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We establish a comparison between the notion of derived Deligne–Mumford stack in the sense of Toën and Vezzosi and the one introduced by Lurie. It is folklore that the two theories yield essentially the same objects, but it is difficult to locate in the literature a precise result, despite it sometimes being useful to be able to switch between the two frameworks.

Introduction

This short paper is devoted to establishing in a precise way the folklore equivalence between the theory of derived Deligne–Mumford stacks introduced by B. Toën and G. Vezzosi [2008] and the one defined by J. Lurie [2011b]. The main comparison result will be stated in the next section. See Theorem 1.7. Even though many of the results used to achieve the proof of the main theorem can be found scattered through the DAG series of J. Lurie, the precise form of Theorem 1.7 has not appeared anywhere in the literature, to the best of my knowledge.

As certain problems are easier to approach from the point of view of the functor of points, and others from the point of view of structured spaces, a precise comparison result can be useful. Moreover, this note can be helpful for someone who is trying to approach the subject of derived algebraic geometry for the first time. For this last reason, I preferred to be lengthy and to give thorough explanations even where perhaps they would not have been necessary.

Conventions. Throughout this paper we will work freely with the language of $(\infty, 1)$-categories. We will call them simply ∞-categories and our basic reference on the subject is [Lurie 2009]. Occasionally, it will be necessary to consider $(n, 1)$-categories. We will refer to such objects as n-categories, and we redirect the reader to [Lurie 2009, §2.3.4] for the definitions and the basic properties. There is no chance of confusion with the theory of (∞, n)-categories, since it plays no role in this note. The notation $\mathcal{S}$ will be reserved for the ∞-categories of spaces.

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Whenever categorical constructions are used (such as limits, colimits, etc.), we mean the corresponding ∞-categorical notion. For the reader with a model categorical background, this means that we are always considering *homotopy* limits, *homotopy* colimits, etc. See [Lurie 2009, 4.2.4.1].

In [Lurie 2009] and more generally in the DAG series, whenever $\mathcal{C}$ is a 1-category the notation $\mathrm{N}(\mathcal{C})$ denotes $\mathcal{C}$ viewed (trivially) as an ∞-category. This notation stands for the nerve of the category $\mathcal{C}$ (and this is because an ∞-category in [Lurie 2009] is defined to be a quasicategory, that is a simplicial set with special lifting properties). In this note, we will systematically suppress this notation, and we encourage the reader to think of ∞-categories as model-independently as possible. For this reason, if k is a (discrete) commutative ring we chose to denote by CAlg_k the ∞-category underlying the category of simplicial commutative k-algebras and by $\mathrm{CAlg}_k^\heartsuit$ the 1-category of discrete k-algebras.

1. Statement of the comparison result

Let us begin by quickly reviewing the two theories.

HAG II framework. In [Toën and Vezzosi 2008], the authors work within the setting previously introduced in [Toën and Vezzosi 2005], where the theory of model topoi is introduced and extensively explored. In particular, model categories are used continuously throughout the whole paper. In order to compare their constructions with the ones of [Lurie 2011b], it will be convenient to rethink the paper in purely ∞-categorical language. This is essentially no more than an easy exercise, and we use this opportunity in this review to explain how it can be done.

Let k be a commutative ring (with unit). We will denote by sMod_k the category of simplicial k-modules. There is an adjunction

$$U : \mathrm{sMod}_k \rightleftarrows \mathrm{sSet} : F \qquad (F \dashv U)$$

which satisfies the hypothesis of the lifting principle (see [Schwede and Shipley 2000]) and therefore it allows us to lift the (Kan) model structure on sSet to a simplicial model structure on sMod_k. Moreover, with respect to this model structure, sMod_k becomes a monoidal model category (whose tensor product is computed objectwise). We set $\mathrm{sAlg}_k := \mathrm{Com}(\mathrm{sMod}_k)$. Using the fact that every object in sMod_k is fibrant, it is possible to establish that the adjunction

$$V : \mathrm{sAlg}_k \rightleftarrows \mathrm{sMod}_k : \mathrm{Sym}_k \qquad (\mathrm{Sym}_k \dashv V),$$

satisfies again the lifting principle (see [Schwede and Shipley 2000, §5]), and therefore the (simplicial) model structure on sMod_k induces a simplicial model structure on sAlg_k. We will simply denote by CAlg_k the ∞-category underlying sAlg_k, which can be explicitly thought as the coherent nerve [Lurie 2009, §1.1.5] of the category of fibrant cofibrant objects in sAlg_k. It is customary to denote the opposite of this ∞-category by dAff_k (the ∞-category of "affine derived schemes").

This ∞-category admits another description which is more useful for our purposes. Let $\mathcal{T}_{\mathrm{disc}}(k)$ the full subcategory of ordinary schemes over $\mathrm{Spec}(k)$ spanned by the relative finite-dimensional affine spaces $\mathbb{A}^n_k$. We can think of $\mathcal{T}_{\mathrm{disc}}(k)$ as a (one-sorted) Lawvere theory; equally, in the language of [Lurie 2011b], we can say that $\mathcal{T}_{\mathrm{disc}}(k)$ is a *discrete pregeometry*. The ∞-category of product-preserving functors with values in the ∞-category of spaces can be identified with the *sifted completion* of $\mathcal{T}_{\mathrm{disc}}(k)$ and we will denote it by $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$ (see [Lurie 2009, Definition 5.5.8.8]). This is a presentable ∞-category and therefore it admits a presentation by a model category [ibid., A.3.7.6], which can be easily obtained as follows: consider the category of simplicial presheaves on $\mathcal{T}_{\mathrm{disc}}(k)$ endowed with the global projective model structure. Then the underlying ∞-category of the Bousfield localization of this model category at the collection of maps $y(\mathbb{A}^n_k) \coprod y(\mathbb{A}^m_k) \to y(\mathbb{A}^{n+m}_k)$ (where y denotes the Yoneda embedding) precisely coincides with $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$. It is somehow remarkable that $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$ admits a much stricter presentation. Consider in fact the category of functors $\mathcal{T}_{\mathrm{disc}}(k) \to \mathrm{sSet}$ which *strictly* preserve products. It follows from a theorem of Quillen [ibid., 5.5.9.1] that this simplicial category admits a global projective model structure. Moreover, a theorem of J. Bergner [ibid., 5.5.9.2] shows that the underlying ∞-category coincides precisely with $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$. However, the category of product-preserving functors $\mathcal{T}_{\mathrm{disc}}(k) \to \mathrm{sSet}$ is precisely equivalent to sAlg_k, and the two model structures agree. Therefore, we have a categorical equivalence

$$\mathrm{CAlg}_k \simeq \mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k)).$$

The reader might want to consult also [Lurie 2011b, Remark 4.1.2] for another discussion of this equivalence.

The next step is to introduce the étale topology on the model category sAlg_k. As this notion only depends on the homotopy category of sAlg_k [Toën and Vezzosi 2005, Definition 4.3.1], it also defines a Grothendieck topology on the ∞-category CAlg_k [Lurie 2009, 6.2.2.3]. We briefly recall that a morphism $f : A \to B$ in sAlg_k is said to be étale if $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is étale and the canonical map

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$$

is an isomorphism (that is, the morphism is *strong*). Similarly, a morphism $f : A \to B$ is smooth if it is strong and $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is smooth. We denote by $\tau_{\text{ét}}$ the étale topology and by $\mathbf{P}_{\text{ét}}$ (resp. $\mathbf{P}_{\mathrm{sm}}$) the collection of étale (resp. smooth) morphisms. Using these data, one can form the model category of hypersheaves with respect to the étale topology. Recall that this is obtained in two steps:

(1) considering the global projective model structure on $\mathrm{Fun}(\mathrm{sAlg}_k, \mathrm{sSet})$;

(2) considering next the Bousfield localization of this model structure at the collection of hypercovers (see [Toën and Vezzosi 2005, §4.4 and §4.5] or [Lurie 2009, §6.5.3]).

