-4}$$

where the cones $\mathcal{C}(G_0), \mathcal{C}(G_1), \mathcal{C}(G_2)$ are of the same dimension and $\mathcal{C}(G_3)$ is a facet of both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$.

For convenience we include a proof of Lemma 3.1 here. It is an adaptation of the proof from [Mészáros 2011], where it was written for acyclic graphs.

Proof. Let the edges of G_0 be $f_1 = (i, j)$, $f_2 = (j, k)$, $f_3, \dots, f_k$. Let $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$ denote the vectors that the edges of G_0 correspond to under the correspondence $\mathrm{v} : (i, j) \mapsto e_i - e_j$, where $i < j$. By equations (3-2),

$$\begin{aligned}
\mathcal{C}(G_0) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)),\\
\mathcal{C}(G_1) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1), \mathrm{v}(f_1)+\mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)),\\
\mathcal{C}(G_2) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1)+\mathrm{v}(f_2), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)),\\
\mathcal{C}(G_3) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1)+\mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)).
\end{aligned}$$

Thus, if $\mathcal{C}(G_0)$ is d-dimensional, so are the cones $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$, while cone $\mathcal{C}(G_3)$ is at least $(d-1)$-dimensional (and at most d-dimensional). We note that $\dim(\mathcal{C}(G_3)) \neq d$ because G_0 is a good graph and f_1 and f_2 are a good pair of edges.

Clearly, $\mathcal{C}(G_1) \cup \mathcal{C}(G_2) \subset \mathcal{C}(G_0)$. Given an expression of a vector $v \in \mathcal{C}(G_0)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$ it satisfies either that the coefficient of $\mathrm{v}(f_1)$ in such an expression is greater than or equal to the coefficient of $\mathrm{v}(f_2)$ in the expression, or it is not. In the former case we see that $v \in \mathcal{C}(G_1)$ and in the latter case $v \in \mathcal{C}(G_2)$. Therefore, $\mathcal{C}(G_0) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2)$.

Clearly, $\mathcal{C}(G_3) \subset \mathcal{C}(G_1) \cap \mathcal{C}(G_2)$. Given an expression of a vector $v \in \mathcal{C}(G_1)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$, the coefficient of $\mathrm{v}(f_1)$ is greater than or equal to the coefficient of $\mathrm{v}(f_2)$. Similarly, given an expression of a vector $v \in \mathcal{C}(G_2)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$, the coefficient of $\mathrm{v}(f_1)$ is less than or equal to the coefficient of $\mathrm{v}(f_2)$. Thus, there is an expression of $v \in \mathcal{C}(G_1) \cap \mathcal{C}(G_2)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$

such that the coefficient of $\mathrm{v}(f_1)$ is equal to the coefficient of $\mathrm{v}(f_2)$. Therefore, $\mathcal{C}(G_1)\cap\mathcal{C}(G_2)\subset\mathcal{C}(G_3)$, leading to $\mathcal{C}(G_1)\cap\mathcal{C}(G_2)=\mathcal{C}(G_3)$. □

The subdivision algebra, Greene's theorem and the integer point transform of a root cone. In this subsection we explain the subdivision algebra and show how it yields a slick proof for Greene's theorem and its generalization.

A graph G can be encoded by the monomial $m[G]=\prod_{(i,j)\in E(G),i<j} x_{ij}$ and the reduction rule going from G_0 to G_1, G_2 and G_3 can be encoded by the equation $x_{ij}x_{jk}=x_{ik}(x_{ij}+x_{jk}+\beta)$. We define the *subdivision algebra* $\mathcal{S}_n$ of root polytopes as the commutative algebra generated by the variables x_{ij}, $1\le i<j\le n$, subject to the relations $x_{ij}x_{jk}=x_{ik}(x_{ij}+x_{jk}+\beta)$ for $1\le i<j<k\le n$.

Let us explain the connection of the subdivision algebra to Greene's function. If we set $\beta=0$, then the relation $x_{ij}x_{jk}=x_{ik}(x_{ij}+x_{jk})$ of $\mathcal{S}_n$ is satisfied by $x_{ij}:=1/(x_i-x_j)$, which are the kind of terms appearing in Greene's function. If instead, we set $\beta=-1$, then the relation $x_{ij}x_{jk}=x_{ik}(x_{ij}+x_{jk}-1)$ of $\mathcal{S}_n$ is satisfied by $x_{ij}:=1/(1-x_i/x_j)$. The latter will play a part in calculating the *integer point transform* $\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})$ *of the root cone* $K_P^{\mathrm{root}}\subset\mathbb{Z}^d$ defined as

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x}):=\sum_{\boldsymbol{m}\in K_P^{\mathrm{root}}\cap\mathbb{Z}^d}\boldsymbol{x}^{\boldsymbol{m}}. \tag{3-5}$$

The function $\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})$ can be seen as a finer invariant of the cone than $\Psi_P(\boldsymbol{x})$, as explained in [Boussicault et al. 2012, Section 2.4]. We note that in that paper the integer point transform $\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})$ is denoted as $H(K_P^{\mathrm{root}};\boldsymbol{X})$ and is referred to as the Hilbert series of the affine semigroup ring of the root cone. We chose to follow the more geometric name and notation of [Beck and Robins 2007, Section 3.2].

We are now ready to prove the following generalization of Greene's theorem via the subdivision algebra, which first appeared in [Boussicault et al. 2012]:

Theorem 3.2 [Boussicault et al. 2012, Corollary 8.10]. *For any* (*connected*) *strongly planar poset P on* $[n]$ *we have*

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})=\frac{\prod_{\rho\in b(P)}(1-x_{\min(\rho)}/x_{\max(\rho)})}{\prod_{i\lessdot_P j}(1-x_i/x_j)} \tag{3-6}$$

and

$$\Psi_P(\boldsymbol{x})=\frac{\prod_{\rho\in b(P)}(x_{\min(\rho)}-x_{\max(\rho)})}{\prod_{i\lessdot_P j}(x_i-x_j)}, \tag{3-7}$$

where ρ runs through all bounded regions of the Hasse diagram.

Proof. Since P is a connected strongly planar poset, it follows that its Hasse diagram is a good graph on the vertex set $[n]$ such that for every cycle C of G the only alternating vertices of C (considered within C), that is vertices that have

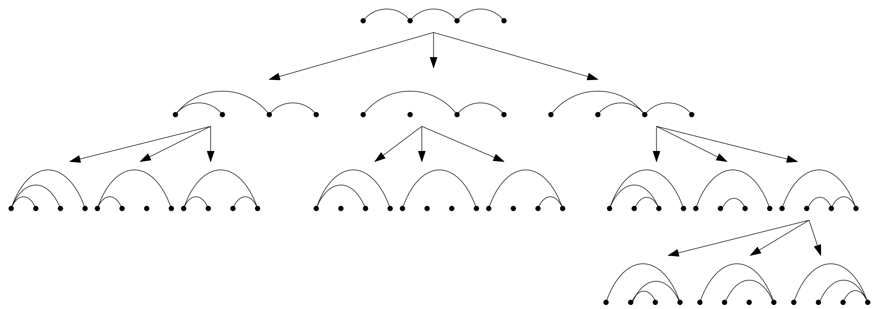

Figure 2. In reducing an increasing path we always pick the topmost leftmost edges in the path and its offsprings to do reductions on. For a graph G_0 the arrow to the left points to G_1, the middle arrow to G_3, and the right arrow to G_2, as in equations (3-2).

only incoming or only outgoing edges, are its minimal and maximal vertices. Thus we have that $K_P^{\text{root}} = \mathcal{C}(G)$ for a good graph G. Note that a root cone $\mathcal{C}(H)$ is unimodular if and only if $g(H)$ is acyclic. We will use the cone reduction lemma to write $\mathcal{C}(G)$ as a union of unimodular cones. Note that the cone reduction lemma applies to good graphs, and thus if we want to repeatedly apply it to the outcome cones $\mathcal{C}(G_i)$, $i \in [3]$, we need to apply it to $g(G_i)$, $i \in [3]$.

We claim that we can apply the cone reduction lemma repeatedly in such a fashion that at the end we have trees $T_1, \ldots, T_k$ (with $n-1$ edges), and forests F^j_{n-i}, $2 \le i \le n-1$, $j \in I_{n-i}$ (for some index sets I_{n-i}), with $n-i$ edges, where $\mathcal{C}(T_1), \ldots, \mathcal{C}(T_k)$ are unimodular cones triangulating $\mathcal{C}(G)$ and the $\mathcal{C}(F^j_{n-i})$ are their intersections.

We now prove the above claim. When G has no cycles, the claim is obvious. Suppose that G has $m > 0$ linearly independent cycles. Fix a strongly planar drawing of P. In it there are m bounded regions, and the boundaries of these regions are m linearly independent cycles in G. Let C be one of these cycles, such that it bounds a region in the drawing of P which is adjacent to the infinite region. The cycle C consists of two increasing paths p and p' from i to j for some $i < j$. Let $p = (i = i_0 \to i_1 \to i_2 \to \cdots \to i_l = j)$ be the path bordering the infinite region in the drawing of P. We can perform consecutive reductions on the edges of the path p and its offsprings, ultimately obtaining all noncrossing alternating forests on the vertices $\{i_0, i_1, \ldots, i_l\}$ containing the edge (i_0, i_l). We do this by picking the topmost leftmost edges that we can do a reduction on in p and its offsprings in the reduction process. See Figure 2 for an illustration. (A proof of the previous claim can be obtained by induction on the length of the path and is given in detail in [Mészáros 2011].) Until we arrive at the aforementioned noncrossing alternating forests on the vertices $\{i_0, i_1, \ldots, i_l\}$ containing the edge (i_0, i_l) all

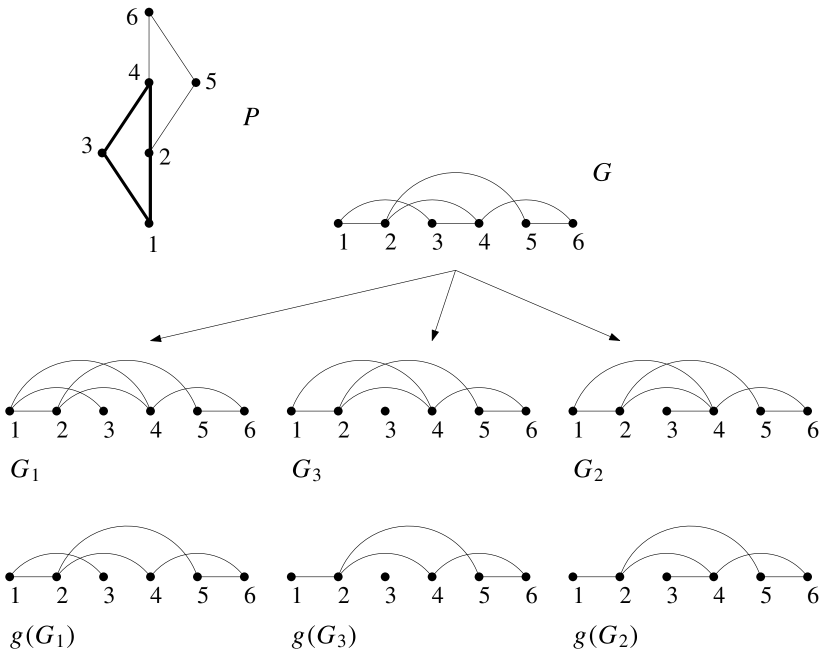

Figure 3. Top left shows a strongly planar drawing of our poset, with the cycle C in bold. The path p is $(1 \to 3 \to 4)$ and p' is $(1 \to 2 \to 4)$. Top right shows the graph G. Below are the graphs G_1, G_3, G_2 obtained by applying the reduction on the topmost leftmost edges of p, which are $(1, 3)$, $(3, 4)$. The last row shows $g(G_1)$, $g(G_3)$, $g(G_2)$ (which are G_1, G_3, G_2 with the edge $(1, 4)$ removed since there is an increasing path $1 \to 2 \to 4$), on which we can keep applying the cone reduction lemma as in the proof of Theorem 3.2.

graphs obtained in this fashion from G are good graphs. We can see that once we obtain the noncrossing alternating forests on the vertices $\{i_0, i_1, \dots, i_l\}$ containing the edge (i_0, i_l) the offspring of G is not good anymore, as there is still p' in it, which is an increasing path between the vertices i_0 and i_l. We need to now remove the edge $(i_0, i_l) = (i, j)$ from all the aforementioned offsprings in order to obtain good graphs and be able to apply the cone reduction lemma further. However, once we remove the edge (i, j) from all these offsprings we will have good graphs with the number of bounded regions one less than it was for G. We can now repeat the same process we just described for each of these graphs and their offsprings until they are all acyclic. We demonstrate the basic idea of this argument in Figure 3.

If we inspect what edges we had to drop in the process to make sure we always apply the cone reduction lemma to good graphs and obtain the acyclic graphs described in the previous paragraph, we find the following relation in the subdivision algebra:

$$m[G] = \prod_{\rho \in b(P)} x_{\min(\rho),\max(\rho)} \Bigg(\sum_{T_i} m[T_i] + \sum_{F^j_{n-i}} \beta^{i-1} m[F^j_{n-i}] \Bigg). \tag{3-8}$$

Note that

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x}) = \Bigg(\sum_{T_i} m[T_i] + \sum_{F^j_{n-i}} (-1)^{i-1} m[F^j_{n-i}] \Bigg) \Bigg|_{x_{ij}=1/(1-x_i x_j{}^{-1})} \tag{3-9}$$

and

$$\Psi_P(\boldsymbol{x}) = \sum_{T_i} m[T_i] \Big|_{x_{ij}=1/(x_i-x_j)}. \tag{3-10}$$

Equations (3-8), (3-9) and (3-10) together with the observations that $x_{ij} = 1/(1 - x_i x_j{}^{-1})$ satisfies $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk} - 1)$ and that $x_{ij} = 1/(x_i - x_j)$ satisfies $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk})$ immediately yield equations (3-6) and (3-7). □

We can see (3-8) is the main theorem of this section, so we bestow it with that title:

Theorem 3.3. *Let $G = \mathcal{H}(P)$ for a naturally labeled connected strongly planar poset P. Then*, *using the notation of the proof of Theorem 3.2*, *we have that*

$$m[G] = \prod_{\rho \in b(P)} x_{\min(\rho),\max(\rho)} \Bigg(\sum_{T_i} m[T_i] + \sum_{F^j_{n-i}} \beta^{i-1} m[F^j_{n-i}] \Bigg)$$

holds in the subdivision algebra.

Both statements of Theorem 3.2 are special cases of Theorem 3.3 as shown in the proof of Theorem 3.2.

4. Generalizing Greene's theorem beyond strongly planar posets

In this section we will examine a special family of posets for which Greene's function factors linearly. These posets were first identified by Boussicault, Féray, Lascoux and Reiner [Boussicault et al. 2012], who proved the aforementioned result by studying the affine semigroup ring of the root cone. We will give a short alternative proof via root polytopes.

We give some definitions following the exposition of [Boussicault et al. 2012]. In a finite poset P, say that a triple of elements (a, b, c) forms a notch of $\vee$ shape (dually, a notch of $\wedge$ shape) if $a \lessdot_P b, c$ (dually, $b, c \lessdot_P a$), and in addition, b, c lie in different connected components of the poset $P \setminus P_{\leq a}$ (dually, $P \setminus P_{\geq a}$). When (a, b, c) forms a notch of either shape in a poset P, say that the quotient poset $\overline{P} := P/\{b = c\}$, having one fewer element and one fewer Hasse diagram edge, is obtained from P by closing the notch, and that P is obtained from $\overline{P}$ by opening a notch.

Theorem 4.1. *Let P be a connected poset in which (a, b, c) forms a notch, and let $\overline{P} := P/\{b = c\}$. We assume without loss of generality that P and $\overline{P}$ are naturally labeled. Then the root polytope $\tilde{Q}_{\mathcal{H}(P)}$ has a triangulation with top-dimensional simplices $\tilde{\Delta}_{T_1}, \ldots, \tilde{\Delta}_{T_k}$, and $\tilde{Q}_{\mathcal{H}(\bar{P})}$ has a triangulation with top-dimensional simplices $\tilde{\Delta}_{T'_1}, \ldots, \tilde{\Delta}_{T'_k}$, where $(a, b) \in T'_i$, $(a, b), (a, c) \in T_i$, $i \in [k]$, and moreover $T_i|_{b=c} = T'_i$ (we ignore multiple edges).*

To prove Theorem 4.1 we use the following criterion.

Lemma 4.2 (cf. [Postnikov 2009, Lemma 12.6]). *For two trees T and T' on the vertex set $[n]$, the intersection $\tilde{\Delta}_T \cap \tilde{\Delta}_{T'}$ is a common face of the simplices $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ if and only if the directed graph*

$$U(T, T') = \big([n], \big\{(i, j) \mid (i, j) \in E(T),\ i < j\big\} \cup \big\{(j, i) \mid (i, j) \in E(T'),\ i < j\big\}\big)$$

has no directed cycles of length at least 3.

The following proof of Lemma 4.2 is a straightforward adaptation of the proof of [Postnikov 2009, Lemma 12.6] to our more general setting. We include the proof here for convenience.

Proof of Lemma 4.2. Suppose that $U(T, T')$ has a directed cycle C of length at least 3. Let E be the set of edges of T in C and E' be the set of edges of T' in C. Then $\sum_{(i,j)\in E}(e_i - e_j) = \sum_{(i,j)\in E'}(e_i - e_j)$. Let $k = \max(|E|, |E'|)$. Then

$$x := \frac{1}{k} \sum_{(i,j)\in E} (e_i - e_j) = \frac{1}{k} \sum_{(i,j)\in E'} (e_i - e_j) \in \tilde{\Delta}_T \cap \tilde{\Delta}_{T'}.$$

However, the minimal face of the simplex $\tilde{\Delta}_T$ containing x is $\Delta_{([n],E)}$ if $k = |E|$ and $\tilde{\Delta}_{([n],E)}$ if $k > |E|$. Similarly, the minimal face of the simplex $\tilde{\Delta}_{T'}$ containing x is $\Delta_{([n],E')}$ if $k = |E'|$ and $\tilde{\Delta}_{([n],E')}$ if $k > |E'|$. In any case, the minimal faces of $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ containing x are not equal. Thus, $\tilde{\Delta}_T \cap \tilde{\Delta}_{T'}$ is not their common face.

Next, assume that $U(T, T')$ has no directed cycles of length at least 3. Let $F = ([n], E(T) \cap E(T'))$. Since $U(T, T')$ has no directed cycles of length at least 3 we can find a function $h : [n] \to \mathbb{R}$ such that (1) h is constant on connected components of F; and (2) for any directed edge $(a, b) \in U(T, T')$ that joins two different components of F we have $h(a) < h(b)$. Thus, if (a, b) is the edge $(i < j)$ of T then $h(i) < h(j)$, and if (a, b) is the edge $(i < j)$ of T then $h(i) > h(j)$. The function h defines a linear form f_h on the space $\mathbb{R}^n$ with the coordinates $h(1), \ldots, h(n)$ in the standard basis. The above conditions imply (1) for any vertex in the common face $\tilde{\Delta}_F$ of $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ we have $f_h(x) = 0$; (2) for any vertex $x \in \tilde{\Delta}_T \setminus \tilde{\Delta}_F$ we have $f_h(x) < 0$; and (3) for any vertex $x \in \tilde{\Delta}_{T'} \setminus \tilde{\Delta}_F$ we have $f_h(x) > 0$. Thus, the hyperplane $f_h(x) = 0$ intersects $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ at their common face $\tilde{\Delta}_F$ as desired. ☐

Proof of Theorem 4.1. The criterion of Lemma 4.2 is sufficient to establish the above theorem, since we also have that $\tilde{Q}_{\mathcal{H}(\bar{P})}$ has a triangulation with top-dimensional simplices $\tilde{\Delta}_{T_1'}, \ldots, \tilde{\Delta}_{T_k'}$, where $(a, b) \in T_i'$, as $e_a - e_b$ is a vertex of $\tilde{Q}_{\mathcal{H}(\bar{P})}$. □

When we calculate $\sigma_{K_{\bar{P}\mathrm{root}}}(\boldsymbol{x})$ and $\Psi_{\bar{P}}(\boldsymbol{x})$ using triangulations of the root cones as implied by Theorem 4.1, we immediately get:

Corollary 4.3 [Boussicault et al. 2012, Theorem 8.6]. *When $\bar{P}$ is obtained from P by closing a $\vee$-shaped notch (a, b, c), then*

$$\sigma_{K_{\bar{P}\mathrm{root}}}(\boldsymbol{x}) = (1 - x_a {x_b}^{-1})\sigma_{K_{P\mathrm{root}}}(\boldsymbol{x})|_{x_b=x_c} \quad \textit{and} \quad \Psi_{\bar{P}}(\boldsymbol{x}) = (x_a - x_b)\Psi_P(\boldsymbol{x})|_{x_b=x_c}.$$

A consequence of Theorem 4.1 is the following generalization of Greene's theorem pertaining to posets P to which we can repeatedly apply the opening notch operation and obtain a poset whose Hasse diagram has only cycles as biconnected components. Such posets P we call *admissible*. We now recall the definition of biconnected components following [Boussicault et al. 2012]. Given a graph $G = (V, E)$ we say that two edges of it are cycle-equivalent if there is a cycle which contains both edges. Let E_i be the equivalence classes of this relation. Let V_i be the set of vertices which are the endpoint of at least one edge in E_i. Then the biconnected components of G are the graphs $G_i = (V_i, E_i)$.

Theorem 4.4. *Let P be an admissible planar poset. Then, we have*

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x}) = \frac{\prod_{\rho\in b(P)}\bigl(1 - \prod_{i\in\min(\rho)} x_i \prod_{j\in\max(\rho)} x_j^{-1}\bigr)}{\prod_{i\lessdot_P j}(1 - x_i {x_j}^{-1})}$$

and

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_{\rho\in b(P)}\bigl(\sum_{i\in\min(\rho)} x_{\min(i)} - \sum_{j\in\max(\rho)} x_j\bigr)}{\prod_{i\lessdot_P j}(x_i - x_j)},$$

where ρ runs through all bounded regions of the Hasse diagram of P.

Proof. This theorem can be deduced from Corollary 4.3 together with Corollaries 8.2 and 8.3 in [Boussicault et al. 2012]. We note that the latter corollaries also have simple proofs using the root polytope considerations of this paper, and we leave such alternative proofs as an exercise for the interested reader. □

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KAROLA MÉSZÁROS
DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
212 GARDEN AVE.
ITHACA, NY 14853
UNITED STATES
karola@math.cornell.edu

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CLASSIFYING RESOLVING SUBCATEGORIES

WILLIAM SANDERS

We use the theory of Auslander–Buchweitz approximations to classify certain resolving subcategories containing a semidualizing or a dualizing module. In particular, we show that if the ring has a dualizing module, then the resolving subcategories containing maximal Cohen–Macaulay modules are in bijection with grade consistent functions and thus are the precisely the dominant resolving subcategories.

1. Introduction

Classifying various types of subcategories of $\mathrm{mod}(R)$ and $D(R)$ for a commutative ring R has been the subject of much recent research. These classifications are intrinsically connected to spec R or some other topological space. For instance, the Hopkins–Neeman theorem [Hopkins 1987; Neeman 1992] and Gabriel's theorem [1962] give a bijection between the Serre subcategories of $\mathrm{mod}(R)$, the thick subcategories of perfect complexes, and the specialization closed subsets of spec R. Another example is the work regarding the classification of thick subcategories of $\mathrm{mod}(R)$ such as in [Takahashi 2010; Stevenson 2014b].

Recently, much attention has been given to classifying the resolving subcategories of $\mathrm{mod}(R)$. The study of resolving subcategories began with Auslander and

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Bridger's influential work [1969] where they define the category of Gorenstein dimension zero modules, which we denote by GDZ. Also, they generalize the notion of projective dimension by defining Gorenstein dimension through approximations of Gorenstein dimension zero modules. In their paper, they also prove that GDZ has certain homological closure properties which cause Gorenstein dimension to behave similarly to projective dimension. They then take these homological closure properties of GDZ as the definition of resolving subcategories. We can take dimension with respect to a resolving subcategory, and, as in the case of GDZ, these homological closure properties force this dimension function to also behave similarly to projective dimension. See Section 2 for further exposition.

The classification of resolving subcategories was advanced by Dao and Takahashi in [2015], where they give a bijection between the set of resolving subcategories of the category of finite projective dimension modules and the set of grade consistent functions. A function $f : \operatorname{spec} R \to \mathbb{N}$ is called grade consistent if it is increasing (as a morphism of posets) and $f(p) \leq \operatorname{grade}(p)$ for all $p \in \operatorname{spec}(R)$. This result motivated the author to find other situations where a similar bijection exists, furthering the use of grade consistent functions in classifying resolving subcategories. Before the work of Dao and Takahashi, Takahashi [2013] classified, over Cohen–Macaulay rings, resolving subcategories closed under tensor products and Auslander transposes, and in [2011] he classified the contravariantly finite resolving subcategories of a Henselian local Gorenstein ring. Takahashi [2009] also studied resolving subcategories which are free on the punctured spectrum. Auslander and Reiten [1991] discovered a connection between resolving subcategories and tilting theory, and they classified all the contravariantly finite resolving subcategories using cotilting bundles. After the work of Dao and Takahashi, the resolving subcategories of the category of finite projective dimension modules were also classified in [Angeleri Hügel et al. 2014] in terms of descending sequences of specialization closed subsets of spec R, and were also classified in [Angeleri Hügel and Saorín 2014] in terms of certain t-structures.

In this paper, we assume that R is commutative and Noetherian, and we consider only finitely generated modules. Let $\mathcal{P}$ denote the category of projective modules and Γ the set of grade consistent functions. For categories $\mathcal{M}, \mathcal{X} \subseteq \operatorname{mod}(R)$ and $f \in \Gamma$, we define

$$\Lambda_{\mathcal{M}}(f) = \{X \in \operatorname{mod}(R) \mid \operatorname{add}\mathcal{M}_p\text{-dim}\, X_p \leq f(p) \text{ for all } p \in \operatorname{spec} R\}$$

and

$$\begin{aligned}\Phi_{\mathcal{M}}(\mathcal{X}) : \operatorname{spec} R &\to \mathbb{N},\\ p &\mapsto \sup\{\operatorname{add}\mathcal{M}_p\text{-dim}\, X_p \mid X \in \mathcal{X}\},\end{aligned}$$

where $\operatorname{add}\mathcal{M}_p$ is the smallest subcategory of $\operatorname{mod}(R_p)$ closed under direct sums and summands and containing M_p for every $M \in \mathcal{M}$, and where $\operatorname{add}\mathcal{M}_p$-dim X_p is the smallest resolution of X_p by objects in $\operatorname{add}\mathcal{M}_p$. Let $\mathfrak{R}$ denote the collection of resolving subcategories of $\operatorname{mod}(R)$. Set $\Delta(\mathcal{M}) = \{X \in \operatorname{mod}(R) \mid \mathcal{M}\text{-dim}\, X < \infty\}$ for any $\mathcal{M} \subseteq \operatorname{mod}(R)$, and let $\mathfrak{R}(\mathcal{M})$ be the collection of resolving subcategories $\mathcal{X}$ such that $\mathcal{M} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{M})$. Using our new notation, we can restate Dao and Takahashi's result [2015].

Theorem 1.1. *When R is Noetherian,*

$$\mathfrak{R}(\mathcal{P}) \underset{\Phi_{\mathcal{P}}}{\overset{\Lambda_{\mathcal{P}}}{\rightleftarrows}} \Gamma$$

is a bijection, where $\Lambda_{\mathcal{P}}$ and $\Phi_{\mathcal{P}}$ are inverses of each other.

Our first main result is Theorem 4.2, which is the following. Note that throughout this paper, all thick subcategories contain R.

Theorem A. *Let Ψ be a set of increasing functions from* $\operatorname{spec} R$ *to $\mathbb{N}$. Suppose $\mathcal{A} \subseteq \mathcal{M}$ such that $\mathcal{A}$ cogenerates $\mathcal{M}$ and* $\operatorname{add} \mathcal{A}_p$ *is thick in* $\operatorname{add} \mathcal{M}_\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{spec} R$. *Define $\eta_{\mathcal{A}}^{\mathcal{M}} : \mathfrak{R}(\mathcal{A}) \to \mathfrak{R}(\mathcal{M})$ by $\eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \operatorname{res}(\mathcal{X} \cup \mathcal{M})$ and $\rho_{\mathcal{A}}^{\mathcal{M}} : \mathfrak{R}(\mathcal{M}) \to \mathfrak{R}(\mathcal{A})$ by setting $\rho_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \Delta(\mathcal{A}) \cap \mathcal{X}$. If $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverses of each other giving a bijection between $\mathfrak{R}(\mathcal{A})$ and Ψ, then we have the commutative diagram*

$$\begin{tikzcd} \mathfrak{R}(\mathcal{M}) \arrow[dr, "\Phi_{\mathcal{M}}"] & \\ & \Psi \\ \mathfrak{R}(\mathcal{A}) \arrow[uu, "\eta_{\mathcal{A}}^{\mathcal{M}}"] \arrow[ur, "\Phi_{\mathcal{A}}"'] & \end{tikzcd}$$

where $\Phi_{\mathcal{M}}$ is bijective with $\Lambda_{\mathcal{M}}$ its inverse. Moreover, $\rho_{\mathcal{A}}^{\mathcal{M}}$ is the inverse of $\eta_{\mathcal{A}}^{\mathcal{M}}$.

This result allows us to extend the bijection from [Dao and Takahashi 2015] to a plethora of categories. We use it to prove the following result which is essentially Theorem 8.5. Note that $\mathcal{G}_C$ is the category of totally C-reflexive modules where C is a semidualizing module: see Definition 3.1 and Definition 3.4. Define, $\rho_{\mathcal{M}}^{\mathcal{N}}$ and $\eta_{\mathcal{M}}^{\mathcal{N}}$ similarly to $\rho_{\mathcal{A}}^{\mathcal{M}}$ and $\eta_{\mathcal{A}}^{\mathcal{M}}$.

Theorem B. *For any thick subcategory $\mathcal{M}$ of $\mathcal{G}_C$ containing C, $\Lambda_{\mathcal{M}}$ and $\Phi_{\mathcal{M}}$ give a bijection between $\mathfrak{R}(\mathcal{M})$ and Γ. Furthermore, let $\mathfrak{S}$ denote the collection of thick subcategories of $\mathcal{G}_C$ containing C. The following is a bijection:*

$$\Lambda : \mathfrak{S} \times \Gamma \longrightarrow \bigcup_{\mathcal{M} \in \mathfrak{S}} \mathfrak{R}(\mathcal{M}) \subseteq \mathfrak{R}.$$

For any $\mathcal{M}, \mathcal{N} \in \mathfrak{S}$ *with* $\mathcal{M} \subseteq \mathcal{N}$, *then the following diagram commutes*:

$$\begin{tikzcd} \mathfrak{R}(\mathcal{N}) \arrow[dr, "\Phi_{\mathcal{N}}"] & \\ & \Gamma \\ \mathfrak{R}(\mathcal{M}) \arrow[uu, "\eta^{\mathcal{N}}_{\mathcal{M}}"] \arrow[ur, "\Phi_{\mathcal{M}}"'] & \end{tikzcd}$$

In particular, $\rho^{\mathcal{N}}_{\mathcal{M}}$ *and* $\eta^{\mathcal{N}}_{\mathcal{M}}$ *are inverse functions.*

These theorems show that the classification of resolving subcategories is intrinsically linked to the classification of thick subcategories of totally C-reflexive modules and hence to the classification of thick subcategories of $\operatorname{mod}(R)$, a topic of current research. See, for instance, [Takahashi 2010; Neeman 1992]. Applying these results in the Gorenstein case yields Theorem 9.1 which, letting MCM denote the category of maximal Cohen–Macaulay modules, states

Theorem C. *If R is Gorenstein*, *then we have the following commutative diagram of bijections*:

$$\begin{tikzcd} \{\text{Thick subcategories of MCM}\} \times \Gamma \arrow[dd, "\Lambda_{\mathcal{P}}"] \arrow[dr, "\Lambda"] & \\ & \{\mathcal{Z} \in \mathfrak{R} \mid \mathcal{Z} \cap MCM \text{ is thick in MCM}\} \\ \{\text{Thick subcategories of MCM}\} \times \mathfrak{R}(\mathcal{P}) \arrow[ur, "\Xi"'] & \end{tikzcd}$$

where $\Xi(\mathcal{M}, \mathcal{X}) = \operatorname{res}(\mathcal{M} \cup \mathcal{X})$.

Of independent interest, using semidualizing modules, we generalize the famed Auslander transpose. This generalization is similar to but different from the generalizations in [Geng 2013; Huang 1999].

This paper is organized as follows: Section 2 gives general information about resolving subcategories, and Section 3 gives pertinent background regarding semidualizing modules. We prove Theorem A in Section 4. In Section 5, we generalize the Auslander transpose, which we use in Section 6 to classify resolving subcategories which are locally maximal Cohen–Macaulay. In Section 7 we prove a special case of Theorem B. We prove Theorem B in full generality in Section 8 by examining the thick subcategories of maximal Cohen–Macaulay modules containing C. In the last section, these results are applied to the Gorenstein case. Here, Theorem C and several other results are proven.

2. Resolving preliminaries

We proceed with an overview of resolving subcategories. All subcategories considered are full and closed under isomorphisms. For any collection $\mathcal{M} \subseteq \mathrm{mod}(R)$, let $\mathrm{add}(\mathcal{M})$ be the smallest subcategory of $\mathrm{mod}(R)$ containing $\mathcal{M}$ which is closed under direct sums and summands.

Definition 2.1. Given a ring R, a full subcategory $\mathcal{M} \subseteq \mathrm{mod}(R)$ is resolving if the following hold:

(1) R is in $\mathcal{M}$.

(2) $M \oplus N$ is in $\mathcal{M}$ if any only if M and N are in $\mathcal{M}$.

(3) If $0 \to M \to N \to L \to 0$ is exact and $L \in \mathcal{M}$, then $N \in \mathcal{M}$ if and only if $M \in \mathcal{M}$.

By [Yoshino 2005, Lemma 3.2], this is equivalent to saying these conditions hold:

(1) All projectives are in $\mathcal{M}$.

(2) If $M \in \mathcal{M}$, then $\mathrm{add}(M) \subseteq \mathcal{M}$.

(3) $\mathcal{M}$ is closed under extensions.

(4) $\mathcal{M}$ is closed under syzygies.

For a subset $\mathcal{M} \subseteq \mathrm{mod}(R)$, we denote by $\mathrm{res}(\mathcal{M})$ the smallest resolving subcategory containing $\mathcal{M}$. Also, $\mathrm{add}\,\mathcal{M}$ will be the smallest subcategory containing $\mathcal{M}$ which is closed under direct sums and summands. Let $\mathcal{P}$ be the category of finitely generated projective R-modules.

Example 2.2. The following categories are easily seen to be resolving.

(1) $\mathcal{P}$,

(2) $\mathrm{mod}(R)$,

(3) the set of Gorenstein dimension zero modules,

(4) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Ext}^{>n}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$,

(5) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Tor}^{>n}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$,

(6) when R is Cohen–Macaulay, the set of maximal Cohen–Macaulay modules.

A special class of resolving subcategories are thick subcategories.

Definition 2.3. Let $\mathcal{N} \subseteq \mathrm{mod}(R)$. A resolving subcategory $\mathcal{M} \subseteq \mathcal{N}$ is a thick subcategory of $\mathcal{N}$ (or $\mathcal{M}$ is thick in $\mathcal{N}$) if for any exact sequence $0 \to L \to M \to N \to 0$ with $L, M \in \mathcal{M}$, if N is in $\mathcal{N}$, then N is in $\mathcal{M}$ too. A thick subcategory refers to a thick subcategory of $\mathrm{mod}(R)$.

For any $\mathcal{M} \subseteq \mathrm{mod}(R)$, let $\mathrm{Thick}(\mathcal{M})$ be the smallest thick subcategory of $\mathrm{mod}(R)$ containing $\mathcal{M}$.

Example 2.4. The following categories are easily seen to be thick subcategories (moreover, each example is the thick closure of a resolving subcategory from Example 2.2):

(1) the set of modules with finite projective dimension,

(2) $\mathrm{mod}(R)$,

(3) the set of modules with finite Gorenstein dimension,

(4) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Ext}^{\gg 0}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$,

(5) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Tor}^{\gg 0}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$.

Resolving subcategories are studied in part because dimension with respect to a resolving subcategory has nice properties. For a subset $\mathcal{M} \subseteq \mathrm{mod}(R)$ and a module $X \in \mathrm{mod}(R)$, we say that $\mathcal{M}\text{-dim}\, X = n$ if $n \in \mathbb{N}$ is the smallest number such that there is an exact sequence

$$0 \to M_n \to \cdots \to M_0 \to X \to 0$$

with $M_0, \ldots, M_n \in \mathcal{M}$. Projective dimension and Gorenstein dimension are dimensions with respect to resolving subcategories of projective modules and Gorenstein dimension zero modules respectively. The following proposition (see [Auslander and Buchweitz 1989, Proposition 3.3]) causes nice properties to hold for dimension with respect to a resolving subcategory.

Proposition 2.5. *If $\mathcal{M}$ is resolving and $\mathcal{M}\text{-dim}(X) \leq n$, then for any exact sequence*

$$0 \to L \to M_{n-1} \to \cdots \to M_0 \to X \to 0$$

with each $M_i \in \mathcal{M}$, L is in $\mathcal{M}$.

This proposition allows us to prove the following results.

Corollary 2.6. *If $\mathcal{M}$ is resolving, then $\mathcal{M}\text{-dim}(X) = \inf\{n \mid \Omega^n X \in \mathcal{M}\}$.*

Proof. If $\Omega^n X \in \mathcal{M}$, then we have

$$0 \to \Omega^n X \to F_{n-1} \to \cdots \to F_0 \to X \to 0$$

with each F_i projective. This shows that $\mathcal{M}\text{-dim}\, X \leq n$. If $\mathcal{M}\text{-dim}\, X \leq n$, the same sequence and Corollary 2.6 show that $\Omega^n X$ is in $\mathcal{M}$. □

Lemma 2.7. *If $\mathcal{M}$ is resolving, then $\mathcal{M}\text{-dim}\, X \oplus Y = \max\{\mathcal{M}\text{-dim}\, X, \mathcal{M}\text{-dim}\, Y\}$.*

Proof. We have $\Omega^n(X \oplus Y) = \Omega^n X \oplus \Omega^n Y$ for a suitable choice of syzygies. Since $\Omega^n(X \oplus Y)$ is in $\mathcal{M}$ if and only if $\Omega^n X$ and $\Omega^n Y$ are in $\mathcal{M}$, the result follows from Corollary 2.6. Parts (1) and (2) are essentially proved in [Masek 1999, Theorem 18]. □

Lemma 2.8. *If $\mathcal{M}$ is a resolving subcategory, and $0 \to X \to Y \to Z \to 0$ is exact, then the following inequalities hold.*

(1) $\mathcal{M}\text{-dim}\, X \leq \max\{\mathcal{M}\text{-dim}\, Y, \mathcal{M}\text{-dim}\, Z - 1\}$,

(2) $\mathcal{M}\text{-dim}\, Y \leq \max\{\mathcal{M}\text{-dim}\, X, \mathcal{M}\text{-dim}\, Z\}$,

(3) $\mathcal{M}\text{-dim}\, Z \leq \max\{\mathcal{M}\text{-dim}\, X, \mathcal{M}\text{-dim}\, Y\} + 1$.

Proof. For suitable choices of syzygies, we have the following.

$$0 \to \Omega^k X \to \Omega^k Y \to \Omega^k Z \to 0$$

If $k = \max\{\mathcal{M}\text{-dim}\, X, \mathcal{M}\text{-dim}\, Z\}$, then, by Corollary 2.6, $\Omega^k X$ and $\Omega^k Z$ are in $\mathcal{M}$, and thus, so is $\Omega^k Y$, giving us (2). If $k = \max\{\mathcal{M}\text{-dim}\, X, \mathcal{M}\text{-dim}\, Y\}$, then, again by Corollary 2.6, $\Omega^k X$ and $\Omega^k Y$ is in $\mathcal{M}$. Therefore $\mathcal{M}\text{-dim}\, \Omega^k Z \leq 1$, and so $\Omega^{k+1} Z$ is in $\mathcal{M}$. Thus by Corollary 2.6, $\mathcal{M}\text{-dim}\, Z \leq k+1$, proving (3).

Now take $k = \max\{\mathcal{M}\text{-dim}\, Y, \mathcal{M}\text{-dim}\, Z - 1\}$. Then $\Omega^k Y$ and $\Omega^{k+1} Z$ are in $\mathcal{M}$. We take the pushout diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega^{k+1} Z \arrow[r, equal] \arrow[d] & \Omega^{k+1} Z \arrow[d] & \\
0 \arrow[r] & \Omega^k X \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^k X \arrow[r] & \Omega^k Y \arrow[r] \arrow[d] & \Omega^k Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with F free and hence in $\mathcal{M}$. Since, by Corollary 2.6, $\Omega^{k+1} Z$ and $\Omega^k Y$ are in $\mathcal{M}$, so is T. Since $F \in \mathcal{M}$, $\Omega^k X$ has to also be in $\mathcal{M}$. Hence $\mathcal{M}\text{-dim}\, X \leq k$, and we have (1). $\square$

For a subset $\mathcal{M} \subseteq \text{mod}(R)$, let $\Delta(\mathcal{M})$ denote the category of modules X such that $\mathcal{M}\text{-dim}\, X$ is finite. If $\mathcal{M}$ is resolving, then by Corollary 2.6, $\Delta(\mathcal{M}) = \{X \in \text{mod}(R) \mid \Omega^{\gg 0} X \in \mathcal{M}\}$. The next result easily follows from the previous lemma.

Corollary 2.9. *Let $\mathcal{M}$ be resolving. For any n, the set $\{X \in \text{mod}(R) \mid \mathcal{M}\text{-dim}\, X \leq n\}$ is resolving. Furthermore, $\Delta(\mathcal{M})$ is thick, and* $\text{Thick}(\mathcal{M}) = \Delta(\mathcal{M})$.

Through these results, we may construct many resolving and thick subcategories. It is easy to show that the intersection of a collection of resolving subcategories and the intersection of a collection of thick subcategories are resolving and thick respectively. The following lemma allows us to construct even more resolving subcategories. For $\mathcal{M} \subseteq \operatorname{mod}(R)$, we say $\mathcal{M}_p = \{M_p \mid M \in \mathcal{M}\}$.

Lemma 2.10. *Let R and S be rings and $F : \operatorname{mod}(R) \to \operatorname{mod}(S)$ be an exact functor with $F(R) = S$. Then for any resolving subcategory $\mathcal{M} \subseteq \operatorname{mod}(S)$, $F^{-1}(\mathcal{M})$ is a resolving subcategory of* $\operatorname{mod}(R)$.

The proof is elementary and is left to the reader. Applying this lemma to the localization functor, for any $V \subseteq \operatorname{spec} R$, the category of all $M \in \operatorname{mod}(R)$ with M_p free for all $p \in V$ is also resolving. The following lemmas give insight into the behavior of resolving categories under localization. The first lemma is from [Takahashi 2010, Lemma 4.8; Dao and Takahashi 2014, Lemma 3.2(1)], and the second is from [Dao and Takahashi 2015, Proposition 3.3].

Lemma 2.11. *If $\mathcal{M}$ is a resolving subcategory, then so is* $\operatorname{add} \mathcal{M}_\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{spec} R$.

Lemma 2.12. *The following are equivalent for a resolving subcategory $\mathcal{M}$ and a module $M \in \operatorname{mod}(R)$:*

(1) $M \in \mathcal{M}$,

(2) $M_p \in \operatorname{add} \mathcal{M}_p$ *for all* $p \in \operatorname{spec} R$,

(3) $M_\mathfrak{m} \in \operatorname{add} \mathcal{M}_\mathfrak{m}$ *for all maximal ideals* $\mathfrak{m}$.

Recall the definition of Λ and Γ from the introduction. These lemmas show that if $\mathcal{M}$ is resolving, then for all $f \in \Gamma$, $\Lambda_{\mathcal{M}}(f)$ is a resolving subcategory.

Corollary 2.13. *Set*

$$\Lambda_{\mathcal{M}}(f) = \{M \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{M}_p\text{-}\dim M_p \le f(p) \text{ for all } p \in \operatorname{spec} R\}.$$

If $\mathcal{M}$ is resolving, then for all $f \in \Gamma$, $\Lambda_{\mathcal{M}}(f)$ is a resolving subcategory.

Let MCM denote the category of maximal Cohen–Macaulay modules. As noted earlier, when R is Cohen–Macaulay, MCM is resolving. Furthermore, letting $d = \dim R$, $\Omega^d M$ is in MCM for every $M \in \operatorname{mod}(R)$. Hence, $\Delta(\mathrm{MCM}) = \operatorname{mod}(R)$. The following shows that dimension with respect to MCM is very computable.

Lemma 2.14. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ are resolving subcategories. Then $\mathcal{M}$ is thick in $\mathcal{N}$ if and only if for every module $X \in \Delta(\mathcal{M})$, we have $\mathcal{M}\text{-}\dim X = \mathcal{N}\text{-}\dim X$. Furthermore, if R is Cohen–Macaulay, $\mathcal{M}$ is a thick subcategory of MCM if and only if dimension with respect to $\mathcal{M}$ satisfies the Auslander–Buchsbaum formula, i.e., for all $X \in \Delta(\mathcal{M})$ we have*

$$\mathcal{M}\text{-}\dim X + \operatorname{depth} X = \operatorname{depth} R.$$

Proof. Suppose M is thick in N and $X \in \Delta(\mathcal{M})$. Then we may write $0 \to M_d \to \cdots \to M_0 \to X \to 0$ with $M_i \in \mathcal{M}$ and $d = \mathcal{M}\text{-}\dim X$. Since each M_i is also in $\mathcal{N}$, we have $\mathcal{N}$-dim $X \leq d$. Setting $e = \mathcal{N}$-dim $X \leq d$, by Corollary 2.6, there exists a $N \in \mathcal{N}$ such that

$$0 \to M_d \to \cdots \to M_e \to N \to 0 \quad \text{and} \quad 0 \to N \to M_{e-1} \to \cdots \to M_0 \to X \to 0$$

are exact. However, since $\mathcal{M}$ is thick in $\mathcal{N}$, N is also in $\mathcal{M}$, which implies that $e = d$, proving the only if part of the statement.

