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Greene's rational function $\Psi_P(x)$ is a sum of certain rational functions in $x = (x_1, \dots, x_n)$ over the linear extensions of the poset P (which has n elements), which he introduced in his study of the Murnaghan–Nakayama formula for the characters of the symmetric group. In recent work Boussicault, Féray, Lascoux and Reiner showed that $\Psi_P(x)$ equals a valuation on a cone and calculated $\Psi_P(x)$ for several posets this way. In this paper we give an expression for $\Psi_P(x)$ for any poset P. We obtain such a formula using dissections of root polytopes. Moreover, we use the subdivision algebra of root polytopes to show that in certain instances $\Psi_P(x)$ can be expressed as a product formula, thus giving a compact alternative proof of Greene's original result and its generalizations.

1. Introduction

Given a poset P on the set $[n] = \{1, \dots, n\}$, Greene's rational function is defined by

$$\Psi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)}\right), \tag{1-1}$$

where $\mathcal{L}(P)$ denotes the set of linear extensions of P and for $w \in \mathcal{L}(P)$ and a function $f(x_1, \dots, x_n)$ we have that $w(f(x_1, \dots, x_n)) = f(x_{w(1)}, \dots, x_{w(n)})$. It was introduced by Greene [1992] in his work on the Murnaghan–Nakayama formula. Boussicault, Féray, Lascoux and Reiner [Boussicault et al. 2012] showed that

$$\Psi_P(\boldsymbol{x}) = s(K_P^{\mathrm{root}}; \boldsymbol{x}), \tag{1-2}$$

where

$$K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid i <_P j\} = \mathbb{R}_+\{e_i - e_j \mid i \lessdot_P j\} \tag{1-3}$$

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and

$$s(K;\boldsymbol{x}) := \int_K e^{-\operatorname{span}_{\mathbb{R}_+}(\boldsymbol{x},v)}\,dv, \tag{1-4}$$

for K a polyhedral cone in a Euclidean space V with inner product $\operatorname{span}_{\mathbb{R}_+}(\cdot\,,\cdot)$.

Next we explain two important results about calculating $\Psi_P(\boldsymbol{x})$. Further work on $\Psi_P(\boldsymbol{x})$ appeared in [Boussicault 2007; 2009; Boussicault and Féray 2009; Ilyuta 2009].

Greene's theorem. Let P be a *strongly planar* poset, meaning that the Hasse diagram of $P \sqcup \{\hat{0}, \hat{1}\}$ has a planar embedding with all edges directed upward in the plane. For a strongly planar poset P the edges of the Hasse diagram of P dissect the plane into bounded regions ρ such that the set of vertices of P in the boundary of ρ are two chains starting and ending at the same two elements, $\min(\rho)$ and $\max(\rho)$, respectively. Denote by $b(P)$ the set of bounded regions into which the Hasse diagram of P dissects the plane.

Greene's theorem [Greene 1992]. *For any strongly planar poset P,*

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_{\rho\in b(P)}(x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i\lessdot_P j}(x_i - x_j)}. \tag{1-5}$$

Boussicault's, Féray's, Lascoux's and Reiner's theorem. A beautiful theorem appearing in [Boussicault et al. 2012] gives an expression for $\Psi_P(\boldsymbol{x})$ for some posets P whose Hasse diagrams are bipartite graphs in terms of certain lattice paths. The setup is as follows. Let D be a skew Ferrers diagram in English notation, and let us labels its rows from top to bottom by $1, 2, \ldots, r$ and its columns from right to left by $1, 2, \ldots, c$. See the left of Figure 1. With this labeling the northeasternmost point of D is $(1, 1)$ and the southwesternmost is (r, c). The *bipartite poset* P_D is a poset on the set $\{x_1, \ldots, x_r, y_1, \ldots, y_c\}$ with order relations $x_i <_P y_j$ if and only if $(i, j) \in D$.

BFLR theorem [Boussicault et al. 2012]. *For any skew diagram D,*

$$\Psi_{P_D}(\boldsymbol{x}) = \sum_{\pi} \frac{1}{\prod_{(i,j)\in\pi}(x_i - y_j)}, \tag{1-6}$$

where the sum runs over all lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.

Roadmap of the paper. The objective of this paper is to (1) give a combinatorial expression of $\Psi_P(\boldsymbol{x})$ for any poset P, (2) give an alternative proof of the BFLR theorem and (3) generalize Greene's theorem. We accomplish (1) and (2) in Section 2, while we do (3) in Sections 3 and 4. In Sections 3 and 4 we also study the integer point transform of the root cone, which can be seen as a more refined invariant of the cone than Greene's function. The integer point transform of the root cone and generalizations of Greene's theorem were also investigated in [Boussicault et al. 2012]. Our tools will be root polytopes and their subdivision algebras, the latter of which were introduced in [Mészáros 2011] and put to use in [Escobar and Mészáros 2015a; 2015b; Mészáros 2015a; 2015b; 2016a; 2016b; Mészáros and Morales 2015].

2. Greene's function for an arbitrary poset

The purpose of this section is twofold. First we show how to express $\Psi_P(\boldsymbol{x})$ for any poset P in terms of $\Psi_P(\boldsymbol{x})$ for posets P whose Hasse diagrams are alternating graphs. Then we give an expression for $\Psi_P(\boldsymbol{x})$ for posets whose Hasse diagrams are alternating graphs, thereby also obtaining an expression for $\Psi_P(\boldsymbol{x})$ for any poset P. Finally, we show that for certain posets P whose Hasse diagrams are bipartite graphs we can write $\Psi_P(\boldsymbol{x})$ as a nice summation formula. The latter result originally appeared in the work of Boussicault, Féray, Lascoux and Reiner [Boussicault et al. 2012], who used triangulations of order polytopes in their proof. We phrase our proof in terms of root polytopes. The point of view of this paper is that (dissections of) root polytopes (and the root cone) are the unifying approach to the calculation of $\Psi_P(\boldsymbol{x})$.

