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THREE-DIMENSIONAL DISCRETE CURVATURE FLOWS AND DISCRETE EINSTEIN METRICS

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A discrete version of the Einstein–Hilbert functional was introduced by Regge. In this paper, we define the discrete Einstein metrics as critical points of Regge's Einstein–Hilbert functional with normalization on triangulated 3-manifolds. We also introduce some discrete curvature flows, which are closely related to the existence of discrete Einstein metrics.

1. Introduction

For triangulated manifolds, the most natural metrics seem to be the piecewise linear metrics defined on all edges, satisfying some nondegenerate conditions so that each simplex in the triangulation can be realized as a Euclidean or hyperbolic simplex. In his work on constructing hyperbolic metrics on 3-manifolds, Thurston [1980] introduced the circle packing metric on a triangulated surface with prescribed intersection angles and further proved that this metric induces a piecewise linear metric. Similarly, for triangulated 3-manifolds, Cooper and Rivin [1996] introduced a ball (or sphere) packing metric. They endowed each vertex with a notion of combinatorial scalar curvature which is defined to be the angle defect of solid angles. Glickenstein [2005] introduced a type of discrete Yamabe flow, aiming at finding sphere packing metrics with constant combinatorial scalar curvature. In [Ge and Xu 2014], we also defined discrete quasi-Einstein metrics and gave some analytical conditions for the existence of discrete quasi-Einstein metrics by introducing two different discrete scalar curvature flows.

However, on one hand, similar to the 2-dimensional case, the ball packing metrics are special piecewise linear metrics and then too restrictive. On the other hand, the combinatorial curvatures studied above are all defined on vertices and may only be considered as an analogue of scalar curvature. As was pointed out by Regge [1961], the discrete curvatures are concentrated on codimension two simplexes. For these reasons, we want to study the general piecewise linear metrics and discrete curvatures defined on edges for 3-dimensional triangulated manifolds. In this paper,

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we shall study Regge's Einstein–Hilbert functional carefully, and give a definition of discrete Einstein metric. Moreover, we will introduce two types of discrete edge curvature flows; one is of second order, the other is of fourth order. Discrete edge curvature flow of second order may be considered as an analogue of smooth Ricci flow. However, discrete edge curvature flow of fourth order seems to be more powerful than the flow of second order.

2. Discrete Ricci curvature and discrete Einstein metric

Consider a compact 3-dimensional manifold M with a triangulation $\mathcal{T}$. The triangulation is written as $\mathcal{T}=\{V,E,F,T\}$, where V,E,F,T represent the set of vertices, edges, faces and tetrahedrons respectively. Denote $v_1,v_2,\ldots,v_N$ as the vertices of $\mathcal{T}$, where N is the number of vertices. We often write i instead of v_i. A piecewise linear metric (written as PL-metric for short) is a map $l:E\to(0,+\infty)$ such that for any tetrahedron $\tau=\{i,j,k,l\}\in T$, the tetrahedron τ with edge lengths $l_{ij},l_{ik},l_{il},l_{jk},l_{jl},l_{kl}$ can be realized as a Euclidean geometric tetrahedron. We may take PL-metrics as points in $\mathbb{R}^m_{>0}$, m times the Cartesian product of $(0,+\infty)$, where m is the number of edges in E. Not all points in $\mathbb{R}^m_{>0}$ represent PL-metrics and we need some nondegenerate conditions. For a start, the triangle inequality should be satisfied, but this alone is not enough. Consider a Euclidean tetrahedron $\tau=\{i,j,k,l\}\in T$ with edge lengths $l_{ij},l_{ik},l_{il},l_{jk},l_{jl},l_{kl}$, then the volume of the Euclidean tetrahedron $\{i,j,k,l\}$ has the following formula due to Tartaglia in the sixteenth century:

$$\mathrm{V}_\tau^2=\tfrac{1}{288}\det A_{ijkl},$$

where

$$A_{ijkl}=\begin{pmatrix}0&1&1&1&1\\1&0&l_{ij}^2&l_{ik}^2&l_{il}^2\\1&l_{ij}^2&0&l_{jk}^2&l_{jl}^2\\1&l_{ik}^2&l_{jk}^2&0&l_{kl}^2\\1&l_{il}^2&l_{jl}^2&l_{kl}^2&0\end{pmatrix}.$$

So, $\mathrm{V}_\tau>0$ for all tetrahedrons τ is another restriction for l to be a PL-metric.

For fixed 3-manifolds M with triangulation $\mathcal{T}$, denote the space of all admissible PL-metrics as

$$\mathfrak{M}_{\mathcal{T}}\triangleq\{l:E\to(0,+\infty)\text{ is a PL-metric on }(M^3,\mathcal{T})\},$$
$$\mathfrak{M}^2_{\mathcal{T}}\triangleq\{l^2:E\to(0,+\infty)\text{ is a PL-metric on }(M^3,\mathcal{T})\}.$$

Mei, Zhou and Ge proved $\mathfrak{M}^2_{\mathcal{T}}$ is a nonempty connected open convex cone, see Theorem 1.1 in [Ge et al. 2015] (see also Theorem 3.1 in [Schrader 2016]). On the other hand, it is easy to prove that $\mathfrak{M}_\mathcal{T}$ is homeomorphic to $\mathfrak{M}^2_\mathcal{T}$. Hence we know $\mathfrak{M}_\mathcal{T}$ is a simply connected open set. However, this set is not convex, due to an observation from [Rivin 2003].

Discrete Ricci curvature and the Einstein–Hilbert–Regge functional. Given a Euclidean tetrahedron $\{i, j, k, l\} \in T$, the dihedral angle at edge $\{i, j\}$ is denoted by $\beta_{ij,kl}$. If an edge is in the interior of the triangulation, the discrete Ricci curvature at this edge is 2π minus the sum of dihedral angles at the edge. More specifically, denote R_{ij} as the discrete Ricci curvature at the edge $\{i, j\}$, then

$$R_{ij} = 2\pi - \sum_{\{i,j,k,l\}\in T} \beta_{ij,kl}, \tag{2-1}$$

where the sum is taken over all tetrahedrons with $\{i, j\}$ as one of its edges. If this edge is on the boundary of the triangulation, then the discrete Ricci curvature should be $R_{ij} = \pi - \sum_{\{i,j,k,l\}\in T} \beta_{ij,kl}$.