Now suppose that $\mathcal{M}$-dim $X = \mathcal{N}$-dim X for all $X \in \Delta(\mathcal{M})$. Now suppose $0 \to L \to M \to N \to 0$ is exact with $L, M \in \mathcal{M}$ and $N \in \mathcal{N}$. Then $N \in \Delta(\mathcal{M})$ and $\mathcal{M}$-dim $N = \mathcal{N}$-dim $N = 0$. Therefore $N \in \mathcal{M}$, and so $\mathcal{M}$ is thick in $\mathcal{N}$.

Assume R is Cohen–Macaulay. Let $\mathcal{M}$ be a resolving subcategory whose dimension satisfies the Auslander–Buchsbaum formula. Then for any module $M \in \Delta(\mathcal{M}) \cap \mathrm{MCM}$, we have

$$\operatorname{depth} R = \mathcal{M}\text{-}\dim M + \operatorname{depth} M = \mathcal{M}\text{-}\dim M + \operatorname{depth} R.$$

Thus $\mathcal{M}$-dim $M = 0$ forcing M to be in $\mathcal{M}$. Hence $\mathcal{M}$ is contained in MCM.

By what we have proved so far, it suffices to show that dimension with respect to MCM satisfies the Auslander–Buchsbaum formula. But this follows from Corollary 2.6. □

Recall the definition of Φ and Γ from the introduction. If dimension with respect to add $\mathcal{M}_p$ satisfies the Auslander–Buchsbaum formula for all $p \in \operatorname{spec} R$, then for all $\mathcal{X} \subseteq \Delta(\mathcal{M})$, $\Phi_{\mathcal{M}}(\mathcal{X})$ is in Γ. Before proceeding, we need one more definition and a result.

Definition 2.15. Let $\mathcal{A} \subseteq \mathcal{M}$. We say $\mathcal{A}$ cogenerates $\mathcal{M}$, if for every $M \in \mathcal{M}$, there exists an exact sequence $0 \to M \to A \to M' \to 0$ with $M' \in \mathcal{M}$ and $A \in \mathcal{A}$.

The following is an important theorem from [Auslander and Buchweitz 1989, Theorem 1.1].

Theorem 2.16. *Suppose $\mathcal{A}$ and $\mathcal{M}$ are resolving with $\mathcal{A} \subseteq \mathcal{M}$. If $\mathcal{A}$ cogenerates $\mathcal{M}$, then for every $X \in \Delta(\mathcal{M})$ with $\mathcal{M}$-dim $X = n$, there exists an $A \in \Delta(\mathcal{A})$ and $M \in \mathcal{M}$ such that $\mathcal{A}$-dim $A = n$ and $0 \to X \to A \to M \to 0$ is exact.*

3. Preliminaries: semidualizing modules

We fix a module $C \in \operatorname{mod}(R)$ and write $M^\dagger = \operatorname{Hom}(M, C)$.

Definition 3.1. A finitely generated module X is totally C-reflexive if it satisfies the following:

(1) $\operatorname{Ext}^{>0}(X, C) = 0$,

(2) $\operatorname{Ext}^{>0}(X^\dagger, C) = 0$,

(3) The natural homothety map $\eta_X : X \to X^{\dagger\dagger}$ defined by $\mu \mapsto (\varphi \mapsto \varphi(\mu))$ is an isomorphism.

Let $\mathcal{G}_C$ denote the category of totally C-reflexive modules.

The set $\mathcal{G}_C$ is essentially the subcategory over which † is a dualizing functor. The notion of totally C-reflexivity generalizes Gorenstein dimension zero. In fact, when $C = R$, $\mathcal{G}_R$ is simply the category of Gorenstein dimension zero modules, which are also known as totally reflexive modules. See [Masek 1999] for further information on the subject. The following proposition shows us that $\mathcal{G}_C$ is almost resolving.

Lemma 3.2. *The set $\mathcal{G}_C$ is closed under direct sums, summands, and extensions.*

Proof. It is easy to show that $\mathcal{G}_C$ is closed under direct sums and direct summands. Suppose we have

$$0 \to X \to Y \to Z \to 0$$

with $X, Z \in \mathcal{G}_C$. It is easy to check that Y satisfies condition (1) of Definition 3.1. We have

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0 \quad \text{and} \quad 0 \to X^{\dagger\dagger} \to Y^{\dagger\dagger} \to Z^{\dagger\dagger} \to 0.$$

From the first exact sequence, it is easy to see that Y satisfies condition (2) of Definition 3.1. We can then use the five lemma to show that Y satisfies condition (3) of Definition 3.1. □

In general, $\mathcal{G}_C$ is not resolving. For example, if $C = R/xR$ for a regular element $x \in R$, $\mathrm{Ext}^1(R/xR, R/xR) = R/xR \neq 0$. So R cannot be in $\mathcal{G}_{R/xR}$, and thus $\mathcal{G}_{R/xR}$ cannot be resolving. It is clear from the definition that $R \in \mathcal{G}_C$ is a necessary condition for $\mathcal{G}_C$ to be resolving. In fact, this condition is sufficient.

Proposition 3.3. *The subcategory $\mathcal{G}_C$ is resolving if and only if $\mathcal{G}_C$ contains R.*

Proof. If $\mathcal{G}_C$ is resolving, by definition it contains R, so we prove the converse. So suppose R is in $\mathcal{G}_C$. In light of the last lemma, we need only to prove that if $0 \to X \to Y \to Z \to 0$ is exact with $Y, Z \in \mathcal{G}_C$, then X is in $\mathcal{G}_C$ as well. Since Y and Z satisfy condition (1) of Definition 3.1, it is easy to show that X does too. Also, since $\mathrm{Ext}^1(Z, C) = 0$, we have

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0.$$

Hence, we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\eta_X"] & Y \arrow[r] \arrow[d, "\eta_Y"] & Z \arrow[rr] \arrow[d, "\eta_Z"] & & 0 \\
0 \arrow[r] & X^{\dagger\dagger} \arrow[r] & Y^{\dagger\dagger} \arrow[r] & Z^{\dagger\dagger} \arrow[r] & \mathrm{Ext}^1(X^\dagger, C) \arrow[r] & 0
\end{tikzcd}$$

Since η_Y and η_Z are isomorphisms, the five lemma shows that η_X is too, and that $\mathrm{Ext}^1(X^\dagger, C) = 0$. Thus X satisfies condition (3) of Definition 3.1. It is easy to check using the first exact sequence that $\mathrm{Ext}^{>1}(X^\dagger, C) = 0$, showing that X satisfies condition (2) of Definition 3.1. □

Motivated by this proposition, we say that a module, C, is semidualizing if R is in $\mathcal{G}_C$. This is easily seen to be equivalent to the following definition which is standard in the literature.

Definition 3.4. A module C is called *semidualizing* if both $\mathrm{Ext}^{>0}(C, C) = 0$ and $R \cong \mathrm{Hom}(C, C)$ via the map $r \mapsto (c \mapsto rc)$.

For the remainder of the paper, we let C denote a semidualizing module. Semidualizing modules were first discovered by Foxby [1972] and were later rediscovered in different guises by various authors, including Vasconcoles [1974], who called them spherical modules, and Golod, who called them suitable modules. For an excellent treatment of the general theory of semidualizing modules, see [Sather-Wagstaff 2009]. Examples of semidualizing modules include R and dualizing modules. If R is Cohen–Macaulay and D is a dualizing module, then $\mathcal{G}_D$ is simply MCM. Dimension with respect to $\mathcal{G}_C$ is often called Gorenstein C-dimension, or G_C-dimension for short, since it is a generalization of Gorenstein dimension. We would expect G_C and Gorenstein dimension to have similar properties. Thus we have the following lemma, which is an easy exercise, and proposition, which is from [Gerko 2001, Theorem 1.22].

Lemma 3.5. *If $X \in \Delta(\mathcal{G}_C)$, then $\mathcal{G}_C\text{-}\dim X = \min\{n \mid \mathrm{Ext}^{>n}(X, C) = 0\}$.*

Proposition 3.6. *For any semidualizing module C, G_C-dimension satisfies the Auslander–Buchsbaum formula, i.e., for any module $X \in \Delta(\mathcal{G}_C)$, we have*

$$\mathcal{G}_C\text{-}\dim X + \operatorname{depth} X = \operatorname{depth} R.$$

In light of Lemma 2.14, when R is Cohen–Macaulay this means that $\mathcal{G}_C$ is a thick subcategory of MCM. Interest in understanding G_C-dimension and the structure of $\mathcal{G}_C$ is not new. The following conjecture by Gerko [2001, Conjecture 1.23] is equivalent to saying that $\mathcal{G}_R$ is a thick subcategory of $\mathcal{G}_C$.

Conjecture 3.7. If C is semidualizing, then for any module X, $\mathcal{G}_C\text{-}\dim X \leq \mathcal{G}_R\text{-}\dim X$, and equality holds when both are finite.

We give one more construction in this section. Take any $X \in \mathcal{G}_C$. Then we have $0 \to \Omega X^\dagger \to R^n \to X^\dagger \to 0$ is exact. Since $R^\dagger \cong C$, applying † yields the exact sequence

$$0 \to X \to C^n \to (\Omega X^\dagger)^\dagger \to 0.$$

Hence $\mathcal{G}_C$ is cogenerated by $\operatorname{add} C$. Furthermore, if $F_\bullet$ is a projective resolution of $X^\dagger$ with $X \in \mathcal{G}_C$, then $F_\bullet^\dagger$ is an $\operatorname{add} C$ coresolution of X. Splicing this together with a free resolution $G_\bullet$ of X, we get what is called a complete PP_C or a complete P_C-resolution of X. See [White 2010; Sather-Wagstaff 2009] for more on the matter.

Before proceeding, we summarize the notations of this paper.

(1) R is a commutative noetherian ring.

(2) $\mathcal{P}$ is the subcategory of projective R-modules.

(3) Γ is the set of grade consistent functions.

(4) $\mathcal{M}$-dim X is the dimension of X with respect to the category $\mathcal{M} \subseteq \operatorname{mod}(R)$.

(5) $\operatorname{add} \mathcal{M}$ is the smallest category closed under direct sums and summands containing $\mathcal{M} \subseteq \operatorname{mod}(R)$.

(6) $\Lambda_{\mathcal{M}}(f) = \{X \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{M}_p\text{-dim}\, X_p \leq f(p) \text{ for all } p \in \operatorname{spec} R\}$ with $f \in \Gamma$.

(7) $\Phi_{\mathcal{M}}(\mathcal{X})(p) = \sup\{\operatorname{add} \mathcal{M}_p\text{-dim}\, X_p \mid X \in \mathcal{X}\}$ with $\mathcal{M}, \mathcal{X} \subseteq \operatorname{mod}(R)$ subcategories.

(8) $\Delta(\mathcal{M}) = \{X \in \operatorname{mod}(R) \mid \mathcal{M}\text{-dim}\, X < \infty\}$ with $\mathcal{M} \subseteq \operatorname{mod}(R)$ a category.

(9) $\mathfrak{R}(\mathcal{M}) = \{\mathcal{X} \subseteq \operatorname{mod}(R) \mid \mathcal{M} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{M})\mathcal{X} \text{ is resolving}\}$.

(10) $\mathfrak{R}$ the collection of resolving subcategories.

(11) $\operatorname{Thick}_{\mathcal{N}}(\mathcal{M})$ the smallest thick subcategory of $\mathcal{N}$ containing $\mathcal{M}$ with $\mathcal{M} \subseteq \mathcal{N} \subseteq \operatorname{mod}(R)$ subcategories.

(12) C is a semidualizing module .

(13) $\mathcal{G}_C$ the collection of totally C-reflexive modules.

(14) $X^\dagger = \operatorname{Hom}(X, C)$.

(15) For a resolving subcategory $\mathcal{A}$ and a module $M \in \operatorname{mod}(R)$, set $\operatorname{res}_{\mathcal{A}} M = \operatorname{res}(\mathcal{A} \cup \{M\})$.

4. Comparing resolving subcategories

For the entirety of this section, let $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{N}$ be resolving subcategories. Recall that $\mathfrak{R}(\mathcal{A})$ is the collection of resolving subcategories $\mathcal{X}$ such that $\mathcal{A} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{A})$. In this section, we compare $\mathfrak{R}(\mathcal{A})$ and $\mathfrak{R}(\mathcal{M})$ when $\mathcal{A}$ is contained in $\mathcal{M}$. If $\mathcal{A} \subseteq \mathcal{M}$, we may define $\eta_{\mathcal{A}}^{\mathcal{M}} : \mathfrak{R}(\mathcal{A}) \to \mathfrak{R}(\mathcal{M})$ by $\mathcal{X} \mapsto \operatorname{res}(\mathcal{X} \cup \mathcal{M})$ and $\rho_{\mathcal{A}}^{\mathcal{M}} : \mathfrak{R}(\mathcal{M}) \to \mathfrak{R}(\mathcal{A})$ by $\mathcal{X} \mapsto \mathcal{X} \cap \Delta(\mathcal{A})$. Note that if $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$, then $\eta_{\mathcal{A}}^{\mathcal{N}} = \eta_{\mathcal{M}}^{\mathcal{N}} \eta_{\mathcal{A}}^{\mathcal{M}}$ and $\rho_{\mathcal{A}}^{\mathcal{N}} = \rho_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{M}}^{\mathcal{N}}$.

Proposition 4.1. *If $\mathcal{A}$ cogenerates $\mathcal{M}$, then the map $\rho_{\mathcal{A}}^{\mathcal{M}}$ is injective.*

Proof. Suppose that $\rho_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \rho_{\mathcal{A}}^{\mathcal{M}}(\mathcal{Y})$ for $\mathcal{X}, \mathcal{Y} \in \mathfrak{R}(\mathcal{M})$, i.e., $\mathcal{X} \cap \Delta(\mathcal{A}) = \mathcal{Y} \cap \Delta(\mathcal{A})$. Take any $X \in \mathcal{X}$. Since $X \in \Delta(\mathcal{M})$ and $\mathcal{A}$ cogenerates $\mathcal{M}$, by Theorem 2.16, there exists $A \in \Delta(\mathcal{A})$ and $M \in \mathcal{M}$ such that $0 \to X \to A \to M \to 0$ is exact. Since $M \in \mathcal{M} \subseteq \mathcal{X}$ and $X \in \mathcal{X}$, we know that A is also in $\mathcal{X}$. But then A is in $\mathcal{X} \cap \Delta(\mathcal{A}) = \mathcal{Y} \cap \Delta(\mathcal{A})$ and thus also in $\mathcal{Y}$. Since $M \in \mathcal{M} \subseteq \mathcal{Y}$, we know that X must also be in $\mathcal{Y}$. Hence $\mathcal{X} \subseteq \mathcal{Y}$, and, by symmetry, we have equality. Therefore, $\rho_{\mathcal{A}}^{\mathcal{M}}$ is injective. □

In certain circumstances, this map is a bijection. The following is Theorem A from the introduction.

Theorem 4.2. *Let* Ψ *be a set of increasing functions from* spec R *to* $\mathbb{N}$. *Suppose*, $\mathcal{A} \subseteq \mathcal{M}$ *such that* $\mathcal{A}$ *cogenerates* $\mathcal{M}$ *and* $\operatorname{add} \mathcal{A}_p$ *is thick in* $\operatorname{add} \mathcal{M}_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{spec} R$. *If* $\Phi_{\mathcal{A}}$ *and* $\Lambda_{\mathcal{A}}$ *are inverse functions giving a bijection between* $\mathfrak{R}(\mathcal{A})$ *and* Ψ, *then the following diagram commutes*:

$$\begin{tikzcd} \mathfrak{R}(\mathcal{M}) \arrow[dr, "\Phi_{\mathcal{M}}"] & \\ & \Psi \\ \mathfrak{R}(\mathcal{A}) \arrow[uu, "\eta_{\mathcal{A}}^{\mathcal{M}}"] \arrow[ur, "\Phi_{\mathcal{A}}"'] & \end{tikzcd}$$

Furthermore, $\Lambda_{\mathcal{M}}$ *and* $\rho_{\mathcal{A}}^{\mathcal{M}}$ *are the respective inverses of* $\Phi_{\mathcal{M}}$ *and* $\eta_{\mathcal{A}}^{\mathcal{M}}$.

The proof of this result will be given after this brief lemma.

Lemma 4.3. *If* $\mathcal{X}$ *and* $\mathcal{Y}$ *are subcategories and* $\mathcal{M}$ *is resolving*, *then*

$$\Phi_{\mathcal{M}}(\operatorname{res}(\mathcal{X} \cup \mathcal{Y})) = \Phi_{\mathcal{M}}(\mathcal{X}) \vee \Phi_{\mathcal{M}}(\mathcal{Y}).$$

Proof. Since every element in $\operatorname{res}(\mathcal{X} \cup \mathcal{Y})$ is obtained by taking extensions, syzygies, and direct summands a finite number of times, and since these operations never increase the $\mathcal{M}$ dimension, we have $\Phi_{\mathcal{M}}(\operatorname{res}(\mathcal{X} \cup \mathcal{Y})) \leq \Phi_{\mathcal{M}}(\mathcal{X}) \vee \Phi_{\mathcal{M}}(\mathcal{Y})$. However, since $\mathcal{X}, \mathcal{Y} \subseteq \operatorname{res}(\mathcal{X} \cup \mathcal{Y})$, we actually have equality. □

Proof of Theorem 4.2. First, we show that $\rho_{\mathcal{A}}^{\mathcal{M}}$ and $\eta_{\mathcal{A}}^{\mathcal{M}}$ are inverse functions and are thus both bijections. Proposition 4.1 shows that $\rho_{\mathcal{A}}^{\mathcal{M}}$ is injective. Fix $\mathcal{X} \in \mathfrak{R}(\mathcal{A})$ and let $\mathcal{Z} = \rho_{\mathcal{A}}^{\mathcal{M}} \eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \operatorname{res}(\mathcal{X} \cup \mathcal{M}) \cap \Delta(\mathcal{A})$. It suffices to show that $\mathcal{Z} = \mathcal{X}$. Setting $f = \Phi_{\mathcal{A}}(\mathcal{X})$, this is equivalent to showing that $\Phi_{\mathcal{A}}(\mathcal{Z}) = f$, since $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverse functions. Since $\mathcal{X} \subseteq \mathcal{Z}$, we know that $\Phi_{\mathcal{A}}(\mathcal{Z}) \geq f$. From Lemma 4.3,

$$\Phi_{\mathcal{M}}(\operatorname{res}(\mathcal{X} \cup \mathcal{M})) = \Phi_{\mathcal{M}}(\mathcal{X}) \vee \Phi_{\mathcal{M}}(\mathcal{M}) = \Phi_{\mathcal{M}}(\mathcal{X}).$$

Furthermore, since $\operatorname{add} \mathcal{A}_p$ is thick in $\operatorname{add} \mathcal{M}_{\mathfrak{p}}$ for all $p \in \operatorname{spec} R$, Lemma 2.14 implies that $\operatorname{add} \mathcal{A}_p$-dim A and $\operatorname{add} \mathcal{M}_p$-dim A are the same for all $p \in \operatorname{spec} R$ and

$A \in \Delta(\mathcal{A})$. Hence $\Phi_{\mathcal{A}}(\mathcal{X}) = \Phi_{\mathcal{M}}(\mathcal{X})$ and $\Phi_{\mathcal{A}}(\mathcal{Z}) = \Phi_{\mathcal{M}}(\mathcal{Z})$. Therefore,

$$f \leq \Phi_{\mathcal{A}}(\mathcal{Z}) = \Phi_{\mathcal{M}}(\mathcal{Z}) \leq \Phi_{\mathcal{M}}(\operatorname{res}(\mathcal{X} \cup \mathcal{M})) = \Phi_{\mathcal{M}}(\mathcal{X}) = \Phi_{\mathcal{A}}(\mathcal{X}) = f$$

and so, $\Phi_{\mathcal{A}}(\mathcal{Z}) = f$. Hence, $\rho_{\mathcal{A}}^{\mathcal{M}}$ and $\eta_{\mathcal{A}}^{\mathcal{M}}$ are inverse functions. Also, this argument shows that $\Phi_{\mathcal{A}}(\mathcal{X}) = \Phi_{\mathcal{M}}(\operatorname{res}(\mathcal{X} \cup \mathcal{M})) = \Phi_{\mathcal{M}}(\eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}))$, showing that the diagram commutes and that $\Phi_{\mathcal{M}}$ is also a bijection.

It remains to show that $\Lambda_{\mathcal{M}} = \Phi_{\mathcal{M}}{}^{-1}$. For any $f \in \Psi$, $\eta_{\mathcal{A}}^{\mathcal{M}}(\Lambda_{\mathcal{A}}(f))$ is contained in $\Lambda_{\mathcal{M}}(f)$. Because $\Phi_{\mathcal{M}}$ is an increasing function and both $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverse functions, we have

$$f = \Phi_{\mathcal{A}}\Lambda_{\mathcal{A}}(f) = \Phi_{\mathcal{M}}(\eta_{\mathcal{A}}^{\mathcal{M}}(\Lambda_{\mathcal{A}}(f)) \leq \Phi_{\mathcal{M}}\Lambda_{\mathcal{M}}(f) \leq f.$$

Thus we have $\Phi_{\mathcal{M}}\Lambda_{\mathcal{M}}(f) = f$, and we are done. □

For a resolving subcategory $\mathcal{A}$, let $\mathfrak{S}(\mathcal{A})$ be the collection of resolving subcategories $\mathcal{M}$ such that $\mathcal{M}$ and $\mathcal{A}$ satisfy the hypotheses of Theorem 4.2, i.e., $\mathcal{A}$ cogenerates $\mathcal{M}$ and $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is thick in $\operatorname{add} \mathcal{M}_{\mathfrak{p}}$ for all $p \in \operatorname{spec} R$. The following theorem shows that we can patch together the bijections in Theorem 4.2.

Theorem 4.4. *Let $\mathcal{A}$ be a resolving subcategory and Ψ be a set of increasing functions from* spec R *to* $\mathbb{N}$. *If $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverse functions giving a bijection between $\mathfrak{R}(\mathcal{A})$ and Ψ, then*

$$\Lambda : \mathfrak{S}(\mathcal{A}) \times \Psi \longrightarrow \bigcup_{\mathcal{M} \in \mathfrak{S}(\mathcal{A})} \mathfrak{R}(\mathcal{M}) \subseteq \mathfrak{R}.$$

is a bijection. Furthermore, for any $\mathcal{M}, \mathcal{N} \in \mathfrak{S}(\mathcal{A})$ with $\mathcal{M} \subseteq \mathcal{N}$, the map $\rho_{\mathcal{M}}^{\mathcal{N}}$ is the inverse of $\eta_{\mathcal{M}}^{\mathcal{N}}$, and the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{R}(\mathcal{N}) \arrow[dr, "\Phi_{\mathcal{N}}"] & \\
\mathfrak{R}(\mathcal{M}) \arrow[u, "\eta_{\mathcal{M}}^{\mathcal{N}}"] \arrow[r, "\Phi_{\mathcal{M}}"] & \Psi \\
\mathfrak{R}(\mathcal{A}) \arrow[u, "\eta_{\mathcal{A}}^{\mathcal{M}}"] \arrow[ur, "\Phi_{\mathcal{A}}"'] &
\end{tikzcd} \tag{1}$$

Before we proceed with the proof of Theorem 4.4, we need a lemma.

Lemma 4.5. *The set $\mathfrak{S}(\mathcal{A})$ is closed under intersections.*

Proof. Let $\mathcal{M}, \mathcal{N} \in \mathfrak{S}(\mathcal{A})$. Take any $p \in \operatorname{spec} R$. Suppose $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence of R_p-modules with $A_1, A_2 \in \operatorname{add} \mathcal{A}_p$ and $A_3 \in \operatorname{add}(\mathcal{M} \cap \mathcal{N})_p$. Then A_3 is in $\operatorname{add} \mathcal{M}_p$. Therefore, since $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is thick in $\operatorname{add} \mathcal{M}_p$ by assumption, A_3 is in $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$. Since $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is resolving and contained in $\operatorname{add}(\mathcal{M} \cap \mathcal{N})_p$, $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is thick in $\operatorname{add}(\mathcal{M} \cap \mathcal{N})_p$.

It remains to show that $\mathcal{A}$ cogenerates $\mathcal{M}\cap\mathcal{N}$. Take $X\in\mathcal{M}\cap\mathcal{N}$. We have

$$0\to X\to A\to M\to 0 \quad\text{and}\quad 0\to X\to A'\to N\to 0,$$

with $M\in\mathcal{M}$, $N\in\mathcal{N}$, and A, $A'\in\mathcal{A}$. Consider the following pushout diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

It is easy to see $T\in\mathcal{M}\cap\mathcal{N}$. We also have the exact sequence

$$0\to X\to A\oplus A'\to T\to 0.$$

Since $A\oplus A'\in\mathcal{A}$, this completes the proof. □

Proof of Theorem 4.4. Suppose $\mathcal{M},\mathcal{N}\in\mathfrak{S}$ with $\mathcal{M}\subseteq\mathcal{N}$. From Theorem 4.2, the following diagrams commute:

$$\begin{tikzcd}
\mathfrak{R}(\mathcal{M}) \arrow[r, "\Phi_{\mathcal{M}}"] & \Psi \\
\mathfrak{R}(\mathcal{A}) \arrow[u, "\eta^{\mathcal{M}}_{\mathcal{A}}"] \arrow[ur, "\Phi_{\mathcal{A}}"'] &
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathfrak{R}(\mathcal{N}) \arrow[r, "\Phi_{\mathcal{N}}"] & \Psi \\
\mathfrak{R}(\mathcal{A}) \arrow[u, "\eta^{\mathcal{N}}_{\mathcal{A}}"] \arrow[ur, "\Phi_{\mathcal{A}}"'] &
\end{tikzcd}$$

From here, it is easy to show that diagram (1) commutes and $\Phi_{\mathcal{N}}$ and $\eta^{\mathcal{N}}_{\mathcal{M}}$ are bijections with $(\eta^{\mathcal{N}}_{\mathcal{M}})^{-1}=\rho^{\mathcal{N}}_{\mathcal{M}}$.

Also, Theorem 4.2 shows that $\operatorname{Im}(\Lambda)=\bigcup_{\mathcal{M}\in\mathfrak{S}}\mathfrak{R}(\mathcal{M})$. It remains to show that Λ is injective. Suppose $\mathcal{X}=\Lambda_{\mathcal{M}}(f)=\Lambda_{\mathcal{N}}(g)$. Then $\mathcal{M},\mathcal{N}\subseteq\mathcal{X}$; hence, $\mathcal{M}\cap\mathcal{N}\subseteq\mathcal{X}$. For any $X\in\mathcal{X}$ and any n greater than $\mathcal{M}$-dim X and $\mathcal{N}$-dim X, $\Omega^n X$ is in $\mathcal{M}\cap\mathcal{N}$ by Corollary 2.6. Therefore, $\mathcal{X}$ is contained in $\Delta(\mathcal{M}\cap\mathcal{N})$ and thus $\mathcal{X}\in\mathfrak{R}(\mathcal{M}\cap\mathcal{N})$. By the previous lemma, $\mathcal{M}\cap\mathcal{N}$ is in $\mathfrak{S}(\mathcal{A})$, so $\Lambda_{\mathcal{M}\cap\mathcal{N}}:\Psi\to\mathfrak{R}(\mathcal{M}\cap\mathcal{N})$ is a bijection, by Theorem 4.2. So there exists an $h\in\Psi$ such that $\Lambda_{\mathcal{M}\cap\mathcal{N}}(h)=\mathcal{Z}=\Lambda_{\mathcal{M}}(f)=\Lambda_{\mathcal{N}}(g)$. Therefore, we may assume that $\mathcal{M}$ is contained in $\mathcal{N}$.

Since $\mathcal{X} \in \mathfrak{R}(M)$ and $\mathcal{X} \in \mathfrak{R}(\mathcal{N})$, we have $\mathcal{N} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{M})$. Thus, because $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$,

$$\mathcal{N} = \mathcal{N} \cap \Delta(\mathcal{M}) = \rho_{\mathcal{M}}^{\mathcal{N}}(\mathcal{N}) = \eta_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{M}}^{\mathcal{N}}(\mathcal{N}) = \eta_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{A}}^{\mathcal{N}}(\mathcal{N}) = \eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{A}) = \mathcal{M}.$$

Since $\Lambda_{\mathcal{M}}$ is injective, we then also have $f = g$. □

As mentioned earlier, it is shown in [Dao and Takahashi 2015] that we have $\Lambda_{\mathcal{P}}$ is a bijection from Γ to $\mathfrak{R}(\mathcal{P})$. In Sections 8 and 9 we apply Theorem 4.4 when $\mathcal{A} = \mathcal{P}$, and show that $\mathfrak{S}(\mathcal{P})$ contains the collection of thick subcategories of $\mathcal{G}_R$. The following results gives an alternative way of viewing Theorem 4.4.

Proposition 4.6. *In the situation of Theorem 4.4, if $\Psi = \Gamma$ and $\mathcal{P}$ is thick in $\mathcal{M}$, then the following diagram commutes*:

$$\begin{tikzcd}
\mathfrak{S}(\mathcal{A}) \times \Gamma \arrow[d, "\mathrm{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda_{\mathcal{P}}"'] \arrow[dr, "\Lambda"] & \\
 & \mathfrak{R} \\
\mathfrak{S}(\mathcal{A}) \times \mathfrak{R}(\mathcal{P}) \arrow[ur, "\Xi"'] &
\end{tikzcd}$$

where $\Xi(\mathcal{M}, \mathcal{X}) = \operatorname{res}(\mathcal{M} \cup \mathcal{X})$. Furthermore, $\mathrm{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda_{\mathcal{P}}$ is bijective and Ξ is injective.

Proof. Since $\Lambda_{\mathcal{P}}$ is bijective, $\mathrm{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda_{\mathcal{P}}$ is too. It suffices to show that for any $(\mathcal{M}, f) \in \mathfrak{S}(\mathcal{A}) \times \Gamma$ we have $\Xi(\mathcal{M}, \Lambda_{\mathcal{P}}(f)) = \Lambda_{\mathcal{M}}(f)$. Set $\mathcal{Z} = \Xi(\mathcal{M}, \Lambda_{\mathcal{P}}(f))$. First note that $\mathcal{Z}$ is in $\mathfrak{R}(\mathcal{M})$. Since $\mathcal{P}$ is thick in $\mathcal{M}$ and hence in $\mathcal{M}$, by Lemma 4.3,

$$\Phi_{\mathcal{M}}(\mathcal{Z}) = \Phi_{\mathcal{M}}(\operatorname{res}(\mathcal{M} \cup \Lambda_{\mathcal{P}}(f))) = \Phi_{\mathcal{M}}(\mathcal{M}) \vee \Phi_{\mathcal{M}}(\Lambda_{\mathcal{P}}(f)) = \Phi_{\mathcal{P}}(\Lambda(\mathcal{P})(f)) = f$$

and thus $\Lambda_{\mathcal{M}}(f) = \mathcal{Z}$, proving the claim. □

5. A generalization of the Auslander transpose

Let C be a semidualizing module, and set $-^{\dagger} = \operatorname{Hom}(-, C)$. For the entirety of this section, $\mathcal{A}$ denotes a thick subcategory of $\mathcal{G}_C$ that is closed under †. Recalling Proposition 3.6, $\mathcal{A}$-dim satisfies the Auslander–Buchsbaum formula. We set $\operatorname{res}_{\mathcal{A}} M = \operatorname{res}(\{M\} \cup \mathcal{A})$.

The Auslander transpose has been an invaluable tool in both representation theory and commutative algebra. In this section, we generalize the notion of the Auslander transpose using semidualizing modules and list some properties which we will use. The Auslander transpose has previously been generalized in [Geng 2013; Huang 1999], but the construction here is different.

Definition 5.1. An *$\mathcal{A}$-presentation* of X is an exact sequence $A_1 \xrightarrow{\varphi} A_0 \to X \to 0$ with $A_1, A_0 \in \mathcal{A}$. Set $\operatorname{Tr}_{\mathcal{A}} X = \operatorname{coker} \varphi^{\dagger}$.

When $C = R$, we get the usual Auslander transpose of X which we denote by $\operatorname{Tr} X$. The "functor" $\operatorname{Tr}_{\mathcal{A}}$ is not well defined up to isomorphism or even stable isomorphism, motivating a new equivalence relation. Finding the correct equivalence relation is actually a subtle affair. The equivalence relation must make $\operatorname{Tr}_{\mathcal{A}} X$ be well defined, but it must also detect resolving subcategories. For modules X and Y, we write $X \sim' Y$ and $Y \sim' X$ if there exists an $A \in \mathcal{A}$ such that $0 \to X \to Y \to A \to 0$ is exact. Let $\mathcal{A}$-equivalence, denoted by $\sim$, be the transitive closure of the relation $\sim'$. Since $\sim'$ is symmetric and reflexive, $\sim$ is an equivalence relation. Stable equivalence implies $\mathcal{A}$-equivalence, and when $\mathcal{A} = \mathcal{P}$, they are the same. We will see in a moment that $\operatorname{Tr}_{\mathcal{A}} X$ has the desired properties.

Remark 5.2. We would like to think of $\operatorname{Tr}_{\mathcal{A}}$ as a functor. However, $\operatorname{mod}(R)$ modulo $\mathcal{A}$-equivalence does not form a sensible category. However, a very similar construction is functorial. Let $P \xrightarrow{\varphi} Q \to X \to 0$ be a projective presentation. Set $\operatorname{Tr}_C X = \operatorname{coker} \varphi^{\dagger}$. A similar construction is given in [Geng 2013; Huang 1999]. We will briefly show that $\operatorname{Tr}_C : \operatorname{mod}(R)/\mathcal{A} \to \operatorname{mod}(R)/\mathcal{A}$ is a functor. We thank the referee for bringing the following construction to our attention.

We give some definitions first.

(1) $\mathcal{X}/\mathcal{Y}$ is the category whose objects are $\mathcal{X}$, and whose morphisms are

$$\operatorname{Hom}_{\mathcal{X}/\mathcal{Y}}(X_1, X_2) := \operatorname{Hom}_{\mathcal{X}}(X_1, X_2)/F_{\mathcal{Y}}(X_1, X_2),$$

where $X_1, X_2 \in \mathcal{X}$ and $F_{\mathcal{Y}}(X_1, X_2)$ is the subgroup of morphisms in $\mathcal{X}$ which factor through an object in $\mathcal{Y}$.

(2) $\operatorname{Morph} \mathcal{X}$ is the category whose objects are morphisms $f : X_1 \to X_2$. A morphism (g_1, g_2) between objects $f : X_1 \to X_2$ and $f' : X_1' \to X_2'$ in $\operatorname{Morph} \mathcal{X}$ is a pair of morphisms $g_1 : X \to X'$ and $g_2 : Y \to Y'$ such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "g_1"] \arrow[d, "f"] & X_1' \arrow[d, "f'"] \\ X_2 \arrow[r, "g_2"] & X_2' \end{tikzcd}$$

(3) For $f, f' \in \operatorname{Morph} \mathcal{X}$, a morphism $(g_1, g_2) : f \to f'$ is homotopically trivial if there exists an $h : X_2 \to X_1'$ such that $f'hf = g_2 f = f'g_1$. Let $H_{\mathcal{X}}(f, f')$ denote the subgroup of homotopically trivial maps.

(4) H-$\operatorname{Morph} \mathcal{X}$ is the category whose objects are the same as $\operatorname{Morph} \mathcal{X}$ but whose morphisms are

$$\operatorname{Hom}_{H\text{-}\operatorname{Morph} \mathcal{X}}(f, f') = \operatorname{Hom}_{\operatorname{Morph} \mathcal{X}}(f, f')/H_{\mathcal{X}}(f, f').$$

We now mimic the construction of the Auslander transpose [1971, Chapter 3, Section 1]. Set $\mathcal{C} = \operatorname{add} C$. The functor † restricts to a functor $^\dagger : \mathcal{P} \to \mathcal{C}$ which induces a contravariant functor

$$\begin{aligned} {}^\dagger : \operatorname{Morph} \mathcal{P} &\to \operatorname{Morph} \mathcal{C}, \\ f &\mapsto f^\dagger. \end{aligned}$$

It is easy to check that the group homomorphism

$$\begin{aligned} {}^\dagger : \operatorname{Hom}_{\operatorname{Morph} \mathcal{P}}(f, f') &\to \operatorname{Hom}_{\operatorname{Morph} \mathcal{C}}(f'^\dagger, f^\dagger), \\ (g_1, g_2) &\mapsto (g_2^\dagger, g_1^\dagger), \end{aligned}$$

maps the subgroup $H_{\mathcal{P}}(f, f')$ to $H_{\mathcal{C}}(f'^\dagger, f^\dagger)$. Therefore, † induces a functor $H\text{-}\operatorname{Morph} \mathcal{P} \to H\text{-}\operatorname{Morph} \mathcal{C}$. Furthermore, it is easy to check that the map

$$\begin{aligned} \operatorname{coker} : H\text{-}\operatorname{Morph} \mathcal{C} &\to \operatorname{mod}(R)/\mathcal{C}, \\ f &\mapsto \operatorname{coker} f, \end{aligned}$$

is a well defined functor. The discussion in [loc. cit.] indicates that there is a functor $\rho : \operatorname{mod}(R)/\mathcal{P} \to H\text{-}\operatorname{Morph} \mathcal{P}$ which sends a module to a projective presentation. We summarize these discussions with the following commutative diagram:

$$\begin{tikzcd} & \operatorname{Morph} \mathcal{P} \arrow[r, "\dagger"] \arrow[d] & \operatorname{Morph} \mathcal{C} \arrow[d] & \\ \operatorname{mod}(R)/\mathcal{P} \arrow[r, "\rho"] & H\text{-}\operatorname{Morph} \mathcal{P} \arrow[r, "\dagger"] & H\text{-}\operatorname{Morph} \mathcal{C} \arrow[r, "\operatorname{coker}"] & \operatorname{mod}(R)/\mathcal{C} \end{tikzcd}$$

The composition of the bottom row is Tr_C. Since $\mathcal{A}$ is closed under † and is thick in MCM, Tr_C fixes $\mathcal{A}$. Thus, since $\mathcal{P}, \mathcal{C} \subseteq \mathcal{A}$, it follows that Tr_C induces a functor $\operatorname{mod}(R)/\mathcal{A} \to \operatorname{mod}(R)/\mathcal{A}$, as desired.

This approach has two deficiencies. First of all, we cannot compute Tr_C using $\mathcal{A}$-resolutions. We will use $\mathcal{A}$-resolutions, for example in Lemma 5.4(4), to show that $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X \sim X$. Second of all, if X and Y are isomorphic in $\operatorname{mod}(R)/\mathcal{A}$, it is not clear if $\operatorname{res}_{\mathcal{A}} X = \operatorname{res}_{\mathcal{A}} Y$. Because of these issues, Tr_C cannot take the place of $\operatorname{Tr}_{\mathcal{A}}$ in this work.

We proceed to show that $\mathcal{A}$-equivalence is sufficient for our purposes.

Proposition 5.3. *For a module X, the module $\operatorname{Tr}_{\mathcal{A}} X$ is unique up to $\mathcal{A}$-equivalence.*

Proof. Let π be the projective presentation $P_1 \to P_0 \to X \to 0$, and let ρ be the $\mathcal{A}$-presentation $A_1 \to A_0 \to X \to 0$. Suppose there is an epimorphism $\pi \to \rho$.

Then there exists the commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B_2 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & 0 & \\
 & & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
 & & A_1 \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}
\tag{2}
$$

whose columns are exact, with B_0, B_1, B_2 in $\mathcal{A}$. Applying † to the diagram yields

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X^\dagger \arrow[r] \arrow[d, equal] & A_0^\dagger \arrow[r] \arrow[d] & A_1^\dagger \arrow[r] \arrow[d] & \mathrm{Tr}_{\mathcal{A}}^{\rho} X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^\dagger \arrow[r] & P_0^\dagger \arrow[r] \arrow[d] & P_1^\dagger \arrow[r] \arrow[d] & \mathrm{Tr}_{\mathcal{A}}^{\pi} X \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & B_0^\dagger \arrow[r] \arrow[d] & B_1^\dagger \arrow[r] \arrow[d] & B_2^\dagger \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}
$$

where $\mathrm{Tr}_{\mathcal{A}}^{\rho} X$ and $\mathrm{Tr}_{\mathcal{A}}^{\pi} X$ denote $\mathrm{Tr}_{\mathcal{A}} X$ computed using ρ and π, respectively. Since the rows are exact, and the middle two columns are exact, the snake lemma shows the last column is exact. Since $B_2^\dagger \in \mathcal{A}$, we see that $\mathrm{Tr}_{\mathcal{A}}^{\rho} X \sim \mathrm{Tr}_{\mathcal{A}}^{\pi} X$.

Consider any two $\mathcal{A}$-presentations, ρ and ρ'. It is easy to construct projective presentations ψ and ψ' with epimorphisms $\psi \to \rho$ and $\psi' \to \rho'$. In the proof of [Masek 1999, Proposition 4], it is shown that there is a projective presentation of π and epimorphisms $\pi \to \psi$ and $\pi \to \psi'$. Using our work so far, we know that $\mathrm{Tr}_{\mathcal{A}}^{\rho} X \sim \mathrm{Tr}_{\mathcal{A}}^{\pi} X \sim \mathrm{Tr}_{\mathcal{A}}^{\rho'}$. □

Lemma 5.4. *For any $X, Y \in \mathrm{mod}(R)$ such that $X \sim Y$, the following are true*:

(1) $\mathrm{res}_{\mathcal{A}} X = \mathrm{res}_{\mathcal{A}} Y$,

(2) $\Omega X \sim \Omega Y$,

(3) $\mathrm{Tr}_{\mathcal{A}} X \sim \mathrm{Tr}_{\mathcal{A}} Y$,

(4) $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X \sim X$.

Proof. It suffices to assume that $0 \to X \to Y \to A \to 0$ with $A \in \mathcal{A}$. Proving (1) is trivial. For suitable choices of syzygies, we have $0 \to \Omega X \to \Omega Y \to \Omega A \to 0$. Since ΩA is in $\mathcal{A}$, and since syzygies are unique up to stable, and hence $\mathcal{A}$-equivalence, this proves (2).

Now we show (3). Consider the diagram with exact rows

$$\begin{tikzcd}
 & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega A \arrow[r] & Q \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

with Q, P_0, P_1 projective and surjective vertical arrows. Using the snake lemma, we can extend this to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & B_1 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega A \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that B_1, B_0 are in $\mathcal{A}$. Applying † to this diagram gives the following:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A^{\dagger} \arrow[r] & Q^{\dagger} \arrow[r] \arrow[d] & (\Omega A)^{\dagger} \arrow[r] \arrow[d] & \operatorname{Ext}^1(A, C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y^{\dagger} \arrow[r] & P_0^{\dagger} \arrow[r] \arrow[d] & P_1^{\dagger} \arrow[r] \arrow[d] & \operatorname{Tr}_{\mathcal{A}} Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{\dagger} \arrow[r] & B_0^{\dagger} \arrow[r] \arrow[d] & B_1^{\dagger} \arrow[r] \arrow[d] & \operatorname{Tr}_{\mathcal{A}} X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Since $\mathrm{Ext}^1(A, C) = 0$, applying the snake lemma to the middle two columns yields $\mathrm{Tr}_{\mathcal{A}} X \cong \mathrm{Tr}_{\mathcal{A}} Y$. This proves (3).

To see (4), consider the projective presentation

$$P_1 \to P_0 \to X \to 0.$$

Then

$$P_0^{\dagger} \to P_1^{\dagger} \to \mathrm{Tr}_{\mathcal{A}} X \to 0$$

is an $\mathcal{A}$-presentation, which we use to compute $\mathrm{Tr}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}} X$, giving the result. □

We close this section with an example of a property shared by $\mathrm{Tr}_{\mathcal{A}}$ and Tr.

