Pacific Journal of Mathematics

Volume 287 No. 1 March 2017

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.1

OPERATOR IDEALS RELATED TO ABSOLUTELY SUMMING AND COHEN STRONGLY SUMMING OPERATORS

GERALDO BOTELHO, JAMILSON R. CAMPOS AND JOEDSON SANTOS

We study the ideals of linear operators between Banach spaces determined by the transformation of vector-valued sequences involving the new sequence space introduced by Karn and Sinha and the classical spaces of absolutely, weakly and Cohen strongly summable sequences. As applications, we prove a new factorization theorem for absolutely summing operators and a contribution to the existence of infinite-dimensional spaces formed by nonabsolutely summing operators is given.

Introduction and background

In the theory of ideals of linear operators between Banach spaces (operator ideals), a central role is played by classes of operators that improve the convergence of series, which are usually defined or characterized by the transformation of vector-valued sequences. The most famous of such classes is the ideal of absolutely p-summing linear operators, which are the ones that send weakly p-summable sequences to absolutely p-summable sequences. The celebrated monograph [Diestel et al. 1995] is devoted to the study of absolutely summing operators.

For a Banach space E, let $\ell_p(E)$, $\ell_p^w(E)$ and $\ell_p\langle E\rangle$ denote the spaces of absolutely, weakly and Cohen strongly p-summable E-valued sequences, respectively. Karn and Sinha [2014] recently introduced a space $\ell_p^{\mathrm{mid}}(E)$ of E-valued sequences such that

$$\ell_p\langle E\rangle \subseteq \ell_p(E) \subseteq \ell_p^{\mathrm{mid}}(E) \subseteq \ell_p^w(E). \tag{1}$$

In the realm of the theory of operator ideals, it is a natural step to study the classes of operators $T : E \to F$ that send: (i) sequences in $\ell_p^w(E)$ to sequences in $\ell_p^{\mathrm{mid}}(F)$, (ii) sequences in $\ell_p^{\mathrm{mid}}(E)$ to sequences in $\ell_p(F)$, (iii) sequences in $\ell_p^{\mathrm{mid}}(E)$ to sequences in $\ell_p\langle F\rangle$. This is the basic motivation of this paper.

Botelho was supported by CNPq Grant 305958/2014-3 and Fapemig Grant PPM-00490-15. Campos was supported by a CAPES Postdoctoral scholarship. Santos was supported by CNPq (Edital Universal 14/2012).

MSC2010: primary 46B45; secondary 47B10, 47L20.

Keywords: Banach sequence spaces, operator ideals, summing operators.

We start by taking a closer look at the space $\ell_p^{\rm mid}(E)$ in Section 1. First we give it a norm that makes it a Banach space. Next we consider the relationship with the space $\ell_p^u(E)$ of unconditionally p-summable E-valued sequences. We show that, although (1) and $\ell_p(E) \subseteq \ell_p^u(E) \subseteq \ell_p^w(E)$ hold for every E, in general $\ell_p^{\rm mid}(E)$ and $\ell_p^u(E)$ are not comparable. It is also proved that the correspondence $E \mapsto \ell_p^{\rm mid}(E)$ enjoys a couple of desired properties in the context of operator ideals.

In Section 2 we prove that the classes of operators described in (i), (ii) and (iii) above are Banach operator ideals. Characterizations of each class and their corresponding norms are given and properties of each ideal are proved. We establish a factorization theorem for absolutely summing operators and a question left open in [Karn and Sinha 2014] is settled. In both Sections 1 and 2 we study Banach spaces E for which $\ell_p(E) = \ell_p^{\rm mid}(E)$ or $\ell_p^{\rm mid}(E) = \ell_p^w(E)$.

In Section 3 we give an application to the existence of infinite-dimensional Banach spaces formed, up to the null operator, by nonabsolutely summing linear operators on nonsuperreflexive spaces.

Let us define the classical sequences spaces we shall work with:

- $\ell_p(E)$ = absolutely p-summable E-valued sequences with the usual norm $\|\cdot\|_p$.
- $\ell_p^w(E)$ = weakly p-summable E-valued sequences with the norm
$$\|(x_j)_{j=1}^\infty\|_{w,p} = \sup_{x^* \in B_{E^*}} \|(x^*(x_j))_{j=1}^\infty\|_p.$$
- $\ell_p^u(E) = \left\{(x_j)_{j=1}^\infty \in \ell_p^w(E) : \lim_k \|(x_j)_{j=k}^\infty\|_{w,p} = 0\right\}$ with the norm inherited from $\ell_p^w(E)$ (unconditionally p-summable sequences, see [Defant and Floret 1993, 8.2]).
- $\ell_p\langle E\rangle = \left\{(x_j)_{j=1}^\infty \in E^{\mathbb{N}} : \|(x_j)_{j=1}^\infty\|_{C,p} := \sup_{(x_j^*)_{j=1}^\infty \in B_{\ell_{p^*}^w(E^*)}} \|(x_j^*(x_j))_{j=1}^\infty\|_1 < \infty\right\}$, where $1/p + 1/p^* = 1$, (Cohen strongly p-summable sequences or strongly p-summable sequences, see, e.g., [Cohen 1973]).

The letters E, F shall denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The closed unit ball of E is denoted by B_E and its topological dual by E^*. The symbol $E \overset{1}{\hookrightarrow} F$ means that E is a linear subspace of F and $\|x\|_F \leq \|x\|_E$ for every $x \in E$. By $\mathcal{L}(E\,;\,F)$ we denote the Banach space of all continuous linear operators $T : E \to F$ endowed with the usual sup norm. By $\Pi_{p;q}$ we denote the ideal of absolutely $(p\,;q)$-summing linear operators [Defant and Floret 1993; Diestel et al. 1995]. If $p = q$ we simply write Π_p. The ideal of Cohen strongly p-summing linear operators [Campos 2013; Cohen 1973] shall be denoted by $\mathcal{D}_p$. We use the standard notation of the theory of operator ideals [Defant and Floret 1993; Pietsch 1980].

1. The space $l_p^{\mathrm{mid}}(E)$

In this section we give the space of sequences defined by Karn and Sinha [2014] a norm that makes it a Banach space and establish some useful properties of this space.

The vector-valued sequences introduced in [Karn and Sinha 2014] are called *operator p-summable sequences*. This term is quite inconvenient for our purposes, and considering the intermediate position of the space formed by such sequences between $\ell_p(E)$ and $\ell_p^w(E)$ (see (1)), we shall use the term *mid-p-summable sequences*. Instead of the original definition, we shall use a characterization proved in [Karn and Sinha 2014, Lemma 2.3 and Proposition 2.4]:

Definition 1.1. A sequence $(x_j)_{j=1}^\infty$ in a Banach space E is said to be *mid-p-summable*, $1 \le p < \infty$, if $((x_n^*(x_j))_{j=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p)$ whenever $(x_n^*)_{n=1}^\infty \in \ell_p^w(E^*)$. The space of all such sequences shall be denoted by $\ell_p^{\mathrm{mid}}(E)$.

Observe that $\ell_p(E) \subseteq \ell_p^{\mathrm{mid}}(E) \subseteq \ell_p^w(E)$. The following extreme cases will be important throughout the paper:

Theorem 1.2 [Karn and Sinha 2014, Proposition 3.1 and Theorem 4.5]. *Let E be a Banach space and $1 \le p < \infty$. Then*:

(i) $\ell_p^{\mathrm{mid}}(E) = \ell_p^w(E)$ *if and only if* $\Pi_p(E; \ell_p) = \mathcal{L}(E; \ell_p)$.

(ii) $\ell_p^{\mathrm{mid}}(E) = \ell_p(E)$ *if and only if E is a subspace of $L_p(\mu)$ for some Borel measure μ.*

We say that a Banach space E is a *weak mid-p-space* if $\ell_p^{\mathrm{mid}}(E) = \ell_p^w(E)$; and it is a *strong mid-p-space* if $\ell_p^{\mathrm{mid}}(E) = \ell_p(E)$.

The space $\ell_p^{\mathrm{mid}}(E)$ is not endowed with a norm in [Karn and Sinha 2014]. Our first goal in this section is to give it a useful complete norm. Let us see first that the norm inherited from $\ell_p^w(E)$ is unhelpful. We believe the next lemma is folklore; we give a short proof because we have found no reference to quote. As usual, $c_{00}(E)$ means the space of finite (or possibly null) E-valued sequences.

Lemma 1.3. *If E is infinite-dimensional, then the norms $\|\cdot\|_p$ and $\|\cdot\|_{w,p}$ are not equivalent on $c_{00}(E)$. In particular, $\ell_p(E)$ is not closed in $\ell_p^w(E)$.*

Proof. It is clear that $c_{00}(E)$ is dense in $(\ell_p(E), \|\cdot\|_p)$, and the definition of $\ell_p^u(E)$ makes clear that $c_{00}(E)$ is dense in $(\ell_p^u(E), \|\cdot\|_{w,p})$ as well. Assume that the norms $\|\cdot\|_p$ and $\|\cdot\|_{w,p}$ are equivalent on $c_{00}(E)$. Then

$$\ell_p(E) = \overline{c_{00}(E)}^{\|\cdot\|_p} = \overline{c_{00}(E)}^{\|\cdot\|_{w,p}} = \ell_p^u(E).$$

It follows that the identity operator in E is absolutely p-summing [Defant and Floret 1993, Proposition 11.1(c)], hence E is finite-dimensional. Now the second assertion follows from the open mapping theorem and the inclusion $c_{00}(E) \subseteq \ell_p(E)$. □

From Theorem 1.2(ii) we know that $\ell_p^{\rm mid}(\ell_p) = \ell_p(\ell_p)$, so $\ell_p^{\rm mid}(\ell_p)$ *is not* closed in $\ell_p^w(\ell_p)$ by Lemma 1.3, proving $\|\cdot\|_{w,p}$ does not make $\ell_p^{\rm mid}(E)$ complete in general.

Proposition 1.4. *The expression*

$$\|(x_j)_{j=1}^\infty\|_{{\rm mid},p} := \sup_{(x_n^*)_{n=1}^\infty \in B_{\ell_p^w(E^*)}} \left(\sum_{n=1}^\infty \sum_{j=1}^\infty |x_n^*(x_j)|^p \right)^{1/p}, \tag{2}$$

is a norm that makes $\ell_p^{\rm mid}(E)$ a Banach space and $\ell_p(E) \overset{1}{\hookrightarrow} \ell_p^{\rm mid}(E) \overset{1}{\hookrightarrow} \ell_p^w(E)$.

Proof. Let $x = (x_j)_{j=1}^\infty \in \ell_p^{\rm mid}(E)$. By definition, the double series in (2) is convergent (this is why we chose this condition to be the definition of $\ell_p^{\rm mid}(E)$). The map

$$T_x : \ell_p^w(E^*) \to \ell_p(\ell_p), \quad T_x((x_n^*)_{n=1}^\infty) = ((x_n^*(x_j))_{j=1}^\infty)_{n=1}^\infty,$$

is a well-defined linear operator. By the closed graph theorem, it is continuous. Therefore,

$$\left(\sum_{n=1}^\infty \sum_{j=1}^\infty |x_n^*(x_j)|^p \right)^{1/p} = \|T_x((x_n^*)_{n=1}^\infty)\| \le \|T_x\| \cdot \|(x_n^*)_{n=1}^\infty\|_{w,p}$$

for every $(x_n^*)_{n=1}^\infty \in \ell_p^w(E^*)$, showing that the supremum in (2) is finite. Straightforward computations prove that $\|\cdot\|_{{\rm mid},p}$ is a norm and a canonical argument shows that $(\ell_p^{\rm mid}(E), \|\cdot\|_{{\rm mid},p})$ is a Banach space.

For every $\varphi \in B_{E^*}$, it is clear that $(\varphi, 0, 0, \ldots) \in B_{\ell_p^w(E^*)}$, so $\|\cdot\|_{w,p} \le \|\cdot\|_{{\rm mid},p}$ in $\ell_p^{\rm mid}(E)$.

Let $(x_j)_{j=1}^\infty \in \ell_p(E)$ and $(x_n^*)_{n=1}^\infty \in \ell_p^w(E^*)$. Since B_E, regarded as a subspace of E^{**}, is a norming subset of E^{**}, we have $\|(x_n^*)_{n=1}^\infty\|_{w,p}^p = \sup_{x \in B_E} \sum_{n=1}^\infty |x_n^*(x)|^p$. Putting $J = \{j \in \mathbb{N} : x_j \neq 0\}$, we have

$$\begin{aligned}\sum_{j=1}^\infty \sum_{n=1}^\infty |x_n^*(x_j)|^p &= \sum_{j\in J} \left(\|x_j\|^p \cdot \left(\sum_{n=1}^\infty \left| x_n^*\left(\frac{x_j}{\|x_j\|} \right) \right|^p \right) \right) \\ &\le \|(x_n^*)_{n=1}^\infty\|_{w,p}^p \cdot \sum_{j\in J} \|x_j\|^p = \|(x_n^*)_{n=1}^\infty\|_{w,p}^p \cdot \sum_{j=1}^\infty \|x_j\|^p,\end{aligned}$$

from which the inequality $\|\cdot\|_{{\rm mid},p} \le \|\cdot\|_p$ follows. ☐

Proposition 1.5. *The following are equivalent for a weak mid-p-space E:*

(a) *$\ell_p(E)$ is closed in $\ell_p^{\rm mid}(E)$.*

(b) *The norms $\|\cdot\|_p$ and $\|\cdot\|_{{\rm mid},p}$ are equivalent on $\ell_p(E)$.*

(c) *The norms $\|\cdot\|_p$ and $\|\cdot\|_{{\rm mid},p}$ are equivalent on $c_{00}(E)$.*

(d) *E is finite-dimensional.*

Proof. (a) $\Rightarrow$ (b) follows from the open mapping theorem, (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a) are obvious. Let us prove (c) $\Rightarrow$ (d): Since E is a weak mid-p-space, the norms $\|\cdot\|_{\mathrm{mid},p}$ and $\|\cdot\|_{w,p}$ are equivalent on $\ell_p^{\mathrm{mid}}(E)$ by the open mapping theorem, hence they are equivalent on $c_{00}(E)$. The assumption gives that the norms $\|\cdot\|_p$ and $\|\cdot\|_{w,p}$ are equivalent on $c_{00}(E)$. By Lemma 1.3 it follows that E is finite-dimensional. $\square$

Analogously, we have:

Proposition 1.6. *The following are equivalent for a strong mid-p-space E:*

(a) $\ell_{\mathrm{mid}}(E)$ *is closed in* $\ell_p^w(E)$.

(b) *The norms* $\|\cdot\|_{\mathrm{mid},p}$ *and* $\|\cdot\|_{w,p}$ *are equivalent on* $\ell_p^{\mathrm{mid}}(E)$.

(c) *The norms* $\|\cdot\|_{\mathrm{mid},p}$ *and* $\|\cdot\|_{w,p}$ *are equivalent on* $c_{00}(E)$.

(d) E *is finite-dimensional.*

The next examples show that the spaces $\ell_p^u(E)$ and $\ell_p^{\mathrm{mid}}(E)$ are incomparable in general.

Example 1.7. On the one hand, combining Theorem 1.2(i) with [Diestel et al. 1995, Theorem 3.7] we have $\ell_2^{\mathrm{mid}}(c_0) = \ell_2^w(c_0)$. Since $\ell_2^u(c_0)$ is a proper subspace of $\ell_2^w(c_0)$ [Defant and Floret 1993, page 93], it follows that $\ell_2^{\mathrm{mid}}(c_0) \not\subseteq \ell_2^u(c_0)$. On the other hand,

$$\ell_1^u(\ell_1) = \ell_1^w(\ell_1) \not\subseteq \ell_1(\ell_1) = \ell_1^{\mathrm{mid}}(\ell_1),$$

where the first equality follows from the fact that bounded linear operators from c_0 to ℓ_1 are compact combined with [Defant and Floret 1993, Proposition 8.2(1)], and the last equality is a consequence of Theorem 1.2(ii).

We saw that $\ell_p^{\mathrm{mid}}(E)$ is not contained in $\ell_p^u(E)$ in general. But sometimes this happens:

Proposition 1.8. *If E is a strong mid-p-space, then* $\ell_p^{\mathrm{mid}}(E) \overset{1}{\hookrightarrow} \ell_p^u(E)$.

Proof. The norms $\|\cdot\|_p$ and $\|\cdot\|_{\mathrm{mid},p}$ are equivalent on $\ell_p^{\mathrm{mid}}(E) = \ell_p(E)$ by the open mapping theorem. Let $x = (x_j)_{j=1}^\infty \in \ell_p^{\mathrm{mid}}(E)$. Since $(x_j)_{j=1}^k \xrightarrow{k} x$ in $\ell_p(E)$, we have $(x_j)_{j=1}^k \xrightarrow{k} x$ in $\ell_p^{\mathrm{mid}}(E)$, by the equivalence of the norms. As $\ell_p^{\mathrm{mid}}(E) \overset{1}{\hookrightarrow} \ell_p^w(E)$, we have

$$\|(x_j)_{j=k}^\infty\|_{w,p} = \big\|(x_j)_{j=1}^\infty - (x_j)_{j=1}^{k-1}\big\|_{w,p} \le \big\|(x_j)_{j=1}^\infty - (x_j)_{j=1}^{k-1}\big\|_{\mathrm{mid},p} \xrightarrow{k\to\infty} 0,$$

proving that $x \in \ell_p^u(E)$. $\square$

The purpose of the next section is to study the operator ideals determined by the transformation of vector-valued sequences belonging to the sequence spaces in the chain (1). A usual approach, proving all the desired properties using the definitions of the underlying sequence spaces, would lead to long and boring proofs. Alternatively, we shall apply the abstract framework constructed in [Botelho and

Campos 2016] to deal with operators of this kind. In this fashion we will end up with short and concise proofs. Instead of its definition, we shall use that the class of mid-p-summable sequences enjoys the two properties we prove below. For the definitions of finitely determined and linearly stable sequence classes, see [Botelho and Campos 2016].

Proposition 1.9. *The correspondence $E \mapsto \ell_p^{\mathrm{mid}}(E)$ is a finitely determined sequence class.*

Proof. It is plain that $c_{00}(E) \subseteq \ell_p^{\mathrm{mid}}(E)$ and $\|e_j\|_{\mathrm{mid},p} = 1$, where e_j is the j-th canonical unit scalar-valued sequence. Since $\ell_p^{\mathrm{mid}}(E) \overset{1}{\hookrightarrow} \ell_p^w(E)$ and $\ell_p^w(E) \overset{1}{\hookrightarrow} \ell_\infty(E)$, we have $\ell_p^{\mathrm{mid}}(E) \overset{1}{\hookrightarrow} \ell_\infty(E)$. Let $(x_j)_{j=1}^\infty$ be an E-valued sequence. The equality

$$\sup_{(x_n^*)_{n=1}^\infty \in B_{\ell_p^w(E^*)}} \left(\sum_{n=1}^\infty \sum_{j=1}^\infty |x_n^*(x_j)|^p \right)^{1/p} = \sup_k \sup_{(x_n^*)_{n=1}^\infty \in B_{\ell_p^w(E^*)}} \left(\sum_{n=1}^\infty \sum_{j=1}^k |x_n^*(x_j)|^p \right)^{1/p}$$

shows that $(x_j)_{j=1}^\infty \in \ell_p^{\mathrm{mid}}(E)$ if and only if $\sup_k \|(x_j)_{j=1}^k\|_{\mathrm{mid},p} < \infty$ and that $\|(x_j)_{j=1}^\infty\|_{\mathrm{mid},p} = \sup_k \|(x_j)_{j=1}^k\|_{\mathrm{mid},p}$. □

Proposition 1.10. *The correspondence $E \mapsto \ell_p^{\mathrm{mid}}(E)$ is linearly stable.*

Proof. Let $T \in \mathcal{L}(E\,;F)$. By the linear stability of $\ell_p^w(\,\cdot\,)$ [Botelho and Campos 2016, Theorem 3.3], $(T^*(y_n^*))_{n=1}^\infty = (y_n^* \circ T)_{n=1}^\infty \in \ell_p^w(E^*)$ for every $(y_n^*)_{n=1}^\infty \in \ell_p^w(F^*)$, where $T^* : F^* \to E^*$ is the adjoint of T. Therefore,

$$((y_n^*(T(x_j)))_{j=1}^\infty)_{n=1}^\infty = ((y_n^* \circ T(x_j))_{j=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p),$$

hence $(T(x_j))_{j=1}^\infty \in \ell_p^{\mathrm{mid}}(F)$ for every $(x_j)_{j=1}^\infty \in \ell_p^{\mathrm{mid}}(E)$. Defining $\widehat{T} : \ell_p^{\mathrm{mid}}(E) \to \ell_p^{\mathrm{mid}}(F)$ by $\widehat{T}((x_j)_{j=1}^\infty) = (T(x_j))_{j=1}^\infty$, a standard calculation shows that $\|T\| = \|\widehat{T}\|$, completing the proof. □

2. Mid-summing operators

Following the classical line of studying operators that improve the summability of sequences, in this section we investigate the obvious classes of operators, involving mid-p-summable sequences, determined by the chain

$$\ell_p\langle E\rangle \subseteq \ell_p(E) \subseteq \ell_p^{\mathrm{mid}}(E) \subseteq \ell_p^w(E).$$

From now on in this section, $1 \le q \le p < \infty$ are real numbers and $T \in \mathcal{L}(E\,;F)$ is a continuous linear operator.

Definition 2.1. The operator T is said to be

(i) *absolutely mid-$(p\,;q)$-summing* if

$$(T(x_j))_{j=1}^\infty \in \ell_p(F) \text{ whenever } (x_j)_{j=1}^\infty \in \ell_q^{\mathrm{mid}}(E); \tag{3}$$

(ii) *weakly mid-$(p\,;q)$-summing* if

$$(T(x_j))_{j=1}^{\infty} \in \ell_p^{\mathrm{mid}}(F) \text{ whenever } (x_j)_{j=1}^{\infty} \in \ell_q^{w}(E); \tag{4}$$

(iii) *Cohen mid-p-summing* if

$$(T(x_j))_{j=1}^{\infty} \in \ell_p\langle F\rangle \text{ whenever } (x_j)_{j=1}^{\infty} \in \ell_p^{\mathrm{mid}}(E). \tag{5}$$

The spaces formed by the operators above are denoted $\Pi_{p;q}^{\mathrm{mid}}(E\,;F)$, $W_{p;q}^{\mathrm{mid}}(E\,;F)$ and $\mathcal{D}_p^{\mathrm{mid}}(E\,;F)$, respectively. When $p=q$ we simply write mid-p-summing instead of mid-$(p\,;p)$-summing and use symbols Π_p^{mid} and W_p^{mid}. A standard calculation shows that if $p<q$ then $\Pi_{p;q}^{\mathrm{mid}}(E\,;F)=W_{p;q}^{\mathrm{mid}}(E\,;F)=\{0\}$. From the definitions it is clear that

$$\Pi_{p;q} \subseteq W_{p;q}^{\mathrm{mid}} \cap \Pi_{p;q}^{\mathrm{mid}} \text{ and } \mathcal{D}_p^{\mathrm{mid}} \subseteq \mathcal{D}_p \cap \Pi_p^{\mathrm{mid}}.$$

Having in mind the properties of $\ell_p^{\mathrm{mid}}(E)$ proved in the previous section, the following three results are straightforward consequences of [Botelho and Campos 2016, Proposition 1.4] (with the exception of the equivalences in Theorem 2.3 involving $\ell_p^{u}(E)$, which follow from [Botelho and Campos 2016, Corollary 1.6]). Recall that any map $T: E\to F$ induces a map $\widetilde{T}$ between E-valued sequences and F-valued sequences given by $\widetilde{T}((x_j)_{j=1}^{\infty})=(T(x_j))_{j=1}^{\infty}$.

Theorem 2.2. *The following are equivalent*:

(i) $T\in \Pi_{p;q}^{\mathrm{mid}}(E\,;F)$.

(ii) *The induced map $\widetilde{T}:\ell_q^{\mathrm{mid}}(E)\to \ell_p(F)$ is a well-defined continuous linear operator.*

(iii) *There is a constant $A>0$ such that $\|(T(x_j))_{j=1}^{k}\|_p \le A\|(x_j)_{j=1}^{k}\|_{\mathrm{mid},q}$ for every $k\in\mathbb{N}$ and all $x_j\in E,\ j=1,\dots,k$.*

(iv) *There is a constant $A>0$ such that $\|(T(x_j))_{j=1}^{\infty}\|_p \le A\|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},q}$ for every $(x_j)_{j=1}^{\infty}\in \ell_q^{\mathrm{mid}}(E)$.*

Moreover,

$$\|T\|_{\Pi_{p;q}^{\mathrm{mid}}} := \|\widetilde{T}\| = \inf\{A : \text{(iii) } \textit{holds}\} = \inf\{A : \text{(iv) } \textit{holds}\}.$$

Theorem 2.3. *The following are equivalent*:

(i) $T\in W_{p;q}^{\mathrm{mid}}(E\,;F)$.

(ii) *The induced map $\widetilde{T}:\ell_q^{w}(E)\to \ell_p^{\mathrm{mid}}(F)$ is a well-defined continuous linear operator.*

(iii) $(T(x_j))_{j=1}^{\infty}\in \ell_p^{\mathrm{mid}}(F)$ *whenever* $(x_j)_{j=1}^{\infty}\in \ell_q^{u}(E)$.

(iv) *The induced map $\widehat{T}:\ell_q^{u}(E)\to \ell_p^{\mathrm{mid}}(F)$ is a well-defined continuous linear operator.*

(v) *There is a constant* $B>0$ *such that* $\|(T(x_j))_{j=1}^k\|_{\mathrm{mid},p}\leq B\|(x_j)_{j=1}^k\|_{w,q}$ *for every* $k\in\mathbb{N}$ *and all* $x_j\in E,\ j=1,\ldots,k$.

(vi) *There is a constant* $B>0$ *such that*

$$\left(\sum_{n=1}^{\infty}\sum_{j=1}^{k}|y_n^*(T(x_j))|^p\right)^{1/p}\leq B\|(x_j)_{j=1}^k\|_{w,q}\cdot\|(y_n^*)_{n=1}^{\infty}\|_{w,p}$$

for every $k\in\mathbb{N}$, *all* $x_j\in E,\ j=1,\ldots,k$, *and every* $(y_n^*)_{n=1}^{\infty}\in\ell_p^w(F^*)$.

(vii) *There is a constant* $B>0$ *such that*

$$\left(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}|y_n^*(T(x_j))|^p\right)^{1/p}\leq B\|(x_j)_{j=1}^{\infty}\|_{w,q}\cdot\|(y_n^*)_{n=1}^{\infty}\|_{w,p}$$

for all $(x_j)_{j=1}^{\infty}\in\ell_q^w(E)$ *and* $(y_n^*)_{n=1}^{\infty}\in\ell_p^w(F^*)$.

Moreover,

$$\|T\|_{W_{p;q}^{\mathrm{mid}}}:=\|\widetilde{T}\|=\|\widehat{T}\|=\inf\{B:\text{(v) } holds\}=\inf\{B:\text{(vi) } holds\}=\inf\{B:\text{(vii) } holds\}.$$

Theorem 2.4. *The following are equivalent*:

(i) $T\in\mathcal{D}_p^{\mathrm{mid}}(E\,;\,F)$.

(ii) *The induced map* $\widetilde{T}:\ell_p^{\mathrm{mid}}(E)\to\ell_p\langle F\rangle$ *is a well-defined continuous linear operator.*

(iii) *There is a constant* $C>0$ *such that* $\|(T(x_j))_{j=1}^k\|_{C,p}\leq C\|(x_j)_{j=1}^k\|_{\mathrm{mid},p}$ *for every* $k\in\mathbb{N}$ *and all* $x_j\in E,\ j=1,\ldots,k$.

(iv) *There is a constant* $C>0$ *such that*

$$\sum_{j=1}^{k}|y_j^*(T(x_j))|\leq C\|(x_j)_{j=1}^k\|_{\mathrm{mid},p}\cdot\|(y_j^*)_{j=1}^k\|_{w,p^*}$$

for every $k\in\mathbb{N}$, *all* $x_j\in E$ *and* $y_j^*\in F^*$, $j=1,\ldots,k$.

(v) *There is a constant* $C>0$ *such that*

$$\sum_{j=1}^{\infty}|y_j^*(T(x_j))|\leq C\|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},p}\cdot\|(y_j^*)_{j=1}^{\infty}\|_{w,p^*}$$

for all $(x_j)_{j=1}^{\infty}\in\ell_p^{\mathrm{mid}}(E)$ *and* $(y_j^*)_{j=1}^{\infty}\in\ell_{p^*}^w(F^*)$.

Moreover,

$$\|T\|_{\mathcal{D}_p^{\mathrm{mid}}}:=\|\widetilde{T}\|=\inf\{C:\text{(iii) } holds\}=\inf\{C:\text{(iv) } holds\}=\inf\{C:\text{(v) } holds\}.$$

Theorem 2.5. *The classes* $(\Pi_{p;q}^{\mathrm{mid}},\|\cdot\|_{\Pi_{p;q}^{\mathrm{mid}}})$, $(W_{p;q}^{\mathrm{mid}},\|\cdot\|_{W_{p;q}^{\mathrm{mid}}})$ *and* $(\mathcal{D}_p^{\mathrm{mid}},\|\cdot\|_{\mathcal{D}_p^{\mathrm{mid}}})$ *are Banach operator ideals.*

Proof. We use the notation and the language of [Botelho and Campos 2016]. All involved sequence classes are linearly stable. Comparing Definition 2.1 and [Botelho and Campos 2016, Definition 2.1], a linear operator T is mid-$(p\,;q)$-summing if and only if it is $(\ell_q^{\mathrm{mid}}(\,\cdot\,)\,;\ell_p(\,\cdot\,))$-summing. Since $\ell_q^{\mathrm{mid}}(\mathbb{K}) \overset{1}{\hookrightarrow} \ell_p = \ell_p(\mathbb{K})$, from [Botelho and Campos 2016, Theorem 2.6] it follows that $\Pi_{p;q}^{\mathrm{mid}}$ is a Banach operator ideal. The other cases are similar. □

The following characterizations of weak and strong mid-p-spaces complement the ones proved in [Karn and Sinha 2014, Theorems 3.7 and 4.5].

Theorem 2.6. *The following are equivalent for a Banach space E and $1 \le p < \infty$:*

(a) *E is a weak mid-p-space.*

(b) $\Pi_p^{\mathrm{mid}}(E\,;F) = \Pi_p(E\,;F)$ *for every Banach space F.*

(c) $\Pi_p^{\mathrm{mid}}(E;\ell_p) = \Pi_p(E;\ell_p) = \mathcal{L}(E;\ell_p)$.

(d) $W_p^{\mathrm{mid}}(F\,;E) = \mathcal{L}(F\,;E)$ *for every Banach space F.*

(e) $\mathrm{id}_E \in W_p^{\mathrm{mid}}(E\,;E)$.

Proof. The implications (a) ⇒ (b), (d) ⇒ (e) ⇒ (a), and (b) ⇒ the first equality in (c) are obvious. Let us see that the first equality in (c) implies (a): Given $x^* = (x_k^*)_{k=1}^\infty \in \ell_p^w(E^*)$, the identification $\ell_p^w(E^*) = \mathcal{L}(E,\ell_p)$ (see the proof of Proposition 2.12) yields that the map

$$S_{x^*} : E \to \ell_p, \quad S_{x^*}(x) = (x_k^*(x))_{k=1}^\infty,$$

is a bounded linear operator. By the definition of $\ell_p^{\mathrm{mid}}(E)$,

$$(S_{x^*}(x_n))_{n=1}^\infty = ((x_k^*(x_n))_{k=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p),$$

for every $(x_n)_{n=1}^\infty \in \ell_p^{\mathrm{mid}}(E)$. This means that $S_{x^*} \in \Pi_p^{\mathrm{mid}}(E;\ell_p)$, hence $S_{x^*} \in \Pi_p(E;\ell_p)$ by assumption, for every $x^* = (x_k^*)_{k=1}^\infty \in \ell_p^w(E^*)$. Therefore, given $(x_n)_{n=1}^\infty \in \ell_p^w(E)$, it follows that $(S_{x^*}(x_n))_{n=1}^\infty = ((x_k^*(x_n))_{k=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p)$ for every $x^* = (x_k^*)_{k=1}^\infty \in \ell_p^w(E^*)$; proving that $(x_n)_{n=1}^\infty \in \ell_p^{\mathrm{mid}}(E)$.

That (a) is equivalent to the second equality in (c) is precisely Theorem 1.2(i).

To complete the proof, let us check that (a) ⇒ (d): Let $T \in \mathcal{L}(F\,;E)$ and $(x_j)_{j=1}^\infty \in \ell_p^w(F)$ be given. The linear stability of $\ell_p^w(\,\cdot\,)$ and the assumption give $(T(x_j))_{j=1}^\infty \in \ell_p^w(E) = \ell_p^{\mathrm{mid}}(E)$. This proves that $T \in W_p^{\mathrm{mid}}(F\,;E)$. □

The corresponding characterizations of strong mid-p-spaces are less interesting. We state them just for the record:

Theorem 2.7. *The following are equivalent for a Banach space F and $1 \le p < \infty$:*

(a) *F is a strong mid-p-space.*

(b) $\Pi_p^{\mathrm{mid}}(E\,;F) = \mathcal{L}(E\,;F)$ *for every Banach space E.*

(c) $id_F \in \Pi_p^{mid}(F\,;F)$.

(d) *F is a subspace of $L_p(\mu)$ for some Borel measure μ.*

Recall that an operator ideal $\mathcal{I}$ is

- *injective* if $u \in \mathcal{I}(E,F)$ whenever $v \in \mathcal{L}(F,G)$ is a metric injection ($\|v(y)\| = \|y\|$ for every $y \in F$) such that $v \circ u \in \mathcal{I}(E,G)$;
- *regular* if $u \in \mathcal{I}(E,F)$ whenever $J_F \circ u \in \mathcal{I}(E,F^{**})$, where $J_F \colon F \to F^{**}$ is the canonical embedding.

Proposition 2.8. *The operator ideal $\Pi_{p;q}^{mid}$ is injective and the operator ideals $W_{p;q}^{mid}$ and $\mathcal{D}_p^{mid}$ are regular.*

Proof. The injectivity of $\Pi_{p;q}^{mid}$ is clear. To prove the regularity of $W_{p;q}^{mid}$, let $(y_j)_{j=1}^\infty \subseteq F$ be such that $(J_F(y_j))_{j=1}^\infty \in \ell_p^{mid}(F^{**})$. We have

$$(y_n^{***}(J_F(y_j)))_{j,n=1}^\infty \in \ell_p(\ell_p) \text{ for every } (y_n^{***})_{n=1}^\infty \in B_{\ell_p^w(F^{***})}. \tag{6}$$

In order to prove that $(y_j)_{j=1}^\infty \in \ell_p^{mid}(F)$, let $(y_n^*)_{n=1}^\infty \in B_{\ell_p^w(F^*)}$ be given. Then

$$\sum_{n=1}^\infty |J_F(y)(y_n^*)|^p = \sum_{n=1}^\infty |y_n^*(y)|^p \leq 1 \text{ for every } y \in B_F. \tag{7}$$

Let us see that, defining $y_n^{***} := J_{F^*}(y_n^*) \in F^{***}$ for each n, we have $(y_n^{***})_{n=1}^\infty \in B_{\ell_p^w(F^{***})}$. To accomplish this task, let $y^{**} \in B_{F^{**}}$ be given. By Goldstine's theorem, there is a net $(y_\lambda)_\lambda$ in B_F such that $J_F(y_\lambda) \xrightarrow{w^*} y^{**}$, that is,

$$y^*(y_\lambda) = J_F(y_\lambda)(y^*) \to y^{**}(y^*) \text{ for every } y^* \in F^*. \tag{8}$$

From (7) it follows that $\sum_{n=1}^\infty |y_n^*(y_\lambda)|^p \leq 1$ for every λ, in particular

$$\sum_{n=1}^k |y_n^*(y_\lambda)|^p \leq 1 \text{ for every } k \text{ and every } \lambda. \tag{9}$$

On the other hand, from (8) we have $|y_n^*(y_\lambda)|^p \xrightarrow{\lambda} |y^{**}(y_n^*)|^p$ for every n, hence

$$\sum_{n=1}^k |y_n^*(y_\lambda)|^p \xrightarrow{\lambda} \sum_{n=1}^k |y^{**}(y_n^*)|^p$$

for every k. So, for every $y^{**} \in B_{F^{**}}$,

$$\begin{aligned}\sum_{n=1}^\infty |y_n^{***}(y^{**})|^p &= \sum_{n=1}^\infty |J_{F^*}(y_n^*)(y^{**})|^p = \sum_{n=1}^\infty |y^{**}(y_n^*)|^p = \sup_k \sum_{n=1}^k |y^{**}(y_n^*)|^p \\ &= \sup_k \lim_\lambda \sum_{n=1}^k |y_n^*(y_\lambda)|^p \leq 1,\end{aligned}$$

the last inequality being a consequence of (9). This proves that $(y_n^{***})_{n=1}^{\infty} \in B_{\ell_p^w(F^{***})}$. From (6) we get

$$(y_n^*(y_j))_{j,n=1}^{\infty} = (J_F(y_j)(y_n^*))_{j,n=1}^{\infty} = \big([J_{F_*}(y_n^*)](J_F(y_j))\big)_{j,n=1}^{\infty}$$
$$= (y_n^{***}(J_F(y_j)))_{j,n=1}^{\infty} \in \ell_p(\ell_p).$$

This holds for arbitrary $(y_n^*)_{n=1}^{\infty} \in B_{\ell_p^w(F^*)}$, which allows us to conclude that $(y_j)_{j=1}^{\infty} \in \ell_p^{\mathrm{mid}}(F)$. Thus far we have proved that $(y_j)_{j=1}^{\infty} \in \ell_p^{\mathrm{mid}}(F)$ whenever $(J_F(y_j))_{j=1}^{\infty} \in \ell_p^{\mathrm{mid}}(F^{**})$. Now the regularity of $W_{p;q}^{\mathrm{mid}}$ follows easily.

An adaptation of the argument above shows that $(y_j)_{j=1}^{\infty} \in \ell_p\langle F\rangle$ whenever $(J_F(y_j))_{j=1}^{\infty} \in \ell_p\langle F^{**}\rangle$. The regularity of $\mathcal{D}_p^{\mathrm{mid}}$ follows. □

Remark 2.9. The final part of the proof above also proves that the ideal $\mathcal{D}_p$ of Cohen strongly p-summing operators is regular. We also know that it is surjective because it is the dual of the injective ideal Π_{p^*} [Cohen 1973].

It is clear from the definitions that $\Pi_{p,r}^{\mathrm{mid}} \circ W_{r,q}^{\mathrm{mid}} \subseteq \Pi_{p,q}$ for $q \le r \le p$. Next we show that the equality holds if $p = q$, which gives a new factorization theorem for absolutely p-summing operators:

Theorem 2.10. *Every absolutely p-summing linear operator factors through absolutely and weakly mid-p-summing linear operators, that is,* $\Pi_p = \Pi_p^{\mathrm{mid}} \circ W_p^{\mathrm{mid}}$.

Proof. We already know that $\Pi_p^{\mathrm{mid}} \circ W_p^{\mathrm{mid}} \subseteq \Pi_p$. Let $u \in \Pi_p(E\,;F)$. By Pietsch's factorization theorem ([Defant and Floret 1993, Corollary 1, page 130] or [Diestel et al. 1995, Theorem 2.13]), there are a Borel–Radon measure μ on (B_{E^*}, w^*), a closed subspace X of $L_p(\mu)$ and an operator $\hat{u} : X \to F$ such that the following diagram commutes (i_E and j_p are the canonical operators and j_p^E is the restriction of j_p to $i_E(E)$):

$$\begin{tikzcd}
E \arrow[r, "u"] \arrow[d, bend right, "i_E"'] & F \\
i_E(E) \arrow[r, "j_p^E"] & X \arrow[u, bend right, "\hat{u}"'] \\
\cap & \cap \\
\mathcal{C}(K) \arrow[r, "j_p"] & L_p(\mu)
\end{tikzcd}$$

Let $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. By the continuity of i_E and the linear stability of $\ell_p^w(\,\cdot\,)$, we have $(i_E(x_j))_{j=1}^{\infty} \in \ell_p^w(i_E(E))$. Since j_p is absolutely p-summing, it follows that $(j_p^E(i_E(x_j)))_{j=1}^{\infty} \in \ell_p(X) \subseteq \ell_p^{\mathrm{mid}}(X)$, proving $j_p^E \circ i_E \in W_p^{\mathrm{mid}}(E, X)$. Now, let $(y_j)_{j=1}^{\infty} \in \ell_p^{\mathrm{mid}}(X)$. As X is a closed subspace of $L_p(\mu)$, from Theorem 1.2(ii) we have $(y_j)_{j=1}^{\infty} \in \ell_p(X)$. Thus, as $\hat{u}$ is bounded and $\ell_p(\,\cdot\,)$ is linearly stable, $(\hat{u}(y_j))_{j=1}^{\infty} \in \ell_p(F)$, proving that $\hat{u} \in \Pi_p^{\mathrm{mid}}(X, F)$. □

Corollary 2.11. *Let $p > 1$, $u \in \mathcal{L}(E,F)$ and $v \in \mathcal{L}(F,G)$. If u^* is absolutely mid-p^*-summing and v^* is weakly mid-p^*-summing, then $v \circ u$ is Cohen strongly p-summing.*

Proof. Denoting, as usual, by $\mathcal{I}^{\text{dual}}$ the ideal of all operators u such that $u^* \in \mathcal{I}$, we have

$$(W_{p^*}^{\text{mid}})^{\text{dual}} \circ (\Pi_{p^*}^{\text{mid}})^{\text{dual}} \subseteq (\Pi_{p^*}^{\text{mid}} \circ W_{p^*}^{\text{mid}})^{\text{dual}} = \Pi_{p^*}^{\text{dual}} = \mathcal{D}_p,$$

where the inclusion is clear, the first equality follows from Theorem 2.10 and the second from [Cohen 1973]. □

We finish this section solving a question left open in the last section of [Karn and Sinha 2014]. There, the authors prove the following characterization in their Theorem 4.4: an operator $T \in \mathcal{L}(E,F)$ is weakly mid-p-summing if and only if $S \circ T \in \Pi_p(E,\ell_p)$ for every $S \in \mathcal{L}(F,\ell_p)$. They define

$$\operatorname{lt}_p(T) = \sup\bigl\{\pi_p(S \circ T) : S \in \mathcal{L}(F,\ell_p) \text{ and } \|S\| \le 1\bigr\},$$

and prove that $(W_p^{\text{mid}}, \operatorname{lt}_p(\cdot))$ is a normed operator ideal. The question whether or not this ideal is a Banach ideal is left open there, and now we solve it in the affirmative:

Proposition 2.12. *Since $\operatorname{lt}_p(T) = \|T\|_{W_{p;q}^{\text{mid}}}$ for every $T \in W_p^{\text{mid}}(E;F)$, we have $(W_p^{\text{mid}}, \operatorname{lt}_p(\cdot))$ is a Banach operator ideal.*

Proof. Let $T \in W_p^{\text{mid}}(E;F)$ and $S \in \mathcal{L}(F,\ell_p)$ with $\|S\| \le 1$. Here we use that the spaces $\ell_p^w(F^*)$ and $\mathcal{L}(F,\ell_p)$ are canonically isometrically isomorphic via the correspondence $x^* = (x_k^*)_{k=1}^\infty \in \ell_p^w(F^*) \mapsto S_{x^*} \in \mathcal{L}(F,\ell_p)$, $S_{x^*}(x) = (x_k^*(x))_{k=1}^\infty$ [Defant and Floret 1993, Proposition 8.2(2)]. So there exists $(y_k^*)_{k=1}^\infty \in B_{\ell_p^w(F^*)}$ such that $S(y) = (y_k^*(y))_{k=1}^\infty$ for every $y \in F$. Thus

$$\left(\sum_{j=1}^\infty \|S \circ T(x_j)\|_p^p\right)^{1/p} = \left(\sum_{j=1}^\infty \sum_{k=1}^\infty |y_k^*(T(x_j))|^p\right)^{1/p} \le \|T\|_{W_{p;q}^{\text{mid}}} \cdot \|(x_j)_{j=1}^\infty\|_{w,p},$$

for every $(x_j)_{j=1}^\infty \in \ell_p^w(E)$. Therefore $S \circ T \in \Pi_p(E;\ell_p)$ and $\pi_p(S \circ T) \le \|T\|_{W_{p;q}^{\text{mid}}}$. From

$$\left(\sum_{j=1}^\infty \sum_{n=1}^\infty |y_n^*(T(x_j))|^p\right)^{1/p} = \left(\sum_{j=1}^\infty \|S \circ T(x_j)\|_p^p\right)^{1/p} \le \pi_p(S \circ T) \cdot \|(x_j)_{j=1}^\infty\|_{w,p},$$

we obtain $\|T\|_{W_{p;q}^{\text{mid}}} \le \pi_p(S \circ T)$, proving that $\operatorname{lt}_p(T) = \|T\|_{W_{p;q}^{\text{mid}}}$. The second assertion follows now from Theorem 2.5. □

3. Infinite-dimensional Banach spaces formed by non-summing operators

We say that the subset A of an infinite-dimensional vector space X is *lineable* if $A \cup \{0\}$ contains an infinite-dimensional subspace. If $A \cup \{0\}$ contains a closed infinite-dimensional subspace than we say that A is *spaceable* (see [Bernal-González et al. 2014] and references therein).

Let us give a contribution to this fashionable subject. Improving a result of [Botelho et al. 2009], in [Kitson and Timoney 2011] it is proved, among other things, that if E is an infinite-dimensional superreflexive Banach space, then, regardless of the infinite-dimensional Banach space F, there exists an infinite-dimensional Banach space formed, up to the null operator, by non-p-summing linear operators from E to F. Very little is known for spaces of operators on nonsuperreflexive spaces. We shall give a contribution in this direction.

The next lemma is left as Exercise 9.10(b) in [Defant and Floret 1993]. We give a short proof for the sake of completeness.

Lemma 3.1. *An operator ideal $\mathcal{I}$ is injective if and only if the following condition holds*: *if $u \in \mathcal{I}(E\,;F)$, $v \in \mathcal{L}(E\,;G)$ and there exists a constant $C > 0$ (possibly depending on E, F, G, u and v) such that $\|v(x)\| \le C\|u(x)\|$ for every $x \in E$, then $v \in \mathcal{I}(E\,;G)$.*

Proof. Assume that $\mathcal{I}$ is injective and let $u \in \mathcal{I}(E\,;F)$, $v \in \mathcal{L}(E\,;G)$ be such that $\|v(x)\| \le C\|u(x)\|$ for every $x \in E$. This inequality guarantees that the map

$$w : u(E) \subseteq F \to G, \quad w(u(x)) = v(x),$$

is a well-defined continuous linear operator. Considering the canonical metric injection $J_G : G \to \ell_\infty(B_{G^*})$, by the extension property of $\ell_\infty(B_{G^*})$ [Pietsch 1980, Proposition C.3.2] there is an extension $\widetilde{w} \in \mathcal{L}(F\,;\ell_\infty(B_{G^*}))$ of $J_G \circ w$ to the whole of F. From $\widetilde{w} \circ u = J_G \circ v$ we conclude that $J_G \circ v$ belongs to $\mathcal{I}$, and the injectivity of $\mathcal{I}$ gives $v \in \mathcal{I}(E\,;G)$. The converse is obvious. □

Henceforth, all Banach spaces are supposed to be infinite-dimensional. Recall that a sequence in a Banach space E is *overcomplete* if the linear span of each of its subsequences is dense in E (see, e.g., [Chalendar and Partington 2007; Fonf and Zanco 2014]). We need a weaker condition:

Definition 3.2. A sequence in a Banach space E is *weakly overcomplete* if the closed linear span of each of its subsequences is isomorphic to E.

Example 3.3. The sequence $(e_j)_{j=1}^\infty$ formed by the canonical unit vectors is a weakly overcomplete unconditional basis in the spaces c_0 and ℓ_p, $1 \le p < \infty$ [Fabian et al. 2011, Proposition 4.45].

Proposition 3.4. *Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a normed operator ideal, $\mathcal{J}$ be an injective operator ideal and suppose that F contains an isomorphic copy of a space X with a* *weakly overcomplete unconditional basis. If* $\mathcal{I}(E\,;\,X)-\mathcal{J}(E\,;\,X)$ *is nonvoid, then* $\mathcal{I}(E\,;\,F)-\mathcal{J}(E\,;\,F)$ *is spaceable (in* $(\mathcal{I}(E\,;\,F),\|\cdot\|_{\mathcal{I}})$*).*

Proof. Let $(e_n)_{n=1}^{\infty}$ be a weakly overcomplete unconditional basis of X with unconditional basis constant ϱ. Split $\mathbb{N}=\bigcup_{j=1}^{\infty}A_j$ into infinitely many infinite pairwise disjoint subsets. For each $j\in\mathbb{N}$, define $X_j=\overline{\operatorname{span}\{e_n : n\in A_j\}}$ and let $P_j : X\to X_j$ be the canonical projection. It is known that $\|P_j\|\le\varrho$ [Megginson 1998, Corollary 4.2.26]. For $x_j\in X_j$ we have $P_i(x_j)=\delta_{ij}x_j$ because the sets $(A_j)_{j=1}^{\infty}$ are pairwise disjoint. Let $I_j : X\to X_j$ be an isomorphism, $T_j : X_j\to X$ denote the formal inclusion and $T : X\to F$ be an isomorphism into. Let $u\in\mathcal{I}(E\,;\,X)-\mathcal{J}(E\,;\,X)$. Defining

$$u_j : E\to F,\quad u_j=T\circ T_j\circ I_j\circ u,$$

we have $u_j\in\mathcal{I}(E,F)$. Using that $\mathcal{J}$ is injective, $u\notin\mathcal{J}(E\,;\,X)$ and

$$\|u_j(x)\|=\left\|T(T_j\circ I_j\circ u(x))\right\|\ge\frac{1}{\|T^{-1}\|}\left\|T_j\circ I_j\circ u(x)\right\|\ge\frac{1}{\|T^{-1}\|\cdot\|I_j^{-1}\|}\|u(x)\|$$

for every $x\in E$, we conclude by Lemma 3.1 that each $u_j\notin\mathcal{J}(E\,;\,F)$. In particular, we have $u_j\neq 0$. Let $Y:=\overline{\operatorname{span}\{u_j : j\in\mathbb{N}\}}^{\|\cdot\|_{\mathcal{I}}}\subseteq\mathcal{I}(E\,;\,F)$. Given $0\neq v\in Y$, let $(v_n)_{n=1}^{\infty}\subseteq\operatorname{span}\{u_j : j\in\mathbb{N}\}$ be such that $v_n\xrightarrow{\|\cdot\|_{\mathcal{I}}}v$. For each n, write $v_n=\sum_{j=1}^{\infty}a_j^n u_j$, where $a_j^n\neq 0$ for only finitely many j. Let $x_0\in E$ be such that $v(x_0)\neq 0$. It is plain that $v(E)\subseteq T(X)$, so $T^{-1}(v(x_0))\neq 0$, and in this case there is $k\in\mathbb{N}$ such that $P_k(T^{-1}(v(x_0)))\neq 0$. Since $\|\cdot\|\le\|\cdot\|_{\mathcal{I}}$, we have $v_n(x)\to v(x)$ for all $x\in E$. So,

$$\begin{aligned}a_k^n T_k(I_k(u(x_0)))&=\sum_{j=1}^{\infty}P_k(a_j^n T_j(I_j(u(x_0))))=\sum_{j=1}^{\infty}P_k(T^{-1}(a_j^n\,T(T_j(I_j(u(x_0))))))\\&=P_k\circ T^{-1}(v_n(x_0))\to P_k\circ T^{-1}\circ v(x_0)\neq 0.\end{aligned}$$

It follows that

$$0\neq T\circ P_k\circ T^{-1}\circ v(x_0)=\lim_n T(a_k^n\,T_k(I_k(u(x_0))))=\lim_n a_k^n u_k(x_0)=(\lim_n a_k^n)u_k(x_0).$$

Setting $\lambda:=\lim_n a_k^n\neq 0$, we have

$$\begin{aligned}\|u_k(x)\|&=\frac{1}{|\lambda|}\cdot\lim_n\|a_k^n u_k(x)\|\le\frac{\|T\|}{|\lambda|}\cdot\lim_n\left\|P_k\left(\sum_{j=1}^{\infty}T_j\circ I_j\circ u(a_j^n x)\right)\right\|\\&\le\frac{\varrho\|T\|}{|\lambda|}\cdot\lim_n\left\|\sum_{j=1}^{\infty}T_j\circ I_j\circ u(a_j^n x)\right\|\\&\le\frac{\varrho\|T\|\cdot\|T^{-1}\|}{|\lambda|}\cdot\lim_n\left\|T\left(\sum_{j=1}^{\infty}T_j\circ I_j\circ u(a_j^n x)\right)\right\|\end{aligned}$$

$$= \frac{\varrho\|T\|\cdot\|T^{-1}\|}{|\lambda|}\cdot\lim_n\left\|\sum_{j=1}^{\infty}a_j^n u_j(x)\right\| = \frac{\varrho\|T\|\cdot\|T^{-1}\|}{|\lambda|}\|v(x)\|$$

for every $x \in E$. Since u_k does not belong to the injective ideal $\mathcal{J}$, it follows from Lemma 3.1 that $v \notin \mathcal{J}(E\,;F)$. This proves that $Y \subseteq (\mathcal{I}(E\,;F) - \mathcal{J}(E\,;F) \cup \{0\})$.

Given $n \in \mathbb{N}$, scalars $a_1, \ldots, a_n$ such that $\sum_{j=1}^n a_j u_j = 0$ and $k \in \{1, \ldots, n\}$, let $x_k \in E$ be such that $u_k(x_k) \neq 0$ (recall that $u_k \neq 0$). From

$$\begin{aligned}
0 &= \left\|\sum_{j=1}^{n} a_j u_j(x_k)\right\| \geq \frac{1}{\|T^{-1}\|}\left\|\sum_{j=1}^{n} a_j (T_j \circ I_j \circ u)(x_k)\right\| \\
&\geq \frac{1}{\varrho\|T^{-1}\|}\left\|P_k\left(\sum_{j=1}^{n} a_j (T_j \circ I_j \circ u)(x_k)\right)\right\| = \frac{1}{\varrho\|T^{-1}\|}\|a_k(T_k \circ I_k \circ u)(x_k)\| \\
&\geq \frac{1}{\varrho\|T^{-1}\|\cdot\|T\|}\|T(a_k(T_k \circ I_k \circ u)(x_k))\| = \frac{1}{\varrho\|T^{-1}\|\cdot\|T\|}|a_k|\cdot\|u_k(x_k)\|,
\end{aligned}$$

it follows that $a_k = 0$, proving that the set $\{u_j : j \in \mathbb{N}\}$ is linearly independent. $\square$

Remark 3.5. (a) Proposition 3.4 is not a consequence of [Kitson and Timoney 2011, Proposition 2.4] because we are not assuming neither that $(\mathcal{I} \cap \mathcal{J})(E\,;F)$ is not closed in $\mathcal{I}(E\,;F)$ nor that $\mathcal{I}(E\,;F)$ is complete.

(b) A result related to Proposition 3.4, with different assumptions, has appeared recently in [Hernández et al. 2015, Theorem 3.5].

Recall that $\mathrm{Space}(\mathcal{I})$ denotes the class of all Banach spaces E such that the identity operator on E belongs to the operator ideal $\mathcal{I}$ (cf. [Pietsch 1980, 2.1.2]).

Theorem 3.6. *Let E be isomorphic to a subspace of $L_1(\mu)$ for some Borel measure μ, let F contain an isomorphic copy of ℓ_1 and let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a Banach operator ideal such that $\ell_1 \in \mathrm{Space}(\mathcal{I})$. Then there exists an infinite-dimensional Banach space formed, up to the null operator, by non-1-summing linear operators from E to F belonging to $\mathcal{I}$.*

Proof. By Theorem 1.2(ii), $\mathrm{id}_E \in \Pi_1^{\mathrm{mid}}(E\,;E)$. Since id_E fails to be 1-summing, because E is infinite-dimensional, by Theorem 2.10 we have $\mathrm{id}_E \notin W_1^{\mathrm{mid}}(E\,;E)$. From Theorem 1.2(i), there is a non-1-summing linear operator $u : E \to \ell_1$. Of course $u \in \mathcal{I}(E\,;\ell_1)$. Taking into account that the canonical unit vectors form a weakly overcomplete unconditional basis of ℓ_1 (Example 3.3) and that the ideal of absolutely p-summing linear operators is injective, from Proposition 3.4 we have that $\mathcal{I}(E\,;F) - \Pi_1(E\,;F)$ is spaceable. The completeness of $(\mathcal{I}(E\,;F), \|\cdot\|_{\mathcal{I}})$ finishes the proof. $\square$

Examples of Banach operator ideals $\mathcal{I}$ for which $\ell_1 \in \mathrm{Space}(\mathcal{I})$ are the following: separable operators, completely continuous operators, cotype 2 operators, absolutely

(r, q)-summing operators with $\frac{1}{r} \le \frac{1}{q} - \frac{1}{2}$ [Defant and Floret 1993, Corollary 8.9] (in particular, absolutely $(r, 1)$-summing operators for every $r \ge 2$).

Acknowledgement

We thank Professor A. Pietsch for pointing out Lemma 3.1 to us.

References

[Bernal-González et al. 2014] L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, "Linear subsets of nonlinear sets in topological vector spaces", *Bull. Amer. Math. Soc. (N.S.)* **51**:1 (2014), 71–130. MR Zbl

[Botelho and Campos 2016] G. Botelho and J. R. Campos, "On the transformation of vector-valued sequences by multilinear operators", *Monatsh. Math.* (online publication July 2016).

[Botelho et al. 2009] G. Botelho, D. Diniz, and D. Pellegrino, "Lineability of the set of bounded linear non-absolutely summing operators", *J. Math. Anal. Appl.* **357**:1 (2009), 171–175. MR Zbl

[Campos 2013] J. R. Campos, "An abstract result on Cohen strongly summing operators", *Linear Algebra Appl.* **439**:12 (2013), 4047–4055. MR Zbl

[Chalendar and Partington 2007] I. Chalendar and J. R. Partington, "Applications of moment problems to the overcompleteness of sequences", *Math. Scand.* **101**:2 (2007), 249–260. MR Zbl

[Cohen 1973] J. S. Cohen, "Absolutely p-summing, p-nuclear operators and their conjugates", *Math. Ann.* **201**:3 (1973), 177–200. MR Zbl

[Defant and Floret 1993] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies **176**, North-Holland, Amsterdam, 1993. MR Zbl

[Diestel et al. 1995] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics **43**, Cambridge University Press, 1995. MR Zbl

[Fabian et al. 2011] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory: the basis for linear and nonlinear analysis*, Springer, 2011. MR Zbl

[Fonf and Zanco 2014] V. P. Fonf and C. Zanco, "Almost overcomplete and almost overtotal sequences in Banach spaces", *J. Math. Anal. Appl.* **420**:1 (2014), 94–101. MR Zbl

[Hernández et al. 2015] F. L. Hernández, C. Ruiz, and V. M. Sánchez, "Spaceability and operator ideals", *J. Math. Anal. Appl.* **431**:2 (2015), 1035–1044. MR Zbl

[Karn and Sinha 2014] A. K. Karn and D. P. Sinha, "An operator summability of sequences in Banach spaces", *Glasg. Math. J.* **56**:2 (2014), 427–437. MR Zbl

[Kitson and Timoney 2011] D. Kitson and R. M. Timoney, "Operator ranges and spaceability", *J. Math. Anal. Appl.* **378**:2 (2011), 680–686. MR Zbl

[Megginson 1998] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics **183**, Springer, 1998. MR Zbl

[Pietsch 1980] A. Pietsch, *Operator ideals*, North-Holland Mathematical Library **20**, North-Holland, Amsterdam, 1980. MR Zbl

Received February 6, 2016.

GERALDO BOTELHO

Faculdade de Matemática
Universidade Federal de Uberlândia
38400-902 Uberlândia
Brazil

botelho@ufu.br

Jamilson R. Campos
Departamento de Ciências Exatas
Universidade Federal da Paraíba
58.297-000 Rio Tinto
Brazil

jamilson@dcx.ufpb.br

jamilsonrc@gmail.com

Joedson Santos
Departamento de Matemática
Universidade Federal da Paraíba
58.051-900 João Pessoa
Brazil

joedsonmat@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.19

HOMOLOGY FOR QUANDLES WITH PARTIAL GROUP OPERATIONS

SCOTT CARTER, ATSUSHI ISHII, MASAHICO SAITO AND KOKORO TANAKA

A quandle is a set that has a binary operation satisfying three conditions corresponding to the Reidemeister moves. Homology theories of quandles have been developed in a way similar to group homology, and have been applied to knots and knotted surfaces. In this paper, a homology theory is defined that unifies group and quandle homology theories. A quandle that is a union of groups with the operation restricting to conjugation on each group component is called a multiple conjugation quandle (MCQ, defined rigorously within). In this definition, compatibilities between the group and quandle operations are imposed which are motivated by considerations on colorings of handlebody-links. The homology theory defined here for MCQs takes into consideration both group and quandle operations, as well as their compatibility. The first homology group is characterized, and the notion of extensions by 2-cocycles is provided. Degenerate subcomplexes are defined in relation to simplicial decompositions of prismatic (products of simplices) complexes and group inverses. Cocycle invariants are also defined for handlebody-links.

1. Introduction

In this paper, a homology theory is proposed that contains aspects of both group and quandle homology theories, for algebraic structures that have both operations and certain compatibility conditions between them.

The notion of a quandle [Joyce 1982; Matveev 1982] was introduced in knot theory as a generalization of the fundamental group. Briefly, a quandle is a set with a binary operation that is idempotent and self-distributive, and a bijective corresponding right action. The axioms correspond to the Reidemeister moves, and quandles have been used extensively to construct knot invariants. They have been considered in various other contexts, for example as symmetries of geometric objects [Takasaki 1943], and with different names, such as distributive groupoids [Matveev 1982] and automorphic sets [Brieskorn 1988]. A typical example is a

MSC2010: primary 57M15, 57M25, 57M27, 57Q45; secondary 55N99, 18G99.
Keywords: quandle, homology, handlebody-link.

group conjugation $a * b = b^{-1}ab$ which is an expression of the Wirtinger relation for the fundamental group of the knot complement. The same structure but without idempotency is called a rack, and is used in the study of framed links [Fenn and Rourke 1992].

In [Fenn et al. 1995] a chain complex was introduced for racks. The resulting homology theory was modified in [Carter et al. 2003] by defining a quotient complex that reflected the quandle idempotence axiom. The motivation for this homology was to construct the quandle cocycle invariants for links and surface-links. Since then a variety of applications have been found. The quandle cocycle invariants were generalized to handlebody-links in [Ishii and Iwakiri 2012]. When a set has multiple quandle operations that are parametrized by a group, the structure is called a G-family of quandles; this notion, with its associated homology theory, was introduced in [Ishii et al. 2013] and it too was motivated from handlebody-knots. This homology theory is called *IIJO*. In particular, cocycle invariants were introduced that distinguished mirror images of some handlebody-knots. These G-families were further generalized to an algebraic system called a multiple conjugation quandle (MCQ) in [Ishii 2015b] for colorings of handlebody-knots. An MCQ has a quandle operation and partial group operations, all linked by compatibility conditions.

This paper proposes to unify the group and quandle homology theories for MCQs. The definition of an MCQ is recalled in Section 2 as a generalization of a G-family of quandles. A homology theory is defined (in Section 3) that simultaneously encompasses the group and quandle homologies of the interrelated structures. As in the case of [Carter et al. 2003], some subcomplexes are defined in order to compensate for the topological motivation of the theory. The first homology group is characterized, and the notion of extensions by 2-cocycles is provided in Section 4.

The homology theory for MCQs is well suited for handlebody-links such that each toroidal component has its core circle oriented, as defined in Section 5. When considering colorings for unoriented handlebody-links, we also need to take into consideration issues about the inverse elements in the group (Section 6). Prismatic sets (products of simplices) are decomposed into subsimplices that are higher-dimensional duals of graph moves; Section 7 defines a subcomplex that compensates for these subdivisions. In Sections 8 and 9, we relate this homology theory with group and quandle homology theories. Finally, in Section 10, we discuss approaches to finding new 2-cocycles of our homology theory.

2. Multiple conjugation quandles

First, recall a *quandle* [Joyce 1982; Matveev 1982] is a nonempty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms:

(1) For any $a \in X$, we have $a * a = a$.

(2) For any $a \in X$, the map $S_a : X \to X$ defined by $S_a(x) = x * a$ is a bijection.

(3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

Definition 1 [Ishii 2015b]. A *multiple conjugation quandle (MCQ) X* is the disjoint union of groups G_λ, where λ is an element of an index set Λ, with a binary operation $* : X \times X \to X$ satisfying the following axioms:

(1) For any $a, b \in G_\lambda$, we have $a * b = b^{-1}ab$.

(2) For any $x \in X$ and $a, b \in G_\lambda$, we have $x * e_\lambda = x$ and $x * (ab) = (x * a) * b$, where e_λ is the identity element of G_λ.

(3) For any $x, y, z \in X$, we have $(x * y) * z = (x * z) * (y * z)$.

(4) For any $x \in X$ and $a, b \in G_\lambda$, we have $(ab) * x = (a * x)(b * x)$ in some group G_μ.

We call the group G_λ a *component* of the MCQ. An MCQ is a type of quandle that can be decomposed as a union of groups, and the quandle operation in each component is given by conjugation. Moreover, there are compatibilities, (2) and (4), between the group and quandle operations.

Note that the quandle axiom $a * a = a$ follows immediately since the operation in any component is given by conjugation. The second quandle axiom also follows, since for the map $S_a : X \to X$ defined by $S_a(x) = x * a$, the inverse map is given by $S_{a^{-1}}$. The second axiom of MCQs implies that the map $\phi : G_\lambda \to \mathrm{Aut}_{\mathrm{Qnd}} X$ defined by $\phi(a) = S_a$ is a group homomorphism, where $\mathrm{Aut}_{\mathrm{Qnd}} X$ is the set of quandle automorphisms of X and is the group with the multiplication defined by $S_a S_b := S_b \circ S_a$. The last axiom (4) may be replaced by the following:

(4′) For any $x \in X$ and $\lambda \in \Lambda$, there is a unique element $\mu \in \Lambda$ such that $S_x(G_\lambda) = G_\mu$ and that $S_x : G_\lambda \to G_\mu$ is a group isomorphism.

The axiom (4) immediately follows from (4′). Conversely, (4′) follows from (4): the condition (4) contains the condition that for any $a, b \in G_\lambda$ and $x \in X$, there exists a unique $\mu \in \Lambda$ such that $a * x, b * x \in G_\mu$. Hence we have $S_x(G_\lambda) \subset G_\mu$, which implies that $S_x : G_\lambda \to G_\mu$ is a well-defined group homomorphism by the condition $(ab) * x = (a * x)(b * x)$. The homomorphism $S_x : G_\lambda \to G_\mu$ is a group isomorphism, since $S_{x^{-1}} : G_\mu \to G_\lambda$ gives its inverse.

A multiple conjugation quandle can be obtained from a G-family of quandles as follows.

