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We study cells in generalized Bott–Samelson varieties for type C_n. These cells are parametrized by certain galleries in the affine building. We define a set of *readable galleries* — we show that the closure in the affine Grassmannian of the image of the cell associated to a gallery in this set is an MV cycle. This then defines a map from the set of readable galleries to the set of MV cycles, which we show to be a morphism of crystals. We further compute the fibers of this map in terms of the Littelmann path model.

1. Introduction

This paper is part of a project started by Gaussent and Littelmann [2005] the aim of which is to establish an explicit relationship between the path model and the set of MV cycles used by Mirković and Vilonen for the Geometric Satake equivalence proven in [Mirković and Vilonen 2007].

1A. We consider a complex connected reductive algebraic group G and its affine Grassmannian $\mathscr{G} = \mathrm{G}(\mathbb{C}((t)))/\mathrm{G}(\mathbb{C}[\![t]\!])$. We fix a maximal torus $\mathrm{T} \subset \mathrm{G}$. The coweight lattice $\mathrm{X}^\vee = \mathrm{Hom}(\mathbb{C}^\times, \mathrm{T})$ can be seen as a subset of $\mathscr{G}$. For a coweight λ, which we may assume dominant with respect to some choice of Borel subgroup containing T, the closure X_λ of the $\mathrm{G}(\mathbb{C}[\![t]\!])$-orbit of λ in $\mathscr{G}$ is an algebraic variety which is usually singular. The Geometric Satake equivalence identifies the complex irreducible highest weight module $\mathrm{L}(\lambda)$ for the Langlands dual group $\mathrm{G}^\vee$ with the intersection cohomology of X_λ, a basis of which is given by the classes of certain subvarieties of X_λ called MV cycles. The set of these subvarieties is denoted by $\mathscr{Z}(\lambda)$. The Geometric Satake equivalence implies that the elements of $\mathscr{Z}(\lambda)$ are in one to one correspondence with the vertices of the crystal $\mathrm{B}(\lambda)$. Braverman

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and Gaitsgory [2001], endow the set $\mathscr{Z}(\lambda)$ with a crystal structure and show the existence of a crystal isomorphism $\varphi : \mathrm{B}(\lambda) \xrightarrow{\sim} \mathscr{Z}(\lambda)$.

1B. Gaussent and Littelmann [2005] define a set $\Gamma(\gamma_\lambda)^{\mathrm{LS}}$ of LS galleries, which are galleries in the affine building $\mathscr{J}^{\mathrm{aff}}$ associated to G, and they endow this set with a crystal structure and an isomorphism of crystals $\mathrm{B}(\lambda) \xrightarrow{\sim} \Gamma(\gamma_\lambda)^{\mathrm{LS}}$. They view the latter as a subset of the T-fixed points in a desingularization $\Sigma_{\gamma_\lambda} \xrightarrow{\pi} \mathrm{X}_\lambda$. To each of these particular fixed points $\delta \in \Gamma(\gamma_\lambda)^{\mathrm{LS}}$ corresponds a Białynicki-Birula cell $\mathrm{C}_\delta \subset \Sigma_{\gamma_\lambda}$. Gaussent and Littelmann [2005] show that the closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle, and Baumann and Gaussent [2008] show that the map

$$\Gamma(\gamma_\lambda)^{\mathrm{LS}} \to \mathscr{Z}(\lambda), \quad \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a crystal isomorphism with respect to the crystal structure on $\mathscr{Z}(\lambda)$ described by Braverman and Gaitsgory [2001]. It is natural to ask whether the closures $\overline{\pi(\mathrm{C}_\delta)}$ are still MV cycles for a more general choice of fixed point δ.

1C. Gaussent and Littelmann [2012] consider *one skeleton* galleries, which are piecewise linear paths in $\mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. Such galleries can be interpreted in terms of Young tableaux for types A, B and C. For $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$, Gaussent et al. [2013] show that for any fixed point $\delta \in \Sigma_{\gamma_\lambda}^{\mathrm{T}}$, the closure $\overline{\pi(\mathrm{C}_\delta)}$ is in fact an MV cycle. They achieve this using combinatorics of Young tableaux such as word reading and the well known Knuth relations, and by relating them to the Chevalley relations for root subgroups which hold in the affine Grassmannian $\mathscr{G}$. In [Torres 2016] it is observed that word reading is a crystal morphism, and this allows one to prove that in this case, the map from all galleries to MV cycles is in fact a morphism of crystals. It was conjectured in [Gaussent et al. 2013] that generalizations of their results hold for arbitrary complex semisimple algebraic groups, in terms of the plactic algebra defined by Littelmann [1996]. It is with this in mind that we formulate and state our results.

1D. ***Results.*** We work with $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$. We define a set $\Gamma(\gamma_\lambda)^{\mathrm{R}} \supset \Gamma(\gamma_\lambda)^{\mathrm{LS}}$ of *readable* galleries, which have an explicit formulation in terms of Young tableaux. These galleries correspond to all galleries in type A. They are called keys in [Gaussent et al. 2013]. Type C combinatorics related to LS galleries has been developed by De Concini [1979], Kashiwara and Nakashima [1994], King [1976], Lakshmibai [1987] (in the context of standard monomial theory), Proctor [1990], Sheats [1999] and Lecouvey [2002], among others. We use the description of LS galleries of fundamental type given by Lakshmibai in [1987; 1986]. We use the formulation given by Lecouvey [2002]. There is a certain word reading described in [Lecouvey 2002] which we show to be a crystal morphism when restricted to readable galleries. We obtain results similar to those obtained in [Gaussent et al.

2013] concerning the defining relations of the *symplectic plactic monoid*, described explicitly by Lecouvey [2002], as well as words of readable galleries. These results together with the work of Gaussent and Littelmann [2005; 2012], and Baumann and Gaussent [2008] allow us to show in Theorem 6.2 that given a readable gallery $\delta \in \Gamma(\gamma_\lambda)^{\mathrm{R}}$ there is an associated dominant coweight $\nu_\delta \leq \lambda$ such that:

(1) The closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in X_{ν_δ}.

(2) The map

$$\Gamma(\gamma_\lambda)^{\mathrm{R}} \xrightarrow{\varphi_{\gamma_\lambda}} \bigoplus_{\delta \in \Gamma(\gamma_\lambda)^{\mathrm{R}}/\sim} \mathscr{Z}(\mu_{\delta^+}), \quad \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a morphism of crystals.

Here $\Gamma(\gamma_\lambda)^{\mathrm{R}}/\sim$ is some set of representatives for a certain equivalence relation on the set of readable galleries. We compute the fibers of this map in terms of the Littelmann path model. Moreover, this map induces an isomorphism when restricted to each connected component. We then provide some examples of galleries $\delta \in \Sigma^{\mathrm{T}}_{\gamma_\lambda} - \Gamma(\gamma_\lambda)^{\mathrm{R}}$ for which $\overline{\pi(\mathrm{C}_\delta)}$ is not an MV cycle in $\mathscr{Z}(\nu_\delta)$.

1E. This paper is organized as follows. In Section 2 we introduce our notation and recall several general facts about affine Grassmannians, MV cycles, galleries in the affine building, generalized Bott–Samelson varieties, and concrete descriptions of the cells C_δ in them. In Section 3 we introduce the crystal structure on combinatorial galleries, motivating our results with the Littelmann path model, and define readable galleries as concatenations of LS galleries of fundamental type and "zero lumps." From Section 4 on we work with $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$, where we recall some type C combinatorics and build up to our main result, which we state and prove in Section 6. However, the main ingredients of the proof, stated in Section 5, are proven in Section 7. In Section 8 we exhibit some examples in special cases where the image of a certain cell cannot be an MV cycle. In the Appendix we show a technical result that we need.

2. Preliminaries

2A. ***Notation.*** Throughout this section, we consider G to be a complex connected reductive algebraic group associated to a root datum $(\mathrm{X}, \mathrm{X}^\vee, \Phi, \Phi^\vee)$, and we denote its Langlands dual by $\mathrm{G}^\vee$. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus of G with character group $\mathrm{X} = \mathrm{Hom}(\mathrm{T}, \mathbb{C}^\times)$ and cocharacter group $\mathrm{X}^\vee = \mathrm{Hom}(\mathbb{C}^\times, \mathrm{T})$. We will call elements of X weights, and elements of $\mathrm{X}^\vee$ coweights. We identify the Weyl group W with the quotient $\mathrm{N_G(T)/T}$, where $\mathrm{N_G(T)}$ denotes the normalizer of T in G. We will abuse notation by denoting a representative in $\mathrm{N_G(T)}$ of an element $w \in \mathrm{W}$ in the Weyl group by the same symbol, w, that we use to denote the element itself. We fix a choice of positive roots Φ^+ (this determines a set $\Phi^{\vee,+}$ of positive coroots), and denote the dominance order on X and $X^\vee$ determined by this choice by $\leq$. We will denote the corresponding set of dominant weights and coweights by $X^+ \subset X$ and $X^{\vee,+} \subset X^\vee$ respectively. Let $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi^+$ be the basis or set of simple roots of Φ that is determined by Φ^+. The number n is called the rank of the root datum. Then the set $\Delta^\vee$ of all coroots $\alpha_i^\vee$ of elements $\alpha_i \in \Delta$ forms a basis of the root system $\Phi^\vee$. Let $\langle -,- \rangle$ be the nondegenerate pairing between X and $X^\vee$, and denote the half sum of positive roots and coroots by ρ and $\rho^\vee$ respectively. Note that if $\lambda = \sum_{\alpha\in\Delta} n_\alpha \alpha$, respectively $\lambda = \sum_{\alpha^\vee \in \Delta^\vee} n_\alpha \alpha^\vee$, is a sum of positive roots then $\langle \lambda, \rho^\vee \rangle = \sum_{\alpha\in\Delta} n_\alpha$, respectively $\langle \rho, \lambda \rangle = \sum_{\alpha^\vee\in\Delta^\vee} n_\alpha$).

Let $B \subset G$ be the Borel subgroup of G containing T that is determined by the choice of positive roots Φ^+, and let $U \subset B$ be its unipotent radical. The group U is generated by the elements $U_\alpha(b)$ for $b \in \mathbb{C}$, $\alpha \in \Phi^+$, where for each root α, U_α is the one-parameter group it determines. For each coweight $\lambda \in X^\vee$ and each nonzero complex number $a \in \mathbb{C}^\times$, we denote its image $\lambda(a) \in T$ by a^λ.

The following identities hold in G (See [Steinberg 1968, §6]):

- For any $\lambda \in X^\vee$, $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$, and $\alpha \in \Phi$,

$$a^\lambda U_\alpha(b) = U_\alpha(a^{\langle\alpha,\lambda\rangle} b) a^\lambda. \tag{1}$$

- (Chevalley's commutator formula) Given linearly independent roots $\alpha, \beta \in \Phi$, there exist numbers $c^{i,j}_{\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ such that, for all $a, b \in \mathbb{C}$,

$$U_\alpha(a)^{-1} U_\beta(b)^{-1} U_\alpha(a) U_\beta(b) = \prod_{i,j\in\mathbb{N}^{>0}} U_{i\alpha+j\beta}\big(c^{i,j}_{\alpha,\beta}(-a)^i b^j\big). \tag{2}$$

  The product is taken in some fixed order. The $c^{i,j}_{\alpha,\beta}$ are integers which apart from depending on i and j depend also on α, β and on the chosen order in the product.

2B. ***Affine Grassmannians.*** Let $\mathscr{O} = \mathbb{C}[[t]]$ denote the ring of complex formal power series and let $\mathscr{K} = \mathbb{C}((t))$ denote its field of fractions; it is the field of complex Laurent power series. For any $\mathbb{C}$-algebra $\mathscr{R}$, we denote the set of $\mathscr{R}$-valued points of G by $G(\mathscr{R})$. The set

$$\mathscr{G} = G(\mathscr{K})/G(\mathscr{O})$$

is called the *affine Grassmannian* associated to G. We will denote the class in $\mathscr{G}$ of an element $g \in G(\mathscr{K})$ by $[g]$. A coweight $\lambda : \mathbb{C}^\times \to T \subset G$ determines a point $t^\lambda \in G(\mathscr{K})$ and hence a class $[t^\lambda] \in \mathscr{G}$. This map is injective, and we may therefore consider $X^\vee$ as a subset of $\mathscr{G}$.

$\mathrm{G}(\mathscr{O})$-orbits in $\mathscr{G}$ are determined by the Cartan decomposition

$$\mathscr{G} = \bigsqcup_{\lambda \in \mathrm{X}^{\vee,+}} \mathrm{G}(\mathscr{O})[t^{\lambda}].$$

Each $\mathrm{G}(\mathscr{O})$-orbit has the structure of an algebraic variety induced from the progroup structure of $\mathrm{G}(\mathscr{O})$ and for a dominant coweight $\lambda \in \mathrm{X}^{\vee,+}$,

$$\overline{\mathrm{G}(\mathscr{O})[t^{\lambda}]} = \bigsqcup_{\substack{\mu \in \mathrm{X}^{\vee,+} \\ \mu \leq \lambda}} \mathrm{G}(\mathscr{O})[t^{\mu}].$$

We call the closure $\overline{\mathrm{G}(\mathscr{O})[t^{\lambda}]}$ a *generalized Schubert variety* and we denote it by X_{λ}. This variety is usually singular. We will review certain resolutions of singularities of it in Section 2E. The $\mathrm{U}(\mathscr{K})$-orbits in $\mathscr{G}$ are given by the Iwasawa decomposition

$$\mathscr{G} = \bigsqcup_{\lambda \in \mathrm{X}^{\vee}} \mathrm{U}(\mathscr{K})[t^{\lambda}].$$

These orbits are indvarieties, and their closures can be described by

$$\overline{\mathrm{U}(\mathscr{K})[t^{\lambda}]} = \bigcup_{\mu \leq \lambda} \mathrm{U}(\mathscr{K})[t^{\mu}]$$

for any $\lambda \in \mathrm{X}^{\vee}$ (see Proposition 3.1(a) of [Mirković and Vilonen 2007]).

2C. ***MV cycles and crystals.*** Let $\lambda \in \mathrm{X}^{\vee,+}$ and $\mu \in \mathrm{X}^{\vee}$ be a dominant integral coweight and any coweight, respectively. Let $\mathrm{L}(\lambda)$ be the irreducible representation of $\mathrm{G}^{\vee}$ of highest weight λ. Then by Theorem 3.2 in [Mirković and Vilonen 2007], the intersection $\mathrm{U}(\mathscr{K})[t^{\mu}] \cap \mathrm{G}(\mathscr{O})[t^{\lambda}]$ is nonempty if and only if μ is a weight of $\mathrm{L}(\lambda)$, and in that case its closure is pure dimensional of dimension $\langle \rho, \lambda + \mu \rangle$ and has the same number of irreducible components as the dimension of the μ-weight space $\mathrm{L}(\lambda)_{\mu}$ [Mirković and Vilonen 2007, Corollary 7.4]. Moreover, $\mathrm{X}^{\vee} \cong \mathrm{Hom}(\mathrm{T}^{\vee}, \mathbb{C}^{\times})$, where $\mathrm{T}^{\vee}$ is the Langlands dual of T, which is a maximal torus of $\mathrm{G}^{\vee}$ (see [Mirković and Vilonen 2007, §7]).

We denote the set of all irreducible components of a given topological space Y by Irr(Y). Consider the sets

$$\mathscr{Z}(\lambda)_{\mu} = \mathrm{Irr}(\overline{\mathrm{U}(\mathscr{K})[t^{\mu}] \cap \mathrm{G}(\mathscr{O})[t^{\lambda}]}) \quad \text{and} \quad \mathscr{Z}(\lambda) = \bigsqcup_{\mu \in \mathrm{X}^{\vee}} \mathscr{Z}(\lambda)_{\mu}.$$

Elements of these sets are called *MV cycles*. Braverman and Gaitsgory [2001, §3.3] have endowed the set $\mathscr{Z}(\lambda)$ with a crystal structure and have shown the existence of an isomorphism of crystals $\mathrm{B}(\lambda) \xrightarrow{\sim} \mathscr{Z}(\lambda)$. We do not use the definition of this crystal structure, but we denote by $\tilde{f}_{\alpha_i}$ (respectively $\tilde{e}_{\alpha_i}$) the corresponding root operators for $i \in \{1, \dots, n\}$, where n is the rank of the root system Φ. See Section 3A below for the definition of a crystal.

2D. *Galleries in the affine building.* Let $\mathscr{J}^{\text{aff}}$ be the affine building associated to G and $\mathscr{K}$. It is a union of simplicial complexes called *apartments*, each of which is isomorphic to the Coxeter complex of the same type as the extended Dynkin diagram associated to G. We refer the reader to [Ronan 2009] for a thorough account of building theory. The affine Grassmannian $\mathscr{G}$ can be $\mathrm{G}(\mathscr{K})$-equivariantly embedded into the building $\mathscr{J}^{\text{aff}}$, which also carries a $\mathrm{G}(\mathscr{K})$ action. Denote by Φ^{aff} the set of real affine roots associated to Φ; we identify it with the set $\Phi \times \mathbb{Z}$.

Let $\mathbb{A} = \mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. For each $(\alpha, m) \in \Phi^{\text{aff}}$, consider the associated hyperplane and the positive and negative half spaces:

$$\mathrm{H}_{(\alpha,m)} = \{x \in \mathbb{A} : \langle \alpha, x\rangle = m\},$$
$$\mathrm{H}^+_{(\alpha,m)} = \{x \in \mathbb{A} : \langle \alpha, x\rangle \geq m\},$$
$$\mathrm{H}^-_{(\alpha,m)} = \{x \in \mathbb{A} : \langle \alpha, x\rangle \leq m\}.$$

The affine Weyl group W^{aff} is generated by all the affine reflections $s_{(\alpha,m)}$ with respect to the affine hyperplanes $\mathrm{H}_{(\alpha,m)}$. We have an embedding $\mathrm{W} \hookrightarrow \mathrm{W}^{\text{aff}}$ given by $s_\alpha \mapsto s_{(\alpha,0)}$, where $s_\alpha \in \mathrm{W}$ is the simple reflection associated to $\alpha \in \Phi$. (The Weyl group W is minimally generated by the set $\{s_{\alpha_i} : i \in \{1, \dots, n\}\}$.) The *dominant Weyl chamber* is the set

$$\mathrm{C}^+ = \{x \in \mathbb{A} : \langle \alpha, x\rangle\ > 0 \text{ for all } \alpha \in \Delta\},$$

and the *fundamental alcove* is in turn

$$\Delta^{\mathrm{f}} = \{x \in \mathrm{C}^+ : \langle \alpha, x\rangle\ \leq 1 \text{ for all } \alpha \in \Phi^+\}.$$

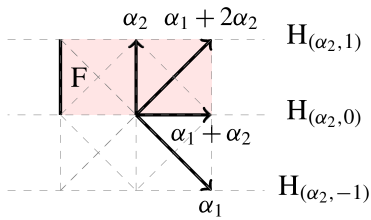
There is a unique apartment in the affine building $\mathscr{J}^{\text{aff}}$ that contains the image of the set of coweights $\mathrm{X}^\vee \subset \mathscr{G}$ under the embedding $\mathscr{G} \hookrightarrow \mathscr{J}^{\text{aff}}$. This apartment is isomorphic to the affine Coxeter complex associated to W^{aff}; its faces are given by all possible intersections of the hyperplanes $\mathrm{H}_{(\alpha,m)}$ and their associated (closed) positive and negative half-spaces $\mathrm{H}^\pm_{(\alpha,m)}$. It is called the *standard ,apartment* in the affine building $\mathscr{J}^{\text{aff}}$. The action on the affine building $\mathscr{J}^{\text{aff}}$ by W^{aff} coincides, when restricted to the standard apartment, with the one induced by the natural action of W^{aff} on $\mathbb{A}$. The fundamental alcove is a fundamental domain for this action.

To each real affine root $(\alpha, m) \in \Phi^{\text{aff}}$ is attached the one-parameter additive *root subgroup* $\mathrm{U}_{(\alpha,m)}$ of $\mathrm{G}(\mathscr{K})$ defined by $b \mapsto \mathrm{U}_\alpha(bt^m)$ for $b \in \mathbb{C}$. Let $\lambda \in \mathrm{X}^\vee$ and $b \in \mathbb{C}$. Identity (1) implies that

$$\mathrm{U}_{(\alpha,m)}(b)[t^\lambda] = [\mathrm{U}_\alpha(bt^m)t^\lambda] = [t^\lambda \mathrm{U}_\alpha(bt^{m-\langle\alpha,\lambda\rangle})], \tag{3}$$

and $[t^{\lambda}\mathrm{U}_{\alpha}(bt^{m-\langle\alpha,\lambda\rangle})]=[t^{\lambda}]$ if and only if $\mathrm{U}_{\alpha}(bt^{m-\langle\alpha,\lambda\rangle})\subset \mathrm{G}(\mathscr{O})$, or, equivalently, $\langle\alpha,\lambda\rangle\leq m$. Hence, the root subgroup $\mathrm{U}_{(\alpha,m)}$ stabilizes the point $[t^{\lambda}]\in\mathscr{G}\hookrightarrow\mathscr{J}^{\mathrm{aff}}$ if and only if $\lambda\in\mathrm{H}^{-}_{(\alpha,m)}$. For each face F in the standard apartment, denote by P_{F}, U_{F} and $\mathrm{W}^{\mathrm{aff}}_{\mathrm{F}}$ its stabilizer in $\mathrm{G}(\mathscr{K})$, $\mathrm{U}(\mathscr{K})$ and $\mathrm{W}^{\mathrm{aff}}$ respectively. These subgroups are generated by the torus T, and respectively by the root subgroups $\mathrm{U}_{(\alpha,m)}$ such that $\mathrm{F}\subset\mathrm{H}^{-}_{(\alpha,m)}$, the root subgroups $\mathrm{U}_{(\alpha,m)}\subset\mathrm{P}_{\mathrm{F}}$ such that $\alpha\in\Phi^{+}$, and those affine reflections $s_{(\alpha,m)}\in\mathrm{W}^{\mathrm{aff}}$ such that $\mathrm{F}\subset\mathrm{H}_{(\alpha,m)}$ [Gaussent and Littelmann 2005, §3.3, Example 3; Baumann and Gaussent 2008, Proposition 5.1].

Example 2.1. Let $\mathrm{G}^{\vee}=\mathrm{Sp}(4,\mathbb{C})$, then $\Phi^{+}=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_1+2\alpha_2\}$. In the picture below the shaded region is the upper half-space $\mathrm{H}^{+}_{(\alpha_2,0)}$. Let F be the face in the standard apartment that joins the vertices $-(\alpha_1+\alpha_2)$ and $-\alpha_1$. This is depicted here.



The subgroup P_{F} is generated by the root subgroups associated to the following real roots:

$$\begin{aligned}(\alpha_1,m)&\ m\geq -1,\\ (\alpha_2,m)&\ m\geq 1,\\ (\alpha_1+\alpha_2,m)&\ m\geq -1,\\ (\alpha_1+2\alpha_2,m)&\ m\geq 0,\\ (-\alpha_1,m)&\ m\geq 2,\\ (-\alpha_2,m)&\ m\geq 0,\\ (-(\alpha_1+\alpha_2),m)&\ m\geq 1,\\ (-(\alpha_1+2\alpha_2),m)&\ m\geq 1.\end{aligned}$$

The stabilizer U_{F} is generated by the root subgroups associated to those previously stated roots (α,m) such that $\alpha\in\Phi^{+}$ is a positive root, and $\mathrm{W}^{\mathrm{aff}}_{\mathrm{F}}=\{s_{(\alpha_1+\alpha_2,-1)},1\}$.