Lemma 5.5. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence in* $\mathrm{mod}(R)$. *For suitable choices of* $\mathrm{Tr}_{\mathcal{A}}$, *we have the exact sequence*

$$0 \to Z^{\dagger} \to Y^{\dagger} \to X^{\dagger} \to \mathrm{Tr}_{\mathcal{A}} Z \to \mathrm{Tr}_{\mathcal{A}} Y \to \mathrm{Tr}_{\mathcal{A}} X \to 0.$$

Furthermore, *if* $\mathrm{Ext}^i(X, C) = 0$, *then*

$$0 \to \mathrm{Tr}_{\mathcal{A}} \Omega^i Z \to \mathrm{Tr}_{\mathcal{A}} \Omega^i Y \to \mathrm{Tr}_{\mathcal{A}} \Omega^i X \to 0.$$

Proof. Let θ denote the map from Y to Z. We have the short exact sequence

$$0 \to \Omega^i X \to \Omega^i Y \xrightarrow{\Omega^i \theta} \Omega^i Z \to 0$$

for all $i \geq 0$. We can construct the following short exact sequence of $\mathcal{A}$ presentations.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A_1^0 \arrow[r] \arrow[d] & A_0^0 \arrow[r] \arrow[d] & \Omega^i X \arrow[r] \arrow[d] & 0 \\
A_1^1 \arrow[r] \arrow[d] & A_0^1 \arrow[r] \arrow[d] & \Omega^i Y \arrow[r] \arrow[d, "\Omega^i \theta"] & 0 \\
A_1^2 \arrow[r] \arrow[d] & A_0^2 \arrow[r] \arrow[d] & \Omega^i Z \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Applying † and also the snake lemma yields

$$0 \to (\Omega^i Z)^{\dagger} \xrightarrow{(\Omega^i \theta)^{\dagger}} \Omega^i Y \xrightarrow{\lambda} \Omega^i X^{\dagger} \xrightarrow{\varepsilon} \mathrm{Tr}_{\mathcal{A}} \Omega^i Z \xrightarrow{\eta} \mathrm{Tr}_{\mathcal{A}} \Omega^i Y \to \mathrm{Tr}_{\mathcal{A}} \Omega^i X \to 0.$$

Setting $i = 0$ at this stage gives us the first claim. The short exact sequence $0 \to \Omega^i X \to \Omega^i Y \to \Omega^i Z \to 0$ gives the following long exact sequence of Ext modules:

$$0 \to (\Omega^i Z)^\dagger \xrightarrow{(\Omega^i\theta)^\dagger} \Omega^i Y \xrightarrow{\lambda} \Omega^i X^\dagger \xrightarrow{\delta} \operatorname{Ext}^1(\Omega^i Z, C) \xrightarrow{\operatorname{Ext}^1(\Omega^i\theta, C)} \operatorname{Ext}^1(\Omega^i Y, C) \to \cdots.$$

We also have

$$\cdots \to \operatorname{Ext}^i(X, C) \to \operatorname{Ext}^{i+1}(Z, C) \xrightarrow{\operatorname{Ext}^{i+1}(\theta, C)} \operatorname{Ext}^{i+1}(Y, C) \to \cdots.$$

Since $\operatorname{Ext}^i(X, C) = 0$ by assumption, $\operatorname{Ext}^{i+1}(\theta, C)$ and $\operatorname{Ext}^1(\Omega^i\theta, C)$ are injective, forcing δ to be zero. Thus λ is surjective. Then the first long exact sequence shows that ε is zero, and so η is injective, giving the desired result. $\square$

6. Resolving subcategories which are maximal Cohen–Macaulay on the punctured spectrum

We keep the same conventions used in the previous section, except we also assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring. Recall that since $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$, according to Proposition 3.6, dimension with respect to $\mathcal{A}$ satisfies the Auslander–Buchsbaum formula. Set

$$\begin{aligned} \operatorname{res}_{\mathcal{A}} M &= \operatorname{res}(\{M\} \cup \mathcal{A}), \\ \Delta(\mathcal{A})_0 &= \{M \in \Delta(\mathcal{A}) \mid M_p \in \operatorname{add} \mathcal{A}_{\mathfrak{p}} \text{ for all } p \in \operatorname{spec} R \backslash \mathfrak{m}\}, \\ \Delta(\mathcal{A})_0^i &= \{M \in \Delta(\mathcal{A})_0 \mid \mathcal{A}\text{-dim}\, M \le i\}. \end{aligned}$$

This section is devoted to proving the following:

Theorem 6.1. *If* $(R, \mathfrak{m}, k)$ *is a local ring with* $\dim R = d$, *the filtration*

$$\mathcal{A} = \Delta(\mathcal{A})_0^0 \subsetneq \Delta(\mathcal{A})_0^1 \subsetneq \cdots \subsetneq \Delta(\mathcal{A})_0^d = \Delta(\mathcal{A})_0$$

is a complete list of the resolving subcategories of $\Delta(\mathcal{A})_0$ *containing* $\mathcal{A}$.

This theorem and its proof is a generalization of [Dao and Takahashi 2015, Theorem 2.1]. We now use results from the previous section to make the building blocks of the proof of Theorem 6.1.

Lemma 6.2. *For any module* $X \in \operatorname{mod}(R)$ *and for suitable choices of* $\operatorname{Tr}_{\mathcal{A}} X$ *and* $\Omega \operatorname{Tr}_{\mathcal{A}} \Omega X$,

$$0 \to \operatorname{Ext}^1(X, C) \to \operatorname{Tr}_{\mathcal{A}} X \to \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X \to 0.$$

Proof. With F_0, F_1, F_2 projective, consider the sequence

$$F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0 \to X \to 0.$$

We have $\operatorname{coker} g^\dagger = \operatorname{Tr}_{\mathcal{A}} X$. By the universal property of kernel and cokernel, we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} g^\dagger \arrow[r] \arrow[d, "\iota"] & F_1^\dagger \arrow[r] \arrow[d, equal] & \operatorname{Tr}_{\mathcal{A}} X \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \ker f^\dagger \arrow[r] & F_1^\dagger \arrow[r] & \operatorname{Im} f^\dagger \arrow[r] & 0
\end{tikzcd}$$

The snake lemma yields the exact sequence

$$0 \to \ker \iota \to 0 \to \ker \varepsilon \to \operatorname{Ext}^1(X, C) \to 0 \to \operatorname{coker} \varepsilon \to 0.$$

Thus ε is surjective and $\ker \varepsilon \cong \operatorname{Ext}^1(X, C)$, giving the exact sequence

$$0 \to \operatorname{Ext}^1(X, C) \to \operatorname{Tr}_{\mathcal{A}} X \to \operatorname{Im} f^\dagger \to 0.$$

It remains to show that $\operatorname{Im} f^\dagger \sim \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X$.

We have the short exact sequence $0 \to \operatorname{Im} f^\dagger \to F_2^\dagger \to \operatorname{Tr}_{\mathcal{A}} \Omega X \to 0$. Consider the pushout diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X \arrow[r, equal] \arrow[d] & \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X \arrow[d] & \\
0 \arrow[r] & \operatorname{Im} f^\dagger \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Im} f^\dagger \arrow[r] & F_2^\dagger \arrow[r] \arrow[d] & \operatorname{Tr}_{\mathcal{A}} \Omega X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with G projective. We have $\operatorname{Im} f^\dagger \sim T \sim \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X$, as desired. $\square$

Lemma 6.3. *If* $X \in \Delta(\mathcal{A})_0$, *for all* $0 \leq i < \operatorname{depth} C$, *for suitable choices of* $\operatorname{Tr}_{\mathcal{A}}$, *the following is exact*:

$$0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} X \to \operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Tr}_{\mathcal{A}} \Omega^i X \to \operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Ext}^{i+1}(X, C) \to 0.$$

Proof. Using Lemma 6.2, we have

$$0 \to \operatorname{Ext}^{i+1}(X, C) \to \operatorname{Tr}_{\mathcal{A}} \Omega^i X \to \Omega \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} X \to 0.$$

Since $X \in \Delta(\mathcal{A})_0$, we have $\operatorname{Ext}^{i+1}(X, C)_p = 0$ for every nonmaximal prime p. Thus $\operatorname{Ext}^{i+1}(X, C)$ has finite length, and so

$$\operatorname{Ext}^i(\operatorname{Ext}^{i+1}(X, C), C) = 0$$

for all $0 \le i < \operatorname{depth} C$. Thus, we can apply Lemma 5.5. $\square$

Lemma 6.4. *Let $X \in \Delta(\mathcal{A})_0$ and $0 < n \le \operatorname{depth} C$. Then*

$$\begin{aligned}\operatorname{res}_{\mathcal{A}}\bigl(X, \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C),\ \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C),\ \cdots, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\bigr) \\ = \operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n X,\ \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C), \\ \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C),\ \cdots,\ \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\bigr)\end{aligned}$$

Proof. The previous lemma tells us that

$$\begin{aligned}&\operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X,\ \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C)\bigr)\\ &\quad = \operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X,\ \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C)\bigr),\\ &\operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X,\ \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C)\bigr)\\ &\quad = \operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \Omega^2 \operatorname{Tr}_{\mathcal{A}} \Omega^2 X,\ \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C)\bigr),\\ &\qquad\qquad\vdots\\ &\operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} X,\ \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\bigr)\\ &\quad = \operatorname{res}_{\mathcal{A}}\bigl(\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n X,\ \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\bigr).\end{aligned}$$

Since $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X \sim X$, the result is now clear. $\square$

Lemma 6.5. *Let $0 \le n < \operatorname{depth} R$ and L a nonzero finite length module. There exists an $\mathcal{A}$-resolution $(G_\bullet, \partial^{L,n})$ of $\operatorname{Tr}_{\mathcal{A}} \Omega^n L$ such that $G_i = 0$ for all $i > n+1$ and*

$$\ker \partial_i^{L,n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-i} L$$

for all $1 \le i \le n$. In particular, $\operatorname{Tr}_{\mathcal{A}} \Omega^i L \in \operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} \Omega^n L)$ for all $0 \le i \le n$, $\mathcal{A}\text{-dim}(\operatorname{Tr}_{\mathcal{A}} \Omega^n L) = n+1$, and $\operatorname{Tr}_{\mathcal{A}} \Omega^n L \in \Delta(\mathcal{A})_0^{n+1}$.

Proof. Let $(F_\bullet, \partial)$ be a free resolution of L. Then we have

$$F_{n+1} \to F_n \to \Omega^n L \to 0$$

and

$$0 \to \Omega^n L \to F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to L \to 0.$$

Because L has finite length, and since $\operatorname{depth} C = \operatorname{depth} R$ by Proposition 3.6, we have $\operatorname{Ext}^i(L, C) = 0$ for all $0 \le i \le n$, and so we have the exact sequence

$$0 \to L^\dagger \to F_0^\dagger \xrightarrow{\partial_1^\dagger} F_1^\dagger \xrightarrow{\partial_2^\dagger} \cdots \xrightarrow{\partial_{n-1}^\dagger} F_{n-1}^\dagger \to (\Omega^n L)^\dagger \to 0.$$

Note that $L^\dagger = 0$ since L has finite length. Thus, splicing this exact sequence with

$$0 \to (\Omega^n L)^\dagger \to F_n^\dagger \xrightarrow{\partial_{n+1}^\dagger} F_{n+1}^\dagger \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to 0,$$

we create an $\mathcal{A}$-resolution of $\operatorname{Tr}_{\mathcal{A}} \Omega^n L$. So we set $G_i = F^{\dagger}_{n+1-i}$ for $0 \le i \le n+1$ and $G_i = 0$ for $i > n+1$. Set $\partial_i^{L,n} = \partial^{\dagger}_{n+2-i}$ for $1 \le i \le n+1$ and $\partial_i^{L,n} = 0$ for all $i > n+1$. Using our previous arguments for values less that n, we see that $\ker \partial_i^{L,n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-i} L$ for $0 \le i \le n$. Showing the first two claims.

It is now apparent that $\mathcal{A}$-dim $\operatorname{Tr}_{\mathcal{A}} \Omega^n L \le n+1$. If $\ker \partial_n^{L,n} = \operatorname{Tr}_{\mathcal{A}} L$ is in $\mathcal{A}$, then so is L since $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} L \sim L$. However, this is impossible since $L^{\dagger} = \operatorname{Ext}^0(L, C) = 0$. Therefore we have $\mathcal{A}$-dim $\operatorname{Tr}_{\mathcal{A}} \Omega^n L = n+1$. □

Lemma 6.6. *For all* $0 \le n < \operatorname{depth} R$ *and all nonzero finite length modules* L, $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L = \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n k$.

Proof. Let λ denote the length function for modules. If $L \ne 0$, then we can write $0 \to L' \to L \to k \to 0$ with $\lambda(L') < \lambda(L)$. Since by Proposition 3.6 $n < \operatorname{depth} R = \operatorname{depth} C$, we have $\operatorname{Ext}^n(L', C) = 0$, and so from Lemma 5.5,

$$0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^n k \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L' \to 0.$$

Thus, by induction, $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n k$.

Now we wish to show that $\operatorname{Tr}_{\mathcal{A}} \Omega^n k \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L$. We proceed by double induction, first on $\lambda(L)$ and then on n. The case $L = k$ is trivial, so suppose $\lambda(L) > 1$. Write $0 \to L' \to L \to k \to 0$ again. Since L' has depth zero, we can use Lemma 6.5 to get the resolution $(G_\bullet, \partial^{L'})$. Thus we have the exact sequence

$$0 \to \ker \partial_1^{L',n} \to G_0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to 0.$$

Taking the pullback diagram with our last exact sequence yields the following:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker \partial_1^{L',n} \arrow[r, equal] \arrow[d] & \ker \partial_1^{L',n} \arrow[d] & \\
0 \arrow[r] & \operatorname{Tr}_{\mathcal{A}} \Omega^n k \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Tr}_{\mathcal{A}} \Omega^n k \arrow[r] & \operatorname{Tr}_{\mathcal{A}} \Omega^n L \arrow[r] \arrow[d] & \operatorname{Tr}_{\mathcal{A}} \Omega^n L' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

It is now easy to see that it suffices to show that $\ker \partial_1^{L',n}$ is in $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L$. When $n = 0$, $(G_\bullet, \partial^{L',n})$ is the resolution

$$0 \to G_1 \xrightarrow{\partial_1^{L',0}} G_0 \to \operatorname{Tr}_{\mathcal{A}} L' \to 0,$$

and we are done since $\ker \partial_1^{L',0} = G_1 \in \mathcal{A} \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} L$. So suppose $n > 0$. We have $\ker \partial_1^{L',n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L'$, by Lemma 6.5. By induction, $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L$ and $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L'$ are the same as $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k$. So we have

$$\ker \partial_1^{L',n} \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n} L,$$

where the inclusion follows from Lemma 6.5, and we are done. □

These next proofs are similar to those in [Dao and Takahashi 2015] with the appropriate changes. They are included here for the sake of completeness.

Proposition 6.7. *For every* $0 < n \leq \operatorname{depth} R$, *we have* $\Delta(\mathcal{A})_0^n = \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L$ *for every nonzero finite length module* L.

Proof. By Lemma 6.6, we may assume that $L = k$. By Lemma 6.5, we know that $\mathcal{A}\text{-}\dim(\operatorname{Tr} \Omega^{n-1} k) = n$. Since localization commutes with cokernels, duals and syzygies, we have $\operatorname{Tr} \Omega^n k$ is in $\Delta(\mathcal{A})_0$ and hence in $\Delta(\mathcal{A})_0^n$. Suppose now that $X \in \Delta(\mathcal{A})_0^n$. Then $\Omega^n X \in \mathcal{A}$, and so $\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n X \in \mathcal{A}$. Furthermore, for each $i \geq 0$, the module $\operatorname{Ext}^{i+1}(X, C)$ has finite length. Hence, Lemma 6.6 implies that $\operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Ext}^{i+1}(X, C)$ is in $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^i k \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k$, where the inclusion follows from Lemma 6.5. By Lemma 6.4, we therefore have

$$X \in \operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n X,\ \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C),$$
$$\operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C),\ \cdots,\ \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\big) \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k$$

which concludes the proof. □

We now prove the main result of this section.

Proof of Theorem 6.1. We clearly have the chain

$$\mathcal{A} = \Delta(\mathcal{A})_0^0 \subsetneq \Delta(\mathcal{A})_0^1 \subsetneq \cdots \subsetneq \Delta(\mathcal{A})_0^d = \Delta(\mathcal{A})_0.$$

Take $X \in \Delta(\mathcal{A})_0^n \setminus \Delta(\mathcal{A})_0^{n-1}$ for $d \geq n \geq 1$. We need to show that $\operatorname{res}_{\mathcal{A}} X = \Delta(\mathcal{A})_0^n$, and we have $\operatorname{res}_{\mathcal{A}} X \subseteq \Delta(\mathcal{A})_0^n$. We proceed by induction. When $n = 0$, the statement is trivial. So assume that $n > 0$ and $\operatorname{res}_{\mathcal{A}} \Omega X = \Delta(\mathcal{A})_0^{n-1}$. Since $\operatorname{Ext}^n(X, C)$ has finite length, it suffices to show $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C) \in \operatorname{res}_{\mathcal{A}} X$, by Proposition 6.7.

Since $\Omega^n X \in \mathcal{A}$, the short exact sequence $0 \to \Omega^n X \to P \to \Omega^{n-1} X \to 0$, with P projective, is an $\mathcal{A}$ presentation of $\Omega^{n-1} X$. Using this presentation to compute $\operatorname{Tr}_{\mathcal{A}}$, we see that $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} X \sim \operatorname{Ext}^1(\Omega^{n-1} X, C) \cong \operatorname{Ext}^n(X, C)$. Therefore, $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} X \sim \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)$ by Lemma 5.4. Thus, it suffices to show that $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} X \in \operatorname{res}_{\mathcal{A}} X$, again by Lemma 5.4.

Let $0 < i \leq n - 1$. Since $\operatorname{Ext}^i(X, C)$ has finite length, Lemma 6.5 implies

$$\operatorname{Tr}_{\mathcal{A}} \Omega^{i-1} \operatorname{Ext}^i(X, C) \in \Delta(\mathcal{A})_0^i \subseteq \Delta(\mathcal{A})_0^{n-1} = \operatorname{res}_{\mathcal{A}} \Omega X \subseteq \operatorname{res}_{\mathcal{A}} X.$$

Therefore, Lemma 6.4 implies that

$$\begin{aligned}\operatorname{Tr}_{\mathcal{A}}\Omega^{n-1}\operatorname{Tr}_{\mathcal{A}}\Omega^{n-1}X \in \operatorname{res}_{\mathcal{A}}\big(X,\ \operatorname{Tr}_{\mathcal{A}}\operatorname{Ext}^1(X,C),\ \operatorname{Tr}_{\mathcal{A}}\Omega\operatorname{Ext}^2(X,C),\\ \cdots,\ \operatorname{Tr}_{\mathcal{A}}\Omega^{n-2}\operatorname{Ext}^{n-1}(X,C)\big) = \operatorname{res}_{\mathcal{A}}(X)\end{aligned}$$

as claimed. □

The following corollary is immediate from Theorem 6.1

Corollary 6.8. *If* $X \in \Delta(\mathcal{A})_0^n \setminus \Delta(\mathcal{A})_0^{n-1}$, *then* $\operatorname{res}_{\mathcal{A}} X = \Delta(\mathcal{A})_0^n$.

7. Resolving subcategories and semidualizing modules

In this section, we keep the same notations and conventions as the previous sections, except we do not assume that R is local. In this section, we classify the resolving subcategories of $\Delta(\mathcal{A})$ which contain $\mathcal{A}$. Note that it is easy to check that C_p is a semidualizing R_p-module for all $p \in \operatorname{spec} R$. In Corollary 8.2, we will see that for all $p \in \operatorname{spec} R$, $\operatorname{add}\mathcal{A}_p$ is a thick subcategory of $\mathcal{G}_{C_p}$ closed under $\operatorname{Hom}_{R_p}(-, C_p)$. The following is a modified version of [Dao and Takahashi 2014, Lemma 4.6], which is a generalization of [Takahashi 2009, Proposition 4.2]. For a module X, let $\operatorname{NA}(X) = \{p \in \operatorname{spec} R \mid X_p \notin \operatorname{add}\mathcal{A}_p\}$.

Proposition 7.1. *Suppose* $X \in \Delta(\mathcal{A})$. *For every* $p \in \operatorname{NA}(X)$, *there is a* $Y \in \operatorname{res}_{\mathcal{A}} X$ *such that* $\operatorname{NA}(Y) = \operatorname{V}(p)$ *and* $\operatorname{add}\mathcal{A}_\pi\text{-}\dim Y_\pi = \operatorname{add}\mathcal{A}_\pi\text{-}\dim X_\pi$ *for all* $\pi \in V(p)$.

Proof. If $\operatorname{NA}(X) = \operatorname{V}(p)$ we are done. So fix a $q \in \operatorname{NA}(X)\setminus\operatorname{V}(p)$. As in the proof of [Dao and Takahashi 2014, Lemma 4.6], choose an $x \in p\setminus q$ and consider the following pushout diagram.

$$\begin{tikzcd} 0 \arrow[r] & \Omega X \arrow[r] \arrow[d, "x"] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega X \arrow[r] & Y \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

with F projective. Immediately, we have $Y \in \operatorname{res}_{\mathcal{A}} X$. Furthermore, $Y_{q'} \in \operatorname{res} X_{q'}$ for all $q' \in \operatorname{spec} R$. Therefore, we have $\operatorname{NA}(Y) \subseteq \operatorname{NA}(X)$. The proof of [Dao and Takahashi 2014, Lemma 4.6] tells us that

$$\operatorname{depth}(Y_\pi) = \min\{\operatorname{depth}(X_\pi), \operatorname{depth}(R_\pi)\}$$

for all $\pi \in V(p)$. Thus, by Proposition 3.6, $\operatorname{add}\mathcal{A}_\pi\text{-}\dim Y_\pi = \operatorname{add}\mathcal{A}_\pi\text{-}\dim X_\pi$, for all $\pi \in V(p)$. In particular, this shows that $V(p)$ is contained in $\operatorname{NA}(Y)$.

Localizing at q yields the following:

$$\begin{tikzcd} 0 \arrow[r] & \Omega X_q \arrow[r] \arrow[d, "x"] & F_q \arrow[r] \arrow[d] & X_q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega X_q \arrow[r] & Y_q \arrow[r] & X_q \arrow[r] & 0 \end{tikzcd}$$

Note x is a unit in R_q. Thus, by the five lemma, Y_q is isomorphic to F_p and therefore is projective. So we have $q \notin \mathrm{NA}(Y)$ and hence $\mathrm{NA}(Y) \subsetneq \mathrm{NA}(X)$.

If $\mathrm{NA}(Y) \neq \mathrm{V}(p)$, then we may repeat this process and construct a Y' that, like Y, satisfies all the desired properties except $\mathrm{V}(p) \subseteq \mathrm{NA}(Y') \subsetneq \mathrm{NA}(Y) \subsetneq \mathrm{NA}(X)$. Since spec R is Noetherian, this process must stabilize after some iteration, producing the desired module. □

Lemma 7.2. *Let V be a nonempty finite subset of* spec R. *Let M be a module and $\mathcal{X}$ a resolving subcategory such that $M_p \in \operatorname{add} \mathcal{X}_p$ for some $p \in \operatorname{spec} R$. Then there exist exact sequences*

$$0 \to K \to X \to M \to 0 \quad \textit{and} \quad 0 \to L \to M \oplus K \oplus R^t \to X \to 0$$

with $X \in \mathcal{X}$, $\mathrm{NA}(L) \subseteq \mathrm{NA}(M)$, and $\mathrm{NA}(L) \cap V = \varnothing$.

Proof. The result is essentially contained in the proof of [Takahashi 2010, Proposition 4.7]. It shows the existence of the exact sequences and shows that V is contained in the free locus of L and thus $\mathrm{NA}(L) \cap V = \varnothing$. Furthermore, the last exact sequence in the proof shows that for any $p \in \operatorname{spec} R$, L_p is in $\operatorname{res} M_p$. Hence, if L_p is not in a resolving subcategory, then M_p cannot be in that category either, giving us $\mathrm{NA}(L) \subseteq \mathrm{NA}(M)$. □

These lemmas help to prove the following proposition which is a key component of the proof of Theorem 7.4. This next result is also where we use Corollary 6.8 of the last section.

Proposition 7.3. *Consider a module $M \in \operatorname{mod}(R)$ and a resolving subcategory $\mathcal{X} \in \mathfrak{R}(\mathcal{A})$. If for every $p \in \operatorname{spec} R$, there exists an $X \in \mathcal{X}$ such that*

$$\operatorname{add} \mathcal{A}_{\mathfrak{p}}\text{-}\dim M_p \leq \operatorname{add} \mathcal{A}_{\mathfrak{p}}\text{-}\dim X_p,$$

then M is in $\mathcal{X}$.

Proof. Because of Lemma 2.12, we may assume $(R, \mathfrak{m}, k)$ is local. We proceed by induction on $\dim \mathrm{NA}(M)$. If $\dim \mathrm{NA}(M) = -\infty$, then M is in $\mathcal{A}$ and we are done. Suppose $\dim \mathrm{NA}(M) = 0$. Then M is in $\Delta(\mathcal{A})_0^t$ where $t = \mathcal{A}\text{-}\dim X$. By Proposition 7.1, there exists a $Y \in \operatorname{res}_{\mathcal{A}} X \subseteq \mathcal{X}$ with $\mathcal{A}\text{-}\dim Y = t$ and $Y \in \Delta(\mathcal{A})_0$, and thus $Y \in \Delta(\mathcal{A})_0^t \setminus \Delta(\mathcal{A})_0^{t-1}$. By Corollary 6.8, $\operatorname{res}_{\mathcal{A}} Y = \Delta(\mathcal{A})_0^t$, and thus $M \in \operatorname{res}_{\mathcal{A}}(Y) \subseteq \mathcal{X}$.

The rest of the proof uses Lemma 7.2 and is identical to [Dao and Takahashi 2015, Theorem 3.5], except one replaces the nonfree locus of M by $\mathrm{NA}(M)$ and replaces projective dimension by $\mathcal{A}$-dim. □

We come to the main theorem of this section. Recall that Γ is the set of grade consistent functions.

Theorem 7.4. *Assume R is Noetherian. If $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ which is closed under †, then $\Lambda_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$ are inverse functions giving a bijection between Γ and $\mathfrak{R}(\mathcal{A})$.*

Proof. The previous proposition shows that $\Lambda_{\mathcal{A}}\Phi_{\mathcal{A}}$ is the identity on $\mathfrak{R}(\mathcal{A})$. Let $f \in \Gamma$ and $p \in \operatorname{spec} R$. Since $\operatorname{add}\mathcal{A}_p$-$\dim X_p \leq f(p)$ for every $X \in \Lambda_{\mathcal{A}}(f)$, we have $\Phi_{\mathcal{A}}(\Lambda_{\mathcal{A}}(f))(p) \leq f(p)$. However, by [Dao and Takahashi 2015, Lemma 5.1] there is an $M \in \Delta(\mathcal{P}) \subseteq \Delta(\mathcal{A})$ such that $\operatorname{pd}_{R_p} M_p = f(p)$ and $\operatorname{pd}_{R_q} M_q \leq f(q)$ for all $q \in \operatorname{spec} R$. Since for all $q \in \operatorname{spec} R$ $\operatorname{pd}_q M_q = \operatorname{add}\mathcal{A}_q$-$\dim M_q$, M is in $\Lambda_{\mathcal{A}}(f)$, and we have $\Phi_{\mathcal{A}}(\Lambda_{\mathcal{A}}(f))(p) = f(p)$. Thus $\Phi_{\mathcal{A}}\Lambda_{\mathcal{A}}$ is the identity on Γ. □

8. Resolving subcategories that are closed under †

We wish to expand upon Theorem 7.4 using the results in Section 4. However, to use Theorem 7.4, we need to understand which thick subcategories of $\mathcal{G}_C$ containing C are closed under duals. In this section, C is a semidualizing module. Since $\mathcal{G}_C$ is cogenerated by $\operatorname{add} C$, as seen at the end of Section 3, it stands to reason that the results of Section 4 are applicable.

Lemma 8.1. *Suppose $\mathcal{M} \subseteq \mathcal{G}_C$ is resolving with $C \in \mathcal{M}$. Then $\mathcal{M}$ is thick in $\mathcal{G}_C$ if and only if for every $M \in \mathcal{M}$, $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$. In particular, $\mathcal{M}$ is thick in $\mathcal{G}_C$ if any only if it is cogenerated by* $\operatorname{add} C$.

Since syzygies are unique up to projective summands, $(\Omega M^\dagger)^\dagger$ is unique up to $\operatorname{add} C$ summands. Thus, for our purposes, our choice of syzygy is inconsequential. When $R = C$, $(\Omega M^\dagger)^\dagger$ is the classical cosyzygy of a Gorenstein dimension zero module M. Thus in this case, the lemma is equivalent to saying that a resolving subcategory $\mathcal{M}$ of $\mathcal{G}_R$ is thick if and only if it is closed under cosyzygies.

Proof. Assume $\mathcal{M}$ is thick, and let $M \in \mathcal{M}$. We have the following exact sequence.

$$0 \to \Omega M^\dagger \to R^n \to M^\dagger \to 0$$

Applying † yields

$$0 \to M \to C^n \to (\Omega M^\dagger)^\dagger \to 0.$$

Since $C \in \mathcal{M}$, if $\mathcal{M}$ is thick in $\mathcal{G}_C$, $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$.

Conversely, suppose for every $M \in \mathcal{M}$, $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$. We wish to show that $\mathcal{M}$ is thick in $\mathcal{G}_C$. Since $\mathcal{M}$ is resolving, it suffices to check that $\mathcal{M}^\dagger$ is also resolving, since † is a duality on $\mathcal{G}_C$. It is also clear that $\mathcal{M}^\dagger$ is extension closed. Since $C \in \mathcal{M}$, we have $R \in \mathcal{M}^\dagger$. Therefore it suffices to check that $\mathcal{M}^\dagger$ is closed under syzygies. Take $Z = M^\dagger \in \mathcal{M}^\dagger$. Then since $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$, $(\Omega M^\dagger)^{\dagger\dagger} \cong \Omega M^\dagger = \Omega Z$ is in $\mathcal{M}^\dagger$, as desired. □

The following corollary, although intuitive, is not obvious, and it is not clear if it holds for other subcategories besides $\mathcal{G}_C$.

Corollary 8.2. *If $\mathcal{M}$ is thick in $\mathcal{G}_C$, then* $\operatorname{add}\mathcal{M}_p$ *is thick in $\mathcal{G}_{C_p}$ for all $p \in \operatorname{spec} R$.*

Proof. Take $p \in \operatorname{spec} R$. From Lemma 2.11, we know that $\operatorname{add}\mathcal{M}_p$ is resolving. By the previous lemma, it suffices to show that for all $M \in \operatorname{add}\mathcal{M}_p$, $(\Omega_{R_p} M^\dagger)^\dagger = \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(M, C_p), C_p)$ is in $\operatorname{add}\mathcal{M}_p$. For every $M \in \operatorname{add}\mathcal{M}_p$, there exists an N such that $M \oplus N = L_p$ for some $L \in \mathcal{M}$. Consider the following:

$$\begin{aligned}
(\Omega L^\dagger)^\dagger{}_p &= \operatorname{Hom}(\Omega_R \operatorname{Hom}(L, C),\ C)_p \\
&= \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(L_p, C_p),\ C_p) \\
&= \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(M \oplus N, C_p),\ C_p) \\
&= \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(M, C_p),\ C_p) \oplus \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(N, C_p),\ C_p)
\end{aligned}$$

By the previous lemma, $(\Omega L^\dagger)^\dagger$ is in $\mathcal{M}$, and so $(\Omega_{R_p} M^\dagger)^\dagger$ is in $\operatorname{add}\mathcal{M}_p$. □

Proposition 8.3. *Let $\mathcal{A}$ be the smallest thick subcategory of $\mathcal{G}_C$ containing C. Then $\mathcal{A}$ is closed under †.*

Since the intersection of thick subcategories of $\mathcal{G}_C$ is thick, it is clear that $\mathcal{A}$ exists.

Proof. First, let $\mathcal{W}$ be the set of modules obtained by applying † and Ω to R successive times. Suppose for a moment that $\operatorname{res}\mathcal{W} = \mathcal{A}$. Let $A \in \mathcal{A}$. We will show that $A^\dagger \in \mathcal{A}$ by inducting on the number of steps needed to construct A from $\mathcal{W}$. See [Takahashi 2009] for a precise definition of the notion of steps with regards to a resolving subcategory. If A takes 0 steps to construct, then A is either R or in $\mathcal{W}$, and the claim is clear. Suppose A is constructed in $n > 0$ steps. Then there exists B_1 and B_0 which can be constructed in $n-1$ steps and satisfy one of the following situations.

(1) $0 \to A \to B_0 \to B_1 \to 0$

(2) $0 \to B_0 \to A \to B_1 \to 0$

(3) $B_0 = A \oplus B_1$

Therefore one of the following is true:

(a) $0 \to B_1^\dagger \to B_0^\dagger \to A^\dagger \to 0$

(b) $0 \to B_1^\dagger \to A^\dagger \to B_0^\dagger \to 0$

(c) $B_0^\dagger = A^\dagger \oplus B_1^\dagger$

By induction, $B_0^\dagger$ and $B_1^\dagger$ are in $\mathcal{A}$. Since $\mathcal{A}$ is thick, each of these situations implies that $A^\dagger$ is in $\mathcal{A}$.

Therefore, it suffices to show that $\operatorname{res}\mathcal{W} = \mathcal{A}$. First, we show that $\operatorname{res}\mathcal{W}$ is a thick subcategory containing C. In light of Lemma 8.1, it suffices to show that for every $A \in \operatorname{res}\mathcal{W}$, we have $(\Omega A^\dagger)^\dagger \in \operatorname{res}\mathcal{W}$. We work as we did in the previous paragraph, and we proceed by induction on the number of steps needed to construct A from $\mathcal{W}$. When it takes 0 steps, then A is either R or in $\mathcal{W}$, in which case the claim is clear.

Suppose A needs $n > 0$ steps to be constructed. Working as we did in the previous paragraph, there exists modules B_1 and B_0 which can be constructed in $n-1$ steps and satisfy one of (1), (2), or (3) above. Therefore, one of the following is true:

(a) $0 \to (\Omega A^\dagger)^\dagger \to (\Omega B_0^\dagger)^\dagger \to (\Omega B_1^\dagger)^\dagger \to 0$

(b) $0 \to (\Omega B_0^\dagger)^\dagger \to (\Omega A^\dagger)^\dagger \to (\Omega B_1^\dagger)^\dagger \to 0$

(c) $(\Omega B_1^\dagger)^\dagger = (\Omega A^\dagger)^\dagger \oplus (\Omega B_0^\dagger)^\dagger$

By induction $(\Omega B_0^\dagger)^\dagger$ and $(\Omega B_1^\dagger)^\dagger$ are in $\operatorname{res} \mathcal{W}$. Since $\mathcal{W}$ is resolving, then so is $(\Omega A^\dagger)^\dagger$ as desired.

It suffices now to show that $\operatorname{res} \mathcal{W} \subseteq \mathcal{A}$. To do this, we show that each $W \in \mathcal{W}$ is in $\mathcal{A}$. We induct on $c(W)$, the smallest number of times it takes to apply Ω and † to R to obtain W. If $c(W) = 0$, then $W = R$, and we are done. If $c(W) = 1$, then W is either 0 or C which are both in $\mathcal{A}$. Therefore, we may assume that $c(W) > 1$. Then one of the following situations must occur.

(1) $A = \Omega^2 B$

(2) $A = B^{\dagger\dagger}$

(3) $A = \Omega(B^\dagger)$

(4) $A = (\Omega B)^\dagger$

where $c(B) = c(A) - 2$. By induction, B is in $\mathcal{A}$. In cases (1) and (2), it is clear that A is in $\mathcal{A}$ too. We have $c(B^\dagger) \leq c(B) + 1 < c(A)$, and so $B^\dagger$ is in $\mathcal{A}$ by induction. Now in case (3), the result is clear. So we assume that we are in case (4). By Lemma 8.1, $(\Omega(B^{\dagger\dagger}))^\dagger \cong (\Omega B)^\dagger = A$ must be in $\mathcal{A}$. □

For the rest of this section, $\mathcal{A}$ will continue to be the smallest thick subcategory of $\mathcal{G}_C$ containing C. It is immediate that $\mathcal{A}$ satisfies the assumptions of Theorem 7.4. We wish to apply the results from the beginning of the paper. Using the notation of Section 4, set $\mathfrak{S}(C) = \mathfrak{S}(\mathcal{A})$, i.e., let $\mathfrak{S}(C)$ be the collection resolving subcategories $\mathcal{M} \subseteq \operatorname{mod}(R)$ such that $\mathcal{A}$ cogenerates $\mathcal{M}$ and $\operatorname{add} \mathcal{A}_p$ is thick in $\operatorname{add} \mathcal{M}_p$ for every $p \in \operatorname{spec} R$.

Lemma 8.4. *Every thick subcategory of $\mathcal{G}_C$ which contains C is in $\mathfrak{S}(C)$. Furthermore, when R is Cohen–Macaulay, every element in $\mathfrak{S}(C)$ is contained in MCM. In particular, when $C = D$ is a dualizing module, $\mathfrak{S}(D)$ is the collection of thick subcategories of MCM containing D.*

Proof. Let $\mathcal{M}$ be a thick subcategory $\mathcal{G}_C$ containing C. It is clear from the definition of $\mathcal{A}$ that $\mathcal{M}$ contains $\mathcal{A}$. By Lemma 8.1, $\mathcal{M}$ is cogenerated by $\operatorname{add} C$, thus also by $\mathcal{A}$. By Corollary 8.2, $\operatorname{add} \mathcal{M}_p$ and $\operatorname{add} \mathcal{A}_p$ are thick in $\mathcal{G}_{C_p}$ for all primes $p \in \operatorname{spec} R$. Therefore, dimension with respect to each of these subcategories satisfies the

Auslander–Buchsbaum formula. It follows from Lemma 2.14 that add $\mathcal{A}_p$ is thick in add $\mathcal{M}_p$

Now suppose that R is Cohen–Macaulay and $\mathcal{X} \in \mathfrak{S}(C)$. Since $\mathcal{A}$ cogenerates $\mathcal{X}$, for any $X \in \mathcal{X}$ there exists $0 \to X \to A_0 \to \cdots \to A_d \to X' \to 0$ with each $A_i \in \mathcal{A}$ and $d = \operatorname{depth} R$. Since $\mathcal{A} \subseteq \mathrm{MCM}$, X is in MCM. The last statement is now clear, since in that case $\mathcal{G}_D = \mathrm{MCM}$. □

We now come to the main results of the paper.

Theorem 8.5. *Let $\mathcal{A}$ denote the smallest thick subcategory of $\mathcal{G}_C$ containing C. For any $\mathcal{M} \in \mathfrak{S}(C)$ (e.g., $\mathcal{M}$ is a thick subcategory of $\mathcal{G}_C$ containing C), $\Lambda_{\mathcal{M}}$ and $\Phi_{\mathcal{M}}$ give a bijection between $\mathfrak{R}(\mathcal{M})$ and Γ.*

Furthermore, the following is a bijection:

$$\Lambda : \mathfrak{S}(C) \times \Gamma \longrightarrow \bigcup_{\mathcal{M} \in \mathfrak{S}(C)} \mathfrak{R}(\mathcal{M}) \subseteq \mathfrak{R}$$

For any $\mathcal{M}, \mathcal{N} \in \mathfrak{S}(C)$ with $\mathcal{M} \subseteq \mathcal{N}$, the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{R}(\mathcal{N}) \arrow[dr, "\Phi_{\mathcal{N}}"] & \\
\mathfrak{R}(\mathcal{M}) \arrow[u, "\eta_{\mathcal{M}}^{\mathcal{N}}"] \arrow[r, "\Phi_{\mathcal{M}}"] & \Gamma \\
\mathfrak{R}(\mathcal{A}) \arrow[u, "\eta_{\mathcal{A}}^{\mathcal{M}}"] \arrow[ur, "\Phi_{\mathcal{A}}"'] &
\end{tikzcd}$$

In particular, $\rho_{\mathcal{M}}^{\mathcal{N}}$ and $\eta_{\mathcal{M}}^{\mathcal{N}}$ are inverse functions.

Proof. Proposition 8.3 states that $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ which contains C and is closed under †. Therefore, by Theorem 7.4, $\Lambda_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$ give a bijection between $\mathfrak{R}(\mathcal{A})$ and Γ. The first statement is an application of Theorem 4.2 and Lemma 8.4. The rest follows from Theorem 4.4. □

A resolving subcategory $\mathcal{X}$ is dominant if for every $p \in \operatorname{spec} R$, there is an $n \in \mathbb{N}$ such that $\Omega^n_{R_p} R_p/pR_p \in \operatorname{add} \mathcal{X}_p$.

Corollary 8.6. *Suppose R is Cohen–Macaulay and has a dualizing module. Then there is a bijection between resolving subcategories containing MCM and grade consistent functions. Furthermore, the following are equivalent for a resolving subcategory $\mathcal{X}$.*

(1) *$\mathcal{X}$ is dominant*

(2) *$MCM \subseteq \mathcal{X}$*

(3) $\Delta(\mathcal{X}) = \operatorname{mod}(R)$

Proof. Letting D be the dualizing module of R, MCM is the same as $\mathcal{G}_D$. Hence, by the previous theorem, $\Lambda_{\mathrm{MCM}} : \Gamma \to \mathfrak{R}(\mathrm{MCM})$ is a bijection, showing the first statement. From [Dao and Takahashi 2015, Theorem 1.3], the following is a bijection:

$$\begin{aligned} \xi : \Gamma &\to \{\text{Dominant resolving subcategories of } \operatorname{mod}(R)\} \\ f &\mapsto \{X \in \operatorname{mod}(R) \mid \operatorname{depth} X\mathfrak{p} \geq \operatorname{ht}\mathfrak{p} - f(\mathfrak{p})\} \end{aligned}$$

It is clear that $\xi(0) = \mathrm{MCM}$; hence, every dominant subcategory contains MCM. Furthermore, we have $\operatorname{mod}(R) = \Delta(\mathrm{MCM})$, and hence every dominant resolving subcategory is an element of $\mathfrak{R}(\mathrm{MCM})$. Then for any $f \in \Gamma$,

$$\begin{aligned} \xi(f) &= \{X \in \operatorname{mod}(R) \mid \operatorname{depth} X_{\mathfrak{p}} \geq \operatorname{ht}\mathfrak{p} - f(\mathfrak{p})\} \\ &= \{X \in \operatorname{mod}(R) \mid \operatorname{add}\mathrm{MCM}_{\mathfrak{p}}\text{-}\dim X_{\mathfrak{p}} \leq f(\mathfrak{p})\} = \Lambda_{\mathrm{MCM}}(f). \end{aligned}$$

Thus ξ equals Λ_{MCM}, showing the equivalence of (1) and (2).

It is clear that (2) implies (3). Now assume (3) and take a $p \in \operatorname{spec} R$. Then $\mathcal{X}\text{-}\dim R/p < \infty$, and this implies that $\Omega^n R/p \in \mathcal{X}$ for some n. Therefore, $\Omega^n_{R_p} R_p/pR_p \in \operatorname{add}\mathcal{X}_p$, so $\mathcal{X}$ is dominant. □

9. Gorenstein rings and vanishing of Ext

In this section, $(R, \mathfrak{m}, k)$ is a local Gorenstein ring. In this case, MCM is the same as $\mathcal{G}_R$, and Lemma 8.4 implies that $\mathfrak{S}(R)$ is merely the collection of thick subcategories of MCM. This gives us the following which recovers [Dao and Takahashi 2015, Theorem 7.4].

Theorem 9.1. *If R is Gorenstein, then we have the following commutative diagram of bijections*:

$$\begin{tikzcd} \{\text{Thick subcategories of MCM}\} \times \Gamma \arrow[dr, "\Lambda"] \arrow[dd, "\Lambda_{\mathcal{P}}"'] & \\ & \{\mathcal{Z} \in \mathfrak{R} \mid \mathcal{Z} \cap MCM \text{ is thick in MCM}\} \\ \{\text{Thick subcategories of MCM}\} \times \mathfrak{R}(\mathcal{P}) \arrow[ur, "\Xi"'] & \end{tikzcd}$$

where $\Xi(\mathcal{M}, \mathcal{X}) = \operatorname{res}(\mathcal{M} \cup \mathcal{X})$.

Proof. Let $\mathfrak{Z}$ be the collection of resolving subcategories whose intersection with MCM is thick in MCM. As observed before the Theorem, $\mathfrak{S}(R)$ is simply the thick subcategories of MCM. Since for any $\mathcal{M} \in \mathfrak{S}(R)$, $\Delta(\mathcal{M}) \cap \mathrm{MCM}$ is $\mathcal{M}$, the image of Λ lies in $\mathfrak{Z}$. Furthermore, for any $\mathcal{Z} \in \mathfrak{Z}$, $\mathcal{Z}$ is in $\mathfrak{R}(\mathcal{Z} \cap \mathrm{MCM})$, thus the result follows from Proposition 4.6 and Theorem 8.5. □

It is natural to ask when the image Λ is all of $\mathfrak{R}$. This happens precisely when every resolving subcategory of MCM is thick. This occurs, by [op. cit., Theorem 6.4], when R is a complete intersection. We will give a necessary condition for $\operatorname{Im}\Lambda = \mathfrak{R}$ by examining the resolving subcategories of the form

$$\mathcal{M}_{\mathcal{B}} = \{M \in \operatorname{mod}(R) \mid \operatorname{Ext}^{>0}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$$

where $\mathcal{B} \subseteq \operatorname{mod}(R)$. Dimension with respect to this category can be calculated in the following manner.

Lemma 9.2. *For all $\mathcal{B} \subseteq \operatorname{mod}(R)$,*

$$\mathcal{M}_{\mathcal{B}}\text{-dim}\, M = \inf\{n \mid \operatorname{Ext}^{>n}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$$

Proof. Let $M \in \operatorname{mod}(R)$. For all $i > 0$ and $j \geq 0$ and each $B \in \mathcal{B}$, we have $\operatorname{Ext}^{i+j}(M, B) = \operatorname{Ext}^i(\Omega^j M, B)$. So $\operatorname{Ext}^{i+n}(M, B) = 0$ for all $i \geq 0$ if and only if $\Omega^n M$ is in $\mathcal{M}_{\mathcal{B}}$. □

Lemma 9.3. *For any $\mathcal{B} \subseteq \operatorname{mod}(R)$, we have $\mathcal{M}_{\mathcal{B}} \cap \Delta(\mathcal{P}) = \mathcal{P}$.*

Proof. To prove this, it suffices to show that if $\operatorname{pd}(X) = n > 0$, then $\operatorname{Ext}^n(X, B) \neq 0$. Take a minimal free resolution

$$0 \to F_n \xrightarrow{d} F_{n-1} \to \cdots \to F_0 \to X \to 0.$$

Note that $\operatorname{Im}(d) \subseteq \mathfrak{m}F_{n-1}$. We then get the complex

$$0 \to \operatorname{Hom}(X, B) \to \operatorname{Hom}(F_0, B) \to \cdots \to \operatorname{Hom}(F_{n-1}, B) \xrightarrow{d^*} \operatorname{Hom}(F_n, B) \to 0.$$

Now $\operatorname{Im}(d^*)$ still lies in $\mathfrak{m}\operatorname{Hom}(F_n, B)$, and thus by Nakayama, d^* cannot be surjective. Hence we have $\operatorname{Ext}^n(X, B) = \operatorname{coker} d^* \neq 0$. □

Araya [2012] defined AB dimension by $\text{AB-dim}\, M = \max\{b_M, \mathcal{G}_R\text{-dim}\, M\}$, where

$$b_M = \min\{n \mid \operatorname{Ext}^{\gg 0}(M, B) = 0 \Rightarrow \operatorname{Ext}^{>n}(M, B) = 0\}.$$

Note that AB dimension satisfies the Auslander–Buchsbaum formula. Also, a ring is AB if and only if every module has finite AB dimension.