For simplicity we will write l_{ij} and R_{ij} as $l_1, \dots, l_m$ and $R_1, \dots, R_m$, respectively, in the following, where m is the number of edges in E, and they are ordered sequentially. Set $l = (l_1, \dots, l_m)^T$, $R = (R_1, \dots, R_m)^T$, to be the transpose of $(l_1, \dots, l_m)$, $(R_1, \dots, R_m)$ respectively. We define the matrix L as

$$L = \frac{\partial(R_1, \dots, R_m)}{\partial(l_1, \dots, l_m)} = \begin{pmatrix} \frac{\partial R_1}{\partial l_1} & \cdots & \frac{\partial R_1}{\partial l_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial R_m}{\partial l_1} & \cdots & \frac{\partial R_m}{\partial l_m} \end{pmatrix}. \tag{2-2}$$

The Einstein–Hilbert–Regge functional was first introduced by Regge [1961] as

$$S = \sum_{i=1}^m R_i l_i, \tag{2-3}$$

and the discrete quadratic energy functional is defined to be

$$\mathcal{C}(l) = \|R\|^2 = \sum_{i=1}^m R_i^2. \tag{2-4}$$

By the Schläfli formula $\sum_{i=1}^m l_i dR_i = 0$, we have

$$dS = \sum_{i=1}^m R_i dl_i + \sum_{i=1}^m l_i dR_i = \sum_{i=1}^m R_i dl_i,$$

so

$$\nabla_l S = R \quad \text{and} \quad \mathrm{Hess}_l\, S = L,$$

which implies that the matrix L is symmetric. It is easy to get

$$\frac{\partial \mathcal{C}}{\partial l_j} = 2\sum_{i=1}^{m} \frac{\partial R_i}{\partial l_j} R_i,$$

so

$$\nabla_l \mathcal{C} = 2L^T R. \tag{2-5}$$

Since $R(tl_1, tl_2, \dots, tl_m) = R(l_1, l_2, \dots, l_m)$, we have the Euler formula

$$Ll = 0. \tag{2-6}$$

Discrete Einstein metric. The curvature R_{ij} is a combinatorial analogue of Ricci curvature in smooth cases. Fixing i, the sum of all R_{ij} with j connected to i is the curvature defined by Cooper and Rivin [1996], which is interpreted as the combinatorial scalar curvature. Inspired by the definition of discrete quasi-Einstein metric in [Ge and Xu 2014], we define the discrete Einstein metric as follows.

Definition 2.1. A PL-metric l is called a discrete Einstein metric, if there exists a constant λ such that $R = \lambda l$.

If l is a discrete Einstein metric, the corresponding PL-metric and curvature will be denoted by l_{DE} and R_{DE}, respectively, in the following. When $R = \lambda l$, or say l is a discrete Einstein metric, $\lambda = S/\|l\|^2$.

Definition 2.1 is a straightforward analogy of the smooth manifold case. R_{ij} is somewhat similar to smooth Ricci curvature Ric, and l_{ij} is somewhat similar to the smooth metric g. Then the Einstein metric g with $\mathrm{Ric} = \lambda g$ on smooth manifolds M can be transformed to a discrete Einstein metric l with $R = \lambda l$ on triangulated manifolds $(M^3, \mathcal{T})$. In this sense, the analogy seems to be only formal. However, for this type of metric, we can develop many more properties which suggest the use of the term discrete Einstein is appropriate.

In [Champion et al. 2011], Champion, Glickenstein and Young studied various normalized Einstein–Hilbert–Regge functionals and related discrete Yamabe invariants on triangulated manifolds with PL-metrics. In this paper, we shall introduce a new type of normalized Einstein–Hilbert–Regge functional, which is different from theirs. Fixing $(M^3, \mathcal{T})$, consider a new type of normalized Einstein–Hilbert–Regge functional

$$Q(l) = \frac{S}{\|l\|}. \tag{2-7}$$

It's easy to calculate

$$\nabla_l Q = \frac{1}{\|l\|}\left(\nabla_l S - \frac{S}{\|l\|^2} l\right) = \frac{1}{\|l\|}\left(R - \frac{S}{\|l\|^2} l\right).$$

Then we have:

Theorem 2.2. *On* $(M^3, \mathcal{T})$ *with* PL*-metric* l, l *is a discrete Einstein metric if and only if* l *is a critical point of the normalized Einstein–Hilbert–Regge functional* Q.

Theorem 2.2 is similar to the smooth case. On smooth manifolds, the metric g is Einstein if and only if it is a critical point of the functional

$$Q(g) = \frac{1}{V^{1/3}} \int_M R \, d\mu_g.$$

Fixing the triangulation, discrete curvatures R_{ij} are uniformly bounded, that is $(2-d)\pi < R_{ij} < 2\pi$, where d is the maximum edge degree of the triangulation. So

$$|Q(l)| = \left| \frac{S}{\|l\|} \right| = \left| \frac{R^T l}{\|l\|} \right| \leq \|R\|.$$

The Cauchy inequality indicates that l is a discrete Einstein metric if and only if $|Q(l)| = \|R\|$.

Using this type of normalized Einstein–Hilbert–Regge functional, we can introduce some new invariants associated to the triangulation $(M^3, \mathcal{T})$. The combinatorial Yamabe invariant with respect to $\mathcal{T}$ is defined as

$$Y_{M,\mathcal{T}} = \inf_{l \in \mathfrak{M}_{\mathcal{T}}} Q(l).$$

The admissible PL-metric space $\mathfrak{M}_{\mathcal{T}}$ for a given triangulated manifold $(M^3, \mathcal{T})$ may be considered as an analogue of the conformal class $[g_0]$ of a Riemannian manifold (M, g_0). Hence we may call $\mathfrak{M}_{\mathcal{T}}$ the combinatorial conformal class for $(M^3, \mathcal{T})$. It is uniquely determined by the triangulation $\mathcal{T}$. Moreover, we can introduce a topology invariant associated to M, i.e., $Y_M = \sup_{\mathcal{T}} Y_{M,\mathcal{T}}$, where the supremum is taken on all triangulations of M.

Similar to [Ge and Xu 2014; Ge and Xu 2016b], we can consider the following combinatorial Yamabe problem.

Question. Given a 3-dimensional manifold M with triangulation $\mathcal{T}$, how many discrete Einstein metrics are there in the combinatorial conformal class $\mathfrak{M}_{\mathcal{T}}$, and how to find them?

Inspired by work on the existence of combinatorial Gauss curvature in [Thurston 1980; Chow and Luo 2003; Luo 2004], we ask the following similar question:

Question. For a manifold M^3, find a suitable triangulation, or find topological and combinatorial obstructions, so that M admits discrete Einstein metrics.

The following is an example of a manifold with a triangulation admitting a discrete Einstein metric.