Example 2. Let G be a group with identity element e, let $(M, \{*^g\}_{g \in G})$ be a G-family of quandles [Ishii et al. 2013]; i.e., a nonempty set M with a family of binary operations $*^g : M \times M \to M$ $(g \in G)$ satisfying

$$x *^g x = x, \quad x *^{gh} y = (x *^g y) *^h y, \quad x *^e y = x,$$
$$(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$$

for $x, y, z \in M$ and $g, h \in G$. Then $\coprod_{x \in M} \{x\} \times G$ is a multiple conjugation quandle with

$$(x, g) * (y, h) = (x *^h y, h^{-1}gh), \qquad (x, g)(x, h) = (x, gh).$$

The following are specific examples of G-families of quandles.

(1) Let M be a group, and G be a subgroup of $\operatorname{Aut} M$. Then for $x, y \in M$ and $g \in G$, $x * y = (xy^{-1})^g y$ gives a G-family of quandles. Here x^g denotes g acting on x. The fact that this is a G-family was pointed out in [Przytycki 2011]; however, that any specific automorphism g yields a quandle was earlier observed in [Joyce 1982; Matveev 1982]. When M is abelian and an element $g \in G$ is fixed, the resulting quandle is called an Alexander quandle.

(2) Let $(X, *)$ be a quandle. We denote $S_b^n(a)$ by $a *^n b$. Put $Z := \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, where $m := \min\{i > 0 \mid x *^i y = x \text{ for any } x, y \in X\}$. Then $(X, \{*^n\}_{n \in Z})$ is a Z-family of quandles.

For a multiple conjugation quandle $X = \coprod_{\lambda \in \Lambda} G_\lambda$, an X-*set* is a nonempty set Y with a map $* : Y \times X \to Y$ satisfying the following axioms, where we use the same symbol $*$ as the binary operation of X.

- For any $y \in Y$ and $a, b \in G_\lambda$, we have $y * e_\lambda = y$ and $y * (ab) = (y * a) * b$, where e_λ is the identity of G_λ.
- For any $y \in Y$ and $a, b \in X$, we have $(y * a) * b = (y * b) * (a * b)$.

Any multiple conjugation quandle X itself is an X-set with its binary operation. Any singleton set $\{y_0\}$ is also an X-set with the map $*$ defined by $y_0 * x = y_0$ for $x \in X$, which is called a trivial X-set. The index set Λ is an X-set with the map $*$ defined by $\lambda * x = \mu$ when $S_x(G_\lambda) = G_\mu$ for $\lambda, \mu \in \Lambda$ and $x \in X$.

3. Homology theory

In this section, we define a chain complex for MCQs that contains aspects of both group and quandle homology theories. A subcomplex is also defined that corresponds to a Reidemeister move for handlebody-links.

Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. In what follows, we denote a sequence of elements of X by a bold symbol such as $\boldsymbol{a}$, and denote by $|\boldsymbol{a}|$ the length of a sequence $\boldsymbol{a}$. For example, $(\boldsymbol{a})$, $\langle \boldsymbol{a} \rangle$, $(y; \boldsymbol{a}; \boldsymbol{b})$ respectively denote

$$(a_1, \ldots, a_{|\boldsymbol{a}|}), \quad \langle a_1, \ldots, a_{|\boldsymbol{a}|} \rangle, \quad (y; a_1, \ldots, a_{|\boldsymbol{a}|}; b_1, \ldots, b_{|\boldsymbol{b}|}).$$

Let $P_n(X)_Y$ be the free abelian group generated by the elements

$$(y; a_{1,1}, \ldots, a_{1,n_1}; \ldots; a_{k,1}, \ldots, a_{k,n_k}) \in \bigcup_{n_1+\cdots+n_k=n} Y \times \prod_{i=1}^{k} \bigcup_{\lambda\in\Lambda} G_\lambda^{n_i}$$

if $n \geq 0$, and let $P_n(X)_Y = 0$ otherwise. The elements of $P_n(X)_Y$ are called *prismatic chains* and $P_n(X)_Y$ is called the *prismatic chain group*. Note that for each j, the elements $a_{j,1}, \ldots, a_{j,n_j}$ belong to one of the G_λ. For example, $P_3(X)_Y$ is generated by the elements $(y; a; b; c)$, $(y; a; e, f)$, $(y; d, e; c)$ and $(y; d, e, f)$ (where $a, b, c \in X,\ d, e, f \in G_\lambda,\ y \in Y$). Here a, b, c may or may not belong to the same G_μ ($\mu \in \Lambda$), but d, e, f belong to the same G_λ. All may belong to the same G_λ.

We represent $(y; \boldsymbol{a}_1; \ldots; \boldsymbol{a}_k)$ using the noncommutative multiplication form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle.$$

We define $\langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle * b := \langle y*b\rangle\langle \boldsymbol{a}_1 * b\rangle \cdots \langle \boldsymbol{a}_k * b\rangle$, where $\langle \boldsymbol{a} * b\rangle$ denotes $\langle a_1 * b, \ldots, a_{|\boldsymbol{a}|} * b\rangle$. We set $|\langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle| := |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_k|$.

We define a boundary homomorphism $\partial_n : P_n(X)_Y \to P_{n-1}(X)_Y$ by

$$\partial\big(\langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle\big) = \sum_{i=1}^{k} (-1)^{|\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_{i-1}\rangle|} \langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \partial\langle \boldsymbol{a}_i\rangle \cdots \langle \boldsymbol{a}_k\rangle,$$

where

$$\partial\langle a_1, \ldots, a_m\rangle = *a_1\langle a_2, \ldots, a_m\rangle + \sum_{i=1}^{m-1} (-1)^i \langle a_1, \ldots, a_i a_{i+1}, \ldots, a_m\rangle + (-1)^m \langle a_1, \ldots, a_{m-1}\rangle.$$

The resulting terms $\partial(\langle a\rangle) = *a\langle\ \rangle - \langle\ \rangle$ for $m = 1$ in the above expression mean that the formal symbol $\langle\ \rangle$ is deleted. For $n = 0$, we define $\partial\langle y\rangle = 0$.

Example 3. The boundary maps in two and three dimensions are computed as follows.

$$\begin{aligned}
\partial_2(\langle y\rangle\langle a\rangle\langle b\rangle) &= \langle y*a\rangle\langle b\rangle - \langle y\rangle\langle b\rangle - \langle y*b\rangle\langle a*b\rangle + \langle y\rangle\langle a\rangle,\\
\partial_2(\langle y\rangle\langle a, b\rangle) &= \langle y*a\rangle\langle b\rangle - \langle y\rangle\langle ab\rangle + \langle y\rangle\langle a\rangle,\\
\partial_3(\langle y\rangle\langle a\rangle\langle b\rangle\langle c\rangle) &= \langle y*a\rangle\langle b\rangle\langle c\rangle - \langle y\rangle\langle b\rangle\langle c\rangle - \langle y*b\rangle\langle a*b\rangle\langle c\rangle\\
&\quad + \langle y\rangle\langle a\rangle\langle c\rangle + \langle y*c\rangle\langle a*c\rangle\langle b*c\rangle - \langle y\rangle\langle a\rangle\langle b\rangle,\\
\partial_3(\langle y\rangle\langle a\rangle\langle b, c\rangle) &= \langle y*a\rangle\langle b, c\rangle - \langle y\rangle\langle b, c\rangle - \langle y*b\rangle\langle a*b\rangle\langle c\rangle\\
&\quad + \langle y\rangle\langle a\rangle\langle bc\rangle - \langle y\rangle\langle a\rangle\langle b\rangle,\\
\partial_3(\langle y\rangle\langle a, b\rangle\langle c\rangle) &= \langle y*a\rangle\langle b\rangle\langle c\rangle - \langle y\rangle\langle ab\rangle\langle c\rangle + \langle y\rangle\langle a\rangle\langle c\rangle\\
&\quad + \langle y*c\rangle\langle a*c, b*c\rangle - \langle y\rangle\langle a, b\rangle,\\
\partial_3(\langle y\rangle\langle a, b, c\rangle) &= \langle y*a\rangle\langle b, c\rangle - \langle y\rangle\langle ab, c\rangle + \langle y\rangle\langle a, bc\rangle - \langle y\rangle\langle a, b\rangle.
\end{aligned}$$

Proposition 4. *$P_*(X)_Y = (P_n(X)_Y, \partial_n)$ is a chain complex.*

Proof. The Leibniz rule

$$\partial(\sigma\tau) = (\partial\sigma)\tau + (-1)^{|\sigma|}\sigma(\partial\tau)$$

is a restatement of the definition when $k = 2$. In fact, the general definition follows from this by induction. Also $\partial(\sigma * a) = (\partial\sigma) * a$, and $\partial \circ \partial = 0$ follows from these two facts. □

We will later define a degeneracy subcomplex that is analogous (albeit more complicated) to the subcomplex of degeneracies for quandle homology. Before its definition, we give a description of simplicial decompositions of products of simplices for motivation. We identify an n-simplex Δ^n with the set

$$\{(x_1, x_2, \dots, x_n) \in [0, 1]^n : 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\},$$

called the *right n-simplex*. Then the n-cube $[0, 1]^n$ can be decomposed into $n!$ sets each of which is congruent to this right n-simplex that has n edges of length 1, and has $(n-k+1)$ edges of length $\sqrt{k}$ for $k = 1, \dots, n$. More specifically, for $\vec{x} \in [0, 1]^n$ consider the permutation $\sigma \in \Sigma_n$ such that $0 \le x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)} \le 1$. If the coordinates of $\vec{x}$ are all distinct, then there is a unique such σ and an n-simplex Δ^n_σ congruent to the right n-simplex such that $\vec{x}$ lies in the interior of Δ^n_σ. Otherwise $\vec{x}$ lies in the boundary of more than one such simplex. Now consider the product of right simplices

$$\Delta^s \times \Delta^t = \left\{ (\vec{x}, \vec{y}) \in [0, 1]^{s+t} \;\middle|\; \begin{array}{c} 0 \le x_1 \le x_2 \le \cdots \le x_s \le 1 \\ 0 \le y_1 \le y_2 \le \cdots \le y_t \le 1 \end{array} \right\},$$

where the notation $(\vec{x}, \vec{y})$ represents $(x_1, \dots, x_s, y_1, \dots, y_t)$. This can be decomposed as a union of simplices of the form given above. For

$$\vec{z} = (\vec{x}, \vec{y}) \in \Delta^s \times \Delta^t \subset [0, 1]^n,$$

where $n = s + t$, there is an associated simplex Δ^n_σ that contains the point $(\vec{x}, \vec{y})$. Suppose all coordinates of $\vec{z}$ are distinct, and let $\sigma \in \Sigma_n$ be a permutation such that $0 < z_{\sigma(1)} < \cdots < z_{\sigma(n)}$. Then the subset $\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, s+t\}$ with $i_1 < i_2 < \cdots < i_s$ is determined from the positions of coordinates of $\vec{x}$, so that $z_{i_k} = x_k$ for $k = 1, \dots, s$. Thus a given subset $\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, s+t\}$ where $i_1 < i_2 < \cdots < i_s$ determines an n-simplex in the decomposition of $\Delta^s \times \Delta^t$. We proceed to the definition of the degeneracy subcomplex.

For an expression of the form $\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle$ in a chain in $P_n(X)_Y$, where $\langle \boldsymbol{a} \rangle = \langle a_1, \dots, a_s \rangle$ and $\langle \boldsymbol{b} \rangle = \langle b_1, \dots, b_t \rangle$ satisfy $a_i, b_j \in G_\lambda$ for all $i = 1, \dots, s$ and $j = 1, \dots, t$, let the notation $\langle \langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle \rangle_{i_1, \dots, i_s}$ represent $(-1)^{\sum_{k=1}^{s}(i_k - k)} \langle c_1, \dots, c_{s+t} \rangle$, where $1 \leq i_1 < \cdots < i_k < \cdots < i_s \leq s+t$, and

$$c_i = \begin{cases} a_k * (b_1 \cdots b_{i-k}) & \text{if } i = i_k, \\ b_{i-k} & \text{if } i_k < i < i_{k+1}. \end{cases}$$

If $i = k$ in the first case, then we regard $(b_1 \cdots b_{i-k})$ to be empty. For example, $\langle\langle a\rangle\langle b\rangle\rangle_1 = \langle a, b\rangle$, $\langle\langle a\rangle\langle b\rangle\rangle_2 = -\langle b, a*b\rangle$, and $\langle\langle a, b\rangle\langle c\rangle\rangle_{1,3} = -\langle a, c, b*c\rangle$. We also define the notation $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle$ by

$$\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle := \sum_{1\leq i_1<\cdots<i_s\leq s+t} \langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s}.$$

Define $D_n(X)_Y$ to be the subgroup of $P_n(X)_Y$ generated by the elements of the form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\cdots\langle \boldsymbol{a}_k\rangle - \langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle\cdots\langle \boldsymbol{a}_k\rangle,$$

where we implicitly assume the linearity of the notations $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s}$ and $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle$, that is,

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle\cdots\langle \boldsymbol{a}_k\rangle = \sum_{1\leq i_1<\cdots<i_{|\boldsymbol{a}|}\leq|\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle|} \langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_{|\boldsymbol{a}|}}\cdots\langle \boldsymbol{a}_k\rangle.$$

The chain group $D_n(X)_Y$ is called the group of *decomposition degeneracies*. We will see that $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$ in Section 7.

We remark that the elements of the form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle a\rangle\langle a\rangle\cdots\langle \boldsymbol{a}_k\rangle$$

belong to $D_n(X)_Y$.

For example, $D_2(X)_Y$ is generated by the elements of the form

$$\langle y\rangle\langle a\rangle\langle b\rangle - \langle y\rangle\langle a, b\rangle + \langle y\rangle\langle b, a*b\rangle,$$

and $D_3(X)_Y$ is generated by the elements of the form

$$\begin{aligned}
&\langle y\rangle\langle a\rangle\langle b\rangle\langle x\rangle - \langle y\rangle\langle a, b\rangle\langle x\rangle + \langle y\rangle\langle b, a*b\rangle\langle x\rangle, \\
&\langle y\rangle\langle x\rangle\langle b\rangle\langle c\rangle - \langle y\rangle\langle x\rangle\langle b, c\rangle + \langle y\rangle\langle x\rangle\langle c, b*c\rangle, \\
&\langle y\rangle\langle a, b\rangle\langle c\rangle - \langle y\rangle\langle a, b, c\rangle + \langle y\rangle\langle a, c, b*c\rangle - \langle y\rangle\langle c, a*c, b*c\rangle, \\
&\langle y\rangle\langle a\rangle\langle b, c\rangle - \langle y\rangle\langle a, b, c\rangle + \langle y\rangle\langle b, a*b, c\rangle - \langle y\rangle\langle b, c, a*(bc)\rangle
\end{aligned}$$

for $a, b, c \in G_\lambda$, $x \in X$.

Definition 5. The quotient complex of $P_*(X)_Y$ modulo decomposition degeneracies $D_*(X)_Y$ is denoted by $C_*(X)_Y = (C_n(X)_Y, \partial_n)$, where $C_n(X)_Y = P_n(X)_Y/D_n(X)_Y$. For an abelian group A, define the cochain complex $C^*(X; A)_Y = \mathrm{Hom}(C_*(X)_Y, A)$. Denote by $H_n(X)_Y$ the n-th homology group of $C_*(X)_Y$.

4. Algebraic aspects of the homology

In this section we study algebraic aspects of the homology theory we defined. Specifically, we characterize the first homology group, and show that a 2-cocycle defines an extension. For simplicity we consider the case $Y = \{y_0\}$ is a singleton, and we suppress the symbols $\langle y_0 \rangle$ whenever possible.

Let X be a multiple conjugation quandle, and $Y = \{y_0\}$ be a singleton. Then $P_0(X)_Y$ is infinite cyclic, generated by $\langle y_0 \rangle$, and $\partial_1(\langle y_0 \rangle \langle a \rangle) = \langle y_0 * a \rangle - \langle y_0 \rangle$ for $a \in X$. Hence $H_0(X)_Y = \mathbb{Z}$. If X is a multiple conjugation quandle consisting of a single group, $H_1(X)_Y \cong X^{\mathrm{ab}}$, since $P_1(X)_Y$ is the free abelian group generated by the elements $\langle y_0 \rangle \langle a \rangle$ ($a \in X$), and

$$\partial_2(\langle y_0 \rangle \langle a, b \rangle) = \langle y_0 \rangle \langle b \rangle - \langle y_0 \rangle \langle ab \rangle + \langle y_0 \rangle \langle a \rangle,$$
$$\partial_2(\langle y_0 \rangle \langle a \rangle \langle b \rangle) = -\langle y_0 \rangle \langle a * b \rangle + \langle y_0 \rangle \langle a \rangle = \partial_2(\langle y_0 \rangle \langle a, b \rangle) - \partial_2(\langle y_0 \rangle \langle b, b^{-1}ab \rangle).$$

Proposition 6. *Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, let $Y = \{y_0\}$ be a singleton, and A an abelian group. A map $\phi : P_2(X)_Y \to A$ is a 2-cocycle of $C^*(X)_Y$ if and only if $X \times A = \coprod_{\lambda \in \Lambda}(G_\lambda \times A)$ with*

$$(a, s) * (b, t) := \big(a * b, s + \phi(\langle a \rangle \langle b \rangle)\big) \text{ for } (a, s), (b, t) \in X \times A,$$
$$(a, s)(b, t) := \big(ab, s + t + \phi(\langle a, b \rangle)\big) \text{ for } (a, s), (b, t) \in G_\lambda \times A$$

is a multiple conjugation quandle, where $\phi(\langle y_0 \rangle \langle a \rangle \langle b \rangle)$ and $\phi(\langle y_0 \rangle \langle a, b \rangle)$ are respectively denoted by $\phi(\langle a \rangle \langle b \rangle)$ and $\phi(\langle a, b \rangle)$ for short. Further, $\big(e_\lambda, -\phi(\langle e_\lambda, e_\lambda \rangle)\big)$ is the identity of the group $G_\lambda \times A$, and $\big(a^{-1}, -s - \phi(\langle a, a^{-1} \rangle) - \phi(\langle e_\lambda, e_\lambda \rangle)\big)$ is the inverse of $(a, s) \in G_\lambda \times A$.

Proof. We show correspondences between cocycle conditions and MCQ conditions for the extension.

(1) *The correspondence between the cocycle condition $\phi\big(\partial_3(\langle a, b, c \rangle)\big) = 0$ and the associativity of a group.*

For $(a, s), (b, t), (c, u) \in G_\lambda \times A$, $\phi(\langle a, b \rangle) + \phi(\langle ab, c \rangle) = \phi(\langle b, c \rangle) + \phi(\langle a, bc \rangle)$ if and only if $\big((a, s)(b, t)\big)(c, u) = (a, s)\big((b, t)(c, u)\big)$, since

$$\big((a, s)(b, t)\big)(c, u) = \big(abc, s + t + u + \phi(\langle a, b \rangle) + \phi(\langle ab, c \rangle)\big),$$
$$(a, s)\big((b, t)(c, u)\big) = \big(abc, s + t + u + \phi(\langle b, c \rangle) + \phi(\langle a, bc \rangle)\big).$$

We note that $\phi(\langle a, b \rangle) + \phi(\langle ab, c \rangle) = \phi(\langle b, c \rangle) + \phi(\langle a, bc \rangle)$, or equivalently $\big((a, s)(b, t)\big)(c, u) = (a, s)\big((b, t)(c, u)\big)$ implies that $\phi(\langle a, e_\lambda \rangle) = \phi(\langle e_\lambda, c \rangle)$ and that $\phi(\langle b^{-1}, b \rangle) = \phi(\langle b, b^{-1} \rangle)$. These equalities respectively imply

$$(a, s) = (a, s)\big(e_\lambda, -\phi(\langle e_\lambda, e_\lambda \rangle)\big) = (e_\lambda, -\phi(\langle e_\lambda, e_\lambda \rangle))(a, s)$$

and

$$\begin{aligned}\big(e_\lambda, -\phi(\langle e_\lambda, e_\lambda\rangle)\big) &= (a,s)\big(a^{-1}, -s-\phi(\langle a, a^{-1}\rangle)-\phi(\langle e_\lambda, e_\lambda\rangle)\big)\\ &= \big(a^{-1}, -s-\phi(\langle a, a^{-1}\rangle)-\phi(\langle e_\lambda, e_\lambda\rangle)\big)(a,s).\end{aligned}$$

It follows that $\big(e_\lambda, -\phi(\langle e_\lambda, e_\lambda\rangle)\big)$ is the identity of the group $G_\lambda \times A$, and that $\big(a^{-1}, -s-\phi(\langle a, a^{-1}\rangle)-\phi(\langle e_\lambda, e_\lambda\rangle)\big)$ is the inverse of $(a,s) \in G_\lambda \times A$.

(2) *The correspondence between the degeneracy of ϕ on $D_2(X)_Y$ and the first axiom of MCQs.*

For $(a,s), (b,t) \in G_\lambda \times A$, $\phi(\langle a\rangle\langle b\rangle)+\phi(\langle b, a*b\rangle) = \phi(\langle a, b\rangle)$ if and only if $(b,t)\big((a,s)*(b,t)\big) = (a,s)(b,t)$, since

$$\begin{aligned}(b,t)\big((a,s)*(b,t)\big) &= \big(b(a*b), s+t+\phi(\langle a\rangle\langle b\rangle)+\phi(\langle b, a*b\rangle)\big),\\ (a,s)(b,t) &= \big(ab, s+t+\phi(\langle a, b\rangle)\big).\end{aligned}$$

(3) *The correspondence between the cocycle condition $\phi\big(\partial_3(\langle x\rangle\langle a, b\rangle)\big) = 0$ and the second axiom of MCQs.*

For $(x,r) \in X \times A$ and $(a,s), (b,t) \in G_\lambda \times A$,

$$\phi(\langle x\rangle\langle ab\rangle) = \phi(\langle x\rangle\langle a\rangle)+\phi(\langle x*a\rangle\langle b\rangle)$$

if and only if $(x,r)*\big((a,s)(b,t)\big) = \big((x,r)*(a,s)\big)*(b,t)$, since

$$\begin{aligned}(x,r)*\big((a,s)(b,t)\big) &= \big(x*(ab), r+\phi(\langle x\rangle\langle ab\rangle)\big),\\ \big((x,r)*(a,s)\big)*(b,t) &= \big((x*a)*b, r+\phi(\langle x\rangle\langle a\rangle)+\phi(\langle x*a\rangle\langle b\rangle)\big).\end{aligned}$$

Note $\phi(\langle x\rangle\langle ab\rangle) = \phi(\langle x\rangle\langle a\rangle)+\phi(\langle x*a\rangle\langle b\rangle)$, or equivalently $(x,r)*\big((a,s)(b,t)\big) = \big((x,r)*(a,s)\big)*(b,t)$, implies that $\phi(\langle x\rangle\langle e_\lambda\rangle) = 0$. Then we have

$$(a,s)*\big(e_\lambda, -\phi(\langle e_\lambda, e_\lambda\rangle)\big) = (a,s).$$

(4) *The correspondence between the cocycle condition $\phi\big(\partial_3(\langle a\rangle\langle b\rangle\langle c\rangle)\big) = 0$ and the third axiom of MCQs.*

For $(a,s), (b,t), (c,u) \in X \times A$,

$$\phi(\langle a\rangle\langle b\rangle)+\phi(\langle a*b\rangle\langle c\rangle) = \phi(\langle a\rangle\langle c\rangle)+\phi(\langle a*c\rangle\langle b*c\rangle)$$

if and only if $\big((a,s)*(b,t)\big)*(c,u) = \big((a,s)*(c,u)\big)*\big((b,t)*(c,u)\big)$, since

$$\begin{aligned}\big((a,s)*(b,t)\big)*(c,u) &= \big((a*b)*c, s+\phi(\langle a\rangle\langle b\rangle)+\phi(\langle a*b\rangle\langle c\rangle)\big),\\ \big((a,s)*(c,u)\big)*\big((b,t)*(c,u)\big) &= \big((a*c)*(b*c), s+\phi(\langle a\rangle\langle c\rangle)+\phi(\langle a*c\rangle\langle b*c\rangle)\big).\end{aligned}$$

(5) *The correspondence between the cocycle condition $\phi\big(\partial_3(\langle a, b\rangle\langle x\rangle)\big) = 0$ and the last axiom of MCQs.*

For $(x,r) \in X \times A$ and $(a,s), (b,t) \in G_\lambda \times A$,

$$\phi(\langle a, b\rangle)+\phi(\langle ab\rangle\langle x\rangle) = \phi(\langle a\rangle\langle x\rangle)+\phi(\langle b\rangle\langle x\rangle)+\phi(\langle a*x, b*x\rangle)$$

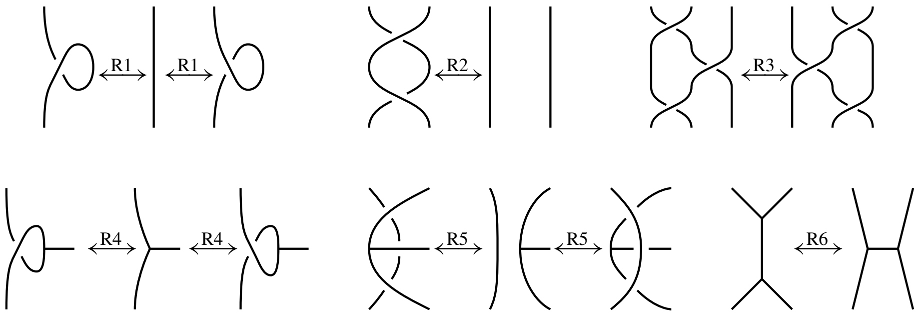


Figure 1. Reidemeister moves for handlebody-links.

if and only if $\big((a,s)(b,t)\big)*(x,r) = \big((a,s)*(x,r)\big)\big((b,t)*(x,r)\big)$, since

$$\begin{aligned}\big((a,s)(b,t)\big)*(x,r) &= \big((ab)*x, s+t+\phi(\langle a,b\rangle)+\phi(\langle ab\rangle\langle x\rangle)\big),\\ \big((a,s)*(x,r)\big)\big((b,t)*(x,r)\big) &= \big((a*x)(b*x),\\ &\qquad s+t+\phi(\langle a\rangle\langle x\rangle)+\phi(\langle b\rangle\langle x\rangle)+\phi(\langle a*x, b*x\rangle)\big).\end{aligned}$$

Therefore ϕ is a 2-cocycle if and only if $X\times A$ is a multiple conjugation quandle. $\square$

5. Quandle cocycle invariants for handlebody-links

The definition of a multiple conjugation quandle is motivated from handlebody-links and their colorings [Ishii 2015b]. A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this paper, a trivalent graph may contain circle components. Two handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of R1–R6 moves depicted in Figure 1 [Ishii 2008].

An S^1-*orientation* of a handlebody-link is an orientation of all genus 1 components of the handlebody-link, where an orientation of a solid torus is an orientation

Figure 2. Y-orientation.

of its core S^1. Two S^1-oriented handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other preserving the S^1-orientation. A *Y-orientation* of a spatial trivalent graph is an orientation of the graph without sources and sinks with respect to the orientation (see Figure 2). We note that the term Y-orientation is a symbolic convention, and has no relation to an X-set Y. A *diagram* of an S^1-oriented handlebody-link is a diagram of a Y-oriented spatial trivalent graph whose regular neighborhood is the S^1-oriented handlebody-link where the S^1-orientation is induced from the Y-orientation by forgetting the orientations except on circle components of the Y-oriented spatial trivalent graph. *Y-oriented* R1–R6 *moves* are R1–R6 moves between two diagrams with Y-orientations which are identical except in the disk where the move applied. Two S^1-oriented handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of Y-oriented R1–R6 moves [Ishii 2015a]. Note that in Figure 1 (R6), if all end points are oriented downward, then either choice of the two possible orientations of the middle edge makes the diagram Y-oriented locally. Thus reversing an orientation of this edge can be regarded as applying Y-oriented R6 moves twice. This is the case whenever both orientations of an edge give Y-orientations.

Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of an S^1-oriented handlebody-link H. We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. In this paper, an orientation of an arc is represented by the normal orientation obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. An *X-coloring* C of a diagram D is an assignment of an element of X to each arc $\alpha \in \mathcal{A}(D)$ satisfying the conditions depicted in the left three diagrams in Figure 3 at each crossing and each vertex of D. An X_Y-coloring C of D is an extension of an X-coloring of D which assigns an element of Y to each region $R \in \mathcal{R}(D)$ satisfying the condition depicted in the rightmost diagram in Figure 3 at each arc. We denote by $\mathrm{Col}_X(D)$ (resp. $\mathrm{Col}_X(D)_Y$) the set of X-colorings (resp. X_Y-colorings) of D. Then we have the following proposition.

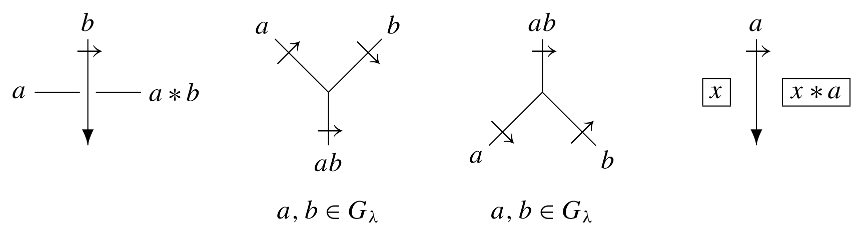
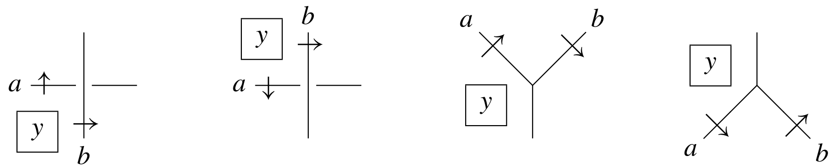


Figure 3. Rules of a coloring.



$\langle y\rangle\langle a\rangle\langle b\rangle$ $\quad -\langle y\rangle\langle a\rangle\langle b\rangle$ $\quad \langle y\rangle\langle a, b\rangle$ $\quad -\langle y\rangle\langle a, b\rangle$

Figure 4. Local chains represented by crossings and vertices.

Proposition 7 [Ishii 2015a]. *Let $X = \coprod_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of the Y-oriented* R1–R6 *moves to the diagram D once. For an X-coloring* (*resp. X_Y-coloring*) *C of D, there is a unique X-coloring* (*resp. X_Y-coloring*) *C' of D' which coincides with C except near a point where the move applied.*

For an X_Y-coloring C of a diagram D of an S^1-oriented handlebody-link, we define the *local chains* $w(\xi; C) \in C_2(X)_Y$ at each crossing ξ and each vertex ξ of D as depicted in Figure 4. We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_{\xi} w(\xi; C),$$

where ξ runs over all crossings and vertices of D. This is similar to the definitions found in [Carter et al. 2001] for links and surface-links, and in [Ishii and Iwakiri 2012] for handlebody-links.

Lemma 8. *The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ, θ' of $C^*(X; A)_Y$, we have*

$$\theta\bigl(W(D; C)\bigr) = \theta'\bigl(W(D; C)\bigr).$$

Proof. It is sufficient to show that $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. We denote by $\mathcal{SA}(D)$ the set of semiarcs of D, where a semiarc is a piece of a curve each of whose endpoints is a crossing or a vertex. We denote by $\mathcal{SA}(D; \xi)$ the set of semiarcs incident to ξ, where ξ is a crossing or a vertex of D.

For a semiarc α, there is a unique region R_α facing α such that the normal orientation of α points from the region R_α to the opposite region with respect to α. For a semiarc α incident to a crossing or a vertex ξ, we define

$$\epsilon(\alpha; \xi) := \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \xi, \\ -1 & \text{otherwise.} \end{cases}$$

Let $\chi_1, \ldots, \chi_4$ and $\omega_1, \omega_2, \omega_3$ be the semiarcs incident to a crossing χ and a vertex

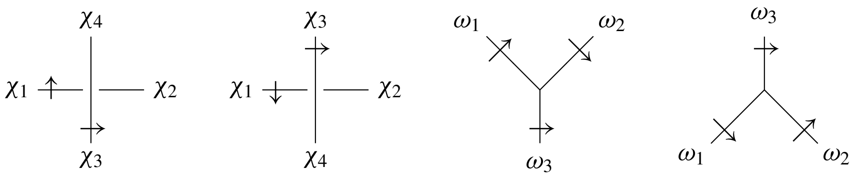


Figure 5. Semiarcs near crossings and vertices.

ω as depicted in Figure 5. From

$$\partial_2\big(w(\chi;C)\big)=\sum_{\alpha\in\mathcal{SA}(D;\chi)}\epsilon(\alpha;\chi)\langle C(R_\alpha)\rangle\langle C(\alpha)\rangle,$$
$$\partial_2\big(w(\omega;C)\big)=\sum_{\alpha\in\mathcal{SA}(D;\omega)}\epsilon(\alpha;\omega)\langle C(R_\alpha)\rangle\langle C(\alpha)\rangle,$$

it follows that

$$\partial_2\big(W(D;C)\big)=\sum_{\chi}\partial_2\big(w(\chi;C)\big)+\sum_{\omega}\partial_2\big(w(\omega;C)\big)=0,$$

where χ and ω, respectively, run over all crossings and vertices of D. $\square$

Lemma 9. *Let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of the Y-oriented* R1–R6 *moves to the diagram D once. Let C be an X_Y-coloring of D, let C' be the unique X_Y-coloring of D' such that C and C' coincide except near a point where the move applied. Then we have $[W(D;C)]=[W(D';C')]$ in $H_2(X)_Y$.*

Proof. We have the invariance under the Y-oriented R1 and R4 moves, since the difference between $[W(D;C)]$ and $[W(D';C')]$ is an element of $D_2(X)_Y$. The invariance under the Y-oriented R2 move follows from the signs of the crossings which appear in the move. We have the invariance under the Y-oriented R3, R5, and R6 moves, since the difference between $[W(D;C)]$ and $[W(D';C')]$ is an image of ∂_3. See Figure 6 for Y-oriented R6 moves, where all arcs are directed from top to bottom. $\square$

For a 2-cocycle θ of $C^*(X;A)_Y$, we define

$$\mathcal{H}(D):=\{[W(D;C)]\in H_2(X)_Y\mid C\in \mathrm{Col}_X(D)_Y\},$$
$$\Phi_\theta(D):=\{\theta(W(D;C))\in A\mid C\in \mathrm{Col}_X(D)_Y\}$$

as multisets. By Lemmas 8 and 9, we have the following theorem.

Theorem 10. *Let D be a diagram of an S^1-oriented handlebody-link H. Then $\mathcal{H}(D)$ and $\Phi_\theta(D)$ are invariants of H.*

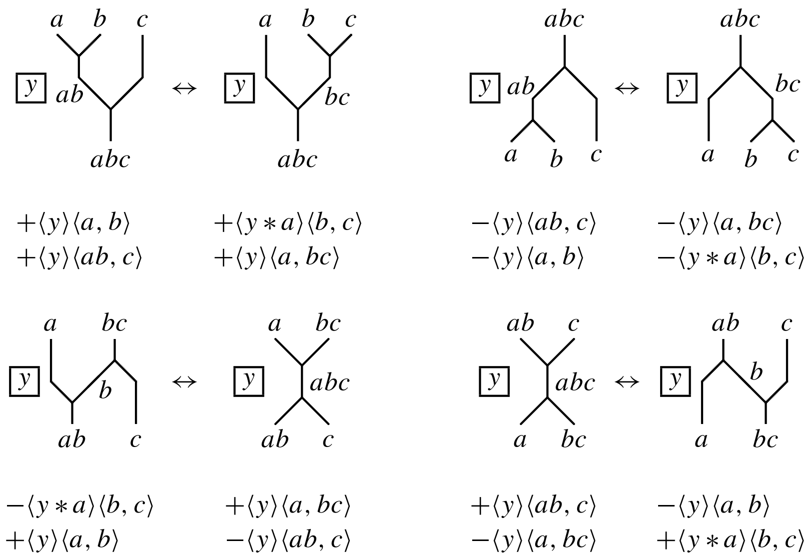


Figure 6. Chains for Y-oriented R6 moves.

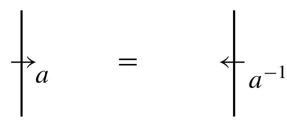


Figure 7. $(X,\uparrow)$-color.

For an S^1-oriented handlebody-link H, let H^* be the mirror image of H, and $-H$ be the S^1-oriented handlebody-link obtained from H by reversing its S^1-orientation. Then we also have

$$\mathcal{H}(-H^*) = -\mathcal{H}(H), \qquad \Phi_\theta(-H^*) = -\Phi_\theta(H),$$

where $-S = \{-a \mid a \in S\}$ for a multiset S. It is desirable to further study these invariants and applications to handlebody-links.

6. For unoriented handlebody-links

Let $X = \coprod_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of an (unoriented) handlebody-link H. An $(X,\uparrow)$-color C_α of an arc $\alpha \in \mathcal{A}(D)$ is a map C_α from the set of orientations of the arc α to X such that $C_\alpha(-o) = C_\alpha(o)^{-1}$, where $-o$ is the inverse of an orientation o. An $(X,\uparrow)$-color C_α is represented by a pair of an orientation o of α and an element $C_\alpha(o) \in X$ on the diagram D. Two pairs (o,a) and $(-o,a^{-1})$ represent the same $(X,\uparrow)$-color (see Figure 7).

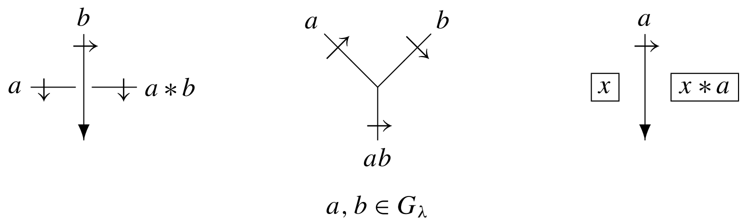


Figure 8. Rules of an unoriented coloring.

An $(X, \uparrow)$-*coloring* C of a diagram D is an assignment of an $(X, \uparrow)$-color C_α to each arc $\alpha \in \mathcal{A}(D)$ satisfying the conditions depicted in the left two diagrams in Figure 8 at each crossing and each vertex of D. An $(X, \uparrow)_Y$-coloring C of D is an extension of an $(X, \uparrow)$-coloring of D which assigns an element of Y to each region $R \in \mathcal{R}(D)$ satisfying the condition depicted in the rightmost diagram in Figure 8 at each arc. We denote by $\mathrm{Col}_{(X,\uparrow)}(D)$ (resp. $\mathrm{Col}_{(X,\uparrow)}(D)_Y$) the set of $(X, \uparrow)$-colorings (resp. $(X, \uparrow)_Y$-colorings) of D. The well-definedness of an $(X, \uparrow)$-coloring (resp. $(X, \uparrow)_Y$-coloring) follows from

$$(a^{-1})^{-1} = a, \qquad a^{-1} * b = (a*b)^{-1}, \quad (a*b)*b^{-1} = a,$$
$$b(ab)^{-1} = a^{-1}, \quad (ab)^{-1}a = b^{-1}.$$

The first three equalities are the defining conditions of a *good involution* considered in [Kamada 2007; Kamada and Oshiro 2010]. They used the notion of a good involution precisely to allow for appropriate changes of orientations. Following their arguments, we can show the following proposition in the same way as Proposition 7.

Proposition 11. *Let $X = \coprod_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of a handlebody-link H. Let D' be a diagram obtained by applying one of the* R1–R6 *moves to the diagram D once. For an $(X, \uparrow)$-coloring (resp. $(X, \uparrow)_Y$-coloring) C of D, there is a unique $(X, \uparrow)$-coloring (resp. $(X, \uparrow)_Y$-coloring) C' of D' which coincides with C except near a point where the move applied.*

Let $D_n^{\uparrow}(X)_Y$ be the subgroup of $P_n(X)_Y$ generated by the elements of the form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}\rangle \cdots \langle \boldsymbol{a}_k\rangle + \langle y\rangle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}\rangle_i^{-1} \cdots \langle \boldsymbol{a}_k\rangle,$$

where $\langle a_1, \dots, a_m\rangle_i^{-1}$ denotes

$$\begin{cases} *a_1\langle a_1^{-1}, a_1a_2, a_3, \dots, a_m\rangle & \text{if } i = 1, \\ \langle a_1, \dots, a_{i-2}, a_{i-1}a_i, a_i^{-1}, a_ia_{i+1}, a_{i+2}, \dots, a_m\rangle & \text{if } i \neq 1, m, \\ \langle a_1, \dots, a_{m-2}, a_{m-1}a_m, a_m^{-1}\rangle & \text{if } i = m. \end{cases}$$

The chain group $D_n^{\uparrow}(X)_Y$ will be called the *group of orientation degeneracies*. For example, $D_1^{\uparrow}(X)_Y$ is generated by the elements of the form

$$\langle y\rangle\langle a\rangle + \langle y*a\rangle\langle a^{-1}\rangle,$$

and $D_2^{\uparrow}(X)_Y$ is generated by the elements of the form

$$\begin{aligned}&\langle y\rangle\langle a\rangle\langle b\rangle + \langle y*a\rangle\langle a^{-1}\rangle\langle b\rangle, &&\langle y\rangle\langle a\rangle\langle b\rangle + \langle y*b\rangle\langle a*b\rangle\langle b^{-1}\rangle,\\ &\langle y\rangle\langle a,b\rangle + \langle y*a\rangle\langle a^{-1},ab\rangle, &&\langle y\rangle\langle a,b\rangle + \langle y\rangle\langle ab,b^{-1}\rangle.\end{aligned}$$

We remark that the elements of the form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle a_1,\ldots,a_m\rangle\cdots\langle \boldsymbol{a}_k\rangle - (-1)^{m(m+1)/2}\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots *(a_1\cdots a_m)\langle a_m^{-1},\ldots,a_1^{-1}\rangle\cdots\langle \boldsymbol{a}_k\rangle$$

belong to $D_n^{\uparrow}(X)_Y$. Furthermore, we can prove that the elements of the form

$$\begin{aligned}&\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle a_1,\ldots,a_m\rangle\cdots\langle \boldsymbol{a}_k\rangle - (-1)^{i(i+1)/2}\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\\ &\qquad\qquad *(a_1\cdots a_i)\langle a_i^{-1},\ldots,a_1^{-1},a_1\cdots a_{i+1},a_{i+2},\ldots,a_m\rangle\cdots\langle \boldsymbol{a}_k\rangle\end{aligned}$$

belong to $D_n^{\uparrow}(X)_Y$ by induction.

Lemma 12. *$D_*^{\uparrow}(X)_Y = (D_n^{\uparrow}(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$.*

Proof. We have $\partial_n(D_n^{\uparrow}(X)_Y) \subset D_{n-1}^{\uparrow}(X)_Y$, since

$$\begin{aligned}&\partial(\langle a_1,\ldots,a_m\rangle + *a_1\langle a_1^{-1}, a_1a_2, a_3,\ldots,a_m\rangle)\\ &\qquad = \langle a_1, a_2a_3, a_4,\ldots,a_m\rangle + *a_1\langle a_1^{-1}, a_1a_2a_3, a_4,\ldots,a_m\rangle\\ &\qquad\quad + \sum_{i=3}^{m-1}(-1)^i\big(\langle a_1,\ldots,a_ia_{i+1},a_{i+2},\ldots,a_m\rangle\\ &\qquad\qquad + *a_1\langle a_1^{-1}, a_1a_2, a_2,\ldots,a_ia_{i+1},a_{i+2},\ldots,a_m\rangle\big)\\ &\qquad\quad + (-1)^m(\langle a_1,\ldots,a_{m-1}\rangle + *a_1\langle a_1^{-1}, a_1a_2, a_3,\ldots,a_{m-1}\rangle)\end{aligned}$$

and

$$\begin{aligned}&\partial(\langle a_1,\ldots,a_m\rangle + \langle a_1,\ldots,a_{i-1}a_i, a_i^{-1}, a_ia_{i+1}, a_{i+2},\ldots,a_m\rangle)\\ &\quad = *a_1\langle a_2,\ldots,a_m\rangle + *a_1\langle a_2,\ldots,a_{i-1}a_i, a_i^{-1}, a_ia_{i+1}, a_{i+2},\ldots,a_m\rangle\\ &\quad + \sum_{j=1}^{i-2}(-1)^j\big(\langle a_1,\ldots,a_ja_{j+1},a_{j+2},\ldots,a_m\rangle\\ &\qquad + \langle a_1,\ldots,a_ja_{j+1},a_{j+2},\ldots,a_{i-1}a_i, a_i^{-1}, a_ia_{i+1},a_{i+2},\ldots,a_m\rangle\big)\\ &\quad + \sum_{j=i+1}^{m-1}(-1)^j\big(\langle a_1,\ldots,a_ja_{j+1},a_{j+2},\ldots,a_m\rangle\\ &\qquad + \langle a_1,\ldots,a_{i-1}a_i, a_i^{-1}, a_ia_{i+1},a_{i+2},\ldots,a_ja_{j+1},\ldots,a_m\rangle\big)\\ &\quad + (-1)^m(\langle a_1,\ldots,a_{m-1}\rangle + \langle a_1,\ldots,a_{i-1}a_i, a_i^{-1}, a_ia_{i+1},a_{i+2},\ldots,a_{m-1}\rangle).\end{aligned}$$

Thus $D_*^{\uparrow}(X)_Y$ is a subcomplex of $P_*(X)_Y$. $\square$

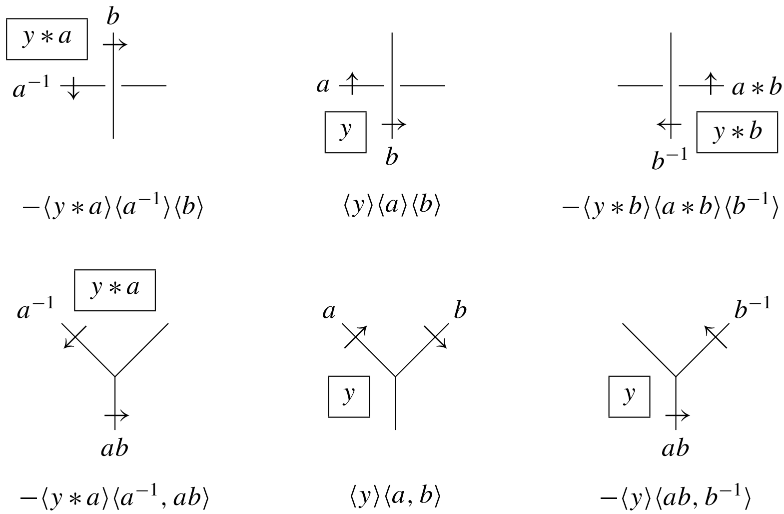


Figure 9. Well-definedness of local chains for unoriented handle-body-links.

Definition 13. We set $C_n^{\uparrow}(X)_Y = P_n(X)_Y/(D_n(X)_Y + D_n^{\uparrow}(X)_Y)$. The quotient complex $(C_n^{\uparrow}(X)_Y, \partial_n)$ is denoted by $C_*^{\uparrow}(X)_Y$. For an abelian group A, we define the cochain complex $C^*_{\uparrow}(X; A)_Y = \mathrm{Hom}(C_*^{\uparrow}(X)_Y, A)$. We denote by $H_n^{\uparrow}(X)_Y$ the n-th homology group of $C_*^{\uparrow}(X)_Y$.

For an $(X, \uparrow)_Y$-coloring C of a diagram D for a handlebody-link, we define the *local chain*s $w(\xi; C)$ at each crossing ξ and each vertex ξ of D as depicted in Figure 4. The local chain is well-defined, since

$$\begin{aligned} -\langle y*a\rangle\langle a^{-1}\rangle\langle b\rangle &= \langle y\rangle\langle a\rangle\langle b\rangle = -\langle y*b\rangle\langle a*b\rangle\langle b^{-1}\rangle, \\ -\langle y*a\rangle\langle a^{-1}, ab\rangle &= \langle y\rangle\langle a, b\rangle \ = -\langle y\rangle\langle ab, b^{-1}\rangle \end{aligned}$$

in $C_2^{\uparrow}(X)_Y$ (see Figure 9). Then we can define the chain $W(D; C) \in C_2^{\uparrow}(X)_Y$ in the same way as $W(D; C) \in C_2(X)_Y$, and obtain invariants $\mathcal{H}(H)$, $\Phi_\theta(H)$ for an (unoriented) handlebody-link H.

7. Simplicial decomposition

The goal of this section is to prove Lemma 15 stating that $D_*(X)_Y$ is a subcomplex. The formula of $D_2(X)_Y$, when $\langle y\rangle$ is omitted, is written as

$$\langle a\rangle\langle b\rangle - \langle a, b\rangle + \langle b, a*b\rangle,$$

and its geometric interpretation is depicted in Figure 10. In (A), a colored triangle representing $\langle a, b\rangle$ is depicted, as well as its dual graph with a trivalent vertex.

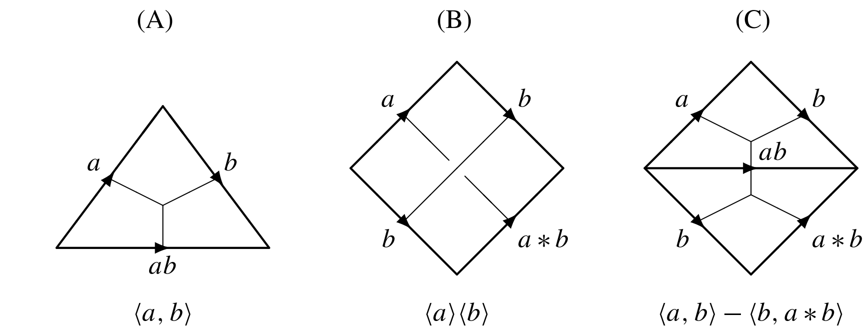


Figure 10. Dividing a square into triangles.

The colorings of such a graph were discussed in Section 5. A colored square representing $\langle a\rangle\langle b\rangle$ is depicted in (B), with the dual graph that corresponds to a crossing. In (C), a triangulation of the square is depicted, and after triangulation it represents $\langle a, b\rangle - \langle b, a * b\rangle$. Thus the triangulation corresponds to the above formula. This decomposition is found in [Carter et al. 2003].

At the same time, this equation corresponds to Y-oriented R4 moves in Figure 1 as follows. In Figure 11, colored diagrams of Y-oriented R4 moves are depicted. In the left diagram, the left-hand side represents the chain $\langle a\rangle\langle b\rangle + \langle b, a * b\rangle$ and the right-hand side represents $\langle a, b\rangle$. In the right diagram, the left-hand side represents the chain $-\langle a\rangle\langle b\rangle - \langle b, a * b\rangle$ and the right-hand side represents $-\langle a, b\rangle$. Thus the above equality is needed for colored diagrams to define equivalent chains in the quotient complex. A geometric interpretation of the last expression of $D_3(X)_Y$

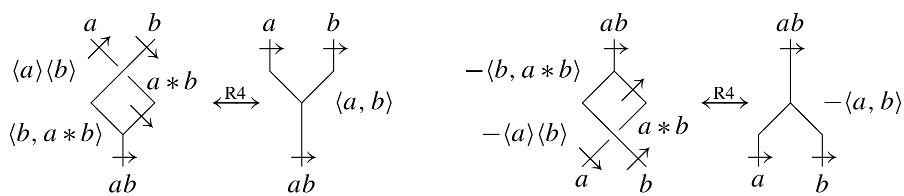


Figure 11. Colors for Y-oriented R4 moves.

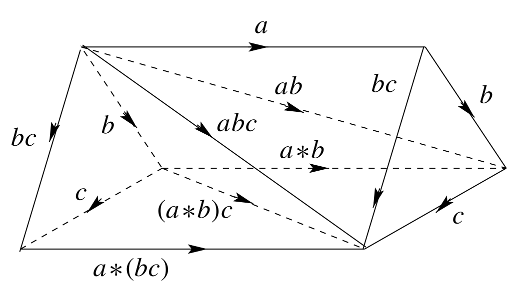


Figure 12. Decomposition of a prism into tetrahedra.

omitting $\langle y \rangle$,

$$\langle a \rangle \langle b, c \rangle - \langle a, b, c \rangle + \langle b, a * b, c \rangle - \langle b, c, a * (bc) \rangle$$

is found in Figure 12. The symbol $\langle a \rangle$ is represented by the horizontal 1-simplex, $\langle b, c \rangle$ is represented by the right triangular face, and $\langle a \rangle \langle b, c \rangle$ is represented by a prism. The term $\langle a, b, c \rangle$ corresponds to the right top tetrahedron in the prism. The expressions of the form $\langle \langle a \rangle \langle b, c \rangle \rangle_i$ provides a triangulation of a product of simplices. Each term corresponds to

$$\begin{aligned} \langle a, b, c \rangle &= \langle \langle a \rangle \langle b, c \rangle \rangle_1, \\ \langle b, a * b, c \rangle &= -\langle \langle a \rangle \langle b, c \rangle \rangle_2, \\ \langle b, c, a * (bc) \rangle &= \langle \langle a \rangle \langle b, c \rangle \rangle_3. \end{aligned}$$

Below we use the notation

$$\begin{aligned} \partial_{(0)} \langle x_1, \dots, x_m \rangle &= * x_1 \langle x_2, \dots, x_m \rangle, \\ \partial_{(i)} \langle x_1, \dots, x_m \rangle &= (-1)^i \langle x_1, \dots, x_i x_{i+1}, \dots, x_m \rangle, \\ \partial_{(m)} \langle x_1, \dots, x_m \rangle &= (-1)^m \langle x_1, \dots, x_{m-1} \rangle. \end{aligned}$$

Then the boundaries of $\langle \langle a \rangle \langle b, c \rangle \rangle_i$ are computed as

$$\langle \langle a \rangle \langle b, c \rangle \rangle_i \overset{\partial}{\longmapsto} \partial_{(0)} \langle \langle a \rangle \langle b, c \rangle \rangle_i + \partial_{(1)} \langle \langle a \rangle \langle b, c \rangle \rangle_i + \partial_{(2)} \langle \langle a \rangle \langle b, c \rangle \rangle_i + \partial_{(3)} \langle \langle a \rangle \langle b, c \rangle \rangle_i$$

and the right-hand sides for $i = 1, 2, 3$ are computed as follows:

$$\begin{aligned} \langle \langle a \rangle \langle b, c \rangle \rangle_1 &\overset{\partial}{\longmapsto} *a \langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle \\ &= \langle (\partial_{(0)} \langle a \rangle) \langle b, c \rangle \rangle_1 + \partial_{(1)} \langle \langle a \rangle \langle b, c \rangle \rangle_1 - \langle \langle a \rangle \partial_{(1)} \langle b, c \rangle \rangle_1 - \langle \langle a \rangle \partial_{(2)} \langle b, c \rangle \rangle_1, \\ \langle \langle a \rangle \langle b, c \rangle \rangle_2 &\overset{\partial}{\longmapsto} - * b \langle a * b, c \rangle + \langle b(a * b), c \rangle - \langle b, (a * b) c \rangle + \langle b, a * b \rangle \\ &= -\langle \langle a \rangle \partial_{(0)} \langle b, c \rangle \rangle_1 - \partial_{(1)} \langle \langle a \rangle \langle b, c \rangle \rangle_1 - \partial_{(2)} \langle \langle a \rangle \langle b, c \rangle \rangle_3 - \langle \langle a \rangle \partial_{(2)} \langle b, c \rangle \rangle_2, \\ \langle \langle a \rangle \langle b, c \rangle \rangle_3 &\overset{\partial}{\longmapsto} *b \langle c, a * (bc) \rangle - \langle bc, a * (bc) \rangle + \langle b, c(a * (bc)) \rangle - \langle b, c \rangle \\ &= -\langle \langle a \rangle \partial_{(0)} \langle b, c \rangle \rangle_2 - \langle \langle a \rangle \partial_{(1)} \langle b, c \rangle \rangle_2 + \partial_{(2)} \langle \langle a \rangle \langle b, c \rangle \rangle_3 + \langle (\partial_{(1)} \langle a \rangle) \langle b, c \rangle \rangle_1, \end{aligned}$$

where $\langle (\partial_{(i)} \langle a \rangle) \langle b, c \rangle \rangle_1$ is regarded as $(\partial_{(i)} \langle a \rangle) \langle b, c \rangle$. The canceling terms of the form $\partial_{(i)} \langle \langle a \rangle \langle b, c \rangle \rangle_j$ in the above boundaries correspond to internal triangles in Figure 12 that are shared by a pair of tetrahedra. Other terms are of the form $\langle \partial_{(i)} \langle a \rangle \langle b, c \rangle \rangle_j$ or $\langle \langle a \rangle \partial_{(i)} \langle b, c \rangle \rangle_j$, and they are outer triangles that constitute the boundary of the prism. The expression $\langle \partial_{(i)} \langle a \rangle \langle b, c \rangle \rangle_j$ represents the two triangles on the right and the left in Figure 12, since this represents

(boundary of the interval represented by $\langle a \rangle$) $\times$ (the triangle represented by $\langle b, c \rangle$).

Thus the outer boundary follows the pattern of Leibniz rule.

In terms of the coloring invariant of graphs, as in the case of the preceding relation for the Y-oriented R4 move, this relation corresponds to an equivalence of colored 2-complexes called foams, which are higher-dimensional analogues of the move depicted in Figure 11. See [Carter and Ishii 2012] for more on colored foams.

Lemma 14. *For $\langle \boldsymbol{a}\rangle = \langle a_1, \dots, a_s\rangle$ and $\langle \boldsymbol{b}\rangle = \langle b_1, \dots, b_t\rangle$ where $a_i, b_j \in G_\lambda$, we have*

$$\partial\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle = \langle(\partial\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle + (-1)^{|\boldsymbol{a}|}\langle\langle \boldsymbol{a}\rangle(\partial\langle \boldsymbol{b}\rangle)\rangle,$$

where $\langle\langle\cdot\rangle\langle\cdot\rangle\rangle$ is linearly extended.

Proof. By definition, we have

$$\partial\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle = \sum_{i=0}^{s+t} \partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle = \sum_{i=0}^{s+t} \sum_{1\le i_1<\cdots<i_s\le s+t} \partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_s}.$$

Direct computations show that

$$\begin{aligned}
&\partial_{(0)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_s}\\
&= \begin{cases} \langle(\partial_{(0)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_2-1,\dots,i_s-1} & \text{if } (i_1 = 1),\\ (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(0)}\langle \boldsymbol{b}\rangle)\rangle_{i_1-1,\dots,i_s-1} & \text{if } (i_1 > 1),\end{cases}\\
&\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_s}\\
&= \begin{cases} \langle(\partial_{(k)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_k,i_{k+2}-1,\dots,i_s-1} & \text{if } (i_k = i < i+1 = i_{k+1}),\\ -\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_{k-1},i_k+1,i_{k+1},\dots,i_s} & \text{if } (i_k = i < i+1 < i_{k+1}),\\ -\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_k,i_{k+1}-1,i_{k+2},\dots,i_s} & \text{if } (i_k < i < i+1 = i_{k+1}),\\ (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(i-k)}\langle \boldsymbol{b}\rangle)\rangle_{i_1,\dots,i_k,i_{k+1}-1,\dots,i_s-1} & \text{if } (i_k < i < i+1 < i_{k+1}),\end{cases}\\
&\partial_{(s+t)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_s}\\
&= \begin{cases} \langle(\partial_{(s)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_{s-1}} & \text{if } (i_s = s+t),\\ (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(t)}\langle \boldsymbol{b}\rangle)\rangle_{i_1,\dots,i_s} & \text{if } (i_s < s+t).\end{cases}
\end{aligned}$$

The terms of the form $-\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_{k-1},i_k+1,i_{k+1},\dots,i_s}$ $(i_k = i < i+1 < i_{k+1})$ and $-\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_k,i_{k+1}-1,i_{k+2},\dots,i_s}$ $(i_k < i < i+1 = i_{k+1})$ cancel in pairs. The other terms are organized as

$$\begin{aligned}
&\sum_{\substack{1\le i_1<\cdots\\<i_{s-1}\le s+t-1}} \sum_{i=0}^{s} \langle(\partial_{(i)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_{s-1}} + \sum_{\substack{1\le i_1<\cdots\\<i_s\le s+t-1}} \sum_{i=0}^{t} (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(i)}\langle \boldsymbol{b}\rangle)\rangle_{i_1,\dots,i_s}\\
&\qquad = \sum_{\substack{1\le i_1<\cdots\\<i_{s-1}\le s+t-1}} \langle(\partial\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_{s-1}} + \sum_{\substack{1\le i_1<\cdots\\<i_s\le s+t-1}} (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial\langle \boldsymbol{b}\rangle)\rangle_{i_1,\dots,i_s}\\
&\qquad = \langle(\partial\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle + (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial\langle \boldsymbol{b}\rangle)\rangle,
\end{aligned}$$

where $\langle\cdot\rangle_{i_1,\dots,i_s}$ is linearly extended. □

Since the Leibniz rule holds (by the preceding Lemma 14), we have the following.

Lemma 15. *$D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$.*

8. Chain map for simplicial decomposition

In this section we examine relations between group and MCQ homology theories.

8.1. ***Simplicial decomposition (general case).*** We observe an associativity of the notation $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle$ defined in Section 3, and extend the notation to multi-tuples. For an expression of the form $\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\langle \boldsymbol{c}\rangle$ in a chain in $P_*(X)_Y$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \bigcup_{m\in\mathbb{N}} G_\lambda^m$, it is easy to see that we have the following.

Lemma 16. $$\langle\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle\langle \boldsymbol{c}\rangle\rangle = \langle\langle \boldsymbol{a}\rangle\langle\langle \boldsymbol{b}\rangle\langle \boldsymbol{c}\rangle\rangle\rangle.$$

By Lemma 16, we can define $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\langle \boldsymbol{c}\rangle\rangle$ by $\langle\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle\langle \boldsymbol{c}\rangle\rangle = \langle\langle \boldsymbol{a}\rangle\langle\langle \boldsymbol{b}\rangle\langle \boldsymbol{c}\rangle\rangle\rangle$. Moreover, for an expression of the form $\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle$ in a chain in $P_*(X)_Y$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k \in \bigcup_{m\in\mathbb{N}} G_\lambda^m$, we can define $\langle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle\rangle$ inductively. By Lemma 14, this notation is compatible with the boundary homomorphism ∂ in the following sense.

Lemma 17. $$\partial\langle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle\rangle = \langle\partial(\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle)\rangle.$$

We give a direct formula (instead of induction) for the notation $\langle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle\rangle$ later in Section 8.3.

8.2. ***Chain map (from MCQ to group).*** Let $X = \coprod_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let $P_n^G(X)_Y$ be the subgroup of $P_n(X)_Y$ generated by the elements of the form $\langle y\rangle\langle \boldsymbol{a}\rangle$. Let $D_n^G(X)_Y$ and $D_n^{G,\uparrow}(X)_Y$ be respectively $P_n^G(X)_Y \cap D_n(X)_Y$ and $P_n^G(X)_Y \cap D_n^{\uparrow}(X)_Y$, which are the subgroups of $P_n^G(X)_Y$. Note that $D_n^G(X)_Y = P_n^G(X)_Y \cap D_n(X)_Y$ is the trivial group. We put

$$\begin{aligned} C_n^G(X)_Y &:= P_n^G(X)_Y / D_n^G(X)_Y = P_n^G(X)_Y, \\ C_n^{G,\uparrow}(X)_Y &:= P_n^G(X)_Y / (D_n^G(X)_Y + D_n^{G,\uparrow}(X)_Y) = P_n^G(X)_Y / D_n^{G,\uparrow}(X)_Y. \end{aligned}$$

Then $C_*^G(X)_Y = (C_n^G(X)_Y, \partial_n)$ and $C_*^{G,\uparrow}(X)_Y = (C_n^{G,\uparrow}(X)_Y, \partial_n)$ are chain complexes. If X is a group (regarded as $X = \coprod_{\lambda\in\Lambda} G_\lambda$ with Λ a singleton) and Y is a singleton, $C_*^G(X)_Y$ is essentially the same as the chain complex of the usual group homology. For an abelian group A, we define the cochain complexes

$$C_G^*(X; A)_Y = \mathrm{Hom}(C_*^G(X)_Y, A) \quad \text{and} \quad C_{G,\uparrow}^*(X; A)_Y = \mathrm{Hom}(C_*^{G,\uparrow}(X)_Y, A).$$

When X is a multiple conjugation quandle consisting of a single group, define homomorphisms $\Delta : P_*(X)_Y \to P_*^G(X)_Y$ by

$$\Delta(\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_m\rangle) := \langle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_m\rangle\rangle.$$

Then by Lemma 17 and from these definitions, we have the following.

Proposition 18. *The homomorphisms $\Delta : P_*(X)_Y \to P^G_*(X)_Y$ give rise to a chain homomorphism. Furthermore, Δ induces the chain homomorphisms $\Delta : C_*(X)_Y \to C^G_*(X)_Y$ and $\Delta : C^{\uparrow}_*(X)_Y \to C^{G,\uparrow}_*(X)_Y$.*

When $n = 0, 1$, the induced homomorphisms $\Delta : C_n(X)_Y \to C^G_n(X)_Y$ and $\Delta : C^{\uparrow}_n(X)_Y \to C^{G,\uparrow}_n(X)_Y$ are identities. Furthermore $H_n(X)_Y \cong H^G_n(X)_Y$ and $H^{\uparrow}_n(X)_Y \cong H^{G,\uparrow}_n(X)_Y$ for $n = 0, 1$. We note that the chain homomorphisms Δ are defined only for an MCQ consisting of a single group. In this case, we also have the cochain homomorphisms $\Delta : C^*_G(X; A)_Y \to C^*(X; A)_Y$ and $\Delta : C^*_{G,\uparrow}(X; A)_Y \to C^*_{\uparrow}(X; A)_Y$ for an abelian group A. Hence, for a given cocycle of group homology theory, we can obtain that of our theory through Δ. This approach will be discussed in Section 10.

Remark 19. We point out here that for a group $X = \mathbb{Z}_3$ and a trivial X-set Y, there is a group 2-cocycle η that satisfies the conditions in $C^2_{G,\uparrow}(X)_Y$ (coming from $D^{G,\uparrow}_n(X)_Y$),

$$\eta\langle a, b\rangle + \eta\langle a^{-1}, ab\rangle = 0 \quad \text{and} \quad \eta\langle a, b\rangle + \eta\langle ab, b^{-1}\rangle = 0.$$

Specifically, let $\eta : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ denote the function that has values $\eta(1, 1) = 1$, $\eta(2, 2) = 2$ and $\eta(g, h) = 0$ otherwise. It is a direct calculation that the condition above is satisfied. Furthermore, to see that η is a cocycle, consider the generating cocycle over $G = \mathbb{Z}_p$ where p is a prime that is defined by

$$\eta_0(x, y) = (1/p)(\bar{x} + \bar{y} - \overline{x + y}) \pmod{p},$$

where $\bar{x}$ is an integer $0 \le \bar{x} < p$ such that $\bar{x} = x \pmod{p}$. It is known that η_0 is a generating 2-cocycle for $H^2_G(\mathbb{Z}_p; \mathbb{Z}_p)$ for prime p. For $p = 3$, let ζ be a 1-chain defined by $\zeta(0) = 0$ and $\zeta(1) + \zeta(2) = 2$. Then one can easily compute that $\eta = \eta_0 + \delta\zeta$. Hence there is a 2-cocycle $\eta \in C^2_{G,\uparrow}(X)_Y$ of our theory that is cohomologous to the standard group 2-cocycle η_0.