The result is what is denoted in [Toën and Vezzosi 2008] by $\mathrm{dAff}^{\sim,\tau_{\text{ét}}}$. It follows from [Lurie 2009, 6.5.2.14, 6.5.2.15] that the underlying ∞-category of $\mathrm{dAff}^{\sim,\tau_{\text{ét}}}$ can be identified with the hypercompletion $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ (we refer the reader to [Lurie 2009, §6.5.2] for a detailed discussion of this notion). We usually refer to the objects in $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ as stacks (for the étale topology). The next step is to consider geometric stacks inside $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$. Since there are many references for this subject [Simpson 1996; Toën and Vezzosi 2008; Toën and Vaquié 2008; Porta and Yu 2016], we do not repeat the full definition here, but we limit ourselves to describing the general idea. Roughly speaking, geometric stacks are stacks X admitting a morphism $p : U \to X$ satisfying the following conditions:

(1) U is an affine derived scheme (seen as a stack via the ∞-categorical Yoneda embedding, see [Lurie 2009, §5.1.3] or [Lurie 2016a, §5.2.1]).

(2) $p : U \to X$ is an effective epimorphism (see [Lurie 2009, §6.2.3 and 7.2.1.14]).

(3) p is either an étale or a smooth morphism.

The notions of étale and smooth morphisms between geometric stacks must be defined with some care, proceeding by induction on the “geometric level” of the stack. See [Porta and Yu 2016, Definition 2.8] or [Toën and Vezzosi 2008, §1.3.3] for a complete review. When p can be chosen to be étale, we refer to X as a (higher) derived Deligne–Mumford stack; if instead p can only be chosen to be smooth, we refer to X as a (higher) derived Artin stack. We are mostly concerned with derived Deligne–Mumford stacks (see however Remark 1.9). We denote the full subcategory of $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ spanned by derived Deligne–Mumford stacks by **DM**. Let us complete the review of [Toën and Vezzosi 2008] with some additional remarks:

(1) Geometric stacks are stable under weak equivalences because only homotopy invariant categorical constructs are used in the definition (i.e., homotopy coproducts, homotopy geometric realizations, etc.). Therefore [Lurie 2009, 4.2.4.1] shows that the notion of geometric stack can be equally formulated at the level of the ∞-category $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$.

(2) The category **DM** is naturally filtered by the notion of geometric level: a stack is said to be (-1)-geometric if it is representable by an object in dAff_k. If $A \in \mathrm{CAlg}_k$, we choose to represent its functor of points by $\mathrm{Spec}(A) \in \mathbf{DM} \subset \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$. Next, proceeding by induction, we say that a stack X is n-geometric if it admits an atlas $p : U \to X$ which is representable by $(n-1)$-geometric stacks in the following precise sense: for every representable stack $\mathrm{Spec}(A)$ and any map $\mathrm{Spec}(A) \to X$

the base change $\mathrm{Spec}(A) \times_X U$ is $(n-1)$-geometric. We say that a derived stack is geometric if it is n-geometric for some n.

(3) We denote by $\mathbf{DM}_n$ the full subcategory of $\mathbf{DM}$ spanned by geometric derived Deligne–Mumford stacks whose restriction to $\mathrm{CAlg}_k^{\heartsuit}$ is an n-truncated stack (i.e., it takes values in n-truncated spaces).

DAG V framework. The point of view taken in [Lurie 2011b] is quite different. We refer the reader to the introduction of [Porta 2015] for an expository account of the role of (pre)geometries (compare [Lurie 2011b, §1.2, 3.1]) in the construction of affine derived objects. Here, we content ourselves with a short review of the theory of $\mathcal{G}$-schemes for a given geometry $\mathcal{G}$ from the point of view of [Lurie 2011b]. Recall either from [Lurie 2011b, Definition 12.8] or from the introduction of [Porta 2015] that a geometry is an ∞-category $\mathcal{G}$ with finite limits and equipped with some extra structure, consisting of a collection of "admissible" morphisms and a Grothendieck topology τ on $\mathcal{G}$ generated by admissible morphisms. If $\mathcal{X}$ is an ∞-topos and $\mathcal{G}$ is a geometry, they define an ∞-category of $\mathcal{G}$-structures, denoted $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$. Recall that a $\mathcal{G}$-structure is a functor $\mathcal{G} \to \mathcal{X}$ which is left exact and takes τ-coverings to effective epimorphisms in $\mathcal{X}$.

Before moving on, it is important to discuss a very important special case. If $\mathcal{X}$ is the ∞-topos of $\mathcal{S}$-valued sheaves on some topological space X, we can think of a $\mathcal{G}$-structure on $\mathcal{X}$ as a sheaf on X with values in the ∞-category $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ having special behavior on the stalks, as the next key example shows:

Example 1.1. Let k be a fixed (discrete) commutative ring. We denote by $\mathcal{G}_{\text{ét}}(k)$ the category $((\mathrm{CRing}_k^{\heartsuit})^{\text{f.p.}})^{\mathrm{op}}$, the opposite of the category of discrete k-algebras of finite presentation. Moreover, we declare a morphism in $\mathcal{G}_{\text{ét}}(k)$ to be an admissible morphism if and only if it is étale, and we endow $\mathcal{G}_{\text{ét}}(k)$ with the usual étale topology. In this case, $\mathrm{Ind}(\mathcal{G}_{\text{ét}}(k)^{\mathrm{op}}) \simeq \mathrm{CAlg}_k^{\heartsuit}$, the category of discrete k-algebras of finite presentation. Then a $\mathcal{G}_{\text{ét}}(k)$-structure $\mathcal{O}$ on $\mathrm{Sh}(X)$ is a sheaf of discrete commutative rings on X whose stalks are strictly henselian local rings. The fact that $\mathcal{O}$ has to be discrete follows from its left-exactness (see [Lurie 2009, §5.5.6] for a general discussion of truncated objects in an ∞-category and more specifically [Lurie 2009, 5.5.6.16] for the needed property). The statement on stalks, instead, is due to the following fact: for every point $x \in X$ (formally seen as a geometric morphism $x^{-1} : \mathrm{Sh}(X) \rightleftarrows \mathcal{S} : x_*$), the stalk $\mathcal{O}_x := x^{-1}\mathcal{O}$ has to take étale coverings of k-algebras of finite presentation to epimorphisms in Set. Unraveling the definitions, this means that for every étale cover $\{A \to A_i\}$ in $\mathcal{G}_{\text{ét}}(k)$ and every solid diagram

$$\begin{tikzcd} & \coprod \mathrm{Spec}(A_i) \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_x) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(A), \end{tikzcd}$$

the lifting exists. This is a possible characterization of strictly henselian local rings (see [de Jong et al. 2005–, Tag 04GG, condition (8)]).

As in the case of locally ringed spaces, we are not really interested in all the transformations of $\mathcal{G}$-structures, but only in those that have good local behavior. This can be made precise by introducing the notion of *local transformation of $\mathcal{G}$-structures*. We recall that a morphism $f : \mathcal{O} \to \mathcal{O}'$ in $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ is said to be local if for every admissible morphism $f : U \to V$ in $\mathcal{G}$ the induced square

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}(V) \arrow[d] \\ \mathcal{O}'(U) \arrow[r] & \mathcal{O}'(V) \end{tikzcd}$$

is a pullback in $\mathcal{X}$. In Example 1.1, morphisms satisfying the above condition simply become local morphisms of local rings.

Precisely as in the case of locally ringed spaces, we can use $\mathcal{G}$-structures and local morphisms of such to build an ∞-category of $\mathcal{G}$-structured topoi, denoted $\mathcal{T}\mathrm{op}(\mathcal{G})$. The actual construction is rather involved, and we refer to [Lurie 2011b, Definition 1.4.8] for the details. Here, we content ourselves with the following rougher idea: the ∞-category $\mathcal{T}\mathrm{op}(\mathcal{G})$ has as objects pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_{\mathcal{X}}$ is a $\mathcal{G}$-structure on $\mathcal{X}$, and as 1-morphisms pairs $(f, \alpha) : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, where f is a geometric morphism $f^{-1} : \mathcal{Y} \rightleftarrows \mathcal{X} : f_*$ and $\alpha : f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is a *local* transformation of $\mathcal{G}$-structures on $\mathcal{X}$.