A *gallery* is a sequence of faces in the affine building $\mathscr{J}^{\mathrm{aff}}$,

$$\gamma=(\mathrm{V}_0=0,\mathrm{E}_0,\mathrm{V}_1,\ldots,\mathrm{E}_k,\mathrm{V}_{k+1}), \tag{4}$$

satisfying these conditions:

1. For each $i \in \{1, \ldots, k\}$, $\mathrm{V}_i \subset \mathrm{E}_i \supset \mathrm{V}_{i+1}$.
2. Each face labeled V_i has dimension zero (is a *vertex*) and each face labeled E_i has dimension one (is an *edge*). In particular, each face in the sequence γ is contained in the one-skeleton of the standard apartment.
3. The last vertex V_{k+1} is a *special vertex*: its stabilizer in the affine Weyl group $\mathrm{W}^{\mathrm{aff}}$ is isomorphic to the finite Weyl group W associated to G.

We denote the set of all galleries in the affine building by Σ. If, in addition, each face in the sequence belongs to the standard apartment, then γ is called a *combinatorial gallery*. We will denote the set of all combinatorial galleries in the affine building by Γ. In this case, the third condition is equivalent to requiring the last vertex V_{k+1} to be a coweight. From now on, if γ is a combinatorial gallery we will denote the coweight corresponding to its final vertex by μ_γ in order to distinguish it from the vertex.

Remark 2.2. The galleries we defined are actually called *one-skeleton* galleries in the literature. The word "gallery" was originally used to describe a more general class of face sequences but since we only work with one-skeleton galleries in this paper, we have left the word "one-skeleton" out.

2E. ***Bott–Samelson varieties.*** Let γ be a combinatorial gallery (as above). The following lemma can be obtained from [Gaussent and Littelmann 2012, Lemma 4.8 and Definition 4.6].

Lemma 2.3. *There exist a unique combinatorial gallery,*

$$\gamma^f = (\mathrm{V}_0^f, \mathrm{E}_0^f, \mathrm{V}_1^f, \ldots, \mathrm{V}_{k+1}^f),$$

with each one of its faces contained in the fundamental alcove, and elements $w_j \in \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_j^f}$ *for each* $j \in \{1, \ldots, k\}$ *such that* $w_0 \cdots w_{r-1}\mathrm{V}_r^f = \mathrm{V}_r$ *for each* $r \in \{0, \ldots, k+1\}$ *and* $w_0 \cdots w_r \mathrm{E}_r^f = \mathrm{E}_r$ *for each* $r \in \{0, \ldots, k\}$.

If two galleries γ and η have the same associated gallery $\nu = \gamma^f = \eta^f$ we say that the two galleries have *the same type*. We will denote the set of combinatorial galleries that have the same type as a given combinatorial gallery γ by $\Gamma(\gamma)$. The map

$$\mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_0} \times \cdots \times \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_k} \to \Gamma(\gamma), \tag{5}$$

$$(w_0, \ldots, w_k) \mapsto (\mathrm{V}_0, w_0\mathrm{E}_0, w_0\mathrm{V}_1, w_0w_1\mathrm{E}_1, \ldots, w_0 \cdots w_k\mathrm{V}_{k+1}), \tag{6}$$

induces a bijection between the set $\prod_{i=0}^r \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_i}/\mathrm{W}^{\mathrm{aff}}_{\mathrm{E}_i}$ and $\Gamma(\gamma)$; it is in particular finite. For a proof see [Gaussent and Littelmann 2012, Lemma 4.8].

Definition 2.4. The *Bott–Samelson variety* of type γ^f is the quotient of

$$G(\mathscr{O}) \times P_{V_1^f} \times \cdots \times P_{V_k^f}$$

by the following right action of $P_{E_0^f} \times \cdots \times P_{E_k^f}$:

$$(q_0, \ldots, q_k) \cdot (p_0, p_1, \ldots, p_k) = (q_0 p_0, p_0^{-1} q_1 p_1, \ldots, p_{k-1}^{-1} q_k p_k).$$

We will denote this quotient by Σ_{γ^f}. The progroup structure of the groups $P_{V_i^f}$ and $P_{E_i^f}$ assures that Σ_{γ^f} is in fact a smooth variety. To each point $(g_0, \ldots, g_k)$ in $G(\mathscr{O}) \times P_{V_1^f} \times \cdots \times P_{V_k^f}$ one can associate a gallery

$$(V_0^f, g_0 E_0^f, g_0 V_1^f, g_0 g_1 V_2^f, \ldots, g_0 \cdots g_k V_{k+1}^f). \tag{7}$$

This induces a well defined injective map $i : \Sigma_{\gamma^f} \hookrightarrow \Sigma$. With respect to this identification, the T-fixed points in Σ_{γ^f} are in natural bijection with the set $\Gamma(\gamma^f)$ of combinatorial galleries of type γ^f.

Let $\omega \in \mathbb{A}$ be a fundamental coweight. We define a particular combinatorial gallery, which starts at 0 and ends at ω. Let $V_1^\omega, \ldots, V_k^\omega$ be the vertices in the standard apartment that lie on the open line segment joining 0 and ω, numbered such that V_{i+1}^ω lies on the open line segment joining V_i^ω and ω. Let further E_i^ω denote the face contained in $\mathbb{A}$ that contains the vertices V_i^ω and V_{i+1}^ω. The gallery

$$\gamma_\omega = (0 = V_0^\omega, E_0^\omega, V_1^\omega, E_1^\omega, \ldots, E_k^\omega, V_{k+1}^\omega = \omega)$$

is called a *fundamental gallery*. Galleries of the same type as a fundamental gallery γ_ω will be called *galleries of fundamental type* ω.

Now let $\lambda \in X^{\vee,+}$ be a dominant integral coweight and let γ_λ be a gallery with endpoint λ and expressible as a concatenation of fundamental galleries, where concatenation of two combinatorial galleries $\gamma_1 * \gamma_2$ is defined by translating γ_2 to the endpoint of γ_1. (Note that it follows from the definition of type that if γ, ν are two galleries of the same type as δ and η respectively, then $\gamma * \nu$ has the same type as $\delta * \eta$. Actually, if $\gamma = \gamma_1 * \cdots * \gamma_r$ then $\Gamma(\gamma) = \{\delta_1 * \cdots * \delta_r : \delta_i \in \Gamma(\gamma_i)\}$.) Then the map

$$\Sigma_{\gamma_\lambda^f} \xrightarrow{\pi} X_\lambda, \quad [g_0, \ldots, g_r] \mapsto g_0 \cdots g_r [t^{\mu_{\gamma^f}}] \tag{8}$$

is a resolution of singularities of the generalized Schubert variety X_λ.

Remark 2.5. That the above map is in fact a resolution of singularities is due to the fact that the gallery γ_λ is minimal (see [Gaussent and Littelmann 2012, §5 and §4.3, Proposition 5]). This resembles the condition for usual Bott–Samelson varieties associated to a reduced expression. See [Gaussent and Littelmann 2005, §9, Proposition 7].

Remark 2.6. The map (8) makes sense for any combinatorial gallery γ. In this generality one has a map $\Sigma_{\gamma^f} \xrightarrow{\pi} \mathcal{G}$ sending $[g_0, \dots, g_r]$ to $g_0, \dots, g_r[t^{\mu_\gamma}]$, which is not necessarily a resolution of singularities. From now on we will write (Σ_{γ^f}, π) to refer to the Bott–Samelson variety together with its map $\Sigma_{\gamma^f} \xrightarrow{\pi} \mathcal{G}$ to the affine Grassmannian.

2F. *Cells and positive crossings.* Let $r_\infty : \mathcal{J}^{\text{aff}} \to \mathbb{A}$ be the retraction at infinity (see [Gaussent and Littelmann 2005, Definition 8]). It extends to a map

$$r_{\gamma^f} : \Sigma_{\gamma^f} \to \Gamma(\gamma^f).$$

To a combinatorial gallery $\delta \in \Gamma(\gamma^f)$ is associated the cell $\mathrm{C}_\delta = r_{\gamma^f}^{-1}(\delta)$ which was explicitly described in [Gaussent and Littelmann 2005; 2012; Baumann and Gaussent 2008]. In this subsection we recollect their results; we will need them later. They are originally formulated in terms of galleries of the same type as γ_λ; we formulate them for any combinatorial gallery. The proofs remain the same, and therefore we do not provide them all, but refer the reader to [Gaussent and Littelmann 2005; 2012].

First consider the subgroup $\mathrm{U}(\mathcal{K})$ of $\mathrm{G}(\mathcal{K})$. It is generated by the elements of the root subgroups $\mathrm{U}_{(\alpha,n)}$ for $\alpha \in \Phi^+$ a positive root and $n \in \mathbb{Z}$. Let $\mathrm{V} \subset \mathrm{E}$ be a vertex and an edge (respectively) in the standard apartment, the vertex contained in the edge. Consider the subset of affine roots

$$\Phi^+_{(\mathrm{V},\mathrm{E})} = \{(\alpha, n) \in \Phi^{\text{aff}} : \alpha \in \Phi^+, \mathrm{V} \in \mathrm{H}_{(\alpha,n)}, \mathrm{E} \not\subseteq \mathrm{H}^-_{(\alpha,n)}\},$$

and let $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$ denote the subgroup of $\mathrm{U}(\mathcal{K})$ generated by $\mathrm{U}_{(\alpha,n)}$ for all $(\alpha, n) \in \Phi^+_{(\mathrm{V},\mathrm{E})}$. The following proposition will be very useful in Section 7. It is stated and proven in [Baumann and Gaussent 2008, Proposition 5.1].

Proposition 2.7. *Let* $\mathrm{V} \subset \mathrm{E}$ *be a vertex and an edge in the standard apartment as above. Then* $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$ *is a set of representatives for the right cosets of* U_E *in* U_V. *For any total order on the set* $\Phi^+_{(\mathrm{V},\mathrm{E})}$, *the map*

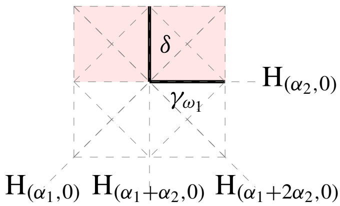
$$(a_\beta)_{\beta\in\Phi^+_{(\mathrm{V},\mathrm{E})}} \mapsto \prod_{\beta\in\Phi^+_{(\mathrm{V},\mathrm{E})}} \mathrm{U}_\beta(a_\beta)$$

is a bijection from $\mathbb{C}^{|\Phi^+_{(\mathrm{V},\mathrm{E})}|}$ *onto* $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$. *The order in the product is the same as the one on the set* $\Phi^+_{(\mathrm{V},\mathrm{E})}$.

Now let γ be a combinatorial gallery with notation as in (4). For each $i \in \{1, \dots, k\}$, let $\mathbb{U}^\gamma_{\mathrm{V}_i} = \mathbb{U}_{(\mathrm{V}_i,\mathrm{E}_i)}$. For later use we fix the notation $\Phi^\gamma_i = \Phi^+_{(\mathrm{V}_i,\mathrm{E}_i)}$.

Example 2.8. Let $\mathrm{G}^\vee = \mathrm{Sp}(4, \mathbb{C})$ as in Example 2.1, and γ_{ω_1} be as in Definition 2.4. Then $\mathbb{U}^{\gamma_{\omega_1}}_{\mathrm{V}_0}$ is generated by the root subgroups associated to the real roots $(\alpha_1, 0)$, $(\alpha_1 + \alpha_2, 0)$, and $(\alpha_1 + 2\alpha_2, 0)$. Let δ be the gallery with one edge and endpoint α_2.

Then $\mathbb{U}^{\delta}_{\mathrm{V}_0}$ is generated by the groups associated to $(\alpha_2, 0)$, $(\alpha_1 + 2\alpha_2, 0)$, as seen here.



Now write $\delta = (\mathrm{V}_0, \mathrm{E}_0, \ldots, \mathrm{E}_k, \mathrm{V}_{k+1}) \in \Gamma(\gamma^f)$ in terms of (7) as $\delta = [\delta_0, \ldots, \delta_k]$. This means $\delta_i \in \mathrm{W}^{\mathrm{aff}}_{\mathrm{V}_i^f}$ and $\delta_0 \cdots \delta_j \mathrm{E}^f_j = \mathrm{E}_j$. A beautiful exposition of the following description (Theorem 2.9) of the cell C_δ can be found in [Gaussent and Littelmann 2012, Proposition 4.19]. We provide an outline of the proof for the benefit of the reader and in order to state Corollary 2.10, which is actually a corollary to its proof.

Theorem 2.9. *The map* $\varphi : \mathbb{U}^\delta = \mathbb{U}^\delta_{\mathrm{V}_0} \times \mathbb{U}^\delta_{\mathrm{V}_1} \times \cdots \times \mathbb{U}^\delta_{\mathrm{V}_k} \to \Sigma_{\gamma^{\mathrm{f}}}$ *given by*

$$(v_0, \ldots, v_k) \mapsto [v_0\delta_0, \delta_0^{-1} v_1 \delta_0 \delta_1, \ldots, (\delta_0 \cdots \delta_{k-1})^{-1} v_k \delta_0 \cdots \delta_k]$$

is injective and has image C_δ.

Proof. Let $\widetilde{\mathbb{U}} = \mathrm{U}_{\mathrm{V}_0} \times \cdots \times \mathrm{U}_{\mathrm{V}_k} / \mathrm{U}_{\mathrm{E}_0} \times \cdots \times \mathrm{U}_{\mathrm{E}_k}$ where

$$(e_0, \ldots, \varepsilon_k) \cdot (v_0, \ldots, v_k) = (v_0 e_0, e_0^{-1} v_1 e_1, \ldots, e_{k-1}^{-1} v_k e_k).$$

The map $(v_0, \ldots, v_k) \mapsto [v_1, \ldots, v_k]$ defines a bijection $\phi : \mathbb{U}^\delta \to \widetilde{\mathbb{U}}$. Indeed, by [Gaussent and Littelmann 2012, Proposition 4.17], $\mathbb{U}_{\mathrm{V}_i}$ is a set of representatives for right cosets of $\mathrm{U}_{\mathrm{E}_j}$ in $\mathrm{U}_{\mathrm{V}_j}$, and hence for $[a_0, \ldots, a_k] \in \widetilde{\mathbb{U}}$ there is a unique $(v_0, \ldots, v_k) \in \mathbb{U}$ such that (for some $e_j \in \mathrm{U}_{\mathrm{E}_j}$) $v_0 e_0 = a_0$, and $v_j e_j = e_{j-1} a_j$, i.e., $\phi((v_0, \ldots, v_k)) = [a_0, \ldots, a_k]$. We use this bijection and consider instead the map $\widetilde{\varphi} := \varphi \circ \phi^{-1}$. Fix $[v_0, \ldots, v_k] \in \widetilde{\mathbb{U}}$. The map $\widetilde{\varphi}$ is well defined because $(\delta_0 \cdots \delta_{j-1})^{-1} v_{ij} (\delta_0 \cdots \delta_j) \in \mathrm{P}_{\mathrm{V}_j^f}$, and if $e_j \in \mathrm{U}_{\mathrm{E}_j}$ then $(\delta_0 \cdots \delta_j)^{-1} e_j (\delta_0 \cdots \delta_j) \in \mathrm{U}_{\mathrm{E}_j^f}$. Since by [Gaussent and Littelmann 2005, Proposition 1] the fibers of r_∞ are $\mathrm{U}(\mathscr{K})$-orbits, an element $p = [p_0, \ldots, p_k] \in \Sigma_{\gamma^f}$ belongs to C_δ if and only if there exist elements $u_0, \ldots, u_k \in \mathrm{U}(\mathscr{K})$ such that

(1) $p_0 \cdots p_j \mathrm{E}^f_j = u_j \mathrm{E}_j$ and

(2) $u_{j-1} \mathrm{V}_j = u_j \mathrm{V}_j$.

Define $u_0 = v_0$ and $u_j = v_0 \cdots v_j$. Then conditions (1) and (2) above hold for

$$p_j = (\delta_0 \cdots \delta_{j-1})^{-1} v_j (\delta_0 \cdots \delta_j).$$

Hence the image of the map is contained in the cell C_δ. For the other inclusion, define $v_j = u_{j-1}^{-1} u_j$ (see [Gaussent and Littelmann 2012, Proposition 4.19]). To show injectivity assume $\widetilde{\varphi}([v_0, \dots, v_k]) = \widetilde{\varphi}([v'_0, \dots, v'_k])$. Then there exist elements $e_j \in \mathrm{U}_{\mathrm{E}_j}$ such that $v_0 \cdots v_j = v'_0 \cdots v'_j e_j$, this implies injectivity. $\square$

The following corollary can be found in [Gaussent et al. 2013, Corollary 3] for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$. Note that in particular it implies that $u\pi(\mathrm{C}_\delta) = \pi(\mathrm{C}_\delta)$ for all $u \in \mathrm{U}_{\mathrm{V}_0}$.

Corollary 2.10. $\pi(\mathrm{C}_\delta) = \mathbb{U}^\delta_{\mathrm{V}_0} \cdots \mathbb{U}^\delta_{\mathrm{V}_k}[t^{\mu_\delta}] = \mathrm{U}_{\mathrm{V}_0} \cdots \mathrm{U}_{\mathrm{V}_k}[t^{\mu_\delta}]$.

Proof. By the arguments in the proof of Theorem 2.9 the image of the map

$$\begin{aligned} \mathrm{U}_{\mathrm{V}_0} \times \cdots \times \mathrm{U}_{\mathrm{V}_k} &\to \Sigma_{\gamma^f} \\ (v_0, \dots, v_k) &\mapsto [v_0\delta_0, \delta_0^{-1} v_1 \delta_0 \delta_1, \dots, \delta_0 \cdots \delta_{r-1}^{-1} v_k \delta_0 \cdots \delta_k] \end{aligned}$$

is contained in and is surjective onto the cell C_δ. In particular conditions (1) and (2) above are satisfied for $p_j = (\delta_0 \cdots \delta_{j-1})^{-1} v_j (\delta_0 \cdots \delta_j)$. The corollary follows since $\delta_0 \cdots \delta_j \mu_{\gamma^f} = \mu_\delta$. $\square$

3. Crystal structure on combinatorial galleries, the Littelmann path model, and Lakshmibai–Seshadri galleries

Let $\lambda \in \mathrm{X}^{+,\vee}$ be a dominant integral coweight and let $\mathrm{L}(\lambda)$ be the corresponding simple module of $\mathrm{G}^\vee$. To $\mathrm{L}(\lambda)$ is associated a certain graph $\mathrm{B}(\lambda)$ that is its "combinatorial model". It is a connected *highest weight* crystal, which means that there exists $b_\lambda \in \mathrm{B}(\lambda)$ such that $e_{\alpha_i}(b_\lambda) = 0$ for all $i \in \{1, \dots, n\}$, where n is the rank of the corresponding root datum. The crystal $\mathrm{B}(\lambda)$ also has the characterizing property that

$$\dim(\mathrm{L}(\lambda)_\mu) = \#\{b \in \mathrm{B}(\lambda) : \mathrm{wt}(b) = \mu\}.$$

See below for definitions. After recalling the notion of a crystal we review the crystal structure on the set Γ of combinatorial galleries.

3A. *Crystals.* A *crystal* is a set B together with maps

$$\begin{aligned} e_{\alpha_i}, f_{\alpha_i} &: \mathrm{B} \to \mathrm{B} \cup \{0\} && \text{(the } \textit{root operators}\text{)}, \\ \mathrm{wt} &: \mathrm{B} \to \mathrm{X}^\vee && \text{(the } \textit{weight function}\text{)}, \end{aligned}$$

for $i \in \{1, \dots, n\}$, such that for every $b, b' \in \mathrm{B}$; $b' = e_{\alpha_i}(b)$ if and only if $b = f_{\alpha_i}(b')$, and, in this case, setting

$$\varepsilon_i(b'') = \max\{n : e^n_{\alpha_i}(b'') \neq 0\} \quad \text{and} \quad \phi_i(b'') = \max\{n : f^n_{\alpha_i}(b'') \neq 0\}$$

for any $b'' \in \mathrm{B}$, we have

$$\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i^\vee \quad \text{and} \quad \phi_i(b) = \varepsilon_i(b) + \langle \alpha_i, \mathrm{wt}(b) \rangle.$$

A crystal is in particular a graph, which we may decompose into the disjoint union of its connected components. Each element $b \in \mathrm{B}$ lies in a unique connected component which we will denote by $\mathrm{Conn}(b)$. A *crystal morphism* is a map $\mathrm{F}: \mathrm{B} \to \mathrm{B}'$ between the underlying sets of crystals B and B′ such that $\mathrm{wt}(\mathrm{F}(b)) = \mathrm{wt}(b)$ and such that it commutes with the action of the root operators. A crystal morphism is an isomorphism if it is bijective.

3B. *Crystal structure on combinatorial galleries.*

Definition 3.1. For each $i \in \{1, \dots, n\}$ and each simple root α_i, we recall the definition of the root operators f_{α_i} and e_{α_i} on the set of combinatorial galleries Γ and endow this set with a crystal structure. We follow [Gaussent and Littelmann 2005, §6; Braverman and Gaitsgory 2001, §1], and refer the reader to [Kashiwara 1995] for a detailed account of the theory of crystals.

Let $\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \dots, \mathrm{E}_k, \mathrm{V}_{k+1})$ be a combinatorial gallery. Define a weight function by $\mathrm{wt}(\gamma) = \mu_\gamma$. Let $m_{\alpha_i} = m \in \mathbb{Z}$ be minimal such that $\mathrm{V}_p \in \mathrm{H}_{(\alpha_i,m)}$ for some $p \in \{0, \dots, k+1\}$. Note that $m \le 0$.

Definition of f_{α_i}. Suppose $\langle \alpha_i, \mu_\gamma \rangle \ge m+1$. Let j be maximal such that $\mathrm{V}_j \in \mathrm{H}_{(\alpha_i,m)}$ and let $j < r \le k+1$ be minimal such that $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i,m+1)}$. Let

$$\mathrm{E}'_p = \begin{cases} \mathrm{E}_p & \text{if } p < j, \\ s_{(\alpha_i,m)}(\mathrm{E}_p) & \text{if } j \le p < r, \\ t_{-\alpha_i^\vee}(\mathrm{E}_p) & \text{if } r \le p. \end{cases}$$

Define $\mathrm{V}'_0 = 0$, and for $1 \le p \le k$, set $\mathrm{V}'_p = \mathrm{E}'_{p-1} \cap \mathrm{E}'_p$, and let V'_{k+1} be the extreme point of the line segment E'_k that is not V'_k. Define

$$f_{\alpha_i}(\gamma) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \dots, \mathrm{E}'_k, \mathrm{V}'_{k+1}),$$

and if $\langle \alpha_i, \mu_\gamma \rangle < m+1$, then $f_{\alpha_i}(\gamma) = 0$.

Definition of e_{α_i}. Suppose $m \le -1$. Let r be minimal such that the $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i,m)}$ and let $0 \le j < r$ be maximal such that $\mathrm{V}_j \in \mathrm{H}_{(\alpha_i,m+1)}$. Let

$$\mathrm{E}'_p = \begin{cases} \mathrm{E}_p & \text{if } p < j, \\ s_{(\alpha_i,m+1)}(\mathrm{E}_p) & \text{if } j \le p < r, \\ t_{\alpha_i^\vee}(\mathrm{E}_p) & \text{if } r \le p, \end{cases}$$

define V'_p as above and define

$$e_{\alpha_i}(\gamma) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \dots, \mathrm{E}'_k, \mathrm{V}'_{k+1}).$$

If $m = 0$ then $e_{\alpha_i}(\gamma) = 0$.

Remark 3.2. It follows from the definitions that the maps e_{α_i}, f_{α_i} and wt define a crystal structure on Γ. Note as well that if γ is a combinatorial gallery then $f_{\alpha_i}(\gamma)$

and $e_{\alpha_i}(\gamma)$ are combinatorial galleries of the same type as γ (as long as they are not zero). We say that the root operators are type preserving. See also [Gaussent and Littelmann 2005, Lemma 6].