A *root polytope* (of type A_{n-1}) is the convex hull of the origin and some of the points $e_i - e_j$ for $1 \le i < j \le n$. Given a graph G on the vertex set $[n]$ we associate to it the root polytope

$$\tilde{Q}_G = \operatorname{ConvHull}\bigl(0, e_i - e_j \mid (i, j) \in E(G),\ i < j\bigr). \tag{2-1}$$

It can be seen that $\tilde{Q}_G$ is a simplex if and only if G is acyclic and to emphasize this we sometimes denote $\tilde{Q}_G$ for acyclic graphs G by $\tilde{\Delta}_G$. In the proof of Lemma 4.2 we will also use the notation

$$\Delta_F = \operatorname{ConvHull}\bigl(e_i - e_j \mid (i, j) \in E(F),\ i < j\bigr) \tag{2-2}$$

for a forest F.

The posets P we work with in this section are on the set $[n]$ and they are labeled naturally; that is to say that if $i <_P j$ then $i < j$ in the order of natural numbers. Note that this does not pose a restriction on the results, it only makes them easier to state. Denote by $\mathcal{H}(P)$ the graph of the Hasse diagram of P. The directed transitive closure of a graph H is denoted by $\overline{H}$, and it is the graph on vertex set $V(G)$ with edges $(i, j) \in \overline{H}$ if there is an increasing path from i to j in H.

$\Psi_P(\boldsymbol{x})$ in terms of alternating posets. This subsection explains how to reduce the computation of $\Psi_P(\boldsymbol{x})$ to the computation of $\Psi_P(\boldsymbol{x})$ for posets P whose Hasse diagram is an alternating graph. A graph G on the vertex set $[n]$ is called *alternating* if there are no edges (i, j) and (j, k) in it with $i < j < k$. We call a poset on $[n]$ an *alternating poset* if its Hasse diagram is an alternating graph.

Proposition 2.1. *For any naturally labeled poset P on $[n]$ we can write*

$$\Psi_P(\boldsymbol{x}) = \sum_{L,R} \Psi_{P_{L,R}}(\boldsymbol{x}), \tag{2-3}$$

where the summation runs over all L, R such that $L \sqcup R = [n]$, and

$$G_{L,R} = \big([n], \{(i, j) \in E(G) \mid i \in L,\ j \in R,\ i < j\}\big)$$

is a connected graph, where $G = \overline{\mathcal{H}(P)}$. Furthermore, $\mathcal{H}(P_{L,R}) = G_{L,R}$ for a naturally labeled alternating poset $P_{L,R}$.

Proof. Recall that $\Psi_P(\boldsymbol{x}) = s(K_P^{\mathrm{root}}; \boldsymbol{x})$. If $K_P^{\mathrm{root}} = \bigcup_{i=1}^{l} K_i$ for interior disjoint cones K_i with $i \in [l]$ then $s(K_P^{\mathrm{root}}; \boldsymbol{x}) = \sum_{i=1}^{l} s(K_i; \boldsymbol{x})$. If $K_i = K_{P_i}^{\mathrm{root}}$ for some posets P_i with $i \in [l]$ then $\Psi_P(\boldsymbol{x}) = \sum_{i=1}^{l} \Psi_{P_i}(\boldsymbol{x})$. Therefore, to prove (2-3), it suffices to show that $K_P^{\mathrm{root}} = \bigcup_{L,R} K_{P_{L,R}}^{\mathrm{root}}$, where the union runs over all L, R such that $L \sqcup R = [n]$, $G_{L,R}$ is a connected graph ($G = \overline{\mathcal{H}(P)}$) and $\mathcal{H}(P_{L,R}) = G_{L,R}$ for a naturally labeled poset $P_{L,R}$.

Since $K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid i <_P j\}$, if $\tilde{Q}_G = \bigcup \tilde{Q}_{G_{L,R}}$ (the $\tilde{Q}_{G_{L,R}}$ are interior disjoint), where the union runs over all L, R such that $L \sqcup R = [n]$, and $G_{L,R}$ is a connected graph, then we also obtain that $K_P^{\mathrm{root}} = \bigcup_{L,R} K_{P_{L,R}}^{\mathrm{root}}$ for interior disjoint cones $K_{P_{L,R}}^{\mathrm{root}}$. The equation $\tilde{Q}_G = \bigcup \tilde{Q}_{G_{L,R}}$ follows from [Postnikov 2009, Proposition 13.3] together with the observation that $G = \overline{G}$ for our choice of G. □

We note that the cones $K_{P_{L,R}}^{\mathrm{root}}$ are generally not simplicial. One way to compute $\Psi_{P_{L,R}}(\boldsymbol{x})$ would be to triangulate $K_{P_{L,R}}^{\mathrm{root}}$ into simplicial cones with rays of the form $e_i - e_j$, since for such a cone the following simple lemma gives the value of Greene's function.

Lemma 2.2 [Boussicault et al. 2012]. *The cone K_P^{root} is simplicial if and only if the Hasse diagram of P contains no cycles. In this case it is also unimodular and*

$$\Psi_P(\boldsymbol{x}) = \frac{1}{\prod_{i \lessdot_P j}(x_i - x_j)}.$$

We remark that a proof of Lemma 2.2 different from that given in [Boussicault et al. 2012] follows immediately using the subdivision algebra of root polytopes defined in [Mészáros 2011].

Calculating $\Psi_P(\boldsymbol{x})$ for an alternating poset P. In light of Proposition 2.1, if we can calculate $\Psi_P(\boldsymbol{x})$ for an alternating poset P, then we can in turn calculate $\Psi_P(\boldsymbol{x})$ for any poset P. In this section we accomplish the former, building on the results of Li and Postnikov [2015]. The next paragraph follows the exposition of that paper.