The category $\mathcal{T}\mathrm{op}(\mathcal{G})$ is too huge to be of any practical interest. Therefore we are going to construct a full subcategory $\mathrm{Sch}(\mathcal{G})$ which intuitively corresponds to the subcategory of $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ spanned by geometric stacks. We will see in discussing Theorem 1.7 that this is not quite a true statement, but until then it is a reasonable analogy. The idea is rather straightforward: as schemes are a full subcategory of locally ringed spaces spanned by those objects which are locally isomorphic to special ones constructed out of commutative rings, so objects $\mathrm{Sch}(\mathcal{G})$ are structured topoi locally equivalent to a collection of special models. As Example 1.1 suggests, it is possible to associate a $\mathcal{G}$-structured topos to every object of $\mathrm{Ind}(\mathcal{G})$. To keep the exposition as elementary as possible, we limit ourselves to considering the case of objects in $\mathcal{G}$, and we refer the reader to [Lurie 2011b, §2.2] for the general discussion.

Let $A \in \mathcal{G}^{\mathrm{op}}$. We will denote by A_{adm} the small admissible site of A. The underlying ∞-category of A_{adm} is the opposite of the full subcategory of $\mathcal{G}^{\mathrm{op}}_{A/}$ spanned by admissible morphisms $A \to B$. We then endow A_{adm} with the Grothendieck topology induced from the one on $\mathcal{G}$, which we still denote τ. Finally, we let $\mathcal{X}_A$ be the *nonhypercomplete* ∞-topos of ($\mathcal{S}$-valued) sheaves on A_{adm}. We next construct the $\mathcal{G}$-structure on $\mathcal{X}_A$. There is a forgetful functor $A_{\mathrm{adm}} \to \mathcal{G}$ which induces a composition

$$A_{\mathrm{adm}}^{\mathrm{op}} \times \mathcal{G} \to \mathcal{G}^{\mathrm{op}} \times \mathcal{G} \xrightarrow{y} \mathcal{S},$$

where y is the functor classifying the Yoneda embedding, see [Lurie 2016a, §5.2.1]. This corresponds to a functor

$$\mathcal{O}_A : \mathcal{G} \to \mathrm{PSh}(A_{\mathrm{adm}}) \xrightarrow{\mathrm{L}} \mathrm{Sh}(A_{\mathrm{adm}}, \tau),$$

where L is the sheafification functor. Note that if the Grothendieck topology on $\mathcal{G}$ were subcanonical, there would not be any need to apply L. Observe further that $\mathcal{O}_A$ is indeed left-exact by its very construction. We leave as an exercise to the reader to prove that $\mathcal{O}_A$ takes τ-coverings in effective epimorphisms (cf., [Lurie 2011b, Proposition 2.2.11]). Therefore the pair $(\mathcal{X}_A, \mathcal{O}_A)$ defines a $\mathcal{G}$-structured topos, which we denote as $\mathrm{Spec}^{\mathcal{G}}(A)$.

Remark 1.2. As often happens in the ∞-categorical world, the construction of the functoriality is the most subtle point in the definition of an ∞-functor. So, to build $\mathrm{Spec}^{\mathcal{G}}(-)$ as an ∞-functor $\mathcal{G} \simeq (\mathcal{G}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{T}\mathrm{op}(\mathcal{G})$, some additional effort is needed. The details are out of the scope of this review, but the rough idea is to prove that $\mathrm{Spec}^{\mathcal{G}}(A)$ enjoys a universal property, which makes $\mathrm{Spec}^{\mathcal{G}}(-)$ a right adjoint to the global section functor $\mathcal{T}\mathrm{op}(\mathcal{G}) \to \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$, informally defined by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto \mathrm{Map}_{\mathcal{X}}(\mathbf{1}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$. Observe that the latter becomes a finite-limit-preserving functor $\mathcal{G} \to \mathcal{S}$ and therefore can be identified with an element of $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. We refer the reader to [Lurie 2011b, §2.2] (and especially to [Lurie 2011b, Theorem 2.2.12]) for a detailed discussion.

With these preparations, it is now easy to define $\mathrm{Sch}(\mathcal{G})$ as a full subcategory of $\mathcal{T}\mathrm{op}(\mathcal{G})$. We will that a $\mathcal{G}$-structured topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a *$\mathcal{G}$-scheme* (resp. a *$\mathcal{G}$-scheme locally of finite presentation*) if there exists a collection of objects $U_i \in \mathcal{X}$ satisfying the following two conditions:

(1) The joint morphism $\coprod U_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism.

(2) For every index i, there exists an object $A_i \in \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ (resp. an object $A_i \in \mathcal{G}^{\mathrm{op}}$) and an equivalence of $\mathcal{G}$-structured topoi $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \simeq \mathrm{Spec}^{\mathcal{G}}(A_i)$.

We conclude this review with two important examples and some discussion about them.

Example 1.3. Let us go back to the geometry $\mathcal{G} := \mathcal{G}_{\text{ét}}(k)$ of Example 1.1. The category $\mathrm{Sch}(\mathcal{G})$ contains a very interesting full subcategory. To describe it, let us briefly recall that an ∞-topos $\mathcal{X}$ is said to be n-localic (for $n \geq -1$ an integer) if it can be thought of as the category of ($\mathcal{S}$-valued) sheaves on some Grothendieck site $(\mathcal{C}, \tau)$ with $\mathcal{G}$ being an n-category (see our conventions on the meaning of this). We refer the reader to [Lurie 2009, §6.4.5] for a more detailed account of this notion. Let $\mathrm{Sch}_{\leq 1}(\mathcal{G})$ be the full subcategory of $\mathrm{Sch}(\mathcal{G})$ spanned by $\mathcal{G}$-schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

such that $\mathcal{X}$ is 1-localic. Then [Lurie 2011b, Theorem 2.6.18] shows that $\mathrm{Sch}_{\leq 1}(\mathcal{G})$ is equivalent to the category of 1-geometric (underived) Deligne–Mumford stacks. More generally, Theorem 1.7 implies $\mathrm{Sch}_{\leq n}(\mathcal{G})$ is equivalent to the ∞-category of n-truncated (underived) Deligne–Mumford stacks.

Example 1.4. Let us define a new geometry $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$ as follows. We let the underlying ∞-category of $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$ to be the opposite of the full subcategory of CAlg_k spanned by compact objects. Observe that $\mathrm{CAlg}_k = \mathrm{Ind}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)^{\mathrm{op}})$. We say that a morphism in $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$ is admissible precisely when it is a (derived) étale morphism (see the previous section for the definition). We will further endow $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$ with the (derived) étale topology, which we will still denote $\tau_{\text{ét}}$ (observe that if $A \to B$ is an étale map in the derived sense and the source is discrete, then so is the target). In this special case, we write $\mathrm{Spec}^{\text{ét}}$ instead of $\mathrm{Spec}^{\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)}$. Following [Lurie 2011b, Definition 4.3.20] (and using the important [Lurie 2011b, Proposition 4.3.15]), we say that a derived Deligne–Mumford stack (in the sense of [Lurie 2011b]) is a $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-scheme.

The following theorem summarizes several results of [Lurie 2011b]. We report them here because it clarifies the relation between the above two examples:

Theorem 1.5. (1) [Lurie 2011b, Proposition 4.3.15] *The natural inclusion*

$$\mathcal{T}_{\acute{e}t}(k) \to \mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k)$$

exhibits the latter as a geometric envelope of $\mathcal{T}_{\acute{e}t}(k)$.

(2) [Lurie 2011b, Remark 4.3.14 and Corollary 4.3.16] *The truncation functor* $\pi_0 : \mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k) \to \mathcal{G}_{\acute{e}t}(k)$ *exhibits the latter as a* 0*-stub for* $\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k)$*. In particular, the composition* $\mathcal{T}_{\acute{e}t}(k) \to \mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k) \to \mathcal{G}_{\acute{e}t}(k)$ *exhibits* $\mathcal{G}_{\acute{e}t}(k)$ *as a* 0*-truncated geometric envelope of* $\mathcal{T}_{\acute{e}t}(k)$.