3C. ***The Littelmann path model and Lakshmibai–Seshadri galleries; readable galleries.*** Let γ be a combinatorial gallery that has each one of its faces contained in the fundamental chamber. We call such galleries *dominant* and denote the set of all dominant combinatorial galleries by Γ^{dom}. By [Littelmann 1995, Theorem 7.1] the crystal of galleries $\mathrm{P}(\gamma)$ generated by γ is isomorphic to the crystal $\mathrm{B}(\mu_\gamma)$ associated to the irreducible highest weight representation $\mathrm{L}(\mu_\gamma)$ of $\mathrm{G}^\vee$. In its original context [Littelmann 1995] it is known as a *Littelmann path model* for the representation $\mathrm{L}(\mu_\gamma)$. We say that a combinatorial gallery γ is a *Littelmann gallery* if there exist indices $i_1, \dots, i_r$ such that $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma) = \gamma^+$ is a dominant gallery. If $\mu_{\gamma^+} = \mu_{\delta^+}$, $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma) = \gamma^+$ and $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\delta) = \delta^+$ for two Littelmann galleries γ and δ, we say that they are *equivalent*. This defines an equivalence relation on the set of Littelmann galleries.

Let $\lambda \in \mathrm{X}^{\vee,+}$ be a dominant integral coweight and γ_λ a gallery that is a concatenation of fundamental galleries and that has endpoint λ (as above). We denote by $\Gamma(\gamma_\lambda)^{\mathrm{LS}}$ the set of *combinatorial LS galleries* of the same type as γ_λ. (LS is short for Lakshmibai–Seshadri. All LS galleries are Littelmann — see [Littelmann 1995, §4] — and Littelmann galleries generalize LS galleries enormously.) The set $\Gamma(\gamma_\lambda)^{\mathrm{LS}}$ is stable under the root operators and has the structure of a crystal isomorphic to $\mathrm{B}(\lambda)$. It was proven by Gaussent and Littelmann [2005] that the resolution in (8) induces a bijection $\Gamma(\gamma_\lambda)^{\mathrm{LS}} \cong \mathscr{Z}(\lambda)$. This bijection was shown to be a crystal isomorphism by Baumann and Gaussent [2008]. We use this heavily in the proof of Theorem 6.2. In [Gaussent and Littelmann 2005] see Definition 18 for a geometric definition of LS galleries, and Definition 23 for an equivalent combinatorial characterization that for one skeleton galleries agrees with the original definition by Lakshmibai, Musili and Seshadri (see [Lakshmibai et al. 1998], for example) in the context of standard monomial theory. We will give a combinatorial characterization of LS galleries of fundamental type in the case $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$, omitting therefore the most general definitions.

We finish this section with a question. Let γ be a dominant gallery (see Section 3C). Consider the map $\Sigma_{\gamma^f} \to \mathscr{G}$ defined by $[g_0, \dots, g_r] \mapsto g_0 \cdots g_r[t^{\mu_{\gamma^f}}]$ (see Remark 2.6).

Question. Does this map induce a crystal isomorphism $\mathrm{P}(\gamma) \cong \mathscr{Z}(\mu_\gamma)$?

This question was answered positively in [Gaussent et al. 2013; Torres 2016] for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$. In the rest of this paper we do so as well for $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$ and γ a *readable* gallery. For $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$ all galleries are readable. This is due to the well known fact that in this case fundamental coweights are all minuscule. In the next sections we will describe readable galleries explicitly for $G^\vee = \mathrm{Sp}(2n, \mathbb{C})$ and show that they are Littelmann galleries. Moreover, we will see there exist readable galleries that are not of the same type as any concatenation of fundamental galleries γ_λ (see Remark 4.9).

Definition 3.3. A *readable* gallery is a concatenation of its *parts*. Its parts are either LS galleries of fundamental type or galleries of the form $(V_0, E_0, V_1, E_1, V_2)$ (we call them *zero lumps*) such that both edges E_0 and E_1 are contained in the dominant chamber and such that the endpoint V_2 is equal to zero. We denote the set of all readable galleries by Γ^R, and if a combinatorial gallery γ is fixed, by $\Gamma(\gamma)^R$, the set of all readable galleries of same type as γ.

Remark 3.4. It follows from [Gaussent and Littelmann 2005, Lemma 8] that readable galleries are stable under root operators.

4. "Type C" combinatorics

4A. *Weights and coweights.* Consider $\mathbb{R}^n$ with canonical basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ and standard inner product $\langle -,- \rangle$. In particular $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. From now on we consider the root datum $(X, \Phi, X^\vee, \Phi^\vee)$ defined by

$$\begin{aligned} \Phi &= \{\pm\varepsilon_i, \varepsilon_i \pm \varepsilon_j\}_{i,j \in \{1,\dots,n\}}, \\ \Phi^\vee &= \left\{\alpha^\vee = \tfrac{2\alpha}{\langle \alpha,\alpha \rangle} : \alpha \in \Phi\right\}, \\ X &= \{v \in \mathbb{R}^n : \langle v, \alpha^\vee \rangle \in \mathbb{Z}\}, \\ X^\vee &= \{v \in \mathbb{R}^n : \langle \alpha, v \rangle \in \mathbb{Z}\}. \end{aligned}$$

Indeed the sets X and $X^\vee$ are free abelian groups which form a root datum together with the pairing $\langle -,- \rangle$ between them and the subsets $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$. We choose a basis $\Delta \subset \Phi$ given by

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : i \in \{1, \dots, n-1\}\} \cup \{\alpha_n = \varepsilon_n\},$$

hence the set

$$\Delta^\vee = \{\alpha_i^\vee = \varepsilon_i - \varepsilon_{i+1} : i \in \{1, \dots, n-1\}\} \cup \{\alpha_n^\vee = 2\varepsilon_n\}$$

is a basis for the root system $\Phi^\vee$. Then $X^\vee$ has a $\mathbb{Z}$-basis given by the set of corresponding fundamental coweights $\{\omega_i\}_{i \in \{1,\dots,n\}}$, where

$$\omega_i = \varepsilon_1 + \dots + \varepsilon_i \quad 1 \le i \le n.$$

Then $G = \mathrm{SO}(2n+1, \mathbb{C})$ and $G^\vee = \mathrm{Sp}(2n, \mathbb{C})$. For later use we introduce the notation $\varepsilon_{\bar{i}} = -\varepsilon_i$.

4B. ***Symplectic keys and words.*** Let $p \in \mathbb{Z}_{\geq 1}$ be an integer, greater than or equal to 1. To it we associate a sequence of positive integers $\underline{p}$ as follows:

$$\underline{p} = \begin{cases} (1) & \text{if } p = 1, \\ (p, p) & \text{if } p \geq 2. \end{cases}$$

Given two sequences of integers $\underline{a} = (a_1, \ldots, a_r)$ and $\underline{b} = (b_1, \ldots, b_s)$ we denote the associated merged list by $\underline{a} * \underline{b} = (a_1, \ldots, a_r, b_1, \ldots, b_s)$. A *symplectic shape* $\underline{d}$ is a sequence of natural numbers of the form $\underline{d} = \underline{p_1} * \cdots * \underline{p_l}$, where $p_i \in \mathbb{Z}_{\geq 1}$. An *arrangement of boxes* of symplectic shape $\underline{d}$ is an arrangement of as many columns of boxes as elements in the sequence $\underline{d}$ such that column j (read from right to left) has p_j boxes.

Example 4.1. An arrangement of boxes of symplectic shape $\underline{1} * \underline{1} * \underline{2} * \underline{1}$.

$$\begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & \ \\ \hline \multicolumn{1}{c|}{} & \ & \ & \multicolumn{2}{c}{} \\ \cline{2-3} \end{array}$$

Consider the ordered alphabet $\mathscr{C}_n = \{1 < 2 < \cdots < n{-}1 < n < \bar{n} < \overline{n{-}1} < \cdots < \bar{1}\}$. A *symplectic key* of (symplectic) shape $\underline{d} = \underline{p_1} * \cdots * \underline{p_l}$ is a filling of an arrangement of boxes of symplectic shape $\underline{d}$ with letters of the alphabet $\mathscr{C}_n$ in such a way that the entries are strictly increasing along each column and such that $p_j \leq n$ for $j \in \{1, \ldots, l\}$.

Example 4.2. A symplectic key, for $n \geq 5$, of symplectic shape $\underline{1} * \underline{2} * \underline{1}$.

$$\begin{array}{|c|c|c|c|} \hline \bar{1} & 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 5 & \bar{2} & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

We denote the *word monoid* on $\mathscr{C}_n$ by $\mathscr{W}_{\mathscr{C}_n}$. To a word $w = w_1 \cdots w_k$ in $\mathscr{W}_{\mathscr{C}_n}$ we associate a symplectic key $\mathscr{K}_w$ that consists of only one row of length k, and with the boxes filled in from right to left with the letters of w read in turn from left to right. For example, the word 12 corresponds to the key $\begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array}$.

4C. ***Readable keys: symplectic keys associated to readable galleries.*** The aim of this section is to assign a symplectic key to every readable gallery. For a subset $\mathrm{Y} \subseteq \mathscr{C}_n$, we denote the corresponding subset of barred elements by $\overline{\mathrm{Y}} = \{\bar{y} : y \in \mathrm{Y}\}$, where, for i unbarred, $\bar{\bar{i}} = i$.

Definition 4.3. Let $\mathscr{B}$ be a symplectic key. We call $\mathscr{B}$ an *LS block* if it is of shape $\underline{p}$ for $p \in \mathbb{Z}_{\geq 1}$ and such that if $p \geq 2$ (which means that $\mathscr{B}$ consists of two columns of size p) there exist positive integers k, r, s with $2k + r + s \leq n$ and disjoint sets of positive integers

$$\begin{aligned}
\mathrm{A} &= \{a_i : 1 \le i \le r,\, a_1 < \cdots < a_r\},\\
\mathrm{B} &= \{b_i : 1 \le i \le s,\, b_1 < \cdots < b_s\},\\
\mathrm{Z} &= \{z_i : 1 \le i \le k,\, z_1 < \cdots < z_k\},\\
\mathrm{T} &= \{t_i : 1 \le i \le k,\, t_1 < \cdots < t_k\},
\end{aligned}$$

such that the right column of $\mathscr{B}$ (respectively the left one) is the column with entries the ordered elements of the set $\overline{\mathrm{T}} \cup \mathrm{Z} \cup \mathrm{A} \cup \overline{\mathrm{B}}$ (respectively $\overline{\mathrm{Z}} \cup \mathrm{T} \cup \mathrm{A} \cup \overline{\mathrm{B}}$), $\mathrm{Z} = \varnothing$ if and only if $\mathrm{T} = \varnothing$, and such that if $\mathrm{Z} \neq \varnothing$ the elements of T are uniquely characterized by the properties

$$t_k = \max\{t \in \mathscr{C}_n : t < z_k,\, t \notin \mathrm{Z} \cup \mathrm{A} \cup \mathrm{B}\}, \tag{9}$$

$$t_{j-1} = \max\{t \in \mathscr{C}_n : t < \min(z_{j-1}, t_j),\, t \notin \mathrm{Z} \cup \mathrm{A} \cup \mathrm{B}\} \text{ for } j \le k. \tag{10}$$

We say that $\mathscr{B}$ is a *zero block* if it is of shape $\underline{k}$ for $k \in \mathbb{Z}_{\ge 1}$ and such that its right column is filled in with the ordered letters $1 < \cdots < k$ and its left one, with $\bar{k} < \cdots < \bar{1}$. A symplectic key is called a *readable block* if it is either an LS block or a zero block. Note that a readable block has symplectic shape $\underline{p}$, where $p \in \mathbb{Z}_{\ge 1}$. A *readable key* is a concatenation of readable blocks. Now assume that $\underline{d} = \underline{p_1} * \cdots * \underline{p_l}$ is such that $p_1 \le \cdots \le p_l$. A symplectic key of shape $\underline{d}$ is called an *LS symplectic key* if its entries are weakly increasing in rows and if it is a concatenation of LS blocks. We denote the set of LS symplectic keys of shape $\underline{d}$ by $\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}}$.

Example 4.4. The symplectic key

1	2
3	3
5	5
$\bar{4}$	$\bar{4}$
$\bar{2}$	$\bar{1}$

is an LS block of shape $\underline{5} = (5, 5)$, with $\mathrm{A} = \{3, 4\}$, $\mathrm{B} = \{4\}$, $\mathrm{Z} = \{2\}$ and $\mathrm{T} = \{1\}$. The first symplectic key immediately below is not an LS block; the second is a zero block.

1	$\bar{2}$
2	$\bar{1}$

$\bar{2}$	1
$\bar{1}$	2

Remark 4.5. A pair of columns that form an LS block is sometimes called a pair of admissible columns. The original definition of admissible columns was given by De Concini [1979], using a slightly different convention than Kashiwara and Nakashima's, which is the one we use here. The map that translates the two can be found in [Lecouvey 2002, §2.2].

To a readable block $\mathscr{B}$ we assign a gallery $\gamma_{\mathscr{B}}$ as follows. If $\mathscr{B}$ consists of only one box filled in with the letter $l \in \mathscr{C}_n$, then we define $\mathrm{V}_0^{\mathscr{B}} = 0$, $\mathrm{V}_1^{\mathscr{B}} = \varepsilon_l$, $\mathrm{E}_0^{\mathscr{B}} = \{t\mathrm{V}_1^{\mathscr{B}} : t \in [0, 1]\}$, and

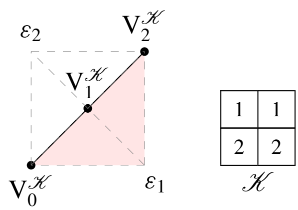
$$\gamma_{\mathscr{B}} = (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}).$$

If the readable block $\mathscr{B}$ has at least two boxes, then its columns are filled in with the letters $l_1^1 < \cdots < l_d^1$ (right column) and $l_1^2 < \cdots < l_d^2$ (left column) respectively. We then define

$$\begin{aligned}
\mathrm{V}_0^{\mathscr{B}} &= 0,\\
\mathrm{V}_1^{\mathscr{B}} &= \tfrac{1}{2}(\varepsilon_{l_1^1} + \cdots + \varepsilon_{l_d^1}),\\
\mathrm{E}_0^{\mathscr{B}} &= \{t\mathrm{V}_1^{\mathscr{T}} : t \in [0, 1]\},\\
\mathrm{V}_2^{\mathscr{B}} &= \varepsilon_{l_1^1} + \cdots + \varepsilon_{l_d^1} + \varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2},\\
\mathrm{E}_1^{\mathscr{B}} &= \{\mathrm{V}_1^{\mathscr{T}} + \tfrac{1}{2}t(\varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2}) : t \in [0, 1]\},\\
\gamma_{\mathscr{B}} &= (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}, \mathrm{E}_1^{\mathscr{B}}, \mathrm{V}_2^{\mathscr{B}}).
\end{aligned}$$

Note that (9) implies that $\mathrm{V}_1^{\mathscr{B}} + \frac{1}{2}(\varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2}) = \mathrm{V}_2^{\mathscr{B}}$ and therefore that $\mathrm{E}_1^{\mathscr{B}}$ is the line segment joining $\mathrm{V}_1^{\mathscr{B}}$ and $\mathrm{V}_2^{\mathscr{B}}$.

Example 4.6. Let $n=2$ and $\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2)$ where $\mathrm{V}_0 = 0$, $\mathrm{V}_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$, $\mathrm{V}_2 = \varepsilon_1 + \varepsilon_2$ and the edges are the line segments joining the vertices in order. See below for a picture of the gallery $\gamma_{\mathscr{B}}$ associated to the symplectic key $\mathscr{B}$.



$$\gamma_{\mathscr{K}} = (\mathrm{V}_0^{\mathscr{K}}, \mathrm{E}_0^{\mathscr{K}}, \mathrm{V}_1^{\mathscr{K}}, \mathrm{E}_1^{\mathscr{K}}, \mathrm{V}_2^{\mathscr{K}})$$

To a readable key $\mathscr{K} = \mathscr{B}_1 \cdots \mathscr{B}_k$ we associate the concatenation

$$\gamma_{\mathscr{K}} = \gamma_{\mathscr{B}_k} * \cdots * \gamma_{\mathscr{B}_1}$$

of the galleries of each of the readable blocks $\mathscr{B}_j$, for $j \in \{1, \ldots, k\}$, that it is a concatenation of (from right to left). To a symplectic shape $\underline{d} = \underline{p}_1 * \cdots * \underline{p}_l$ such that $p_j \le n$ for $j \in \{1, \ldots, l\}$ (once n is fixed, we will only consider such shapes) we associate the dominant coweight $\lambda_{\underline{d}} = \omega_{p_1} + \cdots + \omega_{p_l}$. For example, to the shape $(2, 2)$ is associated the coweight ω_2. We will denote the set of all readable keys of shape $\underline{d}$ by $\mathbf{\Gamma}(\underline{d})^{\mathrm{R}}$.

Remark 4.7. The set $\mathbf{\Gamma}(\underline{d})^{\mathrm{R}}$ is nonempty: since $p_j \leq n$, there is a natural readable key of symplectic shape $\underline{d}$ whose columns are filled in with consecutive integers, starting with 1 at the top. For example, if $\underline{d} = \underline{3} = (3, 3)$ and $n \geq 3$, this is the key

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}\,.$$

It is an LS block, with $\mathrm{A} = \{1, 2, 3\}$ and $\mathrm{B} = \mathrm{Z} = \mathrm{T} = \varnothing$.

The following proposition follows directly from [Gaussent and Littelmann 2012, Lemma 2].

Proposition 4.8. *The map*

$$\bigcup_{\substack{\underline{d}=\underline{p}_1\cdots\underline{p}_l \\ p_j \leq n}} \mathbf{\Gamma}(\underline{d})^{\mathrm{R}} \to \Gamma^{\mathrm{R}}, \qquad \mathscr{K} \mapsto \gamma_{\mathscr{K}}$$

is well defined and is a bijection. Moreover, if $p_1 \leq \cdots \leq p_l$ then this map induces a bijection

$$\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}} \longleftrightarrow \Gamma(\gamma_{\omega_{p_1}} * \cdots * \gamma_{\omega_{p_m}})^{\mathrm{LS}}.$$

Remark 4.9. Zero lumps are not necessarily of fundamental type: this follows from [Gaussent and Littelmann 2012, Lemma 2] for a zero lump with odd k in the above description. This is why readable galleries are not necessarily of the same type as a concatenation of fundamental galleries. This also means that there can be two readable keys of the same shape but such that their associated galleries are not of the same type! For example, take $n > 3$, and consider the keys

$$\mathscr{T} = \begin{array}{|c|c|} \hline \bar{1} & \bar{1} \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \text{and} \quad \mathscr{K} = \begin{array}{|c|c|} \hline 1 & \bar{1} \\ \hline \bar{2} & 2 \\ \hline \bar{3} & 3 \\ \hline \end{array}\,.$$

The first is LS and $\gamma_{\mathscr{T}}$ is of fundamental type ω_3. The second key is a zero block. Its associated gallery, $\gamma_{\mathscr{K}}$, is not of fundamental type.

5. The word of a readable gallery

To a readable key $\mathscr{K}$ we assign a word $w(\mathscr{K})$. The first aim of this section is to state Proposition 5.5, which says that the closure in the affine Grassmannian of the image $\pi(\mathrm{C}_{\gamma_{\mathscr{K}}}) \subset \mathscr{G}$ considered in Section 2F depends only on the word $w(\mathscr{K})$.

Definition 5.1. The *word* of a readable block, $\mathscr{B} = \mathrm{C_L C_R}$ ($\mathrm{C_L}$ is the left column, $\mathrm{C_R}$ the right), is obtained by reading first the unbarred entries in $\mathrm{C_R}$ and then the barred entries in $\mathrm{C_L}$. We denote it by $w(\mathscr{B}) \in \mathscr{W}_{\mathscr{C}_n}$.

Remark 5.2. For an LS block this is the word of the associated single admissible column defined by Kashiwara and Nakashima [Lecouvey 2002, Example 2.2.6].

Definition 5.3. Let $\gamma_{\mathscr{K}}$ be the readable gallery associated to the key $\mathscr{K}$. As before, we may write $\mathscr{K}$ as a concatenation of blocks $\mathscr{K} = \mathscr{B}_1 \cdots \mathscr{B}_k$. The *word* of $\gamma_{\mathscr{K}}$ (or of $\mathscr{K}$) is $w(\mathscr{B}_k) \cdots w(\mathscr{B}_1)$. We denote it by $w(\gamma_{\mathscr{K}})$ (or $w(\mathscr{K})$).

Example 5.4. Let

$$\mathscr{B}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}, \quad \mathscr{B}_2 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}, \quad \text{and} \quad \mathscr{K} = \mathscr{B}_1\mathscr{B}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline \bar{2} & \bar{1} \\ \cline{1-2} \end{array}.$$

Then $w(\mathscr{B}_1) = 2\bar{2}$, $w(\mathscr{B}_2) = 1$, and $w(\mathscr{K}) = 12\bar{2}$.

We have the following result about words of readable galleries, which we prove in Section 7. We will use it in Theorem 6.2. It is in this sense that such galleries are called *readable*.

Proposition 5.5. *Let γ and ν be combinatorial galleries and $\mathscr{K}$ be a readable key. Consider the combinatorial galleries $\gamma * \gamma_{w(\mathscr{K})} * \nu$ and $\gamma * \gamma_{\mathscr{K}} * \nu$. Let $(\Sigma_{(\gamma*\gamma_{w(\mathscr{K})}*\nu)^f}, \pi)$ and $(\Sigma_{(\gamma*\gamma_{\mathscr{K}}*\nu)^f}, \pi')$ be the Bott–Samelson varieties together with their maps to the affine Grassmannian $\mathscr{G}$ (as in Remark 2.6). Then*

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K})}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}}*\nu})}.$$

5A. *Word galleries.* We associate a (readable!) gallery γ_w of the same type as the m-fold product $\gamma_{\omega_1} * \cdots * \gamma_{\omega_1}$ to a word $w \in \mathscr{W}_{\mathscr{C}_n}$ of length m — it is the gallery $\gamma_{\mathscr{K}_w}$ associated to the readable key $\mathscr{K}_w$. We denote the set of word galleries in this case by $\Gamma_{\mathscr{W}_{\mathscr{C}_n}}$. Below we recall the crystal structure on the set $\mathscr{W}_{\mathscr{C}_n}$ as described by Kashiwara and Nakashima [1994, Proposition 2.1.1]. The set of words $\mathscr{W}_{\mathscr{C}_n}$, just like the set $\mathscr{W}_n$, is in one-to-one correspondence with the set of vertices of the crystal of the representation $\bigoplus_{l\in\mathbb{Z}_{\geq 0}} \mathrm{V}_n^{\otimes l}$, where V_n is the natural representation $\mathrm{L}(\omega_1)$ and hence inherits its crystal structure. Proposition 5.7 says that this structure is compatible with the crystal structure defined on galleries in Section 3.

Definition 5.6. Let $w = w_1 \cdots w_l \in \mathscr{C}_n$ be a word and $i \in \{1, \dots, n\}$. Define $\mathrm{wt}(w) = \sum_{i=1}^{l} \varepsilon_i$. To apply the root operators e_{α_i} and f_{α_i} to w one first obtains a word consisting of letters in the alphabet $\{+, -, \varnothing\}$. The word will be obtained from w by replacing every occurrence of i or $\overline{i+1}$ by "$+$", every occurrence of $i+1$ or $\bar{i}$ by "$-$" and all other letters by "$\varnothing$". This word, which we denote by $s_i(w)$ is sometimes called the i-signature of w. To proceed, erase all symbols $\varnothing$ and then all subwords of the form "$+-$". Repeat this process until the i-signature $s_i(w)$ of w has been reduced to a word of the form

$$s_i(w)' = (-)^r(+)^s.$$

To apply f_{α_i} (respectively e_{α_i}) to w, change the letter whose tag corresponds to the leftmost "+" (respectively to the rightmost "−") from i to $i+1$ and from $\overline{i+1}$ to $\bar{i}$ (respectively from $i+1$ to i and from $\bar{i}$ to $\overline{i+1}$). If $s=0$, respectively $r=0$, then $f_{\alpha_i}(w)=0$, respectively $e_{\alpha_i}(w)=0$.