Given an alternating graph G on the vertex set $[n]$, pick a linear order $\mathcal{O}$ on the edges of G. Let T be a spanning tree of G, and let e be an edge that does not belong to T. Let C be the unique cycle contained in the graph $([n], E(T) \cup \{e\})$. Let e^* be the maximal edge in the cycle C in the linear ordering $\mathcal{O}$ of the edges. We say that an edge e is *externally semiactive* if either $e = e^*$ or there is an odd number of edges in C between e and e^*. (Since G is alternating, all cycles in G have an even length.) Let $\mathrm{ext}_G^{\mathcal{O}}(T)$ be the number of externally semiactive edges of G with respect to a spanning tree T.

Theorem 2.3 [Li and Postnikov 2015]. *Given an alternating graph G and a linear ordering $\mathcal{O}$ of its edges, let $\mathcal{T}_G^{\mathcal{O}}$ be the set of spanning trees T with $\mathrm{ext}_G^{\mathcal{O}}(T) = 0$. Then*

$$\tilde{Q}_G = \bigcup_{T \in \mathcal{T}_G^{\mathcal{O}}} \tilde{\Delta}_T, \tag{2-4}$$

where the simplices $\tilde{\Delta}_T$ are interior disjoint.

Corollary 2.4. *For any naturally labeled poset P on $[n]$ we can write*

$$\Psi_P(\boldsymbol{x}) = \sum_{L,R} \sum_{T \in \mathcal{T}_{G_{L,R}}^{\mathcal{O}_{L,R}}} \frac{1}{\prod_{(i,j) \in E(T), i<j} (x_i - x_j)}, \tag{2-5}$$

where the summation runs over all L, R such that $L \sqcup R = [n]$, and

$$G_{L,R} = \big([n], \{(i,j) \in E(G) \mid i \in L,\ j \in R,\ i < j\}\big)$$

is a connected graph, where $G = \overline{\mathcal{H}(P)}$. Furthermore, $\mathcal{O}_{L,R}$ is an arbitrary linear order of the edges of $G_{L,R}$.

Proof. The proof follows from Proposition 2.1, Lemma 2.2 and Theorem 2.3. □

We remark that we obtained Corollary 2.4 from a particular dissection of the root polytope $\mathrm{ConvHull}(0, e_i - e_j \mid e_i <_P e_j)$ into simplices. Such a dissection then induced a dissection of $K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid e_i <_P e_j\}$ into simplicial cones. Since we know that $K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j \mid e_i \lessdot_P e_j\}$, instead of $\mathrm{ConvHull}(0, e_i - e_j \mid e_i <_P e_j)$ one could also dissect $\mathrm{ConvHull}(0, e_i - e_j \mid e_i \lessdot_P e_j)$ into simplices and obtain an expression with fewer terms for $\Psi_P(\boldsymbol{x})$. However, since such a dissection also would not in general yield significantly fewer terms, we find the expression presented in Corollary 2.4 a fine representative of what a general formula for $\Psi_P(\boldsymbol{x})$ for an arbitrary poset P can look like. We devote the next section to particularly nice formulas for $\Psi_P(\boldsymbol{x})$ for special posets P, also demonstrating that in certain instances we can expect the formula presented in Corollary 2.4 to be far better than the formula given in (1-1), although this is not always the case.

An alternative proof of the BFLR theorem. Let P_D be the poset of a connected skew diagram D as in the BFLR theorem. Let G_D be the graph $\mathcal{H}(P_D)$ drawn on a line with vertices from left to right, $x_r, \dots, x_1, y_1, \dots, y_c$, and with edges as arcs above this line. Note that the condition that G_D comes from P_D can be translated into the conditions that G_D is bipartite on parts $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_c\}$ and for each $i \in [r]$, x_i is connected to y_j for $j \in [a_i, b_i]$, $i \in [r]$, where $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_r$ and $[1, c] = \bigcup_{i=1}^{r} [a_i, b_i]$.

Given a drawing of a graph G such that its vertices $v_1, \dots, v_n$ are arranged in this order on a horizontal line and its edges are drawn above this line, we say that G is *noncrossing* if it has no edges (v_i, v_k) and (v_j, v_l) with $i < j < k < l$. A vertex v_i of G is said to be *nonalternating* if it has both an incoming and an outgoing edge; it is called *alternating* otherwise. The graph G is alternating if all its vertices are alternating.

Lemma 2.5. *The root polytope $\tilde{Q}_{G_D}$ decomposes into $\tilde{Q}_{G_D} = \bigcup_T \tilde{\Delta}_T$, where the union runs over all noncrossing alternating trees of G_D and the simplices $\tilde{\Delta}_T$ are interior disjoint.*

Since noncrossing depends on the drawing of the graph it is essential that we remember that we drew G_D with vertices from left to right: $x_r, \dots, x_1, y_1, \dots, y_c$.

Proof of Lemma 2.5. Consider the following ordering $\mathcal{O}$ on the edges of G_D. The edges incident to y_i precede the edges incident to y_j in the ordering $\mathcal{O}$ if $1 \leq i < j \leq c$. Moreover, if edges (x_a, y_k) and (x_b, y_k) are incident to y_k for some $k \in [c]$ with $1 \leq a < b \leq r$, then (x_a, y_k) precedes (x_b, y_k) in the ordering $\mathcal{O}$. We claim that then the spanning trees T of G_D with $\mathrm{ext}_{G_D}^{\mathcal{O}} = 0$ are exactly the noncrossing alternating trees of G_D and then the lemma follows from Theorem 2.3. Indeed, note that given any noncrossing alternating tree T of G_D and an edge $e \in E(G_D) - E(T)$, in the unique cycle C of the graph T with the edge e adjoined, the edge e is always 0 edges away from the largest edge of C in the ordering $\mathcal{O}$. Thus, for any noncrossing alternating tree T of G_D we have $\mathrm{ext}_{G_D}^{\mathcal{O}} = 0$. On the other hand, given a crossing alternating spanning tree T' of G_D (note that all spanning trees of G_D are alternating) let the edges (x_i, y_j) and (x_k, y_l) cross with $k > i$ and $l < j$. Since D is a connected skew diagram, both of the edges (x_k, y_j) or (x_i, y_l) are contained in G_D. Since T' is a spanning tree of G_D, it follows that exactly one of the edges from $\{(x_k, y_j), (x_i, y_l)\}$ is in it. Adjoining the other edge as edge e we see that it is an externally semiactive edge for T, concluding the proof. $\square$