Lemma 9.4. *Taking $\mathcal{B} \subseteq \operatorname{mod}(R)$, if* $\text{AB-dim}\, M < \infty$ *for all $M \in \Delta(\mathcal{M}_{\mathcal{B}})$, then $\mathcal{M}_{\mathcal{B}}$ is a thick subcategory of MCM.*

Proof. Suppose $\text{AB-dim}\, \Delta(\mathcal{M}_{\mathcal{B}}) < \infty$. First, we show that $\mathcal{M}_{\mathcal{B}}$ is contained in MCM. Take any $M \in \mathcal{M}_{\mathcal{B}}$. There is an exact sequence $0 \to M \to Y \to X \to 0$ with $\operatorname{pd}(Y) < \infty$ and $X \in \text{MCM}$. We claim that X has AB dimension zero. Suppose $\operatorname{Ext}^{\gg 0}(X, Z) = 0$. Then $\operatorname{Ext}^{\gg 0}(Y, Z) = 0$ and since $\operatorname{pd} Y = \text{AB-dim}\, Y$, $\operatorname{Ext}^{>\operatorname{pd} Y}(Y, Z)$ is zero. Then we have $\operatorname{Ext}^{\gg 0}(M, Z) = 0$ and thus $\operatorname{Ext}^{>b_M}(M, Z) = 0$. Therefore $\operatorname{Ext}^i(X, Z) = 0$ for all $i > \max\{\operatorname{pd}(Y), b_M\} + 1$. Since R is Gorenstein, that means that X has finite $\mathcal{G}_R$ dimension, and thus X has finite AB dimension.

But since AB dimension satisfies the Auslander Buchsbaum formula, AB-dim X must be zero.

Since $Y \in \Delta(\mathcal{M}_{\mathcal{B}})$, we have $X \in \Delta(\mathcal{M}_B)$. So $\mathrm{Ext}^{\gg 0}(X, B) = 0$ for all $B \in \mathcal{B}$, and we have $\mathrm{Ext}^{>0}(X, B) = 0$ for all $B \in \mathcal{B}$. Hence X is in $\mathcal{M}_{\mathcal{B}}$. Therefore, Y is also in $\mathcal{M}_{\mathcal{B}}$, which, by Lemma 9.3, means that Y is projective and hence in MCM, forcing M to be in MCM as well.

Now to show that $\mathcal{M}_{\mathcal{B}}$ is thick in MCM, it suffices to show that $\mathcal{M}_{\mathcal{B}}$ is closed under cokernels of surjections in MCM. So take $0 \to L \to M \to N \to 0$ with $L, M, N \in \mathrm{MCM}$ and $L, M \in \mathcal{M}_{\mathcal{B}}$. Then $N \in \Delta(\mathcal{M}_{\mathcal{B}})$ and so $\mathrm{Ext}^{\gg 0}(N, B) = 0$ for all $B \in \mathcal{B}$. But then N has finite AB dimension by assumption. Since AB dimension satisfies the Auslander–Buchsbaum formula, AB-dim N is zero. So we have $\mathrm{Ext}^{>0}(N, B) = 0$ for all $B \in \mathcal{B}$, and hence, N is in $\mathcal{M}_{\mathcal{B}}$. □

Now let $d = \dim R$.

Theorem 9.5. *If R is Gorenstein, then the following are equivalent.*

(1) *R is AB.*

(2) *$\mathcal{M}_{\mathcal{B}}$ is a thick subcategory of MCM for all $\mathcal{B} \subseteq \mathrm{mod}(R)$.*

(3) *$\mathrm{MCM} \cap \mathcal{M}_{\mathcal{B}}$ is thick in MCM for every $\mathcal{B} \subseteq \mathrm{mod}(R)$.*

(4) *$\Lambda_{\mathcal{M}_{\mathcal{B}}}$ gives a bijection between $\mathfrak{R}(\mathcal{M}_{\mathcal{B}})$ and Γ for every $\mathcal{B} \subseteq \mathrm{mod}(R)$.*

(5) *For all $\mathcal{B} \subseteq \mathrm{mod}(R)$ and $M \in \mathcal{M}_{\mathcal{B}}$, Γ contains the function $f : \mathrm{spec}\, R \to \mathbb{N}$ defined by*

$$f(p) = \min\{n \mid \mathrm{Ext}^{>n}(M_p, B_p) = 0 \text{ for all } B \in \mathcal{B}\}.$$

Proof. The previous lemma shows that (1) implies (2), and (2) implies (3) is trivial. Assuming (3), we will show (1). Suppose $\mathrm{Ext}^{\gg 0}(M, B) = 0$. Then M is in $\Delta(\mathcal{M}_B)$. Letting $\dim R = d$, we have $\Omega^d M \in \Delta(\mathcal{M}_B) \cap \mathrm{MCM}$. For some $n \geq d$ we have $\Omega^n M \in \mathcal{M}_{\mathcal{B}} \cap \mathrm{MCM}$. But then we have

$$0 \to \Omega^n M \to F_{n-1} \to \cdots \to F_d \to \Omega^d M \to 0,$$

where each F_i is projective. By (3), $\Omega^d M$ is in $\mathcal{M}_{\mathcal{B}}$. So we have $\mathcal{M}_B$-dim $M \leq d$, and so $\mathrm{Ext}^{>d}(M, B) = 0$.

Theorem 4.2 shows that (2) implies (4). Lemma 9.2 shows that (4) implies (5). Since R is local, evaluating f at the maximal ideal shows that (5) implies (1). □

Corollary 9.6. *Set $r = d - \operatorname{depth} M$. If R is AB and $\mathrm{Ext}^{\gg 0}(M, B) = 0$, then $\mathrm{Ext}^r(M, B) \neq 0$. Furthermore, if $\mathrm{Ext}^r(M, B) = 0$ or $\mathrm{Ext}^i(M, B) \neq 0$ for $i > r$, then $\mathrm{Ext}^j(M, B) \neq 0$ for arbitrarily large j.*

Proof. Suppose R is AB. Then (2) holds and so $\mathcal{M}_B$ -dim satisfies the Auslander–Buchsbaum formula. If $\mathrm{Ext}^{\gg 0}(M, B) = 0$ then

$$r = \mathcal{M}_B\text{-dim}\, M = \max\{n \mid \mathrm{Ext}^n(M, B) \neq 0\}.$$

The second statement is just the contrapositive of the first statement. □

Corollary 9.7. *If R is Gorenstein and every resolving subcategory of MCM is thick, then R is AB.*

Proof. The assumption implies (2) in Theorem 9.5. □

Thus if Λ in Theorem 8.5 is a bijection from $\mathfrak{S}(R) \times \Gamma$ to $\mathfrak{R}$, then R is AB. Stevenson [2014a] shows that when R is a complete intersection, every resolving subcategory of MCM is closed under duals. The following gives a necessary condition for this property.

Corollary 9.8. *If R is Gorenstein and every resolving subcategory of MCM is closed under duals, then R is AB.*

Proof. Suppose every resolving subcategory of MCM is closed under duals. Let $\mathcal{M} \subseteq$ MCM be resolving. Let $-^* = \mathrm{Hom}(-, R)$. Then for every $M \in \mathcal{X}$, $(\Omega M^*)^*$ is in $\mathcal{M}$. By Lemma 8.1, $\mathcal{M}$ is thick. The result follows from the previous corollary. □

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WILLIAM SANDERS
DEPARTMENT OF MATHEMATICAL SCIENCES
NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
7491 TRONDHEIM
NORWAY
william.sanders@math.ntnu.no

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THE SYMPLECTIC PLACTIC MONOID, CRYSTALS, AND MV CYCLES

JACINTA TORRES

We study cells in generalized Bott–Samelson varieties for type C_n. These cells are parametrized by certain galleries in the affine building. We define a set of *readable galleries* — we show that the closure in the affine Grassmannian of the image of the cell associated to a gallery in this set is an MV cycle. This then defines a map from the set of readable galleries to the set of MV cycles, which we show to be a morphism of crystals. We further compute the fibers of this map in terms of the Littelmann path model.

1. Introduction

This paper is part of a project started by Gaussent and Littelmann [2005] the aim of which is to establish an explicit relationship between the path model and the set of MV cycles used by Mirković and Vilonen for the Geometric Satake equivalence proven in [Mirković and Vilonen 2007].

1A. We consider a complex connected reductive algebraic group G and its affine Grassmannian $\mathscr{G} = \mathrm{G}(\mathbb{C}((t)))/\mathrm{G}(\mathbb{C}[[t]])$. We fix a maximal torus $\mathrm{T} \subset \mathrm{G}$. The coweight lattice $\mathrm{X}^\vee = \mathrm{Hom}(\mathbb{C}^\times, \mathrm{T})$ can be seen as a subset of $\mathscr{G}$. For a coweight λ, which we may assume dominant with respect to some choice of Borel subgroup containing T, the closure X_λ of the $\mathrm{G}(\mathbb{C}[[t]])$-orbit of λ in $\mathscr{G}$ is an algebraic variety which is usually singular. The Geometric Satake equivalence identifies the complex irreducible highest weight module $\mathrm{L}(\lambda)$ for the Langlands dual group $\mathrm{G}^\vee$ with the intersection cohomology of X_λ, a basis of which is given by the classes of certain subvarieties of X_λ called MV cycles. The set of these subvarieties is denoted by $\mathscr{Z}(\lambda)$. The Geometric Satake equivalence implies that the elements of $\mathscr{Z}(\lambda)$ are in one to one correspondence with the vertices of the crystal $\mathrm{B}(\lambda)$. Braverman

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and Gaitsgory [2001], endow the set $\mathscr{Z}(\lambda)$ with a crystal structure and show the existence of a crystal isomorphism $\varphi : \mathrm{B}(\lambda) \xrightarrow{\sim} \mathscr{Z}(\lambda)$.

1B. Gaussent and Littelmann [2005] define a set $\Gamma(\gamma_\lambda)^{\mathrm{LS}}$ of LS galleries, which are galleries in the affine building $\mathscr{J}^{\mathrm{aff}}$ associated to G, and they endow this set with a crystal structure and an isomorphism of crystals $\mathrm{B}(\lambda) \xrightarrow{\sim} \Gamma(\gamma_\lambda)^{\mathrm{LS}}$. They view the latter as a subset of the T-fixed points in a desingularization $\Sigma_{\gamma_\lambda} \xrightarrow{\pi} \mathrm{X}_\lambda$. To each of these particular fixed points $\delta \in \Gamma(\gamma_\lambda)^{\mathrm{LS}}$ corresponds a Białynicki-Birula cell $\mathrm{C}_\delta \subset \Sigma_{\gamma_\lambda}$. Gaussent and Littelmann [2005] show that the closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle, and Baumann and Gaussent [2008] show that the map

$$\Gamma(\gamma_\lambda)^{\mathrm{LS}} \to \mathscr{Z}(\lambda), \quad \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a crystal isomorphism with respect to the crystal structure on $\mathscr{Z}(\lambda)$ described by Braverman and Gaitsgory [2001]. It is natural to ask whether the closures $\overline{\pi(\mathrm{C}_\delta)}$ are still MV cycles for a more general choice of fixed point δ.

1C. Gaussent and Littelmann [2012] consider *one skeleton* galleries, which are piecewise linear paths in $\mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. Such galleries can be interpreted in terms of Young tableaux for types A, B and C. For $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$, Gaussent et al. [2013] show that for any fixed point $\delta \in \Sigma_{\gamma_\lambda}^{\mathrm{T}}$, the closure $\overline{\pi(\mathrm{C}_\delta)}$ is in fact an MV cycle. They achieve this using combinatorics of Young tableaux such as word reading and the well known Knuth relations, and by relating them to the Chevalley relations for root subgroups which hold in the affine Grassmannian $\mathscr{G}$. In [Torres 2016] it is observed that word reading is a crystal morphism, and this allows one to prove that in this case, the map from all galleries to MV cycles is in fact a morphism of crystals. It was conjectured in [Gaussent et al. 2013] that generalizations of their results hold for arbitrary complex semisimple algebraic groups, in terms of the plactic algebra defined by Littelmann [1996]. It is with this in mind that we formulate and state our results.

1D. ***Results.*** We work with $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$. We define a set $\Gamma(\gamma_\lambda)^{\mathrm{R}} \supset \Gamma(\gamma_\lambda)^{\mathrm{LS}}$ of *readable* galleries, which have an explicit formulation in terms of Young tableaux. These galleries correspond to all galleries in type A. They are called keys in [Gaussent et al. 2013]. Type C combinatorics related to LS galleries has been developed by De Concini [1979], Kashiwara and Nakashima [1994], King [1976], Lakshmibai [1987] (in the context of standard monomial theory), Proctor [1990], Sheats [1999] and Lecouvey [2002], among others. We use the description of LS galleries of fundamental type given by Lakshmibai in [1987; 1986]. We use the formulation given by Lecouvey [2002]. There is a certain word reading described in [Lecouvey 2002] which we show to be a crystal morphism when restricted to readable galleries. We obtain results similar to those obtained in [Gaussent et al.

2013] concerning the defining relations of the *symplectic plactic monoid*, described explicitly by Lecouvey [2002], as well as words of readable galleries. These results together with the work of Gaussent and Littelmann [2005; 2012], and Baumann and Gaussent [2008] allow us to show in Theorem 6.2 that given a readable gallery $\delta \in \Gamma(\gamma_\lambda)^{\mathrm{R}}$ there is an associated dominant coweight $\nu_\delta \leq \lambda$ such that:

(1) The closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in X_{ν_δ}.

(2) The map

$$\Gamma(\gamma_\lambda)^{\mathrm{R}} \xrightarrow{\varphi_{\gamma_\lambda}} \bigoplus_{\delta \in \Gamma(\gamma_\lambda)^{\mathrm{R}}/\sim} \mathscr{Z}(\mu_{\delta^+}), \quad \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a morphism of crystals.

Here $\Gamma(\gamma_\lambda)^{\mathrm{R}}/\sim$ is some set of representatives for a certain equivalence relation on the set of readable galleries. We compute the fibers of this map in terms of the Littelmann path model. Moreover, this map induces an isomorphism when restricted to each connected component. We then provide some examples of galleries $\delta \in \Sigma^{\mathrm{T}}_{\gamma_\lambda} - \Gamma(\gamma_\lambda)^{\mathrm{R}}$ for which $\overline{\pi(\mathrm{C}_\delta)}$ is not an MV cycle in $\mathscr{Z}(\nu_\delta)$.

1E. This paper is organized as follows. In Section 2 we introduce our notation and recall several general facts about affine Grassmannians, MV cycles, galleries in the affine building, generalized Bott–Samelson varieties, and concrete descriptions of the cells C_δ in them. In Section 3 we introduce the crystal structure on combinatorial galleries, motivating our results with the Littelmann path model, and define readable galleries as concatenations of LS galleries of fundamental type and “zero lumps.” From Section 4 on we work with $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$, where we recall some type C combinatorics and build up to our main result, which we state and prove in Section 6. However, the main ingredients of the proof, stated in Section 5, are proven in Section 7. In Section 8 we exhibit some examples in special cases where the image of a certain cell cannot be an MV cycle. In the Appendix we show a technical result that we need.

2. Preliminaries

2A. ***Notation.*** Throughout this section, we consider G to be a complex connected reductive algebraic group associated to a root datum $(\mathrm{X}, \mathrm{X}^\vee, \Phi, \Phi^\vee)$, and we denote its Langlands dual by $\mathrm{G}^\vee$. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus of G with character group $\mathrm{X} = \mathrm{Hom}(\mathrm{T}, \mathbb{C}^\times)$ and cocharacter group $\mathrm{X}^\vee = \mathrm{Hom}(\mathbb{C}^\times, \mathrm{T})$. We will call elements of X weights, and elements of $\mathrm{X}^\vee$ coweights. We identify the Weyl group W with the quotient $\mathrm{N_G(T)/T}$, where $\mathrm{N_G(T)}$ denotes the normalizer of T in G. We will abuse notation by denoting a representative in $\mathrm{N_G(T)}$ of an element $w \in \mathrm{W}$ in the Weyl group by the same symbol, w, that we use to denote the element itself. We fix a choice of positive roots Φ^+ (this determines a set $\Phi^{\vee,+}$ of positive coroots), and denote the dominance order on X and $\mathrm{X}^\vee$ determined by this choice by $\leq$. We will denote the corresponding set of dominant weights and coweights by $\mathrm{X}^+ \subset \mathrm{X}$ and $\mathrm{X}^{\vee,+} \subset \mathrm{X}^\vee$ respectively. Let $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi^+$ be the basis or set of simple roots of Φ that is determined by Φ^+. The number n is called the rank of the root datum. Then the set $\Delta^\vee$ of all coroots $\alpha_i^\vee$ of elements $\alpha_i \in \Delta$ forms a basis of the root system $\Phi^\vee$. Let $\langle -,- \rangle$ be the nondegenerate pairing between X and $\mathrm{X}^\vee$, and denote the half sum of positive roots and coroots by ρ and $\rho^\vee$ respectively. Note that if $\lambda = \sum_{\alpha \in \Delta} n_\alpha \alpha$, respectively $\lambda = \sum_{\alpha^\vee \in \Delta^\vee} n_\alpha \alpha^\vee$, is a sum of positive roots then $\langle \lambda, \rho^\vee \rangle = \sum_{\alpha \in \Delta} n_\alpha$, respectively $\langle \rho, \lambda \rangle = \sum_{\alpha^\vee \in \Delta^\vee} n_\alpha$).

Let $\mathrm{B} \subset \mathrm{G}$ be the Borel subgroup of G containing T that is determined by the choice of positive roots Φ^+, and let $\mathrm{U} \subset \mathrm{B}$ be its unipotent radical. The group U is generated by the elements $\mathrm{U}_\alpha(b)$ for $b \in \mathbb{C}$, $\alpha \in \Phi^+$, where for each root α, U_α is the one-parameter group it determines. For each coweight $\lambda \in \mathrm{X}^\vee$ and each nonzero complex number $a \in \mathbb{C}^\times$, we denote its image $\lambda(a) \in \mathrm{T}$ by a^λ.

The following identities hold in G (See [Steinberg 1968, §6]):

- For any $\lambda \in \mathrm{X}^\vee$, $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$, and $\alpha \in \Phi$,

$$a^\lambda \mathrm{U}_\alpha(b) = \mathrm{U}_\alpha(a^{\langle \alpha, \lambda \rangle} b) a^\lambda. \tag{1}$$

- (Chevalley's commutator formula) Given linearly independent roots $\alpha, \beta \in \Phi$, there exist numbers $c_{\alpha,\beta}^{i,j} \in \{\pm 1, \pm 2, \pm 3\}$ such that, for all $a, b \in \mathbb{C}$,

$$\mathrm{U}_\alpha(a)^{-1} \mathrm{U}_\beta(b)^{-1} \mathrm{U}_\alpha(a) \mathrm{U}_\beta(b) = \prod_{i,j \in \mathbb{N}^{>0}} \mathrm{U}_{i\alpha + j\beta}\big(c_{\alpha,\beta}^{i,j} (-a)^i b^j\big). \tag{2}$$

  The product is taken in some fixed order. The $c_{\alpha,\beta}^{i,j}$ are integers which apart from depending on i and j depend also on α, β and on the chosen order in the product.

2B. ***Affine Grassmannians.*** Let $\mathscr{O} = \mathbb{C}[[t]]$ denote the ring of complex formal power series and let $\mathscr{K} = \mathbb{C}((t))$ denote its field of fractions; it is the field of complex Laurent power series. For any $\mathbb{C}$-algebra $\mathscr{R}$, we denote the set of $\mathscr{R}$-valued points of G by $\mathrm{G}(\mathscr{R})$. The set

$$\mathscr{G} = \mathrm{G}(\mathscr{K}) / \mathrm{G}(\mathscr{O})$$

is called the *affine Grassmannian* associated to G. We will denote the class in $\mathscr{G}$ of an element $g \in \mathrm{G}(\mathscr{K})$ by $[g]$. A coweight $\lambda : \mathbb{C}^\times \to \mathrm{T} \subset \mathrm{G}$ determines a point $t^\lambda \in \mathrm{G}(\mathscr{K})$ and hence a class $[t^\lambda] \in \mathscr{G}$. This map is injective, and we may therefore consider $\mathrm{X}^\vee$ as a subset of $\mathscr{G}$.

$\mathrm{G}(\mathscr{O})$-orbits in $\mathscr{G}$ are determined by the Cartan decomposition

$$\mathscr{G} = \bigsqcup_{\lambda \in \mathrm{X}^{\vee,+}} \mathrm{G}(\mathscr{O})[t^{\lambda}].$$

Each $\mathrm{G}(\mathscr{O})$-orbit has the structure of an algebraic variety induced from the progroup structure of $\mathrm{G}(\mathscr{O})$ and for a dominant coweight $\lambda \in \mathrm{X}^{\vee,+}$,

$$\overline{\mathrm{G}(\mathscr{O})[t^{\lambda}]} = \bigsqcup_{\substack{\mu \in \mathrm{X}^{\vee,+} \\ \mu \leq \lambda}} \mathrm{G}(\mathscr{O})[t^{\mu}].$$

We call the closure $\overline{\mathrm{G}(\mathscr{O})[t^{\lambda}]}$ a *generalized Schubert variety* and we denote it by X_{λ}. This variety is usually singular. We will review certain resolutions of singularities of it in Section 2E. The $\mathrm{U}(\mathscr{K})$-orbits in $\mathscr{G}$ are given by the Iwasawa decomposition

$$\mathscr{G} = \bigsqcup_{\lambda \in \mathrm{X}^{\vee}} \mathrm{U}(\mathscr{K})[t^{\lambda}].$$

These orbits are indvarieties, and their closures can be described by

$$\overline{\mathrm{U}(\mathscr{K})[t^{\lambda}]} = \bigcup_{\mu \leq \lambda} \mathrm{U}(\mathscr{K})[t^{\mu}]$$

for any $\lambda \in \mathrm{X}^{\vee}$ (see Proposition 3.1(a) of [Mirković and Vilonen 2007]).

2C. ***MV cycles and crystals.*** Let $\lambda \in \mathrm{X}^{\vee,+}$ and $\mu \in \mathrm{X}^{\vee}$ be a dominant integral coweight and any coweight, respectively. Let $\mathrm{L}(\lambda)$ be the irreducible representation of $\mathrm{G}^{\vee}$ of highest weight λ. Then by Theorem 3.2 in [Mirković and Vilonen 2007], the intersection $\mathrm{U}(\mathscr{K})[t^{\mu}] \cap \mathrm{G}(\mathscr{O})[t^{\lambda}]$ is nonempty if and only if μ is a weight of $\mathrm{L}(\lambda)$, and in that case its closure is pure dimensional of dimension $\langle \rho, \lambda + \mu \rangle$ and has the same number of irreducible components as the dimension of the μ-weight space $\mathrm{L}(\lambda)_{\mu}$ [Mirković and Vilonen 2007, Corollary 7.4]. Moreover, $\mathrm{X}^{\vee} \cong \mathrm{Hom}(\mathrm{T}^{\vee}, \mathbb{C}^{\times})$, where $\mathrm{T}^{\vee}$ is the Langlands dual of T, which is a maximal torus of $\mathrm{G}^{\vee}$ (see [Mirković and Vilonen 2007, §7]).

We denote the set of all irreducible components of a given topological space Y by Irr(Y). Consider the sets

$$\mathscr{Z}(\lambda)_{\mu} = \mathrm{Irr}(\overline{\mathrm{U}(\mathscr{K})[t^{\mu}] \cap \mathrm{G}(\mathscr{O})[t^{\lambda}]}) \quad \text{and} \quad \mathscr{Z}(\lambda) = \bigsqcup_{\mu \in \mathrm{X}^{\vee}} \mathscr{Z}(\lambda)_{\mu}.$$

Elements of these sets are called *MV cycles*. Braverman and Gaitsgory [2001, §3.3] have endowed the set $\mathscr{Z}(\lambda)$ with a crystal structure and have shown the existence of an isomorphism of crystals $\mathrm{B}(\lambda) \xrightarrow{\sim} \mathscr{Z}(\lambda)$. We do not use the definition of this crystal structure, but we denote by $\tilde{f}_{\alpha_i}$ (respectively $\tilde{e}_{\alpha_i}$) the corresponding root operators for $i \in \{1, \dots, n\}$, where n is the rank of the root system Φ. See Section 3A below for the definition of a crystal.

2D. ***Galleries in the affine building.*** Let $\mathscr{J}^{\mathrm{aff}}$ be the affine building associated to G and $\mathscr{K}$. It is a union of simplicial complexes called *apartments*, each of which is isomorphic to the Coxeter complex of the same type as the extended Dynkin diagram associated to G. We refer the reader to [Ronan 2009] for a thorough account of building theory. The affine Grassmannian $\mathscr{G}$ can be $\mathrm{G}(\mathscr{K})$-equivariantly embedded into the building $\mathscr{J}^{\mathrm{aff}}$, which also carries a $\mathrm{G}(\mathscr{K})$ action. Denote by Φ^{aff} the set of real affine roots associated to Φ; we identify it with the set $\Phi \times \mathbb{Z}$.

Let $\mathbb{A} = \mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. For each $(\alpha, m) \in \Phi^{\mathrm{aff}}$, consider the associated hyperplane and the positive and negative half spaces:

$$\begin{aligned} \mathrm{H}_{(\alpha,m)} &= \{x \in \mathbb{A} : \langle \alpha, x \rangle = m\}, \\ \mathrm{H}^+_{(\alpha,m)} &= \{x \in \mathbb{A} : \langle \alpha, x \rangle \geq m\}, \\ \mathrm{H}^-_{(\alpha,m)} &= \{x \in \mathbb{A} : \langle \alpha, x \rangle \leq m\}. \end{aligned}$$

The affine Weyl group $\mathrm{W}^{\mathrm{aff}}$ is generated by all the affine reflections $s_{(\alpha,m)}$ with respect to the affine hyperplanes $\mathrm{H}_{(\alpha,m)}$. We have an embedding $\mathrm{W} \hookrightarrow \mathrm{W}^{\mathrm{aff}}$ given by $s_\alpha \mapsto s_{(\alpha,0)}$, where $s_\alpha \in \mathrm{W}$ is the simple reflection associated to $\alpha \in \Phi$. (The Weyl group W is minimally generated by the set $\{s_{\alpha_i} : i \in \{1, \dots, n\}\}$.) The *dominant Weyl chamber* is the set

$$\mathrm{C}^+ = \{x \in \mathbb{A} : \langle \alpha, x \rangle \; > 0 \text{ for all } \alpha \in \Delta\},$$

and the *fundamental alcove* is in turn

$$\Delta^{\mathrm{f}} = \{x \in \mathrm{C}^+ : \langle \alpha, x \rangle \; \leq 1 \text{ for all } \alpha \in \Phi^+\}.$$

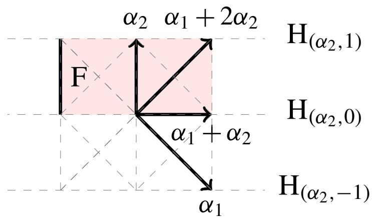
There is a unique apartment in the affine building $\mathscr{J}^{\mathrm{aff}}$ that contains the image of the set of coweights $\mathrm{X}^\vee \subset \mathscr{G}$ under the embedding $\mathscr{G} \hookrightarrow \mathscr{J}^{\mathrm{aff}}$. This apartment is isomorphic to the affine Coxeter complex associated to $\mathrm{W}^{\mathrm{aff}}$; its faces are given by all possible intersections of the hyperplanes $\mathrm{H}_{(\alpha,m)}$ and their associated (closed) positive and negative half-spaces $\mathrm{H}^\pm_{(\alpha,m)}$. It is called the *standard ,apartment* in the affine building $\mathscr{J}^{\mathrm{aff}}$. The action on the affine building $\mathscr{J}^{\mathrm{aff}}$ by $\mathrm{W}^{\mathrm{aff}}$ coincides, when restricted to the standard apartment, with the one induced by the natural action of $\mathrm{W}^{\mathrm{aff}}$ on $\mathbb{A}$. The fundamental alcove is a fundamental domain for this action.

To each real affine root $(\alpha, m) \in \Phi^{\mathrm{aff}}$ is attached the one-parameter additive *root subgroup* $\mathrm{U}_{(\alpha,m)}$ of $\mathrm{G}(\mathscr{K})$ defined by $b \mapsto \mathrm{U}_\alpha(bt^m)$ for $b \in \mathbb{C}$. Let $\lambda \in \mathrm{X}^\vee$ and $b \in \mathbb{C}$. Identity (1) implies that

$$\mathrm{U}_{(\alpha,m)}(b)[t^\lambda] = [\mathrm{U}_\alpha(bt^m)t^\lambda] = [t^\lambda \mathrm{U}_\alpha(bt^{m-\langle \alpha,\lambda \rangle})], \tag{3}$$

and $[t^\lambda \mathrm{U}_\alpha(bt^{m-\langle\alpha,\lambda\rangle})] = [t^\lambda]$ if and only if $\mathrm{U}_\alpha(bt^{m-\langle\alpha,\lambda\rangle}) \subset \mathrm{G}(\mathscr{O})$, or, equivalently, $\langle\alpha, \lambda\rangle \leq m$. Hence, the root subgroup $\mathrm{U}_{(\alpha,m)}$ stabilizes the point $[t^\lambda] \in \mathscr{G} \hookrightarrow \mathscr{J}^{\mathrm{aff}}$ if and only if $\lambda \in \mathrm{H}^-_{(\alpha,m)}$. For each face F in the standard apartment, denote by $\mathrm{P_F}$, $\mathrm{U_F}$ and $\mathrm{W^{aff}_F}$ its stabilizer in $\mathrm{G}(\mathscr{K})$, $\mathrm{U}(\mathscr{K})$ and $\mathrm{W^{aff}}$ respectively. These subgroups are generated by the torus T, and respectively by the root subgroups $\mathrm{U}_{(\alpha,m)}$ such that $\mathrm{F} \subset \mathrm{H}^-_{(\alpha,m)}$, the root subgroups $\mathrm{U}_{(\alpha,m)} \subset \mathrm{P_F}$ such that $\alpha \in \Phi^+$, and those affine reflections $s_{(\alpha,m)} \in \mathrm{W^{aff}}$ such that $\mathrm{F} \subset \mathrm{H}_{(\alpha,m)}$ [Gaussent and Littelmann 2005, §3.3, Example 3; Baumann and Gaussent 2008, Proposition 5.1].

Example 2.1. Let $\mathrm{G}^\vee = \mathrm{Sp}(4, \mathbb{C})$, then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$. In the picture below the shaded region is the upper half-space $\mathrm{H}^+_{(\alpha_2,0)}$. Let F be the face in the standard apartment that joins the vertices $-(\alpha_1+\alpha_2)$ and $-\alpha_1$. This is depicted here.



The subgroup $\mathrm{P_F}$ is generated by the root subgroups associated to the following real roots:

$$\begin{aligned}
(\alpha_1, m)\ & m \geq -1,\\
(\alpha_2, m)\ & m \geq 1,\\
(\alpha_1+\alpha_2, m)\ & m \geq -1,\\
(\alpha_1+2\alpha_2, m)\ & m \geq 0,\\
(-\alpha_1, m)\ & m \geq 2,\\
(-\alpha_2, m)\ & m \geq 0,\\
(-(\alpha_1+\alpha_2), m)\ & m \geq 1,\\
(-(\alpha_1+2\alpha_2), m)\ & m \geq 1.
\end{aligned}$$

The stabilizer $\mathrm{U_F}$ is generated by the root subgroups associated to those previously stated roots (α, m) such that $\alpha \in \Phi^+$ is a positive root, and $\mathrm{W^{aff}_F} = \{s_{(\alpha_1+\alpha_2,-1)}, 1\}$.

A *gallery* is a sequence of faces in the affine building $\mathscr{J}^{\mathrm{aff}}$,

$$\gamma = (\mathrm{V}_0 = 0, \mathrm{E}_0, \mathrm{V}_1, \ldots, \mathrm{E}_k, \mathrm{V}_{k+1}), \tag{4}$$

satisfying these conditions:

1. For each $i \in \{1, \dots, k\}$, $\mathrm{V}_i \subset \mathrm{E}_i \supset \mathrm{V}_{i+1}$.
2. Each face labeled V_i has dimension zero (is a *vertex*) and each face labeled E_i has dimension one (is an *edge*). In particular, each face in the sequence γ is contained in the one-skeleton of the standard apartment.
3. The last vertex V_{k+1} is a *special vertex*: its stabilizer in the affine Weyl group $\mathrm{W}^{\mathrm{aff}}$ is isomorphic to the finite Weyl group W associated to G.

We denote the set of all galleries in the affine building by Σ. If, in addition, each face in the sequence belongs to the standard apartment, then γ is called a *combinatorial gallery*. We will denote the set of all combinatorial galleries in the affine building by Γ. In this case, the third condition is equivalent to requiring the last vertex V_{k+1} to be a coweight. From now on, if γ is a combinatorial gallery we will denote the coweight corresponding to its final vertex by μ_γ in order to distinguish it from the vertex.

Remark 2.2. The galleries we defined are actually called *one-skeleton* galleries in the literature. The word "gallery" was originally used to describe a more general class of face sequences but since we only work with one-skeleton galleries in this paper, we have left the word "one-skeleton" out.

2E. ***Bott–Samelson varieties.*** Let γ be a combinatorial gallery (as above). The following lemma can be obtained from [Gaussent and Littelmann 2012, Lemma 4.8 and Definition 4.6].

Lemma 2.3. *There exist a unique combinatorial gallery*,

$$\gamma^f = (\mathrm{V}_0^f, \mathrm{E}_0^f, \mathrm{V}_1^f, \dots, \mathrm{V}_{k+1}^f),$$

with each one of its faces contained in the fundamental alcove, *and elements* $w_j \in \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_j^f}$ *for each* $j \in \{1, \dots, k\}$ *such that* $w_0 \cdots w_{r-1}\mathrm{V}_r^f = \mathrm{V}_r$ *for each* $r \in \{0, \dots, k+1\}$ *and* $w_0 \cdots w_r \mathrm{E}_r^f = \mathrm{E}_r$ *for each* $r \in \{0, \dots, k\}$.

If two galleries γ and η have the same associated gallery $\nu = \gamma^f = \eta^f$ we say that the two galleries have *the same type*. We will denote the set of combinatorial galleries that have the same type as a given combinatorial gallery γ by $\Gamma(\gamma)$. The map

$$\mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_0} \times \cdots \times \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_k} \to \Gamma(\gamma), \tag{5}$$

$$(w_0, \dots, w_k) \mapsto (\mathrm{V}_0, w_0\mathrm{E}_0, w_0\mathrm{V}_1, w_0w_1\mathrm{E}_1, \dots, w_0 \cdots w_k\mathrm{V}_{k+1}), \tag{6}$$

induces a bijection between the set $\prod_{i=0}^r \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_i}/\mathrm{W}^{\mathrm{aff}}_{\mathrm{E}_i}$ and $\Gamma(\gamma)$; it is in particular finite. For a proof see [Gaussent and Littelmann 2012, Lemma 4.8].

Definition 2.4. The *Bott–Samelson variety* of type γ^f is the quotient of

$$\mathrm{G}(\mathscr{O}) \times \mathrm{P}_{\mathrm{V}_1^f} \times \cdots \times \mathrm{P}_{\mathrm{V}_k^f}$$

by the following right action of $\mathrm{P}_{\mathrm{E}_0^f} \times \cdots \times \mathrm{P}_{\mathrm{E}_k^f}$:

$$(q_0, \ldots, q_k) \cdot (p_0, p_1, \ldots, p_k) = (q_0 p_0, p_0^{-1} q_1 p_1, \ldots, p_{k-1}^{-1} q_k p_k).$$

We will denote this quotient by Σ_{γ^f}. The progroup structure of the groups $\mathrm{P}_{\mathrm{V}_i^f}$ and $\mathrm{P}_{\mathrm{E}_i^f}$ assures that Σ_{γ^f} is in fact a smooth variety. To each point $(g_0, \ldots, g_k)$ in $\mathrm{G}(\mathscr{O}) \times \mathrm{P}_{\mathrm{V}_1^f} \times \cdots \times \mathrm{P}_{\mathrm{V}_k^f}$ one can associate a gallery

$$(\mathrm{V}_0^f, g_0\mathrm{E}_0^f, g_0\mathrm{V}_1^f, g_0g_1\mathrm{V}_2^f, \ldots, g_0 \cdots g_k \mathrm{V}_{k+1}^f). \tag{7}$$

This induces a well defined injective map $i : \Sigma_{\gamma^f} \hookrightarrow \Sigma$. With respect to this identification, the T-fixed points in Σ_{γ^f} are in natural bijection with the set $\Gamma(\gamma^f)$ of combinatorial galleries of type γ^f.

Let $\omega \in \mathbb{A}$ be a fundamental coweight. We define a particular combinatorial gallery, which starts at 0 and ends at ω. Let $\mathrm{V}_1^\omega, \ldots, \mathrm{V}_k^\omega$ be the vertices in the standard apartment that lie on the open line segment joining 0 and ω, numbered such that V_{i+1}^ω lies on the open line segment joining V_i^ω and ω. Let further E_i^ω denote the face contained in $\mathbb{A}$ that contains the vertices V_i^ω and V_{i+1}^ω. The gallery

$$\gamma_\omega = (0 = \mathrm{V}_0^\omega, \mathrm{E}_0^\omega, \mathrm{V}_1^\omega, \mathrm{E}_1^\omega, \ldots, \mathrm{E}_k^\omega, \mathrm{V}_{k+1}^\omega = \omega)$$

is called a *fundamental gallery*. Galleries of the same type as a fundamental gallery γ_ω will be called *galleries of fundamental type ω*.

Now let $\lambda \in \mathrm{X}^{\vee,+}$ be a dominant integral coweight and let γ_λ be a gallery with endpoint λ and expressible as a concatenation of fundamental galleries, where concatenation of two combinatorial galleries $\gamma_1 * \gamma_2$ is defined by translating γ_2 to the endpoint of γ_1. (Note that it follows from the definition of type that if γ, ν are two galleries of the same type as δ and η respectively, then $\gamma * \nu$ has the same type as $\delta * \eta$. Actually, if $\gamma = \gamma_1 * \cdots * \gamma_r$ then $\Gamma(\gamma) = \{\delta_1 * \cdots * \delta_r : \delta_i \in \Gamma(\gamma_i)\}$.) Then the map

$$\Sigma_{\gamma_\lambda^f} \xrightarrow{\pi} \mathrm{X}_\lambda, \quad [g_0, \ldots, g_r] \mapsto g_0 \cdots g_r [t^{\mu_{\gamma^f}}] \tag{8}$$

is a resolution of singularities of the generalized Schubert variety X_λ.

Remark 2.5. That the above map is in fact a resolution of singularities is due to the fact that the gallery γ_λ is minimal (see [Gaussent and Littelmann 2012, §5 and §4.3, Proposition 5]). This resembles the condition for usual Bott–Samelson varieties associated to a reduced expression. See [Gaussent and Littelmann 2005, §9, Proposition 7].

Remark 2.6. The map (8) makes sense for any combinatorial gallery γ. In this generality one has a map $\Sigma_{\gamma^f} \xrightarrow{\pi} \mathcal{G}$ sending $[g_0, \dots, g_r]$ to $g_0, \dots, g_r[t^{\mu_\gamma}]$, which is not necessarily a resolution of singularities. From now on we will write (Σ_{γ^f}, π) to refer to the Bott–Samelson variety together with its map $\Sigma_{\gamma^f} \xrightarrow{\pi} \mathcal{G}$ to the affine Grassmannian.

2F. *Cells and positive crossings.* Let $r_\infty : \mathcal{J}^{\text{aff}} \to \mathbb{A}$ be the retraction at infinity (see [Gaussent and Littelmann 2005, Definition 8]). It extends to a map

$$r_{\gamma^f} : \Sigma_{\gamma^f} \to \Gamma(\gamma^f).$$

To a combinatorial gallery $\delta \in \Gamma(\gamma^f)$ is associated the cell $\mathrm{C}_\delta = r_{\gamma^f}^{-1}(\delta)$ which was explicitly described in [Gaussent and Littelmann 2005; 2012; Baumann and Gaussent 2008]. In this subsection we recollect their results; we will need them later. They are originally formulated in terms of galleries of the same type as γ_λ; we formulate them for any combinatorial gallery. The proofs remain the same, and therefore we do not provide them all, but refer the reader to [Gaussent and Littelmann 2005; 2012].

First consider the subgroup $\mathrm{U}(\mathcal{K})$ of $\mathrm{G}(\mathcal{K})$. It is generated by the elements of the root subgroups $\mathrm{U}_{(\alpha,n)}$ for $\alpha \in \Phi^+$ a positive root and $n \in \mathbb{Z}$. Let $\mathrm{V} \subset \mathrm{E}$ be a vertex and an edge (respectively) in the standard apartment, the vertex contained in the edge. Consider the subset of affine roots

$$\Phi^+_{(\mathrm{V},\mathrm{E})} = \{(\alpha, n) \in \Phi^{\text{aff}} : \alpha \in \Phi^+, \mathrm{V} \in \mathrm{H}_{(\alpha,n)}, \mathrm{E} \not\subseteq \mathrm{H}^-_{(\alpha,n)}\},$$

and let $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$ denote the subgroup of $\mathrm{U}(\mathcal{K})$ generated by $\mathrm{U}_{(\alpha,n)}$ for all $(\alpha, n) \in \Phi^+_{(\mathrm{V},\mathrm{E})}$. The following proposition will be very useful in Section 7. It is stated and proven in [Baumann and Gaussent 2008, Proposition 5.1].

Proposition 2.7. *Let* $\mathrm{V} \subset \mathrm{E}$ *be a vertex and an edge in the standard apartment as above. Then* $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$ *is a set of representatives for the right cosets of* U_E *in* U_V. *For any total order on the set* $\Phi^+_{(\mathrm{V},\mathrm{E})}$, *the map*

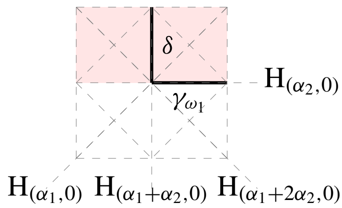
$$(a_\beta)_{\beta \in \Phi^+_{(\mathrm{V},\mathrm{E})}} \mapsto \prod_{\beta \in \Phi^+_{(\mathrm{V},\mathrm{E})}} \mathrm{U}_\beta(a_\beta)$$

is a bijection from $\mathbb{C}^{|\Phi^+_{(\mathrm{V},\mathrm{E})}|}$ *onto* $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$. *The order in the product is the same as the one on the set* $\Phi^+_{(\mathrm{V},\mathrm{E})}$.

Now let γ be a combinatorial gallery with notation as in (4). For each $i \in \{1, \dots, k\}$, let $\mathbb{U}^\gamma_{\mathrm{V}_i} = \mathbb{U}_{(\mathrm{V}_i,\mathrm{E}_i)}$. For later use we fix the notation $\Phi^\gamma_i = \Phi^+_{(\mathrm{V}_i,\mathrm{E}_i)}$.

Example 2.8. Let $\mathrm{G}^\vee = \mathrm{Sp}(4, \mathbb{C})$ as in Example 2.1, and γ_{ω_1} be as in Definition 2.4. Then $\mathbb{U}^{\gamma_{\omega_1}}_{\mathrm{V}_0}$ is generated by the root subgroups associated to the real roots $(\alpha_1, 0)$, $(\alpha_1 + \alpha_2, 0)$, and $(\alpha_1 + 2\alpha_2, 0)$. Let δ be the gallery with one edge and endpoint α_2.

Then $\mathbb{U}^{\delta}_{\mathrm{V}_0}$ is generated by the groups associated to $(\alpha_2, 0)$, $(\alpha_1 + 2\alpha_2, 0)$, as seen here.



Now write $\delta = (\mathrm{V}_0, \mathrm{E}_0, \ldots, \mathrm{E}_k, \mathrm{V}_{k+1}) \in \Gamma(\gamma^f)$ in terms of (7) as $\delta = [\delta_0, \ldots, \delta_k]$. This means $\delta_i \in \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_i^f}$ and $\delta_0 \cdots \delta_j \mathrm{E}_j^f = \mathrm{E}_j$. A beautiful exposition of the following description (Theorem 2.9) of the cell C_δ can be found in [Gaussent and Littelmann 2012, Proposition 4.19]. We provide an outline of the proof for the benefit of the reader and in order to state Corollary 2.10, which is actually a corollary to its proof.

Theorem 2.9. *The map* $\varphi : \mathbb{U}^\delta = \mathbb{U}^\delta_{\mathrm{V}_0} \times \mathbb{U}^\delta_{\mathrm{V}_1} \times \cdots \times \mathbb{U}^\delta_{\mathrm{V}_k} \to \Sigma_{\gamma^{\mathrm{f}}}$ *given by*

$$(v_0, \ldots, v_k) \mapsto [v_0\delta_0, \delta_0^{-1} v_1 \delta_0 \delta_1, \ldots, (\delta_0 \cdots \delta_{k-1})^{-1} v_k \delta_0 \cdots \delta_k]$$

is injective and has image C_δ.