8.3. ***Simplicial decomposition (direct formula).*** We give a direct formula (instead of induction) for the notation $\langle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle\rangle$. To the term $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_s}$, we associate a vector $\boldsymbol{v} = (v_1, \dots, v_n) \in \{1, 2\}^n$ by defining $v_i = 1$ if $i = i_j$ for some j, and otherwise $v_i = 2$, where $n = s + t$. In the term

$$c_i = \begin{cases} a_k * (b_1 \cdots b_{i-k}) & \text{if } i = i_k, \\ b_{i-k} & \text{if } i_k < i < i_{k+1}, \end{cases}$$

the first entry with a_k in it corresponds to $v_i = 1$ and the second with b_{i-k} to $v_i = 2$. We note that the term a_k came from the first part $\langle \boldsymbol{a}\rangle$ in $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\dots,i_s}$ so that $v_i = 1$ is assigned, and the term b_{i-k} belongs to the second part $\langle \boldsymbol{b}\rangle$ receiving $v_i = 2$.

Example 20. For the term $\langle a\rangle\langle b, c\rangle$ discussed for Figure 12, the terms $\langle a, b, c\rangle$, $-\langle b, a * b, c\rangle$, and $\langle b, c, a * (bc)\rangle$ correspond to the vectors $(1, 2, 2)$, $(2, 1, 2)$, and

$(2, 2, 1)$, respectively. Note that $(2, 1, 2)$ is obtained from $(1, 2, 2)$ by a transposition of the first two entries, and this is reflected in Figure 12 by the fact that the tetrahedra represented by these vectors share a triangular internal face. We indicate by an edge between two vectors when one is obtained from the other by a transposition of consecutive entries. In this case we draw the graph:

$$(1, 2, 2) — (2, 1, 2) — (2, 2, 1).$$

For $\langle a, b\rangle\langle c, d\rangle$, the terms $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s}$ are listed as $\langle a, b, c, d\rangle$, $-\langle a, c, b{*}c, d\rangle$, $\langle c, a{*}c, b{*}c, d\rangle$, $\langle a, c, d, b{*}(cd)\rangle$, $-\langle c, a{*}c, d, b{*}(cd)\rangle$, $\langle c, d, a{*}(cd), b{*}(cd)\rangle$, and these correspond to vectors

$$(1, 1, 2, 2),\ (1, 2, 1, 2),\ (2, 1, 1, 2),\ (1, 2, 2, 1),\ (2, 1, 2, 1),\ (2, 2, 1, 1),$$

respectively. They are connected by edges as

$$\begin{array}{ccccc} & & (2,1,1,2) & & \\ & \diagup & & \diagdown & \\ (1,1,2,2) - (1,2,1,2) & & & & (2,1,2,1) - (2,2,1,1) \\ & \diagdown & & \diagup & \\ & & (1,2,2,1) & & \end{array}$$

indicating which simplices share internal faces. Note that from a vector $\boldsymbol{v} = (v_1, \ldots, v_n) \in \{1, 2\}^n$ the subscripts $i_1, \ldots, i_s$ in $\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s}$ are recovered by the condition $v_{i_j} = 1$.

For an expression of the form $\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle$ in a chain in $P_*(X)_Y$, where

$$\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k \in \bigcup_{m\in\mathbb{N}} G_\lambda^m,$$

we put $n = |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_k|$ and consider vectors $\boldsymbol{v} = (v_1, \ldots, v_n) \in \{1, \ldots, k\}^n$, and denote by $\#_j^i \boldsymbol{v}$ the number of j's in $v_1, \ldots, v_i$. Then for a given $\boldsymbol{v}$ define $i(j, 1) < \cdots < i(j, n_j)$ by the condition that $v_{i(j,1)} = \cdots = v_{i(j,n_j)} = j$.

With these notations in hand, we temporarily define $\langle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle\rangle'$ by

$$\sum_{\substack{\boldsymbol{v}\in\{1,\ldots,k\}^n \\ \#_j^n \boldsymbol{v}=n_j\ (j=1,\ldots,k)}} (-1)^{\sum_{j=1}^{k-1}\sum_{t=1}^{n_j}(i(j,t)-t-\sum_{s=1}^{j-1} n_s)}\langle c_1, \ldots, c_n\rangle$$

for $\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle = \langle a_{1,1}, \ldots, a_{1,n_1}\rangle \cdots \langle a_{k,1}, \ldots, a_{k,n_k}\rangle$, where

$$c_i = a_{v_i,\#_{v_i}^i \boldsymbol{v}} * \prod_{s=v_i+1}^{k} \prod_{t=1}^{\#_s^i \boldsymbol{v}} a_{s,t}.$$

Then we have $\langle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle\rangle' = \langle\langle \boldsymbol{a}_1\rangle \cdots \langle \boldsymbol{a}_k\rangle\rangle$, from the fact that simplices of both

$$
\begin{tikzcd}[column sep=tiny]
 & & & (1,2,2,3) \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & (2,1,2,3) \arrow[dl, dash] \arrow[dr, dash] & & (1,2,3,2) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (2,2,1,3) \arrow[d, dash] & & (2,1,3,2) \arrow[d, dash] & & (1,3,2,2) \arrow[d, dash] & \\
 & (2,2,3,1) \arrow[dr, dash] & & (2,3,1,2) \arrow[dl, dash] \arrow[dr, dash] & & (3,1,2,2) \arrow[dl, dash] & \\
 & & (2,3,2,1) \arrow[dr, dash] & & (3,2,1,2) \arrow[dl, dash] & & \\
 & & & (3,2,2,1) & & &
\end{tikzcd}
$$

Figure 13. Boundaries of $\langle a\rangle\langle b, c\rangle\langle d\rangle$.

sides are in one-to-one correspondence with vectors $\boldsymbol{v} = (v_1, \dots, v_n) \in \{1, \dots, k\}^n$, and the signs correspond to the number of transpositions, modulo 2, of a given vector $\boldsymbol{v}$ from the vector $(1, \dots, 1, 2, \dots, 2, \dots, k, \dots, k)$.

Example 21. The terms of $\langle\langle a\rangle\langle b, c\rangle\langle d\rangle\rangle$ consist of

$$
\begin{array}{lll}
\langle a, b, c, d\rangle, & \langle b, a*b, c, d\rangle, & \langle a, b, d, c*d\rangle, \\
\langle b, c, a*(bc), d\rangle, & \langle b, a*b, d, c*d\rangle, & \langle a, d, b*d, c*d\rangle, \\
\langle b, c, d, a*(bcd)\rangle, & \langle b, d, a*(bd), c*d\rangle, & \langle d, a*d, b*d, c*d\rangle, \\
\langle b, d, c*d, a*(bcd)\rangle, & \langle d, b*d, a*(bd), c*d\rangle, & \langle d, b*d, c*d, a*(bcd)\rangle,
\end{array}
$$

which, respectively, correspond to the vectors

$$
\begin{array}{lll}
(1, 2, 2, 3), & (2, 1, 2, 3), & (1, 2, 3, 2), \\
(2, 2, 1, 3), & (2, 1, 3, 2), & (1, 3, 2, 2), \\
(2, 2, 3, 1), & (2, 3, 1, 2), & (3, 1, 2, 2), \\
(2, 3, 2, 1), & (3, 2, 1, 2), & (3, 2, 2, 1).
\end{array}
$$

The graph representing shared faces is depicted in Figure 13.

9. Relationship between MCQ and IIJO

Let $X = \coprod_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let $P_n^{\mathrm{IIJO}}(X)_Y$ be the subgroups of $P_n(X)_Y$ generated by the elements of the form $\langle y\rangle\langle a_1\rangle \cdots \langle a_n\rangle$. Then $P_*^{\mathrm{IIJO}}(X)_Y = (P_n^{\mathrm{IIJO}}(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$. Let $D_n^{\mathrm{IIJO}}(X)_Y$ be the subgroup of $P_n^{\mathrm{IIJO}}(X)_Y$ generated by the elements of the forms

$$\langle y\rangle\langle a_1\rangle \cdots \langle b_1\rangle\langle b_2\rangle \cdots \langle a_n\rangle, \qquad \langle y\rangle\langle a_1\rangle \cdots \partial\langle b_1, b_2\rangle \cdots \langle a_n\rangle$$

IIJO	2-boundary	degenerate $D_2^{\mathrm{IIJO}}(X)_Y$	cancelled by sign	zero by definition
moves	R3	R4($\rightsquigarrow$ R1), R5($\rightsquigarrow$ ori.)	R2	R6

MCQ	2-boundary	degenerate $D_2(X)_Y$	degenerate $D_2^{\uparrow}(X)_Y$	cancelled by sign
moves	R3, R5, R6	R4($\rightsquigarrow$ R1)	orientation	R2

Table 1. Comparison between IIJO theory and MCQ theory

for $a_1, \dots, a_n \in X$ and $b_1, b_2 \in G_\lambda$. We note that the former elements relate to the invariance under the R1 and R4 move, and that the latter elements relate to the invariance under the R5 move and reversing orientation.

Lemma 22. *$D_*^{IIJO}(X)_Y = (D_n^{IIJO}(X)_Y, \partial_n)$ is a subcomplex of $P_*^{IIJO}(X)_Y$.*

Proof. This follows from

$$\partial(\langle b_1\rangle\langle b_2\rangle) = \partial\langle b_1, b_2\rangle - \partial\langle b_2, b_1 * b_2\rangle, \qquad \partial(\partial\langle b_1, b_2\rangle) = 0$$

for $b_1, b_2 \in G_\lambda$. □

We put

$$C_n^{\mathrm{IIJO}}(X)_Y = P_n^{\mathrm{IIJO}}(X)_Y / D_n^{\mathrm{IIJO}}(X)_Y.$$

Then $C_*^{\mathrm{IIJO}}(X)_Y = (C_n^{\mathrm{IIJO}}(X)_Y, \partial_n)$ is a chain complex. If X is obtained from a G-family of quandles as in Example 2, $C_*^{\mathrm{IIJO}}(X)_Y$ is the chain complex defined in [Ishii et al. 2013]. For an abelian group A, we define the cochain complexes

$$C^*_{\mathrm{IIJO}}(X; A)_Y = \mathrm{Hom}(C_*^{\mathrm{IIJO}}(X)_Y, A).$$

We note that a natural projection $\mathrm{pr}_* : P_*(X)_Y \to P_*^{\mathrm{IIJO}}(X)_Y$ does not induce a chain homomorphism $\mathrm{pr}_* : C_*(X)_Y \to C_*^{\mathrm{IIJO}}(X)_Y$, since IIJO homology theory is invariant under the invariance for reversing orientations. (See Table 1.) It is seen, however, that this map induces the chain homomorphism $\mathrm{pr}_* : C_*^{\uparrow}(X)_Y \to C_*^{\mathrm{IIJO}}(X)_Y$ and the cochain homomorphism $\mathrm{pr}^* : C^*_{\mathrm{IIJO}}(X; A)_Y \to C^*_{\uparrow}(X; A)_Y$ for an abelian group A. Hence, for a given cocycle of IIJO homology theory (with some modification for a multiple conjugation quandle as above), we can obtain that of our theory through pr^*. This implies that our invariant is a generalization of the IIJO quandle cocycle invariant.

10. Towards finding 2-cocycles

We discuss approaches to finding 2-cocycles that are not induced from the IIJO (co)homology theory. Let G be a group, M a right G-module, and A an abelian group. The module M and the set $X = M \times G$ $(= \coprod_{x \in M} \{x\} \times G)$ can be considered as a G-family of quandles and a multiple conjugation quandle as in Example 2, respectively.

We take an X-set Y as a singleton $\{y_0\}$ and suppress the notation $\langle y_0 \rangle$. For a 2-cocycle $\psi \in P^2(X; A)_Y$, we denote $\psi\big(\langle (x, g)\rangle \langle (y, h)\rangle\big)$ by $\phi\big((x, g), (y, h)\big)$, and $\psi\big(\langle (x, g), (x, h)\rangle\big)$ by $\eta_x(g, h)$. Then the 2-cocycle conditions are written as

(1) $\eta_x(g, h) + \eta_x(gh, k) = \eta_x(h, k) + \eta_x(g, hk)$,

(2) $\phi((x, g), (y, k)) + \phi((x, h), (y, k)) - \phi((x, gh), (y, k))$
$$= \eta_x(g, h) - \eta_{x *^k y}(g * k, h * k),$$

(3) $\phi((x, g), (y, h)) + \phi((x *^h y, g * h), (y, k)) = \phi((x, g), (y, hk))$,

(4) $\phi((x, g), (y, h)) + \phi((x *^h y, g * h), (z, k))$
$$= \phi((x, g), (z, k)) + \phi((x *^k z, g * k), (y *^k z, h * k)),$$

where $x, y, z \in M$ and $g, h, k \in G$. Furthermore, for a 2-cochain $\psi \in P^2(X; A)_Y$, the condition that ψ is a 2-cochain in $C^2(X; A)_Y$ is written as

(5) $\phi((x, g), (x, h)) = \eta_x(g, h) - \eta_x(h, g * h)$,

where $x \in M$ and $g, h \in G$.

Towards constructing MCQ 2-cocycles that are not from the IIJO homology, first we note that if ϕ above is an IIJO 2-cocycle, then ϕ satisfies the conditions (3),(4), and the condition that the LHS of (2) vanishes. By considering $\psi' = \psi - \phi$, we obtain an MCQ 2-cocycle ψ' that consists only of terms of η_x for $x \in M$. Thus we first consider such a case in Example 23 below. In this case, we can take an approach described in Section 8 for finding MCQ cocycles from group cocycles.

Example 23. For a 2-cochain $\psi \in P^2(X; A)_Y$ with the assumption

(0) $\psi(\langle (x, g)\rangle \langle (y, h)\rangle)$ $(= \phi((x, g), (y, h)))$ $= 0$,

we discuss what conditions are needed for the 2-cochain ψ being a 2-cocycle in $P^2(X; A)_Y$. When we use the notation $\eta_x(g, h)$ for $\psi(\langle (x, g), (x, h)\rangle)$, the 2-cocycle conditions are written as

(1) $\eta_x(g, h) + \eta_x(gh, k) = \eta_x(h, k) + \eta_x(g, hk)$,

(2′) $\eta_x(g, h) - \eta_{x *^k y}(g * k, h * k) = 0$,

where $x, y \in M$ and $g, h, k \in G$. We note that the condition (0) implies (3) and (4). Furthermore, for a 2-cochain $\psi \in P^2(X; A)_Y$ with the assumption (0), the condition that ψ is a 2-cochain in $C^2(X; A)_Y$ are written as

(5′) $\eta_x(g, h) - \eta_x(h, g * h) = 0$, where $x \in M$ and $g, h \in G$. Hence if ψ satisfies (0),(1), (2′) and (5′), then ψ is a 2-cocycle in $C^2(X; A)_Y$ and defines an invariant for handlebody-knots.

If $y = x$, then (2′) implies $\eta_x(g * k, h * k) = \eta_x(g, h)$, called the *right invariance* of η_x. If $x = 0$, then (2′) with right invariance implies $\eta_{y\cdot(1-k)} \equiv \eta_0$, which is another necessary condition for the condition (2′). Hence if any element in M can be represented by the form $y \cdot (1 - k)$ for some $y \in M$ and $k \in G$, then we have $\eta_x \equiv \eta_0$ for any $x \in M$. In this case, we can check that the 2-cocycle ψ in $C^2(X; A)_Y$ comes from the dual of the composition of the chain homomorphisms

$$C_*(X)_Y \xrightarrow{\mathrm{pr}_2} C_*(G)_Y \xrightarrow{\Delta} C_*^G(G)_Y,$$

where a chain homomorphism pr_2 is induced from a natural projection into the second factor and the chain homomorphism Δ was defined in Section 8.2. In this case, ψ assigned at a crossing is decomposed into a pair of weights η corresponding to trivalent vertices as depicted in Figure 10 (B) and (C). Hence the resulting invariant is equivalent to the invariant of the trivalent graph obtained by replacing all crossings with vertices, that is, embedded in the 2-sphere without crossing. Such an embedded graph is equivalent to a circle with small bubbles, and has trivial invariant value ($W(D; C) = 0$ for any coloring C). Thus, in this case, ψ defines a trivial invariant for handlebody-knots by the group 2-cocycle η_0, whose cohomology class may not be zero in $H^2_G(G; A)_Y$.

If the condition that any element in M can be represented by the form $y \cdot (1 - k)$ for some $y \in M$ and $k \in G$ is not satisfied, then ψ satisfying (0), (1), (2′) and (5′) may give rise to a nontrivial invariant for handlebody-links.

Example 24. In contrast to Example 23, next we consider the case when ϕ is not an IIJO 2-cocycle, so that the LHS of (2) does not vanish for ϕ.

For any G-invariant A-bilinear map $f : M^2 \to A$, Theorem 5.2 of [Nosaka 2013] claimed that the map $\phi_f : X^2 \to A$ defined by

$$\phi_f((x, g), (y, h)) := f(x - y, y \cdot (1 - h^{-1}))$$

satisfies the conditions (3) and (4) above. For the G-invariant A-bilinear map f, if we can find maps η_x such that the conditions (1) and (2) are also satisfied, then we obtain a 2-cocycle, which may be new. We remark here that ϕ_f itself can be modified as in [Nosaka 2013, Corollary 4.7] (by using an additive homomorphism form G to some commutative ring) so that the conditions (1) and (2) are also satisfied under the assumption $\eta_x \equiv 0$ for any $x \in M$.

The condition (1) merely says that η_x is a usual group 2-cocycle for any $x \in M$. The condition (2) is equivalent to

$$(2'') \quad f(x - y, y \cdot (1 - k^{-1})) = \eta_x(g, h) - \eta_{x *^k y}(g * k, h * k)$$

from the definition of f. If $y = x$, then (2″) implies that η_x is right invariant in the sense that $\eta_x(g * k, h * k) = \eta_x(g, h)$ as above. If $y = 0$, then (2″) with the right invariance implies $\eta_{x \cdot k} \equiv \eta_x$, called the *orbit dependence* of η_x. Thus we obtain these two necessary conditions for the condition (2″).

We examine the following specific examples. For a prime number p, let G denote $\mathrm{SL}(2, \mathbb{Z}_p)$ that acts on $M = (\mathbb{Z}_p)^2$ from the right. For $A = \mathbb{Z}_p$, the map $f : M^2 \to A$ defined by $f(x, y) := \det \begin{pmatrix} x \\ y \end{pmatrix}$ is a G-invariant A-bilinear map, where $x, y \in M$ are row vectors on which G acts on the right, and det denotes the determinant. This setting is motivated from [Nosaka 2013, Proposition 4.5].

First, we consider the case where $p = 2$. Define $m : M \to A$ by

$$m(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

Then we can check that

$$\phi_f\big((x, g), (y, h)\big) = -m(x) + m(x *^h y)$$

for any $x, y \in M$ and $g, h \in G$. Take $\eta_x(g, h)$ to be $m(x)$ for any $x \in M$ and $g, h \in G$. Then we can show that the 2-cochain ψ, defined by ϕ_f and η_x, is a 2-coboundary as follows. Define a 1-cochain $\tilde{m} \in P^1(X; A)$ by $\tilde{m}(\langle (x, g) \rangle) := m(x)$. Then the 2-coboundary $\delta\tilde{m} \in P^2(X; A)$ is written as

$$\begin{aligned} (\delta\tilde{m})(\langle (x, g) \rangle \langle (y, h) \rangle) &= -m(x) + m(x *^h y), \\ (\delta\tilde{m})(\langle (x, g), (x, h) \rangle) &= m(x), \end{aligned}$$

where $x, y \in M$ and $g, h \in G$. This implies that $\psi = \delta\tilde{m}$.

Second, we consider the case where $p > 2$. If $x = (0, 0)$ and $k = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, the condition (2″) implies $\eta_{2y}(g, h) = \eta_0(g, h)$ for any $y \in M$ and $g, h \in G$. Since p is odd, we have that $\eta_x \equiv \eta_0$ for any $x \in M$. If we substitute $y = (1, 0)$ and $k = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ for (2″), then LHS is 1 and RHS is 0, which turns out to be a contradiction. Hence there is no choice of η_x such that the conditions (1) and (2″) are satisfied.

Although our attempts have not resulted in new nontrivial 2-cocycles, it appears useful to record our approaches and facts we have found, for future endeavors towards constructing new cocycles using these approaches. Further studies are desirable on this homology theory, as it unifies group and quandle homology theories for a structure of multiple conjugation quandles, which have ample interesting examples and applications to handlebody-links.

Acknowledgements

Carter was partially supported by the Simons Foundation. Ishii was partially supported by JSPS KAKENHI Grant Number 24740037. Saito was partially supported by (U.S.) NIH R01GM109459. Tanaka was partially supported by JSPS KAKENHI Grant Number 26400082. The authors are grateful to Daniel Moskovich for valuable comments on exposition, and Yusuke Iijima for pointing out an error in an earlier version.

References

[Brieskorn 1988] E. Brieskorn, "Automorphic sets and braids and singularities", pp. 45–115 in *Braids* (Santa Cruz, CA, 1986), Contemp. Math. **78**, Amer. Math. Soc., Providence, RI, 1988. MR Zbl

[Carter and Ishii 2012] J. S. Carter and A. Ishii, "A knotted 2-dimensional foam with non-trivial cocycle invariant", preprint, 2012. arXiv

[Carter et al. 2001] J. S. Carter, S. Kamada, and M. Saito, "Geometric interpretations of quandle homology", *J. Knot Theory Ramifications* **10**:3 (2001), 345–386. MR Zbl

[Carter et al. 2003] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito, "Quandle cohomology and state-sum invariants of knotted curves and surfaces", *Trans. Amer. Math. Soc.* **355**:10 (2003), 3947–3989. MR Zbl

[Fenn and Rourke 1992] R. Fenn and C. Rourke, "Racks and links in codimension two", *J. Knot Theory Ramifications* **1**:4 (1992), 343–406. MR Zbl

[Fenn et al. 1995] R. Fenn, C. Rourke, and B. Sanderson, "Trunks and classifying spaces", *Appl. Categ. Structures* **3**:4 (1995), 321–356. MR Zbl

[Ishii 2008] A. Ishii, "Moves and invariants for knotted handlebodies", *Algebr. Geom. Topol.* **8**:3 (2008), 1403–1418. MR Zbl

[Ishii 2015a] A. Ishii, "The Markov theorems for spatial graphs and handlebody-knots with Y-orientations", *Internat. J. Math.* **26**:14 (2015), 1550116, 23. MR Zbl

[Ishii 2015b] A. Ishii, "A multiple conjugation quandle and handlebody-knots", *Topology Appl.* **196**:part B (2015), 492–500. MR Zbl

[Ishii and Iwakiri 2012] A. Ishii and M. Iwakiri, "Quandle cocycle invariants for spatial graphs and knotted handlebodies", *Canad. J. Math.* **64**:1 (2012), 102–122. MR Zbl

[Ishii et al. 2013] A. Ishii, M. Iwakiri, Y. Jang, and K. Oshiro, "A G-family of quandles and handlebody-knots", *Illinois J. Math.* **57**:3 (2013), 817–838. MR Zbl

[Joyce 1982] D. Joyce, "A classifying invariant of knots, the knot quandle", *J. Pure Appl. Algebra* **23**:1 (1982), 37–65. MR Zbl

[Kamada 2007] S. Kamada, "Quandles with good involutions, their homologies and knot invariants", pp. 101–108 in *Intelligence of low dimensional topology 2006*, Ser. Knots Everything **40**, World Scientific, Hackensack, NJ, 2007. MR Zbl

[Kamada and Oshiro 2010] S. Kamada and K. Oshiro, "Homology groups of symmetric quandles and cocycle invariants of links and surface-links", *Trans. Amer. Math. Soc.* **362**:10 (2010), 5501–5527. MR Zbl

[Matveev 1982] S. V. Matveev, "Distributive groupoids in knot theory", *Mat. Sb.* (*N.S.*) **119**:1 (1982), 78–88. In Russian; translated in *Math. USSR Sb.* **47**:1 (1984), 73–83. MR Zbl

[Nosaka 2013] T. Nosaka, "Quandle cocycles from invariant theory", *Adv. Math.* **245** (2013), 423–438. MR Zbl

[Przytycki 2011] J. H. Przytycki, "Distributivity versus associativity in the homology theory of algebraic structures", *Demonstratio Math.* **44**:4 (2011), 823–869. MR Zbl

[Takasaki 1943] M. Takasaki, "Abstraction of symmetric transformations", *Tôhoku Math. J.* **49** (1943), 145–207. In Japanese. MR

Received July 8, 2015. Revised July 15, 2016.

SCOTT CARTER
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SOUTH ALABAMA
ILB 325
MOBILE, AL 36688
UNITED STATES
carter@southalabama.edu

ATSUSHI ISHII
INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
1-1-1 TENNODAI
IBARAKI
TSUKUBA 305-8571
JAPAN
aishii@math.tsukuba.ac.jp

MASAHICO SAITO
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SOUTH FLORIDA
4202 E FOWLER AVE.
PHY 114
TAMPA, FL 33620
UNITED STATES
saito@usf.edu

KOKORO TANAKA
DEPARTMENT OF MATHEMATICS
TOKYO GAKUGEI UNIVERSITY
4-1-1 NUKUIKITA-MACHI
KOGANEI-SHI
TOKYO 184-8501
JAPAN
kotanaka@u-gakugei.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.49

THREE-DIMENSIONAL DISCRETE CURVATURE FLOWS AND DISCRETE EINSTEIN METRICS

HUABIN GE, XU XU AND SHIJIN ZHANG

A discrete version of the Einstein–Hilbert functional was introduced by Regge. In this paper, we define the discrete Einstein metrics as critical points of Regge's Einstein–Hilbert functional with normalization on triangulated 3-manifolds. We also introduce some discrete curvature flows, which are closely related to the existence of discrete Einstein metrics.

1. Introduction

For triangulated manifolds, the most natural metrics seem to be the piecewise linear metrics defined on all edges, satisfying some nondegenerate conditions so that each simplex in the triangulation can be realized as a Euclidean or hyperbolic simplex. In his work on constructing hyperbolic metrics on 3-manifolds, Thurston [1980] introduced the circle packing metric on a triangulated surface with prescribed intersection angles and further proved that this metric induces a piecewise linear metric. Similarly, for triangulated 3-manifolds, Cooper and Rivin [1996] introduced a ball (or sphere) packing metric. They endowed each vertex with a notion of combinatorial scalar curvature which is defined to be the angle defect of solid angles. Glickenstein [2005] introduced a type of discrete Yamabe flow, aiming at finding sphere packing metrics with constant combinatorial scalar curvature. In [Ge and Xu 2014], we also defined discrete quasi-Einstein metrics and gave some analytical conditions for the existence of discrete quasi-Einstein metrics by introducing two different discrete scalar curvature flows.

However, on one hand, similar to the 2-dimensional case, the ball packing metrics are special piecewise linear metrics and then too restrictive. On the other hand, the combinatorial curvatures studied above are all defined on vertices and may only be considered as an analogue of scalar curvature. As was pointed out by Regge [1961], the discrete curvatures are concentrated on codimension two simplexes. For these reasons, we want to study the general piecewise linear metrics and discrete curvatures defined on edges for 3-dimensional triangulated manifolds. In this paper,

MSC2010: 53C44.
Keywords: discrete Einstein–Hilbert functional, discrete Ricci curvature, discrete Ricci flow, discrete Einstein metric.

we shall study Regge's Einstein–Hilbert functional carefully, and give a definition of discrete Einstein metric. Moreover, we will introduce two types of discrete edge curvature flows; one is of second order, the other is of fourth order. Discrete edge curvature flow of second order may be considered as an analogue of smooth Ricci flow. However, discrete edge curvature flow of fourth order seems to be more powerful than the flow of second order.

2. Discrete Ricci curvature and discrete Einstein metric

Consider a compact 3-dimensional manifold M with a triangulation $\mathcal{T}$. The triangulation is written as $\mathcal{T} = \{V, E, F, T\}$, where V, E, F, T represent the set of vertices, edges, faces and tetrahedrons respectively. Denote $v_1, v_2, \dots, v_N$ as the vertices of $\mathcal{T}$, where N is the number of vertices. We often write i instead of v_i. A piecewise linear metric (written as PL-metric for short) is a map $l : E \to (0, +\infty)$ such that for any tetrahedron $\tau = \{i, j, k, l\} \in T$, the tetrahedron τ with edge lengths $l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl}$ can be realized as a Euclidean geometric tetrahedron. We may take PL-metrics as points in $\mathbb{R}^m_{>0}$, m times the Cartesian product of $(0, +\infty)$, where m is the number of edges in E. Not all points in $\mathbb{R}^m_{>0}$ represent PL-metrics and we need some nondegenerate conditions. For a start, the triangle inequality should be satisfied, but this alone is not enough. Consider a Euclidean tetrahedron $\tau = \{i, j, k, l\} \in T$ with edge lengths $l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl}$, then the volume of the Euclidean tetrahedron $\{i, j, k, l\}$ has the following formula due to Tartaglia in the sixteenth century:

$$\mathrm{V}_\tau^2 = \tfrac{1}{288} \det A_{ijkl},$$

where

$$A_{ijkl} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_{ij}^2 & l_{ik}^2 & l_{il}^2 \\ 1 & l_{ij}^2 & 0 & l_{jk}^2 & l_{jl}^2 \\ 1 & l_{ik}^2 & l_{jk}^2 & 0 & l_{kl}^2 \\ 1 & l_{il}^2 & l_{jl}^2 & l_{kl}^2 & 0 \end{pmatrix}.$$

So, $\mathrm{V}_\tau > 0$ for all tetrahedrons τ is another restriction for l to be a PL-metric.

For fixed 3-manifolds M with triangulation $\mathcal{T}$, denote the space of all admissible PL-metrics as

$$\mathfrak{M}_{\mathcal{T}} \triangleq \{l : E \to (0, +\infty) \text{ is a PL-metric on } (M^3, \mathcal{T})\},$$
$$\mathfrak{M}_{\mathcal{T}}^2 \triangleq \{l^2 : E \to (0, +\infty) \text{ is a PL-metric on } (M^3, \mathcal{T})\}.$$

Mei, Zhou and Ge proved $\mathfrak{M}_{\mathcal{T}}^2$ is a nonempty connected open convex cone, see Theorem 1.1 in [Ge et al. 2015] (see also Theorem 3.1 in [Schrader 2016]). On the other hand, it is easy to prove that $\mathfrak{M}_{\mathcal{T}}$ is homeomorphic to $\mathfrak{M}^2_{\mathcal{T}}$. Hence we know $\mathfrak{M}_{\mathcal{T}}$ is a simply connected open set. However, this set is not convex, due to an observation from [Rivin 2003].

Discrete Ricci curvature and the Einstein–Hilbert–Regge functional. Given a Euclidean tetrahedron $\{i, j, k, l\} \in T$, the dihedral angle at edge $\{i, j\}$ is denoted by $\beta_{ij,kl}$. If an edge is in the interior of the triangulation, the discrete Ricci curvature at this edge is 2π minus the sum of dihedral angles at the edge. More specifically, denote R_{ij} as the discrete Ricci curvature at the edge $\{i, j\}$, then

$$R_{ij} = 2\pi - \sum_{\{i,j,k,l\}\in T} \beta_{ij,kl}, \tag{2-1}$$

where the sum is taken over all tetrahedrons with $\{i, j\}$ as one of its edges. If this edge is on the boundary of the triangulation, then the discrete Ricci curvature should be $R_{ij} = \pi - \sum_{\{i,j,k,l\}\in T} \beta_{ij,kl}$.

For simplicity we will write l_{ij} and R_{ij} as $l_1, \dots, l_m$ and $R_1, \dots, R_m$, respectively, in the following, where m is the number of edges in E, and they are ordered sequentially. Set $l = (l_1, \dots, l_m)^T$, $R = (R_1, \dots, R_m)^T$, to be the transpose of $(l_1, \dots, l_m)$, $(R_1, \dots, R_m)$ respectively. We define the matrix L as

$$L = \frac{\partial(R_1, \dots, R_m)}{\partial(l_1, \dots, l_m)} = \begin{pmatrix} \frac{\partial R_1}{\partial l_1} & \cdots & \frac{\partial R_1}{\partial l_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial R_m}{\partial l_1} & \cdots & \frac{\partial R_m}{\partial l_m} \end{pmatrix}. \tag{2-2}$$

The Einstein–Hilbert–Regge functional was first introduced by Regge [1961] as

$$S = \sum_{i=1}^{m} R_i l_i, \tag{2-3}$$

and the discrete quadratic energy functional is defined to be

$$\mathcal{C}(l) = \|R\|^2 = \sum_{i=1}^{m} R_i^2. \tag{2-4}$$

By the Schläfli formula $\sum_{i=1}^{m} l_i \, dR_i = 0$, we have

$$dS = \sum_{i=1}^{m} R_i \, dl_i + \sum_{i=1}^{m} l_i \, dR_i = \sum_{i=1}^{m} R_i \, dl_i,$$

so

$$\nabla_l S = R \quad \text{and} \quad \mathrm{Hess}_l \, S = L,$$

which implies that the matrix L is symmetric. It is easy to get

$$\frac{\partial \mathcal{C}}{\partial l_j} = 2\sum_{i=1}^{m} \frac{\partial R_i}{\partial l_j} R_i,$$

so

$$\nabla_l \mathcal{C} = 2L^T R. \tag{2-5}$$

Since $R(tl_1, tl_2, \dots, tl_m) = R(l_1, l_2, \dots, l_m)$, we have the Euler formula

$$Ll = 0. \tag{2-6}$$

Discrete Einstein metric. The curvature R_{ij} is a combinatorial analogue of Ricci curvature in smooth cases. Fixing i, the sum of all R_{ij} with j connected to i is the curvature defined by Cooper and Rivin [1996], which is interpreted as the combinatorial scalar curvature. Inspired by the definition of discrete quasi-Einstein metric in [Ge and Xu 2014], we define the discrete Einstein metric as follows.

Definition 2.1. A PL-metric l is called a discrete Einstein metric, if there exists a constant λ such that $R = \lambda l$.

If l is a discrete Einstein metric, the corresponding PL-metric and curvature will be denoted by l_{DE} and R_{DE}, respectively, in the following. When $R = \lambda l$, or say l is a discrete Einstein metric, $\lambda = S/\|l\|^2$.

Definition 2.1 is a straightforward analogy of the smooth manifold case. R_{ij} is somewhat similar to smooth Ricci curvature Ric, and l_{ij} is somewhat similar to the smooth metric g. Then the Einstein metric g with $\mathrm{Ric} = \lambda g$ on smooth manifolds M can be transformed to a discrete Einstein metric l with $R = \lambda l$ on triangulated manifolds $(M^3, \mathcal{T})$. In this sense, the analogy seems to be only formal. However, for this type of metric, we can develop many more properties which suggest the use of the term discrete Einstein is appropriate.

In [Champion et al. 2011], Champion, Glickenstein and Young studied various normalized Einstein–Hilbert–Regge functionals and related discrete Yamabe invariants on triangulated manifolds with PL-metrics. In this paper, we shall introduce a new type of normalized Einstein–Hilbert–Regge functional, which is different from theirs. Fixing $(M^3, \mathcal{T})$, consider a new type of normalized Einstein–Hilbert–Regge functional

$$Q(l) = \frac{S}{\|l\|}. \tag{2-7}$$

It's easy to calculate

$$\nabla_l Q = \frac{1}{\|l\|}\left(\nabla_l S - \frac{S}{\|l\|^2} l\right) = \frac{1}{\|l\|}\left(R - \frac{S}{\|l\|^2} l\right).$$

Then we have:

Theorem 2.2. *On $(M^3, \mathcal{T})$ with* PL*-metric l, l is a discrete Einstein metric if and only if l is a critical point of the normalized Einstein–Hilbert–Regge functional Q.*

Theorem 2.2 is similar to the smooth case. On smooth manifolds, the metric g is Einstein if and only if it is a critical point of the functional

$$Q(g) = \frac{1}{V^{1/3}} \int_M R \, d\mu_g.$$

Fixing the triangulation, discrete curvatures R_{ij} are uniformly bounded, that is $(2-d)\pi < R_{ij} < 2\pi$, where d is the maximum edge degree of the triangulation. So

$$|Q(l)| = \left| \frac{S}{\|l\|} \right| = \left| \frac{R^T l}{\|l\|} \right| \leq \|R\|.$$

The Cauchy inequality indicates that l is a discrete Einstein metric if and only if $|Q(l)| = \|R\|$.

Using this type of normalized Einstein–Hilbert–Regge functional, we can introduce some new invariants associated to the triangulation $(M^3, \mathcal{T})$. The combinatorial Yamabe invariant with respect to $\mathcal{T}$ is defined as

$$Y_{M,\mathcal{T}} = \inf_{l \in \mathfrak{M}_{\mathcal{T}}} Q(l).$$

The admissible PL-metric space $\mathfrak{M}_{\mathcal{T}}$ for a given triangulated manifold $(M^3, \mathcal{T})$ may be considered as an analogue of the conformal class $[g_0]$ of a Riemannian manifold (M, g_0). Hence we may call $\mathfrak{M}_{\mathcal{T}}$ the combinatorial conformal class for $(M^3, \mathcal{T})$. It is uniquely determined by the triangulation $\mathcal{T}$. Moreover, we can introduce a topology invariant associated to M, i.e., $Y_M = \sup_{\mathcal{T}} Y_{M,\mathcal{T}}$, where the supremum is taken on all triangulations of M.

Similar to [Ge and Xu 2014; Ge and Xu 2016b], we can consider the following combinatorial Yamabe problem.

Question. Given a 3-dimensional manifold M with triangulation $\mathcal{T}$, how many discrete Einstein metrics are there in the combinatorial conformal class $\mathfrak{M}_{\mathcal{T}}$, and how to find them?

Inspired by work on the existence of combinatorial Gauss curvature in [Thurston 1980; Chow and Luo 2003; Luo 2004], we ask the following similar question:

Question. For a manifold M^3, find a suitable triangulation, or find topological and combinatorial obstructions, so that M admits discrete Einstein metrics.

The following is an example of a manifold with a triangulation admitting a discrete Einstein metric.

Example (the 16-cell). Consider the standard 3-dimensional sphere $\mathbb{S}^3$ embedded in $\mathbb{R}^4$. Taking the vertices of $\mathcal{T}$ to be $A_1 = (1,0,0,0)$, $A_2 = (-1,0,0,0)$, $B_1 = (0,1,0,0)$, $B_2 = (0,-1,0,0)$, $C_1 = (0,0,1,0)$, $C_2 = (0,0,-1,0)$, $D_1 = (0,0,0,1)$, $D_2 = (0,0,0,-1)$; the edges of $\mathcal{T}$ are $P_i Q_j$ (where $P \neq Q \in \{A, B, C, D\}$ and $i, j \in \{1,2\}$), the faces of $\mathcal{T}$ are $P_i Q_j R_k$ (where exactly two of $(P, Q, R) \in \{A, B, C, D\}$ are different, with $i, j, k \in \{1,2\}$), and the tetrahedrons of $\mathcal{T}$ are the regular tetrahedrons $A_i B_j C_k D_l$ (with $i, j, k, l \in \{1,2\}$). We know all edges have the same length $\frac{\pi}{2}$. It is easy to calculate that $R_{ij} = 2\pi - 4 \arccos \frac{1}{3}$ for all edges. So $R = (l/\pi)\big(4\pi - 8 \arccos \frac{1}{3}\big)$ and $l = \frac{\pi}{2}\{1, \dots, 1\}^T$ is a discrete Einstein metric associated to $(\mathbb{S}^3, \mathcal{T})$.

It is easy to see that the argument in this example works for any generalization of the platonic solids (uniform polychora) with tetrahedral cells, including the 5-cell (or pentachoron), the 600-cell, etc. In these cases, the PL-metric arises from symmetry and taking the lengths equal. It would be interesting to know whether these are the only triangulations that admit discrete Einstein metrics for the triangulation structure.

3. Combinatorial second order flow

Inspired by combinatorial curvature flow methods, we study discrete Einstein metrics by combinatorial curvature flows in the following sections. The two flows we introduce are negative gradient flows of some discrete functionals. One is the normalized Einstein–Hilbert–Regge functional $Q(l)$, which determines a normalized discrete curvature flow of second order. The other is the discrete quadratic energy $\mathcal{C} = \|R\|^2$, which determines a discrete curvature flow of fourth order.

Definition and evolution equations. We define the combinatorial second order flow as

$$\dot{l}(t)_{ij} = -R_{ij}, \quad \text{or} \quad \dot{l}(t) = -R. \tag{3-1}$$

It is useful to consider the normalized combinatorial second order flow

$$\dot{l}(t)_{ij} = \lambda l_{ij} - R_{ij}, \quad \text{or} \quad \dot{l}(t) = \lambda l - R, \tag{3-2}$$

where $\lambda = S/\|l\|^2$ and $\|l\|^2 = \sum_{i=1}^n l_i^2$.

Flows (3-1) and (3-2) differ from each other only by a change of scale in space and a change of parametrization in time. Let t, l, R, λ denote the variables for the flow (3-1), and $\tilde{t}, \tilde{l}, \widetilde{R}, \tilde{\lambda}$ for the flow (3-2). Suppose $l(t)$, $t \in [0, T)$, is a solution of (3-1). Set $\tilde{l}(\tilde{t}) = \varphi(t) l(t)$, where

$$\varphi(t) = \exp\left(\int_0^t \lambda(\tau)\, d\tau\right), \quad \tilde{t} = \int_0^t \varphi(\tau)\, d\tau.$$

Then we have

$$\tilde{\lambda} = \varphi^{-1}\lambda, \quad \widetilde{R} = R.$$

This gives

$$\frac{d\tilde{l}}{d\tilde{t}} = \frac{d\tilde{l}}{dt}\frac{dt}{d\tilde{t}} = (\lambda\varphi l - \varphi R)\varphi^{-1} = \tilde{\lambda}\tilde{l} - \widetilde{R}.$$

Conversely, if $\tilde{l}(\tilde{t}), \tilde{t} \in [0, \widetilde{T})$, is a solution of (3-2), set $l(t) = \varphi(\tilde{t})\,\tilde{l}(\tilde{t})$, where

$$\varphi(\tilde{t}) = \exp\biggl(-\int_0^{\tilde{t}} \tilde{\lambda}(\tau)\,d\tau\biggr), \quad t = \int_0^{\tilde{t}} \varphi(\tau)\,d\tau,$$

then it is easy to check that $dl/dt = -R$.

Notice that $\nabla_l Q = -(\lambda l - R)/\|l\|$ and $d\|l\|^2/dt = 2l^T\dot{l} = 2l^T(\lambda l - R) = 0$, hence we have:

Theorem 3.1. *Along the flow* (3-2), $\|l\|^2$ *is a constant. Moreover*, *the flow* (3-2) *is a negative gradient flow.*

We can take $\|l\|^2$ as a certain discrete "content" (here we use the word "content" instead of "volume", because the triangulated 3-manifolds have classical volume, that is, the sum of the volume of all tetrahedrons). It plays a similar role to "volume" in smooth cases. We also refer to the second order normalized discrete curvature flow (3-2) as the combinatorial Ricci flow. Moreover, we have the following evolution equations along this flow,

$$\dot{R} = \frac{\partial R}{\partial l}\dot{l} = L(-R + \lambda l) = -LR, \tag{3-3}$$

where we have used the Euler formula $Ll = 0$. So

$$\dot{C} = -2R^T LR, \tag{3-4}$$

and

$$\begin{aligned}\dot{S} &= \sum_{i=1}^{m} \dot{R}_i l_i + R_i \dot{l}_i = -\|R\|^2 + \lambda S \\ &= \frac{S^2 - \|l\|^2\|R\|^2}{\|l\|^2} = \frac{< R, l >^2 - \|l\|^2\|R\|^2}{\|l\|^2} \\ &= -\|R - \lambda l\|^2 = -\left\| R - \frac{S}{\|l\|^2} l \right\|^2 \\ &\leq 0.\end{aligned}$$

Hence

$$\dot{\lambda} = \frac{\dot{S}}{\|l\|^2} = -\left(\frac{\|R\|}{\|l\|}\right)^2 + \lambda^2 = -\frac{\|R - \lambda l\|^2}{\|l\|^2} \leq 0. \tag{3-5}$$

Since $\|l\|^2$ is invariant along the flow (3-2), we can always assume $l(0) \in \mathbb{S}^{m-1}$ and then $l(t) \in \mathbb{S}^{m-1}$ for all $t \in [0, T)$ in the following. Moreover, $\lambda = S/\|l\|^2 \equiv S$ along (3-2). It is easy to derive the following result.

Proposition 3.2. *The quadratic energy functional $\mathcal{C}$ is uniformly bounded on $\mathfrak{M}_{\mathcal{T}}$, where the bound depends only on the triangulation. The Einstein–Hilbert–Regge functional S is uniformly bounded on $\mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$. Moreover, along the discrete flow* (3-2), *S is nonincreasing and bounded.*

Remark. By the Schläfli formula, the differential 1-form $\omega = \sum_{i=1}^{m} R_i \, dl_i = dS$ is exact. Combining this with the fact that $\mathfrak{M}_{\mathcal{T}}$ is simply connected, we have

$$S(l) = \int_a^l \sum_{i=1}^{m} R_i \, dl_i + S(a),$$

where a is an arbitrary point of $\mathfrak{M}_{\mathcal{T}}$.

Nonsingular solution and singularity of solution. To study the convergence of the discrete Ricci flow (3-2), we need to classify the solutions of the flow.

Definition 3.3. A solution $l(t)$ of (3-2) is nonsingular if the solution exists for $t \in [0, +\infty)$ and $\{l(t)\} \Subset \mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$.

In fact, by $\{l(t)\} \Subset \mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$, we know that the solution of (3-2) exists for $t \in [0, +\infty)$. Furthermore, we have the following result for nonsingular solutions of (3-2).

Theorem 3.4. *If there exists a nonsingular solution for the discrete flow* (3-2), *there exists at least one discrete Einstein metric on $(M^3, \mathcal{T})$.*

Proof. Let $l(t)$, $t \in [0, +\infty)$, be a nonsingular solution of the flow (3-2). As S is descending and bounded from below along (3-2), $S(+\infty)$ exists. We can choose $t_n \uparrow +\infty$, such that

$$S'(t_n) = -\|\lambda(t_n) l(t_n) - R(t_n)\|^2 \to 0. \tag{3-6}$$

Using $\{l(t)\} \Subset \mathfrak{M}_{\mathcal{T}}$, we can further choose a subsequence t_{n_k} of t_n, such that $l(t_{n_k}) \to l^*$. Combining this with (3-6), we get $R^* = \lambda^* l^*$ and l^* is a discrete Einstein metric. □

If the solution of flow (3-2) converges to a nondegenerate PL-metric, the unit solution $\lambda(t)/\|\lambda\|$ must be nonsingular. First, assume the maximal time $T < +\infty$. Since $\lambda(T)$ is a nondegenerate PL-metric, the flow can be extended beyond T, so we obtain $T = +\infty$. Second, since $\lim_{t\to\infty} l(t) = l^* \in \mathfrak{M}_{\mathcal{T}}$, there exists $t_0 > 0$ such that $l(t)$ is close to l^* when $t > t_0$, so $l(t) \in \mathfrak{M}_{\mathcal{T}}$. On the other hand, for $t \in [0, t_0]$, $l(t) \in \mathfrak{M}_{\mathcal{T}}$. Hence we know $(\lambda(t)/\|\lambda\|) \Subset \mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}$.

Then we have the following corollary:

Corollary 3.5. *If the solution $l(t)$ of the discrete Ricci flow (3-2) exists for all time and converges to a nondegenerate* PL*-metric $l(+\infty)$, then there exists at least one discrete Einstein metric on $(M, \mathcal{T})$. Moreover, $l(+\infty)$ is a discrete Einstein metric.*

Definition 3.6. A maximal solution $l(t),\ t \in [0, T)$, of (3-2) is said to be singular if

$$\overline{\{l(t)\}} \cap \partial(\mathfrak{M}_{\mathcal{T}} \cap \mathbb{S}^{m-1}) \neq \varnothing.$$

We say the solution develops a type-I singularity at time T if there is an edge l_i and a sequence $t_n \to T$ such that $l_i(t_n) \to 0$. We say the solution develops a type-II singularity at time T if there is a sequence t_n approaching T such that $l_i(t_n)$ remains in a compact set of $\mathbb{R}_{>0}$ for all i and there is a tetrahedron $\tau = \{i, j, k, l\}$ in $\mathcal{T}$ such that $V_\tau \to 0$ as $t_n \to T$.

Remark. In [Bobenko et al. 2015; Ge and Jiang 2016a; Ge and Jiang 2016b; Luo 2011], the authors studied the degeneration of a triangle. In fact, they considered the generalized triangle, that is a topological triangle with three positive edge lengths. While the triangle inequality is not valid, they found that the definition of discrete Gaussian curvatures can be generalized to this case. However, we don't know how to do this degeneration for tetrahedrons, and hence we know very little about the degeneration behavior of a tetrahedron.

The following conjectures are likely to hold for the discrete flow (3-2).

Conjecture. *The normalized discrete Ricci flow (3-2) will not develop type-I singularity in finite time.*

Conjecture. *If no singularity develops along the normalized flow (3-2), the solution converges to a discrete Einstein metric as time approaches infinity.*

Just like Hamilton and Perelman's methods approaching smooth Ricci flow, whenever discrete curvature flow develops type-II singularity, we hope to continue the discrete flow by surgery which changes the combinatorial structure of the triangulation. We hope that discrete curvature flow converges to a discrete Einstein metric after a finite number of surgeries.

Convergence of the combinatorial second order flow. Finding good metrics is always a central topic in Riemannian geometry. In the last section, we proved that if the solution of the flow (3-2) exists for all time and converges to a nondegenerate PL-metric l_∞, the discrete Einstein metric exists. Moreover, l_∞ is such a metric. Conversely, we have:

Theorem 3.7. *Given a nondegenerate metric l, assume there exists a discrete Einstein metric l_{DE} such that $R_{DE} = \lambda l_{DE}$ with*

$$\lambda_{DE}\left(I_m - \frac{l_{DE} l_{DE}^T}{\|l_{DE}\|^2}\right) - L_{DE} \leq 0,$$

where I_m denotes the $m \times m$ identity matrix. Then there exists a constant $\varepsilon > 0$ such that, if

$$\|l(0) - l_{DE}\| < \varepsilon,$$

then the solution of normalized combinatorial second order flow (3-2) with initial metric $l(0) = l$ exists for all time and converges exponentially fast to the discrete Einstein metric l_{DE}.

Proof. We want to prove l_{DE} is a local attractor of the flow. For the evolution equation of the combinatorial two-order flow

$$\dot{l} = \Upsilon(l) = -R + \lambda l,$$

we have

$$\Upsilon(l_{DE}) = -R_{DE} + \lambda_{DE} l_{DE} = 0.$$

The differential of $\Upsilon(l) = -R + \lambda l$ at l is

$$\begin{aligned} D_l \Upsilon(l) &= -D_l R + \lambda D_l l + l D_l \Upsilon \\ &= -L^T + \lambda I_m + l\left(\frac{L^T l + R}{\|l\|^2} - \frac{2Sl}{\|l\|^4}\right)^T \\ &= \lambda I_m - L + \frac{lR^T}{\|l\|^2} - 2S\frac{ll^T}{\|l\|^4} \\ &= \lambda\left(I_m - \frac{ll^T}{\|l\|^2}\right) - L + \frac{l(R-\lambda l)^T}{\|l\|^2}, \end{aligned} \tag{3-7}$$

where we have used the symmetry of L and the Euler formula in the third equality. So

$$D_l \Upsilon(l)|_{l=l_{DE}} = \lambda_{DE}\left(I - \frac{l_{DE} l_{DE}^T}{\|l_{DE}\|^2}\right) - L_{DE} \leq 0,$$

and l_{DE} is a local attractor of the flow. The system is asymptotically stable at l_{DE}. If the initial metric $l(0)$ is close enough to l_{DE}, then the solution $l(t)$ exists for all time and converges to l_{DE} exponentially fast. □

4. Fourth order flow

In this section, we consider the combinatorial fourth order flow

$$\dot{l} = -L^T R, \tag{4-1}$$

where L^T denotes the transpose of L. Combining this with (2-5), we know that the combinatorial fourth order flow (4-1) is in fact a gradient flow of energy $\mathcal{C}$ (which is called discrete Calabi energy in [Ge 2013; Ge and Xu 2016a]), that is:

$$\dot{l} = -\tfrac{1}{2}\nabla_l \mathcal{C} = -L^T R. \tag{4-2}$$

It is easy to obtain the following evolution equations:

(4-3) $\dot{R} = -LL^T R,$

(4-4) $\dot{\mathcal{C}} = -2R^T LL^T R = -2(L^T R)^T (L^T R) = -\frac{1}{2}\|\nabla_l \mathcal{C}\|^2 \le 0,$

$$\dot{S} = (\dot{R})^T l + R^T \dot{l} = -(LL^T R)^T l + R^T(-L^T R) = -R^T LL^T l - R^T L^T R = -R^T L^T R.$$

If there is only a single tetrahedron in the triangulation, then $m = 6$ and it is easy to calculate $\mathrm{rank}(L) = 5$. Thus we guess $\mathrm{rank}(L) = m - 1$ for general triangulations.

Conjecture. $\mathrm{rank}(L) = m - 1$ *for each* $l \in \mathfrak{M}_{\mathcal{T}}$.

The above conjecture is hopefully true. If so, then l is the only solution (up to scaling) of matrix equation $Lx = 0$. Moreover, each nonsingular solution to the fourth order flow (4-1) contains a subsequence converging to a discrete Einstein metric.

Theorem 4.1. *If there exists a discrete Einstein metric* l_{DE} *with* $\mathrm{rank}(L_{DE}) = m-1$ *on* $(M^3, \mathcal{T})$, *then there exists a constant* $\varepsilon > 0$ *such that, for any initial metric* $l(0)$ *with*

$$\|l(0) - l_{DE}\| < \varepsilon,$$

the solution to combinatorial fourth order flow $\dot{l} = L^T(R_{DE} - R)$ *exists for all time* $t \ge 0$ *and converges exponentially fast to the metric* l_{DE}.

Proof. Along the normalized fourth order flow (4-1),

$$\dot{R} = \frac{\partial R}{\partial l}\dot{l} = LL^T(R_{DE} - R),$$
$$\dot{\mathcal{C}} = -2(R_{DE} - R)^T LL^T (R_{DE} - R) \le 0,$$

where $\mathcal{C} = \sum_{i=1}^m ((R_{DE})_i - R_i)^2$. Now we consider the ODE system

$$\dot{l} = \Upsilon(l) = L^T(R_{DE} - R).$$

Then $\Upsilon(l_{DE}) = 0$ and $D_l\Upsilon(l)|_{l=l_{DE}} = -L_{DE}L_{DE}^T \le 0$. As $\mathrm{rank}(L_{DE}) = m-1$, $D_l\Upsilon(l)|_{l=l_{DE}}$ is negative definite up to scaling. Hence l_{DE} is a local attractor and the system is asymptotically stable at l_{DE}. □

5. Discrete curvature flow in non-Euclidean geometry

K-space form triangulation and discrete curvature flow. Assume $K \in \mathbb{R}$ is a constant and, moreover, $K \neq 0$. In this section, we will consider a 3-dimensional compact manifold M^3 with a K-space form triangulation $\mathcal{T}$ on M^3. Let M_K be the space form with constant sectional curvature K. The basic blocks of K-space form triangulation $\mathcal{T}$ are tetrahedrons embedded in M_K.

A tetrahedron embedded in M_K is determined by its six edge lengths. Not every group of six positive numbers can be realized as the six edge lengths of some tetrahedrons embedded in M_K. Similar to the Euclidean case, there are nondegenerate conditions too. All admissible groups of six positive numbers which can be realized as the six edge lengths of some tetrahedrons embedded in M_K form an open connected set in $\mathbb{R}^6_{>0}$. The set is open due to Theorems 3.1 and 4.1 in [Yakut et al. 2009]; the set is connected due to the fact any tetrahedron can be deformed continuously to regular tetrahedrons.

The combinatorial Ricci curvature R_{ij} is defined in the same way as that of the Euclidean PL-manifold. We need to define a new functional S_K corresponding to the total curvature functional S.

Definition 5.1. Set $V = \sum_{\{i,j,k,l\}\in T} \mathrm{V}_{ijkl}$ and define

$$S_K \triangleq 2KV + \sum_{i=1}^{m} R_i l_i.$$

Now we recall the famous Schläfli formula for a K-space form tetrahedrons. For any K-space form tetrahedrons $\{i, j, k, l\} \in T$, one has (see [Milnor 1994; Schlenker 2000])

$$\frac{\partial \mathrm{V}_{ijkl}}{\partial \beta_{pq}} = \frac{l_{pq}}{2K}, \quad p, q \in \{i, j, k, l\},$$

where β_{pq} is the dihedral angle at the edge $\{p, q\}$ in the tetrahedrons $\{i, j, k, l\}$. Using the formula, one can get

$$2K\,dV + \sum_{i=1}^{m} l_i\, dR_i = 0.$$

Hence

$$dS_K = 2K\,dV + \sum_{i=1}^{m}(l_i\, dR_i + R_i\, dl_i) = \sum_{i=1}^{m} R_i\, dl_i,$$

which implies $\partial S_K/\partial l_i = R_i$. Then we have $\nabla_l S_K = R$ and $\mathrm{Hess}_l\, S_K = L$.

Conjecture. *The symmetric matrix L is nonsingular and indefinite.*

We affirm the conjecture for the case of a single tetrahedron, and include the proof in the Appendix, see Theorem A.6.

With K-space form triangulation, we consider discrete curvature flow $\dot{l} = -R$ of second order and flow $\dot{l} = -L^T R$ of fourth order. Most properties are laid out in the following table:

Discrete curvature flow of second order	Discrete curvature flow of fourth order
$\dot{l} = -R = -\nabla_l S_K$	$\dot{l} = -L^T R = \nabla_l \mathcal{C}$
$\dot{R} = -LR$	$\dot{R} = -LL^T R$
$\dot{S}_K = -R^T R = -\mathcal{C} \le 0$	$\dot{S}_K = -R^T L^T R$
$\dot{\mathcal{C}} = -2R^T LR$	$\dot{\mathcal{C}} = -2\|R^T L\|^2 \le 0$

Theorem 5.2. *If the solution to second order flow $\dot{l} = -R$ exists for all time and converges to a nondegenerate metric l_∞, then l_∞ is a discrete Ricci-flat metric.*

Theorem 5.3. *If the solution to fourth order flow $\dot{l} = -L^T R$ exists for all time and converges to a nondegenerate metric l_∞ with L_∞ nonsingular, then l_∞ is a discrete Ricci-flat metric.*

Proof. The limit $\lim_{t\to+\infty} \mathcal{C}(t)$ exists because of the convergence of the flow $\dot{l} = -L^T R$, and $\mathcal{C}(t)$ is nonincreasing along the fourth order discrete curvature flow. So we have

$$\lim_{t\to+\infty} \dot{\mathcal{C}}(t) = 0,$$

which implies that $\lim_{t\to+\infty}(L^T R)^T (L^T R) = 0$. Hence $L^T R = 0$. Since L_∞ is nonsingular, $R_\infty = 0$. □

Theorem 5.4. *If there exists a discrete Ricci-flat metric l_{DE} with L_{DE} nonsingular, then the solution of fourth order discrete curvature flow $\dot{l} = L^T R$ exists for all time and converges to the discrete Einstein metric l_{DE} when the initial discrete Calabi energy $\mathcal{C}(0)$ is small enough.*

Proof. At the point l_{DE}, $D_l(-L^T R) = -LL^T < 0$. Hence l_{DE} is a local attractor of the flow. □

A fourth order flow for hyperbolic 3-manifolds. In the above subsection, we have seen that the matrix L is not so good for evolving a useful curvature flow. This is mainly because of the nondefiniteness of L. For a special type of manifolds and a special kind of triangulations, Feng Luo [2005] introduced a second order combinatorial curvature flow. In this short subsection, we introduce a fourth order flow which is very similar to Luo's flow.

Suppose M is a compact 3-manifold whose boundary is nonempty and is a union of surfaces with negative Euler characteristic. M can be ideally triangulated. The basic building blocks are strictly hyperideal tetrahedrons. For a single strictly hyperideal tetrahedron, let $l_1, \dots, l_6$ be the edge lengths of a strictly hyperideal tetrahedron, and $\beta_1, \dots, \beta_6$ be the dihedral angles at respective edges. Then the volume V is a strictly concave function of its dihedral angles, that is to say, $\operatorname{Hess}_\beta V = -\frac{1}{2}\partial(l_1, \dots, l_6)/\partial(\beta_1, \dots, \beta_6)$ is negative definite.

For M with ideal triangulation, denote $l = (l_1, \dots, l_m)^T$ as the edge lengths, $R = (R_1, \dots, R_m)^T$ as the combinatorial curvatures at all edges. Here the combinatorial curvature R_i at an edge i is 2π minus the sum of dihedral angles at the edge. Denote $\mathcal{C}(l) = \|R\|^2 = \sum_{i=1}^m R_i^2$. Consider the combinatorial curvature flow

$$\dot{l} = -\tfrac{1}{2}\nabla_l \mathcal{C} = -LR, \tag{5-1}$$

where

$$L = \partial(R_1, \dots, R_m)/\partial(l_1, \dots, l_m)$$

is positive definite from [Luo 2005]. The equilibrium points of the combinatorial curvature flow (5-1) are the only flat metric with $R \equiv 0$, that is, the complete hyperbolic metric with totally geodesic boundary. Moreover, by

$$D_l(-LR) = -L^2 < 0,$$

we know that each equilibrium point is a local attractor of this flow. Hence, when the initial discrete energy $\mathcal{C}(0)$ is small enough, the solution of flow (5-1) exists for all time and converges to the flat metric, i.e., the complete hyperbolic metric with totally geodesic boundary.

Appendix

In this appendix we study the matrix L in space forms M_K, where subindex K represents the constant sectional curvature. We conclude that the matrix L is nonsingular and indefinite whenever $K \neq 0$.

Consider a single tetrahedron $\tau = \{A, B, C, D\}$ embedded in M_K. Since τ varies with its six edge lengths, all tetrahedrons can be considered as points of some connected open set in $\mathbb{R}^6_{>0}$. Denote β_{AB} as the dihedral angle at edge $\{A, B\}$. The dihedral angles and the edge lengths are mutually determined. On one hand, six dihedral angles are determined by six edge lengths. On the other hand, each tetrahedron in the space form M_K is determined, up to a motion, by its Gram matrix, which, in turn, is determined by the dihedral angles of the tetrahedron (see Chapter 6 §1 and Chapter 7 §2 in [Alekseevskij et al. 1993]). Therefore the Jacobian of dihedral angles over edges, which is denoted by

$$-L_{ABCD} \triangleq \frac{\partial(\beta_{AB}, \beta_{AC}, \beta_{AD}, \beta_{BC}, \beta_{BD}, \beta_{CD})}{\partial(l_{AB}, l_{AC}, l_{AD}, l_{BC}, l_{BD}, l_{CD})},$$

is nonsingular.

Next we prove that L_{ABCD} is indefinite. A tetrahedron is called regular, if all lengths are equal.

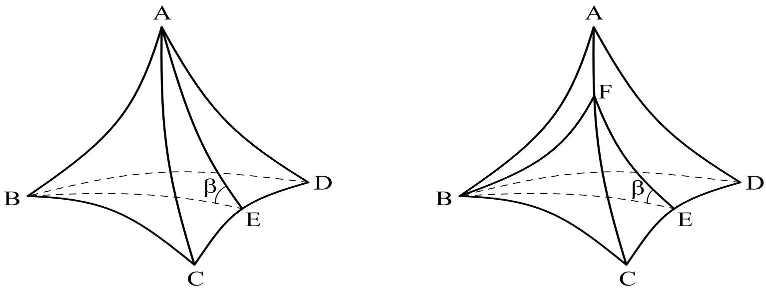


Figure 1

Proposition A.1. *For the regular tetrahedron, we have*

$$-L_{ABCD} = \begin{pmatrix} x & y & y & y & y & z \\ y & x & y & y & z & y \\ y & y & x & z & y & y \\ y & y & z & x & y & y \\ y & z & y & y & x & y \\ z & y & y & y & y & x \end{pmatrix},$$

where

$$x = \frac{\partial \beta_{AB}}{\partial l_{AB}}, \quad y = \frac{\partial \beta_{AB}}{\partial l_{AC}} = \frac{\partial \beta_{AB}}{\partial l_{AD}} = \frac{\partial \beta_{AB}}{\partial l_{BC}} = \frac{\partial \beta_{AB}}{\partial l_{BD}}, \quad z = \frac{\partial \beta_{AB}}{\partial l_{CD}}.$$

Moreover, the eigenvalues of the above $-L_{ABCD}$ are $x-z,\ x+z-2y,\ x+z+4y$ with degree 3, 2, 1 *respectively.*

In the following, we claim that, when $K \neq 0$, the matrix L is nonsingular but not definite. It's enough to determine the sign of $x-z,\ x+z-2y$ and $x+z+4y$.

First, we recall the formula of the cosine law in the 2-dimensional space forms $M^2(K)$ with constant sectional curvature K. Denote

$$S_K(t) = \begin{cases} \sin\big(\sqrt{K}t\big)/\sqrt{K}, & K > 0, \\ t, & K = 0, \\ \sinh\big(\sqrt{-K}t\big)/\sqrt{-K}, & K < 0, \end{cases}$$

$$C_K(t) = \begin{cases} \cos\big(\sqrt{K}t\big), & K > 0, \\ 1, & K = 0, \\ \cosh\big(\sqrt{-K}t\big), & K < 0, \end{cases}$$

$$f_K(r) = \int_0^r S_K(t)\,dt = \begin{cases} (1 - C_K(r))/K, & K \neq 0, \\ r^2/2, & K = 0. \end{cases}$$

Then we have the following identities:

(1) $f_K^{'}(r) = S_K(r), \quad S_K^{'}(r) = C_K(r)$.

(2) $KS_K^2(a) + C_K^2(a) = 1$.

(3) $S_K(a+b) = S_K(a)S_K(b) + C_K(a)C_K(b)$.

(4) $C_K(a+b) = C_K(a)C_K(b) - KS_K(a)S_K(b)$.

(5) $C_K(2a) = 2C_K^2(a) - 1 = 1 - 2KS_K^2(a)$.

So

$$f_K(r) = 2S_K^2(r/2).$$

Proposition A.2 (the cosine law). *For a geodesic triangle $\triangle ABC$ in the space form $M^2(K)$, with side lengths a, b, c opposite to the angles A, B, C, respectively, the cosine law is*

$$f_K(c) = f_K(a-b) + S_K(a)S_K(b)(1-\cos C).$$

For $K \neq 0$, the above formula is equivalent to

$$C_K(c) = C_K(a)C_K(b) + KS_K(a)S_K(b)\cos C.$$

Now, calculating the exact value of a, b, c, we have the following results.

Lemma A.3.
$$z = \frac{\sqrt{2}C_K^2(l_0/2)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}}.$$

Proof. By the definition of L_{16}, we just need to calculate $\partial\beta/\partial l_6$. To calculate it, we assume the length of AB is l_6 and other edges have length l_0 in the hyperbolic tetrahedron in Figure 1 (left). As shown there, E is the midpoint of the edge CD, and the dihedral angle at the edge CD is the angle ∠AEB, i.e., β.