Example (the 16-cell). Consider the standard 3-dimensional sphere $\mathbb{S}^3$ embedded in $\mathbb{R}^4$. Taking the vertices of $\mathcal{T}$ to be $A_1 = (1, 0, 0, 0)$, $A_2 = (-1, 0, 0, 0)$, $B_1 = (0, 1, 0, 0)$, $B_2 = (0, -1, 0, 0)$, $C_1 = (0, 0, 1, 0)$, $C_2 = (0, 0, -1, 0)$, $D_1 = (0, 0, 0, 1)$, $D_2 = (0, 0, 0, -1)$; the edges of $\mathcal{T}$ are $P_i Q_j$ (where $P \neq Q \in \{A, B, C, D\}$ and $i, j \in \{1, 2\}$), the faces of $\mathcal{T}$ are $P_i Q_j R_k$ (where exactly two of $(P, Q, R) \in \{A, B, C, D\}$ are different, with $i, j, k \in \{1, 2\}$), and the tetrahedrons of $\mathcal{T}$ are the regular tetrahedrons $A_i B_j C_k D_l$ (with $i, j, k, l \in \{1, 2\}$). We know all edges have the same length $\frac{\pi}{2}$. It is easy to calculate that $R_{ij} = 2\pi - 4 \arccos \frac{1}{3}$ for all edges. So $R = (l/\pi)\big(4\pi - 8 \arccos \frac{1}{3}\big)$ and $l = \frac{\pi}{2}\{1, \dots, 1\}^T$ is a discrete Einstein metric associated to $(\mathbb{S}^3, \mathcal{T})$.

It is easy to see that the argument in this example works for any generalization of the platonic solids (uniform polychora) with tetrahedral cells, including the 5-cell (or pentachoron), the 600-cell, etc. In these cases, the PL-metric arises from symmetry and taking the lengths equal. It would be interesting to know whether these are the only triangulations that admit discrete Einstein metrics for the triangulation structure.

3. Combinatorial second order flow

Inspired by combinatorial curvature flow methods, we study discrete Einstein metrics by combinatorial curvature flows in the following sections. The two flows we introduce are negative gradient flows of some discrete functionals. One is the normalized Einstein–Hilbert–Regge functional $Q(l)$, which determines a normalized discrete curvature flow of second order. The other is the discrete quadratic energy $\mathcal{C} = \|R\|^2$, which determines a discrete curvature flow of fourth order.

Definition and evolution equations. We define the combinatorial second order flow as

$$\dot{l}(t)_{ij} = -R_{ij}, \quad \text{or} \quad \dot{l}(t) = -R. \tag{3-1}$$

It is useful to consider the normalized combinatorial second order flow

$$\dot{l}(t)_{ij} = \lambda l_{ij} - R_{ij}, \quad \text{or} \quad \dot{l}(t) = \lambda l - R, \tag{3-2}$$

where $\lambda = S/\|l\|^2$ and $\|l\|^2 = \sum_{i=1}^n l_i^2$.

Flows (3-1) and (3-2) differ from each other only by a change of scale in space and a change of parametrization in time. Let t, l, R, λ denote the variables for the flow (3-1), and $\tilde{t}, \tilde{l}, \widetilde{R}, \tilde{\lambda}$ for the flow (3-2). Suppose $l(t)$, $t \in [0, T)$, is a solution of (3-1). Set $\tilde{l}(\tilde{t}) = \varphi(t) l(t)$, where

$$\varphi(t) = \exp\left(\int_0^t \lambda(\tau)\, d\tau\right), \quad \tilde{t} = \int_0^t \varphi(\tau)\, d\tau.$$

Then we have

$$\tilde{\lambda} = \varphi^{-1}\lambda, \quad \widetilde{R} = R.$$

This gives

$$\frac{d\tilde{l}}{d\tilde{t}} = \frac{d\tilde{l}}{dt}\frac{dt}{d\tilde{t}} = (\lambda\varphi l - \varphi R)\varphi^{-1} = \tilde{\lambda}\tilde{l} - \widetilde{R}.$$

Conversely, if $\tilde{l}(\tilde{t}), \tilde{t} \in [0, \widetilde{T})$, is a solution of (3-2), set $l(t) = \varphi(\tilde{t})\,\tilde{l}(\tilde{t})$, where

$$\varphi(\tilde{t}) = \exp\left(-\int_0^{\tilde{t}} \tilde{\lambda}(\tau)\,d\tau\right), \quad t = \int_0^{\tilde{t}} \varphi(\tau)\,d\tau,$$

then it is easy to check that $dl/dt = -R$.

Notice that $\nabla_l Q = -(\lambda l - R)/\|l\|$ and $d\|l\|^2/dt = 2l^T\dot{l} = 2l^T(\lambda l - R) = 0$, hence we have:

Theorem 3.1. *Along the flow (3-2), $\|l\|^2$ is a constant. Moreover, the flow (3-2) is a negative gradient flow.*

We can take $\|l\|^2$ as a certain discrete "content" (here we use the word "content" instead of "volume", because the triangulated 3-manifolds have classical volume, that is, the sum of the volume of all tetrahedrons). It plays a similar role to "volume" in smooth cases. We also refer to the second order normalized discrete curvature flow (3-2) as the combinatorial Ricci flow. Moreover, we have the following evolution equations along this flow,

$$\dot{R} = \frac{\partial R}{\partial l}\dot{l} = L(-R + \lambda l) = -LR, \tag{3-3}$$

where we have used the Euler formula $Ll = 0$. So

$$\dot{C} = -2R^T LR, \tag{3-4}$$

and

$$\begin{aligned}
\dot{S} &= \sum_{i=1}^{m} \dot{R}_i l_i + R_i \dot{l}_i = -\|R\|^2 + \lambda S \\
&= \frac{S^2 - \|l\|^2\|R\|^2}{\|l\|^2} = \frac{< R, l >^2 - \|l\|^2\|R\|^2}{\|l\|^2} \\
&= -\|R - \lambda l\|^2 = -\left\| R - \frac{S}{\|l\|^2} l \right\|^2 \\
&\le 0.
\end{aligned}$$

Hence

$$\dot{\lambda} = \frac{\dot{S}}{\|l\|^2} = -\left(\frac{\|R\|}{\|l\|}\right)^2 + \lambda^2 = -\frac{\|R - \lambda l\|^2}{\|l\|^2} \le 0. \tag{3-5}$$

Since $\|l\|^2$ is invariant along the flow (3-2), we can always assume $l(0) \in \mathbb{S}^{m-1}$ and then $l(t) \in \mathbb{S}^{m-1}$ for all $t \in [0, T)$ in the following. Moreover, $\lambda = S/\|l\|^2 \equiv S$ along (3-2). It is easy to derive the following result.

Proposition 3.2. *The quadratic energy functional $\mathcal{C}$ is uniformly bounded on $\mathfrak{M}_{\mathcal{T}}$, where the bound depends only on the triangulation. The Einstein–Hilbert–Regge functional S is uniformly bounded on $\mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$. Moreover, along the discrete flow* (3-2), *S is nonincreasing and bounded.*

Remark. By the Schläfli formula, the differential 1-form $\omega = \sum_{i=1}^{m} R_i \, dl_i = dS$ is exact. Combining this with the fact that $\mathfrak{M}_{\mathcal{T}}$ is simply connected, we have

$$S(l) = \int_a^l \sum_{i=1}^{m} R_i \, dl_i + S(a),$$

where a is an arbitrary point of $\mathfrak{M}_{\mathcal{T}}$.

Nonsingular solution and singularity of solution. To study the convergence of the discrete Ricci flow (3-2), we need to classify the solutions of the flow.