Proof. Let $\widetilde{\mathbb{U}} = \mathrm{U}_{\mathrm{V}_0} \times \cdots \times \mathrm{U}_{\mathrm{V}_k} / \mathrm{U}_{\mathrm{E}_0} \times \cdots \times \mathrm{U}_{\mathrm{E}_k}$ where

$$(e_0, \ldots, \varepsilon_k) \cdot (v_0, \ldots, v_k) = (v_0 e_0, e_0^{-1} v_1 e_1, \ldots, e_{k-1}^{-1} v_k e_k).$$

The map $(v_0, \ldots, v_k) \mapsto [v_1, \ldots, v_k]$ defines a bijection $\phi : \mathbb{U}^\delta \to \widetilde{\mathbb{U}}$. Indeed, by [Gaussent and Littelmann 2012, Proposition 4.17], $\mathbb{U}_{\mathrm{V}_i}$ is a set of representatives for right cosets of $\mathrm{U}_{\mathrm{E}_j}$ in $\mathrm{U}_{\mathrm{V}_j}$, and hence for $[a_0, \ldots, a_k] \in \widetilde{\mathbb{U}}$ there is a unique $(v_0, \ldots, v_k) \in \mathbb{U}$ such that (for some $e_j \in \mathrm{U}_{\mathrm{E}_j}$) $v_0 e_0 = a_0$, and $v_j e_j = e_{j-1} a_j$, i.e., $\phi((v_0, \ldots, v_k)) = [a_0, \ldots, a_k]$. We use this bijection and consider instead the map $\widetilde{\varphi} := \varphi \circ \phi^{-1}$. Fix $[v_0, \ldots, v_k] \in \widetilde{\mathbb{U}}$. The map $\widetilde{\varphi}$ is well defined because $(\delta_0 \cdots \delta_{j-1})^{-1} v_{ij} (\delta_0 \cdots \delta_j) \in \mathrm{P}_{\mathrm{V}_j^f}$, and if $e_j \in \mathrm{U}_{\mathrm{E}_j}$ then $(\delta_0 \cdots \delta_j)^{-1} e_j (\delta_0 \cdots \delta_j) \in \mathrm{U}_{\mathrm{E}_j^f}$. Since by [Gaussent and Littelmann 2005, Proposition 1] the fibers of r_∞ are $\mathrm{U}(\mathscr{K})$-orbits, an element $p = [p_0, \ldots, p_k] \in \Sigma_{\gamma^f}$ belongs to C_δ if and only if there exist elements $u_0, \ldots, u_k \in \mathrm{U}(\mathscr{K})$ such that

(1) $p_0 \cdots p_j \mathrm{E}_j^f = u_j \mathrm{E}_j$ and

(2) $u_{j-1} \mathrm{V}_j = u_j \mathrm{V}_j$.

Define $u_0 = v_0$ and $u_j = v_0 \cdots v_j$. Then conditions (1) and (2) above hold for

$$p_j = (\delta_0 \cdots \delta_{j-1})^{-1} v_j (\delta_0 \cdots \delta_j).$$

Hence the image of the map is contained in the cell C_δ. For the other inclusion, define $v_j = u_{j-1}^{-1} u_j$ (see [Gaussent and Littelmann 2012, Proposition 4.19]). To show injectivity assume $\widetilde{\varphi}([v_0, \dots, v_k]) = \widetilde{\varphi}([v'_0, \dots, v'_k])$. Then there exist elements $e_j \in \mathrm{U}_{\mathrm{E}_j}$ such that $v_0 \cdots v_j = v'_0 \cdots v'_j e_j$, this implies injectivity. □

The following corollary can be found in [Gaussent et al. 2013, Corollary 3] for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$. Note that in particular it implies that $u\pi(\mathrm{C}_\delta) = \pi(\mathrm{C}_\delta)$ for all $u \in \mathrm{U}_{\mathrm{V}_0}$.

Corollary 2.10. $\pi(\mathrm{C}_\delta) = \mathbb{U}^\delta_{\mathrm{V}_0} \cdots \mathbb{U}^\delta_{\mathrm{V}_k}[t^{\mu_\delta}] = \mathrm{U}_{\mathrm{V}_0} \cdots \mathrm{U}_{\mathrm{V}_k}[t^{\mu_\delta}]$.

Proof. By the arguments in the proof of Theorem 2.9 the image of the map

$$\begin{aligned} \mathrm{U}_{\mathrm{V}_0} \times \cdots \times \mathrm{U}_{\mathrm{V}_k} &\to \Sigma_{\gamma^f} \\ (v_0, \dots, v_k) &\mapsto [v_0\delta_0, \delta_0^{-1} v_1 \delta_0 \delta_1, \dots, \delta_0 \cdots \delta_{r-1}^{-1} v_k \delta_0 \cdots \delta_k] \end{aligned}$$

is contained in and is surjective onto the cell C_δ. In particular conditions (1) and (2) above are satisfied for $p_j = (\delta_0 \cdots \delta_{j-1})^{-1} v_j (\delta_0 \cdots \delta_j)$. The corollary follows since $\delta_0 \cdots \delta_j \mu_{\gamma^f} = \mu_\delta$. □

3. Crystal structure on combinatorial galleries, the Littelmann path model, and Lakshmibai–Seshadri galleries

Let $\lambda \in \mathrm{X}^{+,\vee}$ be a dominant integral coweight and let $\mathrm{L}(\lambda)$ be the corresponding simple module of $\mathrm{G}^\vee$. To $\mathrm{L}(\lambda)$ is associated a certain graph $\mathrm{B}(\lambda)$ that is its "combinatorial model". It is a connected *highest weight* crystal, which means that there exists $b_\lambda \in \mathrm{B}(\lambda)$ such that $e_{\alpha_i}(b_\lambda) = 0$ for all $i \in \{1, \dots, n\}$, where n is the rank of the corresponding root datum. The crystal $\mathrm{B}(\lambda)$ also has the characterizing property that

$$\dim(\mathrm{L}(\lambda)_\mu) = \#\{b \in \mathrm{B}(\lambda) : \mathrm{wt}(b) = \mu\}.$$

See below for definitions. After recalling the notion of a crystal we review the crystal structure on the set Γ of combinatorial galleries.

3A. *Crystals.* A *crystal* is a set B together with maps

$$\begin{aligned} e_{\alpha_i}, f_{\alpha_i} &: \mathrm{B} \to \mathrm{B} \cup \{0\} && \text{(the \emph{root operators})}, \\ \mathrm{wt} &: \mathrm{B} \to \mathrm{X}^\vee && \text{(the \emph{weight function})}, \end{aligned}$$

for $i \in \{1, \dots, n\}$, such that for every $b, b' \in \mathrm{B}$; $b' = e_{\alpha_i}(b)$ if and only if $b = f_{\alpha_i}(b')$, and, in this case, setting

$$\varepsilon_i(b'') = \max\{n : e^n_{\alpha_i}(b'') \neq 0\} \quad \text{and} \quad \phi_i(b'') = \max\{n : f^n_{\alpha_i}(b'') \neq 0\}$$

for any $b'' \in \mathrm{B}$, we have

$$\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i^\vee \quad \text{and} \quad \phi_i(b) = \varepsilon_i(b) + \langle \alpha_i, \mathrm{wt}(b) \rangle.$$

A crystal is in particular a graph, which we may decompose into the disjoint union of its connected components. Each element $b \in \mathrm{B}$ lies in a unique connected component which we will denote by $\mathrm{Conn}(b)$. A *crystal morphism* is a map $\mathrm{F} : \mathrm{B} \to \mathrm{B}'$ between the underlying sets of crystals B and B′ such that $\mathrm{wt}(\mathrm{F}(b)) = \mathrm{wt}(b)$ and such that it commutes with the action of the root operators. A crystal morphism is an isomorphism if it is bijective.

3B. *Crystal structure on combinatorial galleries.*

Definition 3.1. For each $i \in \{1, \dots, n\}$ and each simple root α_i, we recall the definition of the root operators f_{α_i} and e_{α_i} on the set of combinatorial galleries Γ and endow this set with a crystal structure. We follow [Gaussent and Littelmann 2005, §6; Braverman and Gaitsgory 2001, §1], and refer the reader to [Kashiwara 1995] for a detailed account of the theory of crystals.

Let $\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \dots, \mathrm{E}_k, \mathrm{V}_{k+1})$ be a combinatorial gallery. Define a weight function by $\mathrm{wt}(\gamma) = \mu_\gamma$. Let $m_{\alpha_i} = m \in \mathbb{Z}$ be minimal such that $\mathrm{V}_p \in \mathrm{H}_{(\alpha_i, m)}$ for some $p \in \{0, \dots, k+1\}$. Note that $m \leq 0$.

Definition of f_{α_i}. Suppose $\langle \alpha_i, \mu_\gamma \rangle \geq m+1$. Let j be maximal such that $\mathrm{V}_j \in \mathrm{H}_{(\alpha_i, m)}$ and let $j < r \leq k+1$ be minimal such that $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i, m+1)}$. Let

$$\mathrm{E}'_p = \begin{cases} \mathrm{E}_p & \text{if } p < j, \\ s_{(\alpha_i, m)}(\mathrm{E}_p) & \text{if } j \leq p < r, \\ t_{-\alpha_i^\vee}(\mathrm{E}_p) & \text{if } r \leq p. \end{cases}$$

Define $\mathrm{V}'_0 = 0$, and for $1 \leq p \leq k$, set $\mathrm{V}'_p = \mathrm{E}'_{p-1} \cap \mathrm{E}'_p$, and let V'_{k+1} be the extreme point of the line segment E'_k that is not V'_k. Define

$$f_{\alpha_i}(\gamma) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \dots, \mathrm{E}'_k, \mathrm{V}'_{k+1}),$$

and if $\langle \alpha_i, \mu_\gamma \rangle < m+1$, then $f_{\alpha_i}(\gamma) = 0$.

Definition of e_{α_i}. Suppose $m \leq -1$. Let r be minimal such that the $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i, m)}$ and let $0 \leq j < r$ be maximal such that $\mathrm{V}_j \in \mathrm{H}_{(\alpha_i, m+1)}$. Let

$$\mathrm{E}'_p = \begin{cases} \mathrm{E}_p & \text{if } p < j, \\ s_{(\alpha_i, m+1)}(\mathrm{E}_p) & \text{if } j \leq p < r, \\ t_{\alpha_i^\vee}(\mathrm{E}_p) & \text{if } r \leq p, \end{cases}$$

define V'_p as above and define

$$e_{\alpha_i}(\gamma) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \dots, \mathrm{E}'_k, \mathrm{V}'_{k+1}).$$

If $m = 0$ then $e_{\alpha_i}(\gamma) = 0$.

Remark 3.2. It follows from the definitions that the maps e_{α_i}, f_{α_i} and wt define a crystal structure on Γ. Note as well that if γ is a combinatorial gallery then $f_{\alpha_i}(\gamma)$

and $e_{\alpha_i}(\gamma)$ are combinatorial galleries of the same type as γ (as long as they are not zero). We say that the root operators are type preserving. See also [Gaussent and Littelmann 2005, Lemma 6].

3C. ***The Littelmann path model and Lakshmibai–Seshadri galleries; readable galleries.*** Let γ be a combinatorial gallery that has each one of its faces contained in the fundamental chamber. We call such galleries *dominant* and denote the set of all dominant combinatorial galleries by Γ^{dom}. By [Littelmann 1995, Theorem 7.1] the crystal of galleries $\mathrm{P}(\gamma)$ generated by γ is isomorphic to the crystal $\mathrm{B}(\mu_\gamma)$ associated to the irreducible highest weight representation $\mathrm{L}(\mu_\gamma)$ of $\mathrm{G}^\vee$. In its original context [Littelmann 1995] it is known as a *Littelmann path model* for the representation $\mathrm{L}(\mu_\gamma)$. We say that a combinatorial gallery γ is a *Littelmann gallery* if there exist indices $i_1, \ldots, i_r$ such that $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma) = \gamma^+$ is a dominant gallery. If $\mu_{\gamma^+} = \mu_{\delta^+}$, $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma) = \gamma^+$ and $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\delta) = \delta^+$ for two Littelmann galleries γ and δ, we say that they are *equivalent*. This defines an equivalence relation on the set of Littelmann galleries.

Let $\lambda \in \mathrm{X}^{\vee,+}$ be a dominant integral coweight and γ_λ a gallery that is a concatenation of fundamental galleries and that has endpoint λ (as above). We denote by $\Gamma(\gamma_\lambda)^{\mathrm{LS}}$ the set of *combinatorial LS galleries* of the same type as γ_λ. (LS is short for Lakshmibai–Seshadri. All LS galleries are Littelmann — see [Littelmann 1995, §4] — and Littelmann galleries generalize LS galleries enormously.) The set $\Gamma(\gamma_\lambda)^{\mathrm{LS}}$ is stable under the root operators and has the structure of a crystal isomorphic to $\mathrm{B}(\lambda)$. It was proven by Gaussent and Littelmann [2005] that the resolution in (8) induces a bijection $\Gamma(\gamma_\lambda)^{\mathrm{LS}} \cong \mathscr{Z}(\lambda)$. This bijection was shown to be a crystal isomorphism by Baumann and Gaussent [2008]. We use this heavily in the proof of Theorem 6.2. In [Gaussent and Littelmann 2005] see Definition 18 for a geometric definition of LS galleries, and Definition 23 for an equivalent combinatorial characterization that for one skeleton galleries agrees with the original definition by Lakshmibai, Musili and Seshadri (see [Lakshmibai et al. 1998], for example) in the context of standard monomial theory. We will give a combinatorial characterization of LS galleries of fundamental type in the case $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$, omitting therefore the most general definitions.

We finish this section with a question. Let γ be a dominant gallery (see Section 3C). Consider the map $\Sigma_{\gamma^f} \to \mathscr{G}$ defined by $[g_0, \ldots, g_r] \mapsto g_0 \cdots g_r[t^{\mu_{\gamma^f}}]$ (see Remark 2.6).

Question. Does this map induce a crystal isomorphism $\mathrm{P}(\gamma) \cong \mathscr{Z}(\mu_\gamma)$?

This question was answered positively in [Gaussent et al. 2013; Torres 2016] for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$. In the rest of this paper we do so as well for $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$ and γ a *readable* gallery. For $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$ all galleries are readable. This is due to the well known fact that in this case fundamental coweights are all minuscule. In the next sections we will describe readable galleries explicitly for $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$ and show that they are Littelmann galleries. Moreover, we will see there exist readable galleries that are not of the same type as any concatenation of fundamental galleries γ_λ (see Remark 4.9).

Definition 3.3. A *readable* gallery is a concatenation of its *parts*. Its parts are either LS galleries of fundamental type or galleries of the form $(\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2)$ (we call them *zero lumps*) such that both edges E_0 and E_1 are contained in the dominant chamber and such that the endpoint V_2 is equal to zero. We denote the set of all readable galleries by Γ^{R}, and if a combinatorial gallery γ is fixed, by $\Gamma(\gamma)^{\mathrm{R}}$, the set of all readable galleries of same type as γ.

Remark 3.4. It follows from [Gaussent and Littelmann 2005, Lemma 8] that readable galleries are stable under root operators.

4. "Type C" combinatorics

4A. ***Weights and coweights.*** Consider $\mathbb{R}^n$ with canonical basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ and standard inner product $\langle -,- \rangle$. In particular $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. From now on we consider the root datum $(\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ defined by

$$\begin{aligned} \Phi &= \{\pm\varepsilon_i, \varepsilon_i \pm \varepsilon_j\}_{i,j\in\{1,\dots,n\}}, \\ \Phi^\vee &= \left\{\alpha^\vee = \tfrac{2\alpha}{\langle\alpha,\alpha\rangle} : \alpha \in \Phi\right\}, \\ \mathrm{X} &= \{v \in \mathbb{R}^n : \langle v, \alpha^\vee \rangle \in \mathbb{Z}\}, \\ \mathrm{X}^\vee &= \{v \in \mathbb{R}^n : \langle \alpha, v \rangle \in \mathbb{Z}\}. \end{aligned}$$

Indeed the sets X and $\mathrm{X}^\vee$ are free abelian groups which form a root datum together with the pairing $\langle -,- \rangle$ between them and the subsets $\Phi \subset \mathrm{X}$ and $\Phi^\vee \subset \mathrm{X}^\vee$. We choose a basis $\Delta \subset \Phi$ given by

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : i \in \{1, \dots, n-1\}\} \cup \{\alpha_n = \varepsilon_n\},$$

hence the set

$$\Delta^\vee = \{\alpha_i^\vee = \varepsilon_i - \varepsilon_{i+1} : i \in \{1, \dots, n-1\}\} \cup \{\alpha_n^\vee = 2\varepsilon_n\}$$

is a basis for the root system $\Phi^\vee$. Then $\mathrm{X}^\vee$ has a $\mathbb{Z}$-basis given by the set of corresponding fundamental coweights $\{\omega_i\}_{i\in\{1,\dots,n\}}$, where

$$\omega_i = \varepsilon_1 + \dots + \varepsilon_i \;\; 1 \le i \le n.$$

Then $\mathrm{G} = \mathrm{SO}(2n+1, \mathbb{C})$ and $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$. For later use we introduce the notation $\varepsilon_{\bar{i}} = -\varepsilon_i$.

4B. ***Symplectic keys and words.*** Let $p \in \mathbb{Z}_{\geq 1}$ be an integer, greater than or equal to 1. To it we associate a sequence of positive integers $\underline{p}$ as follows:

$$\underline{p} = \begin{cases} (1) & \text{if } p = 1, \\ (p, p) & \text{if } p \geq 2. \end{cases}$$

Given two sequences of integers $\underline{a} = (a_1, \dots, a_r)$ and $\underline{b} = (b_1, \dots, b_s)$ we denote the associated merged list by $\underline{a} * \underline{b} = (a_1, \dots, a_r, b_1, \dots, b_s)$. A *symplectic shape* $\underline{d}$ is a sequence of natural numbers of the form $\underline{d} = \underline{p_1} * \cdots * \underline{p_l}$, where $p_i \in \mathbb{Z}_{\geq 1}$. An *arrangement of boxes* of symplectic shape $\underline{d}$ is an arrangement of as many columns of boxes as elements in the sequence $\underline{d}$ such that column j (read from right to left) has p_j boxes.

Example 4.1. An arrangement of boxes of symplectic shape $\underline{1} * \underline{1} * \underline{2} * \underline{1}$.

$$\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \multicolumn{1}{c|}{} & & & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

Consider the ordered alphabet $\mathscr{C}_n = \{1 < 2 < \cdots < n{-}1 < n < \bar{n} < \overline{n{-}1} < \cdots < \bar{1}\}$. A *symplectic key* of (symplectic) shape $\underline{d} = \underline{p_1} * \cdots * \underline{p_l}$ is a filling of an arrangement of boxes of symplectic shape $\underline{d}$ with letters of the alphabet $\mathscr{C}_n$ in such a way that the entries are strictly increasing along each column and such that $p_j \leq n$ for $j \in \{1, \dots, l\}$.

Example 4.2. A symplectic key, for $n \geq 5$, of symplectic shape $\underline{1} * \underline{2} * \underline{1}$.

$$\begin{array}{|c|c|c|c|} \hline \bar{1} & 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 5 & \bar{2} & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

We denote the *word monoid* on $\mathscr{C}_n$ by $\mathscr{W}_{\mathscr{C}_n}$. To a word $w = w_1 \cdots w_k$ in $\mathscr{W}_{\mathscr{C}_n}$ we associate a symplectic key $\mathscr{K}_w$ that consists of only one row of length k, and with the boxes filled in from right to left with the letters of w read in turn from left to right. For example, the word 12 corresponds to the key $\boxed{2}\boxed{1}$.

4C. ***Readable keys: symplectic keys associated to readable galleries.*** The aim of this section is to assign a symplectic key to every readable gallery. For a subset $\mathrm{Y} \subseteq \mathscr{C}_n$, we denote the corresponding subset of barred elements by $\overline{\mathrm{Y}} = \{\bar{y} : y \in \mathrm{Y}\}$, where, for i unbarred, $\bar{\bar{i}} = i$.

Definition 4.3. Let $\mathscr{B}$ be a symplectic key. We call $\mathscr{B}$ an *LS block* if it is of shape $\underline{p}$ for $p \in \mathbb{Z}_{\geq 1}$ and such that if $p \geq 2$ (which means that $\mathscr{B}$ consists of two columns of size p) there exist positive integers k, r, s with $2k + r + s \leq n$ and disjoint sets of positive integers

$$\begin{aligned} \mathrm{A} &= \{a_i : 1 \le i \le r, a_1 < \cdots < a_r\}, \\ \mathrm{B} &= \{b_i : 1 \le i \le s, b_1 < \cdots < b_s\}, \\ \mathrm{Z} &= \{z_i : 1 \le i \le k, z_1 < \cdots < z_k\}, \\ \mathrm{T} &= \{t_i : 1 \le i \le k, t_1 < \cdots < t_k\}, \end{aligned}$$

such that the right column of $\mathscr{B}$ (respectively the left one) is the column with entries the ordered elements of the set $\bar{\mathrm{T}} \cup \mathrm{Z} \cup \mathrm{A} \cup \bar{\mathrm{B}}$ (respectively $\bar{\mathrm{Z}} \cup \mathrm{T} \cup \mathrm{A} \cup \bar{\mathrm{B}}$), $\mathrm{Z} = \varnothing$ if and only if $\mathrm{T} = \varnothing$, and such that if $\mathrm{Z} \neq \varnothing$ the elements of T are uniquely characterized by the properties

$$t_k = \max\{t \in \mathscr{C}_n : t < z_k, t \notin \mathrm{Z} \cup \mathrm{A} \cup \mathrm{B}\}, \tag{9}$$

$$t_{j-1} = \max\{t \in \mathscr{C}_n : t < \min(z_{j-1}, t_j), t \notin \mathrm{Z} \cup \mathrm{A} \cup \mathrm{B}\} \text{ for } j \le k. \tag{10}$$

We say that $\mathscr{B}$ is a *zero block* if it is of shape $\underline{k}$ for $k \in \mathbb{Z}_{\ge 1}$ and such that its right column is filled in with the ordered letters $1 < \cdots < k$ and its left one, with $\bar{k} < \cdots < \bar{1}$. A symplectic key is called a *readable block* if it is either an LS block or a zero block. Note that a readable block has symplectic shape $\underline{p}$, where $p \in \mathbb{Z}_{\ge 1}$. A *readable key* is a concatenation of readable blocks. Now assume that $\underline{d} = \underline{p}_1 * \cdots * \underline{p}_l$ is such that $p_1 \le \cdots \le p_l$. A symplectic key of shape $\underline{d}$ is called an *LS symplectic key* if its entries are weakly increasing in rows and if it is a concatenation of LS blocks. We denote the set of LS symplectic keys of shape $\underline{d}$ by $\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}}$.

Example 4.4. The symplectic key

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 3 \\ \hline 5 & 5 \\ \hline \bar{4} & \bar{4} \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}$$

is an LS block of shape $\underline{5} = (5, 5)$, with $\mathrm{A} = \{3, 4\}$, $\mathrm{B} = \{4\}$, $\mathrm{Z} = \{2\}$ and $\mathrm{T} = \{1\}$. The first symplectic key immediately below is not an LS block; the second is a zero block.

$$\begin{array}{|c|c|} \hline 1 & \bar{2} \\ \hline 2 & \bar{1} \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline \bar{2} & 1 \\ \hline \bar{1} & 2 \\ \hline \end{array}$$

Remark 4.5. A pair of columns that form an LS block is sometimes called a pair of admissible columns. The original definition of admissible columns was given by De Concini [1979], using a slightly different convention than Kashiwara and Nakashima's, which is the one we use here. The map that translates the two can be found in [Lecouvey 2002, §2.2].

To a readable block $\mathscr{B}$ we assign a gallery $\gamma_{\mathscr{B}}$ as follows. If $\mathscr{B}$ consists of only one box filled in with the letter $l \in \mathscr{C}_n$, then we define $\mathrm{V}_0^{\mathscr{B}} = 0$, $\mathrm{V}_1^{\mathscr{B}} = \varepsilon_l$, $\mathrm{E}_0^{\mathscr{B}} = \{t\mathrm{V}_1^{\mathscr{B}} : t \in [0, 1]\}$, and

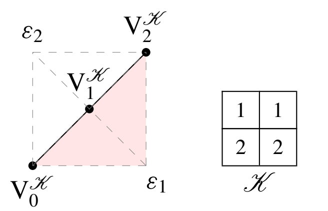
$$\gamma_{\mathscr{B}} = (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}).$$

If the readable block $\mathscr{B}$ has at least two boxes, then its columns are filled in with the letters $l_1^1 < \cdots < l_d^1$ (right column) and $l_1^2 < \cdots < l_d^2$ (left column) respectively. We then define

$$\begin{aligned}
\mathrm{V}_0^{\mathscr{B}} &= 0,\\
\mathrm{V}_1^{\mathscr{B}} &= \tfrac{1}{2}(\varepsilon_{l_1^1} + \cdots + \varepsilon_{l_d^1}),\\
\mathrm{E}_0^{\mathscr{B}} &= \{t\mathrm{V}_1^{\mathscr{T}} : t \in [0, 1]\},\\
\mathrm{V}_2^{\mathscr{B}} &= \varepsilon_{l_1^1} + \cdots + \varepsilon_{l_d^1} + \varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2},\\
\mathrm{E}_1^{\mathscr{B}} &= \{\mathrm{V}_1^{\mathscr{T}} + \tfrac{1}{2}t(\varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2}) : t \in [0, 1]\},\\
\gamma_{\mathscr{B}} &= (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}, \mathrm{E}_1^{\mathscr{B}}, \mathrm{V}_2^{\mathscr{B}}).
\end{aligned}$$

Note that (9) implies that $\mathrm{V}_1^{\mathscr{B}} + \frac{1}{2}(\varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2}) = \mathrm{V}_2^{\mathscr{B}}$ and therefore that $\mathrm{E}_1^{\mathscr{B}}$ is the line segment joining $\mathrm{V}_1^{\mathscr{B}}$ and $\mathrm{V}_2^{\mathscr{B}}$.

Example 4.6. Let $n = 2$ and $\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2)$ where $\mathrm{V}_0 = 0$, $\mathrm{V}_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$, $\mathrm{V}_2 = \varepsilon_1 + \varepsilon_2$ and the edges are the line segments joining the vertices in order. See below for a picture of the gallery $\gamma_{\mathscr{B}}$ associated to the symplectic key $\mathscr{B}$.



$$\gamma_{\mathscr{K}} = (\mathrm{V}_0^{\mathscr{K}}, \mathrm{E}_0^{\mathscr{K}}, \mathrm{V}_1^{\mathscr{K}}, \mathrm{E}_1^{\mathscr{K}}, \mathrm{V}_2^{\mathscr{K}})$$

To a readable key $\mathscr{K} = \mathscr{B}_1 \cdots \mathscr{B}_k$ we associate the concatenation

$$\gamma_{\mathscr{K}} = \gamma_{\mathscr{B}_k} * \cdots * \gamma_{\mathscr{B}_1}$$

of the galleries of each of the readable blocks $\mathscr{B}_j$, for $j \in \{1, \ldots, k\}$, that it is a concatenation of (from right to left). To a symplectic shape $\underline{d} = \underline{p}_1 * \cdots * \underline{p}_l$ such that $p_j \leq n$ for $j \in \{1, \ldots, l\}$ (once n is fixed, we will only consider such shapes) we associate the dominant coweight $\lambda_{\underline{d}} = \omega_{p_1} + \cdots + \omega_{p_l}$. For example, to the shape $(2, 2)$ is associated the coweight ω_2. We will denote the set of all readable keys of shape $\underline{d}$ by $\mathbf{\Gamma}(\underline{d})^{\mathrm{R}}$.

Remark 4.7. The set $\mathbf{\Gamma}(\underline{d})^{\mathrm{R}}$ is nonempty: since $p_j \le n$, there is a natural readable key of symplectic shape $\underline{d}$ whose columns are filled in with consecutive integers, starting with 1 at the top. For example, if $\underline{d} = \underline{3} = (3, 3)$ and $n \ge 3$, this is the key

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}\,.$$

It is an LS block, with $\mathrm{A} = \{1, 2, 3\}$ and $\mathrm{B} = \mathrm{Z} = \mathrm{T} = \varnothing$.

The following proposition follows directly from [Gaussent and Littelmann 2012, Lemma 2].

Proposition 4.8. *The map*

$$\bigcup_{\substack{\underline{d}=\underline{p}_1\cdots\underline{p}_l \\ p_j \le n}} \mathbf{\Gamma}(\underline{d})^{\mathrm{R}} \to \Gamma^{\mathrm{R}}, \quad \mathscr{K} \mapsto \gamma_{\mathscr{K}}$$

is well defined and is a bijection. Moreover, if $p_1 \le \cdots \le p_l$ then this map induces a bijection

$$\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}} \longleftrightarrow \Gamma(\gamma_{\omega_{p_1}} * \cdots * \gamma_{\omega_{p_m}})^{\mathrm{LS}}.$$

Remark 4.9. Zero lumps are not necessarily of fundamental type: this follows from [Gaussent and Littelmann 2012, Lemma 2] for a zero lump with odd k in the above description. This is why readable galleries are not necessarily of the same type as a concatenation of fundamental galleries. This also means that there can be two readable keys of the same shape but such that their associated galleries are not of the same type! For example, take $n > 3$, and consider the keys

$$\mathscr{T} = \begin{array}{|c|c|}\hline \bar{1} & \bar{1} \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \text{and} \quad \mathscr{K} = \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline \bar{2} & 2 \\ \hline \bar{3} & 3 \\ \hline \end{array}\,.$$

The first is LS and $\gamma_{\mathscr{T}}$ is of fundamental type ω_3. The second key is a zero block. Its associated gallery, $\gamma_{\mathscr{K}}$, is not of fundamental type.

5. The word of a readable gallery

To a readable key $\mathscr{K}$ we assign a word $w(\mathscr{K})$. The first aim of this section is to state Proposition 5.5, which says that the closure in the affine Grassmannian of the image $\pi(\mathrm{C}_{\gamma_{\mathscr{K}}}) \subset \mathscr{G}$ considered in Section 2F depends only on the word $w(\mathscr{K})$.

Definition 5.1. The *word* of a readable block, $\mathscr{B} = \mathrm{C_L C_R}$ ($\mathrm{C_L}$ is the left column, $\mathrm{C_R}$ the right), is obtained by reading first the unbarred entries in $\mathrm{C_R}$ and then the barred entries in $\mathrm{C_L}$. We denote it by $w(\mathscr{B}) \in \mathscr{W}_{\mathscr{C}_n}$.

Remark 5.2. For an LS block this is the word of the associated single admissible column defined by Kashiwara and Nakashima [Lecouvey 2002, Example 2.2.6].

Definition 5.3. Let $\gamma_{\mathscr{K}}$ be the readable gallery associated to the key $\mathscr{K}$. As before, we may write $\mathscr{K}$ as a concatenation of blocks $\mathscr{K} = \mathscr{B}_1 \cdots \mathscr{B}_k$. The *word* of $\gamma_{\mathscr{K}}$ (or of $\mathscr{K}$) is $w(\mathscr{B}_k) \cdots w(\mathscr{B}_1)$. We denote it by $w(\gamma_{\mathscr{K}})$ (or $w(\mathscr{K})$).

Example 5.4. Let

$$\mathscr{B}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}, \quad \mathscr{B}_2 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}, \quad \text{and} \quad \mathscr{K} = \mathscr{B}_1\mathscr{B}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline \bar{2} & \bar{1} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Then $w(\mathscr{B}_1) = 2\bar{2}$, $w(\mathscr{B}_2) = 1$, and $w(\mathscr{K}) = 12\bar{2}$.

We have the following result about words of readable galleries, which we prove in Section 7. We will use it in Theorem 6.2. It is in this sense that such galleries are called *readable*.

Proposition 5.5. *Let γ and ν be combinatorial galleries and $\mathscr{K}$ be a readable key. Consider the combinatorial galleries $\gamma * \gamma_{w(\mathscr{K})} * \nu$ and $\gamma * \gamma_{\mathscr{K}} * \nu$. Let $(\Sigma_{(\gamma*\gamma_{w(\mathscr{K})}*\nu)^f}, \pi)$ and $(\Sigma_{(\gamma*\gamma_{\mathscr{K}}*\nu)^f}, \pi')$ be the Bott–Samelson varieties together with their maps to the affine Grassmannian $\mathscr{G}$ (as in Remark 2.6). Then*

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K})}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}}*\nu})}.$$

5A. *Word galleries.* We associate a (readable!) gallery γ_w of the same type as the m-fold product $\gamma_{\omega_1} * \cdots * \gamma_{\omega_1}$ to a word $w \in \mathscr{W}_{\mathscr{C}_n}$ of length m — it is the gallery $\gamma_{\mathscr{K}_w}$ associated to the readable key $\mathscr{K}_w$. We denote the set of word galleries in this case by $\Gamma_{\mathscr{W}_{\mathscr{C}_n}}$. Below we recall the crystal structure on the set $\mathscr{W}_{\mathscr{C}_n}$ as described by Kashiwara and Nakashima [1994, Proposition 2.1.1]. The set of words $\mathscr{W}_{\mathscr{C}_n}$, just like the set $\mathscr{W}_n$, is in one-to-one correspondence with the set of vertices of the crystal of the representation $\bigoplus_{l\in\mathbb{Z}_{\geq 0}} \mathrm{V}_n^{\otimes l}$, where V_n is the natural representation $\mathrm{L}(\omega_1)$ and hence inherits its crystal structure. Proposition 5.7 says that this structure is compatible with the crystal structure defined on galleries in Section 3.

Definition 5.6. Let $w = w_1 \cdots w_l \in \mathscr{C}_n$ be a word and $i \in \{1, \dots, n\}$. Define $\mathrm{wt}(w) = \sum_{i=1}^{l} \varepsilon_i$. To apply the root operators e_{α_i} and f_{α_i} to w one first obtains a word consisting of letters in the alphabet $\{+, -, \varnothing\}$. The word will be obtained from w by replacing every occurrence of i or $\overline{i+1}$ by "+", every occurrence of $i+1$ or $\bar{i}$ by "−" and all other letters by "$\varnothing$". This word, which we denote by $s_i(w)$ is sometimes called the i-signature of w. To proceed, erase all symbols $\varnothing$ and then all subwords of the form "+−". Repeat this process until the i-signature $s_i(w)$ of w has been reduced to a word of the form

$$s_i(w)' = (-)^r(+)^s.$$

To apply f_{α_i} (respectively e_{α_i}) to w, change the letter whose tag corresponds to the leftmost "+" (respectively to the rightmost "−") from i to $i+1$ and from $\overline{i+1}$ to $\bar{i}$ (respectively from $i+1$ to i and from $\bar{i}$ to $\overline{i+1}$). If $s=0$, respectively $r=0$, then $f_{\alpha_i}(w)=0$, respectively $e_{\alpha_i}(w)=0$.

Proposition 5.7. *The crystal structure on words from Definition 5.6 coincides with the one induced from Definition 3.1.*

For a proof, see [Littelmann 1996, §13]. It also follows directly from the definitions.

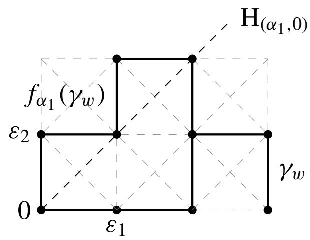
Example 5.8. Let $n=2$ and $w=1121\bar{2}$. We first consider $i=1$, for which $s_1(w)=++-++$, and therefore $s_1'(w)=+++$. Hence $f_{\alpha_1}(w)=2121\bar{2}$ and $e_{\alpha_1}(w)=0$. For $i=2$ we have $s_2(w)=\varnothing\varnothing+\varnothing-$. Therefore s_2' is the empty word and $f_{\alpha_2}(w)=e_{\alpha_2}(w)=0$. Now consider the readable gallery $\gamma_{\mathscr{K}_w}$ associated to w. Explicitly we write it as

$$\gamma_w=(\mathrm{V}_0,\mathrm{E}_0,\mathrm{V}_1,\mathrm{E}_1,\mathrm{V}_2,\mathrm{E}_2,\mathrm{V}_3,\mathrm{E}_3,\mathrm{V}_4,\mathrm{E}_4,\mathrm{V}_5),$$

where $\mathrm{V}_0=0$, $\mathrm{V}_1=\varepsilon_1$, $\mathrm{V}_2=2\varepsilon_1$, $\mathrm{V}_3=2\varepsilon_1+\varepsilon_2$, $\mathrm{V}_4=3\varepsilon_1+\varepsilon_2$, $\mathrm{V}_5=3\varepsilon_1$ and E_j is the line segment joining V_j to V_{j+1} for $j\in\{0,\dots,4\}$. We have $m_{\alpha_1}=0$, so by Definition 3.1, $e_{\alpha_1}(\gamma_w)=0$. We have $s_{(\alpha_1,0)}(\mathrm{E}_0)=\{t\varepsilon_2 : t\in[0,1]\}$, see below. Then $j=1$ (Definition 3.1) and hence

$$f_{\alpha_1}(\gamma_w)=(\mathrm{V}_0',\mathrm{E}_0',\mathrm{V}_1',\mathrm{E}_1',\mathrm{V}_2',\mathrm{E}_2',\mathrm{V}_3',\mathrm{E}_3',\mathrm{V}_4',\mathrm{E}_4',\mathrm{V}_5'),$$

where $\mathrm{V}_0'=0$, $\mathrm{V}_1'=\varepsilon_2$, $\mathrm{V}_2'=\varepsilon_2+\varepsilon_1$, $\mathrm{V}_3'=2\varepsilon_2+\varepsilon_1$, $\mathrm{V}_4'=2\varepsilon_2+2\varepsilon_1$, $\mathrm{V}_5'=\varepsilon_2+2\varepsilon_1$ and E_j' is the line segment joining V_j' and V_{j+1}' for $j\in\{0,\dots,4\}$. For $i=2$ we have $m_{\alpha_2}=0$, which implies that $e_{\alpha_2}(\gamma_w)=0$. We also have $\mu_{\gamma_w}=3\varepsilon_1$, and therefore $\langle\alpha_2,\mu_{\gamma_w}\rangle=0<m_{\alpha_2}+1=1$, so that $f_{\alpha_2}(\gamma_w)=0$ as well. Then $f_{\alpha_1}(\gamma_w)=\gamma_{f_{\alpha_1}(w)}$, $e_{\alpha_1}(\gamma_w)=\gamma_{e_{\alpha_1}(w)}$, $f_{\alpha_2}(\gamma_w)=\gamma_{f_{\alpha_2}(w)}$ and $e_{\alpha_2}(\gamma_w)=\gamma_{e_{\alpha_2}(w)}$.



5B. ***Word reading is a crystal morphism.*** This subsection is the "symplectic" version of [Torres 2016, Proposition 2.5]. Since the root operators are type preserving (see Definition 3.1), the set of words $\mathscr{W}_{\mathscr{C}_n}$ is naturally endowed with a crystal structure. The following proposition will be useful in Theorem 6.2. This result was shown for LS blocks by Kashiwara and Nakashima [1994, Proposition 4.3.2]. They show that word reading induces an isomorphism of crystals from $\mathrm{B}(\omega_k)$ onto the subcrystal of $\bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \mathrm{B}(\omega_1)^{\otimes l}$ generated by the tensor product $\boxed{k} \otimes \cdots \otimes \boxed{1}$. We show that for readable galleries the proof is reduced to this case.

Proposition 5.9. *The map*

$$\Gamma^{\mathrm{R}} \xrightarrow{w} \Gamma_{\mathscr{W}_{\mathscr{C}_n}}, \qquad \gamma_{\mathscr{K}} \mapsto \gamma_{w(\mathscr{K})}$$

is a crystal morphism.

Proof. First note that the map is weight preserving. This follows from the definitions and from the fact that in the definition of a readable block, the sets Z and T do not contribute to the endpoint of the associated gallery. Let γ be a readable gallery and let

$$\gamma_{\mathscr{B}} = (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}, \mathrm{E}_1^{\mathscr{B}}, \mathrm{V}_2^{\mathscr{B}})$$

be one of its parts, associated to some readable block $\mathscr{B}$. We write

$$\gamma_{w(\mathscr{B})} = (\mathrm{V}_0^{\mathscr{K}_{w(\mathscr{B})}}, \mathrm{E}_0^{\mathscr{K}_{w(\mathscr{B})}}, \dots, \mathrm{V}_{r+s}^{\mathscr{K}_{w(\mathscr{B})}}).$$

If

$$w(\mathscr{B}) = g_1 \cdots g_s \bar{h}_k \cdots \bar{h}_1,$$

for g_i and h_i unbarred, then $\mathrm{V}_0^{\mathscr{K}_{w(\mathscr{B})}} = 0$ and $\mathrm{V}_j^{\mathscr{K}_{w(\mathscr{B})}} = \sum_{i=1}^{j} \varepsilon_{x_i}$ for $1 \leq j \leq s+r$, where $x_i = g_i$ for $1 \leq i \leq s$ and $x_{s+i} = \bar{h}_i$ for $1 \leq i \leq k$. Let

$$h(j) = \langle \alpha, \mathrm{V}_j^{\mathscr{B}} \rangle \quad \text{and} \quad h'(j) = \langle \alpha, \mathrm{V}_j^{\mathscr{K}_{w(\mathscr{B})}} \rangle,$$

for $1 \leq j \leq k+s+1$. Then there exist d_1, d_2 with $d_1 \leq s < d_2 \leq s+k$ and such that

$$h'(j) = \begin{cases} h(0) & \text{for } 0 \leq j < d_1, \\ h(1) & \text{for } d_1 \leq j < d_2, \\ h(2) & \text{for } d_2 \leq j \leq k+s+1. \end{cases}$$

From this we conclude that it is enough to consider readable blocks. As mentioned previously, this was shown in [Kashiwara and Nakashima 1994] for LS blocks. Hence let $\mathscr{L}$ be a zero lump — it has word $w(\mathscr{L}) = 1 \cdots k\bar{k} \cdots \bar{1}$ — and let α_i be a simple root. Then, since the galleries associated to $\mathscr{L}$ and $w(\mathscr{L})$ are both dominant, $f_{\alpha_i}(\mathscr{L}) = e_{\alpha_i}(\mathscr{L}) = f_{\alpha_i}(w(\mathscr{L})) = e_{\alpha_i}(w(\mathscr{L})) = 0$. ☐

Example 5.10. Let $n = 2$ and $\mathscr{B}$ be the readable block $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}$. Then $w(\mathscr{B}) = 2\bar{2}$. To calculate $f_{\alpha_1}(\gamma_{\mathscr{B}})$, first consider the gallery,

$$\gamma_{\mathscr{B}} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2),$$

where $\mathrm{V}_0 = 0$, $\mathrm{V}_1 = \frac{1}{2}(\varepsilon_2 - \varepsilon_1)$, $\mathrm{V}_2 = 0$ and E_i is the line segment joining V_i and V_{i+1} for $i \in \{0, 1\}$. Note that $m_{\alpha_1} = -1$, $j = 1$, and $r = 2$ (see Definition 3.1). Therefore

$$f_{\alpha_1}(\gamma_{\mathscr{B}}) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \mathrm{V}'_2),$$

where $\mathrm{V}'_0 = 0$, $\mathrm{E}'_0 = \mathrm{E}_0$, $\mathrm{V}'_1 = \mathrm{V}_1$, $\mathrm{E}'_1 = s_{(\alpha_1,-1)}(\mathrm{E}_1)$ and $\mathrm{V}'_2 = s_{(\alpha_1,-1)}(\mathrm{V}_2) = \varepsilon_2 - \varepsilon_1$. Then $f_{\alpha_1}(\gamma_{\mathscr{B}}) = \gamma_{\mathscr{B}'}$, where

$$\mathscr{B}' = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \bar{1} & \bar{1} \\ \hline \end{array}\,.$$

Similarly, $f_{\alpha_1}(w(\mathscr{B})) = 2\bar{1} = w(f_{\alpha_1}(\gamma_{\mathscr{B}}))$.

5C. *Readable galleries are Littelmann galleries.* We begin with a lemma.

Lemma 5.11. *Let $\gamma_{\mathscr{K}}$ be the readable gallery associated to a readable key $\mathscr{K}$. Then $\gamma_{\mathscr{K}}$ is dominant if and only if $\gamma_{w(\mathscr{K})}$ is dominant.*

Proof. Since the entries in the columns of symplectic keys are strictly increasing, it follows from the definition of word reading (Definition 5.1 and Definition 5.3) that if γ is a dominant readable gallery then $\gamma_{w(\gamma)}$ is also dominant. Now let γ be a nondominant readable gallery. Then there is a readable block $\mathscr{B} = \mathrm{C_L C_R}$ such that $\gamma = \eta_1 * \gamma_{\mathscr{B}} * \eta_2$ with η_1 dominant and $\eta_1 * \gamma_{\mathscr{B}}$ not dominant. This block can't be a zero lump (they are dominant) — so it must be LS. Let A, B, Z and T be the sets from Definition 4.3 that define the LS block $\mathscr{B}$: The entries of its right column $\mathrm{C_R}$ are the letters in $\mathrm{A} \cup \mathrm{Z} \cup \bar{\mathrm{B}} \cup \bar{\mathrm{T}}$ and the entries its left column $\mathrm{C_L}$ are the letters in $\mathrm{A} \cup \mathrm{T} \cup \bar{\mathrm{B}} \cup \bar{\mathrm{Z}}$. Now, $\mu_{\eta_1 * \gamma_{\mathscr{B}}}$ may or may not be dominant. If it is not, then, since $\mu_{\gamma_{w(\eta_1 * \gamma_{\mathscr{B}})}} = \mu_{\eta_1 * \gamma_{\mathscr{B}}}$, the word gallery $\gamma_{w(\eta_1 * \gamma_{\mathscr{B}})}$ is not dominant, and this implies that $\gamma_{w(\mathscr{K})}$ is not dominant either. Now assume that the coweight

$$\mu_{\eta_1 * \gamma_{\mathscr{B}}} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b$$

is dominant, but that the gallery $\eta_1 * \gamma_{\mathscr{B}}$ is not. The last three vertices of this gallery are

$$\mathrm{V}_{l-1} = \mu_{\eta_1} \in \mathrm{C}^+, \tag{11}$$

$$\mathrm{V}_l = \mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z - \sum_{b \in \mathrm{B}} \varepsilon_b - \sum_{t \in \mathrm{T}} \varepsilon_t\Bigg) \notin \mathrm{C}^+, \tag{12}$$

$$\mathrm{V}_{l+1} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b \in \mathrm{C}^+, \tag{13}$$

for some $d \geq 1$. Let $d_1 < \cdots < d_{r+k}$ be the ordered elements of $\mathrm{A} \cup \mathrm{Z}$ and let $f_1 < \cdots < f_{s+k}$ be the ordered elements of $\mathrm{B} \cup \mathrm{Z}$. We have

$$w(\mathscr{B}) = d_1 \cdots d_{r+k} \bar{f}_{s+k} \cdots \bar{f}_1.$$

We claim that the weight

$$\mu_{\eta_1} + \sum_{i=1}^{r+k} \varepsilon_{d_i} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z,$$

which is the endpoint of $\eta_1 * \gamma_{d_1 \cdots d_{r+k}}$ and therefore a vertex of $\eta * \gamma_{w(\mathscr{B})}$, is not dominant. To see this, assume otherwise:

$$\mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z \in \mathrm{C}^+.$$

Since the dominant Weyl chamber C^+ is convex, this means that the line segment that joins μ_{η_1} and $\mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z$ is contained in C^+, in particular the point

$$\mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z\Bigg) \in \mathrm{C}^+ \tag{14}$$

belongs to the dominant Weyl chamber. We will now show

$$\mathrm{V}_l = \mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z - \sum_{b \in \mathrm{B}} \varepsilon_b - \sum_{t \in \mathrm{T}} \varepsilon_t\Bigg) \in \mathrm{C}^+.$$

This would contradict (12) and therefore complete the proof.