(3) [Lurie 2011b, Proposition 4.3.21] *The category of* 1*-localic* $\mathcal{G}_{\acute{e}t}(k)$*-schemes is equivalent to the category of* $\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k)$*-schemes which are* 1*-localic and* 0*-truncated.*

Remark 1.6. The derived Deligne–Mumford stacks of Example 1.4 are locally *connective*. There is a nonconnective variation of such objects, known as *spectral Deligne–Mumford stacks*. This plays a major role in a certain branch of algebraic topology known as *chromatic homotopy theory*. As we are not be concerned with such objects in this note, we invite the interested reader to consult [Lurie 2011c, §2, §8]. Then [Lurie 2011c, Corollary 9.28] completes the task of comparing the category of spectral Deligne–Mumford stacks with the one of Example 1.4. We would like to draw the attention of the reader to the fact that characteristic 0 is needed to have such a comparison. This is a complication that comes from the interaction with power operations in algebraic topology. In this note, no hypothesis on the characteristic is required.

The main theorem. Finally, we are ready to discuss the main comparison result. In order to avoid confusion, we will refer from this moment on to derived Deligne–Mumford stacks as the geometric stacks for the HAG context $(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathbf{P}_{\text{ét}})$ we discussed in Section 1, and to $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-schemes to the derived Deligne–Mumford stacks in the sense of [Lurie 2011b] we introduced in Example 1.4.

Taking inspiration from the comparison discussed in Example 1.3, we introduce the full subcategory $\mathrm{Sch}_{\leq n}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ of $\mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ spanned by $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ whose underlying ∞-topos $\mathcal{X}$ is n-localic. We further let $\mathrm{Sch}_{\mathrm{loc}}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ be the union of the ∞-categories $\mathrm{Sch}_{\leq n}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ as n varies. The comparison result can therefore be stated as follows:

Theorem 1.7. *There exists an equivalence of ∞-categories*

$$\Phi : \mathrm{Sch}_{\mathrm{loc}}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)) \rightleftarrows \mathbf{DM} : \Psi.$$

Moreover, for every $n \geq 1$, this restricts to an equivalence

$$\mathrm{Sch}_{\leq n}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)) \simeq \mathbf{DM}_n.$$

The next section is entirely devoted to the proof of this theorem.

Remark 1.8. The statement Theorem 1.7 is very similar to the one of [Porta 2015, Theorem 3.7]. However, the proof of Theorem 1.7 is somehow subtler. One of the key points is that if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived $\mathbb{C}$-analytic space (cf., [Lurie 2011a, Definition 12.3] or [Porta 2015, Definition 1.3]), then the ∞-topos $\mathcal{X}$ is always hypercomplete (see [Porta 2015, Lemma 3.2]). This is false in the algebraic setting, and the reason is that if $A \in \mathrm{CAlg}_k$, then usually $\mathcal{X}_A := \mathrm{Sh}(A_{\text{ét}})$ itself is not hypercomplete. As a consequence, there is no direct analogue in this setting of [Porta 2015, Corollary 3.4]: one needs to restrict oneself to the case of localic $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-schemes to prove the corresponding statement (see Proposition 2.3).

Another important point that marks the difference is that if $A \in \mathrm{CAlg}_k$ then $\mathcal{X}_A$ is 1-localic instead of 0-localic. Therefore the case of algebraic spaces needs to be dealt with separately and it cannot be uniformly included in an induction proof. This is done in Section 2.

Remark 1.9. Theorem 1.7 actually implies that the two ∞-categories of derived *Artin* stacks considered in [Toën and Vezzosi 2008] and in the DAG series are equivalent. Indeed, it is not possible to deal with Artin stacks from the point of view of structured topoi. Therefore, even in the DAG series and in J. Lurie's Ph.D. thesis [2004], derived Artin stacks are defined as geometric stacks with respect to the context of affine derived Deligne–Mumford stacks in $\mathbf{DM}_n$.

2. The proof of the comparison result

We start with the construction of the two functors Φ and Ψ. [Lurie 2011b, Theorem 2.4.1] provides us with a fully faithful embedding

$$\phi : \mathrm{Sch}(\mathcal{G}_{\text{ét}}^{\text{der}}(k)) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{G}_{\text{ét}}^{\text{der}}(k)^{\text{op}}), \mathcal{S}) = \mathrm{Fun}(\mathrm{dAff}^{\text{op}}, \mathcal{S}),$$

Unraveling the definition of ϕ, we see that for $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{Sch}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))$, the functor $\phi(X)$

$$\phi(X) : \mathrm{CAlg}_k \to \mathcal{S}$$

is defined informally by

$$\phi(X)(A) = \mathrm{Map}_{\mathrm{Sch}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))}(\mathrm{Spec}^{\text{ét}}(A), X).$$

It follows from [Lurie 2011b, Lemma 2.4.13] that this functor factors through $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})$.

To obtain the functor Φ of Theorem 1.7, we are left to show that the restriction of ϕ to $\mathrm{Sch}_{\text{loc}}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))$ factors through **DM**. Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{Sch}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))$. More specifically, the proof of Theorem 1.7 breaks into the following independent steps:

(1) Let $n \geq 1$. If the underlying ∞-topos of X is n-localic, then $\phi(X)$ is hypercomplete.

(2) Let $n \geq 1$. If the underlying ∞-topos of X is n-localic, then $\phi(X)$ is geometric and n-truncated.

(3) The previous two points imply that ϕ factors through a fully faithful functor $\Phi : \mathrm{Sch}(\mathcal{G}_{\text{ét}}^{\text{der}}(k)) \to \mathbf{DM}$. Therefore, to complete the proof, it will be sufficient to show that every object in **DM** arises is of the form $\phi(X)$ for $X \in \mathrm{Sch}_{\text{loc}}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))$.

We deal with the first point in Section 2. In Section 2 we discuss the special case of derived algebraic spaces, which is then used as base for the proof by induction of the second point given in Section 2. Finally, we treat the third point in Section 2, where the proof of Theorem 1.7 will be achieved.

Hypercompleteness. Let us begin with a couple of preliminary lemmas.

Lemma 2.1. *Let $f : B \to A$ be a morphism in* CAlg_k *between finitely presented objects. The following conditions are equivalent*:

(1) *f is étale.*

(2) *The morphism* $\mathrm{Spec}^{\text{ét}}(A) \to \mathrm{Spec}^{\text{ét}}(B)$ *is étale in the sense of* [Lurie 2011b, Definition 2.3.1].

Proof. A proof of this lemma can be formally deduced from [Lurie 2011d, Theorem 1.2.1]. We will propose here a shorter proof that works fine in the connective situation. The implication (1) $\Rightarrow$ (2) is [Lurie 2011b, Example 2.3.8]. Let us show that (2) $\Rightarrow$ (1). Since both A and B are finitely presented, we see that $\pi_0(A) \to \pi_0(B)$ is finitely presented. If we show that $\mathbb{L}_{A/B} \simeq 0$, we will obtain that $B \to A$ is finitely presented (in virtue of [Lurie 2011a, Proposition 8.8][1]

Let

$$f^{-1} : \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}}) \to \mathrm{Sh}(B_{\text{ét}}, \tau_{\text{ét}})$$

be the inverse image functor. Consider the sheaf $\mathbb{L}_{\mathcal{O}_A/f^{-1}\mathcal{O}_B}$ on $A_{\text{ét}}$ defined by

$$C \mapsto \mathbb{L}_{\mathcal{O}_A(C)/f^{-1}\mathcal{O}_B(C)} = \mathbb{L}_{C/f^{-1}\mathcal{O}_B(C)}.$$

Since the morphism of $\mathcal{T}_{\text{ét}}(k)$-structured topoi $\mathrm{Spec}^{\text{ét}}(A) \to \mathrm{Spec}^{\text{ét}}(B)$ is étale in the sense of [Lurie 2011b, Definition 4.3.1], we see that $f^{-1}\mathcal{O}_B \simeq \mathcal{O}_A$. Therefore this sheaf is identically zero.