Definition 3.3. A solution $l(t)$ of (3-2) is nonsingular if the solution exists for $t \in [0, +\infty)$ and $\{l(t)\} \Subset \mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$.

In fact, by $\{l(t)\} \Subset \mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$, we know that the solution of (3-2) exists for $t \in [0, +\infty)$. Furthermore, we have the following result for nonsingular solutions of (3-2).

Theorem 3.4. *If there exists a nonsingular solution for the discrete flow* (3-2), *there exists at least one discrete Einstein metric on $(M^3, \mathcal{T})$.*

Proof. Let $l(t)$, $t \in [0, +\infty)$, be a nonsingular solution of the flow (3-2). As S is descending and bounded from below along (3-2), $S(+\infty)$ exists. We can choose $t_n \uparrow +\infty$, such that

$$S'(t_n) = -\|\lambda(t_n) l(t_n) - R(t_n)\|^2 \to 0. \tag{3-6}$$

Using $\{l(t)\} \Subset \mathfrak{M}_{\mathcal{T}}$, we can further choose a subsequence t_{n_k} of t_n, such that $l(t_{n_k}) \to l^*$. Combining this with (3-6), we get $R^* = \lambda^* l^*$ and l^* is a discrete Einstein metric. ☐

If the solution of flow (3-2) converges to a nondegenerate PL-metric, the unit solution $\lambda(t)/\|\lambda\|$ must be nonsingular. First, assume the maximal time $T < +\infty$. Since $\lambda(T)$ is a nondegenerate PL-metric, the flow can be extended beyond T, so we obtain $T = +\infty$. Second, since $\lim_{t \to \infty} l(t) = l^* \in \mathfrak{M}_{\mathcal{T}}$, there exists $t_0 > 0$ such that $l(t)$ is close to l^* when $t > t_0$, so $l(t) \in \mathfrak{M}_{\mathcal{T}}$. On the other hand, for $t \in [0, t_0]$, $l(t) \in \mathfrak{M}_{\mathcal{T}}$. Hence we know $(\lambda(t)/\|\lambda\|) \Subset \mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$.

Then we have the following corollary:

Corollary 3.5. *If the solution $l(t)$ of the discrete Ricci flow (3-2) exists for all time and converges to a nondegenerate* PL*-metric $l(+\infty)$, then there exists at least one discrete Einstein metric on $(M, \mathcal{T})$. Moreover, $l(+\infty)$ is a discrete Einstein metric.*

Definition 3.6. A maximal solution $l(t),\ t \in [0, T)$, of (3-2) is said to be singular if

$$\overline{\{l(t)\}} \cap \partial(\mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}) \neq \varnothing.$$

We say the solution develops a type-I singularity at time T if there is an edge l_i and a sequence $t_n \to T$ such that $l_i(t_n) \to 0$. We say the solution develops a type-II singularity at time T if there is a sequence t_n approaching T such that $l_i(t_n)$ remains in a compact set of $\mathbb{R}_{>0}$ for all i and there is a tetrahedron $\tau = \{i, j, k, l\}$ in $\mathcal{T}$ such that $V_\tau \to 0$ as $t_n \to T$.

Remark. In [Bobenko et al. 2015; Ge and Jiang 2016a; Ge and Jiang 2016b; Luo 2011], the authors studied the degeneration of a triangle. In fact, they considered the generalized triangle, that is a topological triangle with three positive edge lengths. While the triangle inequality is not valid, they found that the definition of discrete Gaussian curvatures can be generalized to this case. However, we don't know how to do this degeneration for tetrahedrons, and hence we know very little about the degeneration behavior of a tetrahedron.

The following conjectures are likely to hold for the discrete flow (3-2).

Conjecture. *The normalized discrete Ricci flow (3-2) will not develop type-I singularity in finite time.*

Conjecture. *If no singularity develops along the normalized flow (3-2), the solution converges to a discrete Einstein metric as time approaches infinity.*

Just like Hamilton and Perelman's methods approaching smooth Ricci flow, whenever discrete curvature flow develops type-II singularity, we hope to continue the discrete flow by surgery which changes the combinatorial structure of the triangulation. We hope that discrete curvature flow converges to a discrete Einstein metric after a finite number of surgeries.

Convergence of the combinatorial second order flow. Finding good metrics is always a central topic in Riemannian geometry. In the last section, we proved that if the solution of the flow (3-2) exists for all time and converges to a nondegenerate PL-metric l_∞, the discrete Einstein metric exists. Moreover, l_∞ is such a metric. Conversely, we have:

Theorem 3.7. *Given a nondegenerate metric l, assume there exists a discrete Einstein metric l_{DE} such that $R_{DE} = \lambda l_{DE}$ with*

$$\lambda_{DE}\left(I_m - \frac{l_{DE} l_{DE}^T}{\|l_{DE}\|^2}\right) - L_{DE} \leq 0,$$

where I_m denotes the $m \times m$ identity matrix. Then there exists a constant $\varepsilon > 0$ such that, if

$$\|l(0) - l_{DE}\| < \varepsilon,$$

then the solution of normalized combinatorial second order flow (3-2) with initial metric $l(0) = l$ exists for all time and converges exponentially fast to the discrete Einstein metric l_{DE}.

Proof. We want to prove l_{DE} is a local attractor of the flow. For the evolution equation of the combinatorial two-order flow

$$\dot{l} = \Upsilon(l) = -R + \lambda l,$$

we have

$$\Upsilon(l_{DE}) = -R_{DE} + \lambda_{DE} l_{DE} = 0.$$

The differential of $\Upsilon(l) = -R + \lambda l$ at l is

$$\begin{aligned} D_l \Upsilon(l) &= -D_l R + \lambda D_l l + l D_l \Upsilon \\ &= -L^T + \lambda I_m + l\left(\frac{L^T l + R}{\|l\|^2} - \frac{2Sl}{\|l\|^4}\right)^T \\ &= \lambda I_m - L + \frac{lR^T}{\|l\|^2} - 2S\frac{ll^T}{\|l\|^4} \\ &= \lambda\left(I_m - \frac{ll^T}{\|l\|^2}\right) - L + \frac{l(R - \lambda l)^T}{\|l\|^2}, \end{aligned} \tag{3-7}$$

where we have used the symmetry of L and the Euler formula in the third equality. So

$$D_l \Upsilon(l)|_{l=l_{DE}} = \lambda_{DE}\left(I - \frac{l_{DE} l_{DE}^T}{\|l_{DE}\|^2}\right) - L_{DE} \leq 0,$$

and l_{DE} is a local attractor of the flow. The system is asymptotically stable at l_{DE}. If the initial metric $l(0)$ is close enough to l_{DE}, then the solution $l(t)$ exists for all time and converges to l_{DE} exponentially fast. □