Set $\mu_{\eta_1} = \sum_{i=1}^{n} q_i \varepsilon_i$. Recall that $a_1 < \cdots < a_r$, $b_1 < \cdots < b_s$, $z_1 < \cdots < z_k$, and $t_1 < \cdots < t_k$ are the ordered elements of the sets A, B, Z and T, respectively. The dominant Weyl chamber has, in this case, the following description in the coordinates $\varepsilon_1, \ldots, \varepsilon_n$:

$$\mathrm{C}^+ = \Bigg\{\sum_{i=1}^{n} p_i \varepsilon_i : p_i \in \mathbb{R}_{\geq 0} \text{ and } p_1 \geq \cdots \geq p_n\Bigg\}. \tag{15}$$

This description allows us to make the following conclusions. For every $i \in \{1, \cdots, r\}$, we have $t_i < z_i < j$ for every $j \in \{1, \ldots, n\}$ such that $t_i < j$. It follows from (15) and (14) that

$$q_j \leq q_{z_i} + \tfrac{1}{2} \leq q_{t_i}, \tag{16}$$

which implies, since $q_j, q_{t_i}, q_{z_i} \in \mathbb{Z}$, that

$$q_j \leq q_j + \tfrac{1}{2} \leq q_{z_i} + \tfrac{1}{2} \leq q_{t_i} - \tfrac{1}{2}.$$

Now let $b \in \mathrm{B}$, and let $j \in \{1, \ldots, n\}$ such that $b < j$. By (13),

$$\mathrm{V}_{l+1} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b \in \mathrm{C}^+.$$

Together with (15) this implies

$$q_j \le q_j + \tfrac{1}{2} \le q_b - \tfrac{1}{2},$$

particularly so if $j \in (\mathrm{Z} \cup \mathrm{T})^c$. If $j \in \mathrm{Z} \cup \mathrm{T}$ then, as before, by (16) we may assume that $j = t \in \mathrm{T}$. But this means $q_t \le q_b$, therefore $q_t - \frac{1}{2} \le q_b - \frac{1}{2}$. All of these arguments, together with (15), imply

$$\mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z - \sum_{b \in \mathrm{B}} \varepsilon_b - \sum_{t \in \mathrm{T}} \varepsilon_t\Bigg) \in \mathrm{C}^+,$$

which contradicts (12). □

Lemma 5.12. *A readable gallery ν is dominant if and only if $e_{\alpha_i}(\nu) = 0$ for all $i \in \{1, \dots, n\}$.*

Proof. First notice that it follows directly from Definition 5.6 that for a word $w \in \mathscr{W}_{\mathscr{C}_n}$ and α_i a simple root, $e_{\alpha_i}(w) = 0$ if and only if γ_w is dominant. Lemma 5.12 then follows from Lemma 5.11 and Proposition 5.9. □

Proposition 5.13. *Every readable gallery is a Littelmann gallery.*

Proof. Let V_n be the vector representation of $\mathrm{Sp}(2n, \mathbb{C})$. Then the crystal of words $\mathscr{W}_{\mathscr{C}_n}$ is isomorphic to the crystal associated to $\mathrm{T}(\mathrm{V}_n) = \bigoplus_{l \in \mathbb{Z}_{\ge 0}} \mathrm{V}_n^{\otimes l}$, see for example [Lecouvey 2002, §2.1]. Now let γ be any readable gallery. Then there exist indices $i_1, \dots, i_r$ such that $e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma_{w(\gamma)})$ is a highest weight vertex, hence dominant by Lemma 5.12. Since word reading is a morphism of crystals by Proposition 5.9, $\gamma_{w(e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma))} = e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma_{w(\gamma)})$. It follows from Lemma 5.11 that $e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma)$ is dominant. □

Definition 5.14. The *symplectic plactic monoid* $\mathscr{P}_{\mathscr{C}_n}$ is the quotient of the word monoid $\mathscr{W}_{\mathscr{C}_n}$ by the ideal generated by the following relations:

R1. For $z \neq \bar{x}$:

$$y\ x\ z \equiv y\ z\ x \quad \text{for } x \le y < z,$$

$$x\ z\ y \equiv z\ x\ y \quad \text{for } x < y \le z.$$

R2. For $1 < x \le n$ and $x \le y \le \bar{x}$:

$$y\ \overline{x-1}\ x-1 \equiv y\ x\ \bar{x},$$

$$\overline{x-1}\ x-1\ y \equiv x\ \bar{x}\ y.$$

R3. For $a_i, b_i \in \{1, \dots, n\}$, $i \in \{1, \dots, \max\{s, r\}\}$ such that $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_s$, and such that the left-hand side of the next expression is not the word of an LS block:

$$a_1 \cdots a_r\ z\ \ \bar{z}\ \bar{b}_s \cdots \bar{b}_1 \equiv a_1 \cdots a_r\ \bar{b}_s \cdots \bar{b}_1.$$

If two words $w_1, w_2 \in \mathscr{W}_{\mathscr{C}_n}$ are representatives of the same class in $\mathscr{W}_{\mathscr{C}_n}$ we say they are *symplectic plactic equivalent*.

Example 5.15. We have the following equivalences of words:

$$1 2 \bar{2} \bar{1} \equiv 1 \bar{1} \equiv \varnothing,$$
$$112 \equiv 121.$$

Remark 5.16. Relations R1 are the Knuth relations in type A, while relation R3 may be understood as the general relation that specializes to $1\bar{1} \equiv \varnothing$. Note that the gallery γ_w associated to $w = 1\bar{1}$ is a zero lump. This definition of the symplectic plactic monoid is the same as [Lecouvey 2002, Definition 3.1.1] except for relation R3. The equivalence between the relation R3 above and the one in [Lecouvey 2002] is given in the Appendix.

The following Theorem is proven in [Lecouvey 2002].

Theorem 5.17. *Two words $w_1, w_2 \in \mathscr{W}_{\mathscr{C}_n}$ are symplectic plactic equivalent if and only if their associated galleries γ_{w_1} and γ_{w_2} are equivalent.*

Together with the results we have recollected in this section, Theorem 5.17 implies the following proposition.

Proposition 5.18. *Two readable galleries γ and ν are equivalent if and only if the words $w(\gamma)$ and $w(\nu)$ are symplectic plactic equivalent.*

Proof. Two readable galleries γ and ν are equivalent if and only if, by definition, there exist indices $i_1, \ldots, i_r$ such that the galleries $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma)$ and $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\nu)$ are both dominant and have the same endpoint, i.e., $\mu_{e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma)} = \mu_{e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\nu)}$. By Lemma 5.11 and Proposition 5.9 this is true if and only if $\gamma_{w(e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma))}$ and $\gamma_{w(e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\nu))}$ are also both dominant with the same endpoint. By Proposition 5.9, we have $w(e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\delta)) \equiv e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(w(\gamma_\delta))$ for any readable gallery δ. This means that the previous sequence of equivalences is also equivalent to $\gamma_{w(\gamma)} \sim \gamma_{w(\nu)}$ which by Theorem 5.17 is equivalent to $w(\gamma) \equiv w(\nu)$. □

The following theorem is originally due to Kashiwara and Nakashima (see [Kashiwara and Nakashima 1994]). For this particular formulation, see [Lecouvey 2002, Proposition 3.1.2].

Theorem 5.19. *For each word w in $\mathscr{W}_{\mathscr{C}_n}$ there exists a unique symplectic* LS *key $\mathscr{T}$ such that $w \equiv w(\mathscr{T})$.*

The following proposition will be proven in Section 7. Along with Proposition 5.5 it will play a fundamental role in the proof of Theorem 6.2.

Proposition 5.20. *Let γ and ν be combinatorial galleries and let $w_1, w_2 \in \mathscr{W}_{\mathscr{C}_n}$ be two plactic equivalent words. Consider the combinatorial galleries $\gamma * \gamma_{w_1} * \nu$ and*

$\gamma * \gamma_{w_2} * \nu$ as well as their associated Bott–Samelson varieties $(\Sigma_{(\gamma*\gamma_{w_1}*\nu)^f}, \pi)$ and $(\Sigma_{(\gamma*\gamma_{w_2}*\nu)^f}, \pi')$ together with their maps to the affine Grassmannian $\mathscr{G}$. Then

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{w_1}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w_2}*\nu})}.$$

6. Readable galleries and MV cycles

The following result holds in greater generality than is stated here: part (a) is an instance of [Gaussent and Littelmann 2005, Theorem C], and part (b) is an instance of [Baumann and Gaussent 2008, Theorem 5.8].

Theorem 6.1. *Let $\underline{d} = \underline{p}_1 * \cdots * \underline{p}_l$ be a symplectic shape such that $p_1 \leq \cdots \leq p_l$ and consider the desingularization $\pi : \Sigma_{\underline{d}} \to \mathrm{X}_{\lambda_{\underline{d}}}$.*

(a) *If $\delta \in \boldsymbol{\Gamma}(\underline{d})^{\mathrm{LS}}$ is a symplectic LS key, the closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathscr{Z}(\lambda_{\underline{d}})$. This induces a bijection $\boldsymbol{\Gamma}(\underline{d})^{\mathrm{LS}} \xrightarrow{\varphi_{\underline{d}}} \mathscr{Z}(\lambda_{\underline{d}})$.*

(b) *The bijection $\varphi_{\underline{d}}$ is an isomorphism of crystals.*

To formulate our main result we need the following additional notation. Given a readable gallery γ and a dominant coweight $\lambda \in \mathrm{X}^{\vee,+}$, let

$$n^{\lambda}_{\gamma^f} = \#\{\nu \in \Gamma^{\mathrm{dom}} \cap \Gamma(\gamma^f) : \mu_\nu = \lambda\},$$

and let

$$\mathrm{X}^{\vee,+}_{\gamma^f} = \{\lambda \in \mathrm{X}^{\vee,+} : n^{\lambda}_{\gamma^f} \neq 0\}.$$

Further, let $\Gamma(\gamma^f)^{\mathrm{R}}/\sim$ be a set of representatives of the classes for the equivalence relation on Littelmann galleries (and hence on readable galleries by Remark 3.4 and Proposition 5.13) defined in Section 3C.

Theorem 6.2. *Let $\delta \in \Gamma(\gamma^f)^{\mathrm{R}}$ be a readable gallery. Consider the corresponding Bott–Samelson variety (Σ_{γ^f}, π) together with its map π to the affine Grassmannian as in Remark 2.6. Let δ^+ be the gallery that is the highest weight vertex in* $\mathrm{Conn}(\delta)$. (This gallery is dominant and readable by Lemma 5.12 and Remark 3.4, respectively.) *Then*:

(a) *The closed set $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathscr{Z}(\mu_{\delta^+})_{\mu_\delta}$.*

(b) *The map*

$$\Gamma(\gamma^f)^{\mathrm{R}} \xrightarrow{\varphi_{\gamma^f}} \bigoplus_{\nu\in\Gamma(\gamma^f)^{\mathrm{R}}/\sim} \mathscr{Z}(\mu_{\nu^+}), \quad \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a surjective morphism of crystals. The direct sum on the right-hand side is a direct sum of abstract crystals.

(c) *If C is a connected component of $\Gamma(\gamma^f)^{\mathrm{R}}$, then $\varphi|_{\mathrm{C}}$ is an isomorphism onto its image.*

(d) *The number of connected components* C *of* $\Gamma^{\mathrm{R}}(\gamma^f)$ *such that* $\varphi_{\gamma^f}(\mathrm{C}) = \mathscr{Z}(\lambda)$ *is equal to* $n^{\lambda}_{\gamma^f}$.

(e) *Given an MV cycle* $\mathrm{Z} \in \mathscr{Z}(\lambda)_\mu$, *the fiber* $\varphi^{-1}_{\gamma^f}(\mathrm{Z})$ *is given by*

$$\varphi^{-1}_{\gamma^f}(\mathrm{Z}) = \{\delta \in \Gamma^{\mathrm{R}}(\gamma^f) : \varphi_{\gamma^f}(\delta) = \mathrm{Z}\} = \{\delta \in \Gamma^{\mathrm{R}}(\gamma^f) : \gamma \sim \gamma^{\lambda}_{\mu,\mathrm{Z}}\},$$

where $\gamma^{\lambda}_{\mu,\mathrm{Z}}$ *is the unique LS key which exists by Theorem 6.1.*

Proof. Let δ be a readable gallery. Then by Theorem 5.19 there exists a (unique) LS key ν such that $\delta \sim \nu$. By Proposition 5.18, the words $w(\delta)$ and $w(\nu)$ are plactic equivalent. Propositions 5.20 and 5.5 together with Theorem 5.17 then imply that

$$\overline{\pi(\mathrm{C}_\delta)} = \overline{\pi(\mathrm{C}_\nu)},$$

which, by Theorem 6.1 implies that $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathscr{Z}(\mu_{\delta^+})_{\mu_\delta}$. The map φ_{γ^f} in (b) is surjective by Theorems 5.19 and 6.1 above. Now let r be a root operator, and let $\tilde{r}$ be the corresponding root operator that acts on the set of MV cycles. Then by Propositions 5.5, 5.9, 5.20, and Theorem 6.1 we have

$$\overline{\pi(\mathrm{C}_{r(\gamma)})} = \overline{\pi(\mathrm{C}_{\gamma_{w(r(\gamma))}})} = \overline{\pi(\mathrm{C}_{\gamma_{w(r(\nu))}})} = \overline{\pi(\mathrm{C}_{r(\nu)})} = \tilde{r}(\overline{\pi(\mathrm{C}_\nu)}) = \tilde{r}(\overline{\pi(\mathrm{C}_\gamma)}).$$

This completes the proof of (b). Part (c) follows immediately, since every connected component C is crystal isomorphic to the corresponding component consisting of the LS galleries equivalent to those in C. Parts (d) and (e) follow from [Littelmann 1995, Theorem 7.1] (see Section 3C). □

7. Counting positive crossings

We provide proofs of Propositions 5.5 and 5.20. We begin by analyzing the *tail* of a gallery in Section 7A. In Example 7.3 we calculate an example in which it can be seen how to use this proposition. Then in Section 7B we prove Proposition 5.5 and in Section 7C we prove Proposition 5.20. We also wish to establish some notation that we will use throughout. Recall our convention $\varepsilon_{\bar{l}} = -\varepsilon_l$ for $l \in \mathscr{C}_n$ unbarred. For $l, s, d, m \in \mathscr{C}_n$ we will write $c^{i,j}_{ls,dm}$ for the constant $c^{i,j}_{\varepsilon_l+\varepsilon_s,\varepsilon_d+\varepsilon_m}$ in Chevalley's commutator formula (2). Additionally we will write $c^{i,j}_{l,dm}$, and respectively $c^{i,j}_{ls,d}$, for $c^{i,j}_{\varepsilon_l,\varepsilon_d+\varepsilon_m}$, and $c^{i,j}_{\varepsilon_l+\varepsilon_s,\varepsilon_d}$. (Each time we use such notation a total order will be fixed on the set of positive roots.) If $\mathrm{Y} \subseteq \mathscr{C}_n$ and $y \in \mathscr{C}_n$ then we will write $\mathrm{Y}^{\leq y}$ (respectively $\mathrm{Y}^{<y}$, $\mathrm{Y}^{\geq y}$, $\mathrm{Y}^{>y}$) for the subset of elements $x \in \mathrm{Y}$ such that $x \leq y$ (respectively $x < y$, $x \geq y$, $x > y$).

7A. ***Truncated images and tails.*** Let γ be a combinatorial gallery with notation as in (4) with endpoint the coweight μ_γ and let $1 \leq r \leq k+1$ such that V_r is a special vertex; we denote it by $\mu_r \in \mathrm{X}^\vee$. By Corollary 2.10 we know that the image $\pi(\mathrm{C}_\gamma)$ is stable under U_0.

Proposition 7.1. *The r-truncated image of γ,*

$$\mathrm{T}_\gamma^{\geq r} = \mathbb{U}_{\mathrm{V}_r}^\gamma \mathbb{U}_{\mathrm{V}_{r+1}}^\gamma \cdots \mathbb{U}_{\mathrm{V}_k}^\gamma [t^{\mu_\gamma}],$$

is U_{μ_r}-stable, i.e., for any $u \in \mathrm{U}_{\mu_r}$, it follows that $u\mathrm{T}_\gamma^{\geq r} = \mathrm{T}_\gamma^{\geq r}$.

Proof. By (3) we know that $t^{\mu_r}\mathrm{U}_0 t^{-\mu_r} = \mathrm{U}_{\mu_r}$. We consider the *r-truncated gallery*

$$\gamma^{\geq r} = (\mathrm{V}_0', \mathrm{E}_0', \dots, \mathrm{V}_{k-r+1}'),$$

which is the combinatorial gallery obtained from the sequence

$$(\mathrm{V}_r, \mathrm{E}_r, \mathrm{V}_{r+1}, \dots, \mathrm{E}_k, \mathrm{V}_{k+1}),$$

by translating it to the origin. Since V_r is a special vertex, $t^{\mu_r}\mathbb{U}_{\mathrm{V}_i}^{\gamma^{\geq r}} t^{-\mu_r} = \mathbb{U}_{\mathrm{V}_{i+r}}^\gamma$. This gallery has endpoint $\mu_\gamma - \mu_r$ and is in turn a T-fixed point of a Bott–Samelson variety (Σ, π'). Let $u \in \mathrm{U}_{\mu_r}$ and $u' = t^{-\mu_r} u t^{\mu_r} \in \mathrm{U}_0$. Then

$$\begin{aligned} u\mathrm{T}_\gamma^{\geq r} &= u\mathbb{U}_{\mathrm{V}_r}^\gamma \mathbb{U}_{\mathrm{V}_{r+1}}^\gamma \cdots \mathbb{U}_{\mathrm{V}_k}^\gamma [t^{\mu_\gamma}] \\ &= t^{\mu_r} u' \mathbb{U}_{\mathrm{V}_0}^{\gamma^{\geq r}} \cdots \mathbb{U}_{\mathrm{V}_{k-r}}^{\gamma^{\geq r}} [t^{\mu_\gamma - \mu_r}] \\ &= t^{\mu_r} \mathbb{U}_{\mathrm{V}_0}^{\gamma^{\geq r}} \cdots \mathbb{U}_{\mathrm{V}_{k-r}}^{\gamma^{\geq r}} [t^{\mu_\gamma - \mu_r}] = \mathrm{T}_\gamma^{\geq r}. \end{aligned}$$

Where the final equality follows from Corollary 2.10. $\square$

For later use let us fix the notation

$$\mathrm{T}_\gamma^{<r} = \mathbb{U}_{\mathrm{V}_0}^\gamma \cdots \mathbb{U}_{\mathrm{V}_{r-1}}^\gamma,$$

so that

$$\pi(\mathrm{C}_\gamma) = \mathrm{T}_\gamma^{<r}\mathrm{T}_\gamma^{\geq r}.$$

Remark 7.2. This Proposition is proven for $\mathrm{SL}(n, \mathbb{C})$ in [Gaussent et al. 2013, Proposition 3]. The proof we have provided is exactly the same, except for the restriction of only being able to truncate at special vertices.

Example 7.3. Let $n = 2$. Consider the symplectic keys

$$\mathscr{K}_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & \bar{1} \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & \bar{2} & \bar{1} \\ \cline{2-3} \end{array},$$

and their words

$$w(\mathscr{K}_1) = \bar{1}12 \quad \text{and} \quad w(\mathscr{K}_2) = 2\bar{2}2.$$

Note that $\gamma_{\omega_1} * \gamma_{\omega_2} \sim \gamma_{\omega_2} * \gamma_{\omega_1}$, since both $\gamma_{\omega_1} * \gamma_{\omega_2}$ and $\gamma_{\omega_2} * \gamma_{\omega_1}$ are contained in the fundamental chamber and have the same endpoint $\omega_1 + \omega_2$. One checks that

$$f_{\alpha_1} f_{\alpha_2} f_{\alpha_1}(\gamma_{\omega_1} * \gamma_{\omega_2}) = \gamma_{\mathscr{K}_1} \quad \text{and} \quad f_{\alpha_1} f_{\alpha_2} f_{\alpha_1}(\gamma_{\omega_2} * \gamma_{\omega_1}) = \gamma_{\mathscr{K}_2}.$$

Therefore $\gamma_{\mathscr{K}_1} \sim \gamma_{\mathscr{K}_2}$. Lemma 5.11 and Proposition 5.9 imply that $\gamma_{w(\mathscr{K}_1)} \sim \gamma_{w(\mathscr{K}_2)}$ (it can also be checked directly using Relation R2 in Theorem 5.17 with $y = x = 2$). Now consider combinatorial galleries γ and ν. The galleries $\gamma * \gamma_{\mathscr{K}_1} * \nu$ and $\gamma * \gamma_{\mathscr{K}_2} * \nu$ are T-fixed points in the Bott–Samelson varieties $(\Sigma_{(\gamma*\gamma_{\mathscr{K}_1}*\nu)^f}, \pi)$, respectively $(\Sigma_{(\gamma*\gamma_{\mathscr{K}_2}*\nu)^f}, \pi')$. The galleries $\gamma_{w(\mathscr{K}_1)}$ and $\gamma_{w(\mathscr{K}_2)}$ that correspond to their words are T-fixed points in $(\Sigma_{(\gamma*\gamma_{\omega_1}*\gamma_{\omega_1}*\gamma_{\omega_1}*\nu)^f}, \pi'')$. We show that

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K}_1)}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K}_2)}*\nu})}.$$

We use the same notation as in (4) for γ. Since for any combinatorial gallery η, $(\alpha, n) \in \Phi_{k+1}^{\gamma*\eta}$ if and only if $(\alpha, n - \langle \alpha, \mu_\gamma \rangle) \in \Phi_0^{\gamma}$, we may assume that $\gamma = \varnothing$. Since $\gamma_{\mathscr{K}_1}$, $\gamma_{\mathscr{K}_2}$, $\gamma_{w(\mathscr{K}_1)}$ and $\gamma_{w(\mathscr{K}_2)}$ have the same endpoint ε_2, this also implies that $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_2}*\nu} = \mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)}*\nu} = \mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_1)}*\nu}$. By Proposition 2.7, for $a', b', c', d' \in \mathbb{C}$,

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1*\nu}}) = \mathrm{U}_{(\varepsilon_1,-1)}(a')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b')\mathrm{U}_{(\varepsilon_2,0)}(c')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d')\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu}.$$

By Chevalley's commutator formula (2) and an application of Proposition 7.1 to $\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e) \in \mathrm{U}_{\varepsilon_2}$, we obtain

$$\begin{aligned}
&\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)*\nu}})\\
&\quad = \mathrm{U}_{(\varepsilon_1,-1)}(a)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b)\cdot\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e)\cdot\mathrm{U}_{(\varepsilon_2,0)}(c)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d)\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_1)}*\nu}\\
&\quad = \mathrm{U}_{(\varepsilon_1,-1)}\big(a + c^{1,1}_{1\bar{2},2}(-e)c\big)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}\big(b + c^{1,1}_{1\bar{2},2}(-e)c^2\big)\\
&\qquad\qquad \cdot\mathrm{U}_{(\varepsilon_2,0)}(c)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d)\cdot\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e)\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu}\\
&\quad = \mathrm{U}_{(\varepsilon_1,-1)}\big(a + c^{1,1}_{1\bar{2},2}(-e)c\big)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}\big(b + c^{1,1}_{1\bar{2},2}(-e)c^2\big)\\
&\qquad\qquad \cdot\mathrm{U}_{(\varepsilon_2,0)}(c)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d)\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu}\\
&\quad \subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}),
\end{aligned}$$

for $a, b, c, d, e \in \mathbb{C}$. Choosing $a = a'$, $b = b'$, $c = c'$, $d = d'$, and $e = 0$, we have $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) \subset \pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)}})$. Hence, in this case $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) = \pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)}})$. Similarly, for $a'', b'', c'', d'', e'' \in \mathbb{C}$,

$$\begin{aligned}
&\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_2)}*\nu})\\
&\quad = \mathrm{U}_{(\varepsilon_2,0)}(a'')\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(b'')\cdot\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e'')\cdot\mathrm{U}_{(\varepsilon_2,0)}(c'')\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d'')\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)}*\nu}\\
&\quad = \mathrm{U}_{(\varepsilon_1,-1)}\big(c^{1\bar{2},2}_{1,1}(-e'')c''\big)\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}\big(c^{1\bar{2},2}_{1,2}(-e'')c''^2\big)\\
&\qquad\qquad \cdot\mathrm{U}_{(\varepsilon_2,0)}(a''+c'')\cdot\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(b''+d'')\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)}*\nu}\\
&\quad \subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_1*\nu}}).
\end{aligned}$$

Hence the open subset of $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1*\nu}})$ given by $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$ is contained in $\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_2)}*\nu})$.

7B. *Proof of Proposition 5.5.* It is enough to show that if γ and ν are combinatorial galleries and $\mathscr{K}$ is a readable block, then

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K})}*\nu})}, \tag{17}$$

where $(\Sigma_{(\gamma*\gamma_{\mathscr{K}}*\nu)^f}, \pi)$ and $(\Sigma_{(\gamma*\gamma_{w(\mathscr{K})}*\nu)^f}, \pi')$ are the Bott–Samelson varieties associated to the galleries $\gamma * \gamma_{\mathscr{K}} * \nu$ and $\gamma * \gamma_{w(\mathscr{K})} * \nu$ respectively.

Proof. We assume $\gamma = \varnothing$; we may do so by the argument given at the beginning of Example 7.3. Let $\mathscr{K}$ be an LS block and let $\mathrm{A} = \{a_1, \dots, a_r\}$, $\mathrm{B} = \{b_1, \dots, b_s\}$, $\mathrm{Z} = \{z_1, \dots, z_k\}$ and $\mathrm{T} = \{t_1, \dots, t_k\}$ be the subsets of $\{1, \dots, n\}$ from Definition 4.3 that determine $\mathscr{K}$. We will use the notation $d_1 < \cdots < d_{r+k}$ to denote the ordered elements of $\mathrm{Z} \cup \mathrm{A}$ and $f_1 < \cdots < f_{s+k}$ the ordered elements of $\mathrm{B} \cup \mathrm{Z}$. We also write

$$\gamma_{\mathscr{K}} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2).$$

The proof is divided into Lemmas 7.4 and 7.5 below.

Lemma 7.4. *Let ν be a combinatorial gallery and $\mathscr{K}$ be a readable block. Then*

$$\overline{\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})}.$$

Proof. We first show that

$$\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}) \subset \mathrm{U}_0 \mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}, \tag{18}$$

where

$$\mathbb{P}'''_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(k_{l\bar{b}}) \prod_{t\in\mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(k_{t\bar{b}}) \prod_{a\in\mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(k_{a\bar{b}}), \tag{19}$$

$$\mathbb{P}'''_{\bar{z}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; \\ l<z}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(k_{l\bar{z}}) \prod_{t\in\mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) \prod_{b\in\mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(k_{b\bar{z}}), \tag{20}$$

for $b \in \mathrm{B}$, $z \in \mathrm{Z}$ and $k_{ij} \in \mathbb{C}$. Indeed, the points of $\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})$ are of the form

$$\mathbb{P}_{d_1} \cdots \mathbb{P}_{d_{r+k}} \mathbb{P}_{\bar{f}_{k+s}} \cdots \mathbb{P}_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}, \tag{21}$$

where

$$\mathbb{P}_d = \mathrm{U}_{(\varepsilon_d,0)}(g_d) \prod_{d<l\leq n} \mathrm{U}_{(\varepsilon_d-\varepsilon_l,0)}(g_{d\bar{l}}) \prod_{l\notin(\mathrm{Z}\cup\mathrm{A})^{<d}} \mathrm{U}_{(\varepsilon_d+\varepsilon_l,0)}(g_{dl}) \prod_{l\in(\mathrm{Z}\cup\mathrm{A})^{<d}} \mathrm{U}_{(\varepsilon_d+\varepsilon_l,1)}(g^1_{dl}),$$

$$\mathbb{P}_{\bar{b}} = \mathrm{S}_{\bar{b}} \mathbb{P}^{iv}_{\bar{b}} \quad \text{with } \mathrm{S}_{\bar{b}} = \prod_{b'\in\mathrm{B}^{<b}} \mathrm{U}_{(\varepsilon_{b'}-\varepsilon_b,0)}(g_{b'\bar{b}}) \prod_{z\in\mathrm{Z}^{<b}} \mathrm{U}_{(\varepsilon_z-\varepsilon_b,1)}(g^1_{z\bar{b}}) \in \mathrm{U}_0 \quad \text{and}$$

$$\mathbb{P}^{iv}_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(g_{l\bar{b}}) \prod_{t\in\mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(g_{t\bar{b}}) \prod_{a\in\mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(g_{a\bar{b}}),$$

and finally

$$\mathbb{P}_{\bar z} = J_{\bar z}\mathbb{P}^{iv}_{\bar z} \quad \text{with } \mathrm{J}_{\bar z} = \prod_{a\in \mathrm{A}^{<z}} \mathrm{U}_{(\varepsilon_a-\varepsilon_z,0)}(g_{a\bar z}) \prod_{z'\in \mathrm{Z}^{<z}} \mathrm{U}_{(\varepsilon_{z'}-\varepsilon_z,0)}(g_{z'\bar z}) \in \mathrm{U}_0 \quad \text{and}$$

$$\mathbb{P}^{iv}_{\bar z} = \mathrm{J}_{\bar z} \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ l<z}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(g_{l\bar z}) \prod_{t\in \mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(g_{t\bar z}) \prod_{b\in \mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(g_{b\bar z}),$$

for $d\in \mathrm{A}\cup\mathrm{Z}$, $z\in \mathrm{Z}$, $b\in \mathrm{B}$, and $g_{ij}\in\mathbb{C}$. All terms in $\mathrm{J}_{\bar z}$ commute with $\mathbb{P}^{iv}_{z'}$ for $z'\in \mathrm{Z}^{>z}$ and with $\mathbb{P}^{iv}_{\bar b}$ for $b\in \mathrm{B}^{>z}$. All terms in $\mathrm{S}_{\bar b}$ commute with $\mathbb{P}^{iv}_{\bar b'}$ for $b'\in \mathrm{B}^{>b}$. For $z'>b$ it commutes with all terms of $\mathbb{P}^{iv}_{\bar z'}$ except for the term $\mathrm{U}_{(\varepsilon_b-\varepsilon_{z'},-1)}(g_{b\bar z'})$. But commuting $\mathrm{S}_{\bar b}$ with this term (using Chevalley's commutator formula (2)) produces terms $\mathrm{U}_{(\varepsilon_z-\varepsilon_{z'},0)}(*)$ and $\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_{z'},-1)}(*)$, of these terms, $\mathrm{U}_{(\varepsilon_z-\varepsilon_{z'},0)}(*)$ commutes with $\mathbb{P}^{iv}_{z'}$ for $z'\in \mathrm{Z}^{>z}$ and with $\mathbb{P}^{iv}_{\bar b}$ for $b\in \mathrm{B}^{>z}$, and $\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_{z'},-1)}(*)$ is a term of the form of those appearing in $\mathbb{P}^{iv}_{\bar z}$.

Since the terms that appear in $\mathbb{P}^{iv}_{\bar b}$ and $\mathbb{P}^{iv}_{\bar z}$ are the same as those in $\mathbb{P}''_{\bar b}$ and $\mathbb{P}''_{\bar z}$ respectively, this justifies (18), concluding the first step in the proof of Lemma 7.4. The second step is this:

Claim. *There is a dense subset of* $\mathbb{P}'''_{\bar f_{k+s}}\cdots\mathbb{P}'''_{\bar f_1}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}$ *contained in the subset*

$$\mathbb{P}_{\mathrm{T},\mathrm{B}}\mathbb{P}_{\mathscr{K},\bar f_s}\cdots\mathbb{P}_{\mathscr{K},\bar f_s}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \subset \overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})}, \tag{22}$$

where

$$\mathbb{P}_{\mathrm{T},\mathrm{B}} = \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ t\in\mathrm{T},\, l<t}} \mathrm{U}_{(\varepsilon_l-\varepsilon_t,0)}(v_{l\bar t}) \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ b\in\mathrm{B},\, l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(v_{l\bar b}) \in \mathrm{U}_{\mathrm{V}_0},$$

$$\mathbb{P}_{\mathscr{K},\bar b} = \prod_{\substack{b\in\mathrm{B}\\ t\in\mathrm{T}^{<b}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(v_{t\bar b}) \prod_{a\in\mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(v_{a\bar b}) \in \mathrm{U}_{\mathrm{V}_1},$$

$$\mathbb{P}_{\mathscr{K},\bar z} = \prod_{t\in\mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar z}) \prod_{b\in\mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(v_{b\bar z}) \in \mathrm{U}_{\mathrm{V}_1},$$

for $v_{ij}\in\mathbb{C}$, $b\in\mathrm{B}$ *and* $z\in\mathrm{Z}$. (The inclusion in (22) holds by Corollary 2.10.)

To prove this we start by noting that $\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$ and that

$$u = \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ t\in\mathrm{T},\, l<t}} \mathrm{U}_{(\varepsilon_l-\varepsilon_t,0)}(v_{l\bar t}) \in \mathrm{U}_{\mu_{\gamma_{\mathscr{K}}}}. \tag{23}$$

We have the equalities

$$\mathbb{P}_{\mathrm{T},\mathrm{B}}\mathbb{P}_{\mathscr{K},\bar f_s}\cdots\mathbb{P}_{\mathscr{K},\bar f_s}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} = \mathbb{P}''_{\bar f_s}\cdots\mathbb{P}''_{\bar f_s}u\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} = \mathbb{P}''_{\bar f_s}\cdots\mathbb{P}''_{\bar f_s}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}, \tag{24}$$

where we have introduced symbols analogous to those of (19) and (20); namely, for $z \in \mathrm{Z}$ and $b \in \mathrm{B}$,

$$\mathbb{P}''_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(\xi_{l\bar{b}}) \prod_{t\in\mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(\xi_{t\bar{b}}) \prod_{a\in\mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(\xi_{a\bar{b}}),$$

$$\mathbb{P}''_{\bar{z}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(\xi_{l\bar{z}}) \prod_{t\in\mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(\xi_{t\bar{z}}) \prod_{b\in\mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(\xi_{b\bar{z}})$$

with $\xi_{t\bar{z}} = v_{t\bar{z}},\ \xi_{b\bar{z}} = v_{b\bar{z}},\ \xi_{t\bar{b}} = v_{t\bar{b}},$

$$\xi_{l\bar{b}} = v_{l\bar{b}} + \sum_{\substack{l<b<b \\ t\in\mathrm{T}}} c^{1,1}_{s\bar{t},t\bar{b}}(-v_{l\bar{t}})v_{t\bar{b}},$$

$$\xi_{l\bar{z}} = \rho_{l\bar{z}} + \sum_{z'\in\mathrm{Z}} c^{1,1}_{l\bar{z}',z'\bar{z}}(-\rho_{l\bar{z}'})v_{z'\bar{z}} + \sum_{\substack{l<b<z \\ b\in\mathrm{B}}} c^{1,1}_{l\bar{b},b\bar{z}}(-\xi_{l\bar{b}})v_{b\bar{z}} \quad \text{for}$$

$$\rho_{l\bar{z}} = \sum_{\substack{l<t<z \\ t\in\mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{z}}(-v_{l\bar{t}})v_{t\bar{z}} \text{ (for } z \in \mathrm{Z}).$$

To complete the proof of the Claim we must set open conditions on the parameters k_{ij} such that the system of equations defined by $v_{ij} = \xi_{ij}$ has a solution in the variables v_{ij}. Setting $v_{t\bar{z}} := k_{t\bar{z}}$ and $v_{b\bar{z}} := k_{b\bar{z}}$ this is reduced to setting conditions on the k_{ij} so that the following system can be solved:

$$k_{l\bar{b}} = v_{l\bar{b}} + \sum_{\substack{l<t<b \\ t\in\mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{b}}(-v_{l\bar{t}})k_{t\bar{b}}, \tag{25}$$

$$k_{l\bar{z}} = \rho_{l\bar{z}} - \sum_{\substack{l<b<z \\ b\in\mathrm{B}}} c^{1,1}_{l\bar{b},b\bar{z}}\left(v_{l\bar{b}} + \sum_{\substack{l<t<b \\ t\in\mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{b}}(-v_{l\bar{t}})k_{t\bar{b}}\right)k_{b\bar{z}}, \tag{26}$$

$$\rho_{l\bar{z}} = \sum_{\substack{l<t<z \\ t\in\mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{z}}(-v_{l\bar{t}})k_{t\bar{z}}. \tag{27}$$

Lines (25) and (26) define a linear system of as many equations as variables. The variables are $\{v_{l\bar{b}}\}_{l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{T},b\in\mathrm{B}^{>l}} \cup \{v_{l\bar{t}}\}_{l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{Z}\cup\mathrm{T},t\in\mathrm{T}^{>l}}$; there is one equation for each $l\bar{b}$ such that $l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}$ and $b \in \mathrm{B}^{>l}$, and one for each $l\bar{z}$ such that $l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}$ and $z \in \mathrm{Z}^{>l}$. Note that by definition of an LS block the sets $\{l\bar{z}, l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; z \in \mathrm{Z}^{>l}\}$ and $\{l\bar{t}, s \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; b \in \mathrm{B}^{>l}\}$ have the same cardinality (t_i is the maximal element of the set $\{l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}, s < t_{i+1}, s < z_i\}$). Therefore the system has a solution as long as the matrix of coefficients has nonzero determinant, which imposes open conditions on the $k'_{ij}s$. Hence the Claim is proven.

To finish the proof of Lemma 7.4, note that if the $k'_{ij}s$ satisfy the open conditions established by the Claim, then

$$\mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathrm{K})} * \nu} \subseteq \pi(\mathrm{C}_{\gamma_{\mathscr{K}} * \nu}),$$

and therefore Proposition 7.1 implies that

$$\mathrm{U}_0 \mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathrm{K})} * \nu} \subseteq \pi(\mathrm{C}_{\gamma_{\mathscr{K}} * \nu}),$$

which implies Lemma 7.4. □

Lemma 7.5. *Let ν be a combinatorial gallery and $\mathscr{K}$ be an LS block. Then*

$$\overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}} * \nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})} * \nu})}. \tag{28}$$

Proof. Recall that

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}} * \nu}) = \mathbb{U}^{\gamma_{\mathscr{K}} * \nu}_{\mathrm{V}_0} \mathbb{U}^{\gamma_{\mathscr{K}} * \nu}_{\mathrm{V}_1} \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}} * \nu}.$$

Notice that $\mathbb{U}^{\gamma_{\mathscr{K}} * \nu}_{\mathrm{V}_0} \subset \mathrm{U}_0$ and that all generators of $\mathbb{U}^{\gamma_{\mathscr{K}} * \nu}_{\mathrm{V}_1}$ also belong to U_0 except for those of the form $\mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(v_{t\bar{z}})$ or $\mathrm{U}_{(\varepsilon_t + \varepsilon_{t'}, -1)}(v_{tt'})$ for $t, t' \in \mathrm{T}$, $z \in \mathrm{Z}^{>t}$, and $v_{t\bar{z}}, v_{tt'} \in \mathbb{C}$. Hence, since $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}} * \nu} = \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu}$, all elements of $\pi(\mathrm{C}_{\gamma_{\mathscr{K}} * \nu})$ belong to

$$\mathrm{U}_0 \prod_{\substack{t \in \mathrm{T} \\ z \in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(v_{t\bar{z}}) \prod_{t, t' \in \mathrm{T}} \mathrm{U}_{(\varepsilon_t + \varepsilon_{t'}, -1)}(v_{tt'}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu}. \tag{29}$$

Now consider

$$\prod_{\substack{t \in \mathrm{T} \\ z \in \mathrm{Z}}} \mathrm{U}_{(\varepsilon_z + \varepsilon_{t'}, 0)}(k_{zt'}) \prod_{t \in \mathrm{T} z \in \mathrm{Z}^{>t}} \mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(k_{t\bar{z}}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu},$$

which is a subset of $\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})} * \nu})$ by virtue of Proposition 7.1 and because

$$\prod_{\substack{z \in \mathrm{Z} \\ t \in \mathrm{T}}} \mathrm{U}_{(\varepsilon_z + \varepsilon_t, 0)}(k_{zt}) \in \mathrm{U}_0 \quad \text{and} \quad \prod_{\substack{t \in \mathrm{T} \\ z \in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(k_{t\bar{z}}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu} \subset \pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})} * \nu}).$$

We have

$$\prod_{\substack{t' \in \mathrm{T} \\ z \in \mathrm{Z}}} \mathrm{U}_{(\varepsilon_z + \varepsilon_{t'}, 0)}(k_{zt'}) \prod_{\substack{t \in \mathrm{T} \\ z \in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(k_{t\bar{z}}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu} \tag{30}$$

$$= \prod_{\substack{t, t' \in \mathrm{T} \\ t \neq t'}} \mathrm{U}_{(\varepsilon_t + \varepsilon_{t'}, -1)}(\xi_{tt'}) \prod_{\substack{t \in \mathrm{T} \\ z \in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(k_{t\bar{z}}) \prod_{\substack{t' \in \mathrm{T} \\ z \in \mathrm{Z}}} \mathrm{U}_{(\varepsilon_z + \varepsilon_{t'}, 0)}(k_{zt'}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu} \tag{31}$$

$$= \prod_{\substack{t, t' \in \mathrm{T} \\ t \neq t'}} \mathrm{U}_{(\varepsilon_t + \varepsilon_{t'}, -1)}(\xi_{tt'}) \prod_{\substack{t \in \mathrm{T} \\ z \in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t - \varepsilon_z, -1)}(k_{t\bar{z}}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})} * \nu}, \tag{32}$$

where

$$\xi_{tt'} = \sum_{z\in Z^{>t'}} c^{1,1}_{zt,t'\bar{z}}(-k_{zt})k_{t'\bar{z}} + \sum_{z\in Z^{>t}} c^{1,1}_{zt',t\bar{z}}(-k_{zt'})k_{t\bar{z}}. \tag{33}$$

The equality between (30) and (31) is due to Chevalley's commutator formula (2) and the equality between (31) and (32) is obtained by using Proposition 7.1 and $U_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \in U_{\mu_{\gamma_{\mathscr{K}}}}$. Now fix an element in (29). Setting $k_{t\bar{z}} = v_{t\bar{z}}$ defines the linear equations

$$v_{tt'} = \sum_{z\in Z^{>t'}} c^{1,1}_{zt,t'\bar{z}}(-k_{zt})v_{t'\bar{z}} + \sum_{z\in Z^{>t}} c^{1,1}_{zt',t\bar{z}}(-k_{zt'})v_{t\bar{z}},$$

in the variables k_{zt}, for $z \in Z$ and $t \in T$. There are more variables than equations. For each equation indexed by a nonordered pair (t_i, t_j) there are the variables v_{zt_i} and $v_{z't_j}$ for $z > t'$ and $z' > t$ (which always exist by definition of an LS block), hence the system has solutions as long as the matrix of coefficients has nonzero determinants. This imposes an open condition on the parameters $v_{t\bar{z}}$. Hence for such $v_{t\bar{z}}$, $v_{tt'}$, $k_{t\bar{z}} = v_{t\bar{z}}$, and solutions k_{ij}, for the latter equations we have

$$\begin{aligned}\prod_{\substack{t\in T\\ z\in Z^{>t}}} U_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in T} U_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'}) T^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}&\\ = \prod_{\substack{t'\in T\\ z\in Z}} U_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \prod_{\substack{t\in T\\ z\in Z^{>t}}} U_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) T^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} &\subset \pi'(C_{\gamma_{w(\mathscr{K})}*\nu}).\end{aligned}$$

Proposition 7.1 then implies,

$$U_0 \prod_{\substack{t\in T\\ z\in Z^{>t}}} U_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in T} U_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'}) T^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} \subset \pi'(C_{\gamma_{w(\mathscr{K})}*\nu}).$$

This completes the proof of Lemma 7.5 and hence of (17) for $\mathscr{K}$ an LS block. □

Now let $\mathscr{K}$ be a zero lump. This means there exists $k > 1$ such that the right (respectively left) column of $\mathscr{K}$ has as entries the integers $1 < \cdots < k$ (respectively $\bar{k} < \cdots < \bar{1}$), its word is therefore $w(\mathscr{K}) = 1\cdots k\bar{k}\cdots\bar{1}$. This means, in particular, that the truncated images $T^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu} = T^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$ are stabilized by U_0, by Proposition 7.1. We have

$$\pi'(C_{\gamma_{w(\mathscr{K})}*\nu}) = \mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_0} \cdots \mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_{2k-1}} T^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu},$$

by Theorem 2.9. Clearly all of the subgroups $\mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_l} \subset U_0$, for $1 \leq l \leq k$. For $0 \leq j \leq k-1$, the generators of $\mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_{k+j}}$ are all of the form $U_{(\varepsilon_s-\varepsilon_{k-j},n_{k-j})}$ for $l < k-j$. In particular the gallery $\gamma_{1\cdots k\bar{k}\cdots\overline{k-j-1}}$ has crossed the hyperplanes

$\mathrm{H}_{(\varepsilon_s-\varepsilon_{k-j},m)}$ once positively at $m=0$ and once negatively at $m=1$, which means that $n_{k-j}=0$, and $\mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j},n_{k-j})}(a)=\mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j},0)}(a)\in \mathrm{U}_0$, for all $a\in\mathbb{C}$. Hence

$$\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w(\mathscr{K})}*\nu}\cdots\mathbb{U}_{\mathrm{V}_{2k-1}}^{\gamma_{w(\mathscr{K})}*\nu}\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu}=\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu}=\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}.$$

In

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}}*\nu}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$$

we have $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}}*\nu}=\{\mathrm{Id}\}$ and $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}}*\nu}\subset \mathrm{U}_0$, therefore

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})=\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}=\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu},$$

since $\mu_{\gamma_{\mathscr{K}}}=\mu_{\gamma_{w(\mathscr{K})}}$. This finishes the proof of (17) and that of Proposition 5.5. □

7C. *Proof of Proposition 5.20.* The remainder of this section, through page 494, is devoted to the proof of Proposition 5.20. Let ν be a combinatorial gallery.