On the other side, if $\eta^{-1} : \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}}) \to \mathcal{S}$ is a geometric point, then

$$\eta^{-1}(\mathbb{L}_{\mathcal{O}_A/f^{-1}\mathcal{O}_B}) \simeq \mathbb{L}_{\eta^{-1}\mathcal{O}_A/\eta^{-1}f^{-1}\mathcal{O}_B}.$$

We can identify $\eta^{-1}f^{-1}\mathcal{O}_B$ with a strictly henselian B-algebra B'. Since the map $B \to B'$ is formally étale, we conclude that

$$\mathbb{L}_{\eta^{-1}\mathcal{O}_A/\eta^{-1}f^{-1}\mathcal{O}_B} \simeq \mathbb{L}_{\eta^{-1}\mathcal{O}_A/B}.$$

This is also the stalk of the sheaf on $A_{\text{ét}}$ defined by

$$C \mapsto \mathbb{L}_{C/B}.$$

Therefore, this sheaf vanishes as well. In particular, $\mathbb{L}_{A/B} \simeq 0$, completing the proof. □

Let us recall the following result from [Lurie 2011d]:

Lemma 2.2. *Let $\mathcal{T}\mathrm{op}_{\leq n}$ be the full subcategory of $\mathcal{R}\mathcal{T}\mathrm{op}$ spanned by n-localic ∞-topoi. Then $\mathcal{T}\mathrm{op}_{\leq n}$ is categorically equivalent to an $(n+1)$-category.*

Proof. This is a direct consequence of [Lurie 2011d, Lemma 1.3.5] and of [Lurie 2009, 2.3.4.18]. □

Proposition 2.3. *Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-scheme and suppose that $\mathcal{X}$ is n-localic, with $n \geq 1$. Then the functor $\phi(X) : \mathcal{C} \to \mathcal{S}$ is a hypercomplete sheaf.*

[1] We warn the reader that there is a small mistake in [Lurie 2011a, Example 8.4], when considering morphisms of finite presentation to order 0. Namely, it is not true that a discrete A-algebra B is finitely generated if the canonical map $\operatorname{colim} \mathrm{Hom}_A(B, C_\alpha) \to \mathrm{Hom}_A(B, \operatorname{colim} C_\alpha)$ is injective for every filtered diagram $\{C_\alpha\}$ of A-algebras, the easiest counterexample being $A = \mathbb{Z}$ and $B = \mathbb{Q}$. However, the converse is true, and this is precisely what is used afterwards. Therefore the subsequent results are not affected by this. This issue has been fixed in [Lurie 2016b].

Proof. Let $U^\bullet \to U$ be an étale hypercover in the category dAff_k. Let $\mathcal{T}\mathrm{op}_{\le n}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))$ be the ∞-category of $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-structured ∞-topoi which are m-localic for some $m \le n$. We claim that the geometric realization of the simplicial object $\mathrm{Spec}^{\text{ét}}(U^\bullet)$ is $\mathcal{T}\mathrm{op}_{\le n}(\mathcal{G}_{\text{ét}}^{\text{der}}(k))$ is precisely $\mathrm{Spec}^{\text{ét}}(U)$. The claim directly implies the lemma, since

$$\begin{aligned}\phi(X)(\mathrm{Spec}^{\text{ét}}(U)) &= \mathrm{Map}_{\mathrm{Sch}(\mathcal{T}_{\text{ét}}(k))}(\mathrm{Spec}^{\text{ét}}(U), X)\\ &= \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\le n}(\mathcal{T}_{\text{ét}}(k))}(\mathrm{Spec}^{\text{ét}}(U), X)\\ &= \lim \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\le n}(\mathcal{T}_{\text{ét}}(k))}(\mathrm{Spec}^{\text{ét}}(U^\bullet), X)\\ &= \lim \phi(X)(\mathrm{Spec}^{\text{ét}}(U^\bullet)).\end{aligned}$$

We are therefore reduced to proving the claim. Let us denote by $\mathcal{X}_U$ the topos of (nonhypercomplete) sheaves on the small étale site of U. It follows from Lemma 2.1 that each face map

$$\mathrm{Spec}^{\text{ét}}(U^n) \to \mathrm{Spec}^{\text{ét}}(U^{n-1})$$

is étale. Thus, we can find objects $V^n \in \mathcal{X}_U$ and identifications $\mathcal{X}_{U^n} \simeq (\mathcal{X}_U)_{/V^n}$. The universal property of étale subtopoi (see [Lurie 2009, 6.3.5.6]), shows that we can arrange the V^n into a simplicial object in $\mathcal{X}_U$. Using statement (3′) in the proof of [Lurie 2011b, Proposition 2.3.5], we are reduced to prove that in $\mathcal{T}\mathrm{op}_{\le n}$ one has an equivalence

$$\mathcal{X}_U \simeq \operatorname{colim} \mathcal{X}_{U^\bullet}.$$

Since $\mathcal{T}\mathrm{op}_{\le n}$ is an n-category in virtue of Lemma 2.2, Proposition A.1 shows that a presheaf with values in $\mathcal{T}\mathrm{op}_{\le n}$ has descent if and only if it has hyperdescent. We are therefore reduced to the case where $U^\bullet$ is the Čech nerve of the map $U^0 \to U$. In this case, the general descent theory for ∞-topoi (see [Lurie 2009, 6.1.3.9]) allows us to conclude. □

The case of algebraic spaces. Let $A \in \mathrm{CAlg}_k$. We denote by $A_{\text{big, ét}}$ the big étale site of A: that is, its underlying ∞-category is the opposite of $(\mathrm{CAlg}_k)_{A/}$, and the Grothendieck topology is the (derived) étale one. There are continuous and cocontinuous morphisms of ∞-sites

$$(A_{\text{ét}}, \tau_{\text{ét}}) \xrightarrow{\ u\ } (A_{\text{big,ét}}, \tau_{\text{ét}}) \xrightarrow{\ v\ } (\mathrm{dAff}_k, \tau_{\text{ét}}) \ .$$

Note that u commutes with finite limits. It follows from [Porta and Yu 2016, Lemma 2.14] that the induced adjunction

$$u_s : \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}}) \rightleftarrows \mathrm{Sh}(A_{\text{big, ét}}, \tau_{\text{ét}}) : u^s$$

is a geometric morphism of ∞-topoi, in other words, u_s commutes with finite limits. Here u^s denotes the restriction functor along u and u_s is obtained via the left Kan extension along u. We refer the reader to [Porta and Yu 2016, §2.4] for a more detailed discussion of the chosen notations. In particular, we can use [Lurie 2009, 5.5.6.16] to conclude that u_s takes n-truncated objects to n-truncated objects.

This is not true for v, because it commutes only with weakly contractible limits. However, we still have an adjunction

$$v_s : \mathrm{Sh}(A_{\mathrm{big,\, ét}}, \tau_{\mathrm{ét}}) \rightleftarrows \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{ét}}) : v^s,$$

which can be identified with the canonical adjunction

$$v_s : \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{ét}})_{/\,\mathrm{Spec}(A)} \rightleftarrows \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{ét}}) : v^s,$$

where $\mathrm{Spec}(A)$ denotes the functor of points associated to A, accordingly to the notation introduced at the end of Section 1.

Definition 2.4. Let k be a commutative ring, A a commutative k-algebra and $X \in \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{ét}})$ any sheaf equipped with a natural transformation $\alpha : X \to \mathrm{Spec}(A)$. We will say that α *exhibits X as an étale derived algebraic space over* $\mathrm{Spec}(A)$ if there exists a 0-truncated sheaf $F \in \mathrm{Sh}(A_{\mathrm{ét}}, \tau_{\mathrm{ét}})$ and an equivalence $X \simeq v_s(u_s(F))$ in $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{ét}})_{/\,\mathrm{Spec}(A)}$.