Using the cosine law in the triangle △AEB, we have

$$f_K(l_6) = f_K(0) + S_K^2(h_0)(1-\cos\beta_1) = S_K^2(h_0)(1-\cos\beta_1),$$

where h_0 is the length of the altitude in the regular triangle with side length l_0. We can get

$$\frac{\partial\beta_1}{\partial l_6} = \frac{f_K^{'}(l_6)}{S_K^2(h_0)\sin\beta} = \frac{S_K(l_6)}{S_K^2(h_0)\sin\beta}.$$

So at the regular point

$$z = \frac{S_K(l_0)}{S_K^2(h_0)\sin\beta},$$

and

$$f_K(l_0) = f_K\Big(h_0 - \frac{l_0}{2}\Big) + S_K(h_0)S_K\Big(\frac{l_0}{2}\Big).$$

Then we have

$$C_K(h_0) = \frac{C_K(l_0)}{C_K(l_0/2)},$$

which implies

$$S_K^2(h_0) = \begin{cases} \dfrac{1 - C_K^2(h_0)}{K}, & K \neq 0, \\ h_0^2, & K = 0. \end{cases}$$

For $K \neq 0$,

$$S_K^2(h_0) = \frac{C_K^2(l_0/2) - C_K^2(l_0)}{K C_K^2(l_0/2)} = \frac{S_K^2(l_0/2)(1 + 2C_K(l_0))}{C_K^2(l_0/2)}.$$

The equation also holds for the case of $K = 0$. By the cosine law,

$$\cos\beta = \frac{S_K^2(h_0) - f_K(l_0)}{S_K^2(h_0)}.$$

If $K = 0$, it is easy to get $\cos\beta = 1 - l_0^2/(2h_0^2) = 1/3$. For the case of $K \neq 0$,

$$\begin{aligned}
\cos\beta &= \frac{(1 - C_K^2(h_0))/K - (1 - C_K(l_0))/K}{(1 - C_K^2(h_0))/K} = \frac{C_K(l_0) - C_K^2(h_0)}{1 - C_K^2(h_0)} \\
&= \frac{C_K(l_0) - C_K^2(l_0)/C_K^2(l_0/2)}{1 - C_K^2(l_0)/C_K^2(l_0/2)} = \frac{C_K(l_0)(C_K^2(l_0/2) - C_K(l_0))}{C_K^2(l_0/2) - C_K^2(l_0)} \\
&= \frac{K C_K(l_0) S_K^2(l_0/2)}{C_K^2(l_0/2) - C_K^2(l_0)} = \frac{K C_K(l_0) S_K^2(l_0/2)}{(1 + C_K(l_0) - 2C_K^2(l_0))/2} \\
&= \frac{K C_K(l_0) S_K^2(l_0/2)}{(1 + 2C_K(l_0))(1 - C_K(l_0))/2} = \frac{K C_K(l_0) S_K^2(l_0/2)}{(1 + 2C_K(l_0)) K S_K^2(l_0/2)} \\
&= \frac{C_K(l_0)}{1 + 2C_K(l_0)}.
\end{aligned}$$

This formula also holds for $K = 0$. Then we have

$$\sin\beta = \frac{\sqrt{(1 + C_K(l_0))(1 + 3C_K(l_0))}}{1 + 2C_K(l_0)} = \frac{\sqrt{2}\, C_K(l_0/2)\sqrt{1 + 3C_K(l_0)}}{1 + 2C_K(l_0)}.$$

Hence

$$z = \frac{S_K(l_0) C_K(l_0/2)}{\sqrt{2} S_K^2(l_0/2)\sqrt{1 + 3C_K(l_0)}} = \frac{\sqrt{2}\, C_K^2(l_0/2)}{S_K(l_0/2)\sqrt{1 + 3C_K(l_0)}}. \qquad \square$$

Lemma A.4. $$x=\frac{\sqrt{2}C_K^2(l_0)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}(1+2C_K(l_0))}.$$

Proof. To calculate it, we assume the length of CD is l_1 and other edges have length l_0 in the tetrahedron shown in Figure 1 (left). As illustrated there, E is the midpoint of the edge CD, the dihedral angle at the edge CD is the angle $\angle$AEB, i.e., β. We assume the length of AE is h. By the cosine law,

$$f_K(l_0)=f_K(0)+S_K^2(h)(1-\cos\beta)=S_K^2(h)(1-\cos\beta),$$

and we have
$$-\frac{\partial\beta}{\partial l_1}=\frac{1-\cos\beta}{S_K^2(h)\sin\beta}\frac{\partial S_K^2(h)}{\partial l_1}.$$

By the cosine law again,

$$f_K(l_0)=f_K(l_1/2-h)+S_K(h)S_K(l_1/2),$$

and we have
$$C_K(h)=C_K(l_0)/C_K(l_1/2).$$

Hence
$$\frac{\partial C_K(h)}{\partial l_1}=-\frac{C_K(l_0)C_K'(l_1/2)}{2C_K^2(l_1/2)}=\frac{KS_K(l_1/2)C_K(l_0)}{2C_K^2(l_1/2)},$$

and
$$S_K^2(h)=\begin{cases}\frac{1-C_K^2(h)}{K}, & K\neq 0,\\ h^2, & K=0,\end{cases}$$

which implies that

$$\frac{\partial S_K^2(h)}{\partial l_1}=\begin{cases}\frac{-2C_K(h)}{K}\frac{KS_K(l_1/2)C_K(l_0)}{2C_K^2(l_1/2)}=-\frac{C_K^2(l_0)S_K(l_1/2)}{C_K^3(l_1/2)}, & K\neq 0,\\ -\frac{l_1}{2}, & K=0.\end{cases}$$

So we obtain
$$\frac{\partial S_K^2(h)}{\partial l_1}=-\frac{C_K^2(l_0)S_K(l_1/2)}{C_K^3(l_1/2)}.$$

At the regular point, we have

$$\begin{aligned}x=\frac{\partial\beta}{\partial l_1}&=\frac{C_K^2(l_0)S_K(l_1/2)}{C_K^3(l_1/2)}\frac{C_K(l_0/2)(1+C_K(l_0))}{\sqrt{2}S_K^2(l_0/2)\sqrt{1+3C_K(l_0)}(1+2C_K(l_0))}\\&=\frac{\sqrt{2}C_K^2(l_0)}{S_K(l_0/2)\sqrt{1+3C_K(l_0)}(1+2C_K(l_0))}.\end{aligned}$$

$\square$

Lemma A.5. $$y=-\frac{\sqrt{2}C_K(l_0)C_K^2(l_0/2)}{S_K(l_0/2)(1+2C_K(l_0))\sqrt{1+3C_K(l_0)}}.$$

Proof. To calculate it, we assume the length of AD is l_2 and other edges have length l_0 in the tetrahedron in Figure 1 (right). As shown there, E is the midpoint of the edge CD, the dihedral angle at the edge CD is the angle $\angle$FEB, i.e., β. For simplicity, we assume $l_2\le l_0$. Assume the length of AF is s, and the length of FE is $\tilde{h}$. So the length of FC and FD are equal to l_0-s. By the cosine law in the triangle $\triangle$CEF,

$$f_K(l_0-s)=f_K(\tilde{h}-l_0/2)+S_K(\tilde{h})S_K(l_0/2).$$

By the cosine law in the triangle $\triangle$AFD,

$$f_K(l_0-s)=f_K(l_2-s)+\frac{S_K(s)}{S_K(l_0)}(f_K(l_0)-f_K(l_2-l_0)).$$

By the cosine law in the triangles $\triangle$ABF and $\triangle$BEF,

$$f_K(l_0-s)+\frac{f_K(l_0)S_K(s)}{S_K(l_0)}=f_K(h_0-\tilde{h})+S_K(h_0)S_K(\tilde{h})(1-\cos\beta).$$

Differentiating the above three equations at the regular point, i.e., $s=0$, $l_2=l_0$, and $\tilde{h}=h_0$, we have

$$\begin{aligned}-S_K(l_0)ds&=(S_K(h_0-l_0/2)+C_K(h_0)S_K(l_0/2))d\tilde{h}\\&=S_K(h_0)C_K(l_0/2)d\tilde{h},\\-S_K(l_0)ds&=-S_K(l_0)ds+S_K(l_0)dl_2+\frac{C_K(0)}{S_K(l_0)}f_K(l_0)ds,\end{aligned}$$

$$\begin{aligned}-S_K(l_0)ds+\frac{f_K(l_0)C_K(0)}{S_K(l_0)}ds=&-S_K(0)d\tilde{h}+S_K(h_0)C_K(h_0)(1-\cos\beta)d\tilde{h}\\&+S_K^2(h_0)\sin\beta d\beta.\end{aligned}$$

Using the fact $S_K(0)=0$, $C_K(0)=1$, we obtain

(1) $ds=-\dfrac{S_K^2(l_0)}{f_K(l_0)}dl_2,$

(2) $d\tilde{h}=-\dfrac{S_K(l_0)}{S_K(h_0)C_K(l_0/2)}ds=\dfrac{S_K^3(l_0)}{f_K(l_0)S_K(h_0)C_K(l_0/2)}dl_2,$

(3) $\dfrac{f_K(l_0)-S_K^2(l_0)}{S_K(l_0)}ds=S_K(h_0)C_K(h_0)(1-\cos\beta)d\tilde{h}+S_K^2(h_0)\sin\beta d\beta.$

Using

$$\cos\beta=\frac{C_K(l_0)}{1+2C_K(l_0)},\quad C_K(h_0)=\frac{C_K(l_0)}{C_K(l_0/2)},$$

we have

$$-\frac{(f_K(l_0)(1+2C_K(l_0))-S_K^2(l_0))S_K(l_0)}{f_K(l_0)(1+2C_K(l_0))}dl_2 = S_K^2(h_0)\sin\beta d\beta.$$

Since $f_K(l_0)=2S_K^2(l_0/2)$, we have

$$-\frac{C_K(l_0)S_K(l_0)}{1+2C_K(l_0)}dl_2 = S_K^2 \sin\beta d\beta.$$

Hence

$$\begin{aligned} y = \frac{\partial\beta}{\partial l_2} &= -\frac{C_K(l_0)S_K(l_0)}{1+2C_K(l_0)}\frac{1}{S_K^2(h_0)\sin\beta} \\ &= -\frac{C_K(l_0)S_K(l_0)}{1+2C_K(l_0)}\frac{C_K(l_0/2)}{\sqrt{2}S_K^2(l_0/2)\sqrt{1+3C_K(l_0)}} \\ &= -\frac{\sqrt{2}C_K(l_0)C_K^2(l_0/2)}{S_K(l_0/2)(1+2C_K(l_0))\sqrt{1+3C_K(l_0)}}. \end{aligned}$$

□

So we have

(1) $x-z=-\dfrac{\sqrt{2}\sqrt{1+3C_K(l_0)}}{2S_K(l_0/2)(1+2C_K(l_0))}<0,$

(2) $x+z-2y=\dfrac{\sqrt{2}\sqrt{1+3C_K(l_0)}}{2S_K(l_0/2)}>0,$

(3) $x+z+4y=\dfrac{\sqrt{2}KS_K(l_0/2)}{(1+2C_K(l_0))\sqrt{1+3C_K(l_0)}}.$

Hence $x+z+4y>0$ when $K>0$, $x+z+4y=0$ when $K=0$, and $x+z+4y<0$ when $K<0$.

Theorem A.6. *When $K\neq 0$, the matrix L of one single tetrahedron $-L_{ABCD}$ embedded in M_K is nonsingular and indefinite. Hence the conjecture on page 60 is true for this case.*

Proof. By the calculations above, we know that the matrix L at regular points is indefinite. Any tetrahedron can be deformed continuously to the regular tetrahedron, so all tetrahedrons have the same properties. □

Acknowledgements

Ge would like to thank Professor Gang Tian and Yuguang Shi for persistent encouragement and would also like to thank Professors Feng Luo and Glickenstein for many helpful conversations. His research is partially supported by NSFC Grant No. 11501027 and the Fundamental Research Funds for the Central Universities (No. 2015JBM103, 2014RC028 and 2016JBM071). The research of Xu is partially supported by NSFC Grants No. 11301402 and No. 11301399. He would like to thank Professor Xiao Zhang for the invitation to AMSS and thank Professor Guofang Wang for the invitation to the Institute of Mathematics of the University of Freiburg. Zhang would like to thank the Math Department of Rutgers University for its hospitality. His research is partially supported by NSFC Grant No. 11301017, the Research Fund for the Doctoral Program of Higher Education of China and the Fundamental Research Funds for the Central Universities and a scholarship from the China Scholarship Council. The authors are grateful to the referee for some helpful suggestions.

References

[Alekseevskij et al. 1993] D. V. Alekseevskij, E. B. Vinberg, and A. S. Solodovnikov, "Geometry of spaces of constant curvature", pp. 1–138 in *Geometry, II: Spaces of constant curvature*, edited by E. B. Vinberg, Encyclopaedia of Mathematical Sciences **29**, Springer, 1993. MR Zbl

[Bobenko et al. 2015] A. I. Bobenko, U. Pinkall, and B. A. Springborn, "Discrete conformal maps and ideal hyperbolic polyhedra", *Geom. Topol.* **19**:4 (2015), 2155–2215. MR Zbl

[Champion et al. 2011] D. Champion, D. Glickenstein, and A. Young, "Regge's Einstein–Hilbert functional on the double tetrahedron", *Differential Geom. Appl.* **29**:1 (2011), 108–124. MR Zbl

[Chow and Luo 2003] B. Chow and F. Luo, "Combinatorial Ricci flows on surfaces", *J. Differential Geom.* **63**:1 (2003), 97–129. MR Zbl

[Cooper and Rivin 1996] D. Cooper and I. Rivin, "Combinatorial scalar curvature and rigidity of ball packings", *Math. Res. Lett.* **3**:1 (1996), 51–60. MR Zbl

[Ge 2013] H. Ge, "Combinatorial Calabi flows on surfaces", preprint, 2013. To appear in *Trans. Amer. Math. Soc.* arXiv

[Ge and Jiang 2016a] H. Ge and W. Jiang, "On the deformation of discrete conformal factors on surfaces", *Calc. Var. Partial Differential Equations* **55**:6 (2016), art. id. 136, 14 pp. arXiv

[Ge and Jiang 2016b] H. Ge and W. Jiang, "On the deformation of inversive distance circle packings", preprint, 2016. arXiv

[Ge and Xu 2014] H. Ge and X. Xu, "Discrete quasi-Einstein metrics and combinatorial curvature flows in 3-dimension", *Adv. Math.* **267** (2014), 470–497. MR Zbl

[Ge and Xu 2016a] H. Ge and X. Xu, "2-dimensional combinatorial Calabi flow in hyperbolic background geometry", *Differential Geom. Appl.* **47** (2016), 86–98. MR Zbl

[Ge and Xu 2016b] H. Ge and X. Xu, "α-curvatures and α-flows on low dimensional triangulated manifolds", *Calc. Var. Partial Differential Equations* **55**:1 (2016), art. id. 12, 16 pp. MR

[Ge et al. 2015] H. Ge, J. Mei, and D. Zhou, "A note on discrete Einstein metrics", preprint, 2015. arXiv

[Glickenstein 2005] D. Glickenstein, "A combinatorial Yamabe flow in three dimensions", *Topology* **44**:4 (2005), 791–808. MR Zbl

[Luo 2004] F. Luo, "Combinatorial Yamabe flow on surfaces", *Commun. Contemp. Math.* **6**:5 (2004), 765–780. MR Zbl

[Luo 2005] F. Luo, "A combinatorial curvature flow for compact 3-manifolds with boundary", *Electron. Res. Announc. Amer. Math. Soc.* **11** (2005), 12–20. MR Zbl

[Luo 2011] F. Luo, "Rigidity of polyhedral surfaces, III", *Geom. Topol.* **15**:4 (2011), 2299–2319. MR Zbl

[Milnor 1994] J. Milnor, "The Schläfli differential equality", pp. 281–295 in *Collected papers*, vol. 1, Publish or Perish, Houston, TX, 1994. MR Zbl

[Regge 1961] T. Regge, "General relativity without coordinates", *Nuovo Cimento* (10) **19** (1961), 558–571. MR

[Rivin 2003] I. Rivin, "Some observations on the simplex", preprint, 2003. arXiv

[Schlenker 2000] J.-M. Schlenker, "Hyperideal polyhedra in hyperbolic manifolds", preprint, 2000. arXiv

[Schrader 2016] R. Schrader, "Piecewise linear manifolds: Einstein metrics and Ricci flows", *J. Phys. A* **49**:20 (2016), 205201, 51. MR Zbl

[Thurston 1980] W. Thurston, "Geometry and topology of three-manifolds", lecture notes, Princeton University, 1980, available at http://www.msri.org/publications/books/gt3m/.

[Yakut et al. 2009] A. T. Yakut, M. Savas, and S. Kader, "On the Schläfli differential formula based on edge lengths of tetrahedron in H^3 and S^3", *Geom. Dedicata* **138** (2009), 99–115. MR Zbl

Received August 11, 2015. Revised June 17, 2016.

HUABIN GE
DEPARTMENT OF MATHEMATICS
BEIJING JIAOTONG UNIVERSITY
BEIJING, 100044
CHINA
hbge@bjtu.edu.cn

XU XU
SCHOOL OF MATHEMATICS AND STATISTICS
WUHAN UNIVERSITY
WUHAN, 430072
CHINA
xuxu2@whu.edu.cn

SHIJIN ZHANG
SCHOOL OF MATHEMATICS AND SYSTEMS SCIENCE
BEIHANG UNIVERSITY
BEIJING, 100191
CHINA
shijinzhang@buaa.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.71

INCLUSION OF CONFIGURATION SPACES IN CARTESIAN PRODUCTS, AND THE VIRTUAL COHOMOLOGICAL DIMENSION OF THE BRAID GROUPS OF $\mathbb{S}^2$ AND $\mathbb{R}P^2$

DACIBERG LIMA GONÇALVES AND JOHN GUASCHI

Let S be a surface, perhaps with boundary, and either compact or with a finite number of points removed from the interior of the surface. We consider the inclusion $\iota : F_n(S) \to \prod_1^n S$ of the n-th configuration space $F_n(S)$ of S into the n-fold Cartesian product of S, as well as the induced homomorphism $\iota_\# : P_n(S) \to \prod_1^n \pi_1(S)$, where $P_n(S)$ is the n-string pure braid group of S. Both ι and $\iota_\#$ were studied initially by J. Birman, who conjectured that $\mathrm{Ker}(\iota_\#)$ is equal to the normal closure of the Artin pure braid group P_n in $P_n(S)$. The conjecture was later proved by C. Goldberg for compact surfaces without boundary different from the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{R}P^2$. In this paper, we prove the conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$. In the case of $\mathbb{R}P^2$, we prove that $\mathrm{Ker}(\iota_\#)$ is equal to the commutator subgroup of $P_n(\mathbb{R}P^2)$, we show that it may be decomposed in a manner similar to that of $P_n(\mathbb{S}^2)$ as a direct sum of a torsion-free subgroup L_n and the finite cyclic group generated by the full twist braid, and we prove that L_n may be written as an iterated semidirect product of free groups. Finally, we show that the groups $B_n(\mathbb{S}^2)$ and $P_n(\mathbb{S}^2)$ (resp. $B_n(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$) have finite virtual cohomological dimension equal to $n-3$ (resp. $n-2$), where $B_n(S)$ denotes the full n-string braid group of S. This allows us to determine the virtual cohomological dimension of the mapping class groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$ with marked points, which in the case of $\mathbb{S}^2$ reproves a result due to J. Harer.

1. Introduction

Let S be a connected surface, perhaps with boundary, and either compact or with a finite number of points removed from the interior of the surface. The *n-th configuration space* of S is defined by

$$F_n(S) = \{(x_1, \dots, x_n) \in S^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

MSC2010: primary 20F36; secondary 20J06.

Keywords: configuration spaces, surface braid groups, group presentations, virtual cohomological dimension.

It is well known that $\pi_1(F_n(S)) \cong P_n(S)$, the *pure braid group* of S on n strings, and that $\pi_1(F_n(S)/S_n) \cong B_n(S)$, the *braid group* of S on n strings, where $F_n(S)/S_n$ is the quotient space of $F_n(S)$ by the free action of the symmetric group S_n given by permuting coordinates [Fadell and Neuwirth 1962; Fox and Neuwirth 1962]. We compose elements of $B_n(S)$ from left to right. If S is the 2-disc $\mathbb{D}^2$ then $B_n(\mathbb{D}^2)$ (resp. $P_n(\mathbb{D}^2)$) is the Artin braid group B_n (resp. the Artin pure braid group P_n). The canonical projection $F_n(S) \to F_n(S)/S_n$ is a regular $n!$-fold covering map, and thus gives rise to the short exact sequence

$$1 \to P_n(S) \to B_n(S) \to S_n \to 1. \tag{1}$$

If $\mathbb{D}^2$ is a topological disc lying in the interior of S and containing the basepoints of the braids then the inclusion $j : \mathbb{D}^2 \to S$ induces a group homomorphism $j_\# : B_n \to B_n(S)$. This homomorphism is injective if S is different from the 2-sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$ [Birman 1969; Goldberg 1973]. Let $j_\#|_{P_n} : P_n \to P_n(S)$ denote the restriction of $j_\#$ to the corresponding pure braid groups. If $\beta \in B_n$ then we shall denote its image $j_\#(\beta)$ in $B_n(S)$ simply by β. It is well known that the centre of B_n and of P_n is infinite cyclic, generated by the full twist braid that we denote by Δ_n^2, and that Δ_n^2, considered as an element of $B_n(\mathbb{S}^2)$ or of $B_n(\mathbb{R}P^2)$, is of order 2 and generates the centre. If G is a group then we denote its commutator subgroup by $\Gamma_2(G)$ and its Abelianisation by G^{Ab}, and if H is a subgroup of G then we denote its normal closure in G by $\langle\langle H \rangle\rangle_G$.

Let $\prod_1^n S = S \times \cdots \times S$ denote the n-fold Cartesian product of S with itself, let $\iota_n : F_n(S) \to \prod_1^n S$ be the inclusion map, and let

$$\iota_{n\#} : \pi_1(F_n(S)) \to \pi_1\biggl(\prod_1^n S\biggr)$$

denote the induced homomorphism on the level of fundamental groups. To simplify the notation, we shall often just write ι and $\iota_\#$ if n is given. The study of $\iota_\#$ was initiated by Birman [1969]. She had conjectured that $\langle\langle \mathrm{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} = \mathrm{Ker}(\iota_\#)$ if S is a compact orientable surface, but states without proof that her conjecture is false if S is of genus greater than or equal to 1 [Birman 1969, page 45]. However, Goldberg [1973, Theorem 1] proved the conjecture several years later in both the orientable and nonorientable cases for compact surfaces without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$. In connection with the study of Vassiliev invariants of surface braid groups, González-Meneses and Paris [2004] showed that $\mathrm{Ker}(\iota_\#)$ is also normal in $B_n(S)$, and that the resulting quotient is isomorphic to the semidirect product $\pi_1\bigl(\prod_1^n S\bigr) \rtimes S_n$, where the action is given by permuting coordinates (their work was within the framework of compact orientable surfaces without boundary, but their construction is valid for any surface S). In the case of $\mathbb{R}P^2$, this result was reproved using geometric methods [Tochimani 2011].

If $S = \mathbb{S}^2$, then $\operatorname{Ker}(\iota_\#)$ is clearly equal to $P_n(\mathbb{S}^2)$, and so by [Gonçalves and Guaschi 2004b, Theorem 4], it may be decomposed as

$$\operatorname{Ker}(\iota_\#) = P_n(\mathbb{S}^2) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \times \mathbb{Z}_2, \tag{2}$$

where the first factor of the direct product is torsion-free, and the $\mathbb{Z}_2$-factor is generated by Δ_n^2.

The aim of this paper is to resolve Birman's conjecture for surfaces without boundary in the remaining cases, namely $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, to determine the cohomological dimension of $B_n(S)$ and $P_n(S)$, where S is one of these two surfaces, and to elucidate the structure of $\operatorname{Ker}(\iota_\#)$ in the case of $\mathbb{R}P^2$. In Section 2, we start by considering the case $S = \mathbb{R}P^2$, we study $\operatorname{Ker}(\iota_\#)$, which we denote by K_n, and we show that it admits a decomposition similar to that of (2).

Proposition 1. *Let $n \in \mathbb{N}$.*

(a) (i) *Up to isomorphism, the homomorphism*

$$\iota_\# : \pi_1(F_n(\mathbb{R}P^2)) \to \pi_1\left(\prod_1^n \mathbb{R}P^2\right)$$

coincides with Abelianisation. In particular, $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$.

(ii) *If $n \geq 2$ then there exists a torsion-free subgroup L_n of K_n such that K_n is isomorphic to the direct sum of L_n and the subgroup $\langle \Delta_n^2 \rangle$ generated by the full twist that is isomorphic to $\mathbb{Z}_2$.*

(b) *If $n \geq 2$ then any subgroup of $P_n(\mathbb{R}P^2)$ that is normal in $B_n(\mathbb{R}P^2)$ and that properly contains K_n possesses an element of order* 4.

Note that if $n = 1$ then $B_1(\mathbb{R}P^2) = P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and Δ_1^2 is the trivial element, so parts (a)(ii) and (b) do not hold. Part (a)(i) will be proved in Proposition 8. We shall see later on in Remark 14 that there are precisely $2^{n(n-2)}$ subgroups that satisfy the conclusions of part (a)(ii), and to prove the statement, we shall exhibit an explicit torsion-free subgroup L_n. We then prove Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$, using Proposition 1(a)(i) in the case of $\mathbb{R}P^2$.

Theorem 2. *Let S be $\mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geq 1$. Then $\langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} = \operatorname{Ker}(\iota_\#)$.*

In Section 3, we analyse L_n in more detail, and we show that it may be decomposed as an iterated semidirect product of free groups.

Theorem 3. *Let $n \geq 3$. Consider the Fadell–Neuwirth short exact sequence*

$$1 \to P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_n(\mathbb{R}P^2) \xrightarrow{q_{2\#}} P_2(\mathbb{R}P^2) \to 1, \tag{3}$$

where $q_{2\#}$ is given geometrically by forgetting the last $n-2$ strings. Then L_n may be identified with the kernel of the composition

$$P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_n(\mathbb{R}P^2) \xrightarrow{\iota_\#} \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}},$$

where the first homomorphism is that appearing in (3). The image of this composition is the product of the last $n-2$ copies of $\mathbb{Z}_2$. In particular, L_n is of index 2^{n-2} in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Further, L_n is isomorphic to an iterated semidirect product of free groups of the form $\mathbb{F}_{2n-3} \rtimes (\mathbb{F}_{2n-5} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3)\cdots))$, where for all $m \in \mathbb{N}$, $\mathbb{F}_m$ denotes the free group of rank m.

In the semidirect product decomposition of L_n, note that every factor acts on each of the preceding factors. This is also the case for $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ (see (13)), and as we shall see in Remark 13(a), this implies an Artin combing-type result for this group. Analysing these semidirect products in more detail, we obtain the following results.

Proposition 4. *If $n \geq 3$ then*

(a) $\left(P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})\right)^{\mathrm{Ab}} \cong \mathbb{Z}^{2(n-2)}$,

(b) $(L_n)^{\mathrm{Ab}} \cong \mathbb{Z}^{n(n-2)}$.

In two papers in preparation, we shall analyse the homotopy fibre of ι, as well as the induced homomorphism $\iota_\#$ when $S = \mathbb{S}^2$ or $\mathbb{R}P^2$ [Gonçalves and Guaschi $\geq$ 2017], and when S is a space form manifold of dimension different from two [Golasiński et al. 2016]. In the first of these papers, we shall also see that L_n is closely related to the fundamental group of an orbit configuration space of the open cylinder.

In Section 4, we study the virtual cohomological dimension of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. Recall from [Brown 1982, page 226] that if a group Γ is virtually torsion-free then all finite index torsion-free subgroups of Γ have the same cohomological dimension by Serre's theorem, and this dimension is defined to be the *virtual cohomological dimension* of Γ. Using (2) and (3), we prove the following result, namely that if $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, the groups $B_n(S)$ and $P_n(S)$ have finite virtual cohomological dimension, and we compute these dimensions.

Theorem 5. (a) *Let $n \geq 4$. Then the virtual cohomological dimension of both $B_n(\mathbb{S}^2)$ and $P_n(\mathbb{S}^2)$ is equal to the cohomological dimension of the group $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. Furthermore, for all $m \geq 1$, the cohomological dimension of the group $P_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is equal to m.*

(b) *Let $n \geq 3$. Then the virtual cohomological dimension of both $B_n(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$ is equal to the cohomological dimension of the group $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Furthermore, for all $m \geq 1$, the cohomological dimension of the group $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ is equal to m.*

The methods of the proof of Theorem 5 have recently been applied to compute the cohomological dimension of the braid groups of all other compact surfaces (orientable and nonorientable) without boundary [Gonçalves et al. 2016]. Theorem 5 also allows us to deduce the virtual cohomological dimension of the punctured mapping class groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. If $n \geq 0$, let $\mathcal{MCG}(S, n)$ denote the mapping class group of a connected, compact surface S relative to an n-point set. If S is orientable then Harer [1986, Theorem 4.1] determined the virtual cohomological dimension of $\mathcal{MCG}(S, n)$. In the case of $\mathbb{S}^2$ and $\mathbb{D}^2$, he obtained the following results:

(a) If $n \geq 3$, the virtual cohomological dimension of $\mathcal{MCG}(\mathbb{S}^2, n)$ is equal to $n-3$.

(b) If $n \geq 2$, the cohomological dimension of $\mathcal{MCG}(\mathbb{D}^2, n)$ is equal to $n-1$ (recall that $\mathcal{MCG}(\mathbb{D}^2, n)$ is isomorphic to B_n [Birman 1974]).

As a consequence of Theorem 5, we are able to compute the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ for $S = \mathbb{S}^2$ and $\mathbb{R}P^2$.

Corollary 6. *Let $n \geq 4$ (resp. $n \geq 3$). Then the virtual cohomological dimension of $\mathcal{MCG}(\mathbb{S}^2, n)$ (resp. $\mathcal{MCG}(\mathbb{R}P^2, n)$) is finite and is equal to $n-3$ (resp. $n-2$).*

If $S = \mathbb{S}^2$ or $\mathbb{R}P^2$ then for the values of n given by Theorem 5 and Corollary 6, the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ is equal to that of $B_n(S)$. If $S = \mathbb{S}^2$, we thus recover the corresponding result of Harer.

2. The structure of K_n, and Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$

Let $n \in \mathbb{N}$. As we mentioned in the Introduction, if S is a surface different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, the kernel of the homomorphism $\iota_\# : P_n(S) \to \pi_1\big(\prod_1^n S\big)$ was studied in [Birman 1969; Goldberg 1973], and if $S = \mathbb{S}^2$ then $\mathrm{Ker}(\iota_\#) = P_n(\mathbb{S}^2)$. In the first part of this section, we recall a presentation of $P_n(\mathbb{R}P^2)$, and we prove Proposition 1(a)(i). The second part of this section is devoted to proving the rest of Proposition 1 and Theorem 2, the latter being Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$.

Consider the model of $\mathbb{R}P^2$ given by identifying antipodal boundary points of $\mathbb{D}^2$. We equip $F_n(\mathbb{R}P^2)$ with a basepoint $(x_1, \ldots, x_n)$. For $1 \leq i < j \leq n$ (resp. $1 \leq k \leq n$), we define the element $A_{i,j}$ (resp. τ_k, ρ_k) of $P_n(\mathbb{R}P^2)$ by the geometric braids depicted on the left side of Figure 1. Note that the arcs represent the projections of the strings onto $\mathbb{R}P^2$, so that all of the strings of the given braid are vertical, with the exception of the j-th (resp. k-th) string that is based at the point x_j (resp. x_k). As may be seen on the right side of Figure 1, the generator $A_{i,j}$ may also be represented by a loop based at the point x_i.

Theorem 7 [Gonçalves and Guaschi 2007, Theorem 4]. *Let $n \in \mathbb{N}$. The following constitutes a presentation of the pure braid group $P_n(\mathbb{R}P^2)$:*

Generators: $A_{i,j}$, $1 \leq i < j \leq n$, *and* τ_k, $1 \leq k \leq n$.

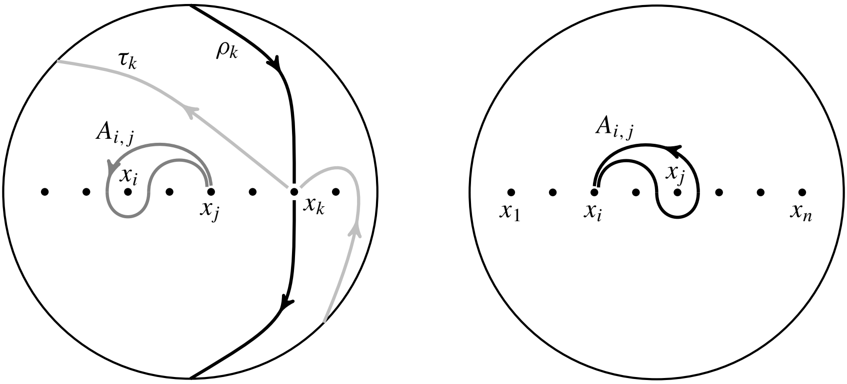


Figure 1. The elements $A_{i,j}$, τ_k and ρ_k of $P_n(\mathbb{R}P^2)$.

Relations:

(a) *The Artin relations between the $A_{i,j}$ emanating from those of P_n*:

$$A_{r,s}A_{i,j}A_{r,s}^{-1} = \tag{4}$$

$$\begin{cases} A_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j,\\ A_{i,j}^{-1}A_{r,j}^{-1}A_{i,j}A_{r,j}A_{i,j} & \text{if } r<i=s<j,\\ A_{s,j}^{-1}A_{i,j}A_{s,j} & \text{if } i=r<s<j,\\ A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{i,j}A_{r,j}^{-1}A_{s,j}^{-1}A_{r,j}A_{s,j} & \text{if } r<i<s<j. \end{cases}$$

(b) *For all $1 \leq i < j \leq n$, $\tau_i\tau_j\tau_i^{-1} = \tau_j^{-1}A_{i,j}^{-1}\tau_j^2$.*

(c) *For all $1 \leq i \leq n$, $\tau_i^2 = A_{1,i}\cdots A_{i-1,i}A_{i,i+1}\cdots A_{i,n}$.*

(d) *For all $1 \leq i < j \leq n$ and $1 \leq k \leq n$ with $k \neq j$,*

$$\tau_k A_{i,j}\tau_k^{-1} = \begin{cases} A_{i,j} & \text{if } j<k \text{ or } k<i,\\ \tau_j^{-1}A_{i,j}^{-1}\tau_j & \text{if } k=i,\\ \tau_j^{-1}A_{k,j}^{-1}\tau_j A_{k,j}^{-1}A_{i,j}A_{k,j}\tau_j^{-1}A_{k,j}\tau_j & \text{if } i<k<j. \end{cases}$$

This enables us to prove that $\iota_\#$ is in fact Abelianisation, which is part (a)(i) of Proposition 1.

Proposition 8. *Let $n \in \mathbb{N}$. The homomorphism $\iota_\# : P_n(\mathbb{R}P^2) \to \pi_1\big(\prod_1^n \mathbb{R}P^2\big)$ is defined on the generators of Theorem 7 by $\iota_\#(A_{i,j}) = (\overline{0}, \ldots, \overline{0})$ for all $1 \leq i < j \leq n$, and $\iota_\#(\tau_k) = (\overline{0}, \ldots, \overline{0}, \overline{1}, \overline{0}, \ldots, \overline{0})$, where $\overline{1}$ is in the k-th position, for all $1 \leq k \leq n$. Further, $\iota_\#$ is Abelianisation, and* $\operatorname{Ker}(\iota_\#) = K_n = \Gamma_2(P_n(\mathbb{R}P^2))$.

Proof. For $1 \leq k \leq n$, let $p_k : F_n(\mathbb{R}P^2) \to \mathbb{R}P^2$ denote projection onto the k-th coordinate. Observe that $\iota_\# = p_{1\#} \times \cdots \times p_{n\#}$, where $p_{k\#} : P_n(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$ is the induced homomorphism on the level of fundamental groups. Identifying

$\pi_1(\mathbb{R}P^2)$ with $\mathbb{Z}_2$ and using the geometric realisation of Figure 1 of the generators of the presentation of $P_n(\mathbb{R}P^2)$ given by Theorem 7, it is straightforward to check that for all $1 \le k, l \le n$ and $1 \le i < j \le n$, we have $p_{k\#}(A_{i,j}) = \bar{0}$, $p_{k\#}(\tau_l) = \bar{0}$ if $l \neq k$ and $p_{k\#}(\tau_k) = \bar{1}$, and this yields the first part of the proposition. The second part follows easily from the presentation of the Abelianisation $(P_n(\mathbb{R}P^2))^{\mathrm{Ab}}$ of $P_n(\mathbb{R}P^2)$ obtained from Theorem 7. More precisely, if we denote the Abelianisation of an element $x \in P_n(\mathbb{R}P^2)$ by $\bar{x}$, relations (b) and (c) imply respectively that for all $1 \le i < j \le n$ and $1 \le k \le n$, $\overline{A_{i,j}}$ and $\overline{\tau_k}^2$ represent the trivial element of $(P_n(\mathbb{R}P^2))^{\mathrm{Ab}}$. Since the remaining relations give no other information under Abelianisation, it follows that $(P_n(\mathbb{R}P^2))^{\mathrm{Ab}} \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, where $\overline{\tau_k} = (\bar{0}, \ldots, \bar{0}, \bar{1}, \bar{0}, \ldots, \bar{0})$ with $\bar{1}$ in the k-th position and $\overline{A_{i,j}} = (\bar{0}, \ldots, \bar{0})$ via this isomorphism, and the Abelianisation homomorphism indeed coincides with $\iota_\#$ on $P_n(\mathbb{R}P^2)$. □

Remark 9. (a) Since $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$, it follows immediately that K_n is normal in $B_n(\mathbb{R}P^2)$, since $\Gamma_2(P_n(\mathbb{R}P^2))$ is characteristic in $P_n(\mathbb{R}P^2)$, and $P_n(\mathbb{R}P^2)$ is normal in $B_n(\mathbb{R}P^2)$.

(b) A presentation of K_n may be obtained by a long but routine computation using the Reidemeister–Schreier method, although it is not clear how to simplify the presentation. In Theorem 3, we will provide an alternative description of K_n using algebraic methods.

(c) In what follows, we shall use Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$ [1966, page 83], whose generating set consists of the standard braid generators $\sigma_1, \ldots, \sigma_{n-1}$ emanating from the 2-disc, as well as the surface generators $\rho_1, \ldots, \rho_n$ depicted in Figure 1. We have the following relation between the elements τ_k and ρ_k:

$$\tau_k = \rho_k^{-1} A_{k,k+1} \cdots A_{k,n} \quad \text{for all } 1 \le k \le n,$$

where for $1 \le i < j \le n$, $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$. In particular, it follows from Proposition 8 that

$$\iota_\#(\rho_k) = \iota_\#(\tau_k) = (\bar{0}, \ldots, \bar{0}, \bar{1}, \bar{0}, \ldots, \bar{0}) \quad \text{for all } 1 \le k \le n, \tag{5}$$

where $\bar{1}$ is in the k-th position.

If $n \ge 2$, the full twist braid Δ_n^2, which may be defined by $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$, is of order 2 [Van Buskirk 1966, page 95], it generates the centre of $B_n(\mathbb{R}P^2)$ [Murasugi 1982, Proposition 6.1], and it is the unique element of $B_n(\mathbb{R}P^2)$ of order 2 [Gonçalves and Guaschi 2004a, Proposition 23]. Since $\Delta_n^2 \in P_n(\mathbb{R}P^2)$, it thus belongs to the centre of $P_n(\mathbb{R}P^2)$, and just as for the Artin braid groups and the braid groups of $\mathbb{S}^2$, it generates the centre of $P_n(\mathbb{R}P^2)$:

Proposition 10. *Let $n \ge 2$. Then the centre $Z(P_n(\mathbb{R}P^2))$ of $P_n(\mathbb{R}P^2)$ is generated by Δ_n^2.*

Proof. We prove the result by induction on n. If $n = 2$ then $P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8$ [Van Buskirk 1966, page 87], the quaternion group of order 8, and the result follows since Δ_2^2 is the element of $P_2(\mathbb{R}P^2)$ of order 2. So suppose that $n \geq 3$. From the preceding remarks, $\langle \Delta_n^2 \rangle \subset Z(P_n(\mathbb{R}P^2))$. Conversely, let $x \in Z(P_n(\mathbb{R}P^2))$, and consider the Fadell–Neuwirth short exact sequence

$$1 \to \pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{n-1}\}) \to P_n(\mathbb{R}P^2) \xrightarrow{q_{(n-1)\#}} P_{n-1}(\mathbb{R}P^2) \to 1,$$

where $q_{(n-1)\#}$ is the surjective homomorphism induced on the level of fundamental groups by the projection $q_{n-1} : F_n(\mathbb{R}P^2) \to F_{n-1}(\mathbb{R}P^2)$ onto the first $n-1$ coordinates. Now $q_{(n-1)\#}(x) \in Z(P_{n-1}(\mathbb{R}P^2))$ by surjectivity, and thus $q_{(n-1)\#}(x) = \Delta_{n-1}^{2\varepsilon}$ for some $\varepsilon \in \{0, 1\}$ by the induction hypothesis. Further, $q_{(n-1)\#}(\Delta_n^2) = \Delta_{n-1}^2$, hence

$$\Delta_n^{-2\varepsilon} x \in \operatorname{Ker}(q_{(n-1)\#}) \cap Z(P_n(\mathbb{R}P^2)),$$

and therefore $\Delta_n^{-2\varepsilon} x \in Z(\operatorname{Ker}(q_{(n-1)\#}))$. But $Z(\operatorname{Ker}(q_{(n-1)\#}))$ is trivial because $\operatorname{Ker}(q_{(n-1)\#})$ is a free group of rank $n-1$. This implies that $x \in \langle \Delta_n^2 \rangle$ as required. $\square$

Proof of Proposition 1. Let $n \geq 3$.

(a) Recall that part (a)(i) of Proposition 1 was proved in Proposition 8, so let us prove part (ii). The projection $q_2 : F_n(\mathbb{R}P^2) \to F_2(\mathbb{R}P^2)$ onto the first two coordinates gives rise to the Fadell–Neuwirth short exact sequence (3). Since $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$ by Proposition 8, the image of the restriction $q_{2\#}|_{K_n}$ of $q_{2\#}$ to K_n is the subgroup $\Gamma_2(P_2(\mathbb{R}P^2)) = \langle \Delta_2^2 \rangle$, and so we obtain the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K_n \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] \arrow[d] & K_n \arrow[r, "q_{2\#}|_{K_n}"] \arrow[d] & \langle \Delta_2^2 \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] & P_n(\mathbb{R}P^2) \arrow[r, "q_{2\#}"] & P_2(\mathbb{R}P^2) \arrow[r] & 1,
\end{tikzcd} \tag{6}$$

where the vertical arrows are inclusions. Now $\langle \Delta_2^2 \rangle \cong \mathbb{Z}_2$, so K_n is an extension of the group $\operatorname{Ker}(q_{2\#}|_{K_n}) = K_n \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ by $\mathbb{Z}_2$. The fact that $q_{2\#}(\Delta_n^2) = \Delta_2^2$ implies that the upper short exact sequence splits, a section being defined by the correspondence $\Delta_2^2 \mapsto \Delta_n^2$, and since $\Delta_n^2 \in Z(P_n(\mathbb{R}P^2))$, the action by conjugation on $\operatorname{Ker}(q_{2\#}|_{K_n})$ is trivial. Part (a) of the proposition follows by taking $L_n = \operatorname{Ker}(q_{2\#}|_{K_n})$ and by noting that $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ is torsion-free.

(b) Recall first that any torsion element in $P_n(\mathbb{R}P^2) \setminus \langle \Delta_n^2 \rangle$ is of order 4 [Gonçalves and Guaschi 2004a, Corollary 19 and Proposition 23], and is conjugate in $B_n(\mathbb{R}P^2)$ to one of a^n or b^{n-1}, where $a = \rho_n \sigma_{n-1} \cdots \sigma_1$ and $b = \rho_{n-1} \sigma_{n-2} \cdots \sigma_1$ satisfy

$$a^n = \rho_n \cdots \rho_1 \quad \text{and} \quad b^{n-1} = \rho_{n-1} \cdots \rho_1 \tag{7}$$

by [Gonçalves and Guaschi 2010b, Proposition 10]. Let N be a normal subgroup of $B_n(\mathbb{R}P^2)$ that satisfies $K_n \subsetneq N \subset P_n(\mathbb{R}P^2)$. We claim that for all $u \in \pi_1\big(\prod_1^n \mathbb{R}P^2\big)$ (which we identify henceforth with $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$), exactly one of the following two conditions holds:

(i) $N \cap \iota_\#^{-1}(\{u\})$ is empty.

(ii) $\iota_\#^{-1}(\{u\})$ is contained in N.

To prove the claim, suppose that $x \in N \cap \iota_\#^{-1}(\{u\}) \neq \varnothing$, and let $y \in \iota_\#^{-1}(\{u\})$. Now $\iota_\#(x) = \iota_\#(y) = u$, so there exists $k \in K_n$ such that $x^{-1}y = k$. Since $K_n \subset N$, it follows that $y = xk \in N$, which proves the claim. Further, $\iota_\#(a^n) = (\bar{1}, \dots, \bar{1})$ and $\iota_\#(b^{n-1}) = (\bar{1}, \dots, \bar{1}, \bar{0})$ by Proposition 8 and equations (5) and (7), so by the claim it suffices to prove that there exists $z \in N$ such that $\iota_\#(z) \in \{(\bar{1}, \dots, \bar{1}), (\bar{1}, \dots, \bar{1}, \bar{0})\}$, for then we are in case (ii) above, and it follows that one of a^n or b^{n-1} belongs to N.

It thus remains to prove the existence of such a z. Let $x \in N \setminus K_n$. Then $\iota_\#(x)$ contains an entry equal to $\bar{1}$ because $K_n = \operatorname{Ker}(\iota_\#)$. If $\iota_\#(x) = (\bar{1}, \dots, \bar{1})$ then we are done. So assume that $\iota_\#(x)$ also contains an entry that is equal to $\bar{0}$. By (5), there exist $1 \le r < n$ and $1 \le i_1 < \cdots < i_r \le n$ such that $\iota_\#(\rho_{i_1} \cdots \rho_{i_r}) = \iota_\#(x)$. It follows from the claim and the fact that $x \in N$ that $\rho_{i_1} \cdots \rho_{i_r} \in N$ also, and so without loss of generality, we may suppose that $x = \rho_{i_1} \cdots \rho_{i_r}$. Further, since $\iota_\#(x)$ contains both a $\bar{0}$ and a $\bar{1}$, there exists $1 \le j \le r$ such that $p_{i_j\#}(x) = \bar{1}$ and $p_{(i_j+1)\#}(x) = \bar{0}$, the homomorphisms $p_{k\#}$ being those defined in the proof of Proposition 8. Note that we consider the indices modulo n, so if $i_j = n$ (so $j = r$) then we set $i_j + 1 = 1$. By [Gonçalves and Guaschi 2004a, page 777], conjugation by a^{-1} permutes cyclically the elements $\rho_1, \dots, \rho_n, \rho_1^{-1}, \dots, \rho_n^{-1}$ of $P_n(\mathbb{R}P^2)$, so the $(n-1)$-st (resp. n-th) entry of $x' = a^{-(n-1-i_j)} x a^{(n-1-i_j)}$ is equal to $\bar{1}$ (resp. $\bar{0}$), and $x' \in N$ because N is normal in $B_n(\mathbb{R}P^2)$. Using the relation $b = \sigma_{n-1} a$, we determine the conjugates of the ρ_i by b^{-1}:

$$b^{-1}\rho_i b = a^{-1}\sigma_{n-1}^{-1}\rho_i\sigma_{n-1}a = a^{-1}\rho_i a = \rho_{i+1} \quad \text{for all } 1 \le i \le n-2,$$

$$\begin{aligned} b^{-1}\rho_{n-1}b &= a^{-1}\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}a = a^{-1}\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}^{-1}.\,\sigma_{n-1}^2 a \\ &= a^{-1}\rho_n a.\,a^{-1}\sigma_{n-1}^2 a = \rho_1^{-1}.\,a^{-1}\sigma_{n-1}^2 a, \end{aligned}$$

where we used the relations $\rho_i\sigma_{n-1} = \sigma_{n-1}\rho_i$ if $1 \le i \le n-2$ and $\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}^{-1} = \rho_n$ of Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$, as well as the effect of conjugation by a^{-1} on the ρ_j. Now $\sigma_{n-1}^2 = A_{n-1,n} \in K_n$ by Proposition 8, so $a^{-1}\sigma_{n-1}^2 a \in K_n$ by Remark 9(a), and hence $\iota_\#(b^{-1}\rho_{n-1}b) = (\bar{1}, \bar{0}, \dots, \bar{0})$. It then follows that $\iota_\#(a^{-1}x'a)$ and $\iota_\#(b^{-1}x'b)$ have the same entries except in the first and last positions, so if $x'' = a^{-1}x'a.\,b^{-1}x'b$, we have $\iota_\#(x'') = (\bar{1}, \bar{0}, \dots, \bar{0}, \bar{1})$. Further, $x'' \in N$ since N is normal in $B_n(\mathbb{R}P^2)$. Let $n = 2m + \varepsilon$, where $m \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. Then setting

$$z = a^{-\varepsilon}x''a^{\varepsilon} \cdot a^{-(2+\varepsilon)}x''a^{2+\varepsilon} \cdots a^{-(2(m-1)+\varepsilon)}x''a^{2(m-1)+\varepsilon},$$

we see once more that $z \in N$, and $\iota_\#(z) = (\overline{1}, \dots, \overline{1})$ if n is even and $\iota_\#(z) = (\overline{1}, \dots, \overline{1}, \overline{0})$ if n is odd, which completes the proof of the existence of z, and thus that of Proposition 1(b). □

Proof of Theorem 2. Let $S = \mathbb{S}^2$ or $\mathbb{R}P^2$. If $n = 1$ then $\iota_\#$ is an isomorphism and $\operatorname{Im}(j_\#|_{P_n})$ is trivial so the result holds. If $n = 2$ and $S = \mathbb{S}^2$ then $P_n(\mathbb{S}^2)$ is trivial, and there is nothing to prove. Now suppose that $S = \mathbb{S}^2$ and $n \geq 3$. As we mentioned in the Introduction, $\operatorname{Ker}(\iota_\#) = P_n(\mathbb{S}^2)$. Let $(A_{i,j})_{1 \leq i < j \leq n}$ be the generating set of P_n, where $A_{i,j}$ has a geometric representative similar to that given in Figure 1. It is well known that the image of this set by $j_\#$ yields a generating set for $P_n(\mathbb{S}^2)$ (see [Scott 1970, page 616]), so $j_\#|_{P_n}$ is surjective, and the statement of the theorem follows. Finally, assume that $S = \mathbb{R}P^2$ and $n \geq 2$. Once more, $\operatorname{Im}(j_\#|_{P_n}) = \langle A_{i,j} \mid 1 \leq i < j \leq n \rangle$, and since $A_{i,j} \in \operatorname{Ker}(\iota_\#)$ by Proposition 8, we conclude that $\langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} \subset \operatorname{Ker}(\iota_\#)$. To prove the converse, first recall from Proposition 8 that $\operatorname{Ker}(\iota_\#) = \Gamma_2(P_n(\mathbb{R}P^2))$. Using the standard commutator identities

$$[x, yz] = [x, y]\big[y, [x, z]\big][x, z]$$

and

$$[xy, z] = \big[x, [y, z]\big][y, z][x, z],$$

$\Gamma_2(P_n(\mathbb{R}P^2))$ is equal to the normal closure in $P_n(\mathbb{R}P^2)$ of the set

$$\Big\{[x, y] \,\Big|\, x, y \in \{A_{i,j},\ \rho_k \mid 1 \leq i < j \leq n \text{ and } 1 \leq k \leq n\}\Big\}.$$

It then follows using the relations of Theorem 7 that the commutators $[x, y]$ belonging to this set also belong to $\langle\langle A_{i,j} \mid 1 \leq i < j \leq n \rangle\rangle_{P_n(\mathbb{R}P^2)}$, which is nothing other than $\langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)}$. We conclude by normality that $\operatorname{Ker}(\iota_\#) \subset \langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)}$. □

3. Some properties of the subgroup L_n

Let $S = \mathbb{S}^2$ or $S = \mathbb{R}P^2$, and for all $m, n \geq 1$, let $\Gamma_{m,n}(S) = P_m(S \setminus \{x_1, \dots, x_n\})$ denote the m-string pure braid group of S with n points removed. In this section, we study $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, which is $\Gamma_{n-2,2}(\mathbb{R}P^2)$, in more detail, and we prove Theorem 3 and Proposition 4, which enable us to understand better the structure of the subgroup L_n defined in the proof of Proposition 1(a)(ii).

We start by exhibiting a presentation of the group $\Gamma_{m,n}(\mathbb{R}P^2)$ in terms of the generators of $P_{m+n}(\mathbb{R}P^2)$ described at the beginning of Section 2. A presentation for $\Gamma_{m,n}(\mathbb{S}^2)$ is given in [Gonçalves and Guaschi 2005, Proposition 7] and will be recalled later in Proposition 15, when we come to proving Theorem 5. For $1 \leq i < j \leq m+n$, let

$$C_{i,j} = A_{j-1,j}^{-1} \cdots A_{i+1,j}^{-1} A_{i,j} A_{i+1,j} \cdots A_{j-1,j} \tag{8}$$

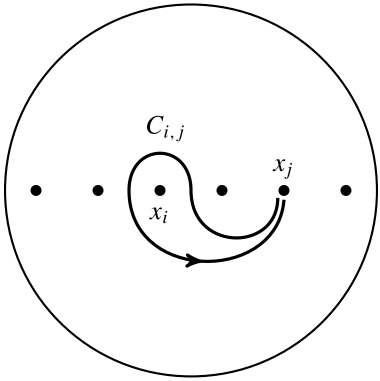


Figure 2. The element $C_{i,j}$ in $P_{m+n}(\mathbb{R}P^2)$.

in $P_{m+n}(\mathbb{R}P^2)$ (see Figure 2). In what follows, any element of the form $A_{i,j}$ or $C_{i,j}$, where $i \geq j$, should be interpreted as the trivial element. The proof of the following proposition is similar in nature to that for $\mathbb{S}^2$, but is a little more involved due to the presence of extra generators that emanate from the fundamental group of $\mathbb{R}P^2$.

Proposition 11. *Let $n, m \geq 1$. The following constitutes a presentation of the group $\Gamma_{m,n}(\mathbb{R}P^2)$:*

Generators: *$A_{i,j}$, ρ_j, where $1 \leq i < j$ and $n+1 \leq j \leq m+n$.*

Relations:

(I) *The Artin relations described by (4) among the generators $A_{i,j}$ of $\Gamma_{m,n}(\mathbb{R}P^2)$.*

(II) *For all $1 \leq i < j$ and $n+1 \leq j < k \leq m+n$, $A_{i,j}\rho_k A_{i,j}^{-1} = \rho_k$.*

(III) *For all $1 \leq i < j$ and $n+1 \leq k < j \leq m+n$,*

$$\rho_k A_{i,j} \rho_k^{-1} = \begin{cases} A_{i,j} & \text{if } k < i, \\ \rho_j^{-1} C_{i,j}^{-1} \rho_j & \text{if } k = i, \\ \rho_j^{-1} C_{k,j}^{-1} \rho_j A_{i,j} \rho_j^{-1} C_{k,j} \rho_j & \text{if } k > i. \end{cases}$$

(IV) *For all $n+1 \leq k < j \leq m+n$, $\rho_k \rho_j \rho_k^{-1} = C_{k,j}\rho_j$.*

(V) *For all $n+1 \leq j \leq m+n$,*

$$\rho_j \left(\prod_{i=1}^{j-1} A_{i,j} \right) \rho_j = \left(\prod_{l=j+1}^{m+n} A_{j,l} \right).$$

The elements $C_{i,j}$ and $C_{k,j}$ appearing in relations (III) and (IV) should be rewritten using (8).

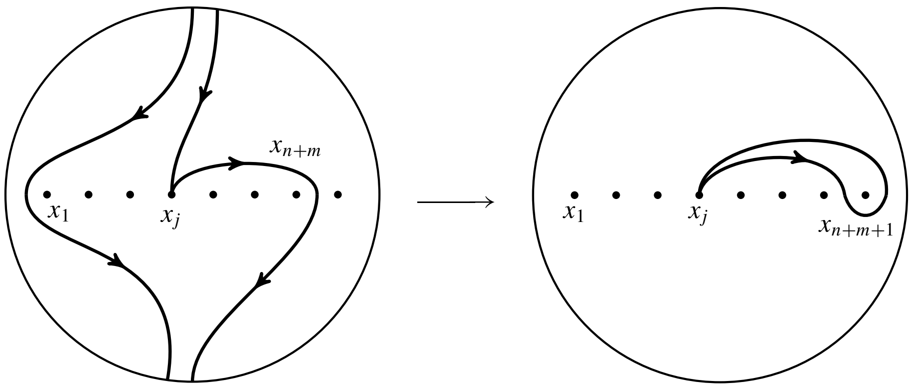


Figure 3. The relation $\left(\prod_{l=j+1}^{m+n} A_{j,l}\right)^{-1} \rho_j \left(\prod_{i=1}^{j-1} A_{i,j}\right) \rho_j = A_{j,n+m+1}$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $n+1 \le j \le n+m+1$.

Proof. If $m, n \ge 1$, we have the following Fadell–Neuwirth short exact sequence of pure braid groups of $\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}$:

$$(9) \quad 1 \to P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}\}) \\ \to \Gamma_{m+1,n}(\mathbb{R}P^2) \xrightarrow{q} \Gamma_{m,n}(\mathbb{R}P^2) \to 1,$$

where the homomorphism q is given geometrically by forgetting the last string. The generators $A_{i,j}$ and ρ_j of $\Gamma_{m,n}(\mathbb{R}P^2)$ given in the statement of the proposition are represented geometrically as in Figure 1, and the basepoints of the m strings of $\Gamma_{m,n}(\mathbb{R}P^2)$ are the points $x_{n+1}, \dots, x_{n+m}$. Using induction on m, we apply standard methods to obtain a group presentation of an extension from presentations of the kernel and the quotient [Johnson 1997, Proposition 1, Chapter 10], using the geometric representations of Figure 1 to derive some of the relations.

Let $n \ge 1$. If $m = 1$ then $\Gamma_{1,n}(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}, x_{n+1})$ is generated by $\{A_{i,n+1}, \rho_{n+1} \mid 1 \le i \le n\}$ subject to the surface relation $\prod_{i=1}^{n} A_{i,n+1} = \rho_{n+1}^{-2}$, which is equivalent to the single relation given by (V). Since the remaining relations (I)–(IV) are empty, the given presentation of $\Gamma_{1,n}(\mathbb{R}P^2)$ is correct.

Now suppose that the given presentation of $\Gamma_{m,n}(\mathbb{R}P^2)$ is correct for some $m \ge 1$. We shall show that we obtain the presentation of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ by applying the above-mentioned methods to the short exact sequence (9). Although $\operatorname{Ker}(q)$ is a free group, it shall be convenient to consider it as the group with generating set

$$Y_{n+m+1} = \{A_{i,n+m+1},\ \rho_{n+m+1} \mid 1 \le i \le n+m\},$$

subject to the single relation $\rho_{n+m+1}\left(\prod_{i=1}^{n+m} A_{i,n+m+1}\right)\rho_{n+m+1} = 1$ (this may be seen by taking $j = n+m+1$ in Figure 3). According to [Johnson 1997, Proposition 1, Chapter 10], $\Gamma_{m+1,n}(\mathbb{R}P^2)$ is generated by the union of Y_{n+m+1} with the set of coset representatives

$$X_{m,n} = \{A_{i,j},\ \rho_j \mid 1 \le i < j \text{ and } n+1 \le j \le m+n\}$$

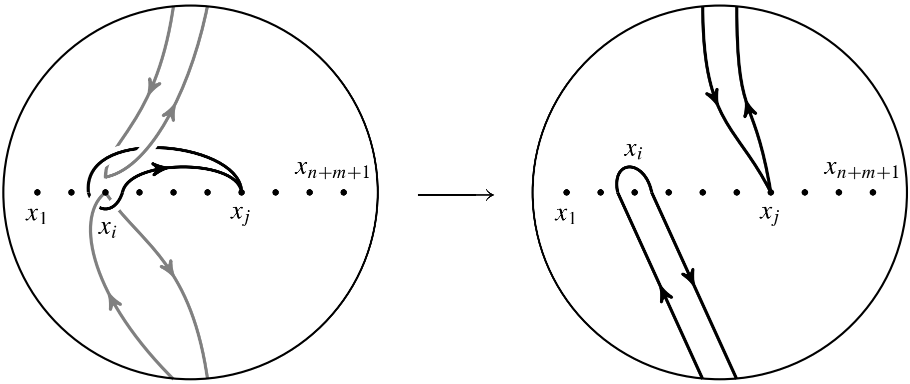


Figure 4. The relation $\rho_i A_{i,j} \rho_i^{-1} = \rho_j^{-1} C_{i,j}^{-1} \rho_j$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $1 \le i < j$ and $n+1 \le j \le n+m+1$.

in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ of the given set of generators of $\Gamma_{m,n}(\mathbb{R}P^2)$. This yields the required set of generators of $\Gamma_{m+1,n}(\mathbb{R}P^2)$. Once more by [Johnson 1997, Proposition 1, Chapter 10], there are three types of relation in $\Gamma_{m+1,n}(\mathbb{R}P^2)$:

(1) the (single) given relation of $\mathrm{Ker}(q)$, which yields the surface relation (V) with $j = n+m+1$;

(2) the relators of $\Gamma_{m,n}(\mathbb{R}P^2)$, rewritten in terms of the elements of Y_{n+m+1};

(3) the conjugates of the elements of Y_{n+m+1} by the elements of $X_{m,n}$, also rewritten in terms of the elements of Y_{n+m+1}.

Let us study the relations of type (2) using the geometric representatives given in Figure 1. The Artin relations (I) of $\Gamma_{m,n}(\mathbb{R}P^2)$ lift directly to relations in $\Gamma_{m+1,n}(\mathbb{R}P^2)$, and yield the relations (I) of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for all $n+1 \le j \le n+m$. The relations (II) (resp. relations (III) with $k < i$) of $\Gamma_{m,n}(\mathbb{R}P^2)$ involve elements that are represented geometrically by disjoint loops. They also lift directly to relations in $\Gamma_{m+1,n}(\mathbb{R}P^2)$, and yield the relations (II) (resp. relations (III) with $k < i$) of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for all $k \le n+m$ (resp. for all $n+1 \le j \le n+m$). The relations (III) with $k = i$ or $k > i$ (resp. relations (IV)) of $\Gamma_{m,n}(\mathbb{R}P^2)$ are represented in Figures 4, 5 and 6 respectively (in $\Gamma_{m,n}(\mathbb{R}P^2)$, the point x_{n+m+1} is unmarked), and from these figures, we see that each of the relations also lifts directly to $\Gamma_{m+1,n}(\mathbb{R}P^2)$. We thus obtain all of the relations (I)–(IV) of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for all $n+1 \le j \le n+m$ in relations (I), (III) and (IV), and for all $k \le n+m$ in relation (II). From Figure 3, we observe that $\bigl(\prod_{l=j+1}^{m+n} A_{j,l}\bigr)^{-1} \rho_j \bigl(\prod_{i=1}^{j-1} A_{i,j}\bigr) \rho_j = A_{j,n+m+1}$ for all $n+1 \le j \le n+m$. Together with the relation of type (1), this yields all of the relations (V) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$. It remains to determine the relations of type (3).

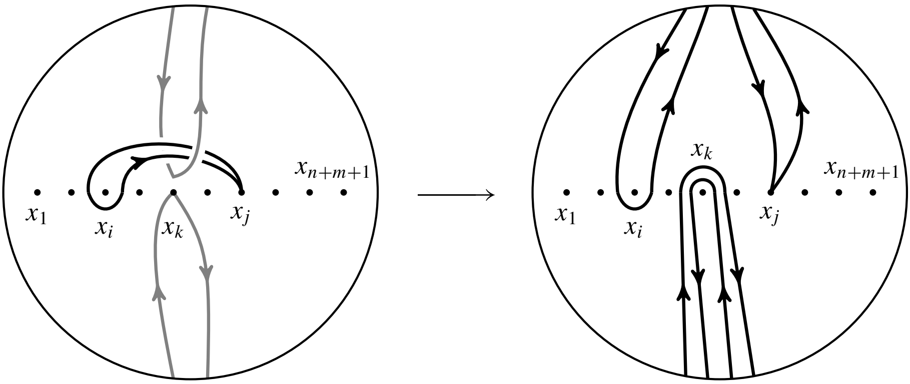


Figure 5. The relation $\rho_k A_{i,j} \rho_k^{-1} = \rho_j^{-1} C_{k,j}^{-1} \rho_j A_{i,j} \rho_j^{-1} C_{k,j} \rho_j$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $1 \leq i < k < j$ and $n+1 \leq k < j \leq n+m+1$.

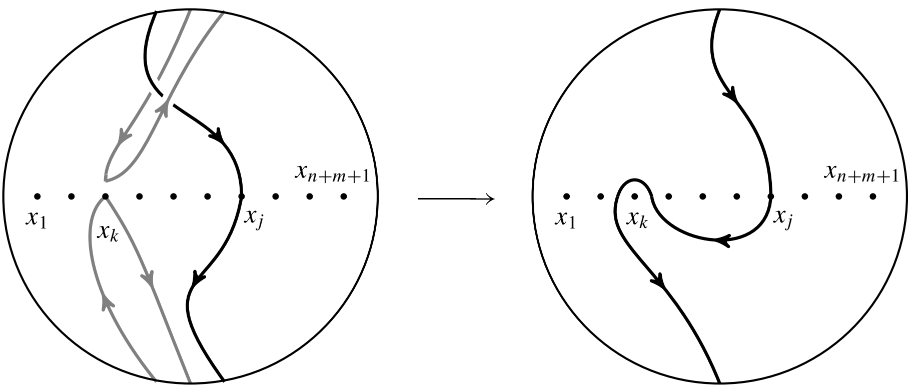


Figure 6. The relation $\rho_k \rho_j \rho_k^{-1} = C_{k,j} \rho_j$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $n+1 \leq k < j \leq n+m+1$.

• If $A_{i,j} \in X_{m,n}$ and $A_{k,n+m+1} \in Y_{n+m+1}$ then $A_{i,j} A_{k,n+m+1} A_{i,j}^{-1}$ is given by the Artin relations (4), and together with the Artin relations of type (2), we obtain all of the relations (I) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$.

• If $A_{i,j} \in X_{m,n}$ and $\rho_{n+m+1} \in Y_{n+m+1}$, then since $j \leq n+m$, $A_{i,j}$ and ρ_{n+m+1} commute since they are represented geometrically by disjoint loops. This yields relations (II) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ with $k = n+m+1$. Together with the corresponding relations of type (2), we obtain all of the relations (II) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$.

• If $\rho_k \in X_{m,n}$ and $A_{i,n+m+1} \in Y_{n+m+1}$, we consider three cases:

(a) If $k < i$, then ρ_k and $A_{i,n+m+1}$ commute since they are represented geometrically by disjoint loops.

(b) If $k = i$, we obtain $\rho_i A_{i,n+m+1} \rho_i^{-1} = \rho_{n+m+1}^{-1} C_{i,n+m+1}^{-1} \rho_{n+m+1}$ by taking $j = n+m+1$ in Figure 4.

(c) If $k > i$, by taking $j = n+m+1$ in Figure 5, we see that $\rho_k A_{i,n+m+1} \rho_k^{-1} = \rho_{n+m+1}^{-1} C_{k,n+m+1}^{-1} \rho_{n+m+1} A_{i,n+m+1} \rho_{n+m+1}^{-1} C_{k,n+m+1} \rho_{n+m+1}$.

Together with the corresponding relations of type (2), we obtain all of the relations (III) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$.