Relation R1. For $z\neq\bar{x}$:

$$\begin{aligned} y\ x\ z &\equiv y\ z\ x \quad \text{for } x\leq y<z,\\ x\ z\ y &\equiv z\ x\ y \quad \text{for } x<y\leq z.\end{aligned}$$

Lemma 7.6. *Let $w_1=y\ x\ z$, $w_2=y\ z\ x$, $w_3=x\ z\ y$, and $w_4=z\ x\ y$ for $z\neq\bar{x}$.*

(a) $\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}=\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}$.

(b) $\overline{\pi(\mathrm{C}_{\gamma_{w_3}*\nu})}=\overline{\pi(\mathrm{C}_{\gamma_{w_4}*\nu})}$.

Proof. Recall the notation $\varepsilon_{\bar{a}}=-\varepsilon_a$ and $\bar{\bar{i}}=i$ for any $i\in\{1,\dots,n\}$. Note that the $\mathrm{T}^{\geq 3}_{\gamma_{w_i}*\nu}$ all coincide for $i\in\{1,2,3,4\}$; we will denote them by T^w. We divide the proof of Lemma 7.6 into three cases.

Case 1: $x<y<z$. We claim that if $z\neq\bar{y}$ and $y\neq\bar{x}$, the following equalities hold:

i. $\pi(\mathrm{C}_{\gamma_{w_1}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w$.

ii. $\pi(\mathrm{C}_{\gamma_{w_2}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{T}^w$.

iii. $\pi(\mathrm{C}_{\gamma_{w_3}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$.

iv. $\pi(\mathrm{C}_{\gamma_{w_4}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$.

Before proving this we remark that, regardless of whether x, y, and z are barred or unbarred, the roots $\varepsilon_x-\varepsilon_z$, $\varepsilon_y-\varepsilon_z$, and $\varepsilon_x-\varepsilon_y$ are positive. Now we recall the notation from Theorem 2.9:

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_i}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_i}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_i}*\nu}\mathrm{T}^w.$$

Assume that $z\neq\bar{y}$ and $y\neq\bar{x}$.

i. We have $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \in \mathbb{U}_1^{\gamma_{w_1}*\nu}$ for any $v_{x\bar{y}} \in \mathbb{C}$, hence

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{l\bar{y}})\mathrm{T}^w \subseteq \pi(\mathrm{C}_{\gamma_{w_1}*\nu}).$$

Out of all generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_1}*\nu}$ for $i \in \{0, 1, 2\}$, the only one that does not belong to U_0 is of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \in \mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_1}*\nu}$, and the ones from $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_1}*\nu}$ that do not commute with it are those of the form $\mathrm{U}_{(\varepsilon_y+\varepsilon_z,1)}(a)$, but in that case Chevalley's commutator formula produces a term $\mathrm{U}_{(\varepsilon_x+\varepsilon_z,0)}(c^{1,1}_{x\bar{y},yz}(-v_{x\bar{y}})a) \in \mathrm{U}_0$. This implies the other inclusion, together with Proposition 2.7, which allows us to write down the generators of each $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_1}*\nu}$ in any order.

ii. The only generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_2}*\nu}$, for $i \in \{0, 1, 2\}$, that do not belong to U_0 are those of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_2}*\nu}$ or the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}}) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_2}*\nu}$. The equality follows by Proposition 2.7, Theorem 2.9, and Proposition 7.1.

iii. All the generators of $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_3}*\nu}$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_3}*\nu}$ belong to U_0, and the only generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_3}*\nu}$ that do not are $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}$. Thus iii follows by Proposition 7.1 and Theorem 2.9.

iv. As in the previous cases, we have

$$\pi(\mathrm{C}_{\gamma_{w_4*\nu}}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_4}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_4}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}\mathrm{T}^w,$$

and $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_4}*\nu} \subset \mathrm{U}_0$. All generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_4}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}$, respectively, belong to U_0 except for $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(a) \in \mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_4}*\nu}$ and $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(b) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}$, respectively, for $\{a, b\} \subset \mathbb{C}$. To prove iv we observe that $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(a)$ commutes with all generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}$ except for $\mathrm{U}_{(\varepsilon_z+\varepsilon_y,1)}(d)$, with $d \in \mathbb{C}$. However, commuting the latter two terms produces elements $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(c^{1,1}_{x\bar{z},zy}(-a)d) \in \mathrm{U}_0$. Therefore

$$\pi(\mathrm{C}_{\gamma_{w_4}*\nu}) \subseteq \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w,$$

and the other inclusion is clear by Proposition 7.1 and the above discussion. This finishes the proof of our claim.

Now we use this to prove Lemma 7.6, assuming $z \neq \bar{y}$ and $y \neq \bar{x}$. For both conclusions (a) and (b) of the lemma, our equalities i–iv immediately imply

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_2}*\nu}) \quad \text{and} \quad \pi(\mathrm{C}_{\gamma_{w_3}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_4}*\nu}).$$

Next we will show that

$$\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}.$$

For this, let $v_{y\bar z}\in\mathbb{C}$ and $v_{x\bar y}\in\mathbb{C}$ with $v_{x\bar y}\neq 0$. Then since $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar z})\in\mathrm{U}_{\mu_w}\cap\mathrm{U}_0$ for any $v_{y\bar z}\in\mathbb{C}$ Chevalley's commutator formula, and Proposition 7.1 imply

$$\begin{aligned}\pi(\mathrm{C}_{\gamma_{w_1}*\nu})&\supset \mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar z})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar y})\mathrm{T}^w\\ &=\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{y\bar z,v\bar y}(-v_{y\bar z})v_{x\bar y})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar y})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar z})\mathrm{T}^w\\ &=\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{y\bar z,v\bar y}(-v_{y\bar z})v_{x\bar y})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar y})\mathrm{T}^w.\end{aligned}$$

Therefore

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar y})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar z})\mathrm{T}^w\subset\pi(\mathrm{C}_{\gamma_{w_1}*\nu}),$$

as long as $v_{x\bar y}\neq 0$, since in that case $c^{1,1}_{y\bar z,v\bar y}(-v_{y\bar z})v_{x\bar y}=v_{x\bar z}$ has a solution in $v_{y\bar z}$. Hence Proposition 7.1 implies

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar y})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar z})\mathrm{T}^w\subset\pi(\mathrm{C}_{\gamma_{w_1}*\nu}).$$

Equalities i and ii then imply that a dense subset of $\pi(\mathrm{C}_{\gamma_{w_2}*\nu})$ is contained in $\pi(\mathrm{C}_{\gamma_{w_1}*\nu})$, which implies Lemma 7.6(a). To finish the proof of Lemma 7.6(b), let $v_{x\bar y}\in\mathbb{C}$ and $v_{y\bar z}\in\mathbb{C}$ with $v_{y\bar z}\neq 0$. Then, just as for (a),

$$\pi(\mathrm{C}_{\gamma_{w_3}*\nu})\supset \mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{x\bar y})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar z})\mathrm{T}^w \tag{34}$$

$$=\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}\big(c^{1,1}_{x\bar y,y\bar z}(-v_{x\bar y})v_{y\bar z}\big)\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar z})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{y\bar z})\mathrm{T}^w \tag{35}$$

$$=\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{x\bar y,y\bar z}(-v_{x\bar y})v_{y\bar z})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar z})\mathrm{T}^w. \tag{36}$$

Therefore the elements of the set

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar z})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar z})\mathrm{T}^w$$

such that $v_{y\bar z}\neq 0$ are contained in (36). By items iii and iv and Proposition 7.1 there is a dense subset of

$$\pi(\mathrm{C}_{\gamma_{w_4}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar z})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar z})\mathrm{T}^w$$

that is contained in $\pi(\mathrm{C}_{\gamma_{w_3}*\nu})$.

The cases $z=\bar y$ and $y=\bar x$ are missing so far. (Note that $z\neq\bar x$ is not allowed. Also note that if $y=\bar x$ then x must be unbarred and if $z=\bar y$ then y must be unbarred.)

Now assume $z=\bar y$. To prove Lemma 7.6(a) in this case, we first show that

$$\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}\subseteq\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}. \tag{37}$$

All of the generators of $\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_1}$ belong to U_0 except for $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar y})$, for $v_{x\bar y}\in\mathbb{C}$. The generators of $\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_1}$ are $\mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar y})$ for $l\neq x$ and $v_{l\bar y}\in\mathbb{C}$, and

$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{x\bar{y}})$ for $v_{x\bar{y}} \in \mathbb{C}$. This last term commutes with $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})$. Therefore, by parallel arguments to those given in the proof of equalities i–iv on page 474,

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \prod_{\substack{l<y\\ l\neq x}} \mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})\mathrm{T}^w.$$

All terms in the product

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \prod_{\substack{l<y\\ l\neq x}} \mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})$$

are at the same time generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_2}}$ as well. Therefore, by Proposition 7.1,

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_2}*\nu}),$$

as wanted. Next we would like to show

$$\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}. \tag{38}$$

To do so we will make use of Proposition 5.5. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|} \hline x & x & y \\ \hline \bar{y} & \bar{y} & \\ \cline{1-2} \end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|} \hline x & \overline{y-1} & y \\ \hline & \bar{y} & \overline{y-1} \\ \cline{2-3} \end{array}.$$

Then we have $w_1 = y\ x\ \bar{y} = w(\mathscr{K}_1)$ and $w_2 = y\ \bar{y}\ x = w(\mathscr{K}_2)$. By Proposition 5.5 it then suffices to show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}})}.$$

First assume $y-1 \neq x$. In this case $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_2}*\nu}$ is generated by terms $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)$ with $a \in \mathbb{C}$, and all generators of $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0. Out of these, the only ones in $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ that do not commute with $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)$ are $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)$ and $\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d)$. Then for every element in $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ there is a $u \in \mathrm{U}_0$ such that it belongs to

$$\begin{aligned} u\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)u'\mathrm{T}^w = uu'\mathrm{U}_{(\varepsilon_{y-1}+\varepsilon_x,-1)}\big(c^{1,1}_{y-1\bar{y},xy}(-a)b\big) \\ \cdot\, \mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}\big(c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d\big)\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w, \end{aligned}$$

where $u' = \mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d)$.

Fix u, a, b, and d such that $abd \neq 0$. Such elements form a dense subset of $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. We will show that

$$\mathrm{U}_{(\varepsilon_{y-1}+\varepsilon_x,-1)}\big(c^{1,1}_{y-1\bar{y},xy}(-a)b\big)\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}\big(c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d\big)\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w$$

is contained in $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$. If this is true, then (38) is implied by Proposition 7.1 applied to $u\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d) \in \mathrm{U}_0$.

First note that for any $\{a_{x\bar{y}}, a_{y-1\bar{y}}, a_{yy-1}\} \subset \mathbb{C}$, both $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1y})$ and $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})$ belong to $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$, and $v = \mathrm{U}_{(\varepsilon_y+\varepsilon_{y-1},0)}(a_{yy-1}) \in \mathrm{U}_{\varepsilon_x} \cap \mathrm{U}_0$ stabilizes the truncated image T^w as well as the whole image $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$. Therefore all elements of

$$v^{-1}\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})v\mathrm{T}^w = \\ \mathrm{U}_{(\varepsilon_x+\varepsilon_{y-1},-1)}(c^{1,1}_{x\bar{y},yy-1}(-a_{x\bar{y}})a_{yy-1})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})\mathrm{T}^w$$

belong to $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ and, since $abd \neq 0$, we may find $a_{x\bar{y}}$, $a_{y-1\bar{y}}$, and a_{yy-1} such that

$$a_{x\bar{y}} = c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d, \quad c^{1,1}_{x\bar{y},yy-1}(-a_{x\bar{y}})a_{yy-1} = c^{1,1}_{y-1\bar{y},xy}(-a)b, \quad a_{y-1\bar{y}} = a.$$

This concludes the proof if $y \neq x-1$. Now assume that $y = x-1$. In this case all generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathrm{K}_2}*\nu}$ commute with $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})$, and therefore all elements in $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w,$$

for some $u \in \mathrm{U}_0$ and $a \in \mathbb{C}$ — but $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a) \in \mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$, which implies (38) by applying Proposition 7.1 to $u \in \mathrm{U}_0$.

Next we prove Lemma 7.6(b), still assuming $z = \bar{y}$. We now have

$$w_3 = x\ \bar{y}\ y = w(\mathscr{K}_3) \quad \text{and} \quad w_4 = \bar{y}\ x\ y = w(\mathscr{K}_4),$$

where

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline y & x & x \\ \hline \end{array}\hspace{-0.3em}\begin{array}{c}\\ \end{array} \quad \text{and} \quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline x & x & \bar{y} \\ \hline y & y & \\ \cline{1-2}\end{array}.$$

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline y & x & x \\ \hline & \bar{y} & \bar{y} \\ \cline{2-3}\end{array}$$

We want to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

First $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_3}*\nu}$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}$ are both contained in U_0. The generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_3}*\nu}$ that do not belong to U_0 are $\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)$, $\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})$, and $\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})$ for $\{\alpha_y, \beta_{yl}, \gamma_{y\bar{s}}\} \subset \mathbb{C}$ and $l \leq n$, $l \neq x$, $y < s \leq n$. All of these are also generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$, hence by Proposition 7.1 and Theorem 2.9 we have

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) \subset \pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}).$$

The discussion above also implies the equality

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{y<s\leq n}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w. \tag{39}$$

There is one more generator of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$ not mentioned above, $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})$.

Since all generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ (which are $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(d') \in \mathrm{U}_0$ for $d' \in \mathbb{C}$) commute with those of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}$, we have by Proposition 7.1,

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})\mathrm{U}_{(\varepsilon_y,-1)}(a_y)\prod_{\substack{l\le n\\ l\ne x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(b_{yl})\prod_{\substack{s\le n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(c_{y\bar{s}})\mathrm{T}^w.$$

We now would like to show

$$\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})} \subset \overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})}.$$

To do this we will see that for complex numbers a_y, b_{yl}, $c_{y\bar{s}}$, and d_{xy}, with $a_y \ne 0$,

$$\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})\mathrm{U}_{(\varepsilon_y,-1)}(a_y)\prod_{\substack{l\le n\\ l\ne x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(b_{yl})\prod_{\substack{s\le n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(c_{y\bar{s}})\mathrm{T}^w \subset \pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}). \tag{40}$$

By (39) we conclude that for any complex numbers α_y, β_{yl}, $\gamma_{y\bar{s}}$, and δ the following set is contained in $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$:

$$\begin{aligned} &v^{-1}\mathrm{U}_{(\varepsilon_x-\varepsilon_y,1)}(\delta)\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\le n\\ l\ne x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{\substack{s\le n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w \\ &= v^{-1}v\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(\rho_{xy})\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\le n\\ l\ne x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{\substack{s\le n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w, \end{aligned} \tag{41}$$

where

$$v = \mathrm{U}_{(\varepsilon_x,0)}\big(c^{1,1}_{x\bar{y},y}(-\delta)\alpha_y\big)\prod_{\substack{l\le n\\ l\ne x}}\mathrm{U}_{(\varepsilon_x+\varepsilon_l,0)}\big(c^{1,1}_{x\bar{y},yl}(-\delta)\beta_{yl}\big)\prod_{\substack{s\le n\\ s>y}}\mathrm{U}_{(\varepsilon_x-\varepsilon_s,0)}\big(c^{1,1}_{x\bar{y},y\bar{s}}(-\delta)\gamma_{y\bar{s}}\big)$$

and $\rho_{xy} = c^{1,2}_{x\bar{y},y}(-\delta)\alpha_y^2$, and where the equality in (41) is obtained by applying Chevalley's commutator formula (2) and Proposition 7.1 to $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,1)}(\delta)$, which stabilizes the truncated image T^w. We will have shown our claim in (40) if we find complex numbers α_y, β_{yl}, $\gamma_{y\bar{s}}$, and δ such that

$$c^{1,2}_{x\bar{y},y}(-\delta)\alpha_y^2 = d_{xy}, \quad \alpha_y = a_y, \quad \beta_{yl} = b_{yl},$$

which we may obtain since $a_y \ne 0$. This concludes the proof in case $z = \bar{y}$.

Lastly assume $y = \bar{x}$. This means that x is necessarily unbarred and therefore $z = \bar{b}$ for some $b < x$.

To prove Lemma 7.6(a) in this case, as before, we use Proposition 5.5. We have

$$w_1 = \bar{x}\ x\ \bar{b} = w(\mathscr{K}_1) \quad\text{and}\quad w_2 = \bar{x}\ \bar{b}\ x = w(\mathscr{K}_2),$$

where

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x & x & \bar{x} \\ \hline \bar{b} & \bar{b} & \\ \cline{1-2}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x & \bar{x} & \bar{x} \\ \hline & \bar{b} & \bar{b} \\ \cline{2-3}\end{array}.$$

First we show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}. \tag{42}$$

To do this, we claim that

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{T}^w. \tag{43}$$

Indeed, $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$ for $s\in\mathscr{C}_n$ such that $s\neq b$ are the generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$ that do not belong to U_0, and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_1}*\nu}$ is the identity, because $\varepsilon_x-\varepsilon_b$ is not a positive root. Therefore (43) follows by Proposition 7.1. The aforementioned terms are also generators (but not all!) of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$; therefore (42) follows. Now we show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})}. \tag{44}$$

To do this, let us first analyze the image

$$\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_2}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}\mathrm{T}^w.$$

In this case $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu} \subset \mathrm{U}_0$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_2}*\nu}$ is the identity, because $-(\varepsilon_x+\varepsilon_b)$ is not a positive root. The generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ are $\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x)$, $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})$ for $s\in\mathscr{C}_n$ such that $s\neq b$ and complex numbers α_x, α_{xs}, and α_{xb}. Therefore

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})\mathrm{T}^w. \tag{45}$$

Let us fix complex numbers α_x, α_{xs}, and α_{xb}, such that $\alpha_x\neq 0$. We will show, as for (43), that

$$\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})\mathrm{T}^w \subset \pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}). \tag{46}$$

To do this we will use Corollary 2.10, which says, in particular, that if we write

$$\gamma_{\mathscr{K}_1} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2, \mathrm{E}_2, \mathrm{V}_3),$$

then

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) \supset \mathrm{U}_{\mathrm{V}_0}\mathrm{U}_{\mathrm{V}_1}\mathrm{U}_{\mathrm{V}_2}\mathrm{T}^w.$$

Therefore, since $u = \mathrm{U}_{(\varepsilon_b-\varepsilon_x,0)}(a) \in \mathrm{U}_{\mathrm{V}_2} \cap \mathrm{U}_0$ for all $a \in \mathbb{C}$, and since $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$, for $s \in \mathscr{C}_n$ and $s \neq b$, are the generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_1}*\nu} \subset \mathrm{U}_{\mathrm{V}_1}$, by using Proposition 7.1 applied to $u \in \mathrm{U}_0$ and $v \in \mathrm{U}_{\mathrm{V}_3}$ (V_3 stabilizes the truncated image T^w, see below for a definition of v), we have the following. For any complex numbers a_{xs} and a_x,

$$\begin{aligned}\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) &\supset u^{-1}\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})u\mathrm{T}^w\\ &= u^{-1}u\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}\big(c^{2,1}_{x,b\bar{x}}(a_x^2)b\big)\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})v\mathrm{T}^w\\ &= \mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}\big(c^{2,1}_{x,b\bar{x}}(a_x^2)b\big)\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{T}^w,\end{aligned}$$

where

$$v = \mathrm{U}_{(\varepsilon_b,-1)}\big(c^{1,1}_{x,b\bar{x}}(-a_x)b\big) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_b+\varepsilon_s,-1)}\big(c^{1,1}_{x,bs}(-a_{xs})b\big) \in \mathrm{U}_{\mathrm{V}_3}.$$

In order to show (46) it suffices to find complex numbers a_x, a_{xs}, and b such that

$$c^{2,1}_{x,b\bar{x}}(a_x^2)b = \alpha_{xb}, \quad a_x = \alpha_x, \quad a_{xs} = \alpha_{xs},$$

and we may do this, since $\alpha_x \neq 0$.

For (b), we again use Proposition 5.5. We have

$$w_3 = x\ \bar{b}\ \bar{x} = w(\mathscr{K}_3) \quad \text{and} \quad w_4 = \bar{b}\ x\ \bar{x} = w(\mathscr{K}_4),$$

where

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline \bar{x} & x & x \\ \hline \end{array}\!\!\begin{array}{c}\\ \end{array} \quad \begin{array}{c|c|c|} & \bar{b} & \bar{b} \\ \cline{2-3}\end{array} \quad \text{and} \quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline x{-}1 & x & \bar{b} \\ \hline \bar{x} & \overline{x{-}1} & \\ \cline{1-2}\end{array}.$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}\,. \tag{47}$$

We analyze both images $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ and $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$ separately and then show (47). First we observe that $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}$ is the identity (this is because $\varepsilon_x - \varepsilon_b$ is not a positive root). Hence

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathrm{U}_0 \prod_{\substack{l<x\\ l\neq b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_x,-1)}(a_{l\bar{x}})\mathrm{U}_{(\varepsilon_b-\varepsilon_x,-2)}(a_{b\bar{x}})\mathrm{T}^w. \tag{48}$$

Now, $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1x})$ for $\alpha_{x-1x} \in \mathbb{C}$, and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by $\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}})$ for $\alpha_{b\overline{x-1}} \in \mathbb{C}$, by $\mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})$ for

$l < x-1$ and $\alpha_{l\overline{x-1}} \in \mathbb{C}$ (this last element stabilizes the truncated image T^w), and by other elements of U_0. Therefore

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4} * \nu}) \tag{49}$$

$$= \mathrm{U}_0 \prod_{\substack{l<x \\ l\neq b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1\bar{x}})\mathrm{T}^w \tag{50}$$

$$= \mathrm{U}_0 \prod_{\substack{l<x, l\neq b \\ l\neq x-1}} \mathrm{U}_{(\varepsilon_l-\varepsilon_x,-1)}(\xi_{l\bar{x}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1\bar{x}})\mathrm{U}_{(\varepsilon_b-\varepsilon_x,-2)}(\xi_{b\bar{x}})\mathrm{T}^w, \tag{51}$$

where

$$\xi_{b\bar{x}} = c^{1,1}_{b\overline{x-1},x-1\bar{x}}(-\alpha_{b\overline{x-1}}\alpha_{x-1\bar{x}}), \quad \xi_{l\bar{x}} = c^{1,1}_{l\overline{x-1},x-1\bar{x}}(-\alpha_{l\overline{x-1}}\alpha_{x-1\bar{x}}),$$

and where the equality between (50) and (51) arises by using (2) and Proposition 7.1 applied to

$$\mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}}) \in \mathrm{U}_{\mu_{\gamma_{\mathscr{K}_4}}}.$$

The sets displayed in (48) and (51) are equal as long as all the parameters are nonzero.

<u>Case 2: $x = y < z$ and $z \neq \bar{x}$.</u> In this case we have $w_1 = y\ y\ z$ and $w_2 = y\ z\ y$. We want to look at

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_0}\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_1}\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_2}\mathrm{T}^w,$$
$$\pi(\mathrm{C}_{\gamma_{w_2}*\nu}) = \mathbb{U}^{\gamma_{w_2}*\nu}_{\mathrm{V}_0}\mathbb{U}^{\gamma_{w_2}*\nu}_{\mathrm{V}_1}\mathbb{U}^{\gamma_{w_2}*\nu}_{\mathrm{V}_2}\mathrm{T}^w.$$

In this case all generators of $\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_i}$ and of $\mathbb{U}^{\gamma_{w_2}*\nu}_{\mathrm{V}_i}$ belong to U_0 for $i \in \{1, 2, 3\}$. Therefore Proposition 7.1 implies in this case that

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathrm{U}_0\mathrm{T}^w = \pi(\mathrm{C}_{\gamma_{w_2}*\nu}).$$

<u>Case 3: $x < y = z$ and $z \neq \bar{x}$.</u> Here it will be convenient to use Proposition 5.5. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|} \hline y & x \\ \hline \end{array}\!\!\!\begin{array}{c} \\ \end{array} \ \text{with } \begin{array}{cc} y & x \\ & y \end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{cc} x & y \\ y & \end{array}.$$

It is then enough to show (by Proposition 5.5) that

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})},$$

since

$$w_1 = x\ y\ y = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = y\ x\ y = w(\mathscr{K}_2).$$

However, this case is now the same as the previous one: all generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_1}*\nu}$ and $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0, therefore, as before,

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = \mathrm{U}_0\mathrm{T}^w = \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}).$$

With this case we conclude the proof of Lemma 7.6. □

Relation R2. For $1 < x \leq n$ and $x \leq y \leq \bar{x}$:

$$y\ \overline{x-1}\ x-1 \equiv y\ x\ \bar{x},$$
$$\overline{x-1}\ x-1\ y \equiv x\ \bar{x}\ y.$$

Lemma 7.7. *Let*

$$w_1 = y\ \overline{x-1}\ x-1, \quad w_2 = y\ x\ \bar{x}, \quad w_3 = \overline{x-1}\ x-1\ y, \quad w_4 = x\ \bar{x}\ y,$$

then

(a) $\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})} = \overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}$,

(b) $\overline{\pi(\mathrm{C}_{\gamma_{w_3}*\nu})} = \overline{\pi(\mathrm{C}_{\gamma_{w_4}*\nu})}$.

Proof. As usual, the proof is divided in some cases. We first consider the case where $y \notin \{x, \bar{x}\}$ and then we analyze $y = x$ and $y = \bar{x}$ separately.

Case 1: $y \notin \{x, \bar{x}\}$.

Note that

$$w_1 = y\ \overline{x-1}\ x-1 = w\left(\begin{array}{|c|c|c|}\hline x-1 & y & y \\ \hline \multicolumn{1}{c|}{} & \overline{x-1} & \overline{x-1} \\ \cline{2-3}\end{array}\right), \quad w_2 = y\ x\ \bar{x} = w\left(\begin{array}{|c|c|c|}\hline x-1 & x & y \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\right).$$

Hence by Proposition 5.5, to show (a) it is enough to show that

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})},$$

where

$$\mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & y \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \text{and} \quad \mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & y & y \\ \hline \multicolumn{1}{c|}{} & \overline{x-1} & \overline{x-1} \\ \cline{2-3}\end{array}.$$

First we check that

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})}.$$

Clearly $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu} \subset \mathrm{U}_0$. The only generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_2}*\nu}$ that do not belong to U_0 are those of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)$, for $a \in \mathbb{C}$, and those in $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ are $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$, for $b \in \mathbb{C}$. This means that every element in $\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}$ belongs to

$$u\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{T}^w,$$

for some $u \in \mathrm{U}_0$. Both $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)$ and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$ belong to $\mathrm{U}_{\varepsilon_y-\varepsilon_{x-1}}$, and this implies the contention by Proposition 7.1 and Corollary 2.10. Now we want to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

By Theorem 2.9, all elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ belong to the set

$$u\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(v_{x-1\bar{y}})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1}) \cdot \prod_{\substack{l>x\\ l\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}}) \prod_{s\neq y} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w, \tag{52}$$

for $u \in \mathrm{U}_0$ and $v_{x-1j} \in \mathbb{C}$. This is because both $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_1}*\nu}$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$ are contained in U_0. Fix such an element such that $v_{x-1\bar{x}} \neq 0$. We know that

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu},$$

and that for any $a_{x\bar{y}} \in \mathbb{C}$, $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}}) \in \mathrm{U}_{\varepsilon_y}$. This means that these elements stabilize both the truncated images $\mathrm{T}^{\geq 3}_{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathrm{T}^{\geq 1}_{\gamma_{\mathscr{K}_2}*\nu}$. Hence the elements in

$$\begin{aligned} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w &\\ = \mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}\big(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}}\big)&\\ \cdot \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w& \end{aligned} \tag{53}$$

all belong to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. More they belong to precisely to $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_1}*\nu}\mathrm{T}^w \subset \mathrm{T}^{\geq 1}_{\gamma_{\mathscr{K}_1}*\nu}$, hence by Proposition 7.1, we may multiply the right side of equation (53) by $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(-v_{x\bar{y}})$ on the left and the product still belongs to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$, hence

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}\big(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}}\big)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w \subset \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}).$$

Now consider the product

$$u = \mathrm{U}_{(\varepsilon_y+\varepsilon_x,1)}(a_{yx})\mathrm{U}_{(\varepsilon_x,0)}(a_x) \prod_{\substack{l>x\\ l\neq y}} \mathrm{U}_{(\varepsilon_x-\varepsilon_l,0)}(a_{x\bar{l}}) \prod_{s\neq y} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,0)}(a_{xs}) \in \mathrm{U}_{\varepsilon_y} \cap \mathrm{U}_0.$$

Proposition 7.1 then implies that

$$\begin{aligned} \pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) &\supset u^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})u\mathrm{T}^w \\ &= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(\rho_{x-1y}) \\ &\quad \cdot \prod_{\substack{l>x\\ l\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1l}) \\ &\quad \cdot \prod_{s\neq y} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w, \end{aligned}$$

with

$$\begin{aligned}
\rho_{x-1x} &= c^{1,2}_{x-1\bar{x},x}(-v_{x-1\bar{x}})a_x^2 - c^{1,1}_{x-1y,yx}c^{1,1}_{x-1\bar{x},x\bar{y}}(v_{x-1\bar{x}})a_{x\bar{y}}a_{yx},\\
\rho_{x-1j} &= c^{1,1}_{x-1\bar{x},xj}(-v_{x-1\bar{x}})a_{xj}\, j \neq y,\ j \in \{\bar{l} : l > x\} \cup \{s : \varepsilon_{x-1}+\varepsilon_s \in \Phi^+\},\\
\rho_{x-1} &= c^{1,1}_{x-1\bar{x},x}(-v_{x-1\bar{x}})a_x.
\end{aligned}$$

The system of equations defined by $v_{x-1} = \rho_{x-1}$ and $v_{x-1j} = \rho_{x-1j}$ does have solutions (the variables are a_x, a_{yx}, $a_{x\bar{l}}$, and a_{xs}) since $v_{x-1,x} \neq 0$. This means that for such solutions we have (see (52))

$$\begin{aligned}
&\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(v_{x-1\bar{y}})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\\
&\qquad\cdot \prod_{\substack{l\geq x\\ l\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}}) \prod_{s\neq y} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w\\
&= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(\rho_{x-1y})\\
&\qquad\cdot \prod_{\substack{l> x\\ l\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1l}) \prod_{s\neq y} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\cdot \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w\\
&\subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}),
\end{aligned}$$

and so by Proposition 7.1 we get that all elements in (52) belong to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. All such elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ form a dense open subset. This finishes the proof in this case.

We turn to (b). Let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline x-1 & x-1 & \overline{x-1}\\ \hline y & y & \\ \cline{1-2}\end{array} \quad\text{and}\quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline y & x-1 & x\\ \hline & \bar{x} & \overline{x-1}\\ \cline{2-3}\end{array}.$$

Then $w_3 = \overline{x-1}\ x-1\ y = w(\mathscr{K}_3)$ and $w_4 = x\ \bar{x}\ y = w(\mathscr{K}_4)$. As in (a), by Proposition 5.5, it is enough to show that

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

To show

$$\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})} \subset \overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})},$$

note first that the only generator of $\mathbb{U}^{\gamma_{\mathscr{K}_4}*\nu}_{\mathrm{V}_i}$ that does not belong to U_0 is

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a) \in \mathbb{U}^{\gamma_{\mathscr{K}_4}*\nu}_{\mathrm{V}_1},\ \text{ for } a \in \mathbb{C}.$$

Of $\mathbb{U}^{\gamma_{\mathscr{K}_4}*\nu}_{\mathrm{V}_2}$, the only generators that do not commute with $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)$ are $\mathrm{U}_{(\varepsilon_y+\varepsilon_x,0)}(b)$, with $b \in \mathbb{C}$. Then Chevalley's commutator formula (2) implies that all elements of $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$ belong to the set

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}\big(c^{1,1}_{x-1\bar{x},xy}(-a)b\big)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w. \tag{54}$$

Since both $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}\big(c^{1,1}_{x-1\bar{x},xy}(-a)b\big)$ and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)$ belong to $\mathbb{U}^{\gamma_{\mathscr{K}_3}*\nu}_{\mathrm{V}_1}$, the desired contention follows by Proposition 7.1. Now we show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}})}\subset\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})}. \tag{55}$$

The proof is similar to that of (a), but there are some subtle differences. First we look at the image $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$. Out of all the generators of $\mathbb{U}^{\gamma_{\mathscr{K}_3}*\nu}_{\mathrm{V}_i}$, the only ones that do not belong to U_0 belong to $\mathbb{U}^{\gamma_{\mathscr{K}_3}*\nu}_{\mathrm{V}_1}$: $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)$, $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})$ for $l\neq x-1$, $s>x$, $s\neq y$, and complex numbers v_{x-1}, v_{x-1s}, and v_{x-1l}. The group $\mathbb{U}^{\gamma_{\mathscr{K}_3}*\nu}_{\mathrm{V}_2}$ has as generators the terms $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,0)}(a)$ (only), and these commute with all the latter terms. Therefore all elements of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\prod_{\substack{s>x-1\\ s\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})\prod_{l\neq x-1}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})\mathrm{T}^w, \tag{56}$$

for some $u\in\mathrm{U}_0$. Fix such a u, and assume $v_{x-1\bar{x}}\neq 0$ and $v_{x-1y}\neq 0$. Such elements as (56) form a dense open subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$. Now, for all complex numbers a, a_{xy}, and $a_{x\bar{y}}$ we have $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\in\mathbb{U}^{\gamma_{\mathscr{K}_4}*\nu}_1$, $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(a_{xy})\in\mathbb{U}^{\gamma_{\mathscr{K}_4}*\nu}_1$, and $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(a_{x\bar{y}})\in\mathrm{U}_0$, which stabilizes the truncated image $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_4}*\nu}$. Therefore, setting $c=\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(a_{xy})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(a_{x\bar{y}})\in\mathrm{U}_0$, all elements in

$$\begin{aligned}
c^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)c\mathrm{T}^w &= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\\
&\qquad\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-1)}\big(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}\big)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w\\
&= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\\
&\qquad\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-1)}\big(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}\big)\mathrm{T}^w\\
&= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\\
&\qquad\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w
\end{aligned}$$

belong to $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$, where

$$\begin{aligned}
\rho_{x-1x} &= c^{1,1}_{x-1y,x\bar{y}}c^{1,1}_{x-1\bar{x},xy}aa_{xy}a_{x\bar{y}},\\
\rho_{x-1y} &= c^{1,1}_{x-1\bar{x},xy}(-a)a_{xy},
\end{aligned}$$

and where the last equality holds because $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-1)}\big(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}\big)\in\mathrm{U}_{\varepsilon_y}$, and all elements of the latter stabilize the truncated image T^w by Proposition 7.1.

Now let

$$c' = \mathrm{U}_{(\varepsilon_x,0)}(a_x) \prod_{\substack{s>x \\ s\neq y}} \mathrm{U}_{(\varepsilon_x-\varepsilon_s,0)}(a_{x\bar{s}}) \prod_{\substack{l\neq x-1 \\ l\neq y}} \mathrm{U}_{(\varepsilon_x+\varepsilon_l,0)}(a_{xl}) \in \mathrm{U}_{\varepsilon_y} \cap \mathrm{U}_0,$$

for a_x, $a_{x\bar{s}}$, and a_{xl} complex numbers; by Proposition 7.1 this element stabilizes the truncated image T^w and the image $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$. Therefore the following are contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})$,

$$\begin{aligned} &c'^{-1}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)c'\mathrm{T}^w \\ &\quad = \mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_x) \\ &\qquad \cdot \prod_{\substack{s>x-1 \\ s\neq y \\ s\neq x}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho'_{x-1x}) \qquad (57) \\ &\qquad \cdot \prod_{l\notin\{x-1,x\}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(\rho_{x-1l})\mathrm{T}^w, \qquad (58) \end{aligned}$$

where

$$\begin{aligned} \rho_{x-1} &= c^{1,1}_{x-1x,x}(-a)a_x, \\ \rho'_{x-1x} &= \rho_{x-1x} + c^{1,2}_{x-1x,x}(-a)a_x^2, \\ \rho_{x-1l} &= c^{1,1}_{x-1\bar{x},xl}(-a)a_{xl}, \\ \rho_{x-1\bar{s}} &= c^{1,1}_{x-1\bar{x},x\bar{s}}(-a)a_{x\bar{s}}. \end{aligned}$$

We want to show that

$$\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1}) \prod_{\substack{s>x-1 \\ s\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s}) \prod_{l\neq x-1} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})\mathrm{T}^w$$

is equal to the product in the last lines (57) and (58) above (see (56)), for some a_x, a_{xl}, and $a_{x\bar{s}}$. This determines a system of equations:

$$\begin{aligned} v_{x-1\bar{x}} &= a, \\ v_{x-1x} &= c^{1,1}_{x-1y,x\bar{y}}c^{1,1}_{x-1\bar{x},xy}aa_{xy}a_{x\bar{y}} + c^{1,2}_{x-1x,x}(-a)a_x^2, \\ v_{x-1} &= c^{1,1}_{x-1x,x}(-a)a_x, \\ v_{x-1\bar{s}} &= c^{1,1}_{x-1\bar{x},x\bar{s}}(-a)a_{x\bar{s}}, \\ v_{x-1l} &= c^{1,1}_{x-1\bar{x},xl}(-a)a_{xl}, \\ v_{x-1y} &= c^{1,1}_{x-1\bar{x},xy}(-a)a_{xy}. \end{aligned}$$

which can always be solved since $v_{x-1y} \neq 0$ and $v_{x\overline{x-1}} \neq 0$. This completes the proof of (b) in this case. $\square$

Case 1. $y = x$.

Proof. As in Case 1, we will make use of Proposition 5.5. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & x & x \\ \hline \multicolumn{1}{c|}{} & \overline{x-1} & \overline{x-1} \\ \cline{2-3}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & x \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}.$$

Then

$$w_1 = x\ \overline{x-1}\ x-1 = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = x\ x\ \bar{x} = w(\mathscr{K}_2).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu})}.$$

First we show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu})}. \tag{59}$$

Since $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2} * \nu}$ is generated by elements of the form $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)$, for $a \in \mathbb{C}$, and the generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_2} * \nu}$ belong to U_0, for $i \in \{1, 2\}$, all elements of $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu})$ are of the form

$$u\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)\mathrm{T}^w$$

for some $u \in \mathrm{U}_0$. Since $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_1} * \nu}$, (59) follows by applying Proposition 7.1 to u. To finish the proof in this case it remains to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu})}. \tag{60}$$

The generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_1} * \nu}$ belong to U_0, for $i \in \{0, 1\}$, and the generators that do not are $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)$, $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})$, $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})$, for $n \geq l > x$, $s \notin \{x, x-1\}$, and complex numbers v_x, $v_{x-1\bar{l}}$, v_{x-1s}, and $v_{x-1\bar{x}}$. Therefore all elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})\mathrm{T}^w.$$

Fix such $u \in \mathrm{U}_0$ and v_x, $v_{x-1\bar{l}}$, v_{x-1s}, and $v_{x-1\bar{x}}$ complex numbers such that $v_{x-1\bar{x}} \neq 0$. We know for any $a \in \mathbb{C}$, that $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2} * \nu}$. Let

$$q = \mathrm{U}_{(\varepsilon_x,1)}(a_x)\prod_{s>x}\mathrm{U}_{(\varepsilon_x-\varepsilon_s,1)}(a_{x\bar{s}})\prod_{l\neq x}\mathrm{U}_{(\varepsilon_x+\varepsilon_l,1)}(a_{xl}) \in \mathrm{U}_{(\varepsilon_x,1)} \cap \mathrm{U}_0$$

for any complex numbers a_x, $a_{x\bar{s}}$, and a_{xl}. Then by Proposition 7.1,

$$q^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)q\mathrm{T}^w \subset \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu}). \tag{61}$$

As in the previous cases, we want to find a, a_x, $a_{x\bar{s}}$, and a_{xl} such that

$$tU_{(\varepsilon_{x-1},-1)}(v_x)U_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})U_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})U_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})T^w$$

equals (61), for some $t \in U_0$. But

$$\begin{aligned} &q^{-1}U_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)qT^w \\ &\quad = t^{-1}U_{(\varepsilon_{x-1},-1)}(\rho_x)U_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1\bar{l}})U_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})U_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)T^w, \end{aligned}$$

where

$$\begin{aligned} t^{-1} &= U_{(\varepsilon_x+\varepsilon_{x-1},0)}(c^{1,2}_{x-1\bar{x},x})(-a)a_x^2 \in U_0, \\ \rho_x &= c^{1,1}_{x-1\bar{x},x}(-a)a_x, \\ \rho_{x-1\bar{l}} &= c^{1,1}_{x-1\bar{x},x\bar{l}}(-a)a_{x\bar{l}}, \\ \rho_{x-1s} &= c^{1,1}_{x-1\bar{x},xs}(-a)a_{xs}. \end{aligned}$$

The system

$$\begin{aligned} v_{x-1\bar{x}} &= a, \\ v_{x-1\bar{l}} &= \rho_{x-1\bar{l}}, \\ v_{x-1s} &= \rho_{x-1s} \end{aligned}$$

always has a solution since $v_{x-1\bar{x}} \neq 0$. This concludes the proof of Case 2. $\square$

Let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|} \hline x-1 & x-1 & \overline{x-1} \\ \hline x & x & \\ \cline{1-2} \end{array} \quad \text{and} \quad \mathscr{K}_4 = \begin{array}{|c|c|c|} \hline x & x-1 & x \\ \hline & x & \overline{x-1} \\ \cline{2-3} \end{array}.$$

Then

$$w_3 = \overline{x-1}\ x-1\ x = w(\mathscr{K}_3) \quad \text{and} \quad w_4 = x\ \bar{x}\ x = w(\mathscr{K}_4).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu})}.$$

To do this we will describe a common dense subset of $\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu})$ and $\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu})$.

Consider first $\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu}) = \mathbb{U}_{V_0}^{\gamma_{\mathscr{K}_3}*\nu}\mathbb{U}_{V_1}^{\gamma_{\mathscr{K}_3}*\nu}\mathbb{U}_{V_2}^{\gamma_{\mathscr{K}_3}*\nu}T^w$. We have $\mathbb{U}_{V_0}^{\gamma_{\mathscr{K}_3}*\nu} \subset U_0$ and also $\mathbb{U}_{V_2}^{\gamma_{\mathscr{K}_3}*\nu} \subset U_0$, since it is generated by the terms $U_{(\varepsilon_{x-1}+\varepsilon_x,0)}(d)$, for $d \in \mathbb{C}$. These commute with all generators of $\mathbb{U}_{V_1}^{\gamma_{\mathscr{K}_3}*\nu}$, out of which $U_{(\varepsilon_{x-1},-1)}(v_{x-1})$, $U_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})$, and $U_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})$, (for $s \leq n$, $s \neq x-1$, $l > x$, and v_{x-1}, v_{x-1s} and $v_{x-1\bar{l}}$ complex numbers) do not belong to U_0. Therefore $\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu})$

coincides with

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\prod_{\substack{s\leq n\\ s\neq x-1}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\prod_{x<l\leq n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{T}^w, \tag{62}$$

for complex numbers v_{x-1}, v_{x-1s} and $v_{x-1\bar{l}}$. Now we look at elements of

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_4}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}\mathrm{T}^w.$$

Both $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_4}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ are contained in U_0, and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by the elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)$, which belong to $\mathrm{U}_{\varepsilon_x}$ and therefore stabilize the truncated image T^w by Proposition 7.1. Now, by Proposition 2.7, we may write any element k of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ as

$$k = \mathrm{U}_{(\varepsilon_x,0)}(k_x)\prod_{x<l\leq n}\mathrm{U}_{(\varepsilon_x-\varepsilon_l,0)}(k_{x\bar{l}})\prod_{\substack{s\leq n\\ s\neq x}}\mathrm{U}_{(\varepsilon_x+\varepsilon_s,0)}(k_{xs}) \in \mathrm{U}_0$$

for some complex numbers k_x, $k_{x\bar{l}}$, and k_{xs}. Theorem 2.9 and Proposition 7.1 imply that

$$\begin{aligned}\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) &= \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)k\mathrm{T}^w\\ &= \mathrm{U}_0 k\mathrm{U}_{(\varepsilon_{x-1},-1)}(\sigma_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\sigma_{x-1x})\\ &\qquad\cdot\prod_{x<l\leq n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\sigma_{x-1\bar{l}})\\ &\qquad\cdot\prod_{\substack{s\leq n\\ s\neq x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,0)}(\sigma_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)\mathrm{T}^w,\end{aligned} \tag{63}$$

for $k \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ and $d \in \mathbb{C}$, where

$$\begin{aligned}\sigma_{x-1} &= c^{1,1}_{x-1\bar{x},x}(-d)k_x,\\ \sigma_{x-1x} &= c^{1,2}_{x-1\bar{x},x}(-d)k_x^2,\\ \sigma_{x-1\bar{l}} &= c^{1,1}_{x-1\bar{x},x\bar{l}}(-d)k_{x\bar{l}},\\ \sigma_{x-1s} &= c^{1,1}_{x-1\bar{x},xs}(-d)k_{xs}.\end{aligned}$$

The set (63) is clearly contained in (62). Moreover, the system

$$\begin{aligned}v_{x-1} &= \sigma_{x-1},\\ v_{x-1x} &= \sigma_{x-1x},\\ v_{x-1\bar{l}} &= \sigma_{x-1\bar{l}},\\ v_{x-1s} &= \sigma_{x-1s},\end{aligned}$$

has solutions for d, k_x, $k_{x\bar{l}}$, and k_{xs} as long as $\{v_{x-1}, v_{x-1x}, v_{x-1\bar{l}}, v_{x-1s}\} \subset \mathbb{C}^\times$. Proposition 7.1 then implies that a dense subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ is contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$, which finishes the proof of Case 1.