Remark 2.5. The above definition is the analogue of [Lurie 2011b, Definition 2.6.4] in the derived setting. Indeed, let us replace the ∞-category CAlg_k with the 1-category $\mathrm{CAlg}_k^{\heartsuit}$. Keeping the same notations as above, we see that if $G \in \mathrm{Sh}(A_{\mathrm{big,\, ét}}, \tau_{\mathrm{ét}})$ then

$$v_s(G) = \coprod_{\phi : A \to B} G(\phi).$$

If moreover F is an object in $\mathrm{Sh}(A_{\mathrm{ét}}, \tau_{\mathrm{ét}})$, then $(u_s F)(\phi) = \phi^{-1}(F)(B)$. In conclusion, we have

$$v_s(u_s(F))(B) = \{(\phi, \eta) \mid \phi \in \mathrm{Hom}_k(A, B),\ \eta \in (\phi^{-1}F)(B)\}.$$

This coincides precisely with the definition of $\widehat{F}$ given in [Lurie 2011b, Notation 2.6.2]. A similar description holds true in the derived setting. Indeed, there is a natural transformation $v_s(u_s(F)) \to \mathrm{Spec}(A)$. The fiber over a given map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ coincides precisely with the global sections of the discrete object $f^{-1}(F)$.

The following proposition is the analogue of [Lurie 2011b, 2.6.20]. The proof is essentially unchanged.

Proposition 2.6. *Let $\alpha : Y \to \mathrm{Spec}(A)$ be a natural transformation of stacks. Write $\mathrm{Spec}^{ét}(A) = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The following conditions are equivalent*:

(1) *α exhibits Y as an étale derived algebraic space over* $\mathrm{Spec}(A)$.

(2) *Y is representable by a $\mathcal{G}^{\mathrm{der}}_{\mathrm{\acute{e}t}}(k)$-scheme $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and α induces an equivalence $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \simeq (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ for some discrete object $U \in \mathcal{X}$.*

(3) *The morphism α is 0-truncated and 0-representable by étale maps.*

Proof. We first prove the equivalence of (1) and (2). If α exhibits Y as an étale derived algebraic space over $\mathrm{Spec}(A)$, we can find a 0-truncated sheaf $U \in \mathrm{Sh}(A_{\mathrm{\acute{e}t}}, \tau_{\mathrm{\acute{e}t}})$ and an equivalence $Y \simeq v_s(u_s(U))$ in $\mathrm{Sh}(\mathrm{dAff}, \tau_{\mathrm{\acute{e}t}})_{/\,\mathrm{Spec}(A)}$. Now, [Lurie 2011b, Remark 2.3.4] and Remark 2.5 show together that the functor represented by $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ coincides with Y. Conversely, if (2) is satisfied, then U defines an étale derived algebraic space $v_s(u_s(U))$ over $\mathrm{Spec}(A)$, and [Lurie 2011b, Remark 2.3.4] again allows us to identify it with Y.

Let us now prove the equivalence of (1) and (3) First, assume that (3) is satisfied. In this case, we can define a sheaf $U : A_{\mathrm{\acute{e}t}} \to \mathcal{S}$ by sending an étale map $f : A \to B$ to the fiber product

$$\begin{tikzcd} U(B) \arrow[r] \arrow[d] & Y(B) \arrow[d, "\alpha_B"] \\ \{*\} \arrow[r, "f"] & \mathrm{Map}(A, B). \end{tikzcd}$$

Since α is 0-truncated, we see that U takes values in Set. Since it is obviously a sheaf, it defines a 0-truncated object in $\mathrm{Sh}(A_{\mathrm{\acute{e}t}}, \tau_{\mathrm{\acute{e}t}})$. [Lurie 2011b, Remark 2.3.4] shows that $v_s(u_s(U))$ can be canonically identified with Y.

Finally, let us show that (1) implies (3). We already know that, in this situation, α is 0-truncated. Choosing sections $\eta_\alpha \in Y(A_\alpha)$ which generate Y, we obtain an effective epimorphism

$$\coprod \mathrm{Spec}(A_\alpha) \to v_s(u_s(Y))$$

in $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{\acute{e}t}})$. Suppose that there exists a (-1)-truncated morphism

$$v_s(u_s(Y)) \to \mathrm{Spec}(B)$$

for some $B \in \mathrm{CAlg}_k$. In this case, we see that

$$\mathrm{Spec}(A_\alpha) \times_{v_s(u_s(Y))} \mathrm{Spec}(A_\beta) \simeq \mathrm{Spec}(A_\alpha) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A_\beta) \simeq \mathrm{Spec}(A_\alpha \otimes_B A_\beta).$$

In the general case, each fiber product $Y_{\alpha,\beta} := \mathrm{Spec}(A_\alpha) \times_{v_s(u_s(Y))} \mathrm{Spec}(A_\beta)$ is again a derived algebraic space étale over A. We claim moreover that the canonical morphism $Y_{\alpha,\beta} \to \mathrm{Spec}(A_\alpha \otimes_A A_\beta)$ is (-1)-truncated. Assuming the claim, it follows that $Y_{\alpha,\beta} \to \mathrm{Spec}(A)$ is (-1)-representable by étale maps, hence it would follow that the morphism $\mathrm{Spec}(A_\alpha) \to v_s(u_s(Y))$ is 0-representable. Finally, we see that it is representable by étale maps combining the equivalence between (1) and (2) with Lemma 2.1.

We are left to prove the claim. Fix $f_\alpha : A_\alpha \to B$, $f_\beta : A_\beta \to B$ together with a homotopy making the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_\alpha \arrow[d, "f_\alpha"] \\ A_\beta \arrow[r, "f_\beta"] & B \end{tikzcd}$$

commutative. We have pullback squares

$$\begin{tikzcd} Y_{\alpha,\beta} \arrow[r] \arrow[d] & v_s(u_s(Y)) \arrow[d] \\ \mathrm{Spec}(A_\alpha) \times \mathrm{Spec}(A_\beta) \arrow[r] & v_s(u_s(Y)) \times_{\mathrm{Spec}(A)} v_s(u_s(Y)), \end{tikzcd}$$

and since $\alpha : v_s(u_s(Y)) \to \mathrm{Spec}(A)$ is 0-truncated, the statement follows. $\square$

$\phi(X)$ is geometric. We can now prove that if $X \in \mathrm{Sch}_{\leq n+1}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$, then $\phi(X)$ belongs to $\mathbf{DM}_n$. The proof goes by induction, and Proposition 2.6 serves as basis of the induction. Before doing that, however, it is convenient to prove the following lemma:

Lemma 2.7. *Let $n \geq 0$ be an integer. Fix $X = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \in \mathrm{Sch}_{\leq n+1}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ and let $V \in \mathcal{X}$ be an object such that $(\mathcal{X}_{/V}, \mathcal{O}_\mathcal{X}|_V) \simeq \mathrm{Spec}^{\text{ét}}(A)$ for some $A \in \mathrm{CAlg}_k$. Then V is n-truncated.*

Proof. We start by replacing X with $\mathrm{t}_0(X) := (\mathcal{X}, \pi_0\mathcal{O}_X)$, which is a $\mathcal{G}_{\text{ét}}(k)$-scheme in virtue of [Lurie 2011b, Corollary 4.3.30]. We can therefore replace A by $\pi_0(A)$ (observe also that $\mathrm{Spec}^{\text{ét}}(\pi_0(A)) \simeq \mathrm{Spec}^{\mathcal{G}_{\text{ét}}(k)}(\pi_0(A))$).