Lemma 2.6. *The noncrossing alternating spanning trees of G_D are in bijection with the lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.*

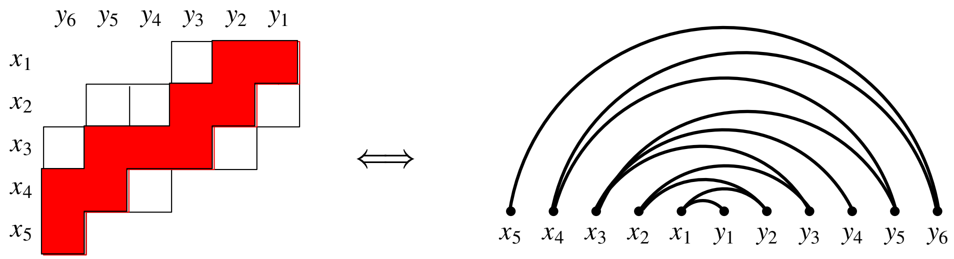

Figure 1. The correspondence between noncrossing alternating spanning trees of G_D and lattice paths from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.

Proof. The bijection is given by the map that takes a noncrossing alternating spanning tree $T = \big(\{x_r, \ldots, x_1, y_1, \ldots, y_c\}, \{(x_i, y_j) \mid (i, j) \in S(T)\}\big)$ of G_D to the path $\pi = S(T)$. See Figure 1. $\square$

Given a graph G on the vertex set $[n]$ such that if $(i, j) \in E(G)$ then the only increasing path from i to j in G is the edge (i, j) itself, we can define the naturally labeled poset P_G to be one on the set $[n]$ with Hasse diagram given by (the edges of) G.

Corollary 2.7 (BFLR theorem). *For any skew diagram D,*

$$\Psi_{P_D}(\boldsymbol{x}) = \sum_{\pi} \frac{1}{\prod_{(i,j)\in\pi}(x_i - y_j)}, \tag{2-6}$$

where the sum runs over all lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west.

Proof. By Lemma 2.5 we have that the cone $K_{P_D}^{\mathrm{root}}$ is triangulated into simplicial cones $K_{P_T}^{\mathrm{root}}$, where the T's run over all noncrossing alternating spanning trees of G_D. By Lemma 2.6 the latter trees are in bijection with lattice paths π from $(1, 1)$ to (r, c) inside D that take steps either one unit south or one unit west, and thus by Lemma 2.2 we obtain the corollary. $\square$

Our proof for Corollary 2.7 is a special case of the proof of Corollary 2.4. We note that the formula for $\Psi_{P_D}(\boldsymbol{x})$ given in Corollary 2.7 is substantially different from the expression given in (1-1). We can see this for example by looking at the number of terms that can appear in each. When D is a diagram in the shape of an $r \times c$ rectangle, then in (1-1) we are summing over all linear extensions of the poset P_D yielding $r!c!$ terms. In comparison, in Corollary 2.7 we have $\binom{r+c-2}{r-1}$ terms corresponding to the lattice paths from $(1, 1)$ to (r, c) inside D. The latter in general can be larger than the former. However, if instead we take D to be the skew shape $D = (n, n-1, \ldots, 1) \setminus (n-2, n-3, \ldots, 1)$, then in Corollary 2.7 we have a single term and in (1-1) we are summing over all linear extensions of the zigzag poset P_D. In this case the number of terms in (1-1) is larger than $n!(n-1)!$, which is many more than the one term in Corollary 2.7.

3. Lifting Greene's theorem to the subdivision algebra

The objective of this section is to generalize Greene's theorem to a relation in the subdivision algebra of root polytopes. Subdivision algebras of root polytopes were introduced and studied in [Mészáros 2011], where they were used for triangulating root polytopes. Subdivision algebras were also utilized for subword complexes and flow polytopes in [Escobar and Mészáros 2015a; Mészáros 2015a; 2015b; 2016a; 2016b; Mészáros and Morales 2015]. We will see in this section that both Greene's theorem and an analogous one for the integer point transform of the root cone are special cases of a relation in the subdivision algebra.

We begin by explaining how to use subdivision algebras to subdivide root cones K_P^{root}. Since Greene's function of a poset P is a valuation on a root cone K_P^{root} and we know its expression for unimodular root cones, if we triangulate K_P^{root} into unimodular root cones, then we obtain a way to calculate Greene's function of P.

Root cones $\mathcal{C}(G)$ and their subdivisions. We establish a simpler notation for root cones here. For an arbitrary loopless graph G, define the *root cone*

$$\mathcal{C}(G) := \operatorname{span}_{\mathbb{R}_+}\big(e_i - e_j \mid (i, j) \in E(G),\ i < j\big). \tag{3-1}$$

In order for $\mathcal{C}(G)$ and $\mathcal{C}(H)$ to be distinct for distinct graphs G and H, we will mostly consider *good graphs* G, which are loopless graphs such that if there is an increasing path from vertex i to vertex j in G, which is not the edge (i, j), then the edge (i, j) is not present in G. (In particular, G contains no multiple edges.) Given a graph H let $g(H)$ be the unique good graph on the vertex set $V(H)$ such that $\mathcal{C}(H) = \mathcal{C}(g(H))$. The graph $g(H)$ can be obtained from H by repeated removal of edges (i, j) for which there is an increasing path between i and j other than the edge (i, j). In particular, all multiple edges are removed in order to obtain $g(H)$. An important property of root cones is given in the cone reduction lemma below, which can be expressed through reduction rules on graphs, as we now explain.