• If $\rho_k \in X_{m,n}$ and $\rho_{n+m+1} \in Y_{n+m+1}$ then by taking $j = n+m+1$ in Figure 6, we see that $\rho_k \rho_{n+m+1} \rho_k^{-1} = C_{k,n+m+1} \rho_{n+m+1}$, which yields relations (IV) with $j = n+m+1$. Together with the corresponding relations of type (2), we obtain all of the relations (IV) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$. □

In the rest of this section, we shall assume that $n = 2$, and we shall focus our attention on the groups $\Gamma_{m,2}(\mathbb{R}P^2)$, where $m \geq 1$, which we interpret as subgroups of $P_{m+2}(\mathbb{R}P^2)$ via the short exact sequence (3). Before proving Theorem 3 and Proposition 4, we introduce some notation that will be used to study the subgroups K_n and L_n. Let $m \geq 2$, and consider the Fadell–Neuwirth short exact sequence

$$(10) \qquad 1 \to \Omega_{m+1} \to P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \xrightarrow{r_{m+1}} P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to 1,$$

where r_{m+1} is given geometrically by forgetting the last string, and where $\Omega_{m+1} = \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_{m+1}\}, x_{m+2})$. From the Fadell–Neuwirth short exact sequences of the form of (3), r_{m+1} is the restriction of $q_{(m+1)\#} : P_{m+2}(\mathbb{R}P^2) \to P_{m+1}(\mathbb{R}P^2)$ to $\mathrm{Ker}(q_{2\#})$. The kernel Ω_{m+1} of r_{m+1} is a free group of rank $m+1$ with a basis $\mathcal{B}_{m+1}$ being given by

$$(11) \qquad \mathcal{B}_{m+1} = \{A_{k,m+2}, \rho_{m+2} \mid 1 \leq k \leq m\}.$$

The group Ω_{m+1} may also be described as the subgroup of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ generated by $\{A_{1,m+2}, \ldots, A_{m+1,m+2}, \rho_{m+2}\}$ subject to the relation

$$(12) \qquad A_{m+1,m+2} = A_{m,m+2}^{-1} \cdots A_{1,m+2}^{-1} \rho_{m+2}^{-2},$$

obtained from relation (V) of Proposition 11. Equations (8) and (12) imply notably that $A_{l,m+2}$ and $C_{l,m+2}$ belong to Ω_{m+1} for all $1 \leq l \leq m+1$. Using geometric methods, for $m \geq 2$, we proved the existence of a section

$$s_{m+1} : P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$$

for r_{m+1} in [Gonçalves and Guaschi 2010a, Theorem 2(a)]. Applying induction to (10), it follows that for all $m \geq 1$,

$$(13) \qquad P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \Omega_{m+1} \rtimes (\Omega_m \rtimes (\cdots \rtimes (\Omega_3 \rtimes \Omega_2) \cdots)).$$

So $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_{m+1} \rtimes (\mathbb{F}_m \rtimes (\cdots \rtimes (\mathbb{F}_3 \rtimes \mathbb{F}_2)\cdots))$, which may be interpreted as the Artin combing operation for $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. It follows from this and (11) that $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ admits $\mathcal{X}_{m+2}$ as a generating set, where

$$\mathcal{X}_{m+2} = \{A_{i,j},\ \rho_j \mid 3 \le j \le m+2,\ 1 \le i \le j-2\}. \tag{14}$$

Remark 12. For what follows, we will need to know an explicit section s_{m+1} for r_{m+1}. Such a section may be obtained as follows: for $m \ge 2$, consider the homomorphism $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by forgetting the string based at x_3. By [Gonçalves and Guaschi 2010a, Theorem 2(a)]), a geometric section is obtained by doubling the second (vertical) string, so that there is a new third string, and renumbering the following strings, which gives rise to an algebraic section for the given homomorphism of the form

$$A_{i,j} \mapsto \begin{cases} A_{1,j+1} & \text{if } i = 1,\\ A_{2,j+1}A_{3,j+1} & \text{if } i = 2,\\ A_{i+1,j+1} & \text{if } 3 \le i < j,\end{cases}$$
$$\rho_j \mapsto \rho_{j+1}$$

for all $3 \le j \le m+1$. However, in view of the nature of r_{m+1}, we would like this new string to be in the $(m+2)$-nd position. We achieve this by composing the above algebraic section with conjugation by $\sigma_{m+1}\cdots\sigma_3$, which gives rise to a section

$$s_{m+1} : P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$$

for r_{m+1} that is defined by

$$\begin{cases} s_{m+1}(A_{i,j}) = \begin{cases} A_{j,m+2}A_{1,j}A_{j,m+2}^{-1} & \text{if } i = 1,\\ A_{j,m+2}A_{2,j} & \text{if } i = 2,\\ A_{i,j} & \text{if } 3 \le i < j,\end{cases}\\ s_{m+1}(\rho_j) = \rho_j A_{j,m+2}^{-1}\end{cases} \tag{15}$$

for all $1 \le i < j$ and $3 \le j \le m+1$. A long but straightforward calculation using the presentation of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by Proposition 11 shows that s_{m+1} does indeed define a section for r_{m+1}.

We now prove Theorem 3, which lets us give a more explicit description of L_n.

Proof of Theorem 3. Let $n \ge 3$. By the commutative diagram (6) of short exact sequences, the restriction of the homomorphism $q_{2\#} : P_n(\mathbb{R}P^2) \to P_2(\mathbb{R}P^2)$ to K_n factors through the inclusion $\langle \Delta_2^2 \rangle \to P_2(\mathbb{R}P^2)$, and the kernel L_n of $q_{2\#}|_{K_n}$ is contained in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. We may then add a third row to this diagram:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & L_n \arrow[r] \arrow[d] & K_n \arrow[r, "q_{2\#}|_{K_n}"] \arrow[d] & \langle \Delta_2^2 \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] \arrow[d, dashed, "\hat{\iota}_{n-2}"] & P_n(\mathbb{R}P^2) \arrow[r, "q_{2\#}"] \arrow[d, "\iota_{n\#}"] & P_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "\iota_{2\#}"] & 1 \\
1 \arrow[r] & \mathbb{Z}_2^{n-2} \arrow[r, "\ell"] \arrow[d, dashed] & \mathbb{Z}_2^n \arrow[r, "\hat{q}_2"] \arrow[d] & \mathbb{Z}_2^2 \arrow[r] \arrow[d] & 1, \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{16}
$$

where $\hat{q}_2 : \mathbb{Z}_2^n \to \mathbb{Z}_2^2$ is projection onto the first two factors, and $\ell : \mathbb{Z}_2^{n-2} \to \mathbb{Z}_2^n$ is the monomorphism defined by

$$\ell(\overline{\varepsilon_1}, \ldots, \overline{\varepsilon_{n-2}}) = (\bar{0}, \bar{0}, \overline{\varepsilon_1}, \ldots, \overline{\varepsilon_{n-2}}).$$

The commutativity of diagram (16) thus induces a homomorphism

$$\hat{\iota}_{n-2} : P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to \mathbb{Z}_2^{n-2}$$

that is the restriction of $\iota_{n\#}$ to $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ that makes the bottom left-hand square commute. To see that $\hat{\iota}_{n-2}$ is surjective, notice that if $x \in \mathbb{Z}_2^{n-2}$ then the first two entries of $\ell(x)$ are equal to $\bar{0}$, and using (5), it follows that there exist $3 \le i_1 < \cdots < i_r \le n$ such that $\iota_{n\#}(\rho_{i_1} \cdots \rho_{i_r}) = \ell(x)$. Furthermore, $\rho_{i_1} \cdots \rho_{i_r} \in \operatorname{Ker}(q_{2\#})$, and by the commutativity of the diagram, we also have $\iota_{n\#}(\rho_{i_1} \cdots \rho_{i_r}) = \ell \circ \hat{\iota}_{n-2}(\rho_{i_1} \cdots \rho_{i_r})$, whence $x = \hat{\iota}_{n-2}(\rho_{i_1} \cdots \rho_{i_r})$ by the injectivity of ℓ. It remains to prove the exactness of the first column. The fact that $L_n \subset \operatorname{Ker}(\hat{\iota}_{n-2})$ follows easily. Conversely, if $x \in \operatorname{Ker}(\hat{\iota}_{n-2})$ then $x \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, and $x \in K_n$ by the commutativity of the diagram, so $x \in L_n$. This proves the first two assertions of the theorem.

To prove the last part of the theorem, let $m \ge 1$, and consider (10). Since $\hat{\iota}_m$ is the restriction of $\iota_{(m+2)\#}$ to $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, we have $\hat{\iota}_m(\rho_j) = (\bar{0}, \ldots, \bar{0}, \bar{1}, \bar{0}, \ldots, \bar{0})$, where $\bar{1}$ is the in the $(j-2)$-nd position, and $\hat{\iota}_m(A_{i,j}) = (\bar{0}, \ldots, \bar{0})$ for all $1 \le i < j$ and $3 \le j \le m+2$. So for each $2 \le l \le m+1$, $\hat{\iota}_m$ restricts to a surjective homomorphism $\hat{\iota}_m|_{\Omega_l} : \Omega_l \to \mathbb{Z}_2$ of each of the factors of (13), with $\mathbb{Z}_2$ being the $(l-1)$-st factor of $\mathbb{Z}_2^m$, and using (11), we see that $\operatorname{Ker}(\hat{\iota}_m|_{\Omega_l})$ is a free group of rank $2l-1$ with basis $\widehat{\mathcal{B}}_l$ given by

$$\widehat{\mathcal{B}}_l = \{ A_{k,l+1}, \rho_{l+1} A_{k,l+1} \rho_{l+1}^{-1}, \rho_{l+1}^2 \mid 1 \le k \le l-1 \}. \tag{17}$$

As we shall now explain, for all $m \geq 2$, the short exact sequence (10) may be extended to a commutative diagram of short exact sequences as follows:

$$
\begin{tikzcd}
& 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \operatorname{Ker}(\hat{\iota}_m|_{\Omega_{m+1}}) \arrow[r] \arrow[d] & L_{m+2} \arrow[r, "r_{m+1}|_{L_{m+2}}"] \arrow[d] & L_{m+1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Omega_{m+1} \arrow[r] \arrow[d, "\hat{\iota}_m|_{\Omega_{m+1}}"] & P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r, shift left, "r_{m+1}"] \arrow[d, "\hat{\iota}_m"] & P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[l, shift left, dashed, "s_{m+1}"] \arrow[r] \arrow[d, "\hat{\iota}_{m-1}"] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathbb{Z}_2^m \arrow[r] \arrow[d] & \mathbb{Z}_2^{m-1} \arrow[r] \arrow[d] & 1. \\
& 1 & 1 & 1 &
\end{tikzcd}
\tag{18}
$$

To obtain this diagram, we start with the commutative diagram that consists of the second and third rows and the three columns (so a priori, the arrows of the first row are missing). The commutativity implies that r_{m+1} restricts to the homomorphism $r_{m+1}|_{L_{m+2}} : L_{m+2} \to L_{m+1}$, which is surjective, since if $w \in L_{m+1}$ is written in terms of the elements of $\mathcal{X}_{m+1}$ then the same word w, considered as an element of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, belongs to L_{m+2}, and satisfies $r_{m+1}(w) = w$. Then the kernel of $r_{m+1}|_{L_{m+2}}$, which is also the kernel of $\hat{\iota}_m|_{\Omega_{m+1}}$, is equal to $L_{m+2} \cap \Omega_{m+1}$. This establishes the existence of the complete commutative diagram (18) of short exact sequences. By induction, it follows from (17) and (18) that for all $m \geq 1$, L_{m+2} is generated by

$$
\widehat{\mathcal{X}}_{m+2} = \bigcup_{j=3}^{m+2} \widehat{\mathcal{B}}_{j-1} = \{A_{i,j},\ \rho_j A_{i,j} \rho_j^{-1},\ \rho_j^2 \mid 3 \leq j \leq m+2,\ 1 \leq i \leq j-2\}. \tag{19}
$$

By (15), for each $x \in \widehat{\mathcal{X}}_{m+1}$, $\hat{\iota}_m \circ s_{m+1}(x)$ is the trivial element of $\mathbb{Z}_2^m$, and thus $s_{m+1}(x) \in L_{m+2}$. Hence s_{m+1} restricts to a section $s_{m+1}|_{L_{m+1}} : L_{m+1} \to L_{m+2}$ for $r_{m+1}|_{L_{m+2}}$. We conclude by induction on the first row of (18) that

$$
L_{m+2} \cong \operatorname{Ker}(\hat{\iota}_m|_{\Omega_{m+1}}) \rtimes L_{m+1} \tag{20}
$$

$$
\cong \operatorname{Ker}(\hat{\iota}_m|_{\Omega_{m+1}}) \rtimes \Big(\operatorname{Ker}(\hat{\iota}_m|_{\Omega_m}) \rtimes \Big(\cdots \rtimes \Big(\operatorname{Ker}(\hat{\iota}_m|_{\Omega_3}) \rtimes \operatorname{Ker}(\hat{\iota}_m|_{\Omega_2})\Big) \cdots\Big)\Big), \tag{21}
$$

the actions being induced by those of (13), so by (17), L_{m+2} is isomorphic to a repeated semidirect product of the form $\mathbb{F}_{2m+1} \rtimes (\mathbb{F}_{2m-1} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots))$. The last part of the statement of Theorem 3 follows by taking $m = n-2$. □

A finer analysis of the actions that appear in (13) and (21) now allows us to determine the Abelianisations of $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and L_n.

Proof of Proposition 4. If $n = 3$ then the two assertions are clear. So assume by induction that they hold for some $n \geq 3$. From the split short exact sequence (10) and (20) with $m = n-1$, we have

$$\begin{cases} P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \Omega_n \rtimes_\psi P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}), \\ \qquad\qquad\qquad L_{n+1} \cong \operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n}) \rtimes_\psi L_n, \end{cases} \tag{22}$$

where ψ denotes the action given by the section s_n, as well as the action induced by the restriction of the section s_n to L_n.

Before going any further, we recall some general considerations from [Gonçalves and Guaschi 2009, pages 3387–88] concerning the Abelianisation of semidirect products. If H and K are groups, and if $\varphi : H \to \operatorname{Aut}(K)$ is an action of H on K, then one may deduce easily from Proposition 3.3 of that paper that

$$(K \rtimes_\varphi H)^{\mathrm{Ab}} \cong \Delta(K) \oplus H^{\mathrm{Ab}}, \tag{23}$$

where

$$\Delta(K) = K/K_1, \quad K_1 = \langle \Gamma_2(K) \cup \widehat{K} \rangle \quad \text{and} \quad \widehat{K} = \langle \varphi(h)(k) \cdot k^{-1} \mid h \in H \text{ and } k \in K \rangle.$$

Recall that $\widehat{K}$ is normal in K (see [Gonçalves and Guaschi 2009, lines 1–4, page 3388]), so K_1 is normal in K, $K_1 = \Gamma_2(K).\widehat{K} = \widehat{K}.\Gamma_2(K)$, and $\Delta(K) \cong K^{\mathrm{Ab}}/p(\widehat{K})$, where $p : K \to K^{\mathrm{Ab}}$ is the canonical projection. If $k \in K$, let $\bar{k} = p(k)$. For all $k, k' \in K$ and $h, h' \in H$, we have

$$\varphi(h^{-1})(k) \cdot k^{-1} = \big(\varphi(h)(\varphi(h^{-1})(k)) \cdot (\varphi(h^{-1})(k))^{-1}\big)^{-1}, \tag{24}$$

$$\varphi(h)(k^{-1}) \cdot k = \big(k^{-1}(\varphi(h)(k) \cdot k^{-1})k\big)^{-1}, \tag{25}$$

$$\begin{aligned} \varphi(hh')(k) \cdot k^{-1} &= \varphi(h)(\varphi(h')(k)) \cdot \varphi(h')(k^{-1}) \cdot \varphi(h')(k) \cdot k^{-1} \\ &= \varphi(h)(k'') \cdot k''^{-1} \cdot \varphi(h')(k) \cdot k^{-1}, \end{aligned} \tag{26}$$

$$\varphi(h)(kk') \cdot (kk')^{-1} = \big(\varphi(h)(k) \cdot k^{-1}\big) \cdot k\big(\varphi(h)(k') \cdot k'^{-1}\big)k^{-1}, \tag{27}$$

where $k'' = \varphi(h')(k)$ belongs to K. Let $\mathcal{H}$ and $\mathcal{K}$ be generating sets for H and K, respectively. By induction on word length relative to the elements of $\mathcal{H}$, (24) and (26) imply that $\widehat{K}$ is generated by elements of the form $\varphi(h)(k) \cdot k^{-1}$, where $h \in \mathcal{H}$ and $k \in K$. A second induction on word length relative to the elements of $\mathcal{K}$ and (25) and (27) imply that $\widehat{K}$ is normally generated by the elements of the form $\varphi(h)(k) \cdot k^{-1}$, where $h \in \mathcal{H}$ and $k \in \mathcal{K}$. It follows that the subgroup $p(\widehat{K})$ of K^{Ab} is generated by the elements of the form $\overline{\varphi(h)(k) \cdot k^{-1}}$, where $h \in \mathcal{H}$ and $k \in \mathcal{K}$, and that a presentation of $\Delta(K)$ may be obtained from a presentation of K^{Ab} by adjoining these elements as relators.

We now take $K = \Omega_n$ (resp. $K = \operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$), $H = P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ (resp. $H = L_n$) and $\varphi = \psi$. Applying the induction hypothesis and (23) to (22), to prove parts (a) and (b), it thus suffices to show that

$$\Delta(\Omega_n) \cong \mathbb{Z}^2, \tag{28}$$

$$\Delta\big(\operatorname{Ker}(\hat{\imath}_{n-1}|_{\Omega_n})\big) \cong \mathbb{Z}^{2n-1}, \tag{29}$$

respectively. We first establish the isomorphism (28). As we saw above, to obtain a presentation of $\Delta(\Omega_n)$, we add the relators of the form $\overline{\psi(\tau)(\omega)\cdot\omega^{-1}}$ to a presentation of $(\Omega_n)^{\mathrm{Ab}}$, where $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$, with $\mathcal{X}_n$ and $\mathcal{B}_n$ as defined in (14) and (11), respectively. In $(\Omega_n)^{\mathrm{Ab}}$, these relators may be written as $\overline{s_n(\tau)\omega(s_n(\tau))^{-1}\omega^{-1}}$, or equivalently in the form

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}}\;\overline{\omega^{-1}}. \tag{30}$$

We claim that it is not necessary to know explicitly the section s_n in order to determine these relators. Indeed, for all $\tau \in \mathcal{X}_n$, we have $r_n(\tau) = \tau$; note that we abuse notation here by letting τ also denote the corresponding element of $\mathcal{X}_{n+1}$ in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Thus $s_n(\tau)\tau^{-1} \in \operatorname{Ker}(r_n)$, and hence there exists $\omega_\tau \in \Omega_n$ such that $s_n(\tau) = \omega_\tau\tau$. Therefore

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}} = \overline{\omega_\tau\tau\omega\tau^{-1}\omega_\tau^{-1}} = \overline{\omega_\tau}\;\overline{\tau\omega\tau^{-1}}\;\overline{\omega_\tau^{-1}} = \overline{\tau\omega\tau^{-1}}$$

in $(\Omega_n)^{\mathrm{Ab}}$, and thus the relators of (30) become

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}\omega^{-1}} = \overline{\tau\omega\tau^{-1}}\;\overline{\omega^{-1}}. \tag{31}$$

This proves the claim. Hence the subgroup $p(\widehat{\Omega}_n)$ of $(\Omega_n)^{\mathrm{Ab}}$ is generated by the elements of the form given by (31), where $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$. In what follows, the relations (I)–(V) refer to those of the presentation of $P_{n-1}(\mathbb{R}P^2\setminus\{x_1, x_2\})$ described by Proposition 11. Using this presentation, we see immediately that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$ in $(\Omega_n)^{\mathrm{Ab}}$ for all $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$, with the following exceptions:

(i) $\tau = \rho_j$ and $\omega = A_{j,n+1}$ for all $3 \le j \le n-1$. Then

$$\overline{\rho_j A_{j,n+1}\rho_j^{-1}} = \overline{C_{j,n+1}^{-1}} = \overline{A_{j,n+1}^{-1}},$$

using relation (III) and (8), which yields the element $(\overline{A_{j,n+1}})^2$ of $p(\widehat{\Omega}_n)$.

(ii) $\tau = \rho_j$ and $\omega = \rho_{n+1}$ for all $3 \le j \le n$. Then

$$\overline{\rho_j\rho_{n+1}\rho_j^{-1}} = \overline{C_{j,n+1}\rho_{n+1}} = \overline{A_{j,n+1}}\;\overline{\rho_{n+1}}$$

by relation (IV) and (8), which yields the element $\overline{A_{j,n+1}}$ of $p(\widehat{\Omega}_n)$.

The relators of (ii) above clearly give rise to those of (i), and so $p(\widehat{\Omega}_n)$ is the subgroup of $(\Omega_n)^{\mathrm{Ab}}$ generated by the elements $\overline{A_{j,n+1}}$, where $3 \le j \le n$. Since by (11), $(\Omega_n)^{\mathrm{Ab}}$ is the free Abelian group with basis

$$\{\overline{A_{j,n+1}}, \overline{\rho_{n+1}} \mid 1 \le j \le n-1\},$$

$\Delta(\Omega_n)$ is the Abelian group generated by this set, subject to the condition that $\overline{A_{j,n+1}}$ is trivial for all $3 \le j \le n$. So in $\Delta(\Omega_n)$, the elements $\overline{A_{j,n+1}}$ are trivial for all $j = 3, \dots, n-1$. Further, $\overline{A_{n,n+1}}$ is also trivial, hence by relation (12), one of the remaining generators $\overline{A_{j,n+1}}$ may be deleted, where $j \in \{1, 2\}$, say $\overline{A_{2,n+1}}$, from which we see that $\Delta(\Omega_n)$ is a free Abelian group of rank 2 with $\{\overline{A_{1,n+1}}, \overline{\rho_{n+1}}\}$ as a basis. This establishes the isomorphism (28), and so proves part (a).

We now prove part (b) by establishing the isomorphism (29). We equip $K = \operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$ (resp. $H = L_n$) with the basis $\widehat{\mathcal{B}}_n$ (resp. the generating set $\widehat{\mathcal{X}}_n$) of (17) (resp. of (19)). Since K is a free group of rank $2n-1$, it suffices to show that $p(\widehat{K})$ is the trivial subgroup of K^{Ab}. The fact that K is normal in Ω_n implies that $A_{l,n+1}$, $\rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}$, $C_{l,n+1}$ and $\rho_{n+1}C_{l,n+1}\rho_{n+1}^{-1}$ belong to K for all $1 \le l \le n$ by (8) and (12). Repeating the argument given between (30) and (31), we see that (31) holds for all $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$, where $\bar{k}$ denotes the element $p(k)$ of K^{Ab} for all $k \in K$. For $\alpha \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, let c_α denote conjugation in K by α (which we consider to be an element of $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$). Since $K = \Omega_n \cap L_{n+1}$ by the commutative diagram (18), K is normal in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, and hence the automorphism c_α is well defined. The fact that $\Gamma_2(K)$ is a characteristic subgroup of K implies that c_α induces an automorphism $\hat{c}_\alpha$ of K^{Ab} (the inverse of $\hat{c}_\alpha$ is $\hat{c}_{\alpha^{-1}}$). In particular, if $\alpha, \alpha' \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and $\omega \in K$ then

$$\hat{c}_{\alpha\alpha'}(\overline{\omega}) = \overline{\alpha\alpha'\omega\alpha'^{-1}\alpha^{-1}} = \overline{c_{\alpha\alpha'}(\omega)} = \hat{c}_\alpha(\hat{c}_{\alpha'}(\overline{\omega})).$$

From the first part of the proof, $p(\widehat{K})$ is generated by the elements $\hat{c}_\tau(\overline{\omega})\overline{\omega^{-1}}$, where $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$. To complete the proof of part (b), it suffices to prove that these elements are trivial in K^{Ab}, or equivalently, that $\hat{c}_\tau(\overline{\omega}) = \overline{\omega}$ for all $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$.

(1) First suppose that $\tau = A_{i,j}$, where $3 \le j \le n$ and $1 \le i \le j-2$.

(i) Let $\omega = A_{l,n+1}$, for $1 \le l \le n-1$. Then

$$\tau\omega\tau^{-1} = \begin{cases} A_{l,n+1} & \text{if } j<l \text{ or if } l<i, \\ A_{l,n+1}^{-1}A_{i,n+1}^{-1}A_{l,n+1}A_{i,n+1}A_{l,n+1} & \text{if } j=l, \\ A_{j,n+1}^{-1}A_{l,n+1}A_{j,n+1} & \text{if } i=l, \\ A_{j,n+1}^{-1}A_{i,n+1}^{-1}A_{j,n+1}A_{i,n+1}A_{l,n+1}A_{i,n+1}^{-1}A_{j,n+1}^{-1}A_{i,n+1}A_{j,n+1} & \text{if } i<l<j \end{cases}$$

by the Artin relations (4), from which we conclude that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

(ii) If $\omega = \rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}$, where $1 \leq l \leq n-1$, then

$$\tau\omega\tau^{-1} = \rho_{n+1}(A_{i,j} A_{l,n+1} A_{i,j}^{-1})\rho_{n+1}^{-1},$$

and from case (i), we deduce also that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

(iii) Let $\omega = \rho_{n+1}^2$. Then $\tau\omega\tau^{-1} = \omega$, hence $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

We conclude that $\hat{c}_{A_{i,j}} = \mathrm{Id}_{K^{\mathrm{Ab}}}$.

(2) Let $\tau = \rho_j A_{i,j} \rho_j^{-1}$, where $3 \leq j \leq n$ and $1 \leq i \leq j-2$. Then for all $\omega \in \widehat{\mathcal{B}}_n$, we have

$$\overline{\tau\omega\tau^{-1}} = \overline{c_\tau(\omega)} = \hat{c}_{\rho_j} \circ \hat{c}_{A_{i,j}} \circ \hat{c}_{\rho_j^{-1}}(\overline{\omega}) = \overline{\omega},$$

since $\hat{c}_{A_{i,j}} = \mathrm{Id}_{K^{\mathrm{Ab}}}$, so $\hat{c}_{\rho_j A_{i,j} \rho_j^{-1}} = \mathrm{Id}_{K^{\mathrm{Ab}}}$.

(3) By (19), it remains to study the elements of the form $\overline{\tau\omega\tau^{-1}}$, where $\tau = \rho_j^2$, $3 \leq j \leq n$, and $\omega \in \widehat{\mathcal{B}}_n$. Since $\hat{c}_{\rho_j^2}(\overline{\omega}) = \overline{\rho_j^2 \omega \rho_j^{-2}} = \hat{c}_{\rho_j}^2(\overline{\omega})$, we first analyse $\hat{c}_{\rho_j}$.

(i) If $\omega = A_{l,n+1}$, where $1 \leq l \leq n-1$, then by relation (III) and (8) and (12), we have

$$
\begin{aligned}
&\hat{c}_{\rho_j}(\overline{\omega}) \\
&= \hat{c}_{\rho_j}(\overline{A_{l,n+1}}) = \overline{\rho_j A_{l,n+1} \rho_j^{-1}} \\
&= \begin{cases} \overline{A_{l,n+1}} & \text{if } j < l, \\ \overline{\rho_{n+1}^{-2} \cdot \rho_{n+1} C_{l,n+1}^{-1} \rho_{n+1}^{-1} \cdot \rho_{n+1}^2} & \text{if } j = l, \\ \overline{\rho_{n+1}^{-2} \cdot \rho_{n+1} C_{j,n+1}^{-1} \rho_{n+1}^{-1} \cdot \rho_{n+1}^2 \cdot A_{l,n+1} \cdot \rho_{n+1}^{-2} \cdot \rho_{n+1} C_{j,n+1} \rho_{n+1}^{-1} \cdot \rho_{n+1}^2} & \text{if } j > l \end{cases} \\
&= \begin{cases} \overline{A_{l,n+1}} & \text{if } j \neq l, \\ \overline{\rho_{n+1} C_{j,n+1}^{-1} \rho_{n+1}^{-1}} = \left(\overline{\rho_{n+1} A_{j,n+1} \rho_{n+1}^{-1}}\right)^{-1} & \text{if } j = l. \end{cases}
\end{aligned} \tag{32}
$$

(ii) Let $\omega = \rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}$, where $1 \leq l \leq n-1$. Relation (IV) implies that $\rho_j \rho_{n+1} \rho_j^{-1} = C_{j,n+1} \rho_{n+1}$, and so by case (i) above, we have

$$\hat{c}_{\rho_j}(\overline{\omega}) = \hat{c}_{\rho_j}\left(\overline{\rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}}\right) = \begin{cases} \overline{\rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}} & \text{if } j \neq l, \\ \overline{C_{j,n+1}^{-1}} = \left(\overline{A_{j,n+1}}\right)^{-1} & \text{if } j = l. \end{cases} \tag{33}$$

Combining (32) and (33), we see that

$$\hat{c}_{\rho_j^2}(\overline{\omega}) = \overline{\omega} \quad \text{for all } \omega \in \widehat{\mathcal{B}}_n \setminus \{\rho_{n+1}^2\}. \tag{34}$$

(iii) Let $\omega = \rho_{n+1}^2$. By relation (IV) and (8), (12), (32) and (33), we have

$$\begin{aligned}\hat{c}_{\rho_j}(\overline{\omega}) = \hat{c}_{\rho_j}(\overline{\rho_{n+1}^2}) = \overline{(\rho_j\rho_{n+1}\rho_j^{-1})^2} = \overline{C_{j,n+1}\cdot\rho_{n+1}C_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2}\\ = \hat{c}_{\rho_j}\big(\overline{\rho_{n+1}A_{j,n+1}^{-1}\rho_{n+1}^{-1}}\big)\cdot\hat{c}_{\rho_j}(\overline{A_{j,n+1}^{-1}})\cdot\overline{\rho_{n+1}^2},\end{aligned}$$

from which we obtain

$$\hat{c}_{\rho_j^2}\Big(\overline{\rho_{n+1}^2}\Big) = \overline{\rho_{n+1}A_{j,n+1}^{-1}\rho_{n+1}^{-1}}\cdot\overline{A_{j,n+1}^{-1}}\cdot\overline{C_{j,n+1}\cdot\rho_{n+1}C_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2} = \overline{\rho_{n+1}^2}$$

using (34). So by (17), we also have $\hat{c}_{\rho_j^2} = \mathrm{Id}_{K^{\mathrm{Ab}}}$.

Hence for all $\tau \in \widehat{\mathfrak{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$, it follows that $\hat{c}_\tau(\overline{\omega}) = \overline{\omega}$, and thus $p(\widehat{K})$ is the trivial subgroup of K^{Ab}. We conclude that $\Delta(K) \cong K^{\mathrm{Ab}} \cong \mathbb{Z}^{2n-1}$, and this completes the proof of part (b). □

Remark 13. (a) An alternative description of $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, similar to that of (13), but with the parentheses in the opposite order, may be obtained as follows. Let $m \geq 2$ and $q \geq 1$, and consider the Fadell–Neuwirth short exact sequence

$$1 \to P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{q+1}\}) \to P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_q\}) \to P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_q\}) \to 1, \tag{35}$$

given geometrically by forgetting the last $m-1$ strings. Since the quotient is a free group $\mathbb{F}_q$ of rank q, the above short exact sequence splits, and so

$$P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_q\}) \cong P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{q+1}\}) \rtimes \mathbb{F}_q,$$

and thus

$$P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})) \cong (\cdots((\mathbb{F}_{n-1} \rtimes \mathbb{F}_{n-2}) \rtimes \mathbb{F}_{n-3}) \rtimes \cdots \rtimes \mathbb{F}_3) \rtimes \mathbb{F}_2 \tag{36}$$

by induction. The ordering of the parentheses thus occurs from the left, in contrast with that of (13). The decomposition given by (13) is in some sense stronger than that of (36), since in the first case, every factor acts on each of the preceding factors, which is not necessarily the case in (36), so (13) gives rise to a decomposition of the form (36). This is a manifestation of the fact that the splitting of the corresponding Fadell–Neuwirth sequence (10) is nontrivial, while that of (35) is obvious.

(b) Note that L_4, the kernel of the homomorphism $\hat{\iota}_2 : P_2(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to \mathbb{Z}_2^2$, is also the subgroup of index 4 of the group $(B_4(\mathbb{R}P^2))^{(3)}$ that appears in [Gonçalves and Guaschi 2011, Theorem 3(d)]. Indeed, by equation (127) of that paper, this subgroup of index 4 is isomorphic to the semidirect product

$$\mathbb{F}_5(A_{1,4}, A_{2,4}, \rho_4^2, \rho_4A_{1,4}\rho_4^{-1}, \rho_4A_{2,4}\rho_4^{-1}) \rtimes \mathbb{F}_3(A_{2,3}, \rho_3^2, \rho_3A_{2,3}\rho_3^{-1}),$$

the action being given by equations (129)–(131) of the same paper (the element $B_{i,j}$ of [Gonçalves and Guaschi 2011] is the element $A_{i,j}$ of this paper).

(c) It follows from the proof of Proposition 4(b) that the induced action of L_n on the Abelianisation of $\mathrm{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$ is trivial. Since $\mathrm{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$ is a free group, its higher homology groups are trivial, and so L_n acts trivially on the homology of $\mathrm{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$.

Remark 14. Using the ideas of the last paragraph of the proof of Proposition 1(b), one may show that L_n is not normal in $B_n(\mathbb{R}P^2)$. Although the subgroup L_n is not unique with respect to the properties of the statement of Proposition 1(a)(ii), there are only a finite number of subgroups, $2^{n(n-2)}$ to be precise, that satisfy these properties. To prove this, we claim that the set of torsion-free subgroups L'_n of K_n such that $K_n = L'_n \oplus \langle \Delta_n^2 \rangle$ is in bijection with the set $\{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}(L_n, \mathbb{Z}_2)\}$. To prove the claim, let $K = K_n$, $L = L_n$, and $q : K \to K/L$ be the canonical surjection, and set

$$\Delta = \{L' \mid L' < K,\, L' \text{ is torsion-free, and } K = L' \oplus \langle \Delta_n^2 \rangle\}.$$

Clearly $L \in \Delta$, so $\Delta \neq \varnothing$. Consider the map $\varphi : \Delta \to \{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}(L, \mathbb{Z}_2)\}$ defined by $\varphi(L') = L \cap L'$. This map is well defined, since if $L' = L$ then $\varphi(L') = L$ is the kernel of the trivial homomorphism of $\mathrm{Hom}(L, \mathbb{Z}_2)$, and if $L' \neq L$ then $L' \not\subset L$ since $[K : L'] = [K : L] = 2$, and so $q|_{L'}$ is surjective as $K/L \cong \mathbb{Z}_2$. Thus $\mathrm{Ker}(q|_{L'}) = \varphi(L')$ is of index 2 in L, and in particular, $\varphi(L')$ is the kernel of some nontrivial element of $\mathrm{Hom}(L, \mathbb{Z}_2)$.

We now prove that φ is surjective. Let $f \in \mathrm{Hom}(L, \mathbb{Z}_2)$, and set $L'' = \mathrm{Ker}(f)$. If $f = 0$ then $L'' = L$, and $\varphi(L) = L''$. So suppose that $f \neq 0$. Then f is surjective, and $L'' = \mathrm{Ker}(f)$ is of index 2 in L. Let $x \in L \setminus L''$. Then

$$L = L'' \amalg xL'', \tag{37}$$

where $\amalg$ denotes the disjoint union. Since $K = L \amalg \Delta_n^2 L$, it follows that

$$K = L'' \amalg xL'' \amalg \Delta_n^2 L'' \amalg x\Delta_n^2 L''. \tag{38}$$

Set $L' = L'' \amalg x\Delta_n^2 L''$. By (37), $x^2 \Delta_n^2 L'' = \Delta_n^2 x^2 L'' = \Delta_n^2 L''$ because Δ_n^2 is central and of order 2, and hence $K = L' \amalg xL'$. Using once more (37), we see that L' is a group, and so the equality $K = L' \amalg xL'$ implies that $[K : L'] = 2$. Further, since the only nontrivial torsion element of K is Δ_n^2, L' is torsion-free by (38), and so the short exact sequence $1 \to L' \to K \to \mathbb{Z}_2 \to 1$ splits. Thus $L' \in \Delta$, and $\varphi(L') = L''$ using (37) and (38).

It remains to prove that φ is injective. Let $L'_1, L'_2 \in \Delta$ be such that $L'_1 \cap L = \varphi(L'_1) = \varphi(L'_2) = L'_2 \cap L$. If one of the L'_i, say L'_1, is equal to L then we must also have $L'_2 = L$ because $L \subset L'_2$ and L and L'_2 have the same index in K. So suppose that $L'_i \neq L$ for all $i \in \{1, 2\}$. If $i \in \{1, 2\}$ then $L'' = \varphi(L'_i) = L \cap L'_i = \mathrm{Ker}(f_i)$ for some nontrivial $f_i \in \mathrm{Hom}(L, \mathbb{Z}_2)$, and thus $[L : L''] = 2$. Let us show that $L'_1 \subset L'_2$.

Let $x \in L'_1$. If $x \in L$ then $x \in L''$, so $x \in L'_2$, and we are done. So assume that $x \notin L$, and suppose that $x \notin L'_2$. Then $q(x)$ is equal to the nontrivial element of K/L, and since $K/L \cong \mathbb{Z}_2$ and $\Delta_n^2 \notin L$, we see that $x\Delta_n^2 \in L$. Further, $K = L'_2 \amalg xL'_2$ since $[K : L'_2] = 2$, and so $x\Delta_n^2 \in L'_2$ (for otherwise $x\Delta_n^2 \in xL'_2$, which implies that $\Delta_n^2 \in L'_2$, which is impossible because L'_2 is torsion-free). Then $x\Delta_n^2 \in L \cap L'_2 = L''$, and hence $x\Delta_n^2 \in L'_1$. But this would imply that $\Delta_n^2 \in L'_1$, which contradicts the fact that L'_1 is torsion-free. We conclude that $L'_1 \subset L'_2$, and exchanging the rôles of L'_1 and L'_2, we see that $L'_1 = L'_2$, which proves that φ is injective, so is bijective, which proves the claim. Therefore the cardinality of Δ is equal to the order of the group $H^1(L, \mathbb{Z}_2)$, which is equal in turn to that of $H_1(L, \mathbb{Z}_2)$. By Proposition 4(b), we have $L^{\mathrm{Ab}} = H_1(L, \mathbb{Z}) \cong \mathbb{Z}^{n(n-2)}$, so $H_1(L, \mathbb{Z}_2) \cong H_1(L, \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2^{n(n-2)}$, and the number of subgroups of K that satisfy the properties of Proposition 1(a) is equal to $2^{n(n-2)}$ as asserted.

4. The virtual cohomological dimension of $B_n(S)$ and $P_n(S)$ for $S = \mathbb{S}^2, \mathbb{R}P^2$

Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), and for all $m, n \geq 1$, let $\Gamma_{n,m}(S) = P_n(S \setminus \{x_1, \ldots, x_m\})$ denote the n-string pure braid group of S with m points removed. In order to study various cohomological properties of the braid groups of S and prove Theorem 5, we shall study $\Gamma_{n,m}(S)$. To prove Theorem 5 in the case $S = \mathbb{S}^2$, by (2), it will suffice to compute the cohomological dimension of $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. We recall the following presentation of $\Gamma_{n,m}(\mathbb{S}^2)$ from [Gonçalves and Guaschi 2005]. The result was stated for $m \geq 3$, but it also holds for $m \leq 2$.

Proposition 15 [Gonçalves and Guaschi 2005, Proposition 7]. *Let $n, m \geq 1$. The following constitutes a presentation of the group $\Gamma_{n,m}(\mathbb{S}^2)$:*

Generators: *$A_{i,j}$, where $1 \leq i < j$ and $m+1 \leq j \leq m+n$.*

Relations:

(i) *The Artin relations described by (4) among the generators $A_{i,j}$ of $\Gamma_{n,m}(\mathbb{S}^2)$.*

(ii) *For all $m+1 \leq j \leq m+n$, $\bigl(\prod_{i=1}^{j-1} A_{i,j}\bigr)\bigl(\prod_{k=j+1}^{m+n} A_{j,k}\bigr) = 1$.*

Let N denote the kernel of the homomorphism $\Gamma_{n,m}(S) \to \Gamma_{n-1,m}(S)$ obtained geometrically by forgetting the last string. If $S = \mathbb{S}^2$ then N is a free group of rank $m+n-2$ and equals $\langle A_{1,m+n}, A_{2,m+n}, \ldots, A_{m+n-1,m+n}\rangle$. If $S = \mathbb{R}P^2$ then N is a free group of rank $m+n-1$ and equals $\langle A_{1,m+n}, A_{2,m+n}, \ldots, A_{m+n-1,m+n}, \rho_{m+n}\rangle$. Clearly N is normal in $\Gamma_{n,m}(S)$. Further, if $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), it follows from relations (i) of Proposition 15 (resp. relations (III) and (IV) of Proposition 11) that the action by conjugation of $\Gamma_{n,m}(S)$ on N induces (resp. does not induce) the trivial action on the Abelianisation of N. In order to determine the virtual cohomological dimension of the braid groups of S and prove Theorem 5, we shall compute the cohomological dimension of a torsion-free finite-index subgroup. In the case of $\mathbb{S}^2$

(resp. $\mathbb{R}P^2$), we choose the subgroup $\Gamma_{n-3,3}(\mathbb{S}^2)$ that appears in the decomposition given in (2) (resp. the subgroup $\Gamma_{n-2,2}(\mathbb{R}P^2)$ that appears in (3)).

Proof of Theorem 5. Let $S=\mathbb{S}^2$ (resp. $S=\mathbb{R}P^2$), let $n>3$ and $k=3$ (resp. $n>2$ and $k=2$), and let $k\le m<n$. Then by (2) (resp. (3)) and (1), $\Gamma_{n-m,m}(S)$ is a subgroup of finite index of both $P_n(S)$ and $B_n(S)$. Further, since $F_{n-m}(S\setminus\{x_1,\dots,x_m\})$ is a finite-dimensional CW-complex and an Eilenberg–Mac Lane space of type $K(\pi,1)$ [Fadell and Neuwirth 1962], the cohomological dimension of $\Gamma_{n-m,m}(S)$ is finite, and the first part follows by taking $m=k$.

We now prove the second part, namely that the cohomological dimension of $\Gamma_{n-k,k}(S)$ is equal to $n-k$ for all $n>k$. We first claim that $\mathrm{cd}(\Gamma_{m,l}(S))\le m$ for all $m\ge 1$ and $l\ge k-1$. The result holds if $m=1$ since $F_1(S\setminus\{x_1,\dots,x_l\})$ has the homotopy type of a bouquet of circles; therefore $H^i(F_1(S\setminus\{x_1,\dots,x_l\}),A)$ is trivial for all $i\ge 2$ and for any local coefficients A, and $H^1(F_1(S\setminus\{x_1,\dots,x_l\}),\mathbb{Z})\neq 0$. Suppose by induction that the result holds for some $m\ge 1$, and consider the Fadell–Neuwirth short exact sequence

$$1\to\Gamma_{1,l+m}(S)\to\Gamma_{m+1,l}(S)\to\Gamma_{m,l}(S)\to 1$$

that emanates from the fibration

$$F_1(S\setminus\{x_1,\dots,x_l,z_1,\dots,z_m\})\to F_{m+1}(S\setminus\{x_1,\dots,x_l\}) \to F_m(S\setminus\{x_1,\dots,x_l\}) \tag{39}$$

obtained by forgetting the last coordinate. By [Brown 1982, Chapter VIII], it follows that

$$\mathrm{cd}(\Gamma_{m+1,l}(S))\le\mathrm{cd}(\Gamma_{m,l}(S))+\mathrm{cd}(\Gamma_{1,l+m}(S))\le m+1,$$

which proves the claim. In particular, taking $l=k$, we have $\mathrm{cd}(\Gamma_{m,k}(S))\le m$.

To conclude the proof of the theorem, it suffices to show that for each $m\ge 1$ there are local coefficients A such that $H^m(\Gamma_{m,l}(S),A)\neq 0$ for all $l\ge k-1$. We will show that this is the case for $A=\mathbb{Z}$. Again by induction suppose that $H^m(\Gamma_{m,l}(S),\mathbb{Z})\neq 0$ for all $l\ge k-1$ and for some $m\ge 1$ (we saw above that this is true for $m=1$). Consider the Serre spectral sequence with integral coefficients associated to the fibration (39). Then we have that

$$E_2^{p,q}=H^p\big(\Gamma_{m,l}(S),H^q(F_1(S\setminus\{x_1,\dots,x_l,z_1,\dots,z_m\}),\mathbb{Z})\big).$$

Since $\mathrm{cd}(\Gamma_{m,l}(S))\le m$ and $\mathrm{cd}(F_1(S\setminus\{x_1,\dots,x_l,z_1,\dots,z_m\}))\le 1$ from above, it follows that this spectral sequence has two horizontal lines whose possible nonvanishing terms occur for $0\le p\le m$ and $0\le q\le 1$. We claim that the group $E_2^{m,1}$ is nontrivial. To see this, first note that $H^1(F_1(S\setminus\{x_1,\dots,x_l,z_1,\dots,z_m\}),\mathbb{Z})$ is isomorphic to the free Abelian group of rank $r=m+l-k+2$, so $r\ge m+1$, and hence $E_2^{m,1} = H^m(\Gamma_{m,l}(S), \mathbb{Z}^r)$, where we identify $\mathbb{Z}^r$ with (the dual of) N^{Ab}. The action of $\Gamma_{m,l}(S)$ on N by conjugation induces an action of $\Gamma_{m,l}(S)$ on N^{Ab}. Let H be the subgroup of N^{Ab} generated by the elements of the form $\alpha(x)x^{-1}$, where $\alpha \in \Gamma_{m,l}(S)$, $x \in N^{\mathrm{Ab}}$, and $\alpha(x)$ represents the action of α on x. Then we obtain a short exact sequence $0 \to H \to N^{\mathrm{Ab}} \to N^{\mathrm{Ab}}/H \to 0$ of Abelian groups, and the long exact sequence in cohomology applied to $\Gamma_{m,l}(S)$ yields

(40)

$$\cdots \to H^m(\Gamma_{m,l}(S), N^{\mathrm{Ab}}) \to H^m(\Gamma_{m,l}(S), N^{\mathrm{Ab}}/H) \to H^{m+1}(\Gamma_{m,l}(S), H) \to \cdots .$$

The last term is zero since $\mathrm{cd}(\Gamma_{m,l}(S)) \leq m$, and so the map between the two remaining terms is surjective. Let us determine N^{Ab}/H. If $S = \mathbb{S}^2$ then from the comments following Proposition 15, the action of $\Gamma_{m,l}(S)$ on N^{Ab} is trivial, so H is trivial, and $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^r$. So suppose that $S = \mathbb{R}P^2$. Choosing the basis

$$\{A_{1,m+l+1}, A_{2,m+l+1}, \ldots, A_{m+l-1,m+l+1}, \rho_{m+l+1}\}$$

of N and using Proposition 11, one sees that the action by conjugation of the generators of $\Gamma_{m,l}(S)$ on the corresponding basis elements of N^{Ab} is trivial, with the exception of that of ρ_i on $A_{i,m+l+1}$ for $l+1 \leq i \leq m+l-1$, which yields elements $A_{i,m+l+1}^2 \in H$ (by abuse of notation, we denote the elements of N^{Ab} in the same way as those of N), and that of ρ_i on ρ_{m+l+1}, where $l+1 \leq i \leq m+l$, which yields elements $A_{i,m+l+1} \in H$. In the quotient N^{Ab}/H the basis elements $A_{l+1,m+l+1}, \ldots, A_{m+l-1,m+l+1}$ thus become zero, and additionally, we have also that $A_{m+l,m+l+1}$ (which is not in the given basis) becomes zero. Hence the relation $\prod_{i=1}^{m+l} A_{i,m+l+1} = \rho_{m+l+1}^{-2}$ is sent to the relation $\prod_{i=1}^{l} A_{i,m+l+1} = \rho_{m+l+1}^{-2}$, and so N^{Ab}/H is generated by (the images of) the elements $A_{1,m+l+1}, \ldots, A_{l,m+l+1}$, ρ_{m+l+1}, subject to this relation (as well as the fact that the elements commute pairwise). It thus follows that $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^l$. Since the induced action of $\Gamma_{m,l}(S)$ on N^{Ab}/H is trivial, we conclude that

$$H^m\big(\Gamma_{m,l}(S), N^{\mathrm{Ab}}/H\big) = \big(H^m(\Gamma_{m,l}(S), \mathbb{Z})\big)^s,$$

where $s = m+l$ if $S = \mathbb{S}^2$ and $s = l$ if $S = \mathbb{R}P^2$. It then follows from (40) that $E_2^{m,1} = H^m(\Gamma_{m,l}(S), N^{\mathrm{Ab}}) \neq 0$. Since $E_2^{p,q} = 0$ for all $p > m$ and $q > 1$, we have $E_2^{m,1} = E_\infty^{m,1}$, thus $E_\infty^{m,1}$ is nontrivial, and hence $H^{m+1}(\Gamma_{m+1,l}(S), \mathbb{Z}) \neq 0$. $\square$

Proof of Corollary 6. Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$). If $n \geq 3$ (resp. $n \geq 2$) then $B_n(S)$ and $\mathcal{MCG}(S, n)$ are closely related by the following short exact sequence [Scott 1970]:

$$1 \to \langle \Delta_n^2 \rangle \to B_n(S) \xrightarrow{\beta} \mathcal{MCG}(S, n) \to 1,$$

where the kernel is isomorphic to $\mathbb{Z}_2$. Now assume that $n \geq 4$ (resp. $n \geq 3$), so that $B_n(S)$ is infinite. If Γ is a torsion-free subgroup of $B_n(S)$ of finite index then $\beta(\Gamma)$, which is isomorphic to Γ, is a torsion-free subgroup of $\mathcal{MCG}(S, n)$ of finite index, and hence the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ is equal to that of $B_n(S)$. The result then follows by Theorem 5. $\square$

Acknowledgements

We would like to thank the referee for carefully reading the manuscript and for constructive comments. This work took place during the visits of Gonçalves to the Laboratoire de Mathématiques Nicolas Oresme during the periods 2–23 December 2012, 29 November–22 December 2013 and 4 October–1 November 2014, and during the visits of Guaschi to the Departamento de Matemática do IME, Universidade de São Paulo, during the periods 10 November–1 December 2012, 1–21 July 2013 and 10 July–2 August 2014, and was supported by the international cooperation Capes-Cofecub project n° Ma 733-12 (France) and n° 1716/2012 (Brazil), and the CNRS/Fapesp programme n° 226555 (France) and n° 2014/50131-7 (Brazil).

References

[Birman 1969] J. S. Birman, "On braid groups", *Comm. Pure Appl. Math.* **22** (1969), 41–72. MR Zbl

[Birman 1974] J. S. Birman, *Braids, links, and mapping class groups*, Annals of Mathematics Studies **82**, Princeton University Press, 1974. MR Zbl

[Brown 1982] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, 1982. MR Zbl

[Fadell and Neuwirth 1962] E. Fadell and L. Neuwirth, "Configuration spaces", *Math. Scand.* **10** (1962), 111–118. MR Zbl

[Fox and Neuwirth 1962] R. Fox and L. Neuwirth, "The braid groups", *Math. Scand.* **10** (1962), 119–126. MR Zbl

[Golasiński et al. 2016] M. Golasiński, D. L. Gonçalves, and J. Guaschi, "On the homotopy fibre of the inclusion map $F_n(X) \hookrightarrow \prod_1^n X$ for some orbit spaces X", *Bol. Soc. Mat. Mex.* (online publication September 2016).

[Goldberg 1973] C. H. Goldberg, "An exact sequence of braid groups", *Math. Scand.* **33** (1973), 69–82. MR Zbl

[Gonçalves and Guaschi 2004a] D. L. Gonçalves and J. Guaschi, "The braid groups of the projective plane", *Algebr. Geom. Topol.* **4** (2004), 757–780. MR Zbl

[Gonçalves and Guaschi 2004b] D. L. Gonçalves and J. Guaschi, "The roots of the full twist for surface braid groups", *Math. Proc. Cambridge Philos. Soc.* **137**:2 (2004), 307–320. MR Zbl

[Gonçalves and Guaschi 2005] D. L. Gonçalves and J. Guaschi, "The braid group $B_{n,m}(\mathbb{S}^2)$ and a generalisation of the Fadell–Neuwirth short exact sequence", *J. Knot Theory Ramifications* **14**:3 (2005), 375–403. MR Zbl

[Gonçalves and Guaschi 2007] D. L. Gonçalves and J. Guaschi, "The braid groups of the projective plane and the Fadell–Neuwirth short exact sequence", *Geom. Dedicata* **130** (2007), 93–107. MR Zbl

[Gonçalves and Guaschi 2009] D. L. Gonçalves and J. Guaschi, "The lower central and derived series of the braid groups of the sphere", *Trans. Amer. Math. Soc.* **361**:7 (2009), 3375–3399. MR Zbl

[Gonçalves and Guaschi 2010a] D. L. Gonçalves and J. Guaschi, "Braid groups of non-orientable surfaces and the Fadell–Neuwirth short exact sequence", *J. Pure Appl. Algebra* **214**:5 (2010), 667–677. MR Zbl

[Gonçalves and Guaschi 2010b] D. L. Gonçalves and J. Guaschi, "Classification of the virtually cyclic subgroups of the pure braid groups of the projective plane", *J. Group Theory* **13**:2 (2010), 277–294. MR Zbl

[Gonçalves and Guaschi 2011] D. L. Gonçalves and J. Guaschi, "The lower central and derived series of the braid groups of the projective plane", *J. Algebra* **331** (2011), 96–129. MR Zbl

[Gonçalves and Guaschi $\geq$ 2017] D. L. Gonçalves and J. Guaschi, "The homotopy fibre of the inclusion $F_n(M) \longrightarrow \prod_1^n M$ for M either $\mathbb{S}^2$ or $\mathbb{R}P^2$ and orbit configuration spaces", in preparation.

[Gonçalves et al. 2016] D. L. Gonçalves, J. Guaschi, and M. Maldonado, "Embeddings and the (virtual) cohomological dimension of the braid and mapping class groups of surfaces", preprint, 2016. arXiv

[González-Meneses and Paris 2004] J. González-Meneses and L. Paris, "Vassiliev invariants for braids on surfaces", *Trans. Amer. Math. Soc.* **356**:1 (2004), 219–243. MR Zbl

[Harer 1986] J. L. Harer, "The virtual cohomological dimension of the mapping class group of an orientable surface", *Invent. Math.* **84**:1 (1986), 157–176. MR Zbl

[Johnson 1997] D. L. Johnson, *Presentations of groups*, 2nd ed., London Mathematical Society Student Texts **15**, Cambridge University Press, 1997. MR Zbl

[Murasugi 1982] K. Murasugi, "Seifert fibre spaces and braid groups", *Proc. London Math. Soc.* (3) **44**:1 (1982), 71–84. MR Zbl

[Scott 1970] G. P. Scott, "Braid groups and the group of homeomorphisms of a surface", *Proc. Cambridge Philos. Soc.* **68** (1970), 605–617. MR Zbl

[Tochimani 2011] A. Tochimani, *Grupos de trenzas de superficies compactas*, master's thesis, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, 2011.

[Van Buskirk 1966] J. Van Buskirk, "Braid groups of compact 2-manifolds with elements of finite order", *Trans. Amer. Math. Soc.* **122** (1966), 81–97. MR Zbl

Received November 12, 2015. Revised July 8, 2016.

DACIBERG LIMA GONÇALVES
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA DA UNIVERSIDADE DE SÃO PAULO
DEPARTAMENTO DE MATEMÁTICA
RUA DO MATÃO, 1010 CEP 05508-090
SÃO PAULO-SP
BRAZIL

dlgoncal@ime.usp.br

JOHN GUASCHI
LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME UMR CNRS 6139
NORMANDIE UNIVERSITÉ
UNIVERSITÉ DE CAEN NORMANDIE
14000 CAEN
FRANCE

john.guaschi@unicaen.fr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.101

GROUPS OF PL-HOMEOMORPHISMS ADMITTING NONTRIVIAL INVARIANT CHARACTERS

DACIBERG LIMA GONÇALVES,
PARAMESWARAN SANKARAN AND RALPH STREBEL

We show that several classes of groups G of PL-homeomorphisms of the real line admit nontrivial homomorphisms $\chi : G \to \mathbb{R}$ that are fixed by every automorphism of G. The classes enjoying the stated property include the generalizations of Thompson's group F studied by K. S. Brown (1992), M. Stein (1992), S. Cleary (1995), and Bieri and Strebel (2016), but also the class of groups investigated by Bieri, Neumann, and Strebel (Theorem 8.1 in *Invent. Math.* 90 (1987), 451–477). It follows that every automorphism of a group in one of these classes has infinitely many associated twisted conjugacy classes.

1. Introduction

This paper stems from two articles [Bleak et al. 2008; Gonçalves and Kochloukova 2010] about twisted conjugacy classes of Thompson's group F. In order to describe the aim of the cited papers, we recall some terminology. Let G be a group and α

MSC2010: primary 20E45; secondary 20E36, 20F28.
Keywords: groups of PL-homeomorphisms of the real line, Bieri–Neumann–Strebel invariants, twisted conjugacy.

an automorphism of G. Then α gives rise to an action $\mu_\alpha : G \times |G| \to |G|$ of G on its underlying set $|G|$, defined by

$$\mu_\alpha(g, x) = g \cdot x \cdot \alpha(g)^{-1}. \tag{1-1}$$

The orbits of this action are called *twisted conjugacy classes*, or *Reidemeister* classes, of α. The twisted conjugacy classes of the identity automorphism, for instance, are nothing but the conjugacy classes.

Two questions now arise, firstly, whether a given automorphism α has infinitely many orbits and, secondly, whether every automorphism of G has infinitely many orbits. As the latter property will be central to this paper, we recall the definition of property R_∞:

Definition 1.1. A group G is said to have *property* R_∞ if the action μ_α has infinitely many orbits for every automorphism $\alpha : G \xrightarrow{\sim} G$.

The problem of determining whether a given group, or a class of groups, satisfies property R_∞ has attracted the attention of several researchers. The problem is rendered particularly interesting by the fact there does not exist a uniform method of solution. Indeed, a variety of techniques and ad hoc arguments from several branches of mathematics have been used to tackle the problem, notably combinatorial group theory in [Gonçalves and Wong 2009], geometric group theory in [Levitt and Lustig 2000], C^*-algebras in [Fel'shtin and Troitsky 2012], and algebraic geometry in [Mubeena and Sankaran 2014b].

Bleak, Fel'shtyn, and Gonçalves [Bleak et al. 2008] show that Thompson's group F enjoys property R_∞, while Gonçalves and Kochloukova [2010] establish the same property for Thompson's group F, but also for many other groups G having the peculiarity that the complement of their BNS-invariant $\Sigma^1(G)$ is made up of finitely many rank 1 points. In this paper, we generalize both approaches and prove in this way that many classes of groups of PL-homeomorphisms have property R_∞.

1A. ***A useful fact.*** The papers by Bleak et al. and by Gonçalves and Kochloukova both exploit the following observation: let α be an automorphism of a group G, let $\psi : G \to B$ be a homomorphism into an *abelian* group, and assume ψ is fixed by α. Then ψ is constant on twisted conjugacy classes of α; indeed, if the elements x and y lie in the same twisted conjugacy class there exists $z \in G$ so that

$$y = z \cdot x \cdot \alpha(z)^{-1};$$

the computation

$$\psi(y) = \psi(z \cdot x \cdot \alpha(z)^{-1}) = \psi(x) \cdot \psi(z) \cdot ((\psi \circ \alpha)(z))^{-1} = \psi(x)$$

proves the claim. A group G therefore *has property* R_∞ if it admits a homomorphism onto an infinite, abelian group that is fixed by every automorphism of G.[1] For $B \subset \mathbb{R}^\times_{>0}$, the classes of groups G admitting such a nontrivial homomorphism include various generalizations of Thompson's group F [Brown 1987a; Stein 1992; Cleary 1995; 2000; Bieri and Strebel 2016].

1B. ***Approach used by Bleak, Fel'shtyn, and Gonçalves.*** The authors [2008] establish that Thompson's group F has property R_∞ by using the mentioned fact. To find the homomorphism ψ, they use a representation of F by piecewise linear homeomorphisms of the real line: F is isomorphic to the group of all piecewise linear homeomorphisms f with supports in the unit interval $I = [0, 1]$, slopes a power of 2, and *break points*, i.e., points where the left and right derivatives differ, in the group $\mathbb{Z}[1/2]$ of dyadic rationals; see, e.g., [Cannon et al. 1996, p. 216, §1]. This representation affords them with two homomorphisms σ_ℓ and σ_r, given by the right derivative in the *left* end point 0 and the left derivative in *right* end point 1 of I, respectively. In formulae,

$$\sigma_\ell(f) = \lim\nolimits_{t \searrow 0} f'(t) \quad \text{and} \quad \sigma_r(f) = \lim\nolimits_{t \nearrow 1} f'(t). \tag{1-2}$$

The images of σ_ℓ and σ_r are both equal to $\operatorname{gp}(2)$, the (multiplicative) cyclic group generated by the natural number 2. Theorem 3.3, the main result of [Bleak et al. 2008], can be rephrased by saying that the homomorphism

$$\psi : F \to \operatorname{gp}(2), \quad f \mapsto \sigma_\ell(f) \cdot \sigma_r(f)$$

is fixed by every automorphism α of F. Its proof uses the very detailed information about $\operatorname{Aut} F$ established by M. Brin [1996].

1C. ***A generalization.*** The stated description of Thompson's group F invites one to introduce generalized groups of type F in the following manner.

Definition 1.2. Let $\mathrm{PL}_o(\mathbb{R})$ denote the group of all increasing PL-homeomorphisms of the real line with only finitely many break points. Fix a closed interval $I \subseteq \mathbb{R}$, a subgroup P of the multiplicative group of positive reals $\mathbb{R}^\times_{>0}$ and a subgroup A of the additive group $\mathbb{R}_{\mathrm{add}}$ of the field $\mathbb{R}$ that is stable under multiplication by P. Define $G(I; A, P)$ to be the subset of $\mathrm{PL}_o(\mathbb{R})$ made up of all PL-homeomorphisms g that satisfy the following conditions:

(a) the support $\operatorname{supp} g = \{t \in \mathbb{R} \mid g(t) \neq t\}$ of g is contained in I,

(b) the slopes of the finitely many line segments forming the graph of g lie in P,

(c) the break points of g lie in A, and

(d) g maps A onto A.

[1] One can find generalizations of, and also many variations on, the stated observation; see, e.g., [Gonçalves and Wong 2003, Formula (2.2)] or [Fel'shtyn and Troitsky 2015, Claim 2 in Theorem 4.4].

Remarks 1.3. (a) The subset $G(I; A, P)$ is closed under composition[2] and inversion. The set $G(I; A, P)$ equipped with these operations is a group; by abuse of notation, it will also be denoted by $G(I; A, P)$.

(b) We shall always require that neither P nor A be reduced to the neutral element. These requirements imply that A contains arbitrary small positive elements and thus A is a dense subgroup of $\mathbb{R}$. As concerns the interval I we shall restrict attention to three types: compact intervals with endpoints 0 and $b \in A_{>0}$, the half-line $[0, \infty[$ and the line $\mathbb{R}$; we refer the reader to [Bieri and Strebel 2016, Sections 2.4 and 16.4] for a discussion of the groups associated to other intervals.

(c) The idea of introducing and studying the groups $G(I; A, P)$ goes back to the papers [Brin and Squier 1985; Bieri and Strebel 1985].

1C1. *The homomorphisms σ_ℓ, σ_r, and ψ.* The definitions of σ_ℓ and σ_r, given in (1-2), admit straightforward extensions to the groups $G(I; A, P)$; note, however, that in case of the half-line $[0, \infty[$, the number $\sigma_r(f)$ will denote the slope of f near $+\infty$, and similarly for $I = \mathbb{R}$ and σ_ℓ, σ_r. The homomorphisms σ_ℓ and σ_r allow one then to introduce an analogue of $\psi : F \to \mathrm{gp}(2)$, namely,

$$\psi : G = G(I; A, P) \to P, \quad g \mapsto \sigma_\ell(g) \cdot \sigma_r(g). \tag{1-3}$$

There remains the question whether this homomorphism ψ is fixed by every automorphism of G. In the case of Thompson's group F the question has been answered in the affirmative by exploiting the detailed information about $\operatorname{Aut} F$ obtained by Brin [1996]. Such a detailed description is not to be expected for every group of the form $G(I; A, P)$; indeed, the results in [Brin and Guzmán 1998] show that the structure of the automorphism group gets considerably more involved if one passes from the group $G([0, 1]; \mathbb{Z}[1/2], \mathrm{gp}(2))$, the group isomorphic to F, to the groups $G([0, 1]; \mathbb{Z}[1/n], \mathrm{gp}(n))$ with n an integer greater than 2.

1C2. *The first main results.* It turns out that one does not need very detailed information about $\operatorname{Aut} G(I; A, P)$ in order to construct a nontrivial homomorphism $\psi : G(I; A, P) \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of the group $G(I; A, P)$; it suffices to go back to the findings in the memoir [Bieri and Strebel 1985] and to supplement them by some auxiliary results based upon them.[3] One outcome is the following theorem.

Theorem 1.4. *Assume the interval I, the group of slopes P and the $\mathbb{Z}[P]$-module A are as in* Definition 1.2 *and in* Remark 1.3(b). *Then there exists an epimorphism*

---

[2]In this article we use *left* actions and the composition of functions familiar to analysts; thus $g_2 \circ g_1$ denotes the function $t \mapsto g_2(g_1(t))$ and ${}^{g_1}g_2$ the homeomorphism $g_1 \circ g_2 \circ g_1^{-1}$.

[3]The memoir [Bieri and Strebel 1985] has recently been published as [Bieri and Strebel 2016].

$\psi : G(I; A, P) \twoheadrightarrow P$ that is fixed by every automorphism of G. Furthermore, the group $G(I; A, P)$ therefore has property R_∞.

Remark 1.5. Let I, A, and P be as before and let $B = B(I; A, P)$ be the subgroup of $G(I; A, P)$ made up of all elements g that are the identity near the endpoints. Then B is a characteristic subgroup of $G(I; A, P)$ and variations of Theorem 1.4 hold for many subgroups G of $G(I; A, P)$ with $B \subset G$. For further details, see Theorems 3.8, 4.4, and 5.5.

1D. ***Route taken by Gonçalves and Kochloukova.*** For details, see [Gonçalves and Kochloukova 2010]. The proof of Theorem 1.4 does not exploit information about $\operatorname{Aut} G(I; A, P)$ that is as precise as that going into the proof of the main result of [Bleak et al. 2008]. It uses, however, nontrivial features of the automorphisms of $G(I; A, P)$. Gonçalves and Kochloukova [loc. cit.] put forward the novel idea of replacing detailed information about $\operatorname{Aut} G$ by information about the form of the BNS-invariant of the group G; they carry out this program for the generalized Thompson group $F_{n,0}$ with $n \geq 2$, a group isomorphic to $G([0, 1]; \mathbb{Z}[1/n], \operatorname{gp}(n))$, and for many other groups, as well.