Let us denote by $F_X : \mathrm{CAlg}_k^\heartsuit \to \mathcal{S}$ the (truncated) functor of points associated to X. Similarly, let $F_V : \mathrm{CAlg}_k^\heartsuit \to \mathcal{S}$ be the functor of points associated to $(\mathcal{X}_{/V}, \mathcal{O}_\mathcal{X}|_V)$. The hypothesis shows that F_V is nothing but the functor of points associated to $\pi_0(A)$ (with the notations of [Toën and Vezzosi 2008], this would be $\mathrm{t}_0(\mathrm{Spec}(\pi_0(A)))$). Reasoning as in the proof of [Lurie 2011b, Theorem 2.6.18], we see that to prove that V is n-truncated is equivalent to prove that for every (discrete) k-algebra B the fibers of $F_V(B) \to F_X(B)$ are n-truncated. [Lurie 2011b, Lemma 2.6.19] shows that $F(B)$ is $(n+1)$-truncated for every k-algebra B. On the other side, $F_V(B)$ is discrete by hypothesis. It follows from the long exact sequence of homotopy groups that the fibers of $F_V(B) \to F_X(B)$ are n-truncated, thus completing the proof. $\square$

Proposition 2.8. *Let $X = (\mathcal{X}, \mathcal{O}_X) \in \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ and suppose that $\mathcal{X}$ is n-localic for $n \geq 1$. Then the stack $\phi(X)$ is $(n+1)$-geometric and moreover its truncation $\mathrm{t}_0(\phi(X))$ is n-truncated.*

Proof. The fact that $\mathrm{t}_0(\phi(X))$ is n-truncated follows directly from [Lurie 2011b, Lemma 2.6.19].

Suppose now that $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an n-localic $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-scheme. By definition, we can find a collection of objects $V_i \in \mathcal{X}$ such that

(1) the morphism $\coprod V_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism, and

(2) the $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-schemes $(\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ are equivalent to $\mathrm{Spec}^{\text{ét}}(U_i)$ for $U_i \in \mathrm{dAff}_k$, and each U_i is of finite presentation.

Set $V := \coprod V_i$. By functoriality, we obtain a map

$$\coprod \phi(V_i) \to \phi(X).$$

We only need to show that this map is $(n-1)$-representable by étale morphisms and that it is an effective epimorphism. The second statement is an immediate consequence of [Lurie 2011b, Lemma 2.4.13].

Suppose first that $X \simeq \mathrm{Spec}^{\text{ét}}(A)$. In this case, the universal property of $\mathrm{Spec}^{\text{ét}}$ proved in [Lurie 2011b, §2.2] shows that $\phi(X) = \mathrm{Spec}(A)$, and therefore $\phi(X)$ is (-1)-geometric. Now suppose that X is a general n-localic $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-scheme. Since ϕ commutes with fiber products and is fully faithful, we see that for every map $\mathrm{Spec}(B) = \phi(\mathrm{Spec}^{\text{ét}}(B)) \to X$, one has

$$\mathrm{Spec}(B) \times_{\phi(X)} \phi(V_i) \simeq \phi(\mathrm{Spec}^{\text{ét}}(B) \times_{(\mathcal{X}, \mathcal{O}_X)} (\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})).$$

Let $(f_*, \varphi) : \mathrm{Spec}^{\text{ét}}(B) \to (\mathcal{X}, \mathcal{O}_X)$ be the given map. Then the fiber product $\mathrm{Spec}^{\text{ét}}(B) \times_{(\mathcal{X}, \mathcal{O}_X)} (\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ is the étale map to $\mathrm{Spec}^{\text{ét}}(B)$ classified by the object $f^{-1}(V_i) \in \mathcal{X}_A$, as it easily follows from [Lurie 2009, 6.3.5.8].

We complete the proof proving by induction on n that each morphism

$$\phi(\mathcal{X}_{/V_i}, \mathcal{O}_{\mathcal{X}}|_{V_i}) \to \phi(X)$$

is $(n-1)$-representable by étale maps. If $n = 1$, Lemma 2.7 shows that each object V_i is 0-truncated. It follows from Proposition 2.6 that the fiber product $\mathrm{Spec}(A) \times_{\phi(X)} \phi(V_i)$ is 1-geometric. Therefore, $\phi(X)$ is 2-geometric. Now suppose that $\mathcal{X}$ is n-localic for $n > 1$. Lemma 2.7 again shows that each V_i is $(n-1)$-truncated, and therefore [Lurie 2011b, Lemma 2.3.16] shows that the underlying ∞-topos of

$$\mathrm{Spec}^{\text{ét}}(A) \times_{(\mathcal{X}, \mathcal{O}_X)} (\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$$

is $(n-1)$-localic. The inductive hypothesis shows therefore that its image via the functor ϕ is n-geometric, and that the map to $\mathrm{Spec}(A)$ is étale. The proof is therefore complete. □

Essential surjectivity. We finally prove that ϕ is essentially surjective. Let $X \in \mathbf{DM}$ be m-geometric and suppose that $\mathrm{t}_0(X)$ is n-truncated. It follows that the small étale site $(t_0(X))_{\text{ét}}$ is equivalent to an n-category. Recall that there is an equivalence of ∞-categories

$$X_{\text{ét}} \leftrightarrows (\mathrm{t}_0(X))_{\text{ét}}$$

(one can proceed as in [Porta 2015, Proposition 3.16] using as base of the induction [Toën and Vezzosi 2008, Corollary 2.2.2.10]). We conclude that $X_{\text{ét}}$ is an n-category. In particular, the ∞-topos $\mathcal{X} := \mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$ is n-localic. Define a $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-structure on $\mathcal{X}$ as follows. Introduce the functor

$$\mathcal{G}_{\text{ét}}^{\text{der}}(k) \times (X_{\text{ét}})^{\text{op}} \to \mathcal{S},$$

defined as

$$(U, V) \mapsto \mathrm{Map}_{\mathrm{dAff}_k}(V, U).$$

Fix $U \in \mathcal{G}_{\text{ét}}^{\text{der}}(k)$. Since the Grothendieck topology on dAff_k is hypersubcanonical, we see that the resulting object of $\mathrm{Fun}((X_{\text{ét}})^{\text{op}}, \mathcal{S})$ is a hypersheaf. In particular, we obtain a well defined functor

$$\mathcal{O}_X : \mathcal{T}_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$$

that in fact factors through the hypercompletion of this category. In order to show that it is a $\mathcal{T}_{\text{ét}}$-structure, we only need to check the following statements:

(1) $\mathcal{O}_X$ is left-exact.

(2) $\mathcal{O}_X$ takes $\tau_{\text{ét}}$-coverings to effective epimorphisms.

Since limits in $\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$ are computed objectwise, the first statement follows directly from the definition of $\mathcal{O}_X$. We are left to show that $\mathcal{O}_X$ takes $\tau_{\text{ét}}$-coverings to effective epimorphisms. Let $\{U_i \to U\}$ be a $\tau_{\text{ét}}$-cover in $\mathcal{T}_{\text{ét}}(k)$. We have to show that the morphism

$$\coprod \mathcal{O}_X(U_i) \to \mathcal{O}_X(U)$$

is an effective epimorphism. In other words, we have to show that

$$\coprod \pi_0\mathcal{O}_X(U_i) \to \pi_0\mathcal{O}_X(U) \tag{2-1}$$

is an epimorphism of sheaves of sets.

Fix $V \in X_{\text{ét}}$ and let $\alpha \in (\pi_0\mathcal{O}_X(U))(V)$. By definition of the sheaf $\pi_0\mathcal{O}_X(U)$, this is equivalent to the given of an étale cover $\{V_j \to V\}$ plus morphisms $V_j \to U$. For every pair of indexes i and j, let

$$V_{ij} := U_i \times_U V_j.$$

Then the collection of morphisms $\{V_{ij} \to V_j\}_i$ for j fixed is an étale cover of V_j. Furthermore, the composition $V_{ij} \to V_j \to U$ can be seen as an element in $\alpha_{ij} \in (\pi_0 \mathcal{O}_X(U))(V_j)$, while the canonical map $V_{ij} \to U_i$ defines an element in $\beta_{ij} \in (\pi_0 \mathcal{O}_X(U_i))(V_{ij})$. The construction shows that the image of β_{ij} via the canonical map

$$(\pi_0 \mathcal{O}_X(U_i))(V_{ij}) \to \pi_0 \mathcal{O}_X(U)(V_{ij})$$

coincides with α_{ij}. Since the collection of maps $\{V_{ij} \to V\}_i$ is an étale cover, we have precisely proven that (2-1) is an epimorphism of sheaves of sets.