The *reduction rule for graphs*: given a graph G_0 on the vertex set $[n]$ and $(i, j), (j, k) \in E(G_0)$ for some $i < j < k$, let G_1, G_2, G_3 be graphs on the vertex set $[n]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0) \setminus \{(j, k)\} \cup \{(i, k)\},\\ E(G_2) &= E(G_0) \setminus \{(i, j)\} \cup \{(i, k)\},\\ E(G_3) &= E(G_0) \setminus \{(i, j), (j, k)\} \cup \{(i, k)\}. \end{aligned} \tag{3-2}$$

We say that G_0 *reduces* to G_1, G_2 and G_3 under the reduction rules defined by equations (3-2).

For a good graph G we define two edges $(i, j), (j, k) \in E(G)$, $i < j < k$, to be a *good pair of edges* of G if they belong to a common cycle in G, or if neither of them belongs to any cycle in G.

Lemma 3.1 (cone reduction lemma; cf. [Mészáros 2011]). *Given a good graph G_0 let $(i, j), (j, k) \in E(G_0)$ be a good pair of edges of G_0 for some $i < j < k$ and G_1, G_2 as described by equations (3-2). Then*

$$\mathcal{C}(G_0) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2) \tag{3-3}$$

and

$$\mathcal{C}(G_3) = \mathcal{C}(G_1) \cap \mathcal{C}(G_2), \tag{3-4}$$

where the cones $\mathcal{C}(G_0), \mathcal{C}(G_1), \mathcal{C}(G_2)$ are of the same dimension and $\mathcal{C}(G_3)$ is a facet of both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$.

For convenience we include a proof of Lemma 3.1 here. It is an adaptation of the proof from [Mészáros 2011], where it was written for acyclic graphs.

Proof. Let the edges of G_0 be $f_1 = (i, j)$, $f_2 = (j, k)$, $f_3, \dots, f_k$. Let $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$ denote the vectors that the edges of G_0 correspond to under the correspondence $\mathrm{v} : (i, j) \mapsto e_i - e_j$, where $i < j$. By equations (3-2),

$$\begin{aligned}
\mathcal{C}(G_0) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)),\\
\mathcal{C}(G_1) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1), \mathrm{v}(f_1)+\mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)),\\
\mathcal{C}(G_2) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1)+\mathrm{v}(f_2), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)),\\
\mathcal{C}(G_3) &= \mathrm{span}_{\mathbb{R}_+}(\mathrm{v}(f_1)+\mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)).
\end{aligned}$$

Thus, if $\mathcal{C}(G_0)$ is d-dimensional, so are the cones $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$, while cone $\mathcal{C}(G_3)$ is at least $(d-1)$-dimensional (and at most d-dimensional). We note that $\dim(\mathcal{C}(G_3)) \neq d$ because G_0 is a good graph and f_1 and f_2 are a good pair of edges.

Clearly, $\mathcal{C}(G_1) \cup \mathcal{C}(G_2) \subset \mathcal{C}(G_0)$. Given an expression of a vector $v \in \mathcal{C}(G_0)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$ it satisfies either that the coefficient of $\mathrm{v}(f_1)$ in such an expression is greater than or equal to the coefficient of $\mathrm{v}(f_2)$ in the expression, or it is not. In the former case we see that $v \in \mathcal{C}(G_1)$ and in the latter case $v \in \mathcal{C}(G_2)$. Therefore, $\mathcal{C}(G_0) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2)$.

Clearly, $\mathcal{C}(G_3) \subset \mathcal{C}(G_1) \cap \mathcal{C}(G_2)$. Given an expression of a vector $v \in \mathcal{C}(G_1)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$, the coefficient of $\mathrm{v}(f_1)$ is greater than or equal to the coefficient of $\mathrm{v}(f_2)$. Similarly, given an expression of a vector $v \in \mathcal{C}(G_2)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$, the coefficient of $\mathrm{v}(f_1)$ is less than or equal to the coefficient of $\mathrm{v}(f_2)$. Thus, there is an expression of $v \in \mathcal{C}(G_1) \cap \mathcal{C}(G_2)$ as a nonnegative linear combination of the vectors $\mathrm{v}(f_1), \mathrm{v}(f_2), \mathrm{v}(f_3), \dots, \mathrm{v}(f_k)$

such that the coefficient of $\mathrm{v}(f_1)$ is equal to the coefficient of $\mathrm{v}(f_2)$. Therefore, $\mathcal{C}(G_1) \cap \mathcal{C}(G_2) \subset \mathcal{C}(G_3)$, leading to $\mathcal{C}(G_1) \cap \mathcal{C}(G_2) = \mathcal{C}(G_3)$. □

The subdivision algebra, Greene's theorem and the integer point transform of a root cone. In this subsection we explain the subdivision algebra and show how it yields a slick proof for Greene's theorem and its generalization.

A graph G can be encoded by the monomial $m[G] = \prod_{(i,j) \in E(G), i<j} x_{ij}$ and the reduction rule going from G_0 to G_1, G_2 and G_3 can be encoded by the equation $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk} + \beta)$. We define the *subdivision algebra* $\mathcal{S}_n$ of root polytopes as the commutative algebra generated by the variables x_{ij}, $1 \le i < j \le n$, subject to the relations $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk} + \beta)$ for $1 \le i < j < k \le n$.

Let us explain the connection of the subdivision algebra to Greene's function. If we set $\beta = 0$, then the relation $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk})$ of $\mathcal{S}_n$ is satisfied by $x_{ij} := 1/(x_i - x_j)$, which are the kind of terms appearing in Greene's function. If instead, we set $\beta = -1$, then the relation $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk} - 1)$ of $\mathcal{S}_n$ is satisfied by $x_{ij} := 1/(1 - x_i/x_j)$. The latter will play a part in calculating the *integer point transform* $\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})$ *of the root cone* $K_P^{\mathrm{root}} \subset \mathbb{Z}^d$ defined as

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x}) := \sum_{\boldsymbol{m} \in K_P^{\mathrm{root}} \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{m}}. \tag{3-5}$$

The function $\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})$ can be seen as a finer invariant of the cone than $\Psi_P(\boldsymbol{x})$, as explained in [Boussicault et al. 2012, Section 2.4]. We note that in that paper the integer point transform $\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})$ is denoted as $H(K_P^{\mathrm{root}}; \boldsymbol{X})$ and is referred to as the Hilbert series of the affine semigroup ring of the root cone. We chose to follow the more geometric name and notation of [Beck and Robins 2007, Section 3.2].