4. Fourth order flow

In this section, we consider the combinatorial fourth order flow

$$\dot{l} = -L^T R, \tag{4-1}$$

where L^T denotes the transpose of L. Combining this with (2-5), we know that the combinatorial fourth order flow (4-1) is in fact a gradient flow of energy $\mathcal{C}$ (which is called discrete Calabi energy in [Ge 2013; Ge and Xu 2016a]), that is:

$$\dot{l} = -\tfrac{1}{2}\nabla_l \mathcal{C} = -L^T R. \tag{4-2}$$

It is easy to obtain the following evolution equations:

$$\dot{R} = -LL^T R, \tag{4-3}$$

$$\dot{\mathcal{C}} = -2R^T LL^T R = -2(L^T R)^T (L^T R) = -\tfrac{1}{2}\|\nabla_l \mathcal{C}\|^2 \le 0, \tag{4-4}$$

$$\begin{aligned}\dot{S} &= (\dot{R})^T l + R^T \dot{l} = -(LL^T R)^T l + R^T(-L^T R) = -R^T LL^T l - R^T L^T R\\ &= -R^T L^T R.\end{aligned}$$

If there is only a single tetrahedron in the triangulation, then $m = 6$ and it is easy to calculate $\operatorname{rank}(L) = 5$. Thus we guess $\operatorname{rank}(L) = m - 1$ for general triangulations.

Conjecture. $\operatorname{rank}(L) = m - 1$ *for each* $l \in \mathfrak{M}_{\mathcal{T}}$.

The above conjecture is hopefully true. If so, then l is the only solution (up to scaling) of matrix equation $Lx = 0$. Moreover, each nonsingular solution to the fourth order flow (4-1) contains a subsequence converging to a discrete Einstein metric.

Theorem 4.1. *If there exists a discrete Einstein metric* l_{DE} *with* $\operatorname{rank}(L_{DE}) = m-1$ *on* $(M^3, \mathcal{T})$, *then there exists a constant* $\varepsilon > 0$ *such that, for any initial metric* $l(0)$ *with*

$$\|l(0) - l_{DE}\| < \varepsilon,$$

the solution to combinatorial fourth order flow $\dot{l} = L^T(R_{DE} - R)$ *exists for all time* $t \ge 0$ *and converges exponentially fast to the metric* l_{DE}.

Proof. Along the normalized fourth order flow (4-1),

$$\begin{aligned}\dot{R} &= \frac{\partial R}{\partial l}\dot{l} = LL^T(R_{DE} - R),\\ \dot{\mathcal{C}} &= -2(R_{DE} - R)^T LL^T (R_{DE} - R) \le 0,\end{aligned}$$

where $\mathcal{C} = \sum_{i=1}^m ((R_{DE})_i - R_i)^2$. Now we consider the ODE system

$$\dot{l} = \Upsilon(l) = L^T(R_{DE} - R).$$

Then $\Upsilon(l_{DE}) = 0$ and $D_l\Upsilon(l)|_{l=l_{DE}} = -L_{DE}L_{DE}^T \le 0$. As $\operatorname{rank}(L_{DE}) = m-1$, $D_l\Upsilon(l)|_{l=l_{DE}}$ is negative definite up to scaling. Hence l_{DE} is a local attractor and the system is asymptotically stable at l_{DE}. □

5. Discrete curvature flow in non-Euclidean geometry

K-space form triangulation and discrete curvature flow. Assume $K \in \mathbb{R}$ is a constant and, moreover, $K \neq 0$. In this section, we will consider a 3-dimensional compact manifold M^3 with a K-space form triangulation $\mathcal{T}$ on M^3. Let M_K be the space form with constant sectional curvature K. The basic blocks of K-space form triangulation $\mathcal{T}$ are tetrahedrons embedded in M_K.

A tetrahedron embedded in M_K is determined by its six edge lengths. Not every group of six positive numbers can be realized as the six edge lengths of some tetrahedrons embedded in M_K. Similar to the Euclidean case, there are nondegenerate conditions too. All admissible groups of six positive numbers which can be realized as the six edge lengths of some tetrahedrons embedded in M_K form an open connected set in $\mathbb{R}^6_{>0}$. The set is open due to Theorems 3.1 and 4.1 in [Yakut et al. 2009]; the set is connected due to the fact any tetrahedron can be deformed continuously to regular tetrahedrons.

The combinatorial Ricci curvature R_{ij} is defined in the same way as that of the Euclidean PL-manifold. We need to define a new functional S_K corresponding to the total curvature functional S.

Definition 5.1. Set $V = \sum_{\{i,j,k,l\}\in T} \mathrm{V}_{ijkl}$ and define

$$S_K \triangleq 2KV + \sum_{i=1}^{m} R_i l_i .$$

Now we recall the famous Schläfli formula for a K-space form tetrahedrons. For any K-space form tetrahedrons $\{i, j, k, l\} \in T$, one has (see [Milnor 1994; Schlenker 2000])

$$\frac{\partial \mathrm{V}_{ijkl}}{\partial \beta_{pq}} = \frac{l_{pq}}{2K}, \quad p, q \in \{i, j, k, l\},$$

where β_{pq} is the dihedral angle at the edge $\{p, q\}$ in the tetrahedrons $\{i, j, k, l\}$. Using the formula, one can get

$$2K\,dV + \sum_{i=1}^{m} l_i\,dR_i = 0.$$

Hence

$$dS_K = 2K\,dV + \sum_{i=1}^{m}(l_i\,dR_i + R_i\,dl_i) = \sum_{i=1}^{m} R_i\,dl_i ,$$

which implies $\partial S_K/\partial l_i = R_i$. Then we have $\nabla_l S_K = R$ and $\mathrm{Hess}_l\, S_K = L$.

Conjecture. *The symmetric matrix L is nonsingular and indefinite.*

We affirm the conjecture for the case of a single tetrahedron, and include the proof in the Appendix, see Theorem A.6.

With K-space form triangulation, we consider discrete curvature flow $\dot{l} = -R$ of second order and flow $\dot{l} = -L^T R$ of fourth order. Most properties are laid out in the following table:

Discrete curvature flow of second order	Discrete curvature flow of fourth order
$\dot{l} = -R = -\nabla_l S_K$	$\dot{l} = -L^T R = \nabla_l \mathcal{C}$
$\dot{R} = -LR$	$\dot{R} = -LL^T R$
$\dot{S}_K = -R^T R = -\mathcal{C} \leq 0$	$\dot{S}_K = -R^T L^T R$
$\dot{\mathcal{C}} = -2R^T LR$	$\dot{\mathcal{C}} = -2\|R^T L\|^2 \leq 0$

Theorem 5.2. *If the solution to second order flow $\dot{l} = -R$ exists for all time and converges to a nondegenerate metric l_∞, then l_∞ is a discrete Ricci-flat metric.*

Theorem 5.3. *If the solution to fourth order flow $\dot{l} = -L^T R$ exists for all time and converges to a nondegenerate metric l_∞ with L_∞ nonsingular, then l_∞ is a discrete Ricci-flat metric.*