Proposition 5.7. *The crystal structure on words from Definition 5.6 coincides with the one induced from Definition 3.1.*

For a proof, see [Littelmann 1996, §13]. It also follows directly from the definitions.

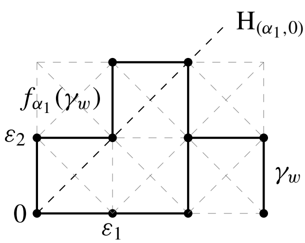
Example 5.8. Let $n=2$ and $w=1121\bar{2}$. We first consider $i=1$, for which $s_1(w)=++-++$, and therefore $s_1'(w)=+++$. Hence $f_{\alpha_1}(w)=2121\bar{2}$ and $e_{\alpha_1}(w)=0$. For $i=2$ we have $s_2(w)=\varnothing\varnothing+\varnothing-$. Therefore s_2' is the empty word and $f_{\alpha_2}(w)=e_{\alpha_2}(w)=0$. Now consider the readable gallery $\gamma_{\mathscr{K}_w}$ associated to w. Explicitly we write it as

$$\gamma_w=(\mathrm{V}_0,\mathrm{E}_0,\mathrm{V}_1,\mathrm{E}_1,\mathrm{V}_2,\mathrm{E}_2,\mathrm{V}_3,\mathrm{E}_3,\mathrm{V}_4,\mathrm{E}_4,\mathrm{V}_5),$$

where $\mathrm{V}_0=0$, $\mathrm{V}_1=\varepsilon_1$, $\mathrm{V}_2=2\varepsilon_1$, $\mathrm{V}_3=2\varepsilon_1+\varepsilon_2$, $\mathrm{V}_4=3\varepsilon_1+\varepsilon_2$, $\mathrm{V}_5=3\varepsilon_1$ and E_j is the line segment joining V_j to V_{j+1} for $j\in\{0,\dots,4\}$. We have $m_{\alpha_1}=0$, so by Definition 3.1, $e_{\alpha_1}(\gamma_w)=0$. We have $s_{(\alpha_1,0)}(\mathrm{E}_0)=\{t\varepsilon_2 : t\in[0,1]\}$, see below. Then $j=1$ (Definition 3.1) and hence

$$f_{\alpha_1}(\gamma_w)=(\mathrm{V}_0',\mathrm{E}_0',\mathrm{V}_1',\mathrm{E}_1',\mathrm{V}_2',\mathrm{E}_2',\mathrm{V}_3',\mathrm{E}_3',\mathrm{V}_4',\mathrm{E}_4',\mathrm{V}_5'),$$

where $\mathrm{V}_0'=0$, $\mathrm{V}_1'=\varepsilon_2$, $\mathrm{V}_2'=\varepsilon_2+\varepsilon_1$, $\mathrm{V}_3'=2\varepsilon_2+\varepsilon_1$, $\mathrm{V}_4'=2\varepsilon_2+2\varepsilon_1$, $\mathrm{V}_5'=\varepsilon_2+2\varepsilon_1$ and E_j' is the line segment joining V_j' and V_{j+1}' for $j\in\{0,\dots,4\}$. For $i=2$ we have $m_{\alpha_2}=0$, which implies that $e_{\alpha_2}(\gamma_w)=0$. We also have $\mu_{\gamma_w}=3\varepsilon_1$, and therefore $\langle\alpha_2,\mu_{\gamma_w}\rangle=0<m_{\alpha_2}+1=1$, so that $f_{\alpha_2}(\gamma_w)=0$ as well. Then $f_{\alpha_1}(\gamma_w)=\gamma_{f_{\alpha_1}(w)}$, $e_{\alpha_1}(\gamma_w)=\gamma_{e_{\alpha_1}(w)}$, $f_{\alpha_2}(\gamma_w)=\gamma_{f_{\alpha_2}(w)}$ and $e_{\alpha_2}(\gamma_w)=\gamma_{e_{\alpha_2}(w)}$.



5B. ***Word reading is a crystal morphism.*** This subsection is the "symplectic" version of [Torres 2016, Proposition 2.5]. Since the root operators are type preserving (see Definition 3.1), the set of words $\mathscr{W}_{\mathscr{C}_n}$ is naturally endowed with a crystal structure. The following proposition will be useful in Theorem 6.2. This result was shown for LS blocks by Kashiwara and Nakashima [1994, Proposition 4.3.2]. They show that word reading induces an isomorphism of crystals from $\mathrm{B}(\omega_k)$ onto the subcrystal of $\bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \mathrm{B}(\omega_1)^{\otimes l}$ generated by the tensor product $\boxed{k} \otimes\cdots\otimes \boxed{1}$. We show that for readable galleries the proof is reduced to this case.

Proposition 5.9. *The map*

$$\Gamma^{\mathrm{R}} \xrightarrow{w} \Gamma_{\mathscr{W}_{\mathscr{C}_n}}, \qquad \gamma_{\mathscr{K}} \mapsto \gamma_{w(\mathscr{K})}$$

is a crystal morphism.

Proof. First note that the map is weight preserving. This follows from the definitions and from the fact that in the definition of a readable block, the sets Z and T do not contribute to the endpoint of the associated gallery. Let γ be a readable gallery and let

$$\gamma_{\mathscr{B}} = (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}, \mathrm{E}_1^{\mathscr{B}}, \mathrm{V}_2^{\mathscr{B}})$$

be one of its parts, associated to some readable block $\mathscr{B}$. We write

$$\gamma_{w(\mathscr{B})} = (\mathrm{V}_0^{\mathscr{K}_{w(\mathscr{B})}}, \mathrm{E}_0^{\mathscr{K}_{w(\mathscr{B})}}, \dots, \mathrm{V}_{r+s}^{\mathscr{K}_{w(\mathscr{B})}}).$$

If

$$w(\mathscr{B}) = g_1\cdots g_s \bar{h}_k \cdots \bar{h}_1,$$

for g_i and h_i unbarred, then $\mathrm{V}_0^{\mathscr{K}_{w(\mathscr{B})}} = 0$ and $\mathrm{V}_j^{\mathscr{K}_{w(\mathscr{B})}} = \sum_{i=1}^{j} \varepsilon_{x_i}$ for $1 \leq j \leq s+r$, where $x_i = g_i$ for $1\leq i\leq s$ and $x_{s+i} = \bar{h}_i$ for $1 \leq i \leq k$. Let

$$h(j) = \langle \alpha, \mathrm{V}_j^{\mathscr{B}}\rangle \quad \text{and} \quad h'(j) = \langle \alpha, \mathrm{V}_j^{\mathscr{K}_{w(\mathscr{B})}}\rangle,$$

for $1 \leq j \leq k+s+1$. Then there exist d_1, d_2 with $d_1 \leq s < d_2 \leq s+k$ and such that

$$h'(j) = \begin{cases} h(0) & \text{for } 0 \leq j < d_1, \\ h(1) & \text{for } d_1 \leq j < d_2, \\ h(2) & \text{for } d_2 \leq j \leq k+s+1. \end{cases}$$

From this we conclude that it is enough to consider readable blocks. As mentioned previously, this was shown in [Kashiwara and Nakashima 1994] for LS blocks. Hence let $\mathscr{L}$ be a zero lump — it has word $w(\mathscr{L}) = 1\cdots k\bar{k}\cdots\bar{1}$ — and let α_i be a simple root. Then, since the galleries associated to $\mathscr{L}$ and $w(\mathscr{L})$ are both dominant, $f_{\alpha_i}(\mathscr{L}) = e_{\alpha_i}(\mathscr{L}) = f_{\alpha_i}(w(\mathscr{L})) = e_{\alpha_i}(w(\mathscr{L})) = 0$. □

Example 5.10. Let $n = 2$ and $\mathscr{B}$ be the readable block $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline\end{array}$. Then $w(\mathscr{B}) = 2\bar{2}$. To calculate $f_{\alpha_1}(\gamma_{\mathscr{B}})$, first consider the gallery,

$$\gamma_{\mathscr{B}} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2),$$

where $\mathrm{V}_0 = 0$, $\mathrm{V}_1 = \frac{1}{2}(\varepsilon_2 - \varepsilon_1)$, $\mathrm{V}_2 = 0$ and E_i is the line segment joining V_i and V_{i+1} for $i \in \{0, 1\}$. Note that $m_{\alpha_1} = -1$, $j = 1$, and $r = 2$ (see Definition 3.1). Therefore

$$f_{\alpha_1}(\gamma_{\mathscr{B}}) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \mathrm{V}'_2),$$

where $\mathrm{V}'_0 = 0$, $\mathrm{E}'_0 = \mathrm{E}_0$, $\mathrm{V}'_1 = \mathrm{V}_1$, $\mathrm{E}'_1 = s_{(\alpha_1,-1)}(\mathrm{E}_1)$ and $\mathrm{V}'_2 = s_{(\alpha_1,-1)}(\mathrm{V}_2) = \varepsilon_2 - \varepsilon_1$. Then $f_{\alpha_1}(\gamma_{\mathscr{B}}) = \gamma_{\mathscr{B}'}$, where

$$\mathscr{B}' = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{1} & \bar{1} \\ \hline \end{array}\,.$$

Similarly, $f_{\alpha_1}(w(\mathscr{B})) = 2\bar{1} = w(f_{\alpha_1}(\gamma_{\mathscr{B}}))$.

5C. ***Readable galleries are Littelmann galleries.*** We begin with a lemma.

Lemma 5.11. *Let $\gamma_{\mathscr{K}}$ be the readable gallery associated to a readable key $\mathscr{K}$. Then $\gamma_{\mathscr{K}}$ is dominant if and only if $\gamma_{w(\mathscr{K})}$ is dominant.*

Proof. Since the entries in the columns of symplectic keys are strictly increasing, it follows from the definition of word reading (Definition 5.1 and Definition 5.3) that if γ is a dominant readable gallery then $\gamma_{w(\gamma)}$ is also dominant. Now let γ be a nondominant readable gallery. Then there is a readable block $\mathscr{B} = \mathrm{C_L C_R}$ such that $\gamma = \eta_1 * \gamma_{\mathscr{B}} * \eta_2$ with η_1 dominant and $\eta_1 * \gamma_{\mathscr{B}}$ not dominant. This block can't be a zero lump (they are dominant) — so it must be LS. Let A, B, Z and T be the sets from Definition 4.3 that define the LS block $\mathscr{B}$: The entries of its right column $\mathrm{C_R}$ are the letters in $\mathrm{A} \cup \mathrm{Z} \cup \bar{\mathrm{B}} \cup \bar{\mathrm{T}}$ and the entries its left column $\mathrm{C_L}$ are the letters in $\mathrm{A} \cup \mathrm{T} \cup \bar{\mathrm{B}} \cup \bar{\mathrm{Z}}$. Now, $\mu_{\eta_1 * \gamma_{\mathscr{B}}}$ may or may not be dominant. If it is not, then, since $\mu_{\gamma_{w(\eta_1 * \gamma_{\mathscr{B}})}} = \mu_{\eta_1 * \gamma_{\mathscr{B}}}$, the word gallery $\gamma_{w(\eta_1 * \gamma_{\mathscr{B}})}$ is not dominant, and this implies that $\gamma_{w(\mathscr{K})}$ is not dominant either. Now assume that the coweight

$$\mu_{\eta_1 * \gamma_{\mathscr{B}}} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b$$

is dominant, but that the gallery $\eta_1 * \gamma_{\mathscr{B}}$ is not. The last three vertices of this gallery are

$$\mathrm{V}_{l-1} = \mu_{\eta_1} \in \mathrm{C}^+, \tag{11}$$

$$\mathrm{V}_l = \mu_{\eta_1} + \frac{1}{2}\Big(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z - \sum_{b \in \mathrm{B}} \varepsilon_b - \sum_{t \in \mathrm{T}} \varepsilon_t\Big) \notin \mathrm{C}^+, \tag{12}$$

$$\mathrm{V}_{l+1} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b \in \mathrm{C}^+, \tag{13}$$

for some $d \geq 1$. Let $d_1 < \cdots < d_{r+k}$ be the ordered elements of $\mathrm{A} \cup \mathrm{Z}$ and let $f_1 < \cdots < f_{s+k}$ be the ordered elements of $\mathrm{B} \cup \mathrm{Z}$. We have

$$w(\mathscr{B}) = d_1 \cdots d_{r+k} \bar{f}_{s+k} \cdots \bar{f}_1.$$

We claim that the weight

$$\mu_{\eta_1} + \sum_{i=1}^{r+k} \varepsilon_{d_i} = \mu_{\eta_1} + \sum_{a\in \mathrm{A}} \varepsilon_a + \sum_{z\in \mathrm{Z}} \varepsilon_z,$$

which is the endpoint of $\eta_1 * \gamma_{d_1\cdots d_{r+k}}$ and therefore a vertex of $\eta * \gamma_{w(\mathscr{B})}$, is not dominant. To see this, assume otherwise:

$$\mu_{\eta_1} + \sum_{a\in \mathrm{A}} \varepsilon_a + \sum_{z\in \mathrm{Z}} \varepsilon_z \in \mathrm{C}^+.$$

Since the dominant Weyl chamber C^+ is convex, this means that the line segment that joins μ_{η_1} and $\mu_{\eta_1} + \sum_{a\in \mathrm{A}} \varepsilon_a + \sum_{z\in \mathrm{Z}} \varepsilon_z$ is contained in C^+, in particular the point

$$\mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a\in \mathrm{A}} \varepsilon_a + \sum_{z\in \mathrm{Z}} \varepsilon_z\Bigg) \in \mathrm{C}^+ \tag{14}$$

belongs to the dominant Weyl chamber. We will now show

$$\mathrm{V}_l = \mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a\in \mathrm{A}} \varepsilon_a + \sum_{z\in \mathrm{Z}} \varepsilon_z - \sum_{b\in \mathrm{B}} \varepsilon_b - \sum_{t\in \mathrm{T}} \varepsilon_t\Bigg) \in \mathrm{C}^+.$$

This would contradict (12) and therefore complete the proof.

Set $\mu_{\eta_1} = \sum_{i=1}^{n} q_i \varepsilon_i$. Recall that $a_1 < \cdots < a_r$, $b_1 < \cdots < b_s$, $z_1 < \cdots < z_k$, and $t_1 < \cdots < t_k$ are the ordered elements of the sets A, B, Z and T, respectively. The dominant Weyl chamber has, in this case, the following description in the coordinates $\varepsilon_1, \ldots, \varepsilon_n$:

$$\mathrm{C}^+ = \Bigg\{\sum_{i=1}^{n} p_i \varepsilon_i : p_i \in \mathbb{R}_{\geq 0} \text{ and } p_1 \geq \cdots \geq p_n\Bigg\}. \tag{15}$$

This description allows us to make the following conclusions. For every $i \in \{1, \cdots, r\}$, we have $t_i < z_i < j$ for every $j \in \{1, \ldots, n\}$ such that $t_i < j$. It follows from (15) and (14) that

$$q_j \leq q_{z_i} + \tfrac{1}{2} \leq q_{t_i}, \tag{16}$$

which implies, since $q_j, q_{t_i}, q_{z_i} \in \mathbb{Z}$, that

$$q_j \leq q_j + \tfrac{1}{2} \leq q_{z_i} + \tfrac{1}{2} \leq q_{t_i} - \tfrac{1}{2}.$$

Now let $b \in \mathrm{B}$, and let $j \in \{1, \ldots, n\}$ such that $b < j$. By (13),

$$\mathrm{V}_{l+1} = \mu_{\eta_1} + \sum_{a\in \mathrm{A}} \varepsilon_a - \sum_{b\in \mathrm{B}} \varepsilon_b \in \mathrm{C}^+.$$

Together with (15) this implies

$$q_j \le q_j + \tfrac{1}{2} \le q_b - \tfrac{1}{2},$$

particularly so if $j \in (\mathrm{Z} \cup \mathrm{T})^c$. If $j \in \mathrm{Z} \cup \mathrm{T}$ then, as before, by (16) we may assume that $j = t \in \mathrm{T}$. But this means $q_t \le q_b$, therefore $q_t - \frac{1}{2} \le q_b - \frac{1}{2}$. All of these arguments, together with (15), imply

$$\mu_{\eta_1} + \frac{1}{2}\Bigg(\sum_{a\in \mathrm{A}} \varepsilon_a + \sum_{z\in \mathrm{Z}} \varepsilon_z - \sum_{b\in \mathrm{B}} \varepsilon_b - \sum_{t\in \mathrm{T}} \varepsilon_t\Bigg) \in \mathrm{C}^+,$$

which contradicts (12). □

Lemma 5.12. *A readable gallery ν is dominant if and only if $e_{\alpha_i}(\nu) = 0$ for all $i \in \{1, \dots, n\}$.*

Proof. First notice that it follows directly from Definition 5.6 that for a word $w \in \mathscr{W}_{\mathscr{C}_n}$ and α_i a simple root, $e_{\alpha_i}(w) = 0$ if and only if γ_w is dominant. Lemma 5.12 then follows from Lemma 5.11 and Proposition 5.9. □

Proposition 5.13. *Every readable gallery is a Littelmann gallery.*

Proof. Let V_n be the vector representation of $\mathrm{Sp}(2n, \mathbb{C})$. Then the crystal of words $\mathscr{W}_{\mathscr{C}_n}$ is isomorphic to the crystal associated to $\mathrm{T}(\mathrm{V}_n) = \bigoplus_{l\in\mathbb{Z}_{\ge 0}} \mathrm{V}_n^{\otimes l}$, see for example [Lecouvey 2002, §2.1]. Now let γ be any readable gallery. Then there exist indices $i_1, \dots, i_r$ such that $e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma_{w(\gamma)})$ is a highest weight vertex, hence dominant by Lemma 5.12. Since word reading is a morphism of crystals by Proposition 5.9, $\gamma_{w(e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma))} = e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma_{w(\gamma)})$. It follows from Lemma 5.11 that $e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma)$ is dominant. □

Definition 5.14. The *symplectic plactic monoid* $\mathscr{P}_{\mathscr{C}_n}$ is the quotient of the word monoid $\mathscr{W}_{\mathscr{C}_n}$ by the ideal generated by the following relations:

R1. For $z \ne \bar{x}$:

$$y\ x\ z \equiv y\ z\ x \quad \text{for } x \le y < z,$$
$$x\ z\ y \equiv z\ x\ y \quad \text{for } x < y \le z.$$

R2. For $1 < x \le n$ and $x \le y \le \bar{x}$:

$$y\ \overline{x{-}1}\ x{-}1 \equiv y\ x\ \bar{x},$$
$$\overline{x{-}1}\ x{-}1\ y \equiv x\ \bar{x}\ y.$$

R3. For $a_i, b_i \in \{1, \dots, n\}$, $i \in \{1, \dots, \max\{s, r\}\}$ such that $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_s$, and such that the left-hand side of the next expression is not the word of an LS block:

$$a_1 \cdots a_r\ z\ \ \bar{z}\ \bar{b}_s \cdots \bar{b}_1 \equiv a_1 \cdots a_r\ \bar{b}_s \cdots \bar{b}_1.$$

If two words $w_1, w_2 \in \mathscr{W}_{\mathscr{C}_n}$ are representatives of the same class in $\mathscr{W}_{\mathscr{C}_n}$ we say they are *symplectic plactic equivalent*.

Example 5.15. We have the following equivalences of words:

$$12\bar{2}\bar{1} \equiv 1\bar{1} \equiv \varnothing,$$
$$112 \equiv 121.$$

Remark 5.16. Relations R1 are the Knuth relations in type A, while relation R3 may be understood as the general relation that specializes to $1\bar{1} \equiv \varnothing$. Note that the gallery γ_w associated to $w = 1\bar{1}$ is a zero lump. This definition of the symplectic plactic monoid is the same as [Lecouvey 2002, Definition 3.1.1] except for relation R3. The equivalence between the relation R3 above and the one in [Lecouvey 2002] is given in the Appendix.

The following Theorem is proven in [Lecouvey 2002].

Theorem 5.17. *Two words $w_1, w_2 \in \mathscr{W}_{\mathscr{C}_n}$ are symplectic plactic equivalent if and only if their associated galleries γ_{w_1} and γ_{w_2} are equivalent.*

Together with the results we have recollected in this section, Theorem 5.17 implies the following proposition.

Proposition 5.18. *Two readable galleries γ and ν are equivalent if and only if the words $w(\gamma)$ and $w(\nu)$ are symplectic plactic equivalent.*

Proof. Two readable galleries γ and ν are equivalent if and only if, by definition, there exist indices $i_1, \dots, i_r$ such that the galleries $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma)$ and $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\nu)$ are both dominant and have the same endpoint, i.e., $\mu_{e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma)} = \mu_{e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\nu)}$. By Lemma 5.11 and Proposition 5.9 this is true if and only if $\gamma_{w(e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma))}$ and $\gamma_{w(e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\nu))}$ are also both dominant with the same endpoint. By Proposition 5.9, we have $w(e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\delta)) \equiv e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(w(\gamma_\delta))$ for any readable gallery δ. This means that the previous sequence of equivalences is also equivalent to $\gamma_{w(\gamma)} \sim \gamma_{w(\nu)}$ which by Theorem 5.17 is equivalent to $w(\gamma) \equiv w(\nu)$. □

The following theorem is originally due to Kashiwara and Nakashima (see [Kashiwara and Nakashima 1994]). For this particular formulation, see [Lecouvey 2002, Proposition 3.1.2].

Theorem 5.19. *For each word w in $\mathscr{W}_{\mathscr{C}_n}$ there exists a unique symplectic* LS *key $\mathscr{T}$ such that $w \equiv w(\mathscr{T})$.*

The following proposition will be proven in Section 7. Along with Proposition 5.5 it will play a fundamental role in the proof of Theorem 6.2.

Proposition 5.20. *Let γ and ν be combinatorial galleries and let $w_1, w_2 \in \mathscr{W}_{\mathscr{C}_n}$ be two plactic equivalent words. Consider the combinatorial galleries $\gamma * \gamma_{w_1} * \nu$ and*

$\gamma * \gamma_{w_2} * \nu$ as well as their associated Bott–Samelson varieties $(\Sigma_{(\gamma*\gamma_{w_1}*\nu)^f}, \pi)$ and $(\Sigma_{(\gamma*\gamma_{w_2}*\nu)^f}, \pi')$ together with their maps to the affine Grassmannian $\mathscr{G}$. Then

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{w_1}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w_2}*\nu})}.$$

6. Readable galleries and MV cycles

The following result holds in greater generality than is stated here: part (a) is an instance of [Gaussent and Littelmann 2005, Theorem C], and part (b) is an instance of [Baumann and Gaussent 2008, Theorem 5.8].

Theorem 6.1. *Let $\underline{d} = \underline{p}_1 * \cdots * \underline{p}_l$ be a symplectic shape such that $p_1 \le \cdots \le p_l$ and consider the desingularization $\pi : \Sigma_{\underline{d}} \to \mathrm{X}_{\lambda_{\underline{d}}}$.*

(a) *If $\delta \in \boldsymbol{\Gamma}(\underline{d})^{\mathrm{LS}}$ is a symplectic LS key, the closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathscr{Z}(\lambda_{\underline{d}})$. This induces a bijection $\boldsymbol{\Gamma}(\underline{d})^{\mathrm{LS}} \xrightarrow{\varphi_{\underline{d}}} \mathscr{Z}(\lambda_{\underline{d}})$.*

(b) *The bijection $\varphi_{\underline{d}}$ is an isomorphism of crystals.*

To formulate our main result we need the following additional notation. Given a readable gallery γ and a dominant coweight $\lambda \in \mathrm{X}^{\vee,+}$, let

$$n^{\lambda}_{\gamma^f} = \#\{\nu \in \Gamma^{\mathrm{dom}} \cap \Gamma(\gamma^f) : \mu_\nu = \lambda\},$$

and let

$$\mathrm{X}^{\vee,+}_{\gamma^f} = \{\lambda \in \mathrm{X}^{\vee,+} : n^{\lambda}_{\gamma^f} \neq 0\}.$$

Further, let $\Gamma(\gamma^f)^{\mathrm{R}}/\sim$ be a set of representatives of the classes for the equivalence relation on Littelmann galleries (and hence on readable galleries by Remark 3.4 and Proposition 5.13) defined in Section 3C.