We are now ready to prove the following generalization of Greene's theorem via the subdivision algebra, which first appeared in [Boussicault et al. 2012]:

Theorem 3.2 [Boussicault et al. 2012, Corollary 8.10]. *For any (connected) strongly planar poset P on $[n]$ we have*

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x}) = \frac{\prod_{\rho \in b(P)} (1 - x_{\min(\rho)}/x_{\max(\rho)})}{\prod_{i \lessdot_P j} (1 - x_i/x_j)} \tag{3-6}$$

and

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_{\rho \in b(P)} (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i \lessdot_P j} (x_i - x_j)}, \tag{3-7}$$

where ρ runs through all bounded regions of the Hasse diagram.

Proof. Since P is a connected strongly planar poset, it follows that its Hasse diagram is a good graph on the vertex set $[n]$ such that for every cycle C of G the only alternating vertices of C (considered within C), that is vertices that have

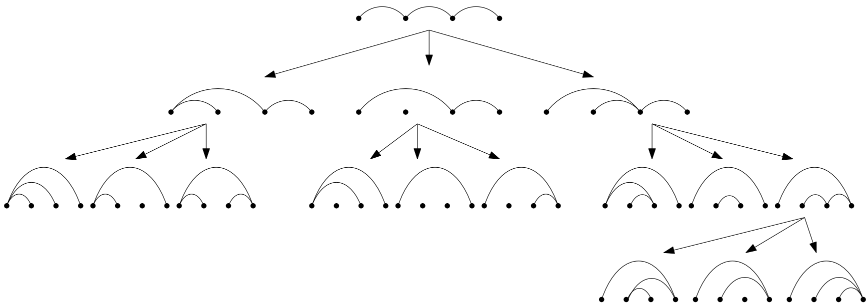

Figure 2. In reducing an increasing path we always pick the topmost leftmost edges in the path and its offsprings to do reductions on. For a graph G_0 the arrow to the left points to G_1, the middle arrow to G_3, and the right arrow to G_2, as in equations (3-2).

only incoming or only outgoing edges, are its minimal and maximal vertices. Thus we have that $K_P^{\mathrm{root}} = \mathcal{C}(G)$ for a good graph G. Note that a root cone $\mathcal{C}(H)$ is unimodular if and only if $g(H)$ is acyclic. We will use the cone reduction lemma to write $\mathcal{C}(G)$ as a union of unimodular cones. Note that the cone reduction lemma applies to good graphs, and thus if we want to repeatedly apply it to the outcome cones $\mathcal{C}(G_i)$, $i \in [3]$, we need to apply it to $g(G_i)$, $i \in [3]$.

We claim that we can apply the cone reduction lemma repeatedly in such a fashion that at the end we have trees $T_1, \dots, T_k$ (with $n-1$ edges), and forests F^j_{n-i}, $2 \le i \le n-1$, $j \in I_{n-i}$ (for some index sets I_{n-i}), with $n-i$ edges, where $\mathcal{C}(T_1), \dots, \mathcal{C}(T_k)$ are unimodular cones triangulating $\mathcal{C}(G)$ and the $\mathcal{C}(F^j_{n-i})$ are their intersections.

We now prove the above claim. When G has no cycles, the claim is obvious. Suppose that G has $m > 0$ linearly independent cycles. Fix a strongly planar drawing of P. In it there are m bounded regions, and the boundaries of these regions are m linearly independent cycles in G. Let C be one of these cycles, such that it bounds a region in the drawing of P which is adjacent to the infinite region. The cycle C consists of two increasing paths p and p' from i to j for some $i < j$. Let $p = (i = i_0 \to i_1 \to i_2 \to \cdots \to i_l = j)$ be the path bordering the infinite region in the drawing of P. We can perform consecutive reductions on the edges of the path p and its offsprings, ultimately obtaining all noncrossing alternating forests on the vertices $\{i_0, i_1, \dots, i_l\}$ containing the edge (i_0, i_l). We do this by picking the topmost leftmost edges that we can do a reduction on in p and its offsprings in the reduction process. See Figure 2 for an illustration. (A proof of the previous claim can be obtained by induction on the length of the path and is given in detail in [Mészáros 2011].) Until we arrive at the aforementioned noncrossing alternating forests on the vertices $\{i_0, i_1, \dots, i_l\}$ containing the edge (i_0, i_l) all