Proof. The limit $\lim_{t\to+\infty} \mathcal{C}(t)$ exists because of the convergence of the flow $\dot{l} = -L^T R$, and $\mathcal{C}(t)$ is nonincreasing along the fourth order discrete curvature flow. So we have

$$\lim_{t\to+\infty} \dot{\mathcal{C}}(t) = 0,$$

which implies that $\lim_{t\to+\infty}(L^T R)^T (L^T R) = 0$. Hence $L^T R = 0$. Since L_∞ is nonsingular, $R_\infty = 0$. □

Theorem 5.4. *If there exists a discrete Ricci-flat metric l_{DE} with L_{DE} nonsingular, then the solution of fourth order discrete curvature flow $\dot{l} = L^T R$ exists for all time and converges to the discrete Einstein metric l_{DE} when the initial discrete Calabi energy $\mathcal{C}(0)$ is small enough.*

Proof. At the point l_{DE}, $D_l(-L^T R) = -LL^T < 0$. Hence l_{DE} is a local attractor of the flow. □

A fourth order flow for hyperbolic 3-manifolds. In the above subsection, we have seen that the matrix L is not so good for evolving a useful curvature flow. This is mainly because of the nondefiniteness of L. For a special type of manifolds and a special kind of triangulations, Feng Luo [2005] introduced a second order combinatorial curvature flow. In this short subsection, we introduce a fourth order flow which is very similar to Luo's flow.

Suppose M is a compact 3-manifold whose boundary is nonempty and is a union of surfaces with negative Euler characteristic. M can be ideally triangulated. The basic building blocks are strictly hyperideal tetrahedrons. For a single strictly hyperideal tetrahedron, let $l_1, \dots, l_6$ be the edge lengths of a strictly hyperideal tetrahedron, and $\beta_1, \dots, \beta_6$ be the dihedral angles at respective edges. Then the volume V is a strictly concave function of its dihedral angles, that is to say, $\mathrm{Hess}_\beta V = -\frac{1}{2}\partial(l_1, \dots, l_6)/\partial(\beta_1, \dots, \beta_6)$ is negative definite.

For M with ideal triangulation, denote $l=(l_1,\dots,l_m)^T$ as the edge lengths, $R=(R_1,\dots,R_m)^T$ as the combinatorial curvatures at all edges. Here the combinatorial curvature R_i at an edge i is 2π minus the sum of dihedral angles at the edge. Denote $\mathcal{C}(l)=\|R\|^2=\sum_{i=1}^m R_i^2$. Consider the combinatorial curvature flow

$$\dot{l}=-\tfrac{1}{2}\nabla_l\mathcal{C}=-LR, \tag{5-1}$$

where

$$L=\partial(R_1,\dots,R_m)/\partial(l_1,\dots,l_m)$$

is positive definite from [Luo 2005]. The equilibrium points of the combinatorial curvature flow (5-1) are the only flat metric with $R\equiv 0$, that is, the complete hyperbolic metric with totally geodesic boundary. Moreover, by

$$D_l(-LR)=-L^2<0,$$

we know that each equilibrium point is a local attractor of this flow. Hence, when the initial discrete energy $\mathcal{C}(0)$ is small enough, the solution of flow (5-1) exists for all time and converges to the flat metric, i.e., the complete hyperbolic metric with totally geodesic boundary.

Appendix

In this appendix we study the matrix L in space forms M_K, where subindex K represents the constant sectional curvature. We conclude that the matrix L is nonsingular and indefinite whenever $K\neq 0$.

Consider a single tetrahedron $\tau=\{A,B,C,D\}$ embedded in M_K. Since τ varies with its six edge lengths, all tetrahedrons can be considered as points of some connected open set in $\mathbb{R}^6_{>0}$. Denote β_{AB} as the dihedral angle at edge $\{A,B\}$. The dihedral angles and the edge lengths are mutually determined. On one hand, six dihedral angles are determined by six edge lengths. On the other hand, each tetrahedron in the space form M_K is determined, up to a motion, by its Gram matrix, which, in turn, is determined by the dihedral angles of the tetrahedron (see Chapter 6 §1 and Chapter 7 §2 in [Alekseevskij et al. 1993]). Therefore the Jacobian of dihedral angles over edges, which is denoted by

$$-L_{ABCD}\triangleq\frac{\partial(\beta_{AB},\beta_{AC},\beta_{AD},\beta_{BC},\beta_{BD},\beta_{CD})}{\partial(l_{AB},l_{AC},l_{AD},l_{BC},l_{BD},l_{CD})},$$

is nonsingular.

Next we prove that L_{ABCD} is indefinite. A tetrahedron is called regular, if all lengths are equal.

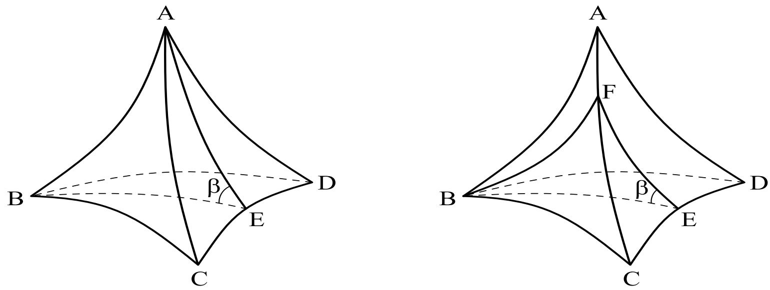

Figure 1

Proposition A.1. *For the regular tetrahedron, we have*

$$-L_{ABCD} = \begin{pmatrix} x & y & y & y & y & z \\ y & x & y & y & z & y \\ y & y & x & z & y & y \\ y & y & z & x & y & y \\ y & z & y & y & x & y \\ z & y & y & y & y & x \end{pmatrix},$$

where

$$x = \frac{\partial \beta_{AB}}{\partial l_{AB}}, \quad y = \frac{\partial \beta_{AB}}{\partial l_{AC}} = \frac{\partial \beta_{AB}}{\partial l_{AD}} = \frac{\partial \beta_{AB}}{\partial l_{BC}} = \frac{\partial \beta_{AB}}{\partial l_{BD}}, \quad z = \frac{\partial \beta_{AB}}{\partial l_{CD}}.$$

Moreover, the eigenvalues of the above $-L_{ABCD}$ *are* $x-z,\ x+z-2y,\ x+z+4y$ *with degree* 3, 2, 1 *respectively.*

In the following, we claim that, when $K \neq 0$, the matrix L is nonsingular but not definite. It's enough to determine the sign of $x-z,\ x+z-2y$ and $x+z+4y$.