We therefore conclude that $\mathcal{O}_X$ is a hypercomplete $\mathcal{T}_{\text{ét}}(k)$-structure on $\mathcal{X}$. Since $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$ is a geometric envelope for $\mathcal{T}_{\text{ét}}(k)$, we can identify $\mathcal{O}_X$ with a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-structure on $\mathcal{X}$. The next step is to prove that the pair $(\mathcal{X}, \mathcal{O}_X)$ is a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-scheme. To do this, we need the following criterion for a morphism of Grothendieck sites to induce an equivalence between the associated hypercomplete ∞-topoi. It is the ∞-categorical analogue of [de Jong et al. 2005–, Tag 039Z], and we refer to [Porta and Yu 2016, Proposition 2.22] for a proof.

Lemma 2.9. *Let $(\mathcal{C}, \tau)$, $(\mathcal{D}, \sigma)$ be two ∞-sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(i) *u is continuous*;

(ii) *u is cocontinuous*;

(iii) *u is fully faithful*;

(iv) *for every object $V \in \mathcal{D}$ there exists a σ-covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i \in I}$*;

(v) *for every object $D \in \mathcal{D}$, the representable presheaf h_D is a hypercomplete sheaf.*

Then the induced adjunction $\mathrm{Sh}(\mathcal{C}, \tau)^\wedge \simeq \mathrm{Sh}(\mathcal{D}, \sigma)^\wedge$ is an equivalence of ∞-categories.

Proposition 2.10. *The pair $(\mathcal{X}, \mathcal{O}_X)$ is a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-scheme.*

Proof. Choose an étale atlas $p : \coprod U_i \to X$ in the category **DM**. Since each morphism $p_i : U_i \to X$ is étale, we see each of them defines an element in the small étale site $(X_{\text{ét}}, \tau_{\text{ét}})$. Since this site is subcanonical, we can identify each U_i with objects $V_i \in \mathcal{X}$. Moreover, the étale subtopos $(\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ is canonically identified with $(\mathrm{Sh}((U_i)_{\text{ét}}, \tau_{\text{ét}}), \mathcal{O}_{U_i})$. The construction of the (absolute) spectrum functor of [Lurie 2011b, §2.2], shows that

$$\mathrm{Spec}^{\text{ét}}(U_i) \simeq (\mathrm{Sh}((U_i)_{\text{ét}}, \tau_{\text{ét}}), \mathcal{O}_{U_i}).$$

It will therefore be sufficient to show that the morphism $\coprod V_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism. In order to do this, it is convenient to replace the small étale site $X_{\text{ét}}$

with the site $((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})$ of étale maps $Y \to X$, where Y is a geometric stack such that $\mathrm{t}_0(Y)$ is n-truncated. We claim that the natural inclusion

$$(X_{\text{ét}}, \tau_{\text{ét}}) \to ((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}}) \tag{2-2}$$

is a Morita equivalence of sites. In other words, we claim that it induces an equivalence of ∞-topoi

$$\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}}) \simeq \mathrm{Sh}((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}}).$$

Lemma 2.9 implies that the morphism (2-2) induces an equivalence of hypercomplete ∞-topoi:

$$\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})^{\wedge} \simeq \mathrm{Sh}((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})^{\wedge}. \tag{2-3}$$

Observe now that the mapping spaces in $(\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}$ are n-truncated, hence [Lurie 2009, 2.3.4.18] implies $(\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}$ is (categorically equivalent to) an n-category. Therefore, the ∞-topos $\mathrm{Sh}((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})$ is n-localic. The same statement holds for $\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$, as we already discussed. Therefore, in order to check that the induced adjunction is an equivalence of ∞-categories, it is enough to check that the restriction to n-truncated object is an equivalence. This follows from equivalence (2-3), since we know from [Lurie 2009, 6.5.2.9] that n-truncated objects are hypercomplete.

In this way, we see that $\mathbf{1}_{\mathcal{X}}$ is the representable sheaf associated to the identity map $\mathrm{id}_X : X \to X$. We are therefore left to show that

$$\coprod \pi_0 \,\mathrm{Map}(-, U_i) \to \pi_0 \,\mathrm{Map}(-, X)$$

is an epimorphism of sheaves on $((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})$. This follows immediately from the fact that the maps $U_i \to X$ were an atlas for X. □

We are left to prove that $\phi(\mathcal{X}, \mathcal{O}_X) \simeq X$. We can proceed by induction on the geometric level m of X. If $m = -1$, the statement is obvious. Otherwise, let $U_i \to X$ be an étale atlas for X. Let $U := \coprod U_i$ and let $U^\bullet$ be the Čech nerve of $U \to X$. Combining the proof of Proposition 2.10, Proposition 2.8 and the induction hypothesis, we see that $U^\bullet$ is a groupoid presentation for both X and $\phi(\mathcal{X}, \mathcal{O}_X)$. We therefore proved that the essential image of the functor

$$\phi : \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)) \to \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})$$

contains all the Deligne–Mumford stacks in the sense of [Toën and Vezzosi 2008].

Appendix: Descent versus hyperdescent

Let $(\mathcal{C}, \tau)$ be an ∞-Grothendieck site. It is well known that for a presheaf on $\mathcal{C}$ with values in a truncated ∞-category, descent and hyperdescent are equivalent conditions. However, we could not locate a precise reference in the literature. For this reason, we decided to include a proof of this fact:

Proposition A.1. *Let* $(\mathcal{C}, \tau)$ *be an* ∞*-Grothendieck site and let* $\mathcal{D}$ *be an* $(n+1, 1)$*-category. Then A functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ *satisfies descent if and only if it satisfies hyperdescent.*

Proof. Let $D \in \mathcal{D}$ be any object and let $c_D : \mathcal{D} \to \mathcal{S}$ be the functor *corepresented* by D. Then F satisfies descent (resp. hyperdescent) if and only if $c_D \circ F$ does. Since $\mathcal{D}$ is an $(n+1, 1)$-category, we see that $c_D \circ F$ takes values in $\tau_{\leq n}\mathcal{S}$. Therefore, we may replace $\mathcal{D}$ with $\mathcal{S}$ and suppose that F takes values in the full subcategory of n-truncated objects. For every $U \in \mathcal{C}$, let us denote by h_U the sheafification of the presheaf associated to U. Since F is an n-truncated object, we see that

$$\mathrm{Map}_{\mathrm{Sh}_{\leq n}(\mathcal{C},\tau)}(\tau_{\leq n} h_U, F) \simeq \mathrm{Map}_{\mathrm{Sh}(\mathcal{C},\tau)}(h_U, F) \simeq F(U),$$

where the last equivalence is obtained combining the universal property of the sheafification with the Yoneda lemma. Therefore, it will be sufficient to show that for every hypercover $U^\bullet \to U$ in $\mathcal{C}$, the augmented simplicial diagram

$$\tau_{\leq n} h_{U^\bullet} \to \tau_{\leq n} h_U$$

is a colimit diagram in $\mathrm{Sh}_{\leq n}(\mathcal{C}, \tau)$. Since $\tau_{\leq n}$ is a left adjoint, we see that in $\mathrm{Sh}_{\leq n}(\mathcal{C}, \tau)$ the relation

$$|\tau_{\leq n} h_{U^\bullet}| \simeq \tau_{\leq n} |h_{U^\bullet}|$$

holds. Moreover, since $U^\bullet \to U$ is an hypercover, the morphism $|h_{U^\bullet}| \to h_U$ is ∞-connected in virtue of [Lurie 2009, 6.5.3.11]. Since $\tau_{\leq n}$ commutes with ∞-connected morphisms, we conclude that

$$\tau_{\leq n} |h_{U^\bullet}| \to \tau_{\leq n} h_U$$

is an ∞-connected morphism between n-truncated objects. Therefore it is an equivalence in $\mathrm{Sh}(\mathcal{C}, \tau)$. In conclusion, the morphism $|\tau_{\leq n} h_{U^\bullet}| \to \tau_{\leq n} h_U$ is an equivalence in $\mathrm{Sh}_{\leq n}(\mathcal{C}, \tau)$. The proof is now complete. □

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