Theorem 6.2. *Let $\delta \in \Gamma(\gamma^f)^{\mathrm{R}}$ be a readable gallery. Consider the corresponding Bott–Samelson variety (Σ_{γ^f}, π) together with its map π to the affine Grassmannian as in Remark 2.6. Let δ^+ be the gallery that is the highest weight vertex in* $\mathrm{Conn}(\delta)$. (This gallery is dominant and readable by Lemma 5.12 and Remark 3.4, respectively.) *Then*:

(a) *The closed set $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathscr{Z}(\mu_{\delta^+})_{\mu_\delta}$.*

(b) *The map*

$$\Gamma(\gamma^f)^{\mathrm{R}} \xrightarrow{\varphi_{\gamma^f}} \bigoplus_{\nu \in \Gamma(\gamma^f)^{\mathrm{R}}/\sim} \mathscr{Z}(\mu_{\nu^+}), \qquad \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a surjective morphism of crystals. The direct sum on the right-hand side is a direct sum of abstract crystals.

(c) *If C is a connected component of $\Gamma(\gamma^f)^{\mathrm{R}}$, then $\varphi|_{\mathrm{C}}$ is an isomorphism onto its image.*

(d) *The number of connected components* C *of* $\Gamma^{\mathrm{R}}(\gamma^f)$ *such that* $\varphi_{\gamma^f}(\mathrm{C}) = \mathscr{Z}(\lambda)$ *is equal to* $n^{\lambda}_{\gamma^f}$.

(e) *Given an MV cycle* $\mathrm{Z} \in \mathscr{Z}(\lambda)_\mu$, *the fiber* $\varphi^{-1}_{\gamma^f}(\mathrm{Z})$ *is given by*

$$\varphi^{-1}_{\gamma^f}(\mathrm{Z}) = \{\delta \in \Gamma^{\mathrm{R}}(\gamma^f) : \varphi_{\gamma^f}(\delta) = \mathrm{Z}\} = \{\delta \in \Gamma^{\mathrm{R}}(\gamma^f) : \gamma \sim \gamma^{\lambda}_{\mu,\mathrm{Z}}\},$$

where $\gamma^{\lambda}_{\mu,\mathrm{Z}}$ *is the unique LS key which exists by Theorem 6.1.*

Proof. Let δ be a readable gallery. Then by Theorem 5.19 there exists a (unique) LS key ν such that $\delta \sim \nu$. By Proposition 5.18, the words $w(\delta)$ and $w(\nu)$ are plactic equivalent. Propositions 5.20 and 5.5 together with Theorem 5.17 then imply that

$$\overline{\pi(\mathrm{C}_\delta)} = \overline{\pi(\mathrm{C}_\nu)},$$

which, by Theorem 6.1 implies that $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathscr{Z}(\mu_{\delta^+})_{\mu_\delta}$. The map φ_{γ^f} in (b) is surjective by Theorems 5.19 and 6.1 above. Now let r be a root operator, and let $\tilde{r}$ be the corresponding root operator that acts on the set of MV cycles. Then by Propositions 5.5, 5.9, 5.20, and Theorem 6.1 we have

$$\overline{\pi(\mathrm{C}_{r(\gamma)})} = \overline{\pi(\mathrm{C}_{\gamma_{w(r(\gamma))}})} = \overline{\pi(\mathrm{C}_{\gamma_{w(r(\nu))}})} = \overline{\pi(\mathrm{C}_{r(\nu)})} = \tilde{r}(\overline{\pi(\mathrm{C}_\nu)}) = \tilde{r}(\overline{\pi(\mathrm{C}_\gamma)}).$$

This completes the proof of (b). Part (c) follows immediately, since every connected component C is crystal isomorphic to the corresponding component consisting of the LS galleries equivalent to those in C. Parts (d) and (e) follow from [Littelmann 1995, Theorem 7.1] (see Section 3C). □

7. Counting positive crossings

We provide proofs of Propositions 5.5 and 5.20. We begin by analyzing the *tail* of a gallery in Section 7A. In Example 7.3 we calculate an example in which it can be seen how to use this proposition. Then in Section 7B we prove Proposition 5.5 and in Section 7C we prove Proposition 5.20. We also wish to establish some notation that we will use throughout. Recall our convention $\varepsilon_{\bar{l}} = -\varepsilon_l$ for $l \in \mathscr{C}_n$ unbarred. For $l, s, d, m \in \mathscr{C}_n$ we will write $c^{i,j}_{ls,dm}$ for the constant $c^{i,j}_{\varepsilon_l+\varepsilon_s,\varepsilon_d+\varepsilon_m}$ in Chevalley's commutator formula (2). Additionally we will write $c^{i,j}_{l,dm}$, and respectively $c^{i,j}_{ls,d}$, for $c^{i,j}_{\varepsilon_l,\varepsilon_d+\varepsilon_m}$, and $c^{i,j}_{\varepsilon_l+\varepsilon_s,\varepsilon_d}$. (Each time we use such notation a total order will be fixed on the set of positive roots.) If $\mathrm{Y} \subseteq \mathscr{C}_n$ and $y \in \mathscr{C}_n$ then we will write $\mathrm{Y}^{\leq y}$ (respectively $\mathrm{Y}^{<y}$, $\mathrm{Y}^{\geq y}$, $\mathrm{Y}^{>y}$) for the subset of elements $x \in \mathrm{Y}$ such that $x \leq y$ (respectively $x < y$, $x \geq y$, $x > y$).

7A. ***Truncated images and tails.*** Let γ be a combinatorial gallery with notation as in (4) with endpoint the coweight μ_γ and let $1 \leq r \leq k+1$ such that V_r is a special vertex; we denote it by $\mu_r \in \mathrm{X}^\vee$. By Corollary 2.10 we know that the image $\pi(\mathrm{C}_\gamma)$ is stable under U_0.

Proposition 7.1. *The r-truncated image of* γ,

$$\mathrm{T}_\gamma^{\geq r} = \mathbb{U}^\gamma_{\mathrm{V}_r}\mathbb{U}^\gamma_{\mathrm{V}_{r+1}}\cdots\mathbb{U}^\gamma_{\mathrm{V}_k}[t^{\mu_\gamma}],$$

is U_{μ_r}*-stable, i.e., for any* $u \in \mathrm{U}_{\mu_r}$, *it follows that* $u\mathrm{T}_\gamma^{\geq r} = \mathrm{T}_\gamma^{\geq r}$.

Proof. By (3) we know that $t^{\mu_r}\mathrm{U}_0 t^{-\mu_r} = \mathrm{U}_{\mu_r}$. We consider the *r-truncated gallery*

$$\gamma^{\geq r} = (\mathrm{V}'_0, \mathrm{E}'_0, \ldots, \mathrm{V}'_{k-r+1}),$$

which is the combinatorial gallery obtained from the sequence

$$(\mathrm{V}_r, \mathrm{E}_r, \mathrm{V}_{r+1}, \ldots, \mathrm{E}_k, \mathrm{V}_{k+1}),$$

by translating it to the origin. Since V_r is a special vertex, $t^{\mu_r}\mathbb{U}^{\gamma^{\geq r}}_{\mathrm{V}_i} t^{-\mu_r} = \mathbb{U}^\gamma_{\mathrm{V}_{i+r}}$. This gallery has endpoint $\mu_\gamma - \mu_r$ and is in turn a T-fixed point of a Bott–Samelson variety (Σ, π'). Let $u \in \mathrm{U}_{\mu_r}$ and $u' = t^{-\mu_r}ut^{\mu_r} \in \mathrm{U}_0$. Then

$$\begin{aligned} u\mathrm{T}_\gamma^{\geq r} &= u\mathbb{U}^\gamma_{\mathrm{V}_r}\mathbb{U}^\gamma_{\mathrm{V}_{r+1}}\cdots\mathbb{U}^\gamma_{\mathrm{V}_k}[t^{\mu_\gamma}] \\ &= t^{\mu_r}u'\mathbb{U}^{\gamma^{\geq r}}_{\mathrm{V}_0}\cdots\mathbb{U}^{\gamma^{\geq r}}_{\mathrm{V}_{k-r}}[t^{\mu_\gamma-\mu_r}] \\ &= t^{\mu_r}\mathbb{U}^{\gamma^{\geq r}}_{\mathrm{V}_0}\cdots\mathbb{U}^{\gamma^{\geq r}}_{\mathrm{V}_{k-r}}[t^{\mu_\gamma-\mu_r}] = \mathrm{T}_\gamma^{\geq r}. \end{aligned}$$

Where the final equality follows from Corollary 2.10. □

For later use let us fix the notation

$$\mathrm{T}_\gamma^{<r} = \mathbb{U}^\gamma_{\mathrm{V}_0}\cdots\mathbb{U}^\gamma_{\mathrm{V}_{r-1}},$$

so that

$$\pi(\mathrm{C}_\gamma) = \mathrm{T}_\gamma^{<r}\mathrm{T}_\gamma^{\geq r}.$$

Remark 7.2. This Proposition is proven for $\mathrm{SL}(n, \mathbb{C})$ in [Gaussent et al. 2013, Proposition 3]. The proof we have provided is exactly the same, except for the restriction of only being able to truncate at special vertices.

Example 7.3. Let $n = 2$. Consider the symplectic keys

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline 1 & 1 & \bar{1} \\ \hline 2 & 2 & \\ \cline{1-2}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline & \bar{2} & \bar{1} \\ \cline{2-3}\end{array},$$

and their words

$$w(\mathscr{K}_1) = \bar{1}12 \quad \text{and} \quad w(\mathscr{K}_2) = 2\bar{2}2.$$

Note that $\gamma_{\omega_1} * \gamma_{\omega_2} \sim \gamma_{\omega_2} * \gamma_{\omega_1}$, since both $\gamma_{\omega_1} * \gamma_{\omega_2}$ and $\gamma_{\omega_2} * \gamma_{\omega_1}$ are contained in the fundamental chamber and have the same endpoint $\omega_1 + \omega_2$. One checks that

$$f_{\alpha_1} f_{\alpha_2} f_{\alpha_1}(\gamma_{\omega_1} * \gamma_{\omega_2}) = \gamma_{\mathscr{K}_1} \quad \text{and} \quad f_{\alpha_1} f_{\alpha_2} f_{\alpha_1}(\gamma_{\omega_2} * \gamma_{\omega_1}) = \gamma_{\mathscr{K}_2}.$$

Therefore $\gamma_{\mathscr{K}_1} \sim \gamma_{\mathscr{K}_2}$. Lemma 5.11 and Proposition 5.9 imply that $\gamma_{w(\mathscr{K}_1)} \sim \gamma_{w(\mathscr{K}_2)}$ (it can also be checked directly using Relation R2 in Theorem 5.17 with $y = x = 2$). Now consider combinatorial galleries γ and ν. The galleries $\gamma * \gamma_{\mathscr{K}_1} * \nu$ and $\gamma * \gamma_{\mathscr{K}_2} * \nu$ are T-fixed points in the Bott–Samelson varieties $(\Sigma_{(\gamma * \gamma_{\mathscr{K}_1} * \nu)^f}, \pi)$, respectively $(\Sigma_{(\gamma * \gamma_{\mathscr{K}_2} * \nu)^f}, \pi')$. The galleries $\gamma_{w(\mathscr{K}_1)}$ and $\gamma_{w(\mathscr{K}_2)}$ that correspond to their words are T-fixed points in $(\Sigma_{(\gamma * \gamma_{\omega_1} * \gamma_{\omega_1} * \gamma_{\omega_1} * \nu)^f}, \pi'')$. We show that

$$\overline{\pi(\mathrm{C}_{\gamma * \gamma_{\mathscr{K}_1} * \nu})} = \overline{\pi''(\mathrm{C}_{\gamma * \gamma_{w(\mathscr{K}_1)} * \nu})} = \overline{\pi'(\mathrm{C}_{\gamma * \gamma_{w(\mathscr{K}_2)} * \nu})}.$$

We use the same notation as in (4) for γ. Since for any combinatorial gallery η, $(\alpha, n) \in \Phi_{k+1}^{\gamma * \eta}$ if and only if $(\alpha, n - \langle \alpha, \mu_\gamma \rangle) \in \Phi_0^{\gamma}$, we may assume that $\gamma = \varnothing$. Since $\gamma_{\mathscr{K}_1}$, $\gamma_{\mathscr{K}_2}$, $\gamma_{w(\mathscr{K}_1)}$ and $\gamma_{w(\mathscr{K}_2)}$ have the same endpoint ε_2, this also implies that $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1} * \nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_2} * \nu} = \mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)} * \nu} = \mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_1)} * \nu}$. By Proposition 2.7, for $a', b', c', d' \in \mathbb{C}$,

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1 * \nu}}) = \mathrm{U}_{(\varepsilon_1, -1)}(a')\mathrm{U}_{(\varepsilon_1+\varepsilon_2, -1)}(b')\mathrm{U}_{(\varepsilon_2, 0)}(c')\mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(d')\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1} * \nu}.$$

By Chevalley's commutator formula (2) and an application of Proposition 7.1 to $\mathrm{U}_{(\varepsilon_1-\varepsilon_2, -1)}(e) \in \mathrm{U}_{\varepsilon_2}$, we obtain

$$\begin{aligned}
&\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1) * \nu}}) \\
&\quad= \mathrm{U}_{(\varepsilon_1, -1)}(a) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, -1)}(b) \cdot \mathrm{U}_{(\varepsilon_1-\varepsilon_2, -1)}(e) \cdot \mathrm{U}_{(\varepsilon_2, 0)}(c) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(d)\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_1)} * \nu} \\
&\quad= \mathrm{U}_{(\varepsilon_1, -1)}\big(a + c^{1,1}_{1\bar{2},2}(-e)c\big) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, -1)}\big(b + c^{1,1}_{1\bar{2},2}(-e)c^2\big) \\
&\qquad\qquad \cdot \mathrm{U}_{(\varepsilon_2, 0)}(c) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(d) \cdot \mathrm{U}_{(\varepsilon_1-\varepsilon_2, -1)}(e)\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1} * \nu} \\
&\quad= \mathrm{U}_{(\varepsilon_1, -1)}\big(a + c^{1,1}_{1\bar{2},2}(-e)c\big) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, -1)}\big(b + c^{1,1}_{1\bar{2},2}(-e)c^2\big) \\
&\qquad\qquad \cdot \mathrm{U}_{(\varepsilon_2, 0)}(c) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(d)\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1} * \nu} \\
&\quad\subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu}),
\end{aligned}$$

for $a, b, c, d, e \in \mathbb{C}$. Choosing $a = a'$, $b = b'$, $c = c'$, $d = d'$, and $e = 0$, we have $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) \subset \pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)}})$. Hence, in this case $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) = \pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)}})$. Similarly, for $a'', b'', c'', d'', e'' \in \mathbb{C}$,

$$\begin{aligned}
&\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_2) * \nu}}) \\
&\quad= \mathrm{U}_{(\varepsilon_2, 0)}(a'') \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(b'') \cdot \mathrm{U}_{(\varepsilon_1-\varepsilon_2, -1)}(e'') \cdot \mathrm{U}_{(\varepsilon_2, 0)}(c'') \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(d'')\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)} * \nu} \\
&\quad= \mathrm{U}_{(\varepsilon_1, -1)}\big(c^{1\bar{2},2}_{1,1}(-e'')c''\big) \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, -1)}\big(c^{1\bar{2},2}_{1,2}(-e'')c''^2\big) \\
&\qquad\qquad \cdot \mathrm{U}_{(\varepsilon_2, 0)}(a'' + c'') \cdot \mathrm{U}_{(\varepsilon_1+\varepsilon_2, 0)}(b'' + d'')\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)} * \nu} \\
&\quad\subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_1 * \nu}}).
\end{aligned}$$

Hence the open subset of $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1 * \nu}})$ given by $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$ is contained in $\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_2)} * \nu})$.

7B. *Proof of Proposition 5.5*. It is enough to show that if γ and ν are combinatorial galleries and $\mathscr{K}$ is a readable block, then

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K})}*\nu})}, \tag{17}$$

where $(\Sigma_{(\gamma*\gamma_{\mathscr{K}}*\nu)^f}, \pi)$ and $(\Sigma_{(\gamma*\gamma_{w(\mathscr{K})}*\nu)^f}, \pi')$ are the Bott–Samelson varieties associated to the galleries $\gamma * \gamma_{\mathscr{K}} * \nu$ and $\gamma * \gamma_{w(\mathscr{K})} * \nu$ respectively.

Proof. We assume $\gamma = \varnothing$; we may do so by the argument given at the beginning of Example 7.3. Let $\mathscr{K}$ be an LS block and let $\mathrm{A} = \{a_1, \dots, a_r\}$, $\mathrm{B} = \{b_1, \dots, b_s\}$, $\mathrm{Z} = \{z_1, \dots, z_k\}$ and $\mathrm{T} = \{t_1, \dots, t_k\}$ be the subsets of $\{1, \dots, n\}$ from Definition 4.3 that determine $\mathscr{K}$. We will use the notation $d_1 < \cdots < d_{r+k}$ to denote the ordered elements of $\mathrm{Z} \cup \mathrm{A}$ and $f_1 < \cdots < f_{s+k}$ the ordered elements of $\mathrm{B} \cup \mathrm{Z}$. We also write

$$\gamma_{\mathscr{K}} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2).$$

The proof is divided into Lemmas 7.4 and 7.5 below.

Lemma 7.4. *Let ν be a combinatorial gallery and $\mathscr{K}$ be a readable block. Then*

$$\overline{\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})}.$$

Proof. We first show that

$$\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}) \subset \mathrm{U}_0 \mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}, \tag{18}$$

where

$$\mathbb{P}'''_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(k_{l\bar{b}}) \prod_{t\in \mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(k_{t\bar{b}}) \prod_{a\in \mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(k_{a\bar{b}}), \tag{19}$$

$$\mathbb{P}'''_{\bar{z}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; \\ l<z}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(k_{l\bar{z}}) \prod_{t\in \mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) \prod_{b\in \mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(k_{b\bar{z}}), \tag{20}$$

for $b \in \mathrm{B}$, $z \in \mathrm{Z}$ and $k_{ij} \in \mathbb{C}$. Indeed, the points of $\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})$ are of the form

$$\mathbb{P}_{d_1} \cdots \mathbb{P}_{d_{r+k}} \mathbb{P}_{\bar{f}_{k+s}} \cdots \mathbb{P}_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}, \tag{21}$$

where

$$\mathbb{P}_d = \mathrm{U}_{(\varepsilon_d,0)}(g_d) \prod_{d<l\leq n} \mathrm{U}_{(\varepsilon_d-\varepsilon_l,0)}(g_{d\bar{l}}) \prod_{l\notin(\mathrm{Z}\cup\mathrm{A})^{<d}} \mathrm{U}_{(\varepsilon_d+\varepsilon_l,0)}(g_{dl}) \prod_{l\in(\mathrm{Z}\cup\mathrm{A})^{<d}} \mathrm{U}_{(\varepsilon_d+\varepsilon_l,1)}(g^1_{dl}),$$

$$\mathbb{P}_{\bar{b}} = \mathrm{S}_{\bar{b}} \mathbb{P}^{iv}_{\bar{b}} \quad \text{with } \mathrm{S}_{\bar{b}} = \prod_{b'\in \mathrm{B}^{<b}} \mathrm{U}_{(\varepsilon_{b'}-\varepsilon_b,0)}(g_{b'\bar{b}}) \prod_{z\in \mathrm{Z}^{<b}} \mathrm{U}_{(\varepsilon_z-\varepsilon_b,1)}(g^1_{z\bar{b}}) \in \mathrm{U}_0 \quad \text{and}$$

$$\mathbb{P}^{iv}_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(g_{l\bar{b}}) \prod_{t\in \mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(g_{t\bar{b}}) \prod_{a\in \mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(g_{a\bar{b}}),$$

and finally

$$\mathbb{P}_{\bar{z}} = J_{\bar{z}}\mathbb{P}^{iv}_{\bar{z}} \quad \text{with } \mathrm{J}_{\bar{z}} = \prod_{a\in \mathrm{A}^{<z}} \mathrm{U}_{(\varepsilon_a-\varepsilon_z,0)}(g_{a\bar{z}}) \prod_{z'\in \mathrm{Z}^{<z}} \mathrm{U}_{(\varepsilon_{z'}-\varepsilon_z,0)}(g_{z'\bar{z}}) \in \mathrm{U}_0 \quad \text{and}$$

$$\mathbb{P}^{iv}_{\bar{z}} = \mathrm{J}_{\bar{z}} \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ l<z}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(g_{l\bar{z}}) \prod_{t\in \mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(g_{t\bar{z}}) \prod_{b\in \mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(g_{b\bar{z}}),$$

for $d\in \mathrm{A}\cup\mathrm{Z}$, $z\in\mathrm{Z}$, $b\in\mathrm{B}$, and $g_{ij}\in\mathbb{C}$. All terms in $\mathrm{J}_{\bar{z}}$ commute with $\mathbb{P}^{iv}_{z'}$ for $z'\in\mathrm{Z}^{>z}$ and with $\mathbb{P}^{iv}_{\bar{b}}$ for $b\in\mathrm{B}^{>z}$. All terms in $\mathrm{S}_{\bar{b}}$ commute with $\mathbb{P}^{iv}_{\bar{b}'}$ for $b'\in\mathrm{B}^{>b}$. For $z'>b$ it commutes with all terms of $\mathbb{P}^{iv}_{\bar{z}'}$ except for the term $\mathrm{U}_{(\varepsilon_b-\varepsilon_{z'},-1)}(g_{b\bar{z}'})$. But commuting $\mathrm{S}_{\bar{b}}$ with this term (using Chevalley's commutator formula (2)) produces terms $\mathrm{U}_{(\varepsilon_z-\varepsilon_{z'},0)}(*)$ and $\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_{z'},-1)}(*)$, of these terms, $\mathrm{U}_{(\varepsilon_z-\varepsilon_{z'},0)}(*)$ commutes with $\mathbb{P}^{iv}_{z'}$ for $z'\in\mathrm{Z}^{>z}$ and with $\mathbb{P}^{iv}_{\bar{b}}$ for $b\in\mathrm{B}^{>z}$, and $\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_{z'},-1)}(*)$ is a term of the form of those appearing in $\mathbb{P}^{iv}_{\bar{z}}$.

Since the terms that appear in $\mathbb{P}^{iv}_{\bar{b}}$ and $\mathbb{P}^{iv}_{\bar{z}}$ are the same as those in $\mathbb{P}''_{\bar{b}}$ and $\mathbb{P}''_{\bar{z}}$ respectively, this justifies (18), concluding the first step in the proof of Lemma 7.4. The second step is this:

Claim. *There is a dense subset of* $\mathbb{P}'''_{\bar{f}_{k+s}}\cdots\mathbb{P}'''_{\bar{f}_1}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}$ *contained in the subset*

$$\mathbb{P}_{\mathrm{T,B}}\mathbb{P}_{\mathscr{K},\bar{f}_s}\cdots\mathbb{P}_{\mathscr{K},\bar{f}_s}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \subset \overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})}, \tag{22}$$

where

$$\mathbb{P}_{\mathrm{T,B}} = \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ t\in\mathrm{T},l<t}} \mathrm{U}_{(\varepsilon_l-\varepsilon_t,0)}(v_{l\bar{t}}) \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ b\in\mathrm{B},l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(v_{l\bar{b}}) \in \mathrm{U}_{\mathrm{V}_0},$$

$$\mathbb{P}_{\mathscr{K},\bar{b}} = \prod_{\substack{b\in\mathrm{B}\\ t\in\mathrm{T}^{<b}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(v_{t\bar{b}}) \prod_{a\in\mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(v_{a\bar{b}}) \in \mathrm{U}_{\mathrm{V}_1},$$

$$\mathbb{P}_{\mathscr{K},\bar{z}} = \prod_{t\in\mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{b\in\mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(v_{b\bar{z}}) \in \mathrm{U}_{\mathrm{V}_1},$$

for $v_{ij}\in\mathbb{C}$, $b\in\mathrm{B}$ *and* $z\in\mathrm{Z}$. (The inclusion in (22) holds by Corollary 2.10.)