First, we recall the formula of the cosine law in the 2-dimensional space forms $M^2(K)$ with constant sectional curvature K. Denote

$$S_K(t) = \begin{cases} \sin\big(\sqrt{K}t\big)/\sqrt{K}, & K > 0, \\ t, & K = 0, \\ \sinh\big(\sqrt{-K}t\big)/\sqrt{-K}, & K < 0, \end{cases}$$

$$C_K(t) = \begin{cases} \cos\big(\sqrt{K}t\big), & K > 0, \\ 1, & K = 0, \\ \cosh\big(\sqrt{-K}t\big), & K < 0, \end{cases}$$

$$f_K(r) = \int_0^r S_K(t)\,dt = \begin{cases} (1 - C_K(r))/K, & K \neq 0, \\ r^2/2, & K = 0. \end{cases}$$

Then we have the following identities:

(1) $f_K^{'}(r) = S_K(r), \quad S_K^{'}(r) = C_K(r)$.

(2) $KS_K^2(a) + C_K^2(a) = 1$.

(3) $S_K(a+b) = S_K(a)S_K(b) + C_K(a)C_K(b)$.

(4) $C_K(a+b) = C_K(a)C_K(b) - KS_K(a)S_K(b)$.

(5) $C_K(2a) = 2C_K^2(a) - 1 = 1 - 2KS_K^2(a)$.

So

$$f_K(r) = 2S_K^2(r/2).$$

Proposition A.2 (the cosine law). *For a geodesic triangle $\triangle ABC$ in the space form $M^2(K)$, with side lengths a, b, c opposite to the angles A, B, C, respectively, the cosine law is*

$$f_K(c) = f_K(a-b) + S_K(a)S_K(b)(1-\cos C).$$

For $K \neq 0$, the above formula is equivalent to

$$C_K(c) = C_K(a)C_K(b) + KS_K(a)S_K(b)\cos C.$$

Now, calculating the exact value of a, b, c, we have the following results.

Lemma A.3.
$$z = \frac{\sqrt{2}C_K^2(l_0/2)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}}.$$

Proof. By the definition of L_{16}, we just need to calculate $\partial\beta/\partial l_6$. To calculate it, we assume the length of AB is l_6 and other edges have length l_0 in the hyperbolic tetrahedron in Figure 1 (left). As shown there, E is the midpoint of the edge CD, and the dihedral angle at the edge CD is the angle ∠AEB, i.e., β.

Using the cosine law in the triangle △AEB, we have

$$f_K(l_6) = f_K(0) + S_K^2(h_0)(1-\cos\beta_1) = S_K^2(h_0)(1-\cos\beta_1),$$

where h_0 is the length of the altitude in the regular triangle with side length l_0. We can get

$$\frac{\partial\beta_1}{\partial l_6} = \frac{f_K^{'}(l_6)}{S_K^2(h_0)\sin\beta} = \frac{S_K(l_6)}{S_K^2(h_0)\sin\beta}.$$

So at the regular point

$$z = \frac{S_K(l_0)}{S_K^2(h_0)\sin\beta},$$

and

$$f_K(l_0) = f_K\Big(h_0 - \frac{l_0}{2}\Big) + S_K(h_0)S_K\Big(\frac{l_0}{2}\Big).$$

Then we have

$$C_K(h_0) = \frac{C_K(l_0)}{C_K(l_0/2)},$$

which implies

$$S_K^2(h_0) = \begin{cases} \dfrac{1-C_K^2(h_0)}{K}, & K \neq 0, \\ h_0^2, & K = 0. \end{cases}$$

For $K \neq 0$,

$$S_K^2(h_0) = \frac{C_K^2(l_0/2) - C_K^2(l_0)}{KC_K^2(l_0/2)} = \frac{S_K^2(l_0/2)(1+2C_K(l_0))}{C_K^2(l_0/2)}.$$

The equation also holds for the case of $K = 0$. By the cosine law,

$$\cos\beta = \frac{S_K^2(h_0) - f_K(l_0)}{S_K^2(h_0)}.$$

If $K = 0$, it is easy to get $\cos\beta = 1 - l_0^2/(2h_0^2) = 1/3$. For the case of $K \neq 0$,

$$\begin{aligned}
\cos\beta &= \frac{(1-C_K^2(h_0))/K - (1-C_K(l_0))/K}{(1-C_K^2(h_0))/K} = \frac{C_K(l_0) - C_K^2(h_0)}{1-C_K^2(h_0)} \\
&= \frac{C_K(l_0) - C_K^2(l_0)/C_K^2(l_0/2)}{1 - C_K^2(l_0)/C_K^2(l_0/2)} = \frac{C_K(l_0)(C_K^2(l_0/2) - C_K(l_0))}{C_K^2(l_0/2) - C_K^2(l_0)} \\
&= \frac{KC_K(l_0)S_K^2(l_0/2)}{C_K^2(l_0/2) - C_K^2(l_0)} = \frac{KC_K(l_0)S_K^2(l_0/2)}{(1+C_K(l_0) - 2C_K^2(l_0))/2} \\
&= \frac{KC_K(l_0)S_K^2(l_0/2)}{(1+2C_K(l_0))(1-C_K(l_0))/2} = \frac{KC_K(l_0)S_K^2(l_0/2)}{(1+2C_K(l_0))KS_K^2(l_0/2)} \\
&= \frac{C_K(l_0)}{1+2C_K(l_0)}.
\end{aligned}$$

This formula also holds for $K = 0$. Then we have

$$\sin\beta = \frac{\sqrt{(1+C_K(l_0))(1+3C_K(l_0))}}{1+2C_K(l_0)} = \frac{\sqrt{2}\,C_K(l_0/2)\sqrt{1+3C_K(l_0)}}{1+2C_K(l_0)}.$$

Hence

$$z = \frac{S_K(l_0)C_K(l_0/2)}{\sqrt{2}S_K^2(l_0/2)\sqrt{1+3C_K(l_0)}} = \frac{\sqrt{2}\,C_K^2(l_0/2)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}}. \qquad \square$$

Lemma A.4.
$$x=\frac{\sqrt{2}C_K^2(l_0)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}(1+2C_K(l_0))}.$$

Proof. To calculate it, we assume the length of CD is l_1 and other edges have length l_0 in the tetrahedron shown in Figure 1 (left). As illustrated there, E is the midpoint of the edge CD, the dihedral angle at the edge CD is the angle $\angle$AEB, i.e., β. We assume the length of AE is h. By the cosine law,

$$f_K(l_0)=f_K(0)+S_K^2(h)(1-\cos\beta)=S_K^2(h)(1-\cos\beta),$$

and we have
$$-\frac{\partial\beta}{\partial l_1}=\frac{1-\cos\beta}{S_K^2(h)\sin\beta}\frac{\partial S_K^2(h)}{\partial l_1}.$$