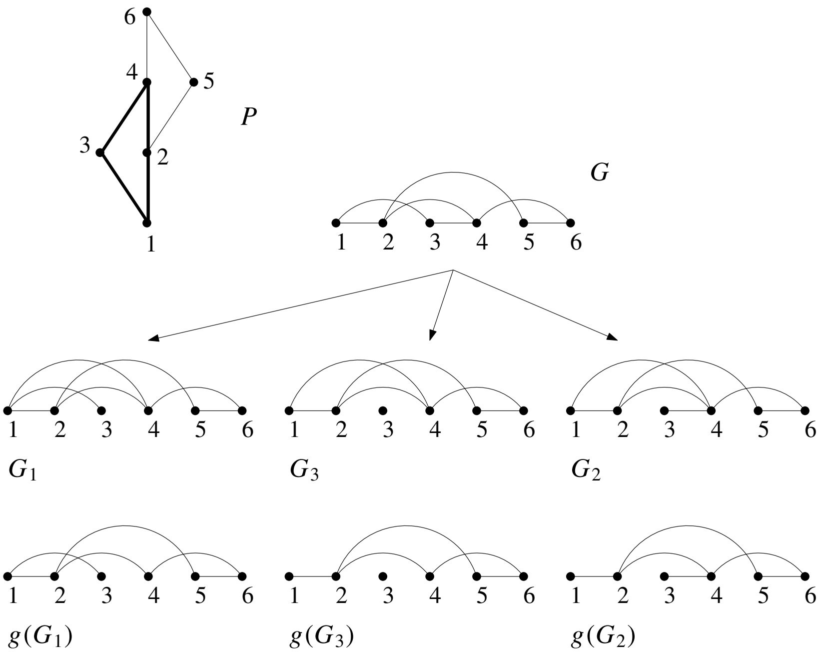

Figure 3. Top left shows a strongly planar drawing of our poset, with the cycle C in bold. The path p is $(1 \to 3 \to 4)$ and p' is $(1 \to 2 \to 4)$. Top right shows the graph G. Below are the graphs G_1, G_3, G_2 obtained by applying the reduction on the topmost leftmost edges of p, which are $(1, 3)$, $(3, 4)$. The last row shows $g(G_1)$, $g(G_3)$, $g(G_2)$ (which are G_1, G_3, G_2 with the edge $(1, 4)$ removed since there is an increasing path $1 \to 2 \to 4$), on which we can keep applying the cone reduction lemma as in the proof of Theorem 3.2.

graphs obtained in this fashion from G are good graphs. We can see that once we obtain the noncrossing alternating forests on the vertices $\{i_0, i_1, \ldots, i_l\}$ containing the edge (i_0, i_l) the offspring of G is not good anymore, as there is still p' in it, which is an increasing path between the vertices i_0 and i_l. We need to now remove the edge $(i_0, i_l) = (i, j)$ from all the aforementioned offsprings in order to obtain good graphs and be able to apply the cone reduction lemma further. However, once we remove the edge (i, j) from all these offsprings we will have good graphs with the number of bounded regions one less than it was for G. We can now repeat the same process we just described for each of these graphs and their offsprings until they are all acyclic. We demonstrate the basic idea of this argument in Figure 3.

If we inspect what edges we had to drop in the process to make sure we always apply the cone reduction lemma to good graphs and obtain the acyclic graphs described in the previous paragraph, we find the following relation in the subdivision algebra:

$$m[G]=\prod_{\rho\in b(P)} x_{\min(\rho),\max(\rho)}\Bigl(\sum_{T_i} m[T_i]+\sum_{F^j_{n-i}} \beta^{i-1} m[F^j_{n-i}]\Bigr). \tag{3-8}$$

Note that

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x})=\Bigl(\sum_{T_i} m[T_i]+\sum_{F^j_{n-i}} (-1)^{i-1} m[F^j_{n-i}]\Bigr)\Big|_{x_{ij}=1/(1-x_i x_j{}^{-1})} \tag{3-9}$$

and

$$\Psi_P(\boldsymbol{x})=\sum_{T_i} m[T_i]\Big|_{x_{ij}=1/(x_i-x_j)}. \tag{3-10}$$

Equations (3-8), (3-9) and (3-10) together with the observations that $x_{ij}=1/(1-x_i x_j{}^{-1})$ satisfies $x_{ij}x_{jk}=x_{ik}(x_{ij}+x_{jk}-1)$ and that $x_{ij}=1/(x_i-x_j)$ satisfies $x_{ij}x_{jk}=x_{ik}(x_{ij}+x_{jk})$ immediately yield equations (3-6) and (3-7). □

We can see (3-8) is the main theorem of this section, so we bestow it with that title:

Theorem 3.3. *Let $G=\mathcal{H}(P)$ for a naturally labeled connected strongly planar poset P. Then, using the notation of the proof of Theorem 3.2, we have that*

$$m[G]=\prod_{\rho\in b(P)} x_{\min(\rho),\max(\rho)}\Bigl(\sum_{T_i} m[T_i]+\sum_{F^j_{n-i}} \beta^{i-1} m[F^j_{n-i}]\Bigr)$$

holds in the subdivision algebra.

Both statements of Theorem 3.2 are special cases of Theorem 3.3 as shown in the proof of Theorem 3.2.

4. Generalizing Greene's theorem beyond strongly planar posets

In this section we will examine a special family of posets for which Greene's function factors linearly. These posets were first identified by Boussicault, Féray, Lascoux and Reiner [Boussicault et al. 2012], who proved the aforementioned result by studying the affine semigroup ring of the root cone. We will give a short alternative proof via root polytopes.

We give some definitions following the exposition of [Boussicault et al. 2012]. In a finite poset P, say that a triple of elements (a, b, c) forms a notch of $\vee$ shape (dually, a notch of $\wedge$ shape) if $a \lessdot_P b, c$ (dually, $b, c \lessdot_P a$), and in addition, b, c lie in different connected components of the poset $P \setminus P_{\leq a}$ (dually, $P \setminus P_{\geq a}$). When (a, b, c) forms a notch of either shape in a poset P, say that the quotient poset $\bar{P} := P/\{b=c\}$, having one fewer element and one fewer Hasse diagram edge, is obtained from P by closing the notch, and that P is obtained from $\bar{P}$ by opening a notch.

Theorem 4.1. *Let P be a connected poset in which (a, b, c) forms a notch, and let $\bar{P} := P/\{b = c\}$. We assume without loss of generality that P and $\bar{P}$ are naturally labeled. Then the root polytope $\tilde{Q}_{\mathcal{H}(P)}$ has a triangulation with top-dimensional simplices $\tilde{\Delta}_{T_1}, \dots, \tilde{\Delta}_{T_k}$, and $\tilde{Q}_{\mathcal{H}(\bar{P})}$ has a triangulation with top-dimensional simplices $\tilde{\Delta}_{T'_1}, \dots, \tilde{\Delta}_{T'_k}$, where $(a, b) \in T'_i$, $(a, b), (a, c) \in T_i$, $i \in [k]$, and moreover $T_i|_{b=c} = T'_i$ (we ignore multiple edges).*

To prove Theorem 4.1 we use the following criterion.

Lemma 4.2 (cf. [Postnikov 2009, Lemma 12.6]). *For two trees T and T' on the vertex set $[n]$, the intersection $\tilde{\Delta}_T \cap \tilde{\Delta}_{T'}$ is a common face of the simplices $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ if and only if the directed graph*

$$U(T, T') = \big([n], \big\{(i, j) \mid (i, j) \in E(T), i < j\big\} \cup \big\{(j, i) \mid (i, j) \in E(T'), i < j\big\}\big)$$

has no directed cycles of length at least 3.