To prove this we start by noting that $\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$ and that

$$u = \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}\\ t\in\mathrm{T},l<t}} \mathrm{U}_{(\varepsilon_l-\varepsilon_t,0)}(v_{l\bar{t}}) \in \mathrm{U}_{\mu_{\gamma_{\mathscr{K}}}}. \tag{23}$$

We have the equalities

$$\mathbb{P}_{\mathrm{T,B}}\mathbb{P}_{\mathscr{K},\bar{f}_s}\cdots\mathbb{P}_{\mathscr{K},\bar{f}_s}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} = \mathbb{P}''_{\bar{f}_s}\cdots\mathbb{P}''_{\bar{f}_s}u\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} = \mathbb{P}''_{\bar{f}_s}\cdots\mathbb{P}''_{\bar{f}_s}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}, \tag{24}$$

where we have introduced symbols analogous to those of (19) and (20); namely, for $z \in \mathrm{Z}$ and $b \in \mathrm{B}$,

$$\mathbb{P}''_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(\xi_{l\bar{b}}) \prod_{t\in \mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(\xi_{t\bar{b}}) \prod_{a\in \mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(\xi_{a\bar{b}}),$$

$$\mathbb{P}''_{\bar{z}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T} \\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(\xi_{l\bar{z}}) \prod_{t\in \mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(\xi_{t\bar{z}}) \prod_{b\in \mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(\xi_{b\bar{z}})$$

with $\xi_{t\bar{z}} = v_{t\bar{z}},\ \xi_{b\bar{z}} = v_{b\bar{z}},\ \xi_{t\bar{b}} = v_{t\bar{b}}$,

$$\xi_{l\bar{b}} = v_{l\bar{b}} + \sum_{\substack{l<b<b \\ t\in \mathrm{T}}} c^{1,1}_{s\bar{t},t\bar{b}}(-v_{l\bar{t}})v_{t\bar{b}},$$

$$\xi_{l\bar{z}} = \rho_{l\bar{z}} + \sum_{z'\in \mathrm{Z}} c^{1,1}_{l\bar{z}',z'\bar{z}}(-\rho_{l\bar{z}'})v_{z'\bar{z}} + \sum_{\substack{l<b<z \\ b\in \mathrm{B}}} c^{1,1}_{l\bar{b},b\bar{z}}(-\xi_{l\bar{b}})v_{b\bar{z}} \quad \text{for}$$

$$\rho_{l\bar{z}} = \sum_{\substack{l<t<z \\ t\in \mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{z}}(-v_{l\bar{t}})v_{t\bar{z}} \text{ (for } z \in \mathrm{Z}).$$

To complete the proof of the Claim we must set open conditions on the parameters k_{ij} such that the system of equations defined by $v_{ij} = \xi_{ij}$ has a solution in the variables v_{ij}. Setting $v_{t\bar{z}} := k_{t\bar{z}}$ and $v_{b\bar{z}} := k_{b\bar{z}}$ this is reduced to setting conditions on the k_{ij} so that the following system can be solved:

$$k_{l\bar{b}} = v_{l\bar{b}} + \sum_{\substack{l<t<b \\ t\in \mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{b}}(-v_{l\bar{t}})k_{t\bar{b}}, \tag{25}$$

$$k_{l\bar{z}} = \rho_{l\bar{z}} - \sum_{\substack{l<b<z \\ b\in \mathrm{B}}} c^{1,1}_{l\bar{b},b\bar{z}}\Bigg(v_{l\bar{b}} + \sum_{\substack{l<t<b \\ t\in \mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{b}}(-v_{l\bar{t}})k_{t\bar{b}}\Bigg)k_{b\bar{z}}, \tag{26}$$

$$\rho_{l\bar{z}} = \sum_{\substack{l<t<z \\ t\in \mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{z}}(-v_{l\bar{t}})k_{t\bar{z}}. \tag{27}$$

Lines (25) and (26) define a linear system of as many equations as variables. The variables are $\{v_{l\bar{b}}\}_{l\notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}, b\in \mathrm{B}^{>l}} \cup \{v_{l\bar{t}}\}_{l\notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{Z}\cup\mathrm{T}, t\in\mathrm{T}^{>l}}$; there is one equation for each $l\bar{b}$ such that $l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}$ and $b \in \mathrm{B}^{>l}$, and one for each $l\bar{z}$ such that $l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}$ and $z \in \mathrm{Z}^{>l}$. Note that by definition of an LS block the sets $\{l\bar{z},\, l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\, z \in \mathrm{Z}^{>l}\}$ and $\{l\bar{t},\, s \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\, b \in \mathrm{B}^{>l}\}$ have the same cardinality (t_i is the maximal element of the set $\{l \notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T},\, s < t_{i+1},\, s < z_i\}$). Therefore the system has a solution as long as the matrix of coefficients has nonzero determinant, which imposes open conditions on the $k'_{ij}s$. Hence the Claim is proven.

To finish the proof of Lemma 7.4, note that if the $k'_{ij}s$ satisfy the open conditions established by the Claim, then

$$\mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathrm{K})}*\nu} \subseteq \pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}),$$

and therefore Proposition 7.1 implies that

$$\mathrm{U}_0\mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathrm{K})}*\nu} \subseteq \pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}),$$

which implies Lemma 7.4. □

Lemma 7.5. *Let ν be a combinatorial gallery and $\mathscr{K}$ be an LS block. Then*

$$\overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})}. \tag{28}$$

Proof. Recall that

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}) = \mathbb{U}^{\gamma_{\mathscr{K}}*\nu}_{\mathrm{V}_0}\mathbb{U}^{\gamma_{\mathscr{K}}*\nu}_{\mathrm{V}_1}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}.$$

Notice that $\mathbb{U}^{\gamma_{\mathscr{K}}*\nu}_{\mathrm{V}_0} \subset \mathrm{U}_0$ and that all generators of $\mathbb{U}^{\gamma_{\mathscr{K}}*\nu}_{\mathrm{V}_1}$ also belong to U_0 except for those of the form $\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}})$ or $\mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'})$ for $t, t' \in \mathrm{T}$, $z \in \mathrm{Z}^{>t}$, and $v_{t\bar{z}}, v_{tt'} \in \mathbb{C}$. Hence, since $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} = \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}$, all elements of $\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})$ belong to

$$\mathrm{U}_0 \prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in\mathrm{T}} \mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}. \tag{29}$$

Now consider

$$\prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}}} \mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \prod_{t\in\mathrm{T} z\in\mathrm{Z}^{>t}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu},$$

which is a subset of $\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})$ by virtue of Proposition 7.1 and because

$$\prod_{\substack{z\in\mathrm{Z}\\ t\in\mathrm{T}}} \mathrm{U}_{(\varepsilon_z+\varepsilon_t,0)}(k_{zt}) \in \mathrm{U}_0 \quad \text{and} \quad \prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \subset \pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}).$$

We have

$$\prod_{\substack{t'\in\mathrm{T}\\ z\in\mathrm{Z}}} \mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \tag{30}$$

$$= \prod_{\substack{t,t'\in\mathrm{T}\\ t\neq t'}} \mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(\xi_{tt'}) \prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) \prod_{\substack{t'\in\mathrm{T}\\ z\in\mathrm{Z}}} \mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \tag{31}$$

$$= \prod_{\substack{t,t'\in\mathrm{T}\\ t\neq t'}} \mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(\xi_{tt'}) \prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}, \tag{32}$$

where

$$\xi_{tt'} = \sum_{z\in Z^{>t'}} c^{1,1}_{zt,t'\bar{z}}(-k_{zt})k_{t'\bar{z}} + \sum_{z\in Z^{>t}} c^{1,1}_{zt',t\bar{z}}(-k_{zt'})k_{t\bar{z}}. \tag{33}$$

The equality between (30) and (31) is due to Chevalley's commutator formula (2) and the equality between (31) and (32) is obtained by using Proposition 7.1 and $U_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \in U_{\mu_{\gamma_{\mathscr{K}}}}$. Now fix an element in (29). Setting $k_{t\bar{z}} = v_{t\bar{z}}$ defines the linear equations

$$v_{tt'} = \sum_{z\in Z^{>t'}} c^{1,1}_{zt,t'\bar{z}}(-k_{zt})v_{t'\bar{z}} + \sum_{z\in Z^{>t}} c^{1,1}_{zt',t\bar{z}}(-k_{zt'})v_{t\bar{z}},$$

in the variables k_{zt}, for $z \in Z$ and $t \in T$. There are more variables than equations. For each equation indexed by a nonordered pair (t_i, t_j) there are the variables v_{zt_i} and $v_{z't_j}$ for $z > t'$ and $z' > t$ (which always exist by definition of an LS block), hence the system has solutions as long as the matrix of coefficients has nonzero determinants. This imposes an open condition on the parameters $v_{t\bar{z}}$. Hence for such $v_{t\bar{z}}$, $v_{tt'}$, $k_{t\bar{z}} = v_{t\bar{z}}$, and solutions k_{ij}, for the latter equations we have

$$\begin{aligned}\prod_{\substack{t\in T\\ z\in Z^{>t}}} U_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in T} U_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'}) T^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \\ = \prod_{\substack{t'\in T\\ z\in Z}} U_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \prod_{\substack{t\in T\\ z\in Z^{>t}}} U_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) T^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \subset \pi'(C_{\gamma_{w(\mathscr{K})}*\nu}).\end{aligned}$$

Proposition 7.1 then implies,

$$U_0 \prod_{\substack{t\in T\\ z\in Z^{>t}}} U_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in T} U_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'}) T^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} \subset \pi'(C_{\gamma_{w(\mathscr{K})}*\nu}).$$

This completes the proof of Lemma 7.5 and hence of (17) for $\mathscr{K}$ an LS block. □

Now let $\mathscr{K}$ be a zero lump. This means there exists $k > 1$ such that the right (respectively left) column of $\mathscr{K}$ has as entries the integers $1 < \cdots < k$ (respectively $\bar{k} < \cdots < \bar{1}$), its word is therefore $w(\mathscr{K}) = 1\cdots k\bar{k}\cdots\bar{1}$. This means, in particular, that the truncated images $T^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu} = T^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$ are stabilized by U_0, by Proposition 7.1. We have

$$\pi'(C_{\gamma_{w(\mathscr{K})}*\nu}) = \mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_0} \cdots \mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_{2k-1}} T^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu},$$

by Theorem 2.9. Clearly all of the subgroups $\mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_l} \subset U_0$, for $1 \leq l \leq k$. For $0 \leq j \leq k-1$, the generators of $\mathbb{U}^{\gamma_{w(\mathscr{K})}*\nu}_{V_{k+j}}$ are all of the form $U_{(\varepsilon_s-\varepsilon_{k-j},n_{k-j})}$ for $l < k-j$. In particular the gallery $\gamma_{1\cdots k\bar{k}\cdots\overline{k-j-1}}$ has crossed the hyperplanes

$\mathrm{H}_{(\varepsilon_s-\varepsilon_{k-j},m)}$ once positively at $m=0$ and once negatively at $m=1$, which means that $n_{k-j}=0$, and $\mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j},n_{k-j})}(a)=\mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j},0)}(a)\in\mathrm{U}_0$, for all $a\in\mathbb{C}$. Hence

$$\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w(\mathscr{K})}*\nu}\cdots\mathbb{U}_{\mathrm{V}_{2k-1}}^{\gamma_{w(\mathscr{K})}*\nu}\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu}=\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu}=\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}.$$

In

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}}*\nu}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$$

we have $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}}*\nu}=\{\mathrm{Id}\}$ and $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}}*\nu}\subset\mathrm{U}_0$, therefore

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})=\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}=\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu},$$

since $\mu_{\gamma_{\mathscr{K}}}=\mu_{\gamma_{w(\mathscr{K})}}$. This finishes the proof of (17) and that of Proposition 5.5. □

7C. *Proof of Proposition 5.20.* The remainder of this section, through page 494, is devoted to the proof of Proposition 5.20. Let ν be a combinatorial gallery.

Relation R1. For $z\neq\bar{x}$:

$$\begin{aligned} y\ x\ z &\equiv y\ z\ x \quad \text{for } x\leq y<z,\\ x\ z\ y &\equiv z\ x\ y \quad \text{for } x<y\leq z.\end{aligned}$$

Lemma 7.6. *Let $w_1=y\ x\ z$, $w_2=y\ z\ x$, $w_3=x\ z\ y$, and $w_4=z\ x\ y$ for $z\neq\bar{x}$.*

(a) $\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}=\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}$.

(b) $\overline{\pi(\mathrm{C}_{\gamma_{w_3}*\nu})}=\overline{\pi(\mathrm{C}_{\gamma_{w_4}*\nu})}$.

Proof. Recall the notation $\varepsilon_{\bar{a}}=-\varepsilon_a$ and $\bar{\bar{i}}=i$ for any $i\in\{1,\dots,n\}$. Note that the $\mathrm{T}^{\geq 3}_{\gamma_{w_i}*\nu}$ all coincide for $i\in\{1,2,3,4\}$; we will denote them by T^w. We divide the proof of Lemma 7.6 into three cases.

<u>Case 1: $x<y<z$.</u> We claim that if $z\neq\bar{y}$ and $y\neq\bar{x}$, the following equalities hold:

i. $\pi(\mathrm{C}_{\gamma_{w_1}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w$.

ii. $\pi(\mathrm{C}_{\gamma_{w_2}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{T}^w$.

iii. $\pi(\mathrm{C}_{\gamma_{w_3}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$.

iv. $\pi(\mathrm{C}_{\gamma_{w_4}*\nu})=\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$.

Before proving this we remark that, regardless of whether x, y, and z are barred or unbarred, the roots $\varepsilon_x-\varepsilon_z$, $\varepsilon_y-\varepsilon_z$, and $\varepsilon_x-\varepsilon_y$ are positive. Now we recall the notation from Theorem 2.9:

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_i}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_i}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_i}*\nu}\mathrm{T}^w.$$

Assume that $z\neq\bar{y}$ and $y\neq\bar{x}$.

i. We have $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \in \mathbb{U}_1^{\gamma_{w_1}*\nu}$ for any $v_{x\bar{y}} \in \mathbb{C}$, hence

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{l\bar{y}})\mathrm{T}^w \subseteq \pi(\mathrm{C}_{\gamma_{w_1}*\nu}).$$

Out of all generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_1}*\nu}$ for $i \in \{0, 1, 2\}$, the only one that does not belong to U_0 is of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \in \mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_1}*\nu}$, and the ones from $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_1}*\nu}$ that do not commute with it are those of the form $\mathrm{U}_{(\varepsilon_y+\varepsilon_z,1)}(a)$, but in that case Chevalley's commutator formula produces a term $\mathrm{U}_{(\varepsilon_x+\varepsilon_z,0)}(c^{1,1}_{x\bar{y},yz}(-v_{x\bar{y}})a) \in \mathrm{U}_0$. This implies the other inclusion, together with Proposition 2.7, which allows us to write down the generators of each $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_1}*\nu}$ in any order.

ii. The only generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_2}*\nu}$, for $i \in \{0, 1, 2\}$, that do not belong to U_0 are those of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_2}*\nu}$ or the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}}) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_2}*\nu}$. The equality follows by Proposition 2.7, Theorem 2.9, and Proposition 7.1.

iii. All the generators of $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_3}*\nu}$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_3}*\nu}$ belong to U_0, and the only generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_3}*\nu}$ that do not are $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}$. Thus iii follows by Proposition 7.1 and Theorem 2.9.

iv. As in the previous cases, we have

$$\pi(\mathrm{C}_{\gamma_{w_4*\nu}}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_4}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_4}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}\mathrm{T}^w,$$

and $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_4}*\nu} \subset \mathrm{U}_0$. All generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_4}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}$, respectively, belong to U_0 except for $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(a) \in \mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_4}*\nu}$ and $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(b) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}$, respectively, for $\{a, b\} \subset \mathbb{C}$. To prove iv we observe that $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(a)$ commutes with all generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_4}*\nu}$ except for $\mathrm{U}_{(\varepsilon_z+\varepsilon_y,1)}(d)$, with $d \in \mathbb{C}$. However, commuting the latter two terms produces elements $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(c^{1,1}_{x\bar{z},zy}(-a)d) \in \mathrm{U}_0$. Therefore

$$\pi(\mathrm{C}_{\gamma_{w_4}*\nu}) \subseteq \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w,$$

and the other inclusion is clear by Proposition 7.1 and the above discussion. This finishes the proof of our claim.

Now we use this to prove Lemma 7.6, assuming $z \neq \bar{y}$ and $y \neq \bar{x}$. For both conclusions (a) and (b) of the lemma, our equalities i–iv immediately imply

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_2}*\nu}) \quad \text{and} \quad \pi(\mathrm{C}_{\gamma_{w_3}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_4}*\nu}).$$

Next we will show that

$$\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}.$$

For this, let $v_{y\bar{z}} \in \mathbb{C}$ and $v_{x\bar{y}} \in \mathbb{C}$ with $v_{x\bar{y}} \neq 0$. Then since $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar{z}}) \in \mathrm{U}_{\mu_w} \cap \mathrm{U}_0$ for any $v_{y\bar{z}} \in \mathbb{C}$ Chevalley's commutator formula, and Proposition 7.1 imply

$$\begin{aligned}\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) &\supset \mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar{z}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w \\ &= \mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{y\bar{z},v\bar{y}}(-v_{y\bar{z}})v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar{z}})\mathrm{T}^w \\ &= \mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{y\bar{z},v\bar{y}}(-v_{y\bar{z}})v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w.\end{aligned}$$

Therefore

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{T}^w \subset \pi(\mathrm{C}_{\gamma_{w_1}*\nu}),$$

as long as $v_{x\bar{y}} \neq 0$, since in that case $c^{1,1}_{y\bar{z},v\bar{y}}(-v_{y\bar{z}})v_{x\bar{y}} = v_{x\bar{z}}$ has a solution in $v_{y\bar{z}}$. Hence Proposition 7.1 implies

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{T}^w \subset \pi(\mathrm{C}_{\gamma_{w_1}*\nu}).$$

Equalities i and ii then imply that a dense subset of $\pi(\mathrm{C}_{\gamma_{w_2}*\nu})$ is contained in $\pi(\mathrm{C}_{\gamma_{w_1}*\nu})$, which implies Lemma 7.6(a). To finish the proof of Lemma 7.6(b), let $v_{x\bar{y}} \in \mathbb{C}$ and $v_{y\bar{z}} \in \mathbb{C}$ with $v_{y\bar{z}} \neq 0$. Then, just as for (a),

$$\pi(\mathrm{C}_{\gamma_{w_3}*\nu}) \supset \mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w \tag{34}$$

$$= \mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}\big(c^{1,1}_{x\bar{y},y\bar{z}}(-v_{x\bar{y}})v_{y\bar{z}}\big)\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{y\bar{z}})\mathrm{T}^w \tag{35}$$

$$= \mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{x\bar{y},y\bar{z}}(-v_{x\bar{y}})v_{y\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w. \tag{36}$$

Therefore the elements of the set

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$$

such that $v_{y\bar{z}} \neq 0$ are contained in (36). By items iii and iv and Proposition 7.1 there is a dense subset of

$$\pi(\mathrm{C}_{\gamma_{w_4}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$$

that is contained in $\pi(\mathrm{C}_{\gamma_{w_3}*\nu})$.

The cases $z = \bar{y}$ and $y = \bar{x}$ are missing so far. (Note that $z \neq \bar{x}$ is not allowed. Also note that if $y = \bar{x}$ then x must be unbarred and if $z = \bar{y}$ then y must be unbarred.)

Now assume $z = \bar{y}$. To prove Lemma 7.6(a) in this case, we first show that

$$\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}. \tag{37}$$

All of the generators of $\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_1}$ belong to U_0 except for $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})$, for $v_{x\bar{y}} \in \mathbb{C}$. The generators of $\mathbb{U}^{\gamma_{w_1}*\nu}_{\mathrm{V}_1}$ are $\mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})$ for $l \neq x$ and $v_{l\bar{y}} \in \mathbb{C}$, and

$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{x\bar{y}})$ for $v_{x\bar{y}} \in \mathbb{C}$. This last term commutes with $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})$. Therefore, by parallel arguments to those given in the proof of equalities i–iv on page 474,

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \prod_{\substack{l<y\\ l\neq x}} \mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})\mathrm{T}^w.$$

All terms in the product

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}}) \prod_{\substack{l<y\\ l\neq x}} \mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})$$

are at the same time generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_2}}$ as well. Therefore, by Proposition 7.1,

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_2}*\nu}),$$

as wanted. Next we would like to show

$$\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}. \tag{38}$$

To do so we will make use of Proposition 5.5. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x & x & y \\ \hline \bar{y} & \bar{y} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x & \overline{y-1} & y \\ \hline \multicolumn{1}{c|}{} & \bar{y} & \overline{y-1} \\ \cline{2-3}\end{array}.$$

Then we have $w_1 = y\ x\ \bar{y} = w(\mathscr{K}_1)$ and $w_2 = y\ \bar{y}\ x = w(\mathscr{K}_2)$. By Proposition 5.5 it then suffices to show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}})}.$$

First assume $y-1 \neq x$. In this case $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_2}*\nu}$ is generated by terms $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)$ with $a \in \mathbb{C}$, and all generators of $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0. Out of these, the only ones in $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ that do not commute with $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)$ are $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)$ and $\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d)$. Then for every element in $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ there is a $u \in \mathrm{U}_0$ such that it belongs to

$$\begin{aligned} u\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)u'\mathrm{T}^w = uu'&\mathrm{U}_{(\varepsilon_{y-1}+\varepsilon_x,-1)}\big(c^{1,1}_{y-1\bar{y},xy}(-a)b\big) \\ &\cdot \mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}\big(c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d\big)\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w, \end{aligned}$$

where $u' = \mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d)$.

Fix u, a, b, and d such that $abd \neq 0$. Such elements form a dense subset of $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. We will show that

$$\mathrm{U}_{(\varepsilon_{y-1}+\varepsilon_x,-1)}\big(c^{1,1}_{y-1\bar{y},xy}(-a)b\big)\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}\big(c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d\big)\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w$$

is contained in $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$. If this is true, then (38) is implied by Proposition 7.1 applied to $u\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d) \in \mathrm{U}_0$.

First note that for any $\{a_{x\bar{y}}, a_{y-1\bar{y}}, a_{yy-1}\} \subset \mathbb{C}$, both $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1y})$ and $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})$ belong to $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$, and $v = \mathrm{U}_{(\varepsilon_y+\varepsilon_{y-1},0)}(a_{yy-1}) \in \mathrm{U}_{\varepsilon_x} \cap \mathrm{U}_0$ stabilizes the truncated image T^w as well as the whole image $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$. Therefore all elements of

$$v^{-1}\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})v\mathrm{T}^w = \mathrm{U}_{(\varepsilon_x+\varepsilon_{y-1},-1)}(c^{1,1}_{x\bar{y},yy-1}(-a_{x\bar{y}})a_{yy-1})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})\mathrm{T}^w$$

belong to $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ and, since $abd \neq 0$, we may find $a_{x\bar{y}}$, $a_{y-1\bar{y}}$, and a_{yy-1} such that

$$a_{x\bar{y}} = c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d, \quad c^{1,1}_{x\bar{y},yy-1}(-a_{x\bar{y}})a_{yy-1} = c^{1,1}_{y-1\bar{y},xy}(-a)b, \quad a_{y-1\bar{y}} = a.$$

This concludes the proof if $y \neq x-1$. Now assume that $y = x-1$. In this case all generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathrm{K}_2}*\nu}$ commute with $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})$, and therefore all elements in $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w,$$

for some $u \in \mathrm{U}_0$ and $a \in \mathbb{C}$ — but $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a) \in \mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$, which implies (38) by applying Proposition 7.1 to $u \in \mathrm{U}_0$.