In a nutshell, their idea is this. Suppose G is a finitely generated group for which the complement of $\Sigma^1(G)$ is *finite*.[4] Then every automorphism of G permutes the finitely many rays in $\Sigma^1(G)^c$. This suggests that it might be possible to construct a new ray $\mathbb{R}_{>0} \cdot \chi_0$ that is fixed by $\operatorname{Aut} G$. If one succeeds in doing so, then $\mathbb{R} \cdot \chi_0$ will be a 1-dimensional subrepresentation of the finite dimensional real vector space $\operatorname{Hom}(G, \mathbb{R})$, acted on by $\operatorname{Aut} G$ via

$$(\alpha, \chi) \mapsto \chi \circ \alpha^{-1}.$$

A priori, this invariant line need not be fixed pointwise.

Gonçalves and Kochloukova detected that the line $\mathbb{R} \cdot \chi_0$ is fixed pointwise by $\operatorname{Aut} G$ if the homomorphism $\chi_0 : G \to \mathbb{R}$ has *rank* 1, i.e., if its image is infinite cyclic. Using this fact they were then able to prove that Thompson's group F, but also many other groups G, admit a rank 1 homomorphism that is fixed by $\operatorname{Aut} G$ and thus satisfy property R_∞.

1E. ***A generalization.*** In the second part of this paper we consider a collection of PL-homeomorphism groups G whose invariant $\Sigma^1(G)^c$ is finite but contains a point of rank greater than 1. One is then confronted with the following problem: Suppose $\mathbb{R}_{>0} \cdot \chi_0$ is a ray that is fixed by $\operatorname{Aut} G$ as a *set*. There may then exist an automorphism α which acts on the ray by multiplication by a positive real number

[4] Recall that $\Sigma^1(G)$ is a certain subset of the space of all half-lines $\mathbb{R}_{>0} \cdot \chi$ emanating from the origin of the real vector space $\operatorname{Hom}(G, \mathbb{R})$, and that $\operatorname{Aut} G$ acts canonically on this subset, as well as on its complement.

$s \neq 1$; if so, the 1-dimensional subspace $\mathbb{R} \cdot \chi_0$ in the real vector space $\mathrm{Hom}(G, \mathbb{R})$ is an eigenline with eigenvalue $s \neq 1$ of the linear transformation α^* induced by α on $\mathrm{Hom}(G, \mathbb{R})$. The existence of an eigenvalue $s \neq 1$ wrecks our attempt to extend the approach adopted in [Bleak et al. 2008] to more general classes of groups of PL-homomorphisms, but — as we shall see — it can be ruled out if the image B of the character χ_0 has only 1 and -1 as *units*, the definition of units being as follows:

Definition 1.6. Given a subgroup B of the additive group $\mathbb{R}_{\text{add}}$ we set

$$U(B) = \{s \in \mathbb{R}^\times \mid s \cdot B = B\} \tag{1-4}$$

and call $U(B)$ the *group of units* of B (inside the multiplicative group of $\mathbb{R}$).

We next explain how the subgroups B will enter into the picture. The groups we shall be interested in will be subgroups of $\mathrm{PL}_o(\mathbb{R})$ with supports in a compact interval $[0, b]$; they are thus subgroups G of $G([0, b]; \mathbb{R}_{\text{add}}, \mathbb{R}^\times_{>0})$. By restricting the homomorphisms σ_ℓ and σ_r, defined in Section 1C1, one obtains homomorphisms of G into the multiplicative group $\mathbb{R}^\times_{>0}$; by composing these with the natural logarithm function one arrives at the homomorphisms $\chi_\ell : G \to \mathbb{R}_{\text{add}}$ and $\chi_r : G \to \mathbb{R}_{\text{add}}$.

The next result lists conditions that allow one to infer that G admits a nontrivial homomorphism into $\mathbb{R}^\times_{>0}$ fixed by every automorphism of G.

Theorem 1.7. *Suppose G is a subgroup of $G([0, b]; \mathbb{R}_{\text{add}}, \mathbb{R}^\times_{>0})$ that satisfies the following conditions*:

(i) *no interior point of the interval $I = [0, b]$ is fixed by G*;

(ii) *the homomorphisms χ_ℓ and χ_r are both nonzero*;

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group*; *and*

(iv) *at least one of the groups $U(\operatorname{im} \chi_\ell)$ or $U(\operatorname{im} \chi_r)$ is reduced to $\{1, -1\}$.*

Then there exists a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of G. The group G has therefore property R_∞.

There remains the problem of finding subgroups $B \subset \mathbb{R}_{\text{add}}$ that have only the units 1 and -1. This problem is addressed in Section 6E. We shall show that a subgroup $B = \ln P$ has this property if the multiplicative group $P \subset \mathbb{R}^\times_{>0}$ is free abelian and generated by algebraic numbers. In addition, we shall construct in Section 8A a collection $\mathcal{G}$ of pairwise nonisomorphic 3-generator groups G_s enjoying the properties that each group G_s satisfies the assumptions of Theorem 1.7 and that the cardinality of $\mathcal{G}$ is that of the continuum.

2. Preliminaries on automorphisms of the groups $G(I; A, P)$

The groups $G(I; A, P)$ form a class of subgroups of the group $\mathrm{PL}_o(\mathbb{R})$, the group of all orientation preserving, piecewise linear homeomorphisms of the real line.

They enjoy some special properties, in particular the following two: each group acts approximately[5] highly transitively on the interior of I, and all its automorphisms are induced by conjugation by homeomorphisms. It is, above all, this second property that will be exploited in the sequel.

In this section, we recall the basic representation theorem for automorphisms of the groups $G(I; A, P)$ and deduce then some consequences.

2A. ***Representation of isomorphisms.*** We begin by fixing the set-up of this section: P is a nontrivial subgroup of $\mathbb{R}^{\times}_{>0}$ and A a nonzero subgroup of $\mathbb{R}_{\text{add}}$ that is stable under multiplication by P. Next, I is a closed interval of positive length; we assume the left end point of I is in A if I is bounded from below and similarly for the right end point.

Remark 2.1. Distinct intervals I_1 and I_2 may give rise to isomorphic groups $G(I_1; A, P)$ and $G(I_2; A, P)$. In particular, it is true that every group $G(I_1; A, P)$ is isomorphic to one whose interval I_2 has one of the three forms

$$[0, b] \text{ with } b \in A, \quad [0, \infty[, \quad \text{and} \quad \mathbb{R}. \tag{2-1}$$

See [Bieri and Strebel 2016, Sections 2.4 and 16.4] for proofs.

We come now to the announced result about isomorphisms of groups $G(I; A, P)$ and $G(\bar{I}; \bar{A}, \bar{P})$. It asserts that each isomorphism of the first group onto the second one is induced by conjugation by a homeomorphism of the interior $\operatorname{int}(I)$ of I onto the interior of $\bar{I}$. This claim holds even for suitably restricted subgroups of $G(I; A, P)$ and of $G(\bar{I}; \bar{A}, \bar{P})$. In order to state the generalized assertion we need the subgroup of "bounded elements".

Definition 2.2. Let $B(I; A, P)$ be the subgroup of $G(I; A, P)$ consisting of all PL-homeomorphisms f that are the identity near the end points or, more formally, that satisfy the inequalities $\inf I < \inf \operatorname{supp} f$ and $\sup \operatorname{supp} f < \sup I$.

We are now in a position to state the representation theorem.

Theorem 2.3. *Assume G is a subgroup of $G(I; A, P)$ that contains the derived subgroup of $B(I; A, P)$, and $\bar{G}$ is a subgroup of $G(\bar{I}; \bar{A}, \bar{P})$ containing the derived group of $B(\bar{I}; \bar{A}, \bar{P})$. Then every isomorphism $\alpha : G \xrightarrow{\sim} \bar{G}$ is induced by conjugation by a unique homeomorphism φ_α of the interior* $\operatorname{int}(I)$ *of I onto the interior of $\bar{I}$; more precisely, the equation*

$$\alpha(g) \restriction \operatorname{int}(\bar{I}) = \varphi_\alpha \circ (g \restriction \operatorname{int}(I)) \circ \varphi_\alpha^{-1} \tag{2-2}$$

holds for every $g \in G$. Moreover, φ_α maps $A \cap \operatorname{int}(I)$ onto $\bar{A} \cap \operatorname{int}(\bar{I})$.

Proof. The result is a restatement of [Bieri and Strebel 2016, Theorem E16.4]. □

[5] See [Bieri and Strebel 2016, Chapter A] for details.

Remarks 2.4. (a) Theorem 2.3 has two simple, but important consequences. First of all, every homeomorphism of intervals is either increasing or decreasing; since the homeomorphism φ_α inducing an isomorphism $\alpha : G \xrightarrow{\sim} \bar{G}$ is uniquely determined by α, there exist therefore two types of isomorphisms: the *increasing* isomorphisms, induced by conjugation by an increasing homeomorphism, and the decreasing ones.

Assume now that $\bar{I} = I$. If the homeomorphism $\varphi_\alpha : \operatorname{int}(I) \xrightarrow{\sim} \operatorname{int}(I)$ is increasing, it extends uniquely to a homeomorphism of I, but this may not be so if it is decreasing. Indeed, φ_α extends if I is a compact interval or the real line, but not if I is a half-line. If the extension exists, it will be denoted by $\tilde{\varphi}_\alpha$.

(b) The increasing automorphisms of a group G form a subgroup $\operatorname{Aut}_+ G$ of $\operatorname{Aut} G$ of index at most 2. It will turn out that is often easier to find a nonzero homomorphism $\psi : G \to B$ that is fixed by the subgroup $\operatorname{Aut}_+ G$ than a nonzero homomorphism fixed by $\operatorname{Aut} G \smallsetminus \operatorname{Aut}_+ G$ (in case this set is nonempty). For this reason, it is useful to dispose of criteria guaranteeing that $\operatorname{Aut} G = \operatorname{Aut}_+ G$.

(c) The derived group of $B(I; A, P)$ is a simple, infinite group (see [Bieri and Strebel 2016, Proposition C10.2]), but $B(I; A, P)$ itself may not be perfect. To date, no characterization of the parameters (I, A, P) corresponding to perfect groups $B(I; A, P)$ is known. The quotient group $G(I; A, P)/B(I; A, P)$, on the other hand, is a metabelian group that can be described explicitly in terms of the triple (I, A, P); see [Bieri and Strebel 2016, Section 5.2]. In the sequel, we shall therefore restrict attention to subgroups G containing $B(I; A, P)$.

(d) The second important consequence of Theorem 2.3 is the fact that $B(I; A, P)$ is a characteristic subgroup of every subgroup G with $B(I; A, P) \subseteq G \subseteq G(I; A, P)$. (The proof is easy; see [Bieri and Strebel 2016, Corollary E16.5] or Corollary 2.7 below.)

In part (a) of the previous remarks the term *increasing isomorphism* has been introduced. In the sequel, this parlance will be used often, and so we declare:

Definition 2.5. Let $\alpha : G \xrightarrow{\sim} \bar{G}$ be an isomorphism induced by the (uniquely determined) homeomorphism $\varphi_\alpha : \operatorname{int}(I) \xrightarrow{\sim} \operatorname{int}(\bar{I})$. If φ is *increasing* then α will be called increasing, and similarly for *decreasing*.

2B. ***The homomorphisms λ and ρ.*** By Remark 2.4(d) the group $B = B(I; A, P)$ is a characteristic subgroup of every group G containing it. Now G has, in addition, subgroups containing B that are invariant under the subgroup $\operatorname{Aut}_+ G$, namely the kernels of the homomorphisms λ and ρ. To set these homomorphisms into perspective, we go back to the homomorphisms

$$\sigma_\ell : G(I; A, P) \to P \quad \text{and} \quad \sigma_r : G(I; A, P) \to P,$$

introduced in Section 1C1. Their images are abelian and coincide with the group of slopes P. If I is not bounded from below, there exist a homomorphism λ, related to σ_ℓ, whose image is contained in $\mathrm{Aff}(A, P)$, the group of all affine maps of $\mathbb{R}$ with slopes in P and displacements $f(0) \in A$. The definition of λ is this:

$$\lambda : G(I; A, P) \to \mathrm{Aff}(A, P), \qquad g \mapsto (\text{affine map coinciding with } g \text{ near } -\infty). \tag{2-3}$$

If the interval I is not bounded from above, then there exists a similarly defined homomorphism

$$\rho : G(I; A, P) \to \mathrm{Aff}(A, P), \qquad g \mapsto (\text{affine map coinciding with } g \text{ near } +\infty). \tag{2-4}$$

The images of λ and ρ are, in general, smaller than $\mathrm{Aff}(A, P)$. They are equal to the entire group $\mathrm{Aff}(A, P)$ if $I = \mathbb{R}$; if I is not bounded from below, but bounded from above, the image of λ is $\mathrm{Aff}(IP \cdot A, P)$ and the analogous statement holds for ρ. In the above, $IP \cdot A$ denotes the submodule of A generated by the products $(p-1) \cdot a$ with $p \in P$ and $a \in A$; see [Bieri and Strebel 2016, Section 4 and Corollary A5.3].

For uniformity of notation, we extend the definition of λ and ρ to compact intervals: if $I = [0, b]$ and $f \in G(I; A, P)$ then $\lambda(g)$ is the linear map $t \mapsto \sigma_\ell(g) \cdot t$ and $\rho(g)$ is the affine map $t \mapsto \sigma_r(f) \cdot (t - b) + b$. Similarly one defines $\lambda(g)$ if I is the half-line $[0, \infty[$.

The homomorphisms λ and ρ allow one to restate the definition of $B(I; A, P)$:

$$B(I; A, P) = \ker \lambda \cap \ker \rho. \tag{2-5}$$

Remark 2.6. In the sequel, we shall often deal with subgroups, denoted G, of a group $G(I; A, P)$ that contain $B(I; A, P)$. For ease of notation, we shall then denote the restrictions of λ and ρ to G again by λ and ρ.

2C. *First consequences of the representation theorem.* Let G be a subgroup of $G(I; A, P)$ that contains the derived subgroup of $B(I; A, P)$ and let $\bar{G}$ be a subgroup of $G(\bar{I}; \bar{A}, \bar{P})$ containing the derived subgroup of $B(\bar{I}; \bar{A}, \bar{P})$. Suppose φ_α is a homeomorphism of $\mathrm{int}(I)$ onto $\mathrm{int}(\bar{I})$ that induces an isomorphism $\alpha : G \xrightarrow{\sim} \bar{G}$. The map φ_α need not be piecewise linear. Theorem 2.3, however, has useful consequences even in such a case. One implication is recorded in this result:

Corollary 2.7. *Assume G and $\bar{G}$ are subgroups of $G(I; A, P)$ both of which contain $B(I; A, P)$, and let*

$$\lambda, \rho : G \to \mathrm{Aff}(A, P) \quad \textit{and} \quad \bar{\lambda}, \bar{\rho} : \bar{G} \to \mathrm{Aff}(A, P)$$

be the obvious restrictions of the homomorphisms λ, ρ *introduced in Section 2B. Consider now an isomorphism* $\alpha : G \xrightarrow{\sim} \bar{G}$ *that is induced by the homeomorphism* $\varphi_\alpha : \operatorname{int}(I) \xrightarrow{\sim} \operatorname{int}(I)$*. If* φ_α *is increasing then*

(i) α *maps* $\ker\lambda$ *onto* $\ker\bar{\lambda}$ *and induces an isomorphism* α_ℓ *of* $G/\ker\lambda$ *onto* $\bar{G}/\ker\bar{\lambda}$;

(ii) α *maps* $\ker\rho$ *onto* $\ker\bar{\rho}$ *and induces an isomorphism* α_r *of* $G/\ker\rho$ *onto* $\bar{G}/\ker\bar{\rho}$.

Proof. (i) If $g \in \ker\lambda$ then g is the identity near $\inf I$. As φ_α is *increasing*, the image $\alpha(g) = \varphi_\alpha \circ g \circ \varphi_\alpha^{-1}$ of g is therefore also the identity near $\inf I$. It follows that $\alpha(\ker\lambda) \subseteq \ker\bar{\lambda}$. This inclusion is actually an equality, for $\alpha^{-1} : \bar{G} \to G$ is an isomorphism and so $\alpha^{-1}(\ker\bar{\lambda}) \subseteq \ker\lambda$. Claim (ii) can be proved similarly. □

2D. ***Automorphisms induced by finitary PL-homeomorphisms.*** Suppose that the group $G \subseteq G(I; A, P)$ is as before, and let α be an automorphism of G. According to Theorem 2.3, α is induced by conjugation by a unique autohomeomorphism φ_α. This autohomeomorphism may not be piecewise linear, but the situation improves if P, the group of slopes, is not cyclic (and hence dense in $\mathbb{R}^\times_{>0}$).

Theorem 2.8. *Suppose* P *is not cyclic. For every automorphism* α *of* G *there exists then a nonzero real number* s *such that* $A = s \cdot A$ *and that the autohomeomorphism* $\varphi_\alpha : \operatorname{int}(I) \xrightarrow{\sim} \operatorname{int}(I)$ *is piecewise linear with slopes in the coset* $s \cdot P$ *of* P*. Moreover,* φ_α *maps the subset* $A \cap \operatorname{int}(I)$ *onto itself and has only finitely many breakpoints in every compact subinterval of* $\operatorname{int}(I)$.

Proof. The result is a special case of [Bieri and Strebel 2016, Theorem E17.1]. □

Theorem 2.8 indicates that automorphisms of groups with a noncyclic group of slopes P are easier to analyze than those of the groups with cyclic P. Note, however, that the conclusion of Theorem 2.8 does not rule out that φ_α has infinitely many breakpoints which accumulate in one or both end points[6] and so φ_α may not be differentiable at the end points. In Section 3A we shall therefore be interested in differentiability criteria.

3. Characters fixed by Aut $G([0, b]; A, P)$

In this section, we prove Theorem 1.4 for the case of a compact interval and various extensions of it. An important ingredient in the proofs of these results is a criterion that allows one to deduce that an autohomeomorphism φ_α inducing an automorphism α of the group is differentiable near one or both of its end points.

[6] The notion of end point is to be interpreted suitably if I is not bounded.

3A. ***A differentiability criterion.*** The proof of the criterion is rather involved. Prior to stating the criterion and giving its proof, we discuss therefore a result that explains the interest in the criterion.

Proposition 3.1. *Let G be a subgroup of $G([a,b]; A, P)$ that contains the derived subgroup of $B(I; A, P)$. Suppose $\tilde{\varphi} : [0,b] \xrightarrow{\sim} [0,b]$ is an autohomeomorphism that induces, by conjugation, an automorphism α of G. Then the following are true*:

(i) *if $\tilde{\varphi}$ is increasing, differentiable at* 0, *and* $\tilde{\varphi}'(0) > 0$, *then α fixes σ_ℓ*;

(ii) *if $\tilde{\varphi}$ is increasing, differentiable at b, with $\tilde{\varphi}'(b) > 0$, then α fixes σ_r*;

(iii) *if $\tilde{\varphi}$ is differentiable both at* 0 *and at b, with nonzero derivatives, then α fixes the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$.*

Proof. (i) and (ii) Suppose the extended autohomeomorphism $\tilde{\varphi} = \tilde{\varphi}_\alpha$ is increasing and fix $g \in G$. If $\tilde{\varphi}$ is differentiable at 0 and $\tilde{\varphi}'(0) > 0$, the chain rule justifies the computation

$$\sigma_\ell(\alpha(g)) = (\tilde{\varphi} \circ g \circ \tilde{\varphi}^{-1})'(0) = \tilde{\varphi}'(0) \cdot g'(0) \cdot (\tilde{\varphi}^{-1})'(0) = \sigma_\ell(g). \tag{3-1}$$

It follows that σ_ℓ is fixed by α. If $\tilde{\varphi}$ admits a left derivative at b and if $\tilde{\varphi}'(b) > 0$, one sees similarly, that σ_r is fixed by α.

(iii) Assume now that $\tilde{\varphi} = \tilde{\varphi}_\alpha$ is differentiable, both at 0 and at b, and that both derivatives are different from 0. If $\tilde{\varphi}$ is *increasing*, parts (i) and (ii) guarantee that σ_ℓ and σ_r are fixed by α, whence so is their product ψ. If, on the other hand, $\tilde{\varphi}$ is *decreasing*, the calculation

$$\sigma_r(\alpha(g)) = (\tilde{\varphi} \circ g \circ \tilde{\varphi}^{-1})'(b) = \tilde{\varphi}'(0) \cdot g'(0) \cdot (\tilde{\varphi}^{-1})'(b) = \sigma_\ell(g) \tag{3-2}$$

holds for every $g \in G$ and establishes the relation $\sigma_r \circ \alpha = \sigma_\ell$.

A similar calculation shows that the relation $\sigma_\ell \circ \alpha = \sigma_r$ is valid. The claim for ψ is then a consequence of the computation

$$(\psi \circ \alpha)(g) = \sigma_\ell(\alpha(g)) \cdot \sigma_r(\alpha(g)) = \sigma_r(g) \cdot \sigma_\ell(g) = \psi(g). \qquad \square$$

3A1. *Statement and proof of the criterion.* We now come to the criterion; we choose a formulation that is slightly more general than what is needed for the case at hand; the extended version will be used in Section 4.

Proposition 3.2. *Suppose I is an interval of one of the forms* $[0,b]$ *or* $[0,\infty[$, *and G as well as $\bar{G}$ are subgroups of $G(I; A, P)$ that contain $B(I; A, P)$. Assume $\tilde{\varphi} : I \xrightarrow{\sim} I$ is an* increasing *autohomeomorphism that induces, by conjugation, an isomorphism α of the group G onto the group $\bar{G}$.*

If the image of $\sigma_\ell : G \to P$ is not cyclic, *then $\tilde{\varphi}$ is linear on a small interval of the form* $[0,\delta]$ *and so $\tilde{\varphi}$ is differentiable at* 0 *with positive derivative.*

Proof. The following argument uses ideas from the proofs of Proposition E16.8 and Supplement E17.3 in [Bieri and Strebel 2016]. The proof will be divided into three parts. In the first one, we show that $\alpha : G \xrightarrow{\sim} \bar{G}$ induces an isomorphism $\alpha_\ell : \operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\bar{\sigma}_\ell$, that takes $p \in \operatorname{im}\sigma_\ell$ to $p^r = \mathrm{e}^{r\cdot\log p}$ for some positive real number r that does *not* depend on p. In the second part, we establish that $\tilde{\varphi}$ satisfies the relation

$$\tilde{\varphi}(p\cdot t) = p^r \cdot \tilde{\varphi}(t) \tag{3-3}$$

for every $p \in (\operatorname{im}\sigma_\ell \cap\,]0,1[)$ and t varying in some small interval $[0,\delta]$. In the last part, we deduce from this relation that $\tilde{\varphi}$ is linear near 0.

We now embark on the first part. Since φ is increasing, Corollary 2.7 applies and shows that α maps the kernel of $\sigma_\ell : G \to P$ onto the kernel of the homomorphism $\bar{\sigma}_\ell : \bar{G} \to P$, and thus induces an isomorphism $\alpha_\ell : \operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\bar{\sigma}_\ell$ that renders the square

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, twoheadrightarrow, "\sigma_\ell"] & \bar{G} \arrow[d, twoheadrightarrow, "\bar{\sigma}_\ell"] \\ \operatorname{im}\sigma_\ell \arrow[r, "\alpha_\ell"] & \operatorname{im}\bar{\sigma}_\ell \end{tikzcd}$$

commutative. We claim α_ℓ maps the set $(\operatorname{im}\sigma_\ell)\cap\,]0,1[$ onto $(\operatorname{im}\bar{\sigma}_\ell)\cap\,]0,1[$. Indeed, let $p \in \operatorname{im}\sigma_\ell$ be a slope with $p<1$ and let $f_p \in G$ be a preimage of p. Then $\alpha(f_p)$ is linear on some interval $[0,\varepsilon_p]$ and has slope $\bar{\sigma}_\ell(\alpha(f_p)) = \alpha_\ell(p)$ there. Since $\tilde{\varphi}$ is continuous at 0, there exists $\delta_p > 0$ so that f_p is linear on $[0,\delta_p]$ and that $\tilde{\varphi}([0,\delta_p]) \subseteq [0,\varepsilon_p]$. Fix $t \in [0,\delta_p]$. The hypothesis that α is induced by conjugation by $\tilde{\varphi}$ then leads to the chain of equalities

$$\tilde{\varphi}(p\cdot t) = (\tilde{\varphi}\circ f_p)(t) = (\alpha(f_p)\circ\tilde{\varphi})(t) = \alpha(f_p)(\tilde{\varphi}(t)) = \alpha_\ell(p)\cdot\tilde{\varphi}(t). \tag{3-4}$$

Since $\tilde{\varphi}$ is increasing and as $p<1$, the chain of equalities implies that $\alpha_\ell(p) < 1$. It follows that α_ℓ maps $(\operatorname{im}\sigma_\ell)\cap\,]0,1[$ into $\operatorname{im}\bar{\sigma}_\ell \cap\,]0,1[$ and then, by applying the preceding argument to φ^{-1}, that

$$\alpha_\ell(\operatorname{im}\sigma_\ell \cap\,]0,1[) = \operatorname{im}\bar{\sigma}_\ell \cap\,]0,1[.$$

We show next that $\alpha_\ell(p) = p^r$ for all $p \in \operatorname{im}\sigma_\ell$ and some positive real number r. We begin by passing from the multiplicative subgroup $\operatorname{im}\sigma_\ell \subset \mathbb{R}^\times_{>0}$ to a subgroup of $\mathbb{R}_{\mathrm{add}}$; to that end, we introduce the homomorphism

$$L_0 = \ln\circ\alpha_\ell\circ\exp : \ln(\operatorname{im}\sigma_\ell) \xrightarrow{\sim} \ln(\operatorname{im}\bar{\sigma}_\ell).$$

The previous verification implies that L_0 is an order preserving isomorphism; by the assumption on $\operatorname{im}\sigma_\ell$ the domain of L_0 is a dense subgroup of $\mathbb{R}_{\mathrm{add}}$. It follows that L_0 extends uniquely to an order preserving automorphism $L : \mathbb{R}_{\mathrm{add}} \to \mathbb{R}_{\mathrm{add}}$.

The homomorphism L is continuous, hence linear, and so given by multiplication by some positive real number r. The isomorphism α_ℓ has therefore the form

$$p \mapsto p^r = \exp(r \cdot \ln p) \quad \text{with } r > 0.$$

We come now to the second part of the proof. Fix a slope $p_1 < 1$ in $\operatorname{im} \sigma_\ell$. Formula (3-4) and the previously found formula for α_ℓ then imply that there exists a small positive number δ_{p_1} such that the equation

$$\tilde{\varphi}(p_1 \cdot t) = p_1^r \cdot \tilde{\varphi}(t) \tag{3-5}$$

holds for every $t \in [0, \delta_{p_1}]$. Consider next another slope $p < 1$. There exists then, as before, a real number $\delta_p > 0$ so that $\tilde{\varphi}(p \cdot t) = p^r \cdot \tilde{\varphi}(t)$ for $t \in [0, \delta_p]$. Choose now $m \in \mathbb{N}$ so large that $p_1^m \cdot \delta_{p_1} \leq \delta_p$. The following chain of equalities then holds for each $t \in [0, \delta_{p_1}]$:

$$p_1^{m \cdot r} \cdot \tilde{\varphi}(p \cdot t) = \tilde{\varphi}(p_1^m \cdot p \cdot t) = \tilde{\varphi}(p \cdot p_1^m \cdot t) = p^r \cdot \tilde{\varphi}(p_1^m \cdot t) = p^r \cdot p_1^{m \cdot r} \cdot \tilde{\varphi}(t).$$

The calculation shows that $\tilde{\varphi}(p \cdot t) = p^r \cdot \tilde{\varphi}(t)$ for every $t \in [0, \delta_1]$. Upon setting $\delta = \delta_{p_1}$ one arrives at (3-3).

The proof is now quickly completed. By assumption, $\operatorname{im} \sigma_\ell$ is not cyclic and so (3-3) holds for a dense set of slopes p and a fixed argument t, say $t = \delta$. Since φ is continuous and increasing, (3-3) continues to hold for every real $x \in \,]0, 1[$. The formula

$$\varphi(x \cdot \delta) = \exp(r \cdot \ln x) \cdot \varphi(\delta) = x^r \cdot \varphi(\delta)$$

is therefore valid for every $x \in \,]0, \delta]$. By Theorem 2.8, on the other hand, φ is piecewise linear on $]0, \delta]$. So the exponent r must be equal to 1, whence φ is linear on $[0, \delta]$ with slope $\varphi(\delta)/\delta > 0$ and so, in particular, differentiable at 0. □

Remark 3.3. Assume I is a compact interval of the form $[0, b]$ with $b \in A_{>0}$ and the images of σ_ℓ and σ_r are both not cyclic. It follows then from Proposition 3.2 that every *increasing* automorphism $\alpha : G \xrightarrow{\sim} G$ is induced by an autohomeomorphism $\tilde{\varphi}$ that is affine near both end points. By [Bieri and Strebel 2016, Proposition E16.9] the homeomorphism $\tilde{\varphi}$ is thus *finitary* piecewise linear.

3A2. *First application.* As a further step towards the main results we give a corollary that combines Propositions 3.1 and 3.2.

Corollary 3.4. *Let G be a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. Assume $I = [0, b]$ and let α be an automorphism of G that is induced by the autohomeomorphism $\tilde{\varphi} : I \xrightarrow{\sim} I$. Then the following statements hold*:

(i) *if α is increasing*[7] *and* $\operatorname{im} \sigma_\ell$ *not cyclic, then σ_ℓ is fixed by α*;

[7]See Definition 2.5.

(ii) *if* α *is increasing and* $\operatorname{im}\sigma_r$ *not cyclic, then* σ_r *is fixed by* α;

(iii) *if* $\tilde{\varphi}$ *is decreasing and* $\operatorname{im}\sigma_\ell$ *is not cyclic, then* $\tilde{\varphi}$ *is affine near both end points and the homomorphism* $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ *is fixed by* α.

Proof. (i) The statement is a direct consequence of Proposition 3.2 and part (i) of Proposition 3.1.

(ii) We invoke Proposition 3.2 for an auxiliary group G_1. Let $\vartheta : I \xrightarrow{\sim} I$ be the reflection in the midpoint of I; set $G_1 = \vartheta \circ G \circ \vartheta^{-1}$ and $\varphi_1 = \vartheta \circ \tilde{\varphi}_\alpha \circ \vartheta^{-1}$. Since $G(I; A, P)$ and $B(I; A, P)$ are both invariant under conjugation by ϑ, and as the image of σ_r is not cyclic, Proposition 3.2 applies to the couple (G_1, φ_1) and shows that φ_1 is linear in a small interval $[0, \delta_1]$ of positive length. But if so, φ_α is affine in the interval $[b - \delta_1, b]$. Now use part (ii) in Proposition 3.1.

(iii) Since $\tilde{\varphi}$ is *decreasing*, the subgroups $\operatorname{im}\sigma_\ell$ and $\operatorname{im}\sigma_r$ are isomorphic by Lemma 3.6 below; the hypothesis on $\operatorname{im}\sigma_\ell$ therefore implies the image of σ_r is not cyclic either. Let $\vartheta : \operatorname{int}(I) \xrightarrow{\sim} \operatorname{int}(I)$ be the reflection in the midpoint of the interval I and set $\tilde{\varphi}_1 = \vartheta \circ \tilde{\varphi}$ and $\bar{G} = \vartheta \circ G \circ \vartheta^{-1}$. Conjugation by $\tilde{\varphi}_1$ induces then an increasing isomorphism $\alpha_1 : G \xrightarrow{\sim} \bar{G}$. Since $G(I; A, P)$ and $B(I; A, P)$ are both invariant under conjugation by ϑ, Proposition 3.2 applies to $\tilde{\varphi}_1$ in the rôle of $\tilde{\varphi}$ and shows that $\tilde{\varphi}_1$ is linear near 0. But if so, $\tilde{\varphi}$ is linear near 0. Consider now the autohomeomorphism $\tilde{\varphi}_2 = \tilde{\varphi} \circ \vartheta$ of I. It induces an isomorphism $\alpha_2 : \bar{G} \xrightarrow{\sim} G$ by conjugation; an argument similar to the preceding one then reveals that $\tilde{\varphi}$ is affine near b. The remainder of the claim follows from part (iii) in Proposition 3.1. □

3B. *Construction of homomorphisms fixed by* $\mathbf{Aut}_+ G$. The first main result holds for all groups G with $B(I; A, P) \subsetneq G \subseteq G(I; A, P)$, but the exhibited homomorphisms may only be fixed by $\operatorname{Aut}_+ G$.

Theorem 3.5. *Suppose* $I = [0, b]$ *with* $b \in A_{>0}$ *and* G *is a subgroup of* $G(I; A, P)$ *that contains* $B(I; A, P)$ *properly. Then the homomorphisms* σ_ℓ *and* σ_r *are fixed by* $\operatorname{Aut}_+ G$, *and at least one of them is nontrivial.*

Proof. Let α be an increasing automorphism of G and let $\tilde{\varphi}$ be the autohomeomorphism of I that induces α. (The map $\tilde{\varphi}$ exists by Theorem 2.3 and Remark 2.4(a).) Since the quotient group $G(I; A, P)/B(I; A, P)$ is isomorphic to the image of $\sigma_\ell \times \sigma_r : G(I; A, P) \to P \times P$ and as G contains $B(I; A, P)$ properly, at least one of the homomorphisms σ_ℓ and σ_r is nonzero.

Assume first that $\psi = \sigma_\ell$ is nonzero. Two cases then arise, depending on whether the image of ψ is cyclic or not. If the image of ψ is *not cyclic* then part (i) in Corollary 3.4 shows that α fixes ψ. If, on the other hand, ψ is cyclic, consider the generator $p \in \operatorname{im}\psi$ with $p < 1$ and pick a preimage $g_p \in G$ of p. Then g_p attracts points in every sufficiently small interval of the form $[0, \delta]$ towards 0; hence so does

$\alpha(g_p) = \tilde{\varphi} \circ g_p \circ \tilde{\varphi}^{-1}$ and thus $p' = (\alpha(g_p))'(0) < 1$. Now p' generates also $\operatorname{im} \sigma_\ell$; being smaller than 1, it therefore coincides with $p = \psi(g_p)$ and so $\psi = \psi \circ \alpha$.

Assume next that $\psi = \sigma_r$ is not zero. If its image is not cyclic, part (ii) of Corollary 3.4 allows us to conclude that α fixes ψ. If $\operatorname{im} \psi$ is cyclic, consider the generator $p \in \operatorname{im} \psi$ with $p < 1$ and pick a preimage $g_p \in G$. Then g_p attracts points in every sufficiently small interval $[b - \delta, b]$ towards b. It then follows, as before, that $\psi(\alpha(g_p)) = p = \psi(g_p)$, whence $\psi \circ \alpha = \alpha$. □

3C. *Existence of decreasing automorphisms.* Theorem 3.5 is very satisfactory in that it produces a nonzero homomorphism ψ onto an infinite abelian group whenever such a homomorphism is likely to exist, i.e., if G contains $B(I; A, P)$ properly. This homomorphism is, however, only guaranteed to be fixed by the subgroup $\operatorname{Aut}_+ G$ of $\operatorname{Aut} G$ which has index 1 or 2 in $\operatorname{Aut} G$. If the index is 1, the conclusion of Theorem 3.5 is as good as we can hope for. So the question arises whether there are useful criteria that force the index to be 1. Here is a very simple observation that leads to such a criterion:

Lemma 3.6. *Assume $I = [0, b]$ with $b \in A_{>0}$ and let G be a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. Then every decreasing automorphism α induces an isomorphism $\alpha_* : \operatorname{im} \sigma_\ell \xrightarrow{\sim} \operatorname{im} \sigma_r$.*

Proof. The kernel of σ_ℓ consists of all elements in G that are the identity near 0. Since α is induced by conjugation by a homeomorphism of I that maps 0 onto b, the image of $\ker \sigma_\ell$ consists of elements that are the identity near b, so $\alpha(\ker \sigma_\ell) \subseteq \ker \sigma_r$. Since α^{-1} is also a decreasing automorphism, the preceding inclusion is actually an equality. So α induces an isomorphism $\alpha_* : \operatorname{im} \sigma_\ell \xrightarrow{\sim} \operatorname{im} \sigma_r$ that renders the square

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, two heads, "\sigma_\ell"] & G \arrow[d, two heads, "\sigma_r"] \\ \operatorname{im} \sigma_\ell \arrow[r, "\alpha_*"] & \operatorname{im} \sigma_r \end{tikzcd} \tag{3-6}$$

commutative. □

Example 3.7. Suppose the slope group P is finitely generated and hence free abelian of finite rank r, say. Choose subgroups Q_ℓ and Q_r of P and set

$$G(Q_\ell, Q_r) = \{g \in G(I; A, P) \mid (\sigma_\ell(g), \sigma_r(g) \in Q_\ell \times Q_r\}. \tag{3-7}$$

Then $\operatorname{im} \sigma_\ell = Q_\ell$ and $\operatorname{im} \sigma_r = Q_r$, and the image of $(\sigma_\ell, \sigma_r) : G \to P \times P$ coincides with $Q_\ell \times Q_r$. (These claims follow from [Bieri and Strebel 2016, Corollary A5.5]).

Now assume that $G(Q_\ell, Q_r)$ admits a decreasing automorphism, say α. By Lemma 3.6 the groups Q_ℓ and Q_r are then isomorphic, and thus have the same rank. But more is true: if $Q_\ell = \operatorname{im} \sigma_\ell$ is *not* cyclic, then Proposition 3.2 and the last line of Proposition 3.1 show that $\sigma_r = \sigma_\ell \circ \alpha$, whence Q_r, the image of σ_r, coincides with Q_ℓ, the image of σ_ℓ. The same conclusion holds if Q_r is not cyclic. Conversely, if $Q_\ell = Q_r$ then $G(Q_\ell, Q_r)$ admits decreasing automorphisms, for instance the automorphism induced by conjugation by the reflection about the mid point of I.

So the only case where the existence of a decreasing automorphism is neither obvious nor easy to rule out by the preceding arguments is that where Q_ℓ and Q_r are both cyclic, but distinct. We shall come back to this exceptional case in Example 3.13.

3D. ***Construction of a homomorphism fixed by* Aut G.** We move on to the construction of a homomorphism fixed by all of $\operatorname{Aut} G$. The following result is our main result.

Theorem 3.8. *Suppose I is a compact interval of the form $[0, b]$ with $b \in A_{>0}$. Let G be a subgroup of $G(I; A, P)$ containing $B(I; A, P)$ and let $\psi : G \to P$ be the homomorphism $g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$. Then ψ is fixed by* $\operatorname{Aut} G$, *except possibly when G satisfies the following three conditions*:

(a) $\operatorname{im}(\sigma_\ell : G \to P)$ *is cyclic*,

(b) *G admits a decreasing automorphism*,

(c) *G does not admit a decreasing automorphism induced by an autohomeomorphism $\vartheta : I \xrightarrow{\sim} I$ that is differentiable at both end points with nonzero values.*

Proof. Let α be an automorphism of G and let φ be the autohomeomorphism of $\operatorname{int}(I)$ that induces α by conjugation. If φ is *increasing* both σ_ℓ and σ_r are fixed by α (see Theorem 3.5) and hence so is ψ. If, on the other hand, α is *decreasing* and the image of σ_ℓ is *not cyclic* then part (iii) of Corollary 3.4 yields the desired conclusion.

Now suppose that G admits an automorphism β that is induced by a decreasing autohomeomorphism $\tilde{\varphi}_\beta$ of I that is differentiable at 0, as well as at b, and has there nonzero derivatives. Then part (iii) of Proposition 3.1 allows us to conclude that ψ is fixed by β. Since β represents the coset $\operatorname{Aut} G \smallsetminus \operatorname{Aut}_+ G$ and as ψ is fixed by $\operatorname{Aut}_+ G$, it follows that ψ is fixed by every decreasing automorphism.

All taken together we have proved that the automorphism α fixes ψ except, possibly, if $\operatorname{im} \sigma_\ell$ is cyclic, α is decreasing and if there does not exists a decreasing automorphism β that is differentiable at the end points and has there nonzero derivatives. $\square$

We state next some consequences of Theorems 3.5 and 3.8. We begin with the special case where G is all of $G(I; A, P)$. Then G is normalized by the reflection in the midpoint of I and so Theorem 3.8 leads to

Corollary 3.9. *If G coincides with $G([0, b]; A, P)$ the homomorphism $\psi : G \to P$ taking $g \in G$ to $\sigma_\ell(g) \cdot \sigma_r(g)$ is surjective, hence nonzero, and fixed by* $\operatorname{Aut} G$.

The second result is a consequence of the proof of Theorem 3.5.

Corollary 3.10. *Suppose that I is the half-line $[0, \infty[$ and G is a subgroup of $G(I; A, P)$ containing $B(I; A, P)$. If G does not admit a decreasing automorphism then $\psi = \sigma_\ell$ is fixed by* $\operatorname{Aut} G$.

Proof. The claim follows from Proposition 3.2 and from the proof of part (i) in Proposition 3.1 upon noting that the cited proof does not presuppose that the interval I be bounded from above. □

3E. *Some examples.* We exhibit some specimens of groups G that possess a homomorphism $\psi : G \to P$ fixed by $\operatorname{Aut} G$. The existence of ψ will be established by recourse to Theorems 3.5 and 3.8 and to Corollary 3.9.

Example 3.11. We begin with variations on Thompson's group F. Assume P is infinite cyclic and A is a (nontrivial) $\mathbb{Z}[P]$-submodule of $\mathbb{R}$. Set $G_0 = G([0, b]; A, P)$ with $b \in A_{>0}$ and consider the following subgroups of G_0:

$$G_1 = \{g \in G_0 \mid \sigma_\ell(g) = 1\}, \tag{3-8}$$

$$G_2 = \{g \in G_0 \mid \sigma_\ell(g) = \sigma_r(g)\}, \tag{3-9}$$

$$G_3 = \{g \in G_0 \mid \sigma_\ell(g) = \sigma_r(g)^{-1}\}. \tag{3-10}$$

The group G_0 is the entire group $G(I; A, P)$ and so Corollary 3.9 tells us that the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is nonzero and fixed by $\operatorname{Aut} G_0$.

The group G_1 is an ascending union of subgroups $H_n = G([a_n, b]; A, P)$ given by a strictly decreasing sequence $n \mapsto a_n$ of elements in A that converges to 0, and so the group G_1 is infinitely generated. It does not admit a decreasing automorphism (for instance because of Lemma 3.6) and so Theorem 3.5 allows us to infer that the epimorphism $\sigma_r : G_1 \twoheadrightarrow P$ is fixed by all of $\operatorname{Aut} G_1$.

The group G_2 is an ascending HNN-extension with a base group that is isomorphic to G_0 (see [Bieri and Strebel 2016, Lemma E18.8]). If G_0 is finitely generated or finitely presented, so is therefore G_2. The group is normalized by the reflection in the midpoint of I and so Theorem 3.8 implies that $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is fixed by $\operatorname{Aut} G_2$. This homomorphism ψ is nonzero, for it coincides with σ_ℓ^2. (Actually, σ_ℓ and σ_r are also fixed by $\operatorname{Aut} G_2$.)

Now to the group G_3. It differs from G_2 in several respects: it cannot be written as an ascending HNN-extension with a finitely generated base group contained in $B(I; A, P)$; it is finitely generated if G_0 is so, but, if finitely generated, it does not admit a finite presentation (see part (ii) of Lemma E18.8 and Remark E18.10 in [Bieri and Strebel 2016]). The group G_3 is normalized by the reflection in the mid point of I and so $\psi : G_3 \to P$ is fixed by $\operatorname{Aut} G_3$; this conclusion, however, is of no interest as ψ is the zero map. Actually, more is true: every homomorphism $\psi' : G_3 \to P$ fixed by ρ and vanishing on the bounded subgroup B_3 of G_3 is the zero-map: by definition (3-10) the group G_3/B_3 is infinite cyclic and so ψ' must be a multiple of σ_ℓ.

Remark 3.12. The previous discussion shows that G_0, G_1 and G_2 admit nontrivial homomorphisms into P that are fixed by the corresponding automorphism groups. This fact and the observation made in Section 1A imply that every automorphism of one of these groups has infinitely many corresponding twisted conjugacy classes. This reasoning does not hold for G_3, for $\psi : G_3 \to P$ is the zero homomorphism.

So the question whether or not an automorphism α of G_3 has infinitely many twisted conjugacy classes has to be tackled by another approach. Note first that the homomorphisms σ_ℓ and σ_r are both nonzero; as G_3 satisfies the assumptions of Theorem 3.5 these homomorphisms are therefore fixed by $\mathrm{Aut}_+ G_3$. It follows that every increasing automorphism α of G_3 has infinitely many α-twisted conjugacy classes. We are thus left with the coset of decreasing automorphisms of G_3.

Consider, for example, the automorphism β induced by conjugation by the reflection ϑ in the midpoint of the interval I. Our aim is to construct an infinite collection of elements $g_n \in G_3$ and to verify then that they represent pairwise distinct β-twisted conjugacy classes. This verification will be based on the fact that β has order 2 and a connection between twisted and ordinary conjugacy classes, available for automorphisms of finite order.[8]

Let f and g be elements of G_3 that lie in the same β-twisted conjugacy class. By definition, there exists then $h \in G_3$ that satisfies the equation $g = h \circ f \circ \beta(h^{-1})$. The calculation

$$\begin{aligned} g \circ \beta(g) &= \bigl(h \circ f \circ \beta(h^{-1})\bigr) \circ \beta\bigl(h \circ f \circ \beta(h^{-1})\bigr) \\ &= h \circ (f \circ \beta(f)) \circ \beta^2(h^{-1}) = {}^{h}(f \circ \beta(f)) \end{aligned}$$

shows then that the elements $f \circ \beta(f)$ and $g \circ \beta(g)$ are conjugate. It suffices therefore to find a sequence of elements $n \mapsto f_n$ with the property that the compositions $f_{n_1} \circ \beta(f_{n_1})$ and $f_{n_2} \circ \beta(f_{n_2})$ represent distinct conjugacy classes whenever $n_1 \neq n_2$.

To obtain such a sequence, we use the fact that G contains $B(I; A, P)$ and that $B(I; A, P)$ consists of all PL-homeomorphisms with slopes in P, breakpoints in the dense subgroup A, and which are the identity near the end points. For every positive integer n there exists therefore a nontrivial element $f_n \in B(I; A, P)$ whose support has n connected components, all contained in the interval $]0, b/2[$. Then

$$h_n = f_n \circ \beta(f_n) = f_n \circ (\vartheta \circ f_n \circ \vartheta^{-1})$$

has $2n$ connected components, so h_{n_1} is not conjugate to h_{n_2} for $n_1 \neq n_2$. It follows that G_3 has infinitely many β-twisted conjugacy classes.

[8] Compare with [Gonçalves and Sankaran 2016, Lemma 2.3].

The previous reasoning allows of some improvements, but it does not seem powerful enough to establish that G_3 has infinitely many α-twisted conjugacy classes for every decreasing automorphism α of G_3.

Example 3.13. Example 3.11 admits a generalization that is worth being brought to the attention of the reader. Assume P is a nontrivial subgroup of the positive reals, A is a (nontrivial) P-submodule of $\mathbb{R}$, and ν is an endomorphism of P. Now fix $b \in A_{>0}$, set $I = [0, b]$, and define

$$G_\nu = \{g \in G([0, b]; A, P) \mid \sigma_r(g) = \nu(\sigma_\ell(g))\}. \tag{3-11}$$

We are interested in finding a nonzero homomorphism $\psi : G_\nu \to P$ that is fixed by $\operatorname{Aut} G_\nu$. Theorem 3.8 implies that the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is fixed by $\operatorname{Aut} G_\nu$ whenever P is not cyclic; this homomorphism is nonzero unless ν is the map that sends $p \in P$ to its inverse p^{-1}.

Assume now that P is cyclic. Then G_ν is isomorphic to one of the groups G_1, G_2, or G_3 discussed in Example 3.11. This claim is clear if ν is the zero map, for G_ν coincides then with $\ker \sigma_r$ and is therefore isomorphic to G_1. Assume now that ν is not zero. The quotient group $G_\nu/B(I; A, P)$ is then an infinite cyclic subgroup of the quotient group $G(I; A, P)/B(I; A, P)$ which is free abelian group of rank 2. By the classification in Section 18.4b of [Bieri and Strebel 2016], the group G_ν is therefore isomorphic, either to G_2 or to G_3. Since the isomorphism $G_\nu \xrightarrow{\sim} G_2$, respectively $G_\nu \xrightarrow{\sim} G_3$, is induced by conjugation by an autohomeomorphism of $]0, b[$ and as conjugation by the reflection in $b/2$ induces decreasing automorphisms in G_2 and in G_3, *the group G_ν admits a decreasing automorphism, say β*; it induces an isomorphism $\beta_* : \operatorname{im} \sigma_\ell \xrightarrow{\sim} \operatorname{im} \sigma_r$ (see Lemma 3.6). Our next aim is to obtain a formula for β_*.

The definition of G_ν shows, first of all, that $\operatorname{im} \sigma_\ell = P$ and that $\operatorname{im} \sigma_r = \nu(P)$. Let p be the generator of P with $p < 1$. Then $\nu(p) = p^m$ for some nonzero integer m (recall that ν is not the zero map). Pick an element $g_p \in G_\nu$ with $\sigma_\ell(g_p) = p$. Then 0 is an attracting fixed point of g_p restricted to a sufficiently small interval of the form $[0, \delta]$, and hence b is an attracting fixed point for the restriction of $\beta(g_p)$ to a sufficiently small interval of the form $[b - \varepsilon, b]$. Thus $\beta(g_p) < 1$. Since $\beta(g_p)$ generates $\operatorname{im} \sigma_r = \nu(P) = \operatorname{gp}(p^m)$ it follows that β_* is given by the formula

$$\beta_* : P \to P, \quad p \mapsto p^{|m|}. \tag{3-12}$$

Consider now the commutative square (3-6), with α replaced by β. It shows that

$$(\sigma_r \circ \beta)(g_p) = \beta_*(\sigma_\ell(g_p)) = \beta_*(p) = p^{|m|} = (\sigma_\ell(g_p))^{|m|} \tag{3-13}$$

and so $\sigma_r \circ \beta = \sigma_\ell^{|m|}$. The preceding reasoning is also valid with β^{-1} in place of β, for β^{-1} is also a decreasing automorphism of G_ν, and so the relation $\sigma_r \circ \beta^{-1} = \sigma_\ell^{|m|}$ holds, hence also the relation $\sigma_\ell^{|m|} \circ \beta = \sigma_r$.

Consider next the homomorphism $\psi : G_\nu \to P$ that takes g to $\sigma_\ell(g)^{|m|} \cdot \sigma_r(g)$. The calculation

$$(\psi \circ \beta)(g) = \sigma_\ell^{|m|}(\beta(g)) \cdot \sigma_r(\beta(g)) = \sigma_r(g) \cdot \sigma_\ell^{|m|}(g) = \psi(g)$$

shows then that ψ is fixed by β. Note, however, that ψ is the zero homomorphism whenever m is negative, for in this case the definition of G_ν implies that

$$\psi(g) = (\sigma_\ell(g))^{|m|} \cdot \sigma_r(g) = (\sigma_\ell(g))^{|m|} \cdot (\sigma_\ell(g))^m = 1$$

for every $g \in G_\nu$, just as it happens with G_3 in Example 3.11.

Remark 3.14. Suppose P is cyclic and $\nu : P \to P$ is neither the identity nor the passage to the inverse. Then G_ν admits decreasing automorphisms β, but none of them can be induced by an autohomeomorphism $\tilde{\varphi} : I \xrightarrow{\sim} I$ that is differentiable at the end points; indeed, (3-13) shows that $\sigma_r \circ \beta \neq \sigma_\ell$, in contrast to what happens if the chain rule can be applied (see Proposition 3.1). It follows, in particular, that the three conditions (a), (b), and (c) stated in Theorem 3.8 can occur simultaneously.

4. Characters fixed by Aut $G([0, \infty[; A, P)$

The results in this section differ from those of Section 3 in two important respects: in many situations several candidates for $\psi : G \to P$ are available and one of the candidates may not be fixed by $\mathrm{Aut}_+ G$.

4A. ***Existence of decreasing automorphisms.*** Every compact interval of the form $[0, b]$, and also the line, is invariant under a reflection. It follows that the groups $G(I; A, P)$ with I one of these intervals, but also many of their subgroups, admit decreasing automorphisms. The case where I is a half-line, say $[0, \infty[$, is different: then $G([0, \infty[; A, P)$ does not admit a decreasing automorphism.

In this section, we first justify this claim and discuss then the extent to which it continues to be valid for subgroups of $G([0, \infty[; A, P)$. We begin with an analogue of Lemma 3.6 in which the homomorphism σ_r is replaced by the homomorphism ρ defined in (2-4).

Lemma 4.1. *Assume I is the half-line $[0, \infty[$ and G is a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. Then every decreasing automorphism α induces an isomorphism $\alpha_* : \mathrm{im}\, \sigma_\ell \xrightarrow{\sim} \mathrm{im}\, \rho$.*

Proof. The proof is very similar to that of Lemma 3.6. The kernel of σ_ℓ consists of all elements in G that are the identity near 0, while the kernel of ρ is made up of the elements in G that are the identity near ∞. Since α is induced by conjugation by a decreasing homeomorphism of $]0,\infty[$, the image of $\ker\sigma_\ell$ consists of elements $\alpha(g)$ that are the identity on a half-line of the form $[t(g),\infty[$, and so $\alpha(\ker\sigma_\ell)\subseteq\ker\rho$. Since α^{-1} is also a decreasing automorphism, the preceding inclusion is actually an equality. It follows that α induces an isomorphism $\alpha_*:\operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\rho$ that renders the square

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, two heads, "\sigma_\ell"] & G \arrow[d, two heads, "\rho"] \\ \operatorname{im}\sigma_\ell \arrow[r, "\alpha_*"] & \operatorname{im}\rho \end{tikzcd} \tag{4-1}$$

commutative. □

The preceding lemma leads directly to a criterion for the nonexistence of decreasing automorphisms. Indeed, the image of σ_ℓ is abelian, while that of ρ is often a nonabelian, metabelian group, and so we obtain

Criterion 4.2. Assume I is the half-line $[0,\infty[$ and G is a subgroup of $G(I;A,P)$ that contains $B(I;A,P)$. If $\operatorname{im}\rho$ is *not* abelian then $\operatorname{Aut} G=\operatorname{Aut}_+ G$.

4B. ***Construction of homomorphisms: part I.*** We turn now to the construction of homomorphisms fixed by $\operatorname{Aut}_+ G$, or even by $\operatorname{Aut} G$. Several homomorphisms are at our disposal. The first of them is σ_ℓ. Corollary 3.10 tells us then:

Proposition 4.3. *Assume that G is a subgroup of $G([0,\infty[;A,P)$ containing $B(I;A,P)$. Then the homomorphisms σ_ℓ is fixed by* $\operatorname{Aut}_+ G$.

We move on to the homomorphism ρ. Here two cases arise, depending on whether its image is abelian or nonabelian. In the second case, a very satisfying conclusion holds. It is enunciated in

Theorem 4.4. *Assume $I=[0,\infty[$ and G is a subgroup of $G(I;A,P)$ containing $B(I;A,P)$. If* $\operatorname{im}\rho$ *is not abelian, σ_r is a nonzero homomorphism fixed by* $\operatorname{Aut} G$.

Proof. Suppose $\operatorname{im}\rho$ is nonabelian. Then Lemma 4.1 forces α to be increasing. Let $\varphi:]0,\infty[\xrightarrow{\sim}]0,\infty[$ be the autohomeomorphism that induces α by conjugation. As it is increasing, it is affine near ∞ by Proposition 4.5 below, and so the following calculation

$$\begin{aligned}\sigma_r(\alpha(g)) &= \lim\nolimits_{t\to\infty}\big(\varphi\circ g\circ\varphi^{-1}\big)'(t)\\ &= \lim\nolimits_{t\to\infty}\big(\varphi'(g\circ\varphi^{-1}(t))\cdot g'(\varphi^{-1}(t))\cdot(\varphi^{-1})'(t)\big)\\ &= \lim\nolimits_{t\to\infty}\big(\varphi'(t)\cdot g'(t)\cdot(\varphi^{-1})'(t)\big)\\ &= \lim\nolimits_{t\to\infty} g'(t)=\sigma_r(g)\end{aligned}$$

is valid for every $g\in G$, which shows that α fixes the homomorphism σ_r. This homomorphism is nonzero. Indeed, $G/\ker\rho \xrightarrow{\sim} \operatorname{im}\rho$ is not abelian by hypothesis, while $\ker\sigma_r/\ker\rho$ is abelian and thus the third term of the extension

$$\ker\sigma_r/\ker\rho \rightarrowtail G/\ker\rho \twoheadrightarrow G/\ker\sigma_r$$

is not zero, whence $\ker\sigma_r \neq G$. □

We are left with proving an analogue of Proposition 3.2. For later use, we state it in greater generality than needed at this point, namely as

Proposition 4.5. *Assume G and $\bar{G}$ are subgroups of $G(I; A, P)$, both containing the subgroup $B(I; A, P)$, and that I is either the half-line* $[0, \infty[$ *or the line* $\mathbb{R}$. *Let $\alpha : G \xrightarrow{\sim} \bar{G}$ be an isomorphism and let φ_α be an autohomeomorphism of* $\operatorname{int}(I)$ *that induces α by conjugation.*

If $\operatorname{im}\rho$ *is not abelian and φ_α is increasing then φ_α is affine near ∞.*

Proof. We adapt the argument of Part 2 in the proof of [Bieri and Strebel 2016, Supplement E17.3] to the case at hand. By assumption, the image of

$$\rho_* : G \to \operatorname{Aff}(A, P) \xrightarrow{\sim} A \rtimes P$$

is not abelian; its derived group is therefore (isomorphic to) a nontrivial submodule A_1 of A which, being nontrivial, contains arbitrary small positive elements and so is dense in $\mathbb{R}$. Let $\bar{\rho}_* : \bar{G} \to A \rtimes P$ be the similarly defined homomorphism; the derived group of its image is then isomorphic to a nontrivial submodule $\bar{A}_1$ of A.

By part (ii) of Corollary 2.7, the isomorphism α induces an isomorphism α_* of $G/\ker\rho$ onto $\bar{G}/\ker\bar{\rho}$; hence an isomorphism of $\operatorname{im}\rho$ onto $\operatorname{im}\bar{\rho}$, and, finally, an isomorphism α_1 of A_1 onto $\bar{A}_1$. They render commutative the following diagram

(4-2)
$$\begin{tikzcd}
G \arrow[r, two heads, "\rho"] \arrow[d, "\alpha"] & \operatorname{im}\rho \arrow[d, "\alpha_*"] & A_1 \arrow[l, tail] \arrow[d, "\alpha_1"] \\
\bar{G} \arrow[r, two heads, "\bar{\rho}"] & \operatorname{im}\bar{\rho} & \bar{A}_1 \arrow[l, tail]
\end{tikzcd}$$

We claim the automorphism $\alpha_1 : A_1 \xrightarrow{\sim} \bar{A}_1$ is strictly *increasing*.

Let $b \in A_1$ be an arbitrary positive element and let $f_b \in G$ be a PL-homeomorphisms that is a translation with amplitude b near ∞, say on $[t_{b,1}, \infty[$. Then $\alpha(f_b)$ is a PL-homeomorphism which is a translation with amplitude $\alpha_1(b)$ near ∞, say for $t \geq \varphi(t_{b,2})$. Since α is induced by conjugation by φ, one has $\alpha(f_b) = {}^{\varphi} f_b$; so $\varphi \circ f_b = \alpha(f_b) \circ \varphi$. By evaluating this equality at $t \geq \max\{t_{b,1}, t_{b,2}\}$ one obtains the chain of equations

$$\varphi(t+b) = (\varphi \circ f_b)(t) = (\alpha(f_b) \circ \varphi)(t) = \alpha_1(b) + \varphi(t).$$

It implies that $\alpha_1(b)$ is positive, for b is so by assumption and φ is increasing.

We show next that α_1 is given by multiplication by a positive real number s_1. As stated in the first paragraph of the proof, A_1 is a dense subgroup of $\mathbb{R}_{\text{add}}$. Since α_1 is strictly increasing it extends to a (unique) strictly increasing automorphism $\tilde{\alpha}_1 : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$. This automorphism is continuous and hence an $\mathbb{R}$-linear map, given by multiplication by some positive real number s_1.

We come now to the final stage of the analysis of φ. In it we show that the restriction of φ to a suitable interval of the form $[t_*, \infty[$ is *affine*. Choose a positive element $b_* \in A_1$ and let $f_{b_*} \in G$ be an element whose image under ρ is a translation with amplitude b_*. It then follows, as before, that there is a positive number t_* so that the equation

$$\varphi(t + b_*) = \alpha_1(b_*) + \varphi(t) = \varphi(t) + s_1 \cdot b_* \tag{4-3}$$

holds for every $t \geq t_*$. Consider now an arbitrary positive element $b \in A_1$. There exists then a positive number t_b such that the calculation

$$\varphi(t + b) = \alpha_1(b) + \varphi(t) = \varphi(t) + s_1 \cdot b$$

is valid for $t \geq t_b$. Choose a positive integer m which is so large that $t_b \leq t_* + m \cdot b_*$. For every $t \geq t_*$ the following calculation is then valid:

$$\begin{aligned} \varphi(t + b) + s_1 \cdot mb_* &= \varphi(t + b + m \cdot b_*) \\ &= \varphi(t + m \cdot b_*) + s_1 \cdot b \\ &= \varphi(t) + s_1 \cdot mb_* + s_1 \cdot b. \end{aligned}$$

It follows, in particular, that the equation

$$\varphi(t_* + b) = \varphi(t_*) + s_1 \cdot b \tag{4-4}$$

holds for every positive element $b \in A_1$ and $t \geq t_*$. Since φ is continuous and increasing and as A_1 is dense in $\mathbb{R}$, this equation allows us to deduce that φ is affine with slope s_1 on the half-line $[t_*, \infty[$, and so the proof is complete. □

The hypotheses of the Theorem 4.4 are satisfied if $G = G([0, \infty[; A, P)$; the theorem, Lemma 4.1 and Corollary 3.10 thus yield the pleasant

Corollary 4.6. *If G coincides with $G([0, \infty[; A, P)$ then both $\sigma_\ell : G \to P$ and $\sigma_r : G \to P$ are surjective homomorphisms fixed by* Aut G.

Corollary 4.6 is the analogue of Corollary 3.9, but with the compact interval I replaced by a half-line. Groups $G(I; A, P)$ with I a half-line have, so far, been investigated less often than groups with I a compact interval; they have, however, their own merits, in particular the following one: to date, finitely generated groups of the form $G(I; A, P)$ with I compact are only known for very special choices of the parameters (A, P).[9] By contrast, finitely generated groups with I a half-line are far more common, as is shown by the following characterization:

Proposition 4.7 [Bieri and Strebel 2016, Theorem B8.2]. *The homeomorphism group $G([0, \infty[; A, P)$ is finitely generated if and only if the following conditions are satisfied*:

(i) *P is finitely generated*,

(ii) *A is a finitely generated $\mathbb{Z}[P]$-module*, *and*

(iii) *$A/(IP \cdot A)$ is finite.*

4C. ***Construction of homomorphisms: part II.*** Theorem 4.4 is very pleasing: it shows that the homomorphism σ_r is fixed by all automorphisms provided merely *the image of $\rho : G \to \operatorname{Aff}(IP \cdot A, P)$ is not abelian*. In this section, we discuss the remaining case.

The image of $G([0, \infty[; A, P)$ under ρ is the affine group

$$\operatorname{Aff}(IP \cdot A, P) \xrightarrow{\sim} (IP \cdot A) \rtimes P$$

(see Section 2B). This group is metabelian and contains two obvious kinds of abelian subgroups: those made up of translations, corresponding to the subgroups of $IP \cdot A$, and the subgroups consisting of homotheties $t \mapsto q \cdot t$ with ratio q varying in a subgroup Q of P. We begin by discussing the second type of abelian subgroups.

4C1. *Image of ρ is made up of homotheties.* Given a subgroup Q of P let G_Q be the subgroup of $G = G([0, \infty[; A, P)$ consisting of the products $f \circ g$ with $g \in B = B([0, \infty[; A, P)$ and f a homothety $t \mapsto q \cdot t$ with $q \in Q$; since B is normal in G the set so defined is actually a subgroup of G. We do not know which of these subgroups G_Q admit decreasing automorphisms, but those with Q cyclic have this peculiarity, as can be seen from

Lemma 4.8. *Assume I is the half-line $[0, \infty[$ and Q is a cyclic subgroup of P. Then the subgroup*

$$G_Q = \{f \circ g \mid f = (t \mapsto q \cdot t) \text{ with } q \in Q \text{ and } g \in B\} \tag{4-5}$$

of the group $G([0, \infty[; A, P)$ does admit a decreasing automorphism.

Proof. Let q_0 be the generator of Q with $q_0 > 1$ and choose a positive element $a_0 \in IP \cdot A$. For each $k \in \mathbb{Z}$ set $t_k = q^k \cdot a_0$ and define $\varphi :]0, \infty[\xrightarrow{\sim}]0, \infty[$ to be the affine interpolation of the assignment $(t_k \mapsto t_{-k})_{k \in \mathbb{Z}}$. Then φ is an infinitary PL-autohomeomorphism of $]0, \infty[$ whose interpolation points lie in $(IP \cdot A) \times (IP \cdot A)$.

[9] See [Bieri and Strebel 2016, p. vii] for the list of the groups known at the end of 2014.

The slopes of the segments forming the graph of φ are the negatives of powers of q_0; indeed,

$$t_{k+1} - t_k = q_0^{k+1} \cdot a_0 - q_0^k \cdot a_0 = (q_0 - 1) \cdot q_0^k \cdot a_0$$

and

$$\varphi(t_{k+1}) - \varphi(t_k) = (1/q_0)^{k+1} \cdot a_0 - (1/q_0)^k \cdot a_0 = (1 - q_0) \cdot q_0^{-k-1} \cdot a_0$$

and so φ has slope $(-1) \cdot q_0^{-2k-1}$ on the interval $[t_k, t_{k+1}]$.

It follows that φ maps $IP \cdot A$ onto itself. Consider now a conjugate ${}^{\varphi}h = \varphi \circ h \circ \varphi^{-1}$ of an element $h \in G_Q$. If $h \in B(I; A, P)$, then h has support contained in some interval of the form $I_{k(h)} = [t_{-k(h)}, t_{k(h)}]$ for some $k(h) > 0$ and so ${}^{\varphi}h$ has support in $\varphi(I_{k(h)}) = I_{k(h)}$, slopes in P, break points in $IP \cdot A$ and is thus an element of $B \subset G_Q$. If, on the other hand, h is the homothety with ratio q_0, then $h(t_k) = t_{k+1}$ for each index $k \in \mathbb{Z}$ and its conjugate ${}^{\varphi}h$ is the PL-function with interpolation points (t_k, t_{k-1}), hence the homothety with center 0 and ratio q_0^{-1} and thus ${}^{\varphi}h = h^{-1}$ lies in G_Q. As G_Q is generated by $B \cup \{(t \mapsto q_0 \cdot t)\}$, the previous reasoning shows that the decreasing autohomeomorphism φ induces by conjugation an automorphism of G_Q and so the lemma is established. □

Remark 4.9. Assume A, P and Q are as in the statement of the lemma. Then the bounded group $B = B([0, \infty[; A, P)$ may be perfect and hence simple; cf. [Bieri and Strebel 2016, Section 12.4]. In such a case, B is the only normal subgroup N of G_Q with G/N *infinite abelian* and so the lemma implies that no homomorphism of G_Q onto an infinite abelian group is fixed by all of $\operatorname{Aut} G_Q$. Note, however, that ρ is fixed by every increasing automorphism of G_Q.