The following proof of Lemma 4.2 is a straightforward adaptation of the proof of [Postnikov 2009, Lemma 12.6] to our more general setting. We include the proof here for convenience.

Proof of Lemma 4.2. Suppose that $U(T, T')$ has a directed cycle C of length at least 3. Let E be the set of edges of T in C and E' be the set of edges of T' in C. Then $\sum_{(i,j)\in E}(e_i - e_j) = \sum_{(i,j)\in E'}(e_i - e_j)$. Let $k = \max(|E|, |E'|)$. Then

$$x := \frac{1}{k} \sum_{(i,j)\in E} (e_i - e_j) = \frac{1}{k} \sum_{(i,j)\in E'} (e_i - e_j) \in \tilde{\Delta}_T \cap \tilde{\Delta}_{T'}.$$

However, the minimal face of the simplex $\tilde{\Delta}_T$ containing x is $\Delta_{([n],E)}$ if $k = |E|$ and $\tilde{\Delta}_{([n],E)}$ if $k > |E|$. Similarly, the minimal face of the simplex $\tilde{\Delta}_{T'}$ containing x is $\Delta_{([n],E')}$ if $k = |E'|$ and $\tilde{\Delta}_{([n],E')}$ if $k > |E'|$. In any case, the minimal faces of $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ containing x are not equal. Thus, $\tilde{\Delta}_T \cap \tilde{\Delta}_{T'}$ is not their common face.

Next, assume that $U(T, T')$ has no directed cycles of length at least 3. Let $F = ([n], E(T) \cap E(T'))$. Since $U(T, T')$ has no directed cycles of length at least 3 we can find a function $h : [n] \to \mathbb{R}$ such that (1) h is constant on connected components of F; and (2) for any directed edge $(a, b) \in U(T, T')$ that joins two different components of F we have $h(a) < h(b)$. Thus, if (a, b) is the edge $(i < j)$ of T then $h(i) < h(j)$, and if (a, b) is the edge $(i < j)$ of T then $h(i) > h(j)$. The function h defines a linear form f_h on the space $\mathbb{R}^n$ with the coordinates $h(1), \dots, h(n)$ in the standard basis. The above conditions imply (1) for any vertex in the common face $\tilde{\Delta}_F$ of $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ we have $f_h(x) = 0$; (2) for any vertex $x \in \tilde{\Delta}_T \setminus \tilde{\Delta}_F$ we have $f_h(x) < 0$; and (3) for any vertex $x \in \tilde{\Delta}_{T'} \setminus \tilde{\Delta}_F$ we have $f_h(x) > 0$. Thus, the hyperplane $f_h(x) = 0$ intersects $\tilde{\Delta}_T$ and $\tilde{\Delta}_{T'}$ at their common face $\tilde{\Delta}_F$ as desired. □

Proof of Theorem 4.1. The criterion of Lemma 4.2 is sufficient to establish the above theorem, since we also have that $\tilde{Q}_{\mathcal{H}(\bar{P})}$ has a triangulation with top-dimensional simplices $\tilde{\Delta}_{T_1'}, \dots, \tilde{\Delta}_{T_k'}$, where $(a, b) \in T_i'$, as $e_a - e_b$ is a vertex of $\tilde{Q}_{\mathcal{H}(\bar{P})}$. □

When we calculate $\sigma_{K_{\bar{P}\mathrm{root}}}(\boldsymbol{x})$ and $\Psi_{\bar{P}}(\boldsymbol{x})$ using triangulations of the root cones as implied by Theorem 4.1, we immediately get:

Corollary 4.3 [Boussicault et al. 2012, Theorem 8.6]. *When $\bar{P}$ is obtained from P by closing a $\vee$-shaped notch (a, b, c), then*

$$\sigma_{K_{\bar{P}\mathrm{root}}}(\boldsymbol{x}) = (1 - x_a {x_b}^{-1})\sigma_{K_{P\mathrm{root}}}(\boldsymbol{x})|_{x_b = x_c} \quad \text{and} \quad \Psi_{\bar{P}}(\boldsymbol{x}) = (x_a - x_b)\Psi_P(\boldsymbol{x})|_{x_b = x_c}.$$

A consequence of Theorem 4.1 is the following generalization of Greene's theorem pertaining to posets P to which we can repeatedly apply the opening notch operation and obtain a poset whose Hasse diagram has only cycles as biconnected components. Such posets P we call *admissible*. We now recall the definition of biconnected components following [Boussicault et al. 2012]. Given a graph $G = (V, E)$ we say that two edges of it are cycle-equivalent if there is a cycle which contains both edges. Let E_i be the equivalence classes of this relation. Let V_i be the set of vertices which are the endpoint of at least one edge in E_i. Then the biconnected components of G are the graphs $G_i = (V_i, E_i)$.

Theorem 4.4. *Let P be an admissible planar poset. Then, we have*

$$\sigma_{K_P^{\mathrm{root}}}(\boldsymbol{x}) = \frac{\prod_{\rho \in b(P)} \bigl(1 - \prod_{i \in \min(\rho)} x_i \prod_{j \in \max(\rho)} x_j^{-1}\bigr)}{\prod_{i \lessdot_P j} (1 - x_i {x_j}^{-1})}$$

and

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_{\rho \in b(P)} \bigl(\sum_{i \in \min(\rho)} x_{\min(i)} - \sum_{j \in \max(\rho)} x_j\bigr)}{\prod_{i \lessdot_P j} (x_i - x_j)},$$

where ρ runs through all bounded regions of the Hasse diagram of P.

Proof. This theorem can be deduced from Corollary 4.3 together with Corollaries 8.2 and 8.3 in [Boussicault et al. 2012]. We note that the latter corollaries also have simple proofs using the root polytope considerations of this paper, and we leave such alternative proofs as an exercise for the interested reader. □

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