Next we prove Lemma 7.6(b), still assuming $z = \bar{y}$. We now have

$$w_3 = x\ \bar{y}\ y = w(\mathscr{K}_3) \quad \text{and} \quad w_4 = \bar{y}\ x\ y = w(\mathscr{K}_4),$$

where

$$\mathscr{K}_3 = \begin{array}{|c|c|c|} \hline y & x & x \\ \hline \multicolumn{1}{c|}{} & \bar{y} & \bar{y} \\ \cline{2-3} \end{array} \quad \text{and} \quad \mathscr{K}_4 = \begin{array}{|c|c|c|} \hline x & x & \bar{y} \\ \hline y & y & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}.$$

We want to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

First $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_3}*\nu}$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}$ are both contained in U_0. The generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_3}*\nu}$ that do not belong to U_0 are $\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)$, $\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})$, and $\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})$ for $\{\alpha_y, \beta_{yl}, \gamma_{y\bar{s}}\} \subset \mathbb{C}$ and $l \leq n$, $l \neq x$, $y < s \leq n$. All of these are also generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$, hence by Proposition 7.1 and Theorem 2.9 we have

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) \subset \pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}).$$

The discussion above also implies the equality

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{y<s\leq n}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w. \tag{39}$$

There is one more generator of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$ not mentioned above, $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})$.

Since all generators of $\mathbb{U}_{V_2}^{\gamma_{\mathscr{K}_4}*\nu}$ (which are $U_{(\varepsilon_x+\varepsilon_y,0)}(d') \in U_0$ for $d' \in \mathbb{C}$) commute with those of $\mathbb{U}_{V_1}^{\gamma_{\mathscr{K}_3}*\nu}$, we have by Proposition 7.1,

$$\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu}) = U_0 U_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy}) U_{(\varepsilon_y,-1)}(a_y) \prod_{\substack{l\le n\\ l\ne x}} U_{(\varepsilon_y+\varepsilon_l,-1)}(b_{yl}) \prod_{\substack{s\le n\\ s>y}} U_{(\varepsilon_y-\varepsilon_s,-1)}(c_{y\bar{s}}) T^w.$$

We now would like to show

$$\overline{\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu})} \subset \overline{\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu})}.$$

To do this we will see that for complex numbers a_y, b_{yl}, $c_{y\bar{s}}$, and d_{xy}, with $a_y \neq 0$,

$$U_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy}) U_{(\varepsilon_y,-1)}(a_y) \prod_{\substack{l\le n\\ l\ne x}} U_{(\varepsilon_y+\varepsilon_l,-1)}(b_{yl}) \prod_{\substack{s\le n\\ s>y}} U_{(\varepsilon_y-\varepsilon_s,-1)}(c_{y\bar{s}}) T^w \subset \pi'''(C_{\gamma_{\mathscr{K}_3}*\nu}). \tag{40}$$

By (39) we conclude that for any complex numbers α_y, β_{yl}, $\gamma_{y\bar{s}}$, and δ the following set is contained in $\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu})$:

$$\begin{aligned} &v^{-1} U_{(\varepsilon_x-\varepsilon_y,1)}(\delta) U_{(\varepsilon_y,-1)}(\alpha_y) \prod_{\substack{l\le n\\ l\ne x}} U_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl}) \prod_{\substack{s\le n\\ s>y}} U_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}}) T^w \\ &= v^{-1} v U_{(\varepsilon_x+\varepsilon_y,-1)}(\rho_{xy}) U_{(\varepsilon_y,-1)}(\alpha_y) \prod_{\substack{l\le n\\ l\ne x}} U_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl}) \prod_{\substack{s\le n\\ s>y}} U_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}}) T^w, \end{aligned} \tag{41}$$

where

$$v = U_{(\varepsilon_x,0)}\big(c^{1,1}_{x\bar{y},y}(-\delta)\alpha_y\big) \prod_{\substack{l\le n\\ l\ne x}} U_{(\varepsilon_x+\varepsilon_l,0)}\big(c^{1,1}_{x\bar{y},yl}(-\delta)\beta_{yl}\big) \prod_{\substack{s\le n\\ s>y}} U_{(\varepsilon_x-\varepsilon_s,0)}\big(c^{1,1}_{x\bar{y},y\bar{s}}(-\delta)\gamma_{y\bar{s}}\big)$$

and $\rho_{xy} = c^{1,2}_{x\bar{y},y}(-\delta)\alpha_y^2$, and where the equality in (41) is obtained by applying Chevalley's commutator formula (2) and Proposition 7.1 to $U_{(\varepsilon_x-\varepsilon_y,1)}(\delta)$, which stabilizes the truncated image T^w. We will have shown our claim in (40) if we find complex numbers α_y, β_{yl}, $\gamma_{y\bar{s}}$, and δ such that

$$c^{1,2}_{x\bar{y},y}(-\delta)\alpha_y^2 = d_{xy}, \quad \alpha_y = a_y, \quad \beta_{yl} = b_{yl},$$

which we may obtain since $a_y \neq 0$. This concludes the proof in case $z = \bar{y}$.

Lastly assume $y = \bar{x}$. This means that x is necessarily unbarred and therefore $z = \bar{b}$ for some $b < x$.

To prove Lemma 7.6(a) in this case, as before, we use Proposition 5.5. We have

$$w_1 = \bar{x}\ x\ \bar{b} = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = \bar{x}\ \bar{b}\ x = w(\mathscr{K}_2),$$

where

$$\mathcal{K}_1 = \begin{array}{|c|c|c|}\hline x & x & \bar{x} \\ \hline \bar{b} & \bar{b} & \\ \hline \end{array} \quad \text{and} \quad \mathcal{K}_2 = \begin{array}{|c|c|c|}\hline x & \bar{x} & \bar{x} \\ \hline & \bar{b} & \bar{b} \\ \hline \end{array}.$$

First we show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathcal{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathcal{K}_2}*\nu})}. \tag{42}$$

To do this, we claim that

$$\pi'(\mathrm{C}_{\gamma_{\mathcal{K}_1}*\nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{T}^w. \tag{43}$$

Indeed, $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$ for $s\in\mathscr{C}_n$ such that $s\neq b$ are the generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_1}*\nu}$ that do not belong to U_0, and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathcal{K}_1}*\nu}$ is the identity, because $\varepsilon_x-\varepsilon_b$ is not a positive root. Therefore (43) follows by Proposition 7.1. The aforementioned terms are also generators (but not all!) of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathcal{K}_2}*\nu}$; therefore (42) follows. Now we show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathcal{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathcal{K}_1}*\nu})}. \tag{44}$$

To do this, let us first analyze the image

$$\pi''(\mathrm{C}_{\gamma_{\mathcal{K}_2}*\nu}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathcal{K}_2}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_2}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathcal{K}_2}*\nu}\mathrm{T}^w.$$

In this case $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathcal{K}_2}*\nu} \subset \mathrm{U}_0$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_2}*\nu}$ is the identity, because $-(\varepsilon_x+\varepsilon_b)$ is not a positive root. The generators of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathcal{K}_2}*\nu}$ are $\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x)$, $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})$ for $s\in\mathscr{C}_n$ such that $s\neq b$ and complex numbers α_x, α_{xs}, and α_{xb}. Therefore

$$\pi(\mathrm{C}_{\gamma_{\mathcal{K}_2}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})\mathrm{T}^w. \tag{45}$$

Let us fix complex numbers α_x, α_{xs}, and α_{xb}, such that $\alpha_x\neq 0$. We will show, as for (43), that

$$\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})\mathrm{T}^w \subset \pi'(\mathrm{C}_{\gamma_{\mathcal{K}_1}*\nu}). \tag{46}$$

To do this we will use Corollary 2.10, which says, in particular, that if we write

$$\gamma_{\mathcal{K}_1} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2, \mathrm{E}_2, \mathrm{V}_3),$$

then

$$\pi'(\mathrm{C}_{\gamma_{\mathcal{K}_1}}) \supset \mathrm{U}_{\mathrm{V}_0}\mathrm{U}_{\mathrm{V}_1}\mathrm{U}_{\mathrm{V}_2}\mathrm{T}^w.$$

Therefore, since $u = \mathrm{U}_{(\varepsilon_b-\varepsilon_x,0)}(a) \in \mathrm{U}_{\mathrm{V}_2} \cap \mathrm{U}_0$ for all $a \in \mathbb{C}$, and since $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$, for $s \in \mathscr{C}_n$ and $s \neq b$, are the generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_1}*\nu} \subset \mathrm{U}_{\mathrm{V}_1}$, by using Proposition 7.1 applied to $u \in \mathrm{U}_0$ and $v \in \mathrm{U}_{\mathrm{V}_3}$ (V_3 stabilizes the truncated image T^w, see below for a definition of v), we have the following. For any complex numbers a_{xs} and a_x,

$$\begin{aligned}\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) &\supset u^{-1}\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})u\mathrm{T}^w\\
&= u^{-1}u\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}\big(c^{2,1}_{x,b\bar{x}}(a_x^2)b\big)\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})v\mathrm{T}^w\\
&= \mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}\big(c^{2,1}_{x,b\bar{x}}(a_x^2)b\big)\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{T}^w,\end{aligned}$$

where

$$v = \mathrm{U}_{(\varepsilon_b,-1)}\big(c^{1,1}_{x,b\bar{x}}(-a_x)b\big) \prod_{\substack{s\neq b\\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_b+\varepsilon_s,-1)}\big(c^{1,1}_{x,bs}(-a_{xs})b\big) \in \mathrm{U}_{\mathrm{V}_3}.$$

In order to show (46) it suffices to find complex numbers a_x, a_{xs}, and b such that

$$c^{2,1}_{x,b\bar{x}}(a_x^2)b = \alpha_{xb}, \quad a_x = \alpha_x, \quad a_{xs} = \alpha_{xs},$$

and we may do this, since $\alpha_x \neq 0$.

For (b), we again use Proposition 5.5. We have

$$w_3 = x\ \bar{b}\ \bar{x} = w(\mathscr{K}_3) \quad \text{and} \quad w_4 = \bar{b}\ x\ \bar{x} = w(\mathscr{K}_4),$$

where

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline \bar{x} & x & x \\ \hline \multicolumn{1}{c|}{} & \bar{b} & \bar{b} \\ \cline{2-3}\end{array} \quad \text{and} \quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline x-1 & x & \bar{b} \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}.$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}. \tag{47}$$

We analyze both images $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ and $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$ separately and then show (47). First we observe that $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}$ is the identity (this is because $\varepsilon_x - \varepsilon_b$ is not a positive root). Hence

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathrm{U}_0 \prod_{\substack{l<x\\ l\neq b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_x,-1)}(a_{l\bar{x}})\mathrm{U}_{(\varepsilon_b-\varepsilon_x,-2)}(a_{b\bar{x}})\mathrm{T}^w. \tag{48}$$

Now, $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1x})$ for $\alpha_{x-1x} \in \mathbb{C}$, and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by $\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}})$ for $\alpha_{b\overline{x-1}} \in \mathbb{C}$, by $\mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})$ for

$l < x-1$ and $\alpha_{l\overline{x-1}} \in \mathbb{C}$ (this last element stabilizes the truncated image T^w), and by other elements of U_0. Therefore

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) \tag{49}$$

$$= \mathrm{U}_0 \prod_{\substack{l<x\\ l\neq b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1\bar{x}})\mathrm{T}^w \tag{50}$$

$$= \mathrm{U}_0 \prod_{\substack{l<x, l\neq b\\ l\neq x-1}} \mathrm{U}_{(\varepsilon_l-\varepsilon_x,-1)}(\xi_{l\bar{x}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1\bar{x}})\mathrm{U}_{(\varepsilon_b-\varepsilon_x,-2)}(\xi_{b\bar{x}})\mathrm{T}^w, \tag{51}$$

where

$$\xi_{b\bar{x}} = c^{1,1}_{b\overline{x-1},x-1\bar{x}}(-\alpha_{b\overline{x-1}}\alpha_{x-1\bar{x}}), \qquad \xi_{l\bar{x}} = c^{1,1}_{l\overline{x-1},x-1\bar{x}}(-\alpha_{l\overline{x-1}}\alpha_{x-1\bar{x}}),$$

and where the equality between (50) and (51) arises by using (2) and Proposition 7.1 applied to

$$\mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}}) \in \mathrm{U}_{\mu_{\gamma_{\mathscr{K}_4}}}.$$

The sets displayed in (48) and (51) are equal as long as all the parameters are nonzero.

<u>Case 2: $x = y < z$ and $z \neq \bar{x}$.</u> In this case we have $w_1 = y\ y\ z$ and $w_2 = y\ z\ y$. We want to look at

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_1}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_1}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_1}*\nu}\mathrm{T}^w,$$
$$\pi(\mathrm{C}_{\gamma_{w_2}*\nu}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{w_2}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{w_2}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{w_2}*\nu}\mathrm{T}^w.$$

In this case all generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_1}*\nu}$ and of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{w_2}*\nu}$ belong to U_0 for $i \in \{1, 2, 3\}$. Therefore Proposition 7.1 implies in this case that

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathrm{U}_0\mathrm{T}^w = \pi(\mathrm{C}_{\gamma_{w_2}*\nu}).$$

<u>Case 3: $x < y = z$ and $z \neq \bar{x}$.</u> Here it will be convenient to use Proposition 5.5. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|}\hline y & x \\ \hline \multicolumn{1}{c|}{} & y \\ \cline{2-2}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|}\hline x & y \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

It is then enough to show (by Proposition 5.5) that

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})},$$

since

$$w_1 = x\ y\ y = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = y\ x\ y = w(\mathscr{K}_2).$$

However, this case is now the same as the previous one: all generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_1}*\nu}$ and $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0, therefore, as before,

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = \mathrm{U}_0\mathrm{T}^w = \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}).$$

With this case we conclude the proof of Lemma 7.6. $\square$

Relation R2. For $1 < x \leq n$ and $x \leq y \leq \bar{x}$:

$$y\ \overline{x-1}\ x-1 \equiv y\ x\ \bar{x},$$
$$\overline{x-1}\ x-1\ y \equiv x\ \bar{x}\ y.$$

Lemma 7.7. *Let*

$$w_1 = y\ \overline{x-1}\ x-1, \quad w_2 = y\ x\ \bar{x}, \quad w_3 = \overline{x-1}\ x-1\ y, \quad w_4 = x\ \bar{x}\ y,$$

then

(a) $\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})} = \overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}$,

(b) $\overline{\pi(\mathrm{C}_{\gamma_{w_3}*\nu})} = \overline{\pi(\mathrm{C}_{\gamma_{w_4}*\nu})}$.

Proof. As usual, the proof is divided in some cases. We first consider the case where $y \notin \{x, \bar{x}\}$ and then we analyze $y = x$ and $y = \bar{x}$ separately.

<u>Case 1: $y \notin \{x, \bar{x}\}$.</u>

Note that

$$w_1 = y\ \overline{x-1}\ x-1 = w\left(\begin{array}{|c|c|c|}\hline x-1 & y & y \\ \hline & \overline{x-1} & \overline{x-1} \\ \hline\end{array}\right), \quad w_2 = y\ x\ \bar{x} = w\left(\begin{array}{|c|c|c|}\hline x-1 & x & y \\ \hline \bar{x} & \overline{x-1} & \\ \hline\end{array}\right).$$

Hence by Proposition 5.5, to show (a) it is enough to show that

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})},$$

where

$$\mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & y \\ \hline \bar{x} & \overline{x-1} & \\ \hline\end{array} \quad \text{and} \quad \mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & y & y \\ \hline & \overline{x-1} & \overline{x-1} \\ \hline\end{array}.$$

First we check that

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})}.$$

Clearly $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu} \subset \mathrm{U}_0$. The only generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_2}*\nu}$ that do not belong to U_0 are those of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)$, for $a \in \mathbb{C}$, and those in $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ are $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$, for $b \in \mathbb{C}$. This means that every element in $\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}$ belongs to

$$u\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{T}^w,$$

for some $u \in \mathrm{U}_0$. Both $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)$ and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$ belong to $\mathrm{U}_{\varepsilon_y-\varepsilon_{x-1}}$, and this implies the contention by Proposition 7.1 and Corollary 2.10. Now we want to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

By Theorem 2.9, all elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ belong to the set

$$(52)\quad u\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(v_{x-1\bar{y}})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1}) \cdot \prod_{\substack{l>x\\ l\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}}) \prod_{s\neq y} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w,$$

for $u \in \mathrm{U}_0$ and $v_{x-1j} \in \mathbb{C}$. This is because both $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_1}*\nu}$ and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_1}*\nu}$ are contained in U_0. Fix such an element such that $v_{x-1\bar{x}} \neq 0$. We know that

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu},$$

and that for any $a_{x\bar{y}} \in \mathbb{C}$, $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}}) \in \mathrm{U}_{\varepsilon_y}$. This means that these elements stabilize both the truncated images $\mathrm{T}^{\geq 3}_{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathrm{T}^{\geq 1}_{\gamma_{\mathscr{K}_2}*\nu}$. Hence the elements in

$$(53)\quad \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w = \mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}\big(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}}\big) \cdot \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w$$

all belong to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. More they belong to precisely to $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_1}*\nu}\mathrm{T}^w \subset \mathrm{T}^{\geq 1}_{\gamma_{\mathscr{K}_1}*\nu}$, hence by Proposition 7.1, we may multiply the right side of equation (53) by $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(-v_{x\bar{y}})$ on the left and the product still belongs to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$, hence

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}\big(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}}\big)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w \subset \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}).$$

Now consider the product

$$u = \mathrm{U}_{(\varepsilon_y+\varepsilon_x,1)}(a_{yx})\mathrm{U}_{(\varepsilon_x,0)}(a_x)\prod_{\substack{l>x\\ l\neq y}}\mathrm{U}_{(\varepsilon_x-\varepsilon_l,0)}(a_{x\bar{l}})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_x+\varepsilon_s,0)}(a_{xs}) \in \mathrm{U}_{\varepsilon_y}\cap\mathrm{U}_0.$$

Proposition 7.1 then implies that

$$\begin{aligned}\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) &\supset u^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})u\mathrm{T}^w \\ &= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(\rho_{x-1y}) \\ &\quad\cdot\prod_{\substack{l>x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1l}) \\ &\quad\cdot\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w,\end{aligned}$$

with

$$\begin{aligned}
\rho_{x-1x} &= c^{1,2}_{x-1\bar{x},x}(-v_{x-1\bar{x}})a_x^2 - c^{1,1}_{x-1y,yx}c^{1,1}_{x-1\bar{x},x\bar{y}}(v_{x-1\bar{x}})a_{x\bar{y}}a_{yx},\\
\rho_{x-1j} &= c^{1,1}_{x-1\bar{x},xj}(-v_{x-1\bar{x}})a_{xj}\, j\neq y,\ j\in\{\bar{l}: l>x\}\cup\{s:\varepsilon_{x-1}+\varepsilon_s\in\Phi^+\},\\
\rho_{x-1} &= c^{1,1}_{x-1\bar{x},x}(-v_{x-1\bar{x}})a_x.
\end{aligned}$$

The system of equations defined by $v_{x-1}=\rho_{x-1}$ and $v_{x-1j}=\rho_{x-1j}$ does have solutions (the variables are a_x, a_{yx}, $a_{x\bar{l}}$, and a_{xs}) since $v_{x-1,x}\neq 0$. This means that for such solutions we have (see (52))

$$\begin{aligned}
&\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(v_{x-1\bar{y}})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\\
&\qquad\cdot\prod_{\substack{l\geq x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w\\
&\ =\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(\rho_{x-1y})\\
&\qquad\cdot\prod_{\substack{l>x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1l})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w\\
&\ \subset\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}),
\end{aligned}$$

and so by Proposition 7.1 we get that all elements in (52) belong to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. All such elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ form a dense open subset. This finishes the proof in this case.

We turn to (b). Let

$$\mathscr{K}_3=\begin{array}{|c|c|c|}\hline x-1 & x-1 & \overline{x-1}\\\hline y & y & \\\cline{1-2}\end{array}\quad\text{and}\quad\mathscr{K}_4=\begin{array}{|c|c|c|}\hline y & x-1 & x\\\hline & \bar{x} & \overline{x-1}\\\cline{2-3}\end{array}.$$

Then $w_3=\overline{x-1}\ x-1\ y=w(\mathscr{K}_3)$ and $w_4=x\ \bar{x}\ y=w(\mathscr{K}_4)$. As in (a), by Proposition 5.5, it is enough to show that

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})}=\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

To show

$$\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}\subset\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})},$$

note first that the only generator of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_4}*\nu}$ that does not belong to U_0 is

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\in\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu},\ \text{ for } a\in\mathbb{C}.$$

Of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$, the only generators that do not commute with $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)$ are $\mathrm{U}_{(\varepsilon_y+\varepsilon_x,0)}(b)$, with $b\in\mathbb{C}$. Then Chevalley's commutator formula (2) implies that all elements of $\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}*\nu})$ belong to the set

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}\big(c^{1,1}_{x-1\bar{x},xy}(-a)b\big)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w. \tag{54}$$

Since both $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}\big(c^{1,1}_{x-1\bar{x},xy}(-a)b\big)$ and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)$ belong to $\mathbb{U}^{\gamma_{\mathcal{K}_3}*\nu}_{\mathrm{V}_1}$, the desired contention follows by Proposition 7.1. Now we show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}})} \subset \overline{\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}})}. \tag{55}$$

The proof is similar to that of (a), but there are some subtle differences. First we look at the image $\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu})$. Out of all the generators of $\mathbb{U}^{\gamma_{\mathcal{K}_3}*\nu}_{\mathrm{V}_i}$, the only ones that do not belong to U_0 belong to $\mathbb{U}^{\gamma_{\mathcal{K}_3}*\nu}_{\mathrm{V}_1}$: $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)$, $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})$ for $l \neq x-1$, $s > x$, $s \neq y$, and complex numbers v_{x-1}, v_{x-1s}, and v_{x-1l}. The group $\mathbb{U}^{\gamma_{\mathcal{K}_3}*\nu}_{\mathrm{V}_2}$ has as generators the terms $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,0)}(a)$ (only), and these commute with all the latter terms. Therefore all elements of $\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\prod_{\substack{s>x-1\\ s\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})\prod_{l\neq x-1}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})\mathrm{T}^w, \tag{56}$$

for some $u \in \mathrm{U}_0$. Fix such a u, and assume $v_{x-1\bar{x}} \neq 0$ and $v_{x-1y} \neq 0$. Such elements as (56) form a dense open subset of $\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu})$. Now, for all complex numbers a, a_{xy}, and $a_{x\bar{y}}$ we have $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a) \in \mathbb{U}_1^{\gamma_{\mathcal{K}_4}*\nu}$, $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(a_{xy}) \in \mathbb{U}_1^{\gamma_{\mathcal{K}_4}*\nu}$, and $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(a_{x\bar{y}}) \in \mathrm{U}_0$, which stabilizes the truncated image $\mathrm{T}^{\geq 2}_{\gamma_{\mathcal{K}_4}*\nu}$. Therefore, setting $c = \mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(a_{xy})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(a_{x\bar{y}}) \in \mathrm{U}_0$, all elements in

$$\begin{aligned}
c^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)c\mathrm{T}^w &= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\\
&\qquad\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-1)}\big(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}\big)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w\\
&= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\\
&\qquad\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-1)}\big(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}\big)\mathrm{T}^w\\
&= \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\\
&\qquad\cdot\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w
\end{aligned}$$

belong to $\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}*\nu})$, where

$$\begin{aligned}
\rho_{x-1x} &= c^{1,1}_{x-1y,x\bar{y}}c^{1,1}_{x-1\bar{x},xy}aa_{xy}a_{x\bar{y}},\\
\rho_{x-1y} &= c^{1,1}_{x-1\bar{x},xy}(-a)a_{xy},
\end{aligned}$$

and where the last equality holds because $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-1)}\big(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}\big) \in \mathrm{U}_{\varepsilon_y}$, and all elements of the latter stabilize the truncated image T^w by Proposition 7.1.