By the cosine law again,

$$f_K(l_0)=f_K(l_1/2-h)+S_K(h)S_K(l_1/2),$$

and we have
$$C_K(h)=C_K(l_0)/C_K(l_1/2).$$

Hence
$$\frac{\partial C_K(h)}{\partial l_1}=-\frac{C_K(l_0)C_K'(l_1/2)}{2C_K^2(l_1/2)}=\frac{KS_K(l_1/2)C_K(l_0)}{2C_K^2(l_1/2)},$$

and
$$S_K^2(h)=\begin{cases}\frac{1-C_K^2(h)}{K}, & K\neq 0,\\ h^2, & K=0,\end{cases}$$

which implies that

$$\frac{\partial S_K^2(h)}{\partial l_1}=\begin{cases}\frac{-2C_K(h)}{K}\frac{KS_K(l_1/2)C_K(l_0)}{2C_K^2(l_1/2)}=-\frac{C_K^2(l_0)S_K(l_1/2)}{C_K^3(l_1/2)}, & K\neq 0,\\ -\frac{l_1}{2}, & K=0.\end{cases}$$

So we obtain
$$\frac{\partial S_K^2(h)}{\partial l_1}=-\frac{C_K^2(l_0)S_K(l_1/2)}{C_K^3(l_1/2)}.$$

At the regular point, we have

$$\begin{aligned}x=\frac{\partial\beta}{\partial l_1}&=\frac{C_K^2(l_0)S_K(l_1/2)}{C_K^3(l_1/2)}\frac{C_K(l_0/2)(1+C_K(l_0))}{\sqrt{2}S_K^2(l_0/2)\sqrt{1+3C_K(l_0)}(1+2C_K(l_0))}\\&=\frac{\sqrt{2}C_K^2(l_0)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}(1+2C_K(l_0))}.\end{aligned}$$

□

Lemma A.5. $$y = -\frac{\sqrt{2}C_K(l_0)C_K^2(l_0/2)}{S_K(l_0/2)(1+2C_K(l_0))\sqrt{1+3C_K(l_0)}}.$$

Proof. To calculate it, we assume the length of AD is l_2 and other edges have length l_0 in the tetrahedron in Figure 1 (right). As shown there, E is the midpoint of the edge CD, the dihedral angle at the edge CD is the angle $\angle$FEB, i.e., β. For simplicity, we assume $l_2 \le l_0$. Assume the length of AF is s, and the length of FE is $\tilde{h}$. So the length of FC and FD are equal to $l_0 - s$. By the cosine law in the triangle $\triangle$CEF,

$$f_K(l_0 - s) = f_K(\tilde{h} - l_0/2) + S_K(\tilde{h})S_K(l_0/2).$$

By the cosine law in the triangle $\triangle$AFD,

$$f_K(l_0 - s) = f_K(l_2 - s) + \frac{S_K(s)}{S_K(l_0)}(f_K(l_0) - f_K(l_2 - l_0)).$$

By the cosine law in the triangles $\triangle$ABF and $\triangle$BEF,

$$f_K(l_0 - s) + \frac{f_K(l_0)S_K(s)}{S_K(l_0)} = f_K(h_0 - \tilde{h}) + S_K(h_0)S_K(\tilde{h})(1 - \cos\beta).$$

Differentiating the above three equations at the regular point, i.e., $s = 0$, $l_2 = l_0$, and $\tilde{h} = h_0$, we have

$$\begin{aligned}
-S_K(l_0)ds &= (S_K(h_0 - l_0/2) + C_K(h_0)S_K(l_0/2))d\tilde{h} \\
&= S_K(h_0)C_K(l_0/2)d\tilde{h}, \\
-S_K(l_0)ds &= -S_K(l_0)ds + S_K(l_0)dl_2 + \frac{C_K(0)}{S_K(l_0)}f_K(l_0)ds,
\end{aligned}$$

$$\begin{aligned}
-S_K(l_0)ds + \frac{f_K(l_0)C_K(0)}{S_K(l_0)}ds = {} & -S_K(0)d\tilde{h} + S_K(h_0)C_K(h_0)(1 - \cos\beta)d\tilde{h} \\
& + S_K^2(h_0)\sin\beta\, d\beta.
\end{aligned}$$

Using the fact $S_K(0) = 0$, $C_K(0) = 1$, we obtain

(1) $ds = -\dfrac{S_K^2(l_0)}{f_K(l_0)}dl_2,$

(2) $d\tilde{h} = -\dfrac{S_K(l_0)}{S_K(h_0)C_K(l_0/2)}ds = \dfrac{S_K^3(l_0)}{f_K(l_0)S_K(h_0)C_K(l_0/2)}dl_2,$

(3) $\dfrac{f_K(l_0) - S_K^2(l_0)}{S_K(l_0)}ds = S_K(h_0)C_K(h_0)(1 - \cos\beta)d\tilde{h} + S_K^2(h_0)\sin\beta\, d\beta.$

Using

$$\cos\beta = \frac{C_K(l_0)}{1+2C_K(l_0)}, \quad C_K(h_0) = \frac{C_K(l_0)}{C_K(l_0/2)},$$

we have

$$-\frac{(f_K(l_0)(1+2C_K(l_0))-S_K^2(l_0))S_K(l_0)}{f_K(l_0)(1+2C_K(l_0))}dl_2 = S_K^2(h_0)\sin\beta d\beta.$$

Since $f_K(l_0) = 2S_K^2(l_0/2)$, we have

$$-\frac{C_K(l_0)S_K(l_0)}{1+2C_K(l_0)}dl_2 = S_K^2 \sin\beta d\beta.$$

Hence

$$\begin{aligned} y = \frac{\partial\beta}{\partial l_2} &= -\frac{C_K(l_0)S_K(l_0)}{1+2C_K(l_0)}\frac{1}{S_K^2(h_0)\sin\beta} \\ &= -\frac{C_K(l_0)S_K(l_0)}{1+2C_K(l_0)}\frac{C_K(l_0/2)}{\sqrt{2}S_K^2(l_0/2)\sqrt{1+3C_K(l_0)}} \\ &= -\frac{\sqrt{2}C_K(l_0)C_K^2(l_0/2)}{S_K(l_0/2)(1+2C_K(l_0))\sqrt{1+3C_K(l_0)}}. \end{aligned}$$

□

So we have

(1) $x - z = -\dfrac{\sqrt{2}\sqrt{1+3C_K(l_0)}}{2S_K(l_0/2)(1+2C_K(l_0))} < 0,$

(2) $x + z - 2y = \dfrac{\sqrt{2}\sqrt{1+3C_K(l_0)}}{2S_K(l_0/2)} > 0,$

(3) $x + z + 4y = \dfrac{\sqrt{2}KS_K(l_0/2)}{(1+2C_K(l_0))\sqrt{1+3C_K(l_0)}}.$

Hence $x+z+4y > 0$ when $K > 0$, $x+z+4y = 0$ when $K = 0$, and $x+z+4y < 0$ when $K < 0$.

Theorem A.6. *When $K \neq 0$, the matrix L of one single tetrahedron $-L_{ABCD}$ embedded in M_K is nonsingular and indefinite. Hence the conjecture on page 60 is true for this case.*

Proof. By the calculations above, we know that the matrix L at regular points is indefinite. Any tetrahedron can be deformed continuously to the regular tetrahedron, so all tetrahedrons have the same properties. □

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