4C2. *Image of ρ consists of translations.* We turn now to the other type of abelian subgroups of $\operatorname{Aff}(IP \cdot A, P)$, but concentrate on a special case. Given a subgroup Q of P and a subgroup $A_0 \subseteq IP \cdot A$, we set

$$G_{Q,A_0} = \bigl\{ g \in G([0, \infty[; A, P) \bigm| \sigma_\ell(g) \in Q \text{ and } \rho(g) \in A_0 \rtimes \{1\} \bigr\}. \tag{4-6}$$

The group G_{Q,A_0} is an extension of $B([0, \infty[; A, P)$ by the abelian group $Q \times A_0$.

The class of groups having the form G_{Q,A_0} is of interest for several reasons. Firstly, if Q and A_0 are *not* isomorphic, every automorphism of G_{Q,A_0} is increasing by Lemma 4.1. This case occurs frequently, as is brought home by the following kind of examples. Suppose Q is finitely generated and contains an integer $p > 1$, while A_0 is a nonzero submodule of $IP \cdot A$. Then A_0 is divisible by p and, in particular, not free abelian.

Some groups of the form G_{Q,A_0} admit decreasing automorphisms, in particular the following ones: Let P be a cyclic group generated by the real number $p > 1$, let A be a $\mathbb{Z}[P]$-submodule of $\mathbb{R}_{\text{add}}$ and choose a positive element $b \in A$. The group $\bar{G} = G([0, b]; A, P)$ admits decreasing automorphisms, for instance the automorphism induced by conjugation by the reflection $\bar{\varphi}$ at the midpoint of $I = [0, b]$.

Consider now the group $G = G_{P,\mathbb{Z}\cdot(p-1)b} \subset G([0, \infty[; P, A)$. It is isomorphic to $\bar{G}$; there exists actually an isomorphism induced by an increasing, infinitary PL-homeomorphism $\varphi_b : [0, \infty[\xrightarrow{\sim} [0, b[$; see [op. cit., Lemma E18.2]. Then the composition $\varphi_b^{-1} \circ \bar{\varphi} \circ \varphi_b$ induces by conjugation a decreasing automorphism of G.

Thirdly, let $\tau_r : G_{Q,A_0} \to \mathbb{R}_{\text{add}}$ be the homomorphism that maps the PL-homeomorphism $g \in G_{Q,A_0}$ to the amplitude of the translation $\rho(g)$. This homomorphism seems to have a good chance of being fixed by $\operatorname{Aut}_+ G_{Q,A_0}$, but this impression is mistaken. Indeed, let $\operatorname{Aut}_P A_0$ be the set of elements $p \in P$ with $p \cdot A_0 = A_0$; this set is a subgroup of P and the semidirect product $A_0 \rtimes \operatorname{Aut}_P A_0$ is a subgroup of $(IP \cdot A) \rtimes P$; let $\tilde{G}$ denote the preimage of $A_0 \rtimes \operatorname{Aut}_P A_0$ under the epimorphism

$$\bar{\rho} : G([0, \infty[; A, P) \overset{\rho}{\twoheadrightarrow} \operatorname{Aff}(IP \cdot A, P) \xrightarrow{\sim} (IP \cdot A) \rtimes P.$$

Then G_{Q,A_0} is a normal subgroup of $\tilde{G}$. The group $\tilde{G}$ contains the homothety $\vartheta_p : t \mapsto p \cdot t$ for every $p \in \operatorname{Aut}_P A_0$, and so conjugation by such a homothety induces an automorphism α_p of G_{Q,A_0}. The calculation

$$\begin{aligned}(\tau_r \circ \alpha_p)(g) &= \tau_r(\vartheta_p \circ g \circ \vartheta_p^{-1}) = (\vartheta_p \circ g \circ \vartheta_p^{-1})(t) - t \\ &= \vartheta_p(g(p^{-1}t)) - t = p \cdot (p^{-1}t + \tau_r(g)) - t = (p \cdot \tau_r)(g),\end{aligned}$$

valid for every sufficiently large real number t, then shows that the formula

$$\tau_r \circ \alpha_p = p \cdot \tau_r \tag{4-7}$$

holds for each $p \in \operatorname{Aut}_P A_0$. We conclude that τ_r can only be fixed by all of $\operatorname{Aut}_+ G_{Q,A_0}$ if $\operatorname{Aut}_P A_0$ is reduced to $1 \in \mathbb{R}^\times_{>0}$. This condition is fulfilled, for instance, if A_0 is infinite cyclic.

Example 4.10. Given a real number $p > 1$, set $P = \operatorname{gp}(p)$ and $A = \mathbb{Z}[P] = \mathbb{Z}[p, p^{-1}]$. Choose $A_0 = A$ and set $G = G_{P,A_0}$. Then $\operatorname{Aut}_P A_0 = P$. Concrete examples are rational integers $p \in \mathbb{N} \smallsetminus \{0, 1\}$, with $A_0 = \mathbb{Z}[1/p]$, or quadratic integers like $\sqrt{2} + 1$ with $A_0 = A = \mathbb{Z}[\sqrt{2}\,]$. We shall come back to the second of these examples in Section 6E1.

5. Characters fixed by Aut $G(\mathbb{R}; A, P)$

Let I denote one of the intervals $[0, b]$, $[0, \infty[$ or $\mathbb{R}$, and let G be a subgroup of $G(I; A, P)$ containing $B(I; A, P)$. In Sections 3 and 4 groups with I a compact interval or a half-line have been studied. In this section we now turn to the line $I = \mathbb{R}$. Finding nonzero homomorphisms $\psi : G \to \mathbb{R}^\times_{>0}$ fixed by $\operatorname{Aut} G$, is then harder than in the previously investigated cases, and this for two reasons. Firstly, subgroups of $G(\mathbb{R}; A, P)$ often admit decreasing automorphisms α, in contrast to what happens if I is a half-line; in the case of a decreasing automorphism, λ (or ρ) is only fixed by α if λ coincides with ρ. Secondly, if the image of λ or that of ρ consists of translations, neither λ nor ρ need be fixed by $\mathrm{Aut}_+ G$.

The plan of our investigation will be similar to that adopted in Section 4. We begin by discussing the existence of decreasing automorphisms (in Section 5A), move on to the main results about the existence of homomorphisms fixed by $\mathrm{Aut}_+ G$ or $\mathrm{Aut}\, G$ (in Section 5B) and complement these results with more special findings in Section 5C. The layout of the middle Section 5B will resemble that of Section 3A.

5A. *Existence of decreasing automorphisms.* As in the cases of a compact interval or a half-line, the existence of a decreasing automorphism has an easily stated consequence, namely

Lemma 5.1. *Assume G is a subgroup of $G(\mathbb{R}; A, P)$ that contains $B(\mathbb{R}; A, P)$. Then every decreasing automorphism α induces an isomorphism $\alpha_* : \mathrm{im}\,\lambda \xrightarrow{\sim} \mathrm{im}\,\rho$ that renders commutative the following square.*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, twoheadrightarrow, "\lambda"] & G \arrow[d, twoheadrightarrow, "\rho"] \\ \mathrm{im}\,\lambda \arrow[r, "\alpha_*"] & \mathrm{im}\,\rho \end{tikzcd} \tag{5-1}$$

Proof. The claim can be established as in the proofs of Lemmata 3.6 and 4.1. $\square$

The images of λ and ρ are both subgroups of the affine group $Q = \mathrm{Aff}_o(I P \cdot A, P)$. It is easy to describe some pairs of subgroups (Q_1, Q_2) that are *not* isomorphic for obvious reasons, for instance if one is abelian, and the other is nonabelian. We are, however, not aware of a classification of the isomorphism types of subgroups of $\mathrm{Aff}_o(I P \cdot A, P)$ for parameters $A \neq \{0\}$ and $P \neq \{1\}$.

5B. *Construction of homomorphisms: part I.* We turn now to the construction of homomorphisms that are fixed by $\mathrm{Aut}_+ G$ or by $\mathrm{Aut}\, G$. The next result is an analogue of Corollary 3.4. The main ingredient in its proof is Proposition 4.5.

Proposition 5.2. *Let G be a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$ and let α be an automorphism of G that is induced by conjugation by the autohomeomorphism $\varphi_\alpha : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$. Then the following statements hold*:

(i) *if α is increasing*[10] *and* $\mathrm{im}\,\rho$ *is not abelian*, *then φ_α is affine near ∞*;

(ii) *if α is increasing and* $\mathrm{im}\,\lambda$ *is not abelian*, *then φ_α is affine near $-\infty$*;

(iii) *if α is decreasing and* $\mathrm{im}\,\rho$ *is not abelian*, *then $\tilde{\varphi}_\alpha$ is affine*, *both near $-\infty$ and near ∞.*

[10] See Definition 2.5.

Proof. Statement (i) is a restatement of the claim of Proposition 4.5. To establish (ii), we show that (ii) can be reduced to (i). Let $\vartheta \colon \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ be the reflection in the origin 0, set $G_1 = \vartheta \circ G \circ \vartheta^{-1}$ and $\varphi_1 = \vartheta \circ \tilde{\varphi}_\alpha \circ \vartheta^{-1}$. We claim that Proposition 4.5 applies to the couple (G_1, φ_1). Indeed, the groups $G(\mathbb{R}; A, P)$ and $B(\mathbb{R}; A, P)$ are invariant under conjugation by ϑ and so G_1 is a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$. Next, Lemma 5.3 below shows that

$$\rho(G_1) = \rho\left(\vartheta \circ G \circ \vartheta^{-1}\right) = \vartheta \circ \lambda(G) \circ \vartheta^{-1}.$$

The group $\vartheta \circ \lambda(G) \circ \vartheta^{-1}$ is isomorphic to $\operatorname{im}\lambda$, which is nonabelian by hypothesis, and so $\rho(G_1)$ is nonabelian. Proposition 4.5 thus applies to G_1 and to φ_1 and implies that $\varphi_1 = \vartheta \circ \varphi_\alpha \circ \vartheta^{-1}$ is affine near $+\infty$, whence φ_α itself is affine near $-\infty$.

To establish (iii), note that since α is *decreasing*, the groups $\operatorname{im}\lambda$ and $\operatorname{im}\rho$ are isomorphic (see Lemma 5.1); the hypothesis on $\operatorname{im}\rho$ implies therefore that the image of λ is not abelian. The idea now is to reduce (iii) to the previously treated cases (i) and (ii). As before, let $\vartheta \colon \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ denote the reflection in the origin 0, and set $\varphi_2 = \vartheta \circ \varphi_\alpha$. Then φ_2 is increasing and conjugation by φ_2 maps G onto $\bar{G} = \vartheta \circ G \circ \vartheta^{-1}$. Proposition 4.5 thus applies and guarantees that φ_2 is affine near ∞. But $\varphi_2 = \vartheta \circ \varphi_\alpha$ and so φ_α itself is affine near ∞. Consider, secondly, $\varphi_3 = \varphi_\alpha \circ \vartheta$. This map is again increasing, and conjugation by it maps $\bar{G} = \vartheta \circ G \circ \vartheta^{-1}$ onto G. Invoking Proposition 4.5 once more, we learn that φ_3 is affine near $+\infty$, and so φ_α itself is affine near $-\infty$. All taken together, we have shown that φ_α is affine, both near $-\infty$ and $+\infty$, as asserted by claim (iii). □

We are left with proving

Lemma 5.3. *Let $\vartheta \colon \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ denote the reflection in 0. Then the formula*

$$\rho\left(\vartheta \circ g \circ \vartheta^{-1}\right) = \vartheta \circ \lambda(g) \circ \vartheta^{-1} \tag{5-2}$$

holds for every $g \in \mathrm{PL}_o(\mathbb{R})$.

Proof. Let μ and ν denote the functions of $\mathrm{PL}_o(\mathbb{R})$ into itself given by the left hand and the right hand side of (5-2); thus $\mu(g) = \rho(\vartheta \circ g \circ \vartheta^{-1})$ for $g \in \mathrm{PL}_o(\mathbb{R})$, and similarly for ν. Both functions are homomorphisms of $\mathrm{PL}_o(\mathbb{R})$ into $\mathrm{Aff}_o(\mathbb{R})$ that vanish on $\ker\lambda$. It suffices therefore to check (5-2) on a complement of $\ker(\lambda : \mathrm{PL}_o(\mathbb{R}) \to \mathrm{Aff}_o(\mathbb{R}))$. Such a complement is $\mathrm{Aff}_o(\mathbb{R})$ and for affine maps h the following calculation holds:

$$\rho(\vartheta \circ h \circ \vartheta^{-1}) = \vartheta \circ h \circ \vartheta^{-1} = \vartheta \circ \lambda(h) \circ \vartheta^{-1}. \qquad \square$$

5B1. *Some corollaries.* The first corollary of Proposition 5.2 deals with homomorphisms fixed by $\mathrm{Aut}_+ G$; the corollary is an analogue of Theorem 3.5.

Theorem 5.4. *Assume G is a subgroup of $G(\mathbb{R}; A, P)$ that contains $B(\mathbb{R}; A, P)$. If $\operatorname{im}\rho$ is not abelian, σ_r is a nonzero homomorphism fixed by $\operatorname{Aut}_+ G$. Similarly, σ_ℓ is a nonzero homomorphism fixed by $\operatorname{Aut}_+ G$ in case $\operatorname{im}\lambda$ is not abelian.*

Proof. Let α be an increasing automorphism of G and let φ_α be the increasing autohomeomorphism of $\mathbb{R}$ inducing α by conjugation. (The map exists thanks to Theorem 2.3.) Assume first that $\operatorname{im}\rho$ is not abelian. By part (i) of Proposition 5.2 the map φ_α is then affine near ∞. On the other hand, the image of ρ, being nonabelian, cannot consist merely of translations; so the homomorphism $\sigma_r : G \to P$ is nonzero. The following calculation then reveals that σ_r is fixed by α:

$$\begin{aligned}(\sigma_r \circ \alpha)(g) &= \sigma_r(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1})\\ &= \lim_{t\to\infty}(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1})'(t)\\ &= \lim_{t\to\infty}\bigl(\varphi_\alpha'(g(\varphi_\alpha^{-1}(t)))\cdot g'(\varphi_\alpha^{-1}(t))\cdot(\varphi_\alpha^{-1})'(t)\bigr)\\ &= \lim_{t\to\infty} g'(\varphi_\alpha^{-1}(t)) = \sigma_r(g).\end{aligned}$$

In this calculation the facts that the derivatives of φ_α and of g are constant on a half-line of the form $[t_*, \infty[$ and that φ_α is an increasing homeomorphism, have been used.

Assume next that $\operatorname{im}\lambda$ is not abelian. By part (ii) of Proposition 5.2 the map φ_α is then affine near $-\infty$. and the homomorphism $\sigma_\ell : G \to P$ is nonzero. Since the derivatives of every element $g \in G$ and of φ_α are constant near $-\infty$, a calculation similar to the preceding one will show that λ is fixed by α. □

As a second application of Proposition 5.2, we present a result that furnishes a homomorphism ψ that is fixed by every automorphism. Note, however, that the hypotheses of the result do not imply that ψ is nontrivial.

Theorem 5.5. *Assume G is a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$ and let $\psi : G \to P$ be the homomorphism $g \mapsto \sigma_\ell(g)\cdot\sigma_r(g)$. If the images of λ and of ρ are both nonabelian, then the homomorphism $\psi : G \to P$ is fixed by $\operatorname{Aut} G$.*

Proof. Let α be an automorphism of G and let φ_α be the autohomeomorphism of $\mathbb{R}$ that induces α by conjugation. If φ_α is *increasing* both σ_ℓ and σ_r are fixed by α (see Theorem 5.4) and hence so is ψ.

Assume now that α is *decreasing*. Part (iii) of Corollary 3.4 then guarantees that φ_α is affine near $-\infty$ and also near ∞. These facts imply the relations

$$\sigma_\ell \circ \alpha = \sigma_r \quad \text{and} \quad \sigma_r \circ \alpha = \sigma_\ell \tag{5-3}$$

(see below) and so $\psi = \sigma_\ell \cdot \sigma_r$ is fixed by α.

We are left with verifying relations (5-3). The following calculation uses the fact that both φ_α and g have constant derivatives near $-\infty$ and $+\infty$:

$$\begin{aligned}(\sigma_\ell \circ \alpha)(g) &= \sigma_\ell(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1})\\ &= \lim_{t \to -\infty} (\varphi_\alpha \circ g \circ \varphi_\alpha^{-1})'(t)\\ &= \lim_{t \to -\infty} \big(\varphi_\alpha'(g(\varphi_\alpha^{-1}(t))) \cdot g'(\varphi_\alpha^{-1}(t)) \cdot (\varphi_\alpha^{-1})'(t)\big)\\ &= \lim_{t \to -\infty} g'(\varphi_\alpha^{-1}(t)) = \sigma_r(g).\end{aligned}$$

A similar calculation establishes the second relation in (5-3). □

We continue with an easy consequence of Theorem 5.5. If the group G is all of $G(I; A, P)$ the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is surjective; in addition, $\operatorname{im} \lambda$ and $\operatorname{im} \rho$ both coincide with $\mathrm{Aff}_o(A, P)$ and thus are nonabelian. Therefore, Theorem 5.5 implies the following:

Corollary 5.6. *If $G = G(\mathbb{R}; A, P)$, the homomorphism $\psi : G \to P$, mapping $g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$, is nonzero and fixed by* $\operatorname{Aut} G$.

Corollary 5.6 is an analogue of Corollaries 3.9 and 4.6. Groups of the form $G(\mathbb{R}; A, P)$ have been investigated, so far, less often than groups with I a compact interval; they have, however, their own merits if it comes to finite generation. There exists, first of all, a characterization of the finitely generated groups of the form $G(\mathbb{R}; A, P)$, namely the following result:

Proposition 5.7 [Bieri and Strebel 2016, Theorem B7.1]. *The group $G(\mathbb{R}; A, P)$ is finitely generated if and only if P is finitely generated and A is a finitely generated $\mathbb{Z}[P]$-module.*

Remark 5.8. Proposition 5.7 implies that *there are continuously many, pairwise nonisomorphic, finitely generated groups of the form $G(\mathbb{R}; A, P)$.*

To prove this assertion, we recall the following result: *if two groups of the form $G(\mathbb{R}; A, P)$ and $G(\mathbb{R}; \bar{A}, \bar{P})$ are isomorphic and if P is not cyclic, then $P = \bar{P}$.*[11]

It suffices therefore to find a collection of finitely generated, pairwise distinct subgroups $\{P_j \mid j \in J\}$ of $\mathbb{R}^\times_{>0}$ with J an index set having the cardinality of $\mathbb{R}$, and to set $A_j = \mathbb{Z}[P_j]$ for each $j \in J$. Such a collection of subgroups can be obtained as follows: first one constructs a family of irrational real numbers $\{x_j \mid j \in J\}$ such that the extended family $\{1\} \cup \{x_j \mid j \in J\}$ is linearly independent (over $\mathbb{Q}$) and then sets $P_j = \exp(\mathrm{gp}(\{1, x_j\}))$. Then each group P_j is free abelian of rank two, hence not cyclic, and for indices $j_1 \neq j_2$ the groups P_{j_1} and P_{j_2} are distinct.

[11] see [Bieri and Strebel 2016, Theorem E17.1].

5C. ***Construction of homomorphisms: part II.*** In this final part of Section 5, we consider subgroups G of $G(\mathbb{R}; A, P)$, containing $B(\mathbb{R}; A, P)$, with $\operatorname{im}\lambda$ and $\operatorname{im}\rho$ both abelian.[12] The most interesting subcase seems to be that where the images of λ and ρ consists only of translations. Then two homomorphisms τ_ℓ and τ_r of G into $\mathbb{R}_{\text{add}}$ can be defined: they associate to $g \in G$ the amplitudes of the translations $\lambda(g)$ and $\rho(g)$, respectively. One sees, as in Section 4C2, that neither of these homomorphisms need be fixed by $\operatorname{Aut}_+ G$.

An exception occurs if the image of ρ or of λ is *infinite cyclic*. Suppose, for instance, that $\operatorname{im}\rho$ is infinite cyclic, and let $f \in G$ be an element that maps onto the positive generator, say x_f, of $\operatorname{im}\tau_r$. Consider an increasing automorphism α of G and let φ_α be the homeomorphism of $\mathbb{R}$ that induces α by conjugation. Then $\tau_r(\alpha(f))$ generates $\operatorname{im}\tau_r$, too, and so $\tau_r(\alpha(f)) = \pm x_f$. Near $+\infty$, the map f is a translation with positive amplitude, hence so is $\alpha(f) = \varphi_\alpha \circ f \circ \varphi_\alpha^{-1}$, and so $\tau_r(\alpha(f)) > 0$. Thus $\tau_r(f) = (\alpha \circ \tau_r)(f)$. We conclude that τ_r is fixed by α. An analogous argument shows that τ_ℓ is fixed by every increasing automorphism of G.

All taken together we have thus established the following result:

Proposition 5.9. *Let G be a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$. Now assume that the images of λ and ρ contain only translations and that these images are infinite cyclic. Then τ_ℓ and τ_r are both nonzero homomorphisms that are fixed by* $\operatorname{Aut}_+ G$.

Example 5.10. Suppose P is an infinite cyclic group, A a (nonzero) $\mathbb{Z}[P]$-module and b a positive element of A. Set $\bar{G} = G([0, b]; A, P)$. Then there exists a homeomorphism $\vartheta :]0, b[\xrightarrow{\sim} \mathbb{R}$ that induces, by conjugation, an embedding

$$\mu : G([0, b]; A, P) \rightarrowtail G(\mathbb{R}; A, P)$$

whose image contains $B(\mathbb{R}; A, P)$.[13] Let G denote the image of μ. The images of $\lambda \restriction G$ and $\rho \restriction G$ are both infinite cyclic and consist of translations. The images of τ_ℓ and τ_r are therefore infinite cyclic, too, and so the previous lemma applies.

Let's now consider the special case where P is generated by an integer $n \geq 2$, where $A = \mathbb{Z}[P] = \mathbb{Z}[1/n]$ and $b = 1$. For a suitably chosen homeomorphism ϑ the image G of μ consists then of all elements $g \in G(\mathbb{R}; \mathbb{Z}[1/n], \operatorname{gp}(n))$ fulfilling the conditions

$$\sigma_\ell(g) = \sigma_r(g) = 1 \quad \text{and} \quad \tau_\ell(g), \tau_r(g) \in \mathbb{Z}(n-1); \tag{5-4}$$

[12] If exactly one of $\operatorname{im}\lambda$ and $\operatorname{im}\rho$ is abelian, the group does not admit a decreasing automorphism (by Lemma 5.1) and so Theorem 5.4 yields a nonzero homomorphism fixed by $\operatorname{Aut} G$.

[13] In special cases, for instance if $\bar{G}$ is Thompson's group F, this fact is well known (see, e.g., [Belk and Brown 2005, Proposition 3.1.1]); the general claim is established in [Bieri and Strebel 2016] (see Lemma E18.4).

see [Bieri and Strebel 2016, Lemma E18.4]. This group G is called $F_{n,\infty}$ in [Brin and Guzmán 1998, p. 298].

By relaxing conditions (5-4) one obtains supergroups of $F_{n,\infty}$, in particular the group called F_n in [op. cit., p. 298] and defined by the requirements

$$\text{(5-5)}\quad \sigma_\ell(g) = \sigma_r(g) = 1 \quad \text{and} \quad \tau_\ell(g), \tau_r(g) \in \mathbb{Z}, \quad \tau_r(g) - \tau_\ell(g) \in \mathbb{Z}(n-1);$$

see [op. cit., Proposition 2.2.6]. Proposition 5.9 applies to the groups $F_{n,\infty}$, but also to the larger groups F_n. Now, the groups F_n and $F_{n,\infty}$ both admit decreasing automorphisms, in particular the automorphism induced by the reflection in the origin. The homomorphisms τ_ℓ and τ_r are therefore not fixed by the full automorphism group of the groups $F_{n,\infty}$ and F_n, but the difference $\tau_r - \tau_\ell$ is a nonzero homomorphism, with infinite cyclic image, that enjoys this property.

6. Characters fixed by Aut G with G a subgroup of $\mathrm{PL}_o([0, b])$

In this section we prove Theorem 1.7. For the convenience of the reader we restate this result here.

Theorem 6.1. *Suppose $I = [0, b]$ is a compact interval of positive length and G is subgroup of $\mathrm{PL}_o(I)$ that satisfies the following conditions*:

(i) *no interior point of the interval $I = [0, b]$ is fixed by G*;

(ii) *the characters χ_ℓ and χ_r are both nonzero*;

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group*; *and*

(iv) *at least one of the group of units $U(\operatorname{im} \chi_\ell)$ or $U(\operatorname{im} \chi_r)$ is reduced to $\{1, -1\}$.*

Then there exists a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of G. The group G therefore has property R_∞.

Next we explain the layout of Section 6. We begin by recalling the definition of the invariant Σ^1 and stating some basic results concerning it. In Section 6C, we prove Theorem 6.1. The hypotheses of the theorem allow of variations that deserve some comments. This topic is taken care of in sections 6D through 6F.

6A. ***Review of Σ^1.*** Given an infinite group G, consider the real vector space $\mathrm{Hom}(G, \mathbb{R})$ made up of all homomorphisms $\chi : G \to \mathbb{R}_{\mathrm{add}}$ into the additive group of $\mathbb{R}$. These homomorphisms will be referred to as *characters*. Two nonzero characters χ_1 and χ_2 are called equivalent, if one is a positive real multiple of the other. Geometrically speaking, the associated equivalence classes are (open) rays emanating from the origin. The space of all rays is denoted by $S(G)$ and called the *character sphere* of G. In case the abelianization $G_{\mathrm{ab}} = G/[G, G]$ of G is finitely generated, the vector space $\mathrm{Hom}(G, \mathbb{R})$ is finite dimensional and carries a unique topology, induced by its norms; the sphere $S(G)$ equipped with the quotient topology is then homeomorphic to the spheres in a Euclidean vector space of dimension $\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) = \dim_{\mathbb{Q}}(G_{ab} \otimes \mathbb{Q})$.

The invariant $\Sigma^1(G)$ is a subset of $S(G)$. It admits several equivalent definitions; in the sequel, we use the definition in terms of Cayley graphs.[14] Fix a generating set $\mathcal{X}$ of G and define $\Gamma = \Gamma(G, \mathcal{X})$ to be the associated Cayley graph of G. This graph can be equipped with G-actions; as we want to work with *left* G-actions we define the set of positive edges of the Cayley graph like this:

$$E_+(\Gamma) = \{(g, g \cdot x) \in G \times G \mid (g, x) \in G \times \mathcal{X}\}.$$

We move on to the *definition of* $\Sigma^1(G)$. Given a nonzero character χ, consider the submonoid $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$ of G and define $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ to be the full subgraph of $\Gamma(G; \mathcal{X})$ with vertex set G_χ. Both the submonoid G_χ and the subgraph Γ_χ remain the same if χ is replaced by a positive multiple; so these objects depend only on the ray $[\chi] = \mathbb{R}_{>0} \cdot \chi$ represented by χ. The Cayley graph Γ is connected, but its subgraph Γ_χ may not be so; the invariant $\Sigma^1(G)$ records the rays for which the subgraph $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ *is connected*. In symbols,

$$\Sigma^1(G, \mathcal{X}) = \{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X})_\chi \text{ is connected}\}. \tag{6-1}$$

One now faces the problem, familiar from homological algebra, that the definition of $\Sigma^1(G, \mathcal{X})$ involves an arbitrary choice and that one wants to construct an object that does not depend on this choice.

Suppose, first, that G is *finitely generated* and let $\mathcal{X}_f$ be a *finite* generating set. Then the subgraph $\Gamma(G, \mathcal{X}_f)_\chi$ is connected if and only if all the subgraphs $\Gamma(G, \mathcal{X})_\chi$, are connected (see, e.g., [Strebel 2013, Lemma C2.1]) and so the following definition is licit:

Definition 6.2. Let G be a finitely generated group and $\mathcal{X}_f$ a *finite* generating set of G. Then $\Sigma^1(G)$ is defined to be the subset

$$\{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X}_f)_\chi \text{ is connected}\}. \tag{6-2}$$

The fact that the set (6-2) does not depend on the choice of the finite set $\mathcal{X}_f$, allows one to select $\mathcal{X}_f$ in accordance with the problem at hand; see [Strebel 2013, Sections A2.3a and A2.3b] for some consequences of this fact.

Now suppose that G is an arbitrary group. A useful subset of $S(G)$ can then be obtained by defining

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X})_\chi \text{ is connected for every generating set } \mathcal{X}\}; \tag{6-3}$$

see [Strebel 2013, Definition C2.2]. If G happens to be finitely generated, the sets (6-2) and (6-3) are equal; for an arbitrary group, the set $\Sigma^1(G)$ coincides with the

[14] See, e.g., [Strebel 2013, Chapter C] for alternate definitions.

invariant $\Sigma(G)$ defined by Ken Brown in [Brown 1987b, p. 489] *up to a sign*; in other words,

$$\Sigma(G) = -\Sigma^1(G). \tag{6-4}$$

The sign in this formula is caused by the fact that Brown uses right actions on $\mathbb{R}$-trees, whereas *left* actions are employed in our definition of Σ^1.

The subset $\Sigma^1(G)$ of $S(G)$ is traditionally called the Σ^1-*invariant*. The epithet "invariant" is justified by a fact that we explain next. Suppose $\alpha : G \xrightarrow{\sim} \bar{G}$ is an isomorphism of groups. Then α induces, first of all, a linear isomorphism of vector spaces $\operatorname{Hom}(\alpha, \mathbb{R}) : \operatorname{Hom}(\bar{G}, \mathbb{R}) \xrightarrow{\sim} \operatorname{Hom}(G, \mathbb{R})$, and so an isomorphism of spheres

$$\alpha^* : S(\bar{G}) \xrightarrow{\sim} S(G), \qquad [\bar{\chi}] \mapsto [\bar{\chi} \circ \alpha]. \tag{6-5}$$

This second isomorphism maps the subset $\Sigma^1(\bar{G}) \subseteq S(\bar{G})$ onto $\Sigma^1(G) \subseteq S(G)$. [Strebel 2013, Section B1.2a] has more details.

Later, the case where α is *automorphism* will be crucial. The assignment

$$\Sigma^1(G) \xrightarrow{\sim} \Sigma^1(G), \qquad \alpha \mapsto (\alpha^{-1})^*, \tag{6-6}$$

defines a homomorphism from the automorphism group of G into the group of bijections of $\Sigma^1(G)$, and hence also one into that of its complement $\Sigma^1(G)^c$.

Remarks 6.3. (a) Historically speaking, the invariant Σ^1 is a descendent of the invariant $\Sigma_A(G)$, introduced by R. Bieri and R. Strebel [1980]. Here the group G is abelian, A is a finitely generated $\mathbb{Z}G$-module, and $\Sigma_A(G)$ is a subset of the sphere $S(G)$ depending both on A and on G. The motivation for introducing this invariant stems from a question posed by G. Baumslag [1974], namely: *Is there any way of discerning finitely presented metabelian groups from the other finitely generated metabelian groups?*

(b) The invariant Σ^1 is a member of a sequence of invariants Σ^m introduced by B. Renz in his thesis [1988]. The definition of these higher Σ-invariants is considerably more involved than that of Σ^1 and so we shall not give it here; we refer the interested reader to Section 8 of K.-U. Bux's paper [2004] for a survey of various equivalent definitions given in the literature. Suffice it to say here that these invariants form a descending chain

$$\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots \supseteq \Sigma^m(G) \supseteq \cdots$$

of open subsets in $S(G)$, and that, so far, there are very few groups whose higher Σ-invariants are completely known. In the case of PL-homeomorphism groups, the most general result known today is due to M. Zaremsky [2016]; it deals with the sequence of groups $G([0, 1]; \mathbb{Z}[1/n], \operatorname{gp}(n))$, with $n \geq 2$ an integer.

6B. Σ^1 *of subgroups of* $\mathrm{PL}_o([0, b])$. Given a subgroup G of $\mathrm{PL}_o([0, b])$, let σ_ℓ be the homomorphism that assigns to a function $g \in G$ the value of its (right) derivative in the *left* end point 0; similarly, define $\sigma_r : G \to \mathbb{R}^{\times}_{>0}$ to be the homomorphism given by the formula $\sigma_r(g) = \lim_{t \to b} g'(t)$. The homomorphisms σ_ℓ and σ_r generalize the maps with the same names studied in Section 3. By composing them with the natural logarithm function, one obtains characters of G, namely

(6-7) $$\chi_\ell = \ln \circ \sigma_\ell \quad \text{and} \quad \chi_r = \ln \circ \sigma_r.$$

The invariant $\Sigma^1(G)^c$ turns out to consist of precisely two points, represented by the characters χ_ℓ and χ_r, provided G satisfies certain restrictions. The first of them rules out that G is a direct product of subgroups G_1, G_2 with supports in two disjoint open subintervals I_1, I_2, and more general decompositions; the second requires that χ_ℓ, χ_r be nonzero and hence represent points of $S(G)$; the third condition is natural in the sense that it holds for all groups of the form $G([0; b]; A, P)$ investigated in Section 3.

Theorem 6.4. *Let I be a compact interval of positive length and G a subgroup of* $\mathrm{PL}_o(I)$. *Assume the following requirements are satisfied*:

(i) *no interior point of I is fixed by G*;

(ii) *the characters χ_ℓ and χ_r are both nonzero*; *and*

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi)$ is a torsion group.*

Then $\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}$.

Remarks 6.5. (a) Theorem 6.4 generalizes [Bieri et al. 1987, Theorem 8.1]; in that work, G is assumed to be finitely generated and condition (iii) is sharpened to $G = \ker \chi_\ell \cdot \ker \chi_r$. The theorem improves also on a result stated in [Brown 1987b, Remark on p. 502]. A proof of Theorem 6.4, based on the Cayley graph definition of $\Sigma^1(G)$, can be found in [Strebel 2015, Theorem 1.1].

(b) We continue with a comment that seems overdue. In [Bieri et al. 1987] an invariant $\Sigma_{G'}(G)$ is introduced for finitely generated groups G; in the sequel, this invariant will be called $\Sigma^{BNS}(G)$. It is defined in terms of a generation property that uses *right* conjugation, while left action is employed in the definition of $\Sigma^1(G)$. There is, however, a close connection between the two invariants: if G is finitely generated, then

(6-8) $$\Sigma^{BNS}(G) = -\Sigma^1(G),$$

similar to the what happens for Brown's invariant $\Sigma(G)$; see (6-4).

Now, PL-homeomorphism groups are examples of groups made up of permutations, and for such a group G the underlying set can be equipped with two familiar compositions. Suppose the composition in the group G is the one familiar to analysts (and used in this paper); to emphasize this fact call the group temporarily G_{ana}. The assignment $g \mapsto g^{-1}$ defines then an antiautomorphism of G_{ana} and hence an isomorphism $\iota : G_{\text{ana}} \xrightarrow{\sim} G_{\text{gt}}$ onto the group obtained by equipping the set underlying G_{ana} with the composition defined by $f \circ g : t \mapsto f(t) \mapsto g(f(t))$ and preferred by many group theorists (hence, the subscript "gt"). The invariants of the groups G_{ana} and G_{gt} are then related by the formulae

$$\Sigma^1(G_{\text{ana}}) = -\Sigma^1(G_{\text{gt}}) \quad \text{and} \quad \Sigma^{BNS}(G_{\text{ana}}) = -\Sigma^{BNS}(G_{\text{gt}}).$$

The analogous formula holds for the invariant Σ studied in [Brown 1987b].

The two parts of the comment, taken together, lead to the following formulae for groups made up of bijections:

$$G_{\text{gt}} \text{ arbitrary} \implies \Sigma(G_{\text{gt}}) = \Sigma^1(G_{\text{ana}}), \tag{6-9}$$

$$G_{\text{gt}} \text{ is finitely generated} \implies \Sigma^{BNS}(G_{\text{gt}}) = \Sigma^1(G_{\text{ana}}). \tag{6-10}$$

6C. ***Proof of Theorem 6.1.*** Let $I = [0, b]$ be an interval of positive length and G a subgroup of $\mathrm{PL}_o(I)$ that satisfies hypotheses (i) through (iv) stated in Theorem 6.1. Hypotheses (i), (ii), and (iii) allow one to invoke Theorem 6.4 and so

$$\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}.$$

In view of the remarks made at the end of Section 6A, every automorphism α of G will therefore permute the set $\{[\chi_\ell], [\chi_r]\}$. Two cases now arise, depending on whether or not the automorphism group of G acts by the identity on $\Sigma^1(G)^c$.

Suppose first that $\operatorname{Aut} G$ *acts trivially on* $\Sigma^1(G)^c$. By hypothesis (iv), one of the characters χ_ℓ and χ_r, say χ_ℓ, has an image B with $U(B) = \{1, -1\}$. We assert that χ_ℓ is fixed by $\operatorname{Aut} G$. Consider an automorphism α of G. It fixes the ray $\mathbb{R}_{>0} \cdot \chi_\ell$ and so $\chi_\ell \circ \alpha = s \cdot \chi_\ell$ for some positive real s. The relation $\chi_\ell \circ \alpha = s \cdot \chi_\ell$ implies next that

$$\operatorname{im} \chi_\ell = \operatorname{im}(\chi_\ell \circ \alpha) = s \cdot \operatorname{im} \chi_\ell.$$

So s is a positive element of $U(\operatorname{im} \chi_\ell) = \{1, -1\}$ and thus $s = 1$.

So far we have assumed that $U(\chi_\ell)$ equals $\{1, -1\}$; if $U(\operatorname{im} \chi_r)$ is so, one proves in the same way that χ_r is fixed by $\operatorname{Aut} G$. The homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ can thus be chosen to be σ_ℓ if $U(\operatorname{im} \chi_\ell) = \{1, -1\}$ and to be σ_r if $U(\operatorname{im} \chi_r) = \{1, -1\}$.

Assume now that $\operatorname{Aut} G$ *interchanges the points* $[\chi_\ell]$ and $[\chi_r]$. Pick an automorphism, say α_-, that interchanges these points (and hence is decreasing) and denote, as in Remark 2.4(b), by $\operatorname{Aut}_+ G$ the subgroup of $\operatorname{Aut} G$ made up of the increasing automorphisms. Then $\chi_r \circ \alpha_- = s \cdot \chi_\ell$ for some positive real s and so $\operatorname{im} \chi_r = s \cdot \operatorname{im} \chi_\ell$. This relation implies that $U(\operatorname{im} \chi_\ell) = U(\operatorname{im} \chi_r) = \{1, -1\}$.

We claim that the homomorphism

$$\psi = \sigma_\ell \cdot (\sigma_\ell \circ \alpha_-) = \sigma_\ell \cdot (s \cdot \sigma_r)$$

is fixed by Aut G. Two cases arise. If $\alpha \in \mathrm{Aut}_+ G$ then σ_ℓ is fixed by α in view of the first part of the proof. Moreover, $\alpha' = \alpha_- \circ \alpha \circ (\alpha_-)^{-1} \in \mathrm{Aut}_+ G$ and so the calculation

$$\psi \circ \alpha = (\sigma_\ell \circ \alpha) \cdot (\sigma_\ell \circ \alpha_-) \circ \alpha = \sigma_\ell \cdot (\sigma_\ell \circ \alpha') \circ \alpha_- = \sigma_\ell \cdot (\sigma_\ell \circ \alpha_-) = \psi$$

holds. If $\alpha = \alpha_-$ then $\alpha_-^2 \in \mathrm{Aut}_+ G$ and so $\psi \circ \alpha_- = (\sigma_\ell \circ \alpha_-) \cdot (\sigma_\ell \circ \sigma_-^2) = \psi$. It follows that ψ is fixed by $\mathrm{Aut}_+ G \cup \{\alpha_-\}$ and hence by Aut G.

6D. ***Discussion of the hypotheses of Theorem 6.1.*** This section and the next two contain various remarks on the hypotheses of Theorem 6.1.

6D1. *Irreducibility.* Let G be a subgroup of $\mathrm{PL}_o([0, b])$. The union of the supports of the elements of G is then an open subset of $I = [0, b]$, and hence a union of disjoint intervals J_k for k running over some index set K. For each $k \in K$ the assignment $g \mapsto g \restriction J_k$ defines an epimorphism π_k onto a quotient group G_k so G itself is isomorphic to a subgroup of the cartesian product $\prod\{G_k \mid k \in K\}$; more precisely, G is a subdirect product of the quotient groups G_k. Hypothesis (i) requires that K be a singleton, and so the group G does not admit such obvious decompositions. This fact prompted the authors of [Bieri et al. 1987] to call a group G *irreducible* if $\mathrm{card}(K) = 1$.

If the group G is not irreducible it may be a direct product $G_1 \times G_2$ with each factor G_k an irreducible subgroup of $\mathrm{PL}_o(I_k)$ where I_k is the closure of J_k. Then $\Sigma^1(G)^c$ can contain more than 2 points (for more details, see [Strebel 2015, Section 4.1]).

6D2. *Nontriviality of the characters χ_ℓ and χ_r.* In Theorem 6.1 the characters χ_ℓ and χ_r are assumed to be nonzero. They represent therefore points of $S(G)$; the remaining hypotheses and Theorem 6.4 then guarantee that $\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}$ and so every automorphism of G must permute the points $[\chi_\ell]$ and $[\chi_r]$.

There exists a variant of Theorem 6.1 in which only one of the characters, say χ_ℓ, is nonzero, the remaining hypotheses being as before. Then $\Sigma^1(G)^c = \{[\chi_\ell]\}$ (see [Strebel 2015, Theorem 1.1]) and so the argument in the first part of the proof of Theorem 6.1 applies and shows that $\psi = \chi_\ell$ is fixed by every automorphism of G.

Note that hypothesis (iii) holds automatically if χ_ℓ or χ_r vanishes.

6D3. *Almost independence of χ_ℓ and χ_r.* Among the assumptions of [Bieri et al. 1987, Theorem 8.1], a sharper form of hypothesis (iii) is assumed, namely that $G = \ker \chi_\ell \cdot \ker \chi_r$; in addition, G is assumed to be finitely generated. The authors of that reference refer to this stronger condition by saying that "χ_ℓ and χ_r are independent". In what follows, we exhibit various versions of this stronger requirement and explain then the reason that led the authors to adopt the mentioned language.

We start out with a general result.

Lemma 6.6. *Let $\psi_1 : G \twoheadrightarrow H_1$ and $\psi_2 : G \twoheadrightarrow H_2$ be epimorphisms of groups. Then the following statements are equivalent*:

(i) $H_1 = \psi_1(\ker \psi_2)$,

(ii) $H_2 = \psi_2(\ker \psi_1)$,

(iii) $G = \ker \psi_1 \cdot \ker \psi_2$,

(iv) $(\psi_1, \psi_2) : G \to H_1 \times H_2$ *is surjective.*

Proof. Note first that the product $\ker \psi_1 \cdot \ker \psi_2$ is a normal subgroup of G. Next, note that ψ_1 maps G onto H_1 and $\ker \psi_1 \cdot \ker \psi_2$ onto $\psi_1(\ker \psi_2)$ and induces thus an isomorphism

$$(\psi_1)_* : G/(\ker \psi_1 \cdot \ker \psi_2) \xrightarrow{\sim} H_1/\psi_1(\ker \psi_2). \tag{6-11}$$

It follows, in particular, that statements (i) and (iii) are equivalent. By exchanging the rôles of the indices 1 and 2, one sees that statements (ii) and (iii) are equivalent.

Assume now that statements (i) and (ii) hold and consider $(h_1, h_2) \in H_1 \times H_2$. Since ψ_1 is surjective, h_1 has a preimage $g_1 \in G$; as statement (i) holds, this preimage can actually be chosen in $\ker \psi_2$. If this is done, one sees that $(\psi_1, \psi_2)(g_1) = (h_1, 1)$. One finds similarly that there exists $g_2 \in \ker \psi_1$ with $(\psi_1, \psi_2)(g_2) = (1, h_2)$. The product $g_1 \cdot g_2$ is therefore a preimage of (h_1, h_2) under (ψ_1, ψ_2).

The preceding argument proves that the conjunction of (i) and (ii) implies statement (iv). Assume, finally, that (iv) holds. Given $h_1 \in H_1$, there exists then $g_1 \in G$ with $(\psi_1, \psi_2)(g_1) = (h_1, 1)$; so g_1 is a preimage of h_1 lying in $\ker \psi_2$. The implication (iv) $\Rightarrow$ (i) is thus valid, and so the proof is complete. □

Remark 6.7. Lemma 6.6 allows one to understand why the phrase "χ_ℓ and χ_r are independent" is used in [Bieri et al. 1987] to express the requirement that $G = \ker \chi_\ell \cdot \ker \chi_r$, the group G being a finitely generated, irreducible subgroup of $\mathrm{PL}_o([0, b])$. Let ψ_1 denote the epimorphism $G \twoheadrightarrow \operatorname{im} \chi_\ell$ obtained by restricting the domain of $\chi_\ell : G \to \mathbb{R}$ to $\operatorname{im} \chi_\ell$, and let ψ_2 be defined analogously. If statement (iii) holds, then the implication (iii) $\Rightarrow$ (iv) of Lemma 6.6 shows that the image of $(\chi_\ell, \chi_r) : G \to \mathbb{R}_{\text{add}} \times \mathbb{R}_{\text{add}}$ is $\operatorname{im} \chi_\ell \times \operatorname{im} \chi_r$. This fact amounts to saying that the values of the characters χ_ℓ and χ_r can be prescribed *independently* (within $\operatorname{im} \chi_\ell \times \operatorname{im} \chi_r$), in contrast to what happens, for instance, if the characters satisfy a relation like $\chi_2 = -\chi_1$.[15]

By analyzing the proof of Theorem 8.1 in [op. cit.] one finds that it suffices to require that the normal subgroup $\ker \chi_\ell \cdot \ker \chi_r$ has finite index in the finitely generated group G; a condition that we shall paraphrase by saying that χ_ℓ and χ_r are *almost independent*. Theorem 6.4 extends this result to possibly infinitely generated groups G; the new form of hypothesis (iii) will likewise be referred to by saying

[15] Example 3.13 considers more general relations.

that χ_ℓ and χ_r are *almost independent*. This form of almost independence is used in the proof Theorem 6.4 to find commuting elements of a certain type; see, e.g., [Strebel 2015, Section 3.3]. It remains unclear what $\Sigma^1(G)^c$ looks like if χ_ℓ and χ_r are not almost independent.[16]

6E. *Group of units.* In Section 1E, the *group of units* $U(B)$ of a subgroup B of $\mathbb{R}_{\text{add}}$ is introduced. This notion allows one to state a very simple condition that implies, in conjunction with the hypotheses of Theorem 6.4, that $\operatorname{Aut} G$ fixes the character χ_ℓ if it fixes the ray $[\chi_\ell] = \mathbb{R} \cdot \chi_\ell$.

In this section, we discuss the group of units of some concrete examples of subgroups B of $\mathbb{R}_{\text{add}}$, then study two types of subgroups B of $\mathbb{R}_{\text{add}}$ where methods taken from the theory of transcendental numbers allow one to establish that B has only trivial units.

6E1. *Elementary examples.* We begin with an observation: *a subgroup B and a nonzero real multiple $s \cdot B$ of B have the same group of units*. If B is not reduced to 0, we may therefore assume that $1 \in B$.

(a) If B is infinite cyclic, it is a positive multiple of $\mathbb{Z}$. Clearly $U(\mathbb{Z}) = \{1, -1\}$.

(b) If B is free abelian of rank 2, we may assume that it is generated by 1 and an irrational number ϑ; so $B = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \vartheta$. If u is a unit of B then $u = u \cdot 1 \in B$, say $u = a + b \cdot \vartheta$ with $(a, b) \in \mathbb{Z}^2$. The condition $u \cdot B \subseteq B$ implies next that $u \cdot \vartheta = a \cdot \vartheta + b \cdot \vartheta^2$ lies in B. If $b \neq 0$, the real ϑ is thus a quadratic algebraic number; if $b = 0$, the condition that $u \cdot B = B$ forces a to 1 or -1. It follows that $U(B) = \{1, -1\}$ if ϑ is an irrational, but not a quadratic algebraic number.

(c) Let B be the additive group of a subring R of $\mathbb{R}$, for instance the additive group of the ring $\mathbb{Z}[P]$ generated by a subgroup P of $\mathbb{R}^\times_{>0}$ or of a ring of algebraic integers. Then $U(B)$ is nothing but the group of units $U(R)$ of R; if R is a ring of the form $\mathbb{Z}[P]$ its group of units contains, of course, $P \cup -P$, but it may be considerably larger; moreover, rings of algebraic integers have also often units of infinite order. Note, however, that not every subring $R \neq \mathbb{Z}$ of $\mathbb{R}$ has nontrivial units, an example being the polynomial ring $\mathbb{Z}[s]$ generated by a transcendental number s.

6E2. *Transcendental subgroups.* Many of the familiar examples of subgroups of $\mathrm{PL}_o([0, b])$ consist of PL-homeomorphisms with rational slopes; this is true for Thompson's group F, but also for its generalizations $G_m = G([0,1]; \mathbb{Z}[1/m], \mathrm{gp}(m))$ with $m \geq 3$ an integer and for many of the groups studied by Stein [1992].

The values of the characters χ_ℓ are then natural logarithms of rational numbers, so either transcendental numbers or 0 (see, e.g., [Niven 1956, Theorem 9.11c]). We are thus led to study the unit groups $U(B)$ of subgroups $B \subset \mathbb{R}$ that contain

[16] Sections 4.2 and 4.3 in [Strebel 2015] have some preliminary results.

transcendental numbers; in view of the fact that $U(B) = U(s \cdot B)$ for every $s \neq 0$, it is not so much the nature of the elements of B that is important, but the nature of the quotients b_1/b_2 of nonzero elements in B. The following definition singles out a class of subgroups B that turn out to be significant.

Definition 6.8. (a) Let $B \neq \{0\}$ be a subgroup of the additive group $\mathbb{R}_{\text{add}}$ of the reals. We say B is *transcendental* if, for each ordered pair (b_1, b_2) of nonzero elements in B, the quotient b_1/b_2 is either rational or transcendental.

(b) We call a nonzero character $\chi : G \to \mathbb{R}$ *transcendental* if its image in $\mathbb{R}_{\text{add}}$ is transcendental.

The next result explains why transcendental subgroups are welcome in our study.

Proposition 6.9. *If B is a nontrivial, finitely generated, transcendental subgroup of $\mathbb{R}_{\text{add}}$, then $U(B) = \{1, -1\}$.*

Proof. Suppose u is a unit of B. Then $u \cdot B = B$. Pick $b \in B \smallsetminus 0$; this is possible since B is not reduced to 0. The assignment $1 \mapsto b$ extends to a homomorphism $\mathbb{Z}[u] \to B$ of $\mathbb{Z}[u]$-modules; it is injective since $\mathbb{R}$ has no zero-divisors. The fact that B is finitely generated implies next that the additive group of the integral domain $\mathbb{Z}[u]$ is finitely generated and so u is an algebraic integer; as B is transcendental by assumption, u must therefore be an algebraic integer and also a rational number, hence an integer. Finally, u^{-1} satisfies also the relation $u^{-1} \cdot B = B$, and so u^{-1} is an integer, too. □

We continue with a combination of Theorem 6.1 and Proposition 6.9.

Corollary 6.10. *Suppose $I = [0, b]$ is a compact interval of positive length and G is subgroup of $\mathrm{PL}_o(I)$ that satisfies the following conditions:*

(i) *no interior point of the interval $I = [0, b]$ is fixed by G;*

(ii) *the characters χ_ℓ and χ_r are both nonzero;*

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group G; and*

(iv) *the image of σ_ℓ or that of σ_r is finitely generated and transcendental.*

Then there exists a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of G.

6E3. *Examples of transcendental subgroups of* $\mathbb{R}_{\text{add}}$. In order to make use of Proposition 6.9, one needs a supply of transcendental subgroups of $\mathbb{R}$. The simplest ones are the cyclic subgroups; noncyclic subgroups are harder to come by.

Example 6.12 below describes a first collection of transcendental subgroups. It is based on the following theorem, established independently by A. O. Gelfond in 1934 and by T. Schneider in 1935:

Theorem 6.11 (Gelfond–Schneider theorem). *If p_1 and p_2 are nonzero* (*real or complex*) *algebraic numbers and if $p_2 \neq 1$, then $\ln p_1 / \ln p_2$ is either a rational or a transcendental number.*

Proof. See, e.g., [Niven 1956, Theorem 10.2]. □

Example 6.12. Let P denote a subgroup of $\mathbb{R}^\times_{>0}$ generated by a set $\mathcal{P}$ of algebraic numbers and define $B = \ln P$ to be its image in $\mathbb{R}_{\text{add}}$ under the natural logarithm. Then every element in P is a positive algebraic number, so the Gelfond–Schneider theorem implies that every quotient $\ln p_1 / \ln p_2$ of elements in $P \smallsetminus \{1\}$ is either rational or transcendental.

In Example 6.12 the set $\mathcal{P}$ is allowed to be infinite; for such a choice, the group $B = \ln(\text{gp}(\mathcal{P}))$ is not finitely generated and so neither Proposition 6.9 nor its Corollary 6.10 applies. Now, in Proposition 6.9 the finite generation of B is only used to infer that a unit u of B — which, by the transcendence of B, is either rational or transcendental — is also an algebraic integer, and hence a rational integer.

Proposition 6.14 below furnishes examples of infinitely generated, transcendental groups that have only 1 and -1 as units. Its proof makes use of the following result, due to C. L. Siegel and rediscovered by S. Lang; see [Lang 1966, Theorem II.1] or [Lang 1971, Theorem (1.6)]:

Theorem 6.13 (Siegel–Lang theorem). *Suppose β_1, β_2 and z_1, z_2, z_3 are nonzero complex numbers. If the subsets $\{\beta_1, \beta_2\}$ and $\{z_1, z_2, z_3\}$ are both $\mathbb{Q}$-linearly independent then at least one of the six numbers*

$$\exp(\beta_i \cdot z_j), \text{ with } (i, j) \in \{1, 2\} \times \{1, 2, 3\},$$

is transcendental.

Here then is the announced result:

Proposition 6.14. *Suppose that $\mathcal{P}$ is a set of positive algebraic numbers and set $B = \ln \text{gp}(\mathcal{P})$. If B is free abelian of positive rank, then $U(B) = \{1, -1\}$.*

Proof. Note first that every element of $P = \text{gp}(\mathcal{P})$ is a positive algebraic number. Consider now a unit u of B. Since B has positive rank, it contains a nonzero element $b_1 = \ln q_1$. Then $u \cdot b \in B \smallsetminus \{0\}$; so $b_2 = u \cdot b_1$ has the form $\ln q_2$ and thus u is either rational or transcendental (by the Gelfond–Schneider theorem).

Assume first that u is rational, say $u = m/n$, where m and n are relatively prime integers. The hypothesis $(m/n) \cdot B = B$ implies then that $mB = nB$. As B is free abelian of positive rank this equality can only hold if $|m| = |n| = 1$. So $u \in \{1, -1\}$.

Assume now that u is transcendental. Fix $p \in P \smallsetminus \{1\}$. Then $u \cdot \ln p \in B = \ln P$; so there exists $q \in P$ with $\ln q = u \cdot \ln p$; put differently, $\exp(u \cdot \ln p)$ lies in P and is thus an algebraic number. As the powers of u are again units of B it follows that $\exp(u^\ell \cdot \ln p) \in P$ for every $\ell \in \mathbb{N}$. Set

$$\beta_1 = \ln p, \quad \beta_2 = u \cdot \ln p, \quad \text{and} \quad z_j = u^j \text{ for } j = 1, 2, 3.$$

Then the sets $\{\beta_1, \beta_2\}$ and $\{z_1, z_2, z_3\}$ fulfill the hypotheses of Theorem 6.13; its conclusion, however, is contradicted by the previous calculation. This state of affairs shows that the unit u cannot be transcendental. □

Example 6.15. Let $\mathcal{P}$ be a nonempty set of (rational) prime numbers and let P denote the subgroup of $\mathbb{Q}^\times_{>0}$ generated by $\mathcal{P}$. Then P is free abelian with basis $\mathcal{P}$ (by the unique factorization in $\mathbb{N}^\times_{>0}$) and so $U(\ln P) = \{1, -1\}$.

More generally, every nontrivial subgroup P of $\mathbb{Q}^\times_{>0}$ is a free abelian group and hence $B = \ln P$ has only the units 1 and -1.

6E4. *Some properties of transcendental subgroups and transcendental characters.* The transcendence of a character is a property that has not yet been discussed in the literature on the invariant Σ^1. In this section, we therefore assemble a few useful properties of this notion.

Assume $B \subset \mathbb{R}_{\text{add}}$ is a transcendental subgroup. Then:

(a) every nontrivial subgroup $B' \subseteq B$ is transcendental (immediate from the definition);

(b) if $\chi : G \to \mathbb{R}$ is a character whose image is a nontrivial subgroup of B then χ is transcendental (by (a)), and so are all the compositions $\chi \circ \pi$ with $\pi : \tilde{G} \twoheadrightarrow G$ an epimorphism of groups (immediate from the first part);

(c) if χ, χ' are characters of G with images equal to B, the image of $\chi + \chi'$ is contained in B, and so the character $\chi + \chi'$ is transcendental, unless it is 0;

(d) if χ is transcendental and $\alpha_1, \ldots, \alpha_m$ are automorphisms of G the character

$$\eta = \chi \circ \alpha_1 + \cdots + \chi \circ \alpha_m$$

is transcendental, unless it is zero.

A further property is discussed in part (iv) of Proposition 6.16 below.

6F. ***Passage to subgroups of finite index.*** The next proposition shows that the hypotheses stated in Corollary 6.10 are inherited by subgroups of finite index.

Proposition 6.16. *Let G be a subgroup of $\mathrm{PL}_o([0, b])$ and $H \subseteq G$ a subgroup of finite index. Denote the restrictions of χ_ℓ and χ_r to H by χ'_ℓ and χ'_r. Then the following statements are valid*:

(i) *G is irreducible if and only if H is irreducible*;

(ii) *χ_ℓ is nonzero precisely if χ'_ℓ is nonzero, and similarly for χ_r and χ'_r*;

(iii) *the characters* χ_ℓ *and* χ_r *are almost independent if and only if* χ'_ℓ *and* χ'_r *are almost independent*;

(iv) χ_ℓ *is transcendental exactly if* χ'_ℓ *is transcendental, and similarly for* χ_r *and* χ'_r.

Proof. Claim (i) holds since the support of a PL-homeomorphism f coincides with that of its positive powers f^m. Assertion (ii) is valid since the image of a character is a subgroup of $\mathbb{R}_{\text{add}}$ and hence torsion-free. The fact that the quotient b_1/b_2 of nonzero real numbers coincides, for every positive integer m, with the quotient $(mb_1)/(mb_2)$ allows one to see that a nonzero character χ of G is transcendental if its restriction to H is so; the converse is covered by property (a) stated in Section 6E4. We are left with establishing statement (iii).

To achieve this goal, we compare the quotient groups $G/(\ker\chi_\ell \cdot \ker\chi_r)$ and $H/(\ker\chi'_\ell \cdot \ker\chi'_r)$. By (6-11), the first of them is isomorphic to the quotient group $A_1 = \operatorname{im}\chi_\ell/\chi_\ell(\ker\chi_r)$; the second one is isomorphic to $A_2 = \operatorname{im}\chi'_\ell/\chi'_\ell(\ker\chi'_r)$. Clearly, $A_2 = \operatorname{im}\chi'_\ell/\chi_\ell(\ker\chi'_r)$. The groups A_1 and A_2 fit into the short exact sequences

$$A_2 = \operatorname{im}\chi'_\ell/\chi_\ell(\ker\chi'_r) \hookrightarrow A = \operatorname{im}\chi_\ell/\chi_\ell(\ker\chi'_r) \twoheadrightarrow \operatorname{im}\chi_\ell/\operatorname{im}\chi'_\ell, \tag{6-12}$$

$$\chi_\ell(\ker\chi_r)/\chi_\ell(\ker\chi'_r) \hookrightarrow A = \operatorname{im}\chi_\ell/\chi_\ell(\ker\chi'_r) \twoheadrightarrow A_1 = \operatorname{im}\chi_\ell/\chi_\ell(\ker\chi_r). \tag{6-13}$$

The claim now follows from the fact that $\operatorname{im}\chi_\ell/\operatorname{im}\chi'_\ell$ and $\chi_\ell(\ker\chi_r)/\chi_\ell(\ker\chi'_r)$ are finite groups with orders that divide the index of H in G. □

A first application of Proposition 6.16 is the following corollary:

Corollary 6.17. *Let G be a finitely generated, irreducible subgroup of* $\mathrm{PL}_o(I)$. *If the characters* χ_ℓ *and* χ_r *are almost independent and one of them is transcendental, then any group* Γ *commensurable*[17] *with G has property* R_∞.

Proof. Let $H_0 \subset G$ be a finite index subgroup of G that is isomorphic to a finite index subgroup Γ_0 of Γ. There exists then a *finite index subgroup* Γ_1 *of* Γ_0 *that is characteristic in* Γ; see, e.g., [Lyndon and Schupp 1977, Theorem IV.4.7]. Let H_1 be the subgroup of H_0 that corresponds to Γ_1 under an isomorphism $H_0 \xrightarrow{\sim} \Gamma_0$. Then H_1 has finite index in G and thus Proposition 6.16 allows us to infer that H_1 inherits the properties enunciated for G in the statement of Corollary 6.10. This corollary applies therefore to H_1 and shows that H_1 admits a nonzero homomorphism $\psi_1 \colon H_1 \to \mathbb{R}^\times_{>0}$ that is fixed by $\operatorname{Aut} H_1$. So H_1, and hence Γ_1, satisfy property R_∞. Use now the fact that Γ_1 is a characteristic subgroup of Γ and apply [Mubeena and Sankaran 2014a, Lemma 2.2(ii)] to infer that Γ satisfies property R_∞. □

[17] Two groups G_1 and G_2 are called *commensurable* if they contain subgroups H_1 and H_2 that are isomorphic and of finite indices in G_1 and in G_2, respectively.

Remark 6.18. If the group G_1 has property R_∞, then a group G_2 commensurable to G_1 need not have this property, as is shown by the fundamental group G_1 of the Klein bottle and the fundamental group G_2 of a torus: the group G_1 has property R_∞ by [Gonçalves and Wong 2009, Theorem 2.2]), while the automorphism $-\mathbb{1}$ of $G_2 = \mathbb{Z}^2$ has Reidemeister number 4.

7. Miscellaneous examples

In this section we illustrate by various examples the notions of irreducible subgroup, almost independence of χ_ℓ and χ_r, and the group of units.

7A. ***Irreducible subgroups.*** Let b be a positive real number and G a subgroup of $\mathrm{PL}_o([0, b])$. Recall that G is called *irreducible* if no interior point of $I = [0, b]$ is fixed by all of G (see Section 6D1 for the motivation that led to this name).

The group is irreducible if and only if the supports of the elements of G cover the interior $\operatorname{int}(I)$ of I or, equivalently, if the supports of the elements in a generating set $\mathcal{X}$ of G cover $\operatorname{int}(I)$; these claims are easily verified. If G is cyclic, generated by f, say, it is therefore irreducible if f fixes no point in $\operatorname{int}(I)$ or, equivalently, if $f^\varepsilon(t) < t$ for $t \in \operatorname{int}(I)$ and some sign ε. Such a function is often called a *bump*.

Example 7.1. Here is a very simple kind of PL-homeomorphism bump. Given a positive slope $s \neq 1$, set

$$f_s(t) = \begin{cases} \dfrac{1}{s} \cdot t, & \text{if } 0 \le t \le \dfrac{s}{s+1} \cdot b, \\ s\Big(t - \dfrac{s \cdot b}{s+1}\Big) + \dfrac{b}{s+1}, & \text{if } \dfrac{s}{s+1} \cdot b < t \le b. \end{cases} \tag{7-1}$$

Then f_s is continuous at $s/(s+1) \cdot b$; since $f_s(0) = 0$ and $f_s(b) = b$, the function f_s lies in $\mathrm{PL}_o([0, b])$. Let G_s denote the group generated by f_s and let α be the automorphism that sends f_s to its inverse f_s^{-1}. Then

$$(\chi_\ell \circ \alpha)(f_s) = \chi_\ell(f_s^{-1}) = -\chi_\ell(f_s);$$

similarly $(\chi_r \circ \alpha)(f_s) = -\chi_r(f_s)$, whence

$$\chi_\ell \circ \alpha = -\chi_\ell \quad \text{and} \quad \chi_r \circ \alpha = -\chi_r. \tag{7-2}$$

So neither χ_ℓ nor χ_r is fixed by $\operatorname{Aut}(G_s)$. However, Theorem 6.4 cannot be applied, as requirement (iii) is violated; indeed, $\ker \chi_\ell = \ker \chi_r = \{\mathbb{1}\}$, so $G_s/(\ker \chi_\ell \cdot \ker \chi_r)$ is infinite cyclic. The conclusion of Theorem 6.4 is likewise false, for $\Sigma^1(G)^c = \varnothing$ (this follows, e.g., from [Strebel 2013, Example A2.5a]). Property R_∞, finally, does not hold, either; for the Reidemeister number of the automorphism α is 2, as a simple calculation shows.

The groups in the previous example are cyclic; more challenging groups are considered in the following example:

Example 7.2. Let $d > 1$ be an integer and $s_1, \dots, s_d$ pairwise distinct, positive real numbers not equal to 1. For each index $i \in \{1, \dots, d\}$, define f_i by (7-1) with $s = s_i$, and set

$$G = G_{\{s_1,\dots,s_d\}} = \operatorname{gp}(f_1, \dots, f_d).$$

The group G inherits two properties from the group G_s in the previous example: it is irreducible (obvious), and the assignment $f_i \mapsto f_i^{-1}$ extends to an automorphism α; indeed, the special form of the elements f_i implies that conjugation by the reflection in the midpoint of $I = [0, b]$ sends f_i to its inverse. It follows, as before, that the relations (7-2) are valid; so neither χ_ℓ nor χ_r is fixed by $\operatorname{Aut} G$.

Now to another property of the automorphism α. The calculation

$$(\chi_\ell \circ \alpha)(f_i) = \chi_\ell(f_i^{-1}) = -\chi_\ell(f_i) = \chi_r(f_i) \tag{7-3}$$

is valid for every index i. It shows that α exchanges χ_ℓ and χ_r. It follows, in particular, that $\ker \chi_\ell = \ker \chi_r$ and so the quotient

$$G/(\ker \chi_\ell \cdot \ker \chi_r) = G/\ker \chi_\ell \xrightarrow{\sim} \operatorname{im} \chi_\ell = \operatorname{gp}(\ln s_1, \dots, \ln s_d)$$

is a nontrivial free abelian group of rank at most d. Requirement (iii) in Theorem 6.4 is thus violated and so we cannot use that result to determine $\Sigma^1(G)^c$. Actually, only the following meager facts are known about $\Sigma^1(G)^c$: both χ_ℓ and $\chi_r = -\chi_\ell$ represent points of $\Sigma^1(G)^c$ [Strebel 2015, Proposition 2.5]; moreover, the existence and form of the automorphism α and formula (6-6) imply that $\Sigma^1(G)^c$ is invariant under the antipodal map $[\chi] \mapsto [-\chi]$.

Computation (7-3) shows that $\chi_\ell \circ \alpha = -\chi_\ell$. This conclusion holds, actually, for every character $\chi : G \to \mathbb{R}$ and proves that no nonzero character of G is fixed by α.