Case 2. $y = \bar{x}$.

Proof. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & \bar{x} & \bar{x} \\ \hline \multicolumn{1}{c|}{} & \overline{x-1} & \overline{x-1} \\ \cline{2-3}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & \bar{x} \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}.$$

Then

$$w_1 = \bar{x}\ \overline{x-1}\ x-1 = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = \bar{x}\ x\ \bar{x} = w(\mathscr{K}_2).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

In this case we have $\mathbb{U}_0^{\gamma_{\mathscr{K}_1}*\nu} = 1 = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_1}*\nu}$. Proposition 2.7 and Theorem 2.9 then say,

$$\begin{aligned}\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = {} & \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(v_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(v_{x-1x}) \\ & \cdot \prod_{x<l\le n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-1\\ x\ne x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(v_{x-1s})\mathrm{T}^w,\end{aligned} \tag{64}$$

for complex numbers v_{x-1x}, v_{x-1}, v_{x-1x}, v_{x-1l}, and v_{x-1s}. Fix such complex numbers. Now we look at $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}})$. We have that $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ are both contained in U_0, and the latter is generated by elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)$, for $a \in \mathbb{C}$. Out of the generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathrm{K}_2}*\nu}$, the ones that do not belong to U_0 are $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$, $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$, and $\mathrm{U}_{(\varepsilon_x-\varepsilon_l,-1)}(a_{x\bar{l}})$. Therefore, if

$$\mathrm{A} = \mathrm{U}_{(\varepsilon_x,-1)}(a_x)\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{U}_{(\varepsilon_x-\varepsilon_l,-1)}(a_{x\bar{l}}) \in \mathrm{U}_{\varepsilon_{\bar{x}}},$$

we conclude that

$$\begin{aligned}\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) &= \mathrm{U}_0\mathrm{A}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)\mathrm{T}^w \\ &= \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x}) \\ &\quad \cdot \prod_{x<l\le n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})\mathrm{A}\mathrm{T}^w\end{aligned}$$

$$\begin{aligned}&= \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x}) \\ &\quad \cdot \prod_{x<l\le n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})\mathrm{T}^w,\end{aligned} \tag{65}$$

where

$$\begin{aligned}\xi_{x-1} &= c^{1,1}_{x,x-1\bar{x}}(-a_x)a,\\ \xi_{x-1x} &= c^{2,1}_{x,x-1\bar{x}}(a_x^2)a,\\ \xi_{x-1\bar{l}} &= c^{1,1}_{x\bar{l},x-1\bar{x}}(-a_{x\bar{l}})a,\\ \xi_{x-1s} &= c^{1,1}_{xs,x-1\bar{x}}(-a_{xs})a.\end{aligned}$$

Therefore it follows directly that in fact

$$\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) \subseteq \pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}).$$

Now, the system of equations

$$\begin{aligned}v_{x-1} &= \xi_{x-1},\\ v_{x-1x} &= \xi_{x-1x},\\ v_{x-1\bar{l}} &= \xi_{x-1\bar{l}},\\ v_{x-1s} &= \xi_{x-1s},\end{aligned}$$

has solutions as long as $\{v_{x-1}, v_{x-1x}, v_{x-1\bar{l}}, v_{x-1s}\} \subset \mathbb{C}^\times$. For such a set of solutions we conclude

$$\begin{aligned}&\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(v_{x-1x})\\ &\quad\cdot \prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(v_{x-1s})\\ &\qquad = \mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x})\\ &\qquad\quad \cdot \prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-q\\ s\ne x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s}),\end{aligned}$$

and therefore we conclude by Proposition 7.1 (applied to $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(v_{x-1x})$ in (64)) that a dense subset of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ is contained in $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ (see (64), (65)). □

Proof. To prove (b) let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline x-1 & x-1 & \overline{x-1}\\ \hline \bar{x} & \bar{x} & \\ \cline{1-2}\end{array} \quad\text{and}\quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline \bar{x} & x-1 & x\\ \hline \multicolumn{1}{c|}{} & \bar{x} & \overline{x-1}\\ \cline{2-3}\end{array},$$

then

$$w_3 = \overline{x-1}\ x-1\ \bar{x} = w(\mathscr{K}_3) \quad\text{and}\quad w_4 = x\ \bar{x}\ \bar{x} = w(\mathscr{K}_4).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})}.$$

First we claim

$$\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}*\nu}) \subseteq \pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu}).$$

Note that the terms $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$, for $b \in \mathbb{C}$, generate both $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_4}*\nu}$ and are contained in $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_3}*\nu}$. Also, the terms $\mathrm{U}_{(\varepsilon_l-\varepsilon_x,0)}$, which generate $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathcal{K}_4}*\nu}$, commute with $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$. Therefore

$$\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{T}^w \subseteq \pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}}),$$

where the last contention follows by Proposition 7.1. Now we will show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu})} \subseteq \overline{\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}*\nu})}.$$

We claim that

$$\begin{aligned}(66)\quad &\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu})\\ &= \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x-1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w,\end{aligned}$$

for complex numbers v_{x-1}, $v_{x-1\bar{x}}$, and v_{x-1s}. Let us fix such complex numbers. Let

$$D = \mathrm{U}_{(\varepsilon_x,0)}(a_x) \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x-1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{x-1s}) \in \mathrm{U}_0,$$

then by the usual arguments (note that U_0 stabilizes both the image $\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}})$ and the truncated image $\mathrm{T}^{\geq 2}_{\gamma_{\mathcal{K}_4}*\nu}$),

$$\mathrm{D}^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)D\mathrm{T}^w \subset \pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}}),$$

and

$$\begin{aligned}\mathrm{D}^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)D\mathrm{T}^w &= \mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\\ &\quad\cdot \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x+1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_x+\varepsilon_{x-1},-1)}(\rho_{xx-1}),\end{aligned}$$

where

$$\begin{aligned}\rho_{x-1} &= c^{1,1}_{x-1\bar{x},x}(-b)a_x,\\ \rho_{x-1x} &= c^{2,1}_{x-1\bar{x},x}(-b)a_x^2,\\ \rho_{x-1s} &= c^{1,1}_{x-1\bar{x},xs}(-b)a_{xs}.\end{aligned}$$

As usual by requiring that v_{x-1}, $v_{x-1\bar{x}}$, v_{x-1x}, and ρ_{x-1s} be nonzero we may find suitable complex numbers b, a_x, a_{xs} such that

$$\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x-1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s}) \\ = \mathrm{D}^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{D}\mathrm{T}^w.$$

Therefore Proposition 7.1 (see (66)) implies that a dense open subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ is contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$. This completes the proof of Lemma 7.7. □

Relation R3.

Lemma 7.8. *Let $w\in\mathscr{W}_{\mathscr{C}_n}$ be a word and let w_1 be a word that is not of an LS block, and such that it has the form $w_1=a_1\cdots a_r z\bar{z}\bar{b}_s\cdots\bar{b}_1$, and let $w_2=a_1\cdots a_r\bar{b}_s\cdots\bar{b}_1$ with $a_1<\cdots a_r<z>b_s>\cdots>b_1$. Then $\overline{\pi(\mathrm{C}_{\gamma_{w_1w}})}=\overline{\pi'(\mathrm{C}_{\gamma_{w_2w}})}$.*

Proof. Let $\mathrm{A}=\{a_1,\dots,a_r\}$. We have

$$\pi(\mathrm{C}_{\gamma_{w_1w}})=\mathbb{P}_{a_1}\cdots\mathbb{P}_{a_r}\mathbb{P}_z\mathbb{P}_{\bar{z}}\mathbb{P}_{\bar{b}_s}\cdots\mathbb{P}_{\bar{b}_1}\mathrm{T}^{\geq r+s+2}_{\gamma_{w_1w}},$$

where

$$\mathbb{P}_z=\mathrm{U}_{(\varepsilon_z,0)}(v_z)\prod_{l>z}\mathrm{U}_{(\varepsilon_z-\varepsilon_l,0)}(v_{z\bar{l}})\prod_{l\notin\mathrm{A}}\mathrm{U}_{(\varepsilon_z+\varepsilon_l,0)}(v_{zl})\prod_{a_i\in\mathrm{A}}\mathrm{U}_{(\varepsilon_z+\varepsilon_{a_i},1)}(v_{za_i}),$$

$$\mathbb{P}_{\bar{z}}=\prod_{a_i\in\mathrm{A}}\mathrm{U}_{(\varepsilon_{a_i}-\varepsilon_z,0)}(v_{a_i\bar{z}}),$$

and note that $\mu_{\gamma_{w_1}}=\mu_{\gamma_{w_2}}=\sum_{i\in\mathrm{I}_r}\varepsilon_{a_i}-\sum_{j\in\mathrm{I}_s}\varepsilon_{b_j}$. The terms that appear in $\mathbb{P}_z$ all stabilize $\mu_{\gamma_{w_1}}$ and commute with $\mathbb{P}_{\bar{b}_j}$, while the terms in $\mathbb{P}_{\bar{z}}$ all appear in $\mathbb{P}_{a_i}$ and commute with $\mathbb{P}_{a_l}$, for $l>i$. This concludes the proof of Lemma 7.8 with the usual arguments, and therefore of Proposition 5.20. □

8. Nonexamples for nonreadable galleries

Let $n=2$ and $\lambda=\varepsilon_1+\varepsilon_2$, and $(\Sigma_{\gamma_\lambda},\pi)$ the corresponding Bott–Samelson variety, as in (8). Let γ be the gallery corresponding to the block

$$\begin{array}{|c|c|}\hline 1 & \bar{2}\\ \hline 2 & \bar{1}\\ \hline\end{array}\,.$$

Then points in $\pi(\mathrm{C}_\gamma)$ are of the form

$$\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b)[t^0],$$

for $b\in\mathbb{C}$, hence form an affine set of dimension 1. We claim that the set $\mathrm{Z}=\overline{\pi(\mathrm{C}_\gamma)}$ cannot be an MV cycle in $\mathscr{Z}(\mu)$ for any dominant coweight μ. First note that for any $u \in \mathrm{U}(\mathscr{K})$ a necessary condition for ut^0 to lie in the closure $\overline{\mathrm{U}(\mathscr{K})t^\nu \cap G(\mathscr{O})t^\mu}$ is that $0 \leq \nu$, since it would in particular imply that $ut^0 \in \overline{\mathrm{U}(\mathscr{K})t^\nu}$. Also note that it is necessary for $\nu \leq \mu$ in order for the set $\mathscr{Z}(\mu)_\nu$ not to be empty. Any MV cycle in $\mathscr{Z}(\mu)_\nu$ has dimension $\langle \rho, \mu + \nu \rangle$, and the only possibility for the latter to be equal to 1 (since $\mu + \nu$ is a sum of positive coroots) is for either $\mu = 0$ and $\nu = \alpha_i^\vee$, or $\nu = 0$ and $\mu = \alpha_i^\vee$, for some $i \in \mathrm{I}$, and both options are impossible: the first contradicts $\nu \leq \mu$, and the second contradicts the dominance of μ. Note that γ is not a Littelmann gallery.

Appendix

Here we show that relation R3 in Theorem 5.17 is equivalent to relation R_3 in [Lecouvey 2002, Definition 3.1]. For a word $w \in \mathscr{W}_{\mathscr{C}_n}$ and $m \leq n$ define $\mathrm{N}(w, m) = |\{x \in w : x \leq m \text{ or } \bar{m} \leq x\}|$. Lecouvey's relation R_3 is: "Let w be a word that is not the word of an LS block and such that each strict subword is. Let z be the lowest unbarred letter such that the pair $(z, \bar{z})$ occurs in w and $\mathrm{N}(w, z) = z + 1$. Then $w \cong w'$, where w' is the subword obtained by erasing the pair $(z, \bar{z})$ in w." The following Lemma is a translation between R_3 and R3.

Lemma 8.1. *Let w be a word that is not the word of an LS block and such that each strict subword is. Then $w = a_1 \cdots a_r z \bar{z} \bar{b}_s \cdots \bar{b}_1$ for a_i, b_i unbarred and $a_1 < \cdots < a_r$, $b_1 < \cdots < b_s$.*

Proof. By [Lecouvey 2002, Remark 2.2.2], w is the word of an LS block if and only if $\mathrm{N}(w, m) \leq m$ for all $m \leq n$. Let w be as in the statement of Lemma 8.1. Then there exists in w a pair $(z, \bar{z})$ such that $\mathrm{N}(w, z) > z$. Let z be minimal with this property. In particular $\mathrm{N}(w, z) = z + 1$ since if w'' is the word obtained from w by erasing z, then $z \geq \mathrm{N}(w'', z) = \mathrm{N}(w, z) - 1$. We claim that z is the largest unbarred letter to appear in w. If there was a larger letter y then $\mathrm{N}(w''', z) = \mathrm{N}(w, z) = z + 1$ where w''' denotes the word obtained from w by deleting y. This is impossible since by assumption w''' is the word of an LS block. Likewise $\bar{z}$ is the smallest unbarred letter to appear in w. The $a_i's$ and $b_i's$ are then those from Definition 4.3 for the word obtained from w by deleting $z, \bar{z}$ from it. □

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JACINTA TORRES
MAX PLANCK INSTITUTE FOR MATHEMATICS
VIVATSGASSE 7
D-53111 BONN
GERMANY

jtorres@mpim-bonn.mpg.de

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A NOTE ON TORUS ACTIONS AND THE WITTEN GENUS

MICHAEL WIEMELER

We show that the Witten genus of a string manifold M vanishes if there is an effective action of a torus T on M such that $\dim T > b_2(M)$. We apply this result to study group actions on $M \times G/T$, where G is a compact connected Lie group and T a maximal torus of G.

Moreover, we use the methods which are needed to prove these results to the study of torus manifolds. We show that up to diffeomorphism there are only finitely many quasitoric manifolds M with the same cohomology ring as $\#_{i=1}^{k} \pm \mathbb{C}P^n$ with $k < n$.

1. Introduction

In this note we prove a vanishing result for the Witten genus of a string manifold on which a high dimensional torus acts effectively. Concerning the Witten genus of string manifolds on which a compact connected Lie group acts the following is known:

- It has been shown by Liu [1995, discussion after Theorem 4, page 370] that the Witten genus of a string manifold M with $b_2(M) = 0$ vanishes if there is a nontrivial action of S^1 on M.
- Dessai [1999] showed that the Witten genus of a string manifold M vanishes if there is an almost effective action of $\mathrm{SU}(2)$ on M.

Moreover we showed in [Wiemeler 2013] the following stabilizing result: if there is an effective action of a semisimple compact connected Lie group G with $\operatorname{rank} G > \operatorname{rank} H$ on $M \times H/T$, where H is a semisimple compact connected Lie group with maximal torus T, then the Witten genus of M vanishes.

In this note we generalize the first statement in the following way:

Theorem 3.2. *Let M be a spin manifold such that $p_1(M)$ is torsion. If there is an almost effective action of a torus T with* $\operatorname{rank} T > b_2(M)$ *on M then the Witten genus of M vanishes.*

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The main new ingredient to prove this theorem is a spectral sequence argument for actions of tori T on manifolds M with $b_2(M) < \operatorname{rank} T$ (see Lemma 3.1).

If $b_1(M) = 0$, this theorem allows the following generalization, which is also a generalization of the third statement from above.

Theorem 3.3. *Let M be a spin manifold such that $p_1(M)$ is torsion and $b_1(M) = 0$. Moreover, let M' be a $2n$-dimensional spin^c manifold, $n > 0$, with $b_1(M') = 0$ such that there are $x_1, \dots, x_n \in H^2(M'; \mathbb{Z})$ with*

(1) $\sum_{i=1}^n x_i = c_1^c(M')$ *modulo torsion,*

(2) $\sum_{i=1}^n x_i^2 = p_1(M')$ *modulo torsion,*

(3) $\langle \prod_{i=1}^n x_i, [M'] \rangle \neq 0$.

If there is an almost effective action of a torus T on $M \times M'$ such that $\operatorname{rank} T$ *is greater than $b_2(M \times M')$, then the Witten genus of M vanishes. Here $c_1^c(M')$ denotes the first Chern class of the line bundle associated to the spin^c structure on M'.*

To deduce Theorem 3.2 from Theorem 3.3 in the case that $b_1(M) = 0$, let M' be S^2 and x_1 be the Euler class of M'. Then M' satisfies all the assumptions from Theorem 3.3. Moreover there is an almost effective action of $T \times S^1$ on $M \times M'$ which is induced from the T-action on M and the S^1-action on M' given by rotation. Hence, the Witten genus of M vanishes, because

$$\operatorname{rank}(T \times S^1) = \operatorname{rank} T + 1 > b_2(M) + 1 = b_2(M \times M').$$

If H is a semisimple compact connected Lie group with maximal torus T', then the tangent bundle of H/T' splits as a sum of complex line bundles and H/T' has positive Euler characteristic. Therefore H/T' satisfies the assumptions on M' in the above theorem. Hence, we get:

Corollary 4.1. *Let M be a spin manifold with $p_1(M) = 0$ and $b_1(M) = 0$ and H a semisimple compact connected Lie group with maximal torus T' and* $\dim H > 0$. *If there is an almost effective action of a torus T on $M \times H/T'$ such that* $\operatorname{rank} T$ *is greater than* $\operatorname{rank} H + b_2(M)$, *then the Witten genus of M vanishes.*

A torus manifold is a $2n$-dimensional orientable manifold M with an effective action of an n-dimensional torus T such that $M^T \neq \varnothing$. A torus manifold M is called locally standard, if each orbit in M has an invariant neighborhood which is weakly equivariantly diffeomorphic to an open invariant subset of $\mathbb{C}^n$. Here $\mathbb{C}^n$ is equipped with the action of $T = (S^1)^n$ given by componentwise multiplication. If this condition is satisfied, the orbit space of M is naturally a manifold with corners.

A quasitoric manifold is a locally standard torus manifold whose orbit space M/T is face-preserving homeomorphic to a simple convex polytope P. Quasitoric manifolds were introduced by Davis and Januszkiewicz [1991]. Torus manifolds were introduced by Masuda [1999] and Masuda and Hattori [2003].

By combining our results with results of Dessai [1999, 2000] and a recent result of the author [Wiemeler 2015a] on the rigidity of certain torus manifolds, we also get the following finiteness result for simply connected torus manifolds:

Theorem 5.1. *Up to homeomorphism (diffeomorphism, respectively) there are only finitely many simply connected torus manifolds M (quasitoric manifolds, respectively) such that $H^*(M;\mathbb{Z}) \cong H^*\big(\#_{i=1}^k \pm\mathbb{C}P^n;\mathbb{Z}\big)$ with $k < n$.*

For an application of our methods to the study of torus actions on complete intersections and homotopy complex projective spaces, see [Dessai and Wiemeler 2016].

This article is structured as follows. In Section 2 we describe background material on vanishing results for indices of certain twisted Dirac operators on Spin^c manifolds. In Section 3 we prove Theorems 3.2 and 3.3. Then in Section 4 we deduce Corollary 4.1 and give some applications to computations of the degree of symmetry of certain manifolds. In Section 5 we prove Theorem 5.1.

2. Preliminaries

In this section we recall some properties of $2n$-dimensional spin^c manifolds and certain twisted Dirac operators defined on them. For more details on this subject see [Atiyah et al. 1964; Petrie 1972; Hattori 1978; Dessai 1999; 2000].

A spin^c manifold M is an orientable manifold such that the second Stiefel–Whitney class $w_2(M)$ is the reduction of an integral class $c \in H^2(M;\mathbb{Z})$. If this is the case then the tangent bundle of M admits a reduction of structure group to the group Spin$^c(2n)$. We call such a reduction a spin^c structure on M. Associated to a spin^c structure there is a complex line bundle. We denote by $c_1^c(M)$ the first Chern class of this line bundle. Its reduction modulo 2 is $w_2(M)$. For each class $c \in H^2(M;\mathbb{Z})$ with $c \equiv w_2(M) \bmod 2$, there is a spin^c structure on M with $c_1^c(M) = c$.

Now let M be a $2n$-dimensional Spin^c manifold. We assume that S^1 acts on M and that the S^1-action lifts into the spin^c structure. This is the case if and only if the S^1-action lifts into the line bundle associated to the spin^c structure [Wiemeler 2013, Lemma 2.1].

Then we have an S^1-equivariant spin^c Dirac operator ∂_c. Its S^1-equivariant index is an element of the representation ring of S^1 and is defined as

$$\operatorname{ind}_{S^1}(\partial_c) = \ker \partial_c - \operatorname{coker} \partial_c \in R(S^1).$$

We will discuss certain indices of twisted Dirac operators which are related to generalized elliptic genera. Generalized elliptic genera of the type which we discuss here were first studied by Witten [1988].

Let V be an S^1-equivariant complex vector bundle over M and W an even-dimensional S^1-equivariant spin vector bundle over M. From these bundles we construct a power series $R \in K_{S^1}(M)[\![q]\!]$ defined by

$$\bigotimes_{k=1}^{\infty} S_{q^k}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Lambda_{-1}(V^*) \otimes \bigotimes_{k=1}^{\infty} \Lambda_{-q^k}(\widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Delta(W) \otimes \bigotimes_{k=1}^{\infty} \Lambda_{q^k}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}).$$

Here q is a formal variable, $\widetilde{E}$ denotes the reduced vector bundle $E - \dim E$, $\Delta(W)$ is the full complex spinor bundle associated to the spin vector bundle W, and Λ_t (resp. S_t) denotes the exterior (resp. symmetric) power operation. The tensor products are, if not indicated otherwise, taken over the complex numbers.

We extend ind_{S^1} to power series. Then we can define:

Definition 2.1. Let $\varphi^c(M; V, W)_{S^1}$ be the S^1-equivariant index of the spin^c Dirac operator twisted with R:

$$\varphi^c(M; V, W)_{S^1} = \mathrm{ind}_{S^1}(\partial_c \otimes R) \in R(S^1)[\![q]\!].$$

We denote by $\varphi^c(M; V, W)$ the nonequivariant version of this index:

$$\varphi^c(M; V, W) = \mathrm{ind}(\partial_c \otimes R) \in \mathbb{Z}[\![q]\!].$$

With the Atiyah–Singer index theorem [1968], we can calculate $\varphi^c(M; V, W)$ from cohomological data:

$$\varphi^c(M; V, W) = \langle e^{c_1^c(M)/2} \operatorname{ch}(R) \hat{A}(M), [M] \rangle.$$

Here the Chern character of R is a product,

$$\operatorname{ch}(R) = Q_1(TM) Q_2(V) Q_3(W),$$

with

$$Q_1(TM) = \operatorname{ch}\Bigl(\bigotimes_{k=1}^{\infty} S_{q^k}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C})\Bigr) = \prod_i \prod_{k=1}^{\infty} \frac{(1-q^k)^2}{(1-e^{x_i}q^k)(1-e^{-x_i}q^k)},$$

$$\begin{aligned} Q_2(V) &= \operatorname{ch}\Bigl(\Lambda_{-1}(V^*) \otimes \bigotimes_{k=1}^{\infty} \Lambda_{-q^k}(\widetilde{V} \otimes_{\mathbb{R}} \mathbb{C})\Bigr) \\ &= \prod_i (1-e^{-v_i}) \prod_{k=1}^{\infty} \frac{(1-e^{v_i}q^k)(1-e^{-v_i}q^k)}{(1-q^k)^2}, \\ Q_3(W) &= \operatorname{ch}\Bigl(\Delta(W) \otimes \bigotimes_{k=1}^{\infty} \Lambda_{q^k}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C})\Bigr) \\ &= \prod_i (e^{w_i/2} + e^{-w_i/2}) \prod_{k=1}^{\infty} \frac{(1+e^{w_i}q^k)(1+e^{-w_i}q^k)}{(1+q^k)^2}, \end{aligned}$$

where $\pm x_i$ (resp. v_i and $\pm w_i$) denote the formal roots of TM (resp. V and W). If $c_1^c(M)$ coincides with $c_1(V)$, then we have

$$e^{c_1^c(M)/2}Q_2(V) = e(V)\frac{1}{\hat{A}(V)}\prod_i\prod_{k=1}^{\infty}\frac{(1-e^{v_i}q^k)(1-e^{-v_i}q^k)}{(1-q^k)^2} = e(V)Q_2'(V).$$

Note that if M is a spin manifold, then there is a canonical spin^c structure on M. With respect to this spin^c structure the twisted index $\varphi^c(M; 0, TM)$ is equal to the elliptic genus of M. Moreover, our definition of $\varphi^c(M; 0, 0)$ coincides with the index-theoretic definition of the Witten genus of M.

To prove our results we need the following theorem. It was proven first by Liu [1995] for certain twisted elliptic genera of spin manifolds and almost complex manifolds. Later the more general version for spin^c manifolds has been proven by Dessai.

Theorem 2.2 [Dessai 2000, Theorem 3.2, p. 243]. *Assume that the equivariant Pontrjagin class $p_1^{S^1}(V + W - TM)$ restricted to M^{S^1} is equal to $\pi_{S^1}^*(Ix^2)$ modulo torsion, where $\pi_{S^1} : BS^1 \times M^{S^1} \to BS^1$ is the projection on the first factor, $x \in H^2(BS^1; \mathbb{Z})$ is a generator and I is an integer. Assume, moreover, that $c_1^c(M)$ and $c_1(V)$ are equal modulo torsion. If $I < 0$, then $\varphi^c(M; V, W)_{S^1}$ vanishes identically.*

3. Torus actions and the Witten genus

In this section we prove Theorems 3.2 and 3.3. Our methods here are similar to those which were used in Section 4 of [Wiemeler 2013]. We start with a lemma.

Lemma 3.1. *Let M be a T-manifold with* $\operatorname{rank} T > b_2(M)$ *and $a \in H_T^4(M; \mathbb{Q})$ such that $\iota^* a = 0 \in H^4(M; \mathbb{Q})$. Then there is a nontrivial homomorphism $\rho : S^1 \to T$ such that $\rho^* a \in \pi_{S^1}^* H^4(BS^1; \mathbb{Q})$.*

Proof. From the Serre spectral sequence for the fibration $M \to M_T \to BT$ we have the following direct sum decomposition of the $\mathbb{Q}$-vector space $H_T^4(M; \mathbb{Q})$,

$$H_T^4(M; \mathbb{Q}) \cong E_\infty^{0,4} \oplus E_\infty^{2,2} \oplus E_\infty^{4,0}.$$

Moreover, we have

$$E_\infty^{0,4} \subset H^4(M; \mathbb{Q}), \quad E_\infty^{2,2} \subset E_2^{2,2}/d_2(E_2^{0,3}), \quad E_\infty^{4,0} = \pi_{S^1}^* H^4(BT; \mathbb{Q}).$$

Let $a_{0,4}, a_{2,2}, a_{4,0}$ be the components of a according to this decomposition. Then $a_{0,4} = 0$ by assumption. Moreover, there is an $\tilde{a}_{2,2} \in E_2^{2,2}$ such that $a_{2,2} = [\tilde{a}_{2,2}]$.

Now it is sufficient to find a nontrivial homomorphism $\rho : S^1 \to T$ such that $\rho^* \tilde{a}_{2,2} = 0$. We have isomorphisms

$$E_2^{2,2} \cong H^2(BT; \mathbb{Q}) \otimes H^2(M; \mathbb{Q}) \cong (H^2(BT; \mathbb{Q}))^{b_2(M)}.$$

Since $\operatorname{rank} T > b_2(M)$, we can find a nontrivial homomorphism $\phi : H^2(BT;\mathbb{Q}) \to H^2(BS^1;\mathbb{Q}) = \mathbb{Q}$ such that all components of $\tilde{a}_{2,2}$ according to the above decomposition of $E_2^{2,2}$ are mapped to zero by ϕ. After scaling, we may assume that ϕ is induced by a surjective homomorphism $H^2(BT;\mathbb{Z}) \to H^2(BS^1;\mathbb{Z})$. By dualizing we get a homomorphism $\hat{\phi} : H_2(BS^1;\mathbb{Z}) \to H_2(BT;\mathbb{Z})$. Since for any torus, $H_2(BT;\mathbb{Z})$ is naturally isomorphic to the integer lattice in the Lie algebra LT of T, $\hat{\phi}$ defines the desired homomorphism. □

By combining this lemma with the above result of Liu and Dessai (Theorem 2.2), we get the following theorem.

Theorem 3.2. *Let M be a spin manifold such that $p_1(M)$ is torsion. If there is an almost effective action of a torus T with* $\operatorname{rank} T > b_2(M)$ *on M then the Witten genus $\varphi^c(M;0,0)$ of M vanishes.*

Proof. First note that, by replacing the T-action by the action of a double covering group of T, we may assume that the T-action lifts into the spin structure of M.

Therefore, by Theorem 2.2, it is sufficient to show that there is a homomorphism $\rho : S^1 \hookrightarrow T$ such that $\rho^* p_1^T(-TM) = ax^2$, where $x \in H^2(BS^1;\mathbb{Z})$ is a generator and $a \in \mathbb{Z}$, $a < 0$. By Lemma 3.1, there is a homomorphism $\rho : S^1 \to T$ such that

$$p_1^{S^1}(-TM) = \rho^* p_1^T(-TM) = ax^2 \quad \text{with } a \in \mathbb{Z}.$$

Moreover, we have

$$ax^2 = p_1^{S^1}(-TM)|_y = -\sum v_i^2,$$

where $y \in M^T$ is a T fixed point and the $v_i \in H^2(BS^1;\mathbb{Z})$ are the weights of the S^1-representation T_yM. We may assume that such a fixed point y exists because otherwise the Witten genus of M vanishes by an application of the Lefschetz fixed point formula.

Not all of the v_i vanish because the T-action on M is almost effective, which implies that the S^1-action on M is nontrivial. Therefore the theorem is proved. □

We can also deduce the following partial generalization of the above result. Its proof is similar to the proof of Theorems 4.1 and 4.4 in [Wiemeler 2013]. These theorems are concerned with actions of semisimple and simple compact connected Lie groups, whereas the theorem which we present here deals with torus actions.

Theorem 3.3. *Let M be a spin manifold such that $p_1(M)$ is torsion and $b_1(M) = 0$. Moreover, let M' be a $2n$-dimensional spin^c manifold, $n > 0$, with $b_1(M') = 0$ such that there are $x_1, \dots, x_n \in H^2(M';\mathbb{Z})$ with*

1. *$\sum_{i=1}^n x_i = c_1^c(M')$ modulo torsion,*
2. *$\sum_{i=1}^n x_i^2 = p_1(M')$ modulo torsion,*
3. *$\left\langle \prod_{i=1}^n x_i, [M'] \right\rangle \neq 0$.*

If there is an almost effective action of a torus T on $M \times M'$ such that $\operatorname{rank} T$ *is greater than $b_2(M \times M')$, then the Witten genus $\varphi^c(M; 0, 0)$ of M vanishes.*

Proof. Let L_i, $i = 1, \dots, n$, be the line bundle over M' with $c_1(L_i) = x_i$. Because $b_1(M \times M') = 0$, the natural map $\iota^* : H^2_T(M \times M'; \mathbb{Z}) \to H^2(M \times M'; \mathbb{Z})$ is surjective.

Therefore by Corollary 1.2 of [Hattori and Yoshida 1976, page 13] the T-action on $M \times M'$ lifts into $p'^*(L_i)$, $i = 1, \dots, n$. Here $p' : M \times M' \to M'$ is the projection. We can choose these lifts in such a way that the torus action on the fibers of $p'^*(L_i)$, $i = 1, \dots, n$, over a fixed point $y \in (M \times M')^T$ are trivial. Moreover, by the above cited corollary and Lemma 2.1 of [Wiemeler 2013], the action of every $S^1 \subset T$ lifts into the spin^c structure on $M \times M'$ induced by the spin structure on M and the spin^c structure on M'.

By Lemma 3.1 of [Wiemeler 2013], we have

$$\varphi^c\left(M \times M'; \bigoplus_{i=1}^n p'^* L_i, 0\right) = \varphi^c(M; 0, 0)\varphi^c\left(M'; \bigoplus_{i=1}^n L_i, 0\right).$$

By condition (3), we have

$$\begin{aligned}\varphi^c\left(M'; \bigoplus_{i=1}^n L_i, 0\right) &= \left\langle Q_1(TM') \prod_{i=1}^n x_i\, Q_2'\left(\bigoplus_{i=1}^n L_i\right) \hat{A}(M'), [M'] \right\rangle \\ &= \left\langle \prod_{i=1}^n x_i, [M'] \right\rangle \neq 0.\end{aligned}$$

Hence, $\varphi^c(M; 0, 0)$ vanishes if and only if $\varphi^c\big(M \times M'; \bigoplus_{i=1}^n p'^* L_i, 0\big)$ vanishes.

By Theorem 2.2, it is sufficient to show that there is a homomorphism $\rho : S^1 \hookrightarrow T$ such that $\rho^* p_1^T \big(\bigoplus_{i=1}^n p'^* L_i - T(M \times M')\big) = ax^2$, where $x \in H^2(BS^1; \mathbb{Z})$ is a generator and $a \in \mathbb{Z}$, $a < 0$. By Lemma 3.1, there is a homomorphism $\rho : S^1 \to T$ such that

$$p_1^{S^1}\left(\bigoplus_{i=1}^n p'^* L_i - T(M \times M')\right) = \rho^* p_1^T\left(\bigoplus_{i=1}^n p'^* L_i - T(M \times M')\right) = ax^2,$$

with $a \in \mathbb{Z}$.

Moreover, we have

$$ax^2 = p_1^{S^1}\left(\bigoplus_{i=1}^n p'^* L_i - T(M \times M')\right)\bigg|_y = \sum_{i=1}^n a_i^2 - \sum v_i^2,$$

where the $a_i \in H^2(BS^1; \mathbb{Z})$, $i = 1, \dots, n$, are the weights of the S^1-representations $p'^* L_i|_y$ and the $v_i \in H^2(BS^1; \mathbb{Z})$ are the weights of the S^1-representation $T_y(M \times M')$. By our choice of the lifted actions the a_i vanish. Not all of the v_i vanish because the T-action on M is effective, which implies that the S^1-action on M is nontrivial. Therefore the theorem is proved. □

Examples of manifolds M' to which the above theorem applies are manifolds whose tangent bundles split as Whitney sums of complex line bundles and which have nonzero Euler characteristic. In particular, if H is a semisimple compact connected Lie group with maximal torus T' and $\dim H > 0$, then $M' = H/T'$ satisfies these assumptions. We deal with this case in the following section.

4. Torus actions and stabilizing with G/T

In this section we deal with applications of Theorem 3.3 to the particular case where M' is a homogeneous space H/T' with H a semisimple compact connected Lie group and T' a maximal torus of H and $\dim H > 0$.

It has already been noted that the tangent bundle of H/T' splits as a sum of complex line bundles. Therefore H/T' satisfies all the assumptions on M' from Theorem 3.3. Hence we immediately get the following corollary.

Corollary 4.1. *Let M be a spin manifold with $p_1(M) = 0$ and $b_1(M) = 0$ and H a semisimple compact connected Lie group with maximal torus T' and* $\dim H > 0$. *If there is an almost effective action of a torus T on $M \times H/T'$ such that* $\operatorname{rank} T$ *is greater than* $\operatorname{rank} H + b_2(M)$, *then the Witten genus of M vanishes.*

The degree of symmetry $N(M)$ of a manifold M is the maximum of the dimensions of compact connected Lie groups G which act smoothly and almost effectively on M. By combining the above corollary with Corollary 4.2 of [Wiemeler 2013] we get the following bounds for the degree of symmetry of the manifolds $M \times H/T'$. To state our result we have to introduce some notation. For $l \geq 1$ let

$$\alpha_l = \max\left\{ \frac{\dim G}{\operatorname{rank} G} \;\middle|\; G \text{ a simple compact Lie group with } \operatorname{rank} G \leq l \right\}.$$

The values of the α_l are listed in Table 1.

Corollary 4.2. *Let M be a spin manifold with $p_1(M) = 0$ and $b_1(M) = 0$, such that the Witten-genus of M does not vanish and let $H_1, \dots, H_k$ be simple compact connected Lie groups with maximal tori $T_1, \dots, T_k$. Then we have*

$$\sum_{i=1}^{k} \dim H_i \leq N\left(M \times \prod_{i=1}^{k} H_i/T_i\right) \leq \alpha_l \sum_{i=1}^{k} \operatorname{rank} H_i + b_2(M),$$

where $l = \max\{\operatorname{rank} H_i \mid i = 1, \dots, k\}$ *and α_l is defined as above.*

Proof. Let G be a compact connected Lie group which acts almost effectively on $M \times \prod_{i=1}^{k} H_i/T_i$. We may assume that $G = G_{\mathrm{ss}} \times Z$ with a semisimple Lie group G_{ss} and a torus Z.

l	α_l	G_l
1	3	$\mathrm{Spin}(3)$
2	7	G_2
3	7	$\mathrm{Spin}(7)$, $\mathrm{Sp}(3)$
4	13	F_4
5	13	none
6	13	E_6, $\mathrm{Spin}(13)$, $\mathrm{Sp}(6)$
7	19	E_7
8	31	E_8
$9 \leq l \leq 14$	31	none
$l \geq 15$	$2l+1$	$\mathrm{Spin}(2l+1)$, $\mathrm{Sp}(l)$

Table 1. The values of α_l and the simply connected compact simple Lie groups G_l of rank l with $\dim G_l = \alpha_l \cdot l$.

By Corollary 4.1, $\operatorname{rank} G$ is bounded from above by $\sum_{i=1}^k \operatorname{rank} H_i + b_2(M)$. By Corollary 4.2 of [Wiemeler 2013], $\operatorname{rank} G_{\mathrm{ss}}$ is bounded from above by $\sum_{i=1}^k \operatorname{rank} H_i$. Moreover, by the proof of Corollary 4.6 of [Wiemeler 2013] the dimension of G_{ss} is bounded from above by $\alpha_l \operatorname{rank} G_{\mathrm{ss}}$. Since $\alpha_l > 1$, it follows that

$$\begin{aligned}\dim G &= \dim G_{\mathrm{ss}} + \dim Z = \dim G_{\mathrm{ss}} + \operatorname{rank} G - \operatorname{rank} G_{\mathrm{ss}} \\ &\leq (\alpha_l - 1)\operatorname{rank} G_{\mathrm{ss}} + \sum_{i=1}^k \operatorname{rank} H_i + b_2(M) \\ &\leq \alpha_l \sum_{i=1}^k \operatorname{rank} H_i + b_2(M).\end{aligned}$$

This proves the second inequality. The first inequality is trivial. □

Note that if in the situation of Corollary 4.2 the groups H_i are all equal to one of the groups listed in Table 1 and are all isomorphic and $b_2(M) = 0$, then the left and right hand sides of the inequality in Corollary 4.2 are equal. Therefore in this case the degree of symmetry of $M \times \prod_{i=1}^k H_i/T$ is equal to $\dim \prod_{i=1}^k H_i$. This leads to the following corollary.

Corollary 4.3. *Let G be* $\mathrm{Spin}(2l+1)$, $\mathrm{Sp}(l)$ *with* $l \geq 15$, *or an exceptional simple compact connected Lie group with maximal torus T. Moreover, let M be a two-connected manifold with $p_1(M) = 0$ and nonzero Witten genus. Then we have*

$$N\left(M \times \prod_{i=1}^k G/T\right) = k \dim G.$$

5. An application to torus manifolds

In this section we prove the following theorem.

Theorem 5.1. *Up to homeomorphism* (*diffeomorphism*, *respectively*) *there are only finitely many simply connected torus manifolds* M (*quasitoric manifolds*, *respectively*) *such that* $H^*(M;\mathbb{Z}) \cong H^*\big(\#_{i=1}^k \pm\mathbb{C}P^n;\mathbb{Z}\big)$ *with* $k < n$.

Note that if $\dim M < 6$ then this theorem follows directly from the classification of simply connected torus manifolds of dimension four given by Orlik and Raymond [1970] and the fact that the sphere is the only two-dimensional torus manifold.

In higher dimensions the proof of the theorem is subdivided into two lemmas.

Lemma 5.2. *Let* M *be a simply connected torus manifold* (*a quasitoric manifold*, *respectively*) *with* $H^*(M;\mathbb{Z}) \cong H^*\big(\#_{i=1}^k \pm\mathbb{C}P^n;\mathbb{Z}\big)$, $k \in \mathbb{N}$, $n \geq 3$. *Then up to finite ambiguity the homeomorphism type* (*diffeomorphism type*, *respectively*) *is determined by the first Pontrjagin class of* M.

Proof. By Theorem 1.1 of [Wiemeler 2015a], Theorem 2.2 of [Wiemeler 2012] and Theorem 3.6 of [Wiemeler 2015b], it is sufficient to prove that the Poincaré duals of the characteristic submanifolds of M are determined up to finite ambiguity by $p_1(M)$. The characteristic submanifolds of M are codimension two submanifolds which are fixed by circle subgroups of the torus which acts on M. Let

$$u_1, \dots, u_m \in H^2\Big(\mathop{\#}_{i=1}^{k} \pm\mathbb{C}P^n;\mathbb{Z}\Big)$$

be their Poincaré duals. Moreover, we have

$$H^* := H^*\Big(\mathop{\#}_{i=1}^{k} \pm\mathbb{C}P^n;\mathbb{Z}\Big) = \mathbb{Z}[v_1,\dots,v_k]/(v_iv_j,\, v_i^n \pm v_j^n \mid 1 \leq i < j \leq k)$$

with $\deg v_i = 2$ for $i = 1, \dots, k$.

Therefore there are $\alpha_{ij} \in \mathbb{Z}$ such that $u_i = \sum_{j=1}^k \alpha_{ij} v_j$.

Since M is equivariantly formal, it follows from localization in equivariant cohomology that

$$p_1(M) = \sum_{i=1}^m u_i^2 = \sum_{j=1}^k \Big(\sum_{i=1}^m \alpha_{ij}^2\Big) v_j^2.$$

Because the v_j^2 form a basis of H^4 it follows that for fixed $p_1(M)$ there are only finitely many possibilities for the α_{ij}. Therefore the u_i are contained in a finite set which only depends on $p_1(M)$. This proves the lemma. □

Lemma 5.3. *Let* M *be a torus manifold such that* $H^*(M;\mathbb{Z}) \cong H^*\big(\#_{i=1}^k \pm\mathbb{C}P^n;\mathbb{Z}\big)$, *with* $k < n$ *and* $n \geq 3$. *Then with the notation from the proof of the previous lemma* *we have*

$$p_1(M) = \sum_{i=1}^{k} \beta_i v_i^2, \quad \textit{with } 0 < \beta_i \leq n+1.$$

Proof. The inequality $0 < \beta_i$ follows from the formula for $p_1(M)$ given in the proof of the previous lemma. Therefore we only have to show that for all i, $\beta_i \leq n+1$.

Assume the contrary, i.e., $\beta_{i_0} > n+1$ for some $i_0 \in \{1, \dots, k\}$. Since the natural map $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ is surjective, M is a Spin^c manifold. Let $\alpha_i \in \{0, 1\}$, $i = 1, \dots, k$ such that $w_2(M) \equiv \sum_{i=1}^{k} \alpha_i v_i \mod 2$.

Then there are two cases, $\alpha_{i_0} \equiv n+1 \mod 2$ and $\alpha_{i_0} \equiv n \mod 2$.

We first deal with the first case. Choose a Spin^c structure on M such that $c_1^c(M) = (n+1)v_{i_0} + \sum_{i \neq i_0} \alpha_i v_i$. Because $b_1(M) = 0$ every S^1-action on M lifts into this spin^c structure and into all line bundles over M. We can choose these lifts in such a way that the actions on the fiber of a line bundle over a given fixed point $y \in M^{S^1}$ is trivial. By the relation $w_2(M)^2 \equiv p_1(M) \mod 2$, we know that $\beta_i \equiv \alpha_i^2 \mod 2$. Therefore we have $\beta_{i_0} \geq n+3$. Now for $x \in H^2(M;\mathbb{Z})$ let $L(x)$ be the line bundle over M with first Chern class x.

Moreover, let

$$V = L(2v_{i_0}) \oplus L\left(v_{i_0} + \sum_{i \neq i_0} \alpha_i v_i\right) \oplus (n-2)L(v_{i_0}),$$

$$W = \bigoplus_{i \neq i_0} (\beta_i - \alpha_i)L(v_i) \oplus (\beta_{i_0} - n - 3)L(v_{i_0}).$$

Then we have $c_1(V) = c_1^c(M)$, $p_1(V \oplus W \ominus TM) = 0$ and W is a spin bundle.

Therefore, as in the proof of Theorem 3.3, it follows from Theorem 2.2 and Lemma 3.1, that $\varphi^c(M; V, W) = 0$ if $k < n$. This gives a contradiction since a direct computation shows that

$$\varphi^c(M; V, W) = \langle e(V), [M] \rangle = \pm 2 \neq 0.$$

The case where $\alpha_{i_0} \equiv n \mod 2$ is similar. In this case one has to choose a spin^c structure on M such that $c_1^c(M) = nv_{i_0} + \sum_{i \neq i_0} \alpha_i v_i$. Moreover one has to consider the bundles

$$V = L\left(v_{i_0} + \sum_{i \neq i_0} \alpha_i v_i\right) \oplus (n-1)L(v_{i_0}),$$

$$W = \bigoplus_{i \neq i_0} (\beta_i - \alpha_i)L(v_i) \oplus (\beta_{i_0} - n)L(v_{i_0}).$$

The details are left to the reader. □

Now Theorem 5.1 follows directly from Lemmas 5.2 and 5.3.

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MICHAEL WIEMELER
INSTITUT FÜR MATHEMATIK
UNIVERSITÄT AUGSBURG
UNIVERSITÄTSSTRASSE 14
D-86159 AUGSBURG
GERMANY
michael.wiemeler@math.uni-augsburg.de

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