Now let

$$c' = \mathrm{U}_{(\varepsilon_x,0)}(a_x) \prod_{\substack{s>x \\ s\neq y}} \mathrm{U}_{(\varepsilon_x-\varepsilon_s,0)}(a_{x\bar{s}}) \prod_{\substack{l\neq x-1 \\ l\neq y}} \mathrm{U}_{(\varepsilon_x+\varepsilon_l,0)}(a_{xl}) \in \mathrm{U}_{\varepsilon_y} \cap \mathrm{U}_0,$$

for a_x, $a_{x\bar{s}}$, and a_{xl} complex numbers; by Proposition 7.1 this element stabilizes the truncated image T^w and the image $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$. Therefore the following are contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})$,

$$\begin{aligned}
&c'^{-1}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\rho_{x-1y})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)c'\mathrm{T}^w \\
&\qquad = \mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_x) \qquad (57)\\
&\qquad\quad \cdot \prod_{\substack{s>x-1 \\ s\neq y \\ s\neq x}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho'_{x-1x}) \\
&\qquad\quad \cdot \prod_{l\notin\{x-1,x\}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(\rho_{x-1l})\mathrm{T}^w, \qquad (58)
\end{aligned}$$

where

$$\begin{aligned}
\rho_{x-1} &= c^{1,1}_{x-1x,x}(-a)a_x, \\
\rho'_{x-1x} &= \rho_{x-1x} + c^{1,2}_{x-1x,x}(-a)a_x^2, \\
\rho_{x-1l} &= c^{1,1}_{x-1\bar{x},xl}(-a)a_{xl}, \\
\rho_{x-1\bar{s}} &= c^{1,1}_{x-1\bar{x},x\bar{s}}(-a)a_{x\bar{s}}.
\end{aligned}$$

We want to show that

$$\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1}) \prod_{\substack{s>x-1 \\ s\neq y}} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s}) \prod_{l\neq x-1} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})\mathrm{T}^w$$

is equal to the product in the last lines (57) and (58) above (see (56)), for some a_x, a_{xl}, and $a_{x\bar{s}}$. This determines a system of equations:

$$\begin{aligned}
v_{x-1\bar{x}} &= a, \\
v_{x-1x} &= c^{1,1}_{x-1y,x\bar{y}}c^{1,1}_{x-1\bar{x},xy}aa_{xy}a_{x\bar{y}} + c^{1,2}_{x-1x,x}(-a)a_x^2, \\
v_{x-1} &= c^{1,1}_{x-1x,x}(-a)a_x, \\
v_{x-1\bar{s}} &= c^{1,1}_{x-1\bar{x},x\bar{s}}(-a)a_{x\bar{s}}, \\
v_{x-1l} &= c^{1,1}_{x-1\bar{x},xl}(-a)a_{xl}, \\
v_{x-1y} &= c^{1,1}_{x-1\bar{x},xy}(-a)a_{xy}.
\end{aligned}$$

which can always be solved since $v_{x-1y} \neq 0$ and $v_{x\overline{x-1}} \neq 0$. This completes the proof of (b) in this case. $\square$

Case 1. $y = x$.

Proof. As in Case 1, we will make use of Proposition 5.5. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & x & x \\ \hline \multicolumn{1}{c|}{} & \overline{x-1} & \overline{x-1} \\ \cline{2-3}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & x \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}.$$

Then

$$w_1 = x\ \overline{x-1}\ x-1 = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = x\ x\ \bar{x} = w(\mathscr{K}_2).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

First we show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})}. \tag{59}$$

Since $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ is generated by elements of the form $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)$, for $a \in \mathbb{C}$, and the generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0, for $i \in \{1, 2\}$, all elements of $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ are of the form

$$u\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)\mathrm{T}^w$$

for some $u \in \mathrm{U}_0$. Since $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_1}*\nu}$, (59) follows by applying Proposition 7.1 to u. To finish the proof in this case it remains to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}. \tag{60}$$

The generators of $\mathbb{U}_{\mathrm{V}_i}^{\gamma_{\mathscr{K}_1}*\nu}$ belong to U_0, for $i \in \{0, 1\}$, and the generators that do not are $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)$, $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})$, $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})$, for $n \geq l > x$, $s \notin \{x, x-1\}$, and complex numbers v_x, $v_{x-1\bar{l}}$, v_{x-1s}, and $v_{x-1\bar{x}}$. Therefore all elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})\mathrm{T}^w.$$

Fix such $u \in \mathrm{U}_0$ and v_x, $v_{x-1\bar{l}}$, v_{x-1s}, and $v_{x-1\bar{x}}$ complex numbers such that $v_{x-1\bar{x}} \neq 0$. We know for any $a \in \mathbb{C}$, that $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a) \in \mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$. Let

$$q = \mathrm{U}_{(\varepsilon_x,1)}(a_x)\prod_{s>x}\mathrm{U}_{(\varepsilon_x-\varepsilon_s,1)}(a_{x\bar{s}})\prod_{l\neq x}\mathrm{U}_{(\varepsilon_x+\varepsilon_l,1)}(a_{xl}) \in \mathrm{U}_{(\varepsilon_x,1)} \cap \mathrm{U}_0$$

for any complex numbers a_x, $a_{x\bar{s}}$, and a_{xl}. Then by Proposition 7.1,

$$q^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)q\mathrm{T}^w \subset \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}). \tag{61}$$

As in the previous cases, we want to find a, a_x, $a_{x\bar{s}}$, and a_{xl} such that

$$t\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})\mathrm{T}^w$$

equals (61), for some $t \in \mathrm{U}_0$. But

$$\begin{aligned} &q^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)q\mathrm{T}^w \\ &\quad= t^{-1}\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)\mathrm{T}^w, \end{aligned}$$

where

$$\begin{aligned} t^{-1} &= \mathrm{U}_{(\varepsilon_x+\varepsilon_{x-1},0)}(c^{1,2}_{x-1\bar{x},x})(-a)a_x^2 \in \mathrm{U}_0, \\ \rho_x &= c^{1,1}_{x-1\bar{x},x}(-a)a_x, \\ \rho_{x-1\bar{l}} &= c^{1,1}_{x-1\bar{x},x\bar{l}}(-a)a_{x\bar{l}}, \\ \rho_{x-1s} &= c^{1,1}_{x-1\bar{x},xs}(-a)a_{xs}. \end{aligned}$$

The system

$$\begin{aligned} v_{x-1\bar{x}} &= a, \\ v_{x-1\bar{l}} &= \rho_{x-1\bar{l}}, \\ v_{x-1s} &= \rho_{x-1s} \end{aligned}$$

always has a solution since $v_{x-1\bar{x}} \neq 0$. This concludes the proof of Case 2. $\square$

Let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline x-1 & x-1 & \overline{x-1} \\ \hline x & x & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \text{and} \quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline x & x-1 & x \\ \hline \multicolumn{1}{c|}{} & x & \overline{x-1} \\ \cline{2-3}\end{array}.$$

Then

$$w_3 = \overline{x-1}\ x-1\ x = w(\mathscr{K}_3) \quad \text{and} \quad w_4 = x\ \bar{x}\ x = w(\mathscr{K}_4).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

To do this we will describe a common dense subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ and $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$.

Consider first $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_3}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_3}*\nu}\mathrm{T}^w$. We have $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$ and also $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$, since it is generated by the terms $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,0)}(d)$, for $d \in \mathbb{C}$. These commute with all generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_3}*\nu}$, out of which $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})$, $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})$, (for $s \leq n$, $s \neq x-1$, $l > x$, and v_{x-1}, v_{x-1s} and $v_{x-1\bar{l}}$ complex numbers) do not belong to U_0. Therefore $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$

coincides with

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\prod_{\substack{s\le n\\ s\ne x-1}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{T}^w, \tag{62}$$

for complex numbers v_{x-1}, v_{x-1s} and $v_{x-1\bar{l}}$. Now we look at elements of

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})=\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_4}*\nu}\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}\mathrm{T}^w.$$

Both $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_4}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ are contained in U_0, and $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by the elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)$, which belong to $\mathrm{U}_{\varepsilon_x}$ and therefore stabilize the truncated image T^w by Proposition 7.1. Now, by Proposition 2.7, we may write any element k of $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ as

$$k=\mathrm{U}_{(\varepsilon_x,0)}(k_x)\prod_{x<l\le n}\mathrm{U}_{(\varepsilon_x-\varepsilon_l,0)}(k_{x\bar{l}})\prod_{\substack{s\le n\\ s\ne x}}\mathrm{U}_{(\varepsilon_x+\varepsilon_s,0)}(k_{xs})\in\mathrm{U}_0$$

for some complex numbers k_x, $k_{x\bar{l}}$, and k_{xs}. Theorem 2.9 and Proposition 7.1 imply that

$$\begin{aligned}\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})&=\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)k\mathrm{T}^w\\&=\mathrm{U}_0k\mathrm{U}_{(\varepsilon_{x-1},-1)}(\sigma_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\sigma_{x-1x})\\&\qquad\cdot\prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\sigma_{x-1\bar{l}})\\&\qquad\cdot\prod_{\substack{s\le n\\ s\ne x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,0)}(\sigma_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)\mathrm{T}^w,\end{aligned}\tag{63}$$

for $k\in\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_4}*\nu}$ and $d\in\mathbb{C}$, where

$$\begin{aligned}\sigma_{x-1}&=c^{1,1}_{x-1\bar{x},x}(-d)k_x,\\ \sigma_{x-1x}&=c^{1,2}_{x-1\bar{x},x}(-d)k_x^2,\\ \sigma_{x-1\bar{l}}&=c^{1,1}_{x-1\bar{x},x\bar{l}}(-d)k_{x\bar{l}},\\ \sigma_{x-1s}&=c^{1,1}_{x-1\bar{x},xs}(-d)k_{xs}.\end{aligned}$$

The set (63) is clearly contained in (62). Moreover, the system

$$\begin{aligned}v_{x-1}&=\sigma_{x-1},\\ v_{x-1x}&=\sigma_{x-1x},\\ v_{x-1\bar{l}}&=\sigma_{x-1\bar{l}},\\ v_{x-1s}&=\sigma_{x-1s},\end{aligned}$$

has solutions for d, k_x, $k_{x\bar{l}}$, and k_{xs} as long as $\{v_{x-1}, v_{x-1x}, v_{x-1\bar{l}}, v_{x-1s}\} \subset \mathbb{C}^\times$. Proposition 7.1 then implies that a dense subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ is contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$, which finishes the proof of Case 1.

Case 2. $y = \bar{x}$.

Proof. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & \bar{x} & \bar{x} \\ \hline \multicolumn{1}{c|}{} & \overline{x-1} & \overline{x-1} \\ \cline{2-3}\end{array} \quad \text{and} \quad \mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & \bar{x} \\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}.$$

Then

$$w_1 = \bar{x}\ \overline{x-1}\ x-1 = w(\mathscr{K}_1) \quad \text{and} \quad w_2 = \bar{x}\ x\ \bar{x} = w(\mathscr{K}_2).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

In this case we have $\mathbb{U}_0^{\gamma_{\mathscr{K}_1}*\nu} = 1 = \mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_1}*\nu}$. Proposition 2.7 and Theorem 2.9 then say,

$$\begin{aligned}\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = {} & \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(v_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(v_{x-1x}) \\ & \cdot \prod_{x<l\le n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1l}) \prod_{\substack{s\le n\\ s\neq x-1\\ x\neq x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(v_{x-1s})\mathrm{T}^w,\end{aligned} \tag{64}$$

for complex numbers v_{x-1x}, v_{x-1}, v_{x-1x}, v_{x-1l}, and v_{x-1s}. Fix such complex numbers. Now we look at $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}})$. We have that $\mathbb{U}_{\mathrm{V}_0}^{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathscr{K}_2}*\nu}$ are both contained in U_0, and the latter is generated by elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)$, for $a \in \mathbb{C}$. Out of the generators of $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathrm{K}_2}*\nu}$, the ones that do not belong to U_0 are $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$, $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$, and $\mathrm{U}_{(\varepsilon_x-\varepsilon_l,-1)}(a_{x\bar{l}})$. Therefore, if

$$\mathrm{A} = \mathrm{U}_{(\varepsilon_x,-1)}(a_x)\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{U}_{(\varepsilon_x-\varepsilon_l,-1)}(a_{x\bar{l}}) \in \mathrm{U}_{\varepsilon_{\bar{x}}},$$

we conclude that

$$\begin{aligned}\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) &= \mathrm{U}_0\mathrm{A}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)\mathrm{T}^w \\ &= \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x}) \\ &\quad \cdot \prod_{x<l\le n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\le n\\ s\neq x-1\\ s\neq x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})\mathrm{A}\mathrm{T}^w\end{aligned}$$

$$\begin{aligned}&= \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x}) \\ &\quad \cdot \prod_{x<l\le n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\le n\\ s\neq x-1\\ s\neq x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})\mathrm{T}^w,\end{aligned} \tag{65}$$

where

$$\begin{aligned}
\xi_{x-1} &= c^{1,1}_{x,x-1\bar{x}}(-a_x)a,\\
\xi_{x-1x} &= c^{2,1}_{x,x-1\bar{x}}(a_x^2)a,\\
\xi_{x-1\bar{l}} &= c^{1,1}_{x\bar{l},x-1\bar{x}}(-a_{x\bar{l}})a,\\
\xi_{x-1s} &= c^{1,1}_{xs,x-1\bar{x}}(-a_{xs})a.
\end{aligned}$$

Therefore it follows directly that in fact

$$\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) \subseteq \pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}).$$

Now, the system of equations

$$\begin{aligned}
v_{x-1} &= \xi_{x-1},\\
v_{x-1x} &= \xi_{x-1x},\\
v_{x-1\bar{l}} &= \xi_{x-1\bar{l}},\\
v_{x-1s} &= \xi_{x-1s},
\end{aligned}$$

has solutions as long as $\{v_{x-1}, v_{x-1x}, v_{x-1\bar{l}}, v_{x-1s}\} \subset \mathbb{C}^\times$. For such a set of solutions we conclude

$$\begin{aligned}
&\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(v_{x-1x})\\
&\quad\cdot \prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(v_{x-1s})\\
&\qquad= \mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x})\\
&\qquad\quad\cdot \prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\le n\\ s\ne x-q\\ s\ne x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s}),
\end{aligned}$$

and therefore we conclude by Proposition 7.1 (applied to $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(v_{x-1x})$ in (64)) that a dense subset of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ is contained in $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ (see (64), (65)). □

Proof. To prove (b) let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline x-1 & x-1 & \overline{x-1}\\ \hline \bar{x} & \bar{x} & \\ \hline\end{array} \quad\text{and}\quad \mathscr{K}_4 = \begin{array}{|c|c|c|}\hline \bar{x} & x-1 & x\\ \hline & \bar{x} & \overline{x-1}\\ \hline\end{array},$$

then

$$w_3 = \overline{x-1}\ x-1\ \bar{x} = w(\mathscr{K}_3) \quad\text{and}\quad w_4 = x\ \bar{x}\ \bar{x} = w(\mathscr{K}_4).$$

By Proposition 5.5 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})}.$$

First we claim

$$\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}*\nu}) \subseteq \pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu}).$$

Note that the terms $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$, for $b \in \mathbb{C}$, generate both $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_4}*\nu}$ and are contained in $\mathbb{U}_{\mathrm{V}_1}^{\gamma_{\mathcal{K}_3}*\nu}$. Also, the terms $\mathrm{U}_{(\varepsilon_l-\varepsilon_x,0)}$, which generate $\mathbb{U}_{\mathrm{V}_2}^{\gamma_{\mathcal{K}_4}*\nu}$, commute with $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$. Therefore

$$\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{T}^w \subseteq \pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}}),$$

where the last contention follows by Proposition 7.1. Now we will show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu})} \subseteq \overline{\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}*\nu})}.$$

We claim that

$$(66)\quad \pi'''(\mathrm{C}_{\gamma_{\mathcal{K}_3}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x-1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w,$$

for complex numbers v_{x-1}, $v_{x-1\bar{x}}$, and v_{x-1s}. Let us fix such complex numbers. Let

$$D = \mathrm{U}_{(\varepsilon_x,0)}(a_x) \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x-1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{x-1s}) \in \mathrm{U}_0,$$

then by the usual arguments (note that U_0 stabilizes both the image $\pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}})$ and the truncated image $\mathrm{T}^{\geq 2}_{\gamma_{\mathcal{K}_4}*\nu}$),

$$\mathrm{D}^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)D\mathrm{T}^w \subset \pi''''(\mathrm{C}_{\gamma_{\mathcal{K}_4}}),$$

and

$$\mathrm{D}^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)D\mathrm{T}^w = \mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b) \cdot \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x+1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_x+\varepsilon_{x-1},-1)}(\rho_{xx-1}),$$

where

$$\rho_{x-1} = c^{1,1}_{x-1\bar{x},x}(-b)a_x,$$
$$\rho_{x-1x} = c^{2,1}_{x-1\bar{x},x}(-b)a_x^2,$$
$$\rho_{x-1s} = c^{1,1}_{x-1\bar{x},xs}(-b)a_{xs}.$$

As usual by requiring that v_{x-1}, $v_{x-1\bar{x}}$, v_{x-1x}, and ρ_{x-1s} be nonzero we may find suitable complex numbers b, a_x, a_{xs} such that

$$\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \prod_{\substack{s\neq x\\ \varepsilon_s+\varepsilon_{x-1}\in\Phi^+}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s}) = \mathrm{D}^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{D}\mathrm{T}^w.$$

Therefore Proposition 7.1 (see (66)) implies that a dense open subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ is contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$. This completes the proof of Lemma 7.7. □

Relation R3.

Lemma 7.8. *Let $w \in \mathscr{W}_{\mathscr{C}_n}$ be a word and let w_1 be a word that is not of an LS block, and such that it has the form $w_1 = a_1 \cdots a_r z\bar{z}\bar{b}_s \cdots \bar{b}_1$, and let $w_2 = a_1 \cdots a_r \bar{b}_s \cdots \bar{b}_1$ with $a_1 < \cdots a_r < z > b_s > \cdots > b_1$. Then $\overline{\pi(\mathrm{C}_{\gamma_{w_1w}})} = \overline{\pi'(\mathrm{C}_{\gamma_{w_2w}})}$.*

Proof. Let $\mathrm{A} = \{a_1, \dots, a_r\}$. We have

$$\pi(\mathrm{C}_{\gamma_{w_1w}}) = \mathbb{P}_{a_1} \cdots \mathbb{P}_{a_r}\mathbb{P}_z\mathbb{P}_{\bar{z}}\mathbb{P}_{\bar{b}_s} \cdots \mathbb{P}_{\bar{b}_1}\mathrm{T}^{\geq r+s+2}_{\gamma_{w_1w}},$$

where

$$\mathbb{P}_z = \mathrm{U}_{(\varepsilon_z,0)}(v_z)\prod_{l>z}\mathrm{U}_{(\varepsilon_z-\varepsilon_l,0)}(v_{z\bar{l}})\prod_{l\notin\mathrm{A}}\mathrm{U}_{(\varepsilon_z+\varepsilon_l,0)}(v_{zl})\prod_{a_i\in\mathrm{A}}\mathrm{U}_{(\varepsilon_z+\varepsilon_{a_i},1)}(v_{za_i}),$$

$$\mathbb{P}_{\bar{z}} = \prod_{a_i\in\mathrm{A}}\mathrm{U}_{(\varepsilon_{a_i}-\varepsilon_z,0)}(v_{a_i\bar{z}}),$$

and note that $\mu_{\gamma_{w_1}} = \mu_{\gamma_{w_2}} = \sum_{i\in\mathrm{I}_r}\varepsilon_{a_i} - \sum_{j\in\mathrm{I}_s}\varepsilon_{b_j}$. The terms that appear in $\mathbb{P}_z$ all stabilize $\mu_{\gamma_{w_1}}$ and commute with $\mathbb{P}_{\bar{b}_j}$, while the terms in $\mathbb{P}_{\bar{z}}$ all appear in $\mathbb{P}_{a_i}$ and commute with $\mathbb{P}_{a_l}$, for $l > i$. This concludes the proof of Lemma 7.8 with the usual arguments, and therefore of Proposition 5.20. □

8. Nonexamples for nonreadable galleries

Let $n = 2$ and $\lambda = \varepsilon_1 + \varepsilon_2$, and $(\Sigma_{\gamma_\lambda}, \pi)$ the corresponding Bott–Samelson variety, as in (8). Let γ be the gallery corresponding to the block

1	$\bar{2}$
2	$\bar{1}$

.

Then points in $\pi(\mathrm{C}_\gamma)$ are of the form

$$\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b)[t^0],$$

for $b \in \mathbb{C}$, hence form an affine set of dimension 1. We claim that the set $\mathrm{Z} = \overline{\pi(\mathrm{C}_\gamma)}$ cannot be an MV cycle in $\mathscr{Z}(\mu)$ for any dominant coweight μ. First note that for any $u \in \mathrm{U}(\mathscr{K})$ a necessary condition for ut^0 to lie in the closure $\overline{\mathrm{U}(\mathscr{K})t^\nu \cap G(\mathscr{O})t^\mu}$ is that $0 \leq \nu$, since it would in particular imply that $ut^0 \in \overline{\mathrm{U}(\mathscr{K})t^\nu}$. Also note that it is necessary for $\nu \leq \mu$ in order for the set $\mathscr{Z}(\mu)_\nu$ not to be empty. Any MV cycle in $\mathscr{Z}(\mu)_\nu$ has dimension $\langle \rho, \mu + \nu \rangle$, and the only possibility for the latter to be equal to 1 (since $\mu + \nu$ is a sum of positive coroots) is for either $\mu = 0$ and $\nu = \alpha_i^\vee$, or $\nu = 0$ and $\mu = \alpha_i^\vee$, for some $i \in \mathrm{I}$, and both options are impossible: the first contradicts $\nu \leq \mu$, and the second contradicts the dominance of μ. Note that γ is not a Littelmann gallery.

Appendix

Here we show that relation R3 in Theorem 5.17 is equivalent to relation R_3 in [Lecouvey 2002, Definition 3.1]. For a word $w \in \mathscr{W}_{\mathscr{C}_n}$ and $m \leq n$ define $\mathrm{N}(w, m) = |\{x \in w : x \leq m \text{ or } \bar{m} \leq x\}|$. Lecouvey's relation R_3 is: "Let w be a word that is not the word of an LS block and such that each strict subword is. Let z be the lowest unbarred letter such that the pair $(z, \bar{z})$ occurs in w and $\mathrm{N}(w, z) = z + 1$. Then $w \cong w'$, where w' is the subword obtained by erasing the pair $(z, \bar{z})$ in w." The following Lemma is a translation between R_3 and R3.

Lemma 8.1. *Let w be a word that is not the word of an LS block and such that each strict subword is. Then* $w = a_1 \cdots a_r z \bar{z} \bar{b}_s \cdots \bar{b}_1$ *for* a_i, b_i *unbarred and* $a_1 < \cdots < a_r$, $b_1 < \cdots < b_s$.

Proof. By [Lecouvey 2002, Remark 2.2.2], w is the word of an LS block if and only if $\mathrm{N}(w, m) \leq m$ for all $m \leq n$. Let w be as in the statement of Lemma 8.1. Then there exists in w a pair $(z, \bar{z})$ such that $\mathrm{N}(w, z) > z$. Let z be minimal with this property. In particular $\mathrm{N}(w, z) = z + 1$ since if w'' is the word obtained from w by erasing z, then $z \geq \mathrm{N}(w'', z) = \mathrm{N}(w, z) - 1$. We claim that z is the largest unbarred letter to appear in w. If there was a larger letter y then $\mathrm{N}(w''', z) = \mathrm{N}(w, z) = z + 1$ where w''' denotes the word obtained from w by deleting y. This is impossible since by assumption w''' is the word of an LS block. Likewise $\bar{z}$ is the smallest unbarred letter to appear in w. The $a_i's$ and $b_i's$ are then those from Definition 4.3 for the word obtained from w by deleting $z, \bar{z}$ from it. □

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