7B. ***Independence of χ_ℓ and χ_r.*** As before, let G be a subgroup of $\mathrm{PL}_o([0, b])$ with b a positive real number. Recall that the characters χ_ℓ and χ_r are called *independent* if $G = \ker \chi_\ell \cdot \ker \chi_r$; see Section 6D1. It follows that χ_ℓ and χ_r are independent if and only if G admits a generating set $\mathcal{X} = \mathcal{X}_\ell \cup \mathcal{X}_r$ in which the elements of $\mathcal{X}_\ell$ have slope 1 near b and those of $\mathcal{X}_r$ have slope 1 near 0.

It is thus very easy to manufacture groups for which χ_ℓ and χ_r are independent. In the next example, some very particular specimens are constructed.

Example 7.3. Choose a real number $b_1 \in \,]b/2, b[$. Given a tuple of positive real numbers $s_1, \dots, s_{d_\ell}$ that are pairwise distinct and not equal to 1, let f_i be the bump defined by (7-1) but with $s = s_i$ and $b = b_1$. Next let $s'_1, \dots, s'_{d_r}$ be another sequence of positive reals that are pairwise distinct and different from 1. Use them to define bump functions g_j with supports in $]b - b_1, b[$ like this: let h_j be the function given by (7-1) but with $s = s'_j$ and $b = b_1$, and define then g_j to be h_j conjugated by the translation with amplitude $b - b_1$. Finally set

$$G = G_{\{s_1,\ldots,s_{d_\ell}, s'_1,\ldots s'_{d_r}; b_1\}} = \operatorname{gp}(f_1, \ldots, f_{d_\ell}, g_1, \ldots, g_{d_r}). \tag{7-4}$$

From now on, we assume that d_ℓ and d_r are both positive. Then G is irreducible (since $b_1 > b - b_1$), the characters χ_ℓ, χ_r are nonzero and independent, and thus Theorem 6.4 allows us to conclude that $\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}$.

The character χ_ℓ is transcendental if all the positive reals $s_1, \ldots, s_{d_\ell}$ are algebraic (see Example 6.12). Then G admits a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by $\operatorname{Aut} G$ (see Theorem 6.1). If G does not admit an automorphism α with $\chi_\ell \circ \alpha \in [\chi_r]$ the homomorphism ψ can be chosen to be σ_ℓ (see the second paragraph of Section 6C). The stated condition holds, in particular, if there does not exists a number s with $\operatorname{im} \chi_r = s \cdot \operatorname{im} \chi_\ell$. Similar remarks apply to χ_r.

7B1. *Independence versus almost independence.* The characters χ_ℓ and χ_r are called almost independent if $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group (see Remark 6.7). Statement (iii) of Proposition 6.16 shows that almost independence of χ_ℓ and χ_r is inherited by the restricted characters $\chi'_\ell = \chi_\ell \restriction H$ and $\chi'_r = \chi_r \restriction H$ whenever $H \subseteq G$ is a subgroup of finite index. The next result characterizes those ordered pairs (G, H), with χ_ℓ, χ_r independent whose restrictions χ'_ℓ and χ'_r are again independent.

Lemma 7.4. *Let G be a subgroup of* $\mathrm{PL}_o([0, b])$ *for which χ_ℓ and χ_r are independent and let $H \subset G$ be a subgroup of finite index. Then the restrictions χ'_ℓ and χ'_r of these characters are independent if and only if the homomorphism*

$$\zeta : \chi_\ell(\ker \chi_r)/\chi_\ell(\ker \chi'_r) \longrightarrow \operatorname{im} \chi_\ell / \operatorname{im} \chi'_\ell, \tag{7-5}$$

induced by the inclusions, is injective.

Proof. The justification will be an assemblage of facts extracted from the proof of Lemma 6.6 and from that of Proposition 6.16. Firstly, χ_ℓ and χ_r are independent if and only if the abelian group $A_1 = \operatorname{im} \chi_\ell/\chi_\ell(\ker \chi_r)$ is 0. Similarly, χ'_ℓ and χ'_r are independent precisely if $A_2 = \operatorname{im} \chi'_\ell/\chi_\ell(\ker \chi'_r)$ is the zero group. The groups A_1 and A_2 occur among the groups in the short exact sequences (6-12) and (6-13). Since $A_1 = 0$, these exact sequences lead to the short exact sequence

$$\operatorname{im} \chi'_\ell/\chi_\ell(\ker \chi'_r) \hookrightarrow \chi_\ell(\ker \chi_r)/\chi_\ell(\ker \chi'_r) = \operatorname{im} \chi_\ell/\chi_\ell(\ker \chi'_r) \twoheadrightarrow \operatorname{im} \chi_\ell/\operatorname{im} \chi'_\ell,$$

It shows that $A_2 = \operatorname{im} \chi'_\ell/\chi_\ell(\ker \chi'_r)$ is the kernel of the homomorphism ζ. □

It is now easy to construct independent characters χ_ℓ and χ_r of G whose restrictions to a subgroup of finite index are no longer independent.

Example 7.5. Let G be a subgroup of $\mathrm{PL}_o([0, b])$ and H a subgroup of finite index. Assume the characters χ_ℓ and χ_r are independent. According to Lemma 7.4 the restricted characters χ'_ℓ and χ'_r of H are independent if and only if the obvious homomorphism

$$\zeta : \chi_\ell(\ker \chi_r)/\chi_\ell(\ker \chi'_r) \longrightarrow \operatorname{im} \chi_\ell / \operatorname{im} \chi'_\ell$$

is injective. The characters χ'_ℓ and χ'_r of H will therefore *not* be independent whenever

$$\operatorname{im} \chi'_\ell = \operatorname{im} \chi_\ell \quad \text{but} \quad \chi_\ell(\ker \chi_r \cap H) \subsetneq \chi_\ell(\ker \chi_r). \tag{7-6}$$

Now to some explicit examples. We begin with quotients of the groups we shall ultimately be interested in. Set $\bar{G} = \mathbb{Z}^2$, let $p \geq 2$ be an integer and set

$$\bar{H} = \mathbb{Z}(p, 0) + \mathbb{Z}(1, 1).$$

Then $\bar{H}$ has index p in $\bar{G}$.

Next, let χ_1, χ_2 denote the canonical projections of $\mathbb{Z}^2$ onto its factors. Then

$$\chi_1(\bar{G}) = \mathbb{Z} = \chi_1(\bar{H}), \quad \ker \chi_2 = \mathbb{Z}(1, 0), \text{ and } \ker \chi_2 \cap \bar{H} = \mathbb{Z}(1, 0) \cap \bar{H} = \mathbb{Z}(p, 0),$$

and thus

$$\chi_1(\ker \chi_2 \cap \bar{H}) = \mathbb{Z} \cdot p \subsetneq \chi_1(\ker \chi_2) = \mathbb{Z}.$$

The auxiliary groups $\bar{G}$ and $\bar{H}$ therefore satisfy the relations (7-6).

We are now ready to define the group G; it will be of the kind considered in Example 7.3 with $d_\ell = d_r = 1$. Fix $b > 0$ and $b_1 \in \,]b/2, b[$ and choose positive numbers s_1, s'_1, both different from 1. Define f_1 and g_1 as in Example 7.3 and set

$$G = \operatorname{gp}(f_1, g_1).$$

Then G is an irreducible subgroup of $\mathrm{PL}_0([0, b])$ and the characters χ_ℓ and χ_r of G are independent. Moreover, G_{ab} is free abelian of rank 2, freely generated by the canonical images of f_1 and g_1. Set $H = \operatorname{gp}(f_1^p, f_1 \circ g_1, [G, G])$. The above calculations then imply that

$$\chi_\ell(G) = \mathbb{Z} \cdot \ln s_1 = \chi_\ell(H) \quad \text{and} \quad \chi_\ell(\ker \chi_r \cap H) = \mathbb{Z} \cdot p \cdot \ln s_1 \subsetneq \chi_\ell(\ker \chi_r) = \mathbb{Z} \ln s_1.$$

7C. *Eigenlines.* Let G be an irreducible subgroup of $\mathrm{PL}_o([0, b])$. If the characters χ_ℓ and χ_r are nonzero and almost independent, then $\Sigma^1(G)^c$ consist of the two points $[\chi_\ell]$ and $[\chi_r]$ (by Theorem 6.4). Every automorphism α of G either fixes or exchanges them. Suppose we are in the first case. Then $\chi_r \circ \alpha = s \cdot \chi_r$ for some positive real s, and so $\mathbb{R} \cdot \chi_r$ is an eigenline, with eigenvalue s, in the vector space $\operatorname{Hom}(G, \mathbb{R})$ acted on by α^*. No example with $s \neq 1$ has been found so far.

If the compact interval $[0, b]$ is replaced by the half-line $[0, \infty[$, such examples exist, provided χ_r is replaced by a suitable analogue τ_r. In order to construct examples, we return to the set-up of Section 4. So P is a nontrivial subgroup of

$\mathbb{R}^{\times}_{>0}$ and A is a nontrivial $\mathbb{Z}[P]$ submodule of $\mathbb{R}_{\text{add}}$. Define G to be the kernel of the homomorphism $\sigma_r : G([0, \infty[; A, P) \to \mathbb{R}^{\times}_{>0}$; thus G consists of all the elements of $G([0, \infty[; A, P)$ that are translations near $+\infty$. The analysis in Section 4C2 shows that conjugation by the PL-homeomorphism $f_p : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$, given by $f_p(t) = p \cdot t$ for $t \geq 0$, induces, for every $p \in P$, an automorphism α_p of G that satisfies the relation

$$\tau_r \circ \alpha_p = p \cdot \tau_r; \tag{7-7}$$

here $\tau_r : G \to \mathbb{R}_{\text{add}}$ is the character that sends $g \in G$ to the amplitude of the translation that coincides with g near $+\infty$. This character τ_r shares an important property with the character χ_r: the invariant $\Sigma^1(G)^c$ consists of two points, one represented by χ_ℓ, the other by τ_r; see [Strebel 2015, Theorem 1.2].

The image of τ_r in $\mathbb{R}_{\text{add}}$ is a subgroup B of A, namely

$$B = IP \cdot A = \Big\{\sum (p-1) \cdot a \;\Big|\; p \in P \text{ and } a \in A\Big\}$$

(see assertion (iii) of [Bieri and Strebel 2016, CorollaryeA5.3]). The group of units $U(B)$ of B contains the group P and so it is not reduced to $\{1, -1\}$.

The subgroup B is typically infinitely generated; if so, G is likewise infinitely generated. Examples of finitely generated groups $G = \ker \sigma_r$ are harder to find, and they are so far rare. Suppose the group $G([0, b]; A, P)$ is finitely generated for some $b \in A_{>0}$. Then $G([b, 2b]; A, P)$ is a finitely generated subgroup of the group of bounded elements $B([0, \infty[; A, P)$. Pick now an element $g_0 \in G$ that moves every point of the open interval $]0, \infty[$ to the right and satisfies the inequality $g_0(b) < 2b$. Then translates of the interval $]b, 2b[$ under the powers of g_0 will then cover $]0, \infty[$. It follows that the subgroup

$$N = \operatorname{gp}\big(\{g_0^j \circ G([b, 2b]; A, P) \circ g_0^{-j} \mid j \in \mathbb{Z}\}\big)$$

coincides with the bounded group $B([0, \infty[; A, P)$ (use [Bieri and Strebel 2016, Lemma E18.9]). So the group $B([0, \infty[; A, P) \rtimes \operatorname{gp}(g_0)$ is finitely generated. The group G, finally, is finitely generated if $G/N \xrightarrow{\sim} \operatorname{im} \tau_r = IP \cdot A$ is finitely generated.

To show that finitely generated groups of the form $G = \ker \sigma_r$ exist we need thus an example of a group $G([0, b]; A, P)$ where both $G([0, b]; A, P)$ and the abelian group underlying $B = IP \cdot A$ are finitely generated. The parameters

$$P = \operatorname{gp}\big(\sqrt{2} + 1\big), \quad A = \mathbb{Z}\big[\sqrt{2}\big] = \mathbb{Z}[P], \quad b = 1$$

lead to such a group; see [Cleary 1995].

7D. *Variation on Theorem 6.1*. Among the hypotheses of Theorems 6.1 and 6.4 figures the requirement that G acts irreducibly on the interval $[0, b]$. This requirement rules out, in particular, that G is a product $G_1 \times G_2$ with G_1 acting irreducibly on some interval $I_1 = [0, b_1]$ and G_2 acting irreducibly on an interval $I_2 = [b_2, b]$ that is disjoint from I_1.

Now suppose we are in this excluded case and that the groups G_1, G_2 satisfy the assumptions of Theorem 6.4, suitably interpreted; more explicitly, suppose the characters $\chi_{1,\ell}$ and $\chi_{1,r}$ of G_1 are nonzero and almost independent, and similarly for the characters $\chi_{2,\ell}$ and $\chi_{2,r}$ of G_2. The question then arises whether $G = G_1 \times G_2$ admits a nonzero homomorphism $\psi : G \to \mathbb{R}^{\times}_{>0}$ that is fixed by $\operatorname{Aut} G$. We shall see that this is the case if at least one of the four groups $\operatorname{im} \chi_{1,\ell}$, $\operatorname{im} \chi_{1,r}$ and $\operatorname{im} \chi_{2,\ell}$, $\operatorname{im} \chi_{2,r}$ has a unit group that is reduced to $\{1, -1\}$.

The following proposition is a variation on Theorem 3.2 in [Gonçalves and Kochloukova 2010].

Proposition 7.6. *Let G be a group for which $\Sigma^1(G)^c$ is a nonempty finite set with m elements. Assume the rays $[\chi] \in \Sigma^1(G)^c$ span a subspace of $\operatorname{Hom}(G, \mathbb{R})$ having dimension m over $\mathbb{R}$ and that $U(\operatorname{im} \chi_1) = \{1, -1\}$ for some point $[\chi_1] \in \Sigma^1(G)^c$. Then G admits a nontrivial homomorphism $\psi : G \to \mathbb{R}^{\times}_{>0}$ that is fixed by $\operatorname{Aut} G$.*

Proof. The automorphism group $\operatorname{Aut} G$ acts on $\Sigma^1(G)^c$ via the assignment

$$(\alpha, [\chi]) \mapsto [\chi \circ \alpha^{-1}];$$

let $\{[\chi_1], \ldots, [\chi_n]\}$ be the orbit in $\Sigma^1(G)^c$ containing $[\chi_1]$. If $n = 1$, the point $[\chi_1]$ is fixed by $\operatorname{Aut} G$; hence χ_1 itself is fixed by $\operatorname{Aut} G$ in view of the assumption that $U(\operatorname{im} \chi_1) = \{1, -1\}$, and so we can take $\psi = \exp \circ \chi_1$.

Now suppose that $n > 1$ and choose, for every $i \in \{1, \ldots, n\}$, an automorphism α_i with $[\chi_i] = [\chi_1 \circ \alpha_i]$. Let α be an automorphism of G. For every index $i \in \{1, \ldots, n\}$ there exists then an index j so that $[\chi_i \circ \alpha^{-1}] = [(\chi_1 \circ \alpha_i) \circ \alpha^{-1}]$ is equal to $[\chi_j] = |\chi_1 \circ \alpha_j]$. It follows that there exists a positive real number $s_{i,j}$ so that

$$\chi_1 \circ \alpha_i \circ \alpha^{-1} = s_{i,j} \cdot \chi_1 \circ \alpha_j.$$

But if so, $\beta = \alpha_i \circ \alpha^{-1} \circ \alpha_j^{-1}$ is an automorphism with $\chi_1 \circ \beta = s_{i,j} \cdot \chi_1$. The assumption that $U(\operatorname{im} \chi_1) = \{1, -1\}$ permits one then to deduce that $s_{i,j} = 1$. So $\operatorname{Aut} G$ permutes the set of characters

$$\chi_1 \circ \alpha_1, \quad \chi_1 \circ \alpha_2, \quad \ldots, \quad \chi_1 \circ \alpha_n. \tag{7-8}$$

Their sum η is therefore fixed by $\operatorname{Aut} G$. It is nonzero since the characters displayed in (7-8) are linearly independent over $\mathbb{R}$. Set $\psi = \exp \circ \eta$. □

Corollary 7.7. *Let G_1 be a subgroup of $\mathrm{PL}_0([0, b_1])$ and let G_2 be a subgroup of $\mathrm{PL}_0([b_2, b])$ with $0 < b_1 < b_2 < b$. Assume G_1 and G_2 are irreducible, the characters $\chi_{1,\ell}$ and $\chi_{1,r}$ of G_1 are nonzero and almost independent, and that the characters $\chi_{2,\ell}$ and $\chi_{2,r}$ of G_2 have the same properties. If the image of at least one of the four characters $\chi_{1,\ell}$, $\chi_{1,r}$ and $\chi_{2,\ell}$, $\chi_{2,r}$ has a unit group that is reduced* *to* $\{1,-1\}$ *then* $G = G_1 \times G_2$ *admits a nontrivial homomorphism* $\psi : G \to \mathbb{R}^{\times}_{>0}$ *that is fixed by* $\operatorname{Aut} G$.

Proof. The hypothesis on G_1 and G_2 allow us to apply Theorem 6.4 and so

$$\Sigma^1(G_1)^c = \{[\chi_{1,\ell}], [\chi_{1,r}]\} \quad \text{and} \quad \Sigma^1(G_2)^c = \{[\chi_{2,\ell}], [\chi_{2,r}]\}.$$

The product formula for Σ^1 then implies that $\Sigma^1(G)^c$ consists of the four points represented by

$$\chi_{1,\ell} \circ \pi_1, \quad \chi_{1,r} \circ \pi_1, \quad \chi_{2,\ell} \circ \pi_2, \quad \chi_{1,\ell} \circ \pi_2; \tag{7-9}$$

here $\pi_i : G \twoheadrightarrow G_i$ denotes the canonical projection onto the i-th factor G_i (see, e.g., [Strebel 2013, Proposition C2.55]). These four characters are $\mathbb{R}$-linearly independent since all are nonzero, as $\ker \chi_{1,\ell} \neq \ker \chi_{1,r}$ by the almost independence of $\chi_{1,\ell}$ and $\chi_{1,r}$, as $\ker \chi_{2,\ell} \neq \ker \chi_{2,r}$ by the almost independence of $\chi_{2,\ell}$ and $\chi_{2,r}$, and since $\pi_1^*(\operatorname{Hom}(G_1, \mathbb{R}))$ and $\pi_2^*(\operatorname{Hom}(G_2, \mathbb{R}))$ are complementary subspaces of $\operatorname{Hom}(G, \mathbb{R})$. Finally, at least one of the images of the four characters displayed in (7-9) has an image B with $U(B) = \{1, -1\}$. All the assumptions of Proposition 7.6 are thus satisfied, so the contention of the corollary follows from that proposition. □

Remark 7.8. It is not known whether the direct product of groups G_1, G_2 each of which has property R_∞ has again property R_∞. The previous corollary implies that this will be so if the groups G_1 and G_2 satisfy the assumptions of the corollary.

8. Complements

By Remark 5.8 there exist continuously many pairwise nonisomorphic, finitely generated groups of the form $G(\mathbb{R}; A, P)$, and by Corollary 5.6 each of these groups admits a nonzero homomorphism ψ into P. These facts prompt the question whether there exist similarly large collections of finitely generated subgroups of $\mathrm{PL}_o(I)$ with I a compact interval, say $I = [0, 1]$. Since only countably many *finitely generated* groups of the form $G([0, 1]; A, P)$ have been found so far, we look for finitely generated groups that satisfy the assumptions of Theorem 6.1.

In Section 8A we exhibit a collection $\mathcal{G}$ of 3-generator groups with the desired properties. Checking that each group in $\mathcal{G}$ satisfies the assumptions of Theorem 6.1 is fairly easy; the verification that distinct groups in $\mathcal{G}$ are not isomorphic, however, is more demanding. We shall succeed by exploiting properties of the Σ^1-invariant of the groups in $\mathcal{G}$ in a roundabout manner. In Section 8B we describe then a collection of 2-generator groups which, despite appearances, turn out to be pairwise isomorphic. This indicates once more that criteria which allow one to prove that two given, similarly looking, groups are not isomorphic, are very useful. In the final section, we give such a criterion.

8A. ***A large collection of groups G with characters fixed by* Aut G.** In this section we construct a collection $\mathcal{G}$ of pairwise nonisomorphic groups G_s with the following properties:

(i) each $G_s \in \mathcal{G}$ is an irreducible subgroup of $\mathrm{PL}_o([0,1])$ generated by 3 elements;

(ii) the characters χ_ℓ, χ_r of G_s are independent and have ranks 1, respectively 2;

(iii) for each $G_s \in \mathcal{G}$ the character χ_ℓ is fixed by $\operatorname{Aut} G_s$; and

(iv) the cardinality of $\mathcal{G}$ is that of the continuum.

8A1. *Construction of the groups G_s.* The groups G_s are obtained by the recipe described in Example 7.3. Fix a triple $s = (s_1, s_2 = s_1', s_3 = s_2')$ of real numbers in $]1, \infty[$. Let f_s be the PL-homeomorphism defined by (7-1) with $s = s_1$ and $b = \frac{3}{4}$. Next, let g be the function obtained by putting $s = s_2$, $b = \frac{3}{4}$ and by then conjugating the function so obtained by translation with amplitude $\frac{1}{4}$. Similarly, let h_s be the function obtained by setting $s = s_3$, $b = \frac{3}{4}$ and by conjugating the function so obtained by the translation $t \mapsto t + \frac{1}{4}$. Finally, set

$$G_s = G_{\{s_1, s_2, s_3\}} = \mathrm{gp}(f_s, g_s, h_s) \tag{8-1}$$

The definition of G_s shows that it is an irreducible subgroup of $\mathrm{PL}_o([0,1])$ with nonzero and independent characters χ_ℓ and χ_r. By Theorem 6.4, the complement of $\Sigma^1(G_s)$ consists therefore of the two rays $[\chi_\ell]$ and $[\chi_r]$.

Consider now an automorphism α of G_s. It induces an autohomeomorphism α^* of the sphere $S(G_s)$ that maps the subset $\Sigma^1(G_s)^c$ onto itself. Suppose α^* is the identity on $\Sigma^1(G_s)^c$. Since χ_ℓ has rank 1 and is thus transcendental, the first two paragraphs of Section 6C apply and show that α fixes the character χ_ℓ and hence also the homomorphism $\sigma_\ell : G \to \mathbb{R}^\times_{>0}$. This homomorphism σ_ℓ will therefore be fixed by all of $\operatorname{Aut} G_s$ whenever the images of χ_ℓ and χ_r are not isomorphic.

8A2. *Additional assumptions.* Assume therefore that s_2 and s_3 are *multiplicatively independent*. Then the free abelian group

$$\operatorname{im} \chi_r = \ln \mathrm{gp}(\{s_2, s_3\}) = \mathbb{Z} \ln s_2 + \mathbb{Z} \ln s_3.$$

has rank 2.

Consider now two triples s and s' where $s_2' = s_2$ and where both pairs $\{s_2, s_3\}$ and $\{s_2, s_3'\}$ are multiplicatively independent. Suppose there exists an isomorphism $\beta : G_s \xrightarrow{\sim} G_{s'}$. Then β induces a homeomorphism

$$\beta^* : S(G_{s'}) \xrightarrow{\sim} S(G_s)$$
$$\beta^*\big(\{[\chi_\ell'], [\chi_r']\}\big) = \{[\chi_\ell], [\chi_r]\}.$$

The ranks of the involved characters imply that $\beta^*[\chi_r'] = [\chi_r]$; so there exists a positive real number u with $\chi_r' \circ \beta = u \cdot \chi_r$. It follows that $\operatorname{im} \chi_r' = u \cdot \operatorname{im} \chi_r$ or, equivalently, that

$$\mathbb{Z}(\ln s_3') + \mathbb{Z}(\ln s_2) = u \cdot \big(\mathbb{Z}(\ln s_3) + \mathbb{Z}(\ln s_2)\big).$$

This equality amounts to saying that there exists a matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \ln s_3' \\ \ln s_2 \end{pmatrix} = u \cdot T \cdot \begin{pmatrix} \ln s_3 \\ \ln s_2 \end{pmatrix} = u \cdot \begin{pmatrix} a \cdot \ln s_3 + b \cdot \ln s_2 \\ c \cdot \ln s_3 + d \cdot \ln s_2 \end{pmatrix}.$$

It follows that

$$\frac{\ln s_3'}{\ln s_2} = \frac{a(\ln s_3/\ln s_2) + b}{c(\ln s_3/\ln s_2) + d};$$

alternatively put, the numbers $\ln s_3$, $\ln s_3'$ lie in the same orbit of the group

$$H_{s_2} = \begin{pmatrix} \ln s_2 & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{GL}(2, \mathbb{Z}) \cdot \begin{pmatrix} \ln s_2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{8-2}$$

acting on the extended real line $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations.

8A3. *Consequences.* It is now easy to exhibit a collection of groups $\mathcal{G}$ that enjoy the properties stated at the beginning of Section 8A. Choose first a number $s_1 > 1$; for instance $s_1 = 2$, and select s_2 so that $\ln s_2$ is rational, for instance $s_2 = \exp 1$. The group H_{s_2} is then a subgroup of $\mathrm{GL}(2, \mathbb{Q})$; it acts on $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations. The set $\mathbb{Q} \cup \{\infty\}$ is an orbit; all other orbits are made up of irrational numbers. Use the axiom of choice to find a set of representative $\mathcal{T}$ of the orbits of H_{s_2} contained in $\mathbb{R} \smallsetminus \mathbb{Q}$. For every $t \in \mathcal{T}$ the numbers $\ln s_2$ and t are then $\mathbb{Q}$-linearly independent, and hence s_2 and $\exp t$ are multiplicatively independent. Since $\mathbb{R} \smallsetminus \mathbb{Q}$ has the cardinality of the continuum and H_{s_2} is countable, the set $\mathcal{T}$ likewise has the cardinality of the continuum. The collection

$$\mathcal{G} = \{G_{(s_1, s_2, \exp t)} \mid t \in \mathcal{T}\} \tag{8-3}$$

therefore enjoys properties (i) through (iv) stated at the beginning of Section 8A.

8B. ***Some unexpected isomorphisms.*** Let t_1, t_2 be distinct irrational numbers and consider the groups $G_1 = G_{(2, \exp 1, \exp t_1)}$ and $G_2 = G_{(2, \exp 1, \exp t_2)}$. We don't know under which conditions on t_1 and t_2 the groups G_1 and G_2 are isomorphic. In the construction of the collection $\mathcal{G}$, carried out in Section 8A, we proceeded therefore in a very cautious manner and required that distinct elements in the parameter space $\mathcal{T}$ fail to satisfy a certain condition. The question now arises whether this approach is overly pessimistic. The next example indicates that caution may have been appropriate. We begin with a simple, but surprising, lemma.[18]

[18] Strebel got word of this result in discussions with Matt Brin and Matt Zaremsky.

Lemma 8.1. *Suppose that G is a subgroup of $\mathrm{PL}_o([a,d])$ generated by two PL-homeomorphisms f and g with the following properties*:

(i) $\operatorname{supp} f = \,]a,c[$ *and* $f(t) < t$ *for* $t \in \operatorname{supp} f$,

(ii) $\operatorname{supp} g = \,]b,d[$ *and* $g(t) < t$ *for* $t \in \operatorname{supp} g$,

(iii) $a < b < c < d$ *and* $f(g(c)) \leq b$.

Then G is isomorphic to Thompson's group F.

Proof. Set $h = f \circ g$ and note that $h(t) < t$ for every $t \in \,]a,d[\,$. Property (iii) then implies that $h(c) \leq b$ and so the supports of g and that of ${}^{h}f = h \circ f \circ h^{-1}$ are disjoint, as are the supports of g and that of ${}^{h^2}f$. The first fact implies that g commutes with ${}^{h}f$ and leads to the chain of equations

$$ {}^{h\circ h}f = {}^{f}\big({}^{g\circ h}f\big) = {}^{f}\big({}^{h}f\big) = {}^{f\circ h}f. \tag{8-4}$$

The second fact leads to the equations

$$ {}^{h\circ h^2}f = {}^{f}\big({}^{g\circ h^2}f\big) = {}^{f}\big({}^{h^2}f\big) = {}^{f\circ h^2}f. \tag{8-5}$$

Thompson's group F, on the other hand, has the presentation

$$\big\langle x, x_1 \;\big|\; {}^{x^2}x_1 = {}^{x_1 x}x_1,\ {}^{x^3}x_1 = {}^{x_1 x^2}x_1 \big\rangle;$$

see, e.g., [Bieri and Strebel 2016, Examples D15.11]. The assignments $x \mapsto h$, $x_1 \mapsto f$ extend therefore to an epimorphism $\rho : F \twoheadrightarrow G$. As the derived group of F is simple (see, e.g., [Cannon et al. 1996, Theorem 4.5]) and as G is nonabelian, ρ must be injective, hence an isomorphism, and so the proof is complete. □

Our next result shows that the assumptions of the previous lemma can be satisfied by PL-homeomorphisms with preassigned values for the slopes in the end points.

Lemma 8.2. *Let s_f, s_g be positive reals with $s_f < 1 < s_g$ and let a, b, c, d be real numbers with $a < b < c < d$. Then there exist PL-homeomorphisms f and g that satisfy properties* (i) *through* (iii) *listed in Lemma 8.1 and, in addition*,

(iv) $f'(a) = s_f$ *and* $g'(d) = s_g$.

Proof. The generators f and g will both be affine interpolations of 5 interpolation points. To define them fix numbers t_1, t_2, t_3, t_4 so that

$$a < t_1 < b < t_2 \leq t_3 < c < t_4 < d.$$

Next choose $t_0 \in \,]a, t_1[$ so that $(t_0 - a)/(t_1 - a) = s_f$. Then $a < t_1 < t_3 < c < d$ and $a < t_0 < b < c < d$ and so the affine interpolation, given by the 5 points

$$(a,a), \quad (t_1,t_0), \quad (t_3,b), \quad (c,c), \quad (d,d),$$

exists and is an increasing PL-homeomorphism, say f, with $f'(a) = s_f$. Next, there exists a number $t_5 \in \,]t_4, d[$ so that $(d - t_4)/(d - t_5) = s_g$. Then $a < b < c < t_5 < d$ and $a < b < t_2 < t_4 < d$ and so the affine interpolation, given by the 5 points

$$(a, a), \quad (b, b), \quad (c, t_2), \quad (t_5, t_4), \quad (d, d),$$

exists and is an increasing PL-homeomorphism, say g; the definition of t_5 implies, in addition, that $g'(d) = s_g$. Finally, $f(g(c)) = f(t_2) \le f(t_3) = b$. $\square$

Remarks 8.3. (a) In the statement of Lemma 8.2 the slopes s_f and s_g have been chosen so that $s_f < 1 < s_g$. This requirement can be weakened to $s_f \neq 1$ and $s_g \neq 1$; indeed the four pairs $\{f, g\}$, $\{f, g^{-1}\}$ and $\{f^{-1}, g\}$, $\{f^{-1}, g^{-1}\}$ generate the same group.

(b) The generators f_s, g_s and h_s of the groups G_s, constructed in Section 8A, are simpler then those used in Lemma 8.2 in that they are defined by affine interpolations of 3 rather than of 5 points. But a variant of the Lemma 8.2 holds even in this more restricted setup.

Suppose $s_1 = s_2 = 2$ and $s_3 \ge 2$. The function f_s is then given by the formula

$$f_s(t) = \begin{cases} \frac{1}{2}t, & \text{if } 0 \le t \le \frac{1}{2}, \\ 2\left(t - \frac{1}{2}\right) + \frac{1}{4}, & \text{if } \frac{1}{2} \le t \le \frac{3}{4}, \end{cases} \tag{8-6}$$

and it is the identity outside of $]0, \frac{3}{4}[$, while g_s is defined by

$$g_s(t) = \begin{cases} \frac{1}{2}\left(t - \frac{1}{4}\right) + \frac{1}{4}, & \text{if } \frac{1}{4} \le t \le \frac{3}{4}, \\ 2\left(t - \frac{3}{4}\right) + \frac{1}{2}, & \text{if } \frac{3}{4} \le t \le 1, \end{cases} \tag{8-7}$$

and it is the identity outside of $]\frac{1}{4}, 1[$. The function h_s, finally, is defined by

$$h_s(t) = \begin{cases} \frac{1}{s_3}\left(t - \frac{1}{4}\right) + \frac{1}{4}, & \text{if } \frac{1}{4} \le t \le \frac{3s_3}{4(s_3+1)} + \frac{1}{4}, \\ s_3\left(t - \frac{3s_3}{4(s_3+1)} - \frac{1}{4}\right) + \frac{3}{4(s_3+1)} + \frac{1}{4}, & \text{if } \frac{3s_3}{4(s_3+1)} + \frac{1}{4} \le t \le 1, \end{cases} \tag{8-8}$$

and is the identity outside of $]\frac{1}{4}, 1[$. The function g_s is not differentiable at $\frac{1}{4}$, $\frac{3}{4}$, and 1, while the function h_s has singularities at $\frac{1}{4}$, $t_* = \frac{3s_3}{4(s_3+1)} + \frac{1}{4}$, and 1. Since $s_3 \ge s_2 = 2$, the inequality $t_* \ge \frac{3}{4}$ holds, as one verifies easily. The calculation

$$f_s\left(h_s\left(\tfrac{3}{4}\right)\right) = f_s\left(\tfrac{1}{s_3}\left(\tfrac{3}{4} - \tfrac{1}{4}\right) + \tfrac{1}{4}\right) = f_s\left(\tfrac{1}{2s_3} + \tfrac{1}{4}\right) \le f_s\left(\tfrac{1}{4} + \tfrac{1}{4}\right) \le \tfrac{1}{4}.$$

then shows that the functions f_s and h_s fulfill the assumptions imposed on the functions f and g in Lemma 8.1. It follows that the groups $\mathrm{gp}(f_s, h_s)$ are isomorphic to each other for every $s_3 \ge s_2 = 2$.

8C. *A criterion.* The groups G_s studied in Section 8A are generated by 3 elements; in addition the image of χ_ℓ is infinite cyclic and that of χ_r is free abelian of rank 2. Any isomorphism $\beta : G_s \xrightarrow{\sim} G_{s'}$ between two such groups must therefore induce an homeomorphism $\beta^* : S(G_{s'}) \xrightarrow{\sim} S(G_s)$ with $\beta^*([\chi'_r]) = [\chi_r]$. This consequence amounts to say that there exists a positive real number u so that $\chi'_r \circ \beta = u \cdot \chi_r$ and this new condition implies the equality

$$\operatorname{im} \chi'_r = u \cdot \operatorname{im} \chi_r. \tag{8-9}$$

In Section 8A we did not study this condition in general; we dealt only with the special case where

$$\operatorname{im} \chi_r = \mathbb{Z}(\ln s_3) + \mathbb{Z}(\ln s_2) \quad \text{and} \quad \operatorname{im} \chi'_r = \mathbb{Z}(\ln s'_3) + \mathbb{Z}(\ln s_2)$$

and exploited then the fact that, in this particular case, condition (8-9) involves basically only the two numbers $\ln s_3$ and $\ln s'_3$. In this final section we shall investigate another special case. It is reminiscent of a situation considered in Section 6E.

Let B_1 and B_2 be finitely generated subgroups of $\mathbb{R}_{\mathrm{add}}$ and suppose there exists a positive real number u with $B_2 = u \cdot B_1$. If B_2 coincides with B_1, then u is a unit of B_1 and the results of Section 6E apply. They show, in particular, that $u = 1$ whenever B is the image under ln of a subgroup P of $\mathbb{R}^\times_{>0}$ that is generated by finitely many algebraic numbers. The proof of this consequence relies on Theorem 6.11, the Gelfond–Schneider theorem. Below we give an analogue of this criterion, but dealing with the equation $B_2 = u \cdot B_1$. In the proof, both the Gelfond–Schneider theorem and the Siegel–Lang theorem will be used.

Lemma 8.4. *Let P_1 and P_2 be subgroups $\mathbb{Q}^\times_{>0}$ and set $B_1 = \ln P_1$ and $B_2 = \ln P_2$. Suppose there exists a prime number π that occurs with nonzero power in the factorization of an element in P_1, but not in that of an element of P_2. If the rank of B_1 is at least 3, then B_2 is distinct from $u \cdot B_1$ for every positive real number u.*

Proof. Let $p_1 \in P_1$ be an element with a prime factorization that involves the prime π, and let u be a positive real number. Assume first that $u \in \mathbb{Q}$. Then π occurs in the prime factorization of p_1^u, so $q_1^u \notin P_2$, and thus $u \cdot B_1 = u \ln P_1 \neq \ln P_2 = B_2$. Now suppose that u is irrational and that $u \cdot \ln p_1 \in B_2$. There exists then a rational number $p_2 \in P_2$ with $u = \ln p_2 / \ln p_1$ and so u is transcendental by the Gelfond–Schneider theorem. Choose, finally, three $\mathbb{Q}$-linearly independent elements z_1, z_2 and z_3 in B_1 (this is possible as the rank of B_1 is at least 3) and consider the six numbers

$$\exp(1 \cdot z_j) \text{ with } j = 1, 2, 3, \quad \text{and} \quad \exp(u \cdot z_j) \text{ with } j = 1, 2, 3.$$

The first three of them are in P_1, and hence rational. As the subsets $\{1, u\}$ and $\{z_1, z_2, z_3\}$ are both linearly independent over $\mathbb{Q}$, Theorem 6.13 implies therefore that at least one the remaining three numbers, say $\exp(u \cdot z_{j_*})$, is transcendental. This number is therefore outside of P_2 and so $u \cdot z_{j_*} \in uB_1 \smallsetminus B_2$. $\square$

We end with an application of the preceding lemma.

Example 8.5. Given a nonempty set of prime numbers $\mathcal{P}$, let $G_{\mathcal{P}}$ be a subgroup of $\mathrm{PL}_o([0, 1])$ generated by a set $\{f_p, g_p \mid p \in \mathcal{P}\}$ of elements that satisfy the conditions

(i) $\sigma_\ell(f_p) = p,\ \sigma_\ell(g_p) = 1$ and $\sigma_r(f_p) = 1,\ \sigma_\ell(g_p) = 1$ for every $p \in \mathcal{P}$;

(ii) the union of the supports of the generators f_p and g_p is $]0, 1[$.

The group $G_{\mathcal{P}}$ admits then an epimorphism $\psi : G_{\mathcal{P}} \twoheadrightarrow \mathrm{gp}(\mathcal{P})$ that is fixed by every automorphism of $G_{\mathcal{P}}$ (use Corollary 6.10). Moreover, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct sets of primes of cardinality at least 3, the groups $G_{\mathcal{P}_1}$ and $G_{\mathcal{P}_2}$ are not isomorphic in view of Lemma 8.4 and the considerations at the beginning of Section 8C.

Acknowledgments

Gonçalves has been partially supported by the Fapesp project "Topologia Algébrica, Geométrica e Diferencial-2012/24454-8". Sankaran acknowledges financial support from the Department of Atomic Energy, Government of India, under a XII plan project. The present work was initiated during the visit of Sankaran to the University of São Paulo in August 2012. He thanks Gonçalves for the invitation and the warm hospitality. Strebel is grateful to M. Brin and M. Zaremsky for numerous, helpful discussions. Last, not least, the three authors thank E. Troitsky for pointing out reference [Fel'shtyn and Troitsky 2015] and the referee for her or his suggestions.

References

[Baumslag 1974] G. Baumslag, "Finitely presented metabelian groups", pp. 65–74 in *Proceedings of the Second International Conference on the Theory of Groups* (Canberra, 1973), Lecture Notes in Math. **372**, Springer, Berlin, 1974. MR Zbl

[Belk and Brown 2005] J. M. Belk and K. S. Brown, "Forest diagrams for elements of Thompson's group *F*", *Internat. J. Algebra Comput.* **15**:5-6 (2005), 815–850. MR Zbl

[Bieri and Strebel 1980] R. Bieri and R. Strebel, "Valuations and finitely presented metabelian groups", *Proc. London Math. Soc.* (3) **41**:3 (1980), 439–464. MR Zbl

[Bieri and Strebel 1985] R. Bieri and R. Strebel, "On groups of PL-homeomorphisms of the real line", preprint, 1985.

[Bieri and Strebel 2016] R. Bieri and R. Strebel, *On groups of PL-homeomorphisms of the real line*, Mathematical Surveys and Monographs **215**, American Mathematical Society, Providence, RI, 2016. MR

[Bieri et al. 1987] R. Bieri, W. D. Neumann, and R. Strebel, "A geometric invariant of discrete groups", *Invent. Math.* **90**:3 (1987), 451–477. MR Zbl

[Bleak et al. 2008] C. Bleak, A. Fel'shtyn, and D. L. Gonçalves, "Twisted conjugacy classes in R. Thompson's group *F*", *Pacific J. Math.* **238**:1 (2008), 1–6. MR Zbl

[Brin 1996] M. G. Brin, "The chameleon groups of Richard J. Thompson: automorphisms and dynamics", *Inst. Hautes Études Sci. Publ. Math.* 84 (1996), 5–33. MR Zbl

[Brin and Guzmán 1998] M. G. Brin and F. Guzmán, "Automorphisms of generalized Thompson groups", *J. Algebra* **203**:1 (1998), 285–348. MR Zbl

[Brin and Squier 1985] M. G. Brin and C. C. Squier, "Groups of piecewise linear homeomorphisms of the real line", *Invent. Math.* **79**:3 (1985), 485–498. MR Zbl

[Brown 1987a] K. S. Brown, "Finiteness properties of groups", *J. Pure Appl. Algebra* **44**:1-3 (1987), 45–75. MR Zbl

[Brown 1987b] K. S. Brown, "Trees, valuations, and the Bieri–Neumann–Strebel invariant", *Invent. Math.* **90**:3 (1987), 479–504. MR Zbl

[Bux 2004] K.-U. Bux, "Finiteness properties of soluble arithmetic groups over global function fields", *Geom. Topol.* **8** (2004), 611–644. MR Zbl

[Cannon et al. 1996] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Introductory notes on Richard Thompson's groups", *Enseign. Math.* (2) **42**:3-4 (1996), 215–256. MR Zbl

[Cleary 1995] S. Cleary, "Groups of piecewise-linear homeomorphisms with irrational slopes", *Rocky Mountain J. Math.* **25**:3 (1995), 935–955. MR Zbl

[Cleary 2000] S. Cleary, "Regular subdivision in $\mathbb{Z}[(1+\sqrt{5})/2]$", *Illinois J. Math.* **44**:3 (2000), 453–464. MR Zbl

[Fel'shtin and Troitsky 2012] A. Fel'shtin and E. Troitsky, "Twisted conjugacy classes in residually finite groups", preprint, 2012. arXiv

[Fel'shtyn and Troitsky 2015] A. Fel'shtyn and E. Troitsky, "Aspects of the property R_∞", *J. Group Theory* **18**:6 (2015), 1021–1034. MR Zbl

[Gonçalves and Kochloukova 2010] D. Gonçalves and D. H. Kochloukova, "Sigma theory and twisted conjugacy classes", *Pacific J. Math.* **247**:2 (2010), 335–352. MR Zbl

[Gonçalves and Sankaran 2016] D. Gonçalves and P. Sankaran, "Sigma theory and twisted conjugacy, II: Houghton groups and pure symmetric automorphism groups", *Pacific J. Math.* **280**:2 (2016), 349–369. MR Zbl

[Gonçalves and Wong 2003] D. Gonçalves and P. Wong, "Twisted conjugacy classes in exponential growth groups", *Bull. London Math. Soc.* **35**:2 (2003), 261–268. MR Zbl

[Gonçalves and Wong 2009] D. Gonçalves and P. Wong, "Twisted conjugacy classes in nilpotent groups", *J. Reine Angew. Math.* **633** (2009), 11–27. MR Zbl

[Lang 1966] S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, Reading, MA, 1966. MR Zbl

[Lang 1971] S. Lang, "Transcendental numbers and diophantine approximations", *Bull. Amer. Math. Soc.* **77**:5 (1971), 635–677. MR Zbl

[Levitt and Lustig 2000] G. Levitt and M. Lustig, "Most automorphisms of a hyperbolic group have very simple dynamics", *Ann. Sci. École Norm. Sup.* (4) **33**:4 (2000), 507–517. MR Zbl

[Lyndon and Schupp 1977] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **89**, Springer, Berlin, 1977. MR Zbl

[Mubeena and Sankaran 2014a] T. Mubeena and P. Sankaran, "Twisted conjugacy classes in abelian extensions of certain linear groups", *Canad. Math. Bull.* **57**:1 (2014), 132–140. MR Zbl

[Mubeena and Sankaran 2014b] T. Mubeena and P. Sankaran, "Twisted conjugacy classes in lattices in semisimple Lie groups", *Transform. Groups* **19**:1 (2014), 159–169. MR Zbl

[Niven 1956] I. Niven, *Irrational numbers*, The Carus Mathematical Monographs **11**, Math. Assoc. Amer., New York, 1956. MR Zbl

[Renz 1988] B. Renz, *Geometrische invarianten und endlichkeitseigenschaften von gruppen*, Ph.D. thesis, Goethe-Universität Frankfurt am Main, 1988, https://homepages.thm.de/~hg11260/mat/diss.pdf.

[Stein 1992] M. Stein, "Groups of piecewise linear homeomorphisms", *Trans. Amer. Math. Soc.* **332**:2 (1992), 477–514. MR Zbl

[Strebel 2013] R. Strebel, "Notes on the Sigma-invariants", preprint, 2013. arXiv

[Strebel 2015] R. Strebel, "Sigma 1 of PL-homeomorphism groups", preprint, 2015. arXiv

[Zaremsky 2016] M. C. B. Zaremsky, "On the Σ-invariants of generalized Thompson groups and Houghton groups", *Int. Math. Res. Not.* (online publication August 2016).

Received December 12, 2015. Revised June 28, 2016.

DACIBERG LIMA GONÇALVES
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA DA UNIVERSIDADE DE SÃO PAULO
DEPARTAMENTO DE MATEMÁTICA
RUA DO MATÃO, 1010 CEP 05508-090
SÃO PAULO-SP
BRAZIL

dlgoncal@ime.usp.br

PARAMESWARAN SANKARAN
THE INSTITUTE OF MATHEMATICAL SCIENCES
CIT CAMPUS
CHENNAI 600113
INDIA

sankaran@imsc.res.in

RALPH STREBEL
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE FRIBOURG
CHEMIN DU MUSÉE 23
CH-1700 FRIBOURG
SWITZERLAND

ralph.strebel@unifr.ch

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.159

BERNSTEIN-TYPE THEOREMS FOR SPACELIKE STATIONARY GRAPHS IN MINKOWSKI SPACES

XIANG MA, PENG WANG AND LING YANG

For entire spacelike stationary 2-dimensional graphs in Minkowski spaces, we establish Bernstein-type theorems under specific boundedness assumptions either on the W-function or on the total (Gaussian) curvature. These conclusions imply the classical Bernstein theorem for minimal surfaces in $\mathbb{R}^3$ and Calabi's theorem for spacelike maximal surfaces in $\mathbb{R}^3_1$.

1. Introduction

The classical Bernstein theorem [1915] says that any entire minimal graph in $\mathbb{R}^3$ has to be an affine plane. In other words, suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 0. \tag{1-1}$$

Then f has to be affine linear. This conclusion is generally not true in the higher-codimensional case. The simplest counterexample is the minimal graph $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{C}\} \subset \mathbb{R}^4$ of an arbitrary nonlinear holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

To find a suitable generalization, usually we have to add some boundedness assumptions on the growth rate of the function f. Chern and Osserman [1967] obtained one such Bernstein-type theorem as follows. Suppose that $f = (f_1, \dots, f_m)$ is a smooth vector-valued function from $\mathbb{R}^2$ to $\mathbb{R}^m$. If $M = \operatorname{graph} f$ is a minimal graph, and

$$W := \left[\det\left(\delta_{ij} + \sum_{1\le\alpha\le m} \frac{\partial f_\alpha}{\partial x_i}\frac{\partial f_\alpha}{\partial x_j}\right)\right]^{1/2} \tag{1-2}$$

is uniformly bounded, then M has to be an affine plane.

This W-function is a significant quantity for various reasons.

Ma is supported by NSFC Project 11471021; Wang is supported by NSFC Projects 11571255 and 11471078; Yang is supported by NSFC Project 11471078.
MSC2010: 30F15, 53C24, 58J05.
Keywords: Bernstein problems, spacelike, stationary, entire graph.

For any $f:\mathbb{R}^2\to\mathbb{R}^m$, denote the metric on graph(f) as $g=g_{ij}\,dx_i\,dx_j$ under the global coordinate chart $x=(x_1,x_2)\mapsto(x,\,f(x))\in\text{graph}\,f$. Then the area element is given by $W\,dx_1\wedge dx_2$. Thus W is a geometric measure of the area growth of the graph of f.

Secondly, Chern and Osserman's theorem can be stated in the language of PDEs as below. Namely, the entire solution to the PDE system

$$\begin{aligned}\sum_{1\le i\le 2}\frac{\partial}{\partial x_i}(Wg^{ij})&=0,\qquad j=1,2,\\ \sum_{1\le i,j\le 2}\frac{\partial}{\partial x_i}\Big(Wg^{ij}\frac{\partial f_\alpha}{\partial x_j}\Big)&=0,\qquad \alpha=1,\dots,m\end{aligned}\tag{1-3}$$

has to be affine linear provided that $W\le C$ for a positive constant C, where

$$(g_{ij}):=I_2+J_f^T E J_f\tag{1-4}$$

(I_2 and E denote the identity matrices of size 2 and m separately and J_f is the Jacobian matrix of f), $(g^{ij}):=(g_{ij})^{-1}$ and $W=\det(g_{ij})^{1/2}$. A key point from the analytic viewpoint is that the boundedness of W ensures that (1-3) is a uniformly elliptic PDE system.

For more work on the generalization of Chern and Osserman's theorem in relation to the W-function, see [Barbosa 1979], [Fischer-Colbrie 1980], [Jost et al. 2014; 2015].

Now we consider entire spacelike stationary graphs in Minkowski spaces. They too correspond to solutions to (1-3), the differences being that $f=(f_1,\dots,f_m)$ is now from $\mathbb{R}^2$ to m-dimensional Minkowski space $\mathbb{R}^m_1$, and the E appearing in (1-4) should be replaced by the Minkowski inner product matrix $\operatorname{diag}(1,1,\dots,1,-1)$. Here we need to assume that (g_{ij}) is positive definite everywhere.

When $m=1$, M becomes a spacelike maximal graph in $\mathbb{R}^3_1$, which has to be an affine plane. This is a well-known Bernstein-type result by E. Calabi [1970]. But for higher-codimensional cases, the Bernstein-type result fails to be true even if the W-function is uniformly bounded. Such a counterexample, which can be found in [Ma et al. 2013], is given by the function

$$f(x_1,x_2)=\Big(2\sinh(x_1)\cos\big(-\tfrac{\sqrt{2}}{2}x_2\big),\,2\cosh(x_1)\cos\big(-\tfrac{\sqrt{2}}{2}x_2\big)\Big).$$

So it is a more subtle problem about the value distribution of the W-function for entire spacelike stationary graphs in Minkowski spaces. This is the main topic of the present paper.

As the first step, we generalize Osserman's result [1969, §5] to entire spacelike stationary graphs in the Minkowski space. They are still conformally equivalent to the complex plane (see Theorem 3.1), and have an explicit simple representation formula. Based on these formulas, we establish the following results:

- Let M be an entire spacelike stationary graph in $\mathbb{R}^4_1$. Then the W-function is either constant, or takes each value in $[r^{-1}, r]$ infinitely often, where r can be any positive number strictly bigger than 1. Moreover, W is a constant if and only if M is a flat surface (see Theorem 4.1).
- For any entire spacelike stationary graph M in $\mathbb{R}^4_1$, if $W \le 1$ (or $W \ge 1$) always holds true on M, then M has to be flat (see Corollary 4.2). Note that Calabi's theorem [1970] and the classical Bernstein theorem [1915] can easily be deduced from the above two conclusions, respectively.
- For any entire spacelike stationary graph M in $\mathbb{R}^n_1$ ($n \ge 4$), if $W \le 1$, then M must be flat (see Theorem 5.1). (On the contrary, the same conclusion does not necessarily hold true in the case $W \ge 1$; see Proposition 5.2.)

Another measure of the complexity of a complete stationary surface is its total Gaussian curvature $\int_M |K|\, dM$. This is closely related with its end behavior at infinity; see the generalized Jorge–Meeks formula in [Ma et al. 2013]. Using the Weierstrass representation formula given in the same work, one can compute the integral of the Gauss curvature and the normal curvature of an arbitrary spacelike stationary surface in $\mathbb{R}^4_1$. A Bernstein-type theorem (Theorem 6.1) follows immediately, which states that an entire spacelike stationary graph in $\mathbb{R}^4_1$ has to be flat, provided that $\int_M |K|\, dM < \infty$. (This result cannot be generalized to higher-codimensional cases.)

2. Entire graphs in Minkowski spaces and the W-function

Let $\mathbb{R}^m_1$ denote the m-dimensional Minkowski space. The Minkowski inner product of any $\boldsymbol{u} = (u_1, \dots, u_{m-1}, u_m)$ and $\boldsymbol{v} = (v_1, \dots, v_{m-1}, v_m) \in \mathbb{R}^m_1$ is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \cdots + u_{m-1} v_{m-1} - u_m v_m. \tag{2-1}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^m_1$

$$(x_1, x_2) \mapsto f(x_1, x_2) = (f_1(x_1, x_2), \dots, f_m(x_1, x_2)) \tag{2-2}$$

be a smooth vector-valued function. As in §3 of [Osserman 1969], we introduce the vector notation

$$p := \frac{\partial f}{\partial x_1}, \qquad q := \frac{\partial f}{\partial x_2}. \tag{2-3}$$

Let $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ be the entire graph in $\mathbb{R}^{2+m}_1$ generated by f. Then the metric on M is

$$g = g_{ij}\, dx_i\, dx_j, \tag{2-4}$$

with

$$g_{11} = 1 + \langle p, p\rangle, \quad g_{22} = 1 + \langle q, q\rangle, \quad g_{12} = g_{21} = \langle p, q\rangle. \tag{2-5}$$

According to the properties of positive definite matrices, M is a spacelike surface if and only if $1 + \langle p, p\rangle > 0$ and $\det(g_{ij}) > 0$. Hence

$$W = \det(g_{ij})^{1/2} > 0 \tag{2-6}$$

for any spacelike graph.

Denote by $\mathcal{P}_0$ the orthogonal projection of $\mathbb{R}^{2+m}_1$ onto $\mathbb{R}^2$. Then $w := W^{-1}$ is equivalent to the Jacobian determinant of $\mathcal{P}_0|_M$. Thus $W \le 1$ (resp., $\equiv 1$, ≥ 1) is equivalent to saying that $\mathcal{P}_0|_M$ is an area-increasing (resp., area-preserving, area-decreasing) map.

For entire graphs in Euclidean space, it is well known that the orthogonal projection onto the coordinate plane is a length-decreasing map, which becomes an isometry if and only if the graph is parallel to the coordinate plane. Therefore every entire graph in Euclidean space must be complete. But the following examples show these properties cannot be generalized to entire graphs in Minkowski spaces.

Examples.

- Let $\boldsymbol{y}_0$ be a nonzero lightlike vector in $\mathbb{R}^m_1$, h be a smooth real-valued function on $\mathbb{R}^2$ and $f := h\boldsymbol{y}_0$. Then

$$p = \frac{\partial h}{\partial x_1}\boldsymbol{y}_0 \quad \text{and} \quad q = \frac{\partial h}{\partial x_2}\boldsymbol{y}_0,$$

and hence $g_{ij} = \delta_{ij}$, which implies the projection of $M = \operatorname{graph} f$ onto $\mathbb{R}^2$ is an isometry, but M cannot be an affine plane of $\mathbb{R}^{2+m}_1$ whenever h is nonlinear.

- Let $t \in \mathbb{R} \mapsto \theta(t) \in (-\pi/2, \pi/2)$ be a smooth odd function which satisfies $\lim_{t\to+\infty} \theta(t) = \pi/2$ and $\pi/2 - \theta(t) = O(t^{-2})$. Denote

$$h(t) := \int_0^t \sin\theta(t)\, dt.$$

Then h is a smooth even function on $\mathbb{R}$. Define

$$f(x_1, x_2) = (0, \dots, 0, h(r)) \qquad \left(r = \sqrt{x_1^2 + x_2^2}\right).$$

Then $p = \partial f/\partial x_1 = (0, \dots, 0, h'(r)x_1/r)$, $q = \partial f/\partial x_2 = (0, \dots, 0, h'(r)x_2/r)$ and hence

$$g_{11} = 1 + \langle p, p\rangle = 1 - \frac{h'(r)^2 x_1^2}{r^2} \geq 1 - h'(r)^2 = \cos^2\theta(t) > 0,$$

$$\det(g_{ij}) = \det\begin{pmatrix} 1 - \dfrac{h'(r)^2 x_1^2}{r^2} & -\dfrac{h'(r)^2 x_1 x_2}{r^2} \\ -\dfrac{h'(r)^2 x_1 x_2}{r^2} & 1 - \dfrac{h'(r)^2 x_2^2}{r^2} \end{pmatrix} = 1 - h'(r)^2 > 0.$$

Therefore $M = \operatorname{graph} f$ is an entire spacelike graph. Define $\gamma : \mathbb{R} \to \mathbb{R}^3$ by $\gamma(t) = (t, 0, f(t, 0))$. Then γ is a smooth curve in M tending to infinity. Since $f(t, 0) = (0, \dots, 0, h(t))$,

$$L(\gamma) = \int_{-\infty}^{\infty} \sqrt{1 - h'(t)^2}\, dt = \int_{-\infty}^{\infty} \cos\theta(t)\, dt.$$

But $\cos\theta(t) \sim \pi/2 - |\theta(t)| \sim |t|^{-2}$ when $t \to \infty$. Therefore $L(\gamma) < \infty$ and hence M cannot be complete.

3. Isothermal parameters of spacelike stationary graphs

Let $\boldsymbol{x} : M \to \mathbb{R}_1^{2+m}$ be a spacelike surface in Minkowski space. If the mean curvature vector field $\boldsymbol{H}$ vanishes everywhere, then M is said to be *stationary*. M is stationary if and only if the restriction of any coordinate function on M is harmonic. Namely, $\Delta x_l \equiv 0$ for each $1 \leq l \leq 2+m$, with Δ the Laplace–Beltrami operator with respect to the induced metric on M; see [Ma et al. 2013]. Now we additionally assume M to be an entire graph over $\mathbb{R}^2$. More precisely, there exists $f : \mathbb{R}^2 \to \mathbb{R}_1^m$, such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$. The denotation of p, q, g_{ij}, W is the same as in Section 2. For an arbitrary smooth function F on M,

$$\Delta F = W^{-1}\partial_i(W g^{ij}\partial_j F), \tag{3-1}$$

where

$$(g^{ij}) = (g_{ij})^{-1} = W^{-2}\begin{pmatrix} 1 + \langle q, q\rangle & -\langle p, q\rangle \\ -\langle p, q\rangle & 1 + \langle p, p\rangle \end{pmatrix}. \tag{3-2}$$

The stationarity of M implies x_1 and x_2 are both harmonic functions on M, hence

$$\begin{aligned} 0 = W\Delta x_1 &= \partial_i(W g^{ij}\partial_j x_1) = \partial_i(W g^{ij}\delta_{1j}) = \partial_i(W g^{i1}) \\ &= \frac{\partial}{\partial x_1}\Big(\frac{1 + \langle q, q\rangle}{W}\Big) - \frac{\partial}{\partial x_2}\Big(\frac{\langle p, q\rangle}{W}\Big), \end{aligned} \tag{3-3}$$

and similarly,

$$0 = W\Delta x_2 = \partial_i(Wg^{i2}) = -\frac{\partial}{\partial x_1}\Big(\frac{\langle p, q\rangle}{W}\Big) + \frac{\partial}{\partial x_2}\Big(\frac{1+\langle p, p\rangle}{W}\Big). \tag{3-4}$$

The above two equations imply the existence of smooth functions ξ_1 and ξ_2 such that

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1+\langle p, p\rangle}{W}, \quad \frac{\partial \xi_1}{\partial x_2} = \frac{\langle p, q\rangle}{W}, \quad \frac{\partial \xi_2}{\partial x_1} = \frac{\langle p, q\rangle}{W}, \quad \frac{\partial \xi_2}{\partial x_2} = \frac{1+\langle q, q\rangle}{W}. \tag{3-5}$$

As in §5 of [Osserman 1969], one can define the Lewy's transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L : (x_1, x_2) \mapsto (\eta_1, \eta_2)$ by

$$\eta_i = x_i + \xi_i(x_1, x_2), \quad i = 1, 2. \tag{3-6}$$

Since the Jacobi matrix of L,

$$J_L = I_2 + \left(\frac{\partial \xi_i}{\partial x_j}\right) = I_2 + W^{-1}(g_{ij}), \tag{3-7}$$

is positive definite, L is a local diffeomorphism. Again based on the fact that $\big(\partial\xi_i/\partial x_j\big)$ is positive definite, one can proceed as in [Lewy 1937] or §5 of [Osserman 1969] to show that L is length-increasing, thus L is injective. Let Ω be the image of L. Then Ω is open. If $\Omega \neq \mathbb{R}^2$, take η in the complement of Ω that is nearest to $L(0)$, and find a sequence of points $\{\eta^{(k)} : k \in \mathbb{Z}^+\}$ such that $|\eta^{(k)} - L(0)| < |\eta - L(0)|$ and $\lim_{k\to\infty} \eta^{(k)} = \eta$. Then there exists $x^{(k)} \in \mathbb{R}^2$ such that $\eta^{(k)} = L(x^{(k)})$. Since L is length-increasing, $\{x^{(k)} : k \in \mathbb{Z}^+\}$ lies in a bounded domain of $\mathbb{R}^2$, so there exists an subsequence converging to $x \in \mathbb{R}^2$, which implies $L(x) = \eta$ and causes a contradiction. Therefore $\Omega = \mathbb{R}^2$ and then L is a diffeomorphism of $\mathbb{R}^2$ onto itself.

Denote by λ_1^2, λ_2^2 (where $\lambda_1, \lambda_2 > 0$) the eigenvalues of (g_{ij}). Then $W = \det(g_{ij})^{1/2} = \lambda_1\lambda_2$, and there exists an orthogonal matrix O, such that

$$(g_{ij}) = O^T \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} O.$$

Hence,

$$J_L = I_2 + W^{-1}(g_{ij}) = O^T \begin{pmatrix} 1+\dfrac{\lambda_1}{\lambda_2} & \\ & 1+\dfrac{\lambda_2}{\lambda_1} \end{pmatrix} O = (\lambda_1^{-1}+\lambda_2^{-1})\, O^T \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} O,$$

and furthermore,

$$\begin{aligned} d\eta_1^2 + d\eta_2^2 &= \begin{pmatrix} d\eta_1 & d\eta_2 \end{pmatrix} \begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix} = \begin{pmatrix} dx_1 & dx_2 \end{pmatrix} J_L^T J_L \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \\ &= (\lambda_1^{-1}+\lambda_2^{-1})^2 \begin{pmatrix} dx_1 & dx_2 \end{pmatrix} O^T \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} O \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \end{aligned}$$

$$= (\lambda_1^{-1} + \lambda_2^{-1})^2 \begin{pmatrix} dx_1 & dx_2 \end{pmatrix} (g_{ij}) \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}$$
$$= (\lambda_1^{-1} + \lambda_2^{-1})^2 (g_{ij}\, dx_i\, dx_j),$$

i.e.,

$$g = g_{ij}\, dx_i\, dx_j = (\lambda_1^{-1} + \lambda_2^{-1})^{-2} (d\eta_1^2 + d\eta_2^2). \tag{3-8}$$

This means that (η_1, η_2) are global isothermal parameters on M. Define

$$\zeta := \eta_1 + \sqrt{-1}\eta_2 \tag{3-9}$$

and

$$\beta_l := \frac{\partial x_l}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial x_l}{\partial \eta_1} - \sqrt{-1} \frac{\partial x_l}{\partial \eta_2} \right) \quad \text{for} \quad l = 1, \ldots, 2+m. \tag{3-10}$$

Then the harmonicity of coordinate functions implies

$$0 = \frac{\partial^2 x_l}{\partial \zeta\, \partial \bar{\zeta}} = \frac{\partial \beta_l}{\partial \bar{\zeta}},$$

i.e., $\beta_1, \ldots, \beta_{2+m}$ are all holomorphic functions on M. A straightforward calculation shows $-4\,\mathrm{Im}(\bar{\beta}_1 \beta_2)$ equals the Jacobian of the inverse of Lewy's transformation, which is positive everywhere, thus $\beta_2/\beta_1 = \bar{\beta}_1 \beta_2 / |\beta_1|^2$ is an entire function whose imaginary part is always negative. The classical Liouville's theorem implies $\beta_2/\beta_1 \equiv c := a - b\sqrt{-1}$, where $a, b \in \mathbb{R}$ and $b > 0$. In conjunction with (3-10) we get

$$\frac{\partial x_2}{\partial \eta_1} = a \frac{\partial x_1}{\partial \eta_1} - b \frac{\partial x_1}{\partial \eta_2} \quad \text{and} \quad \frac{\partial x_2}{\partial \eta_2} = b \frac{\partial x_1}{\partial \eta_1} + a \frac{\partial x_1}{\partial \eta_2}. \tag{3-11}$$

Let (u_1, u_2) be global parameters of M, satisfying $x_1 = u_1$ and $x_2 = au_1 + bu_2$. Then (3-11) tells us

$$\frac{\partial u_2}{\partial \eta_1} = -\frac{\partial u_1}{\partial \eta_2} \quad \text{and} \quad \frac{\partial u_2}{\partial \eta_2} = \frac{\partial u_1}{\partial \eta_1}. \tag{3-12}$$

This means the one-to-one map $(\eta_1, \eta_2) \in \mathbb{R}^2 \mapsto (u_1, u_2) \in \mathbb{R}^2$ is biholomorphic. Thereby we arrive at the following conclusion:

Theorem 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^m_1$ be a smooth vector-valued function such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a spacelike stationary surface. Then there exists a nonsingular linear transformation*

$$\begin{aligned} x_1 &= u_1, \\ x_2 &= au_1 + bu_2 \quad (b > 0), \end{aligned} \tag{3-13}$$

such that (u_1, u_2) are global isothermal parameters for M.

Now we introduce the complex coordinate $z := u_1 + \sqrt{-1}u_2$ and define

$$\alpha = (\alpha_1, \dots, \alpha_{2+m}) := \frac{\partial \boldsymbol{x}}{\partial z} = \frac{1}{2}\Big(\frac{\partial \boldsymbol{x}}{\partial u_1} - \sqrt{-1}\frac{\partial \boldsymbol{x}}{\partial u_2}\Big). \tag{3-14}$$

Then α is a holomorphic vector-valued function. The induced metric on M can be written as

$$\begin{aligned} g &= \Big\langle \frac{\partial \boldsymbol{x}}{\partial z}, \frac{\partial \boldsymbol{x}}{\partial z} \Big\rangle dz^2 + \Big\langle \frac{\partial \boldsymbol{x}}{\partial \bar{z}}, \frac{\partial \boldsymbol{x}}{\partial \bar{z}} \Big\rangle d\bar{z}^2 + 2\Big\langle \frac{\partial \boldsymbol{x}}{\partial z}, \frac{\partial \boldsymbol{x}}{\partial \bar{z}} \Big\rangle |dz|^2 \\ &= 2\,\mathrm{Re}\big(\langle \alpha, \alpha\rangle\, dz^2\big) + 2\langle \alpha, \bar{\alpha}\rangle |dz|^2. \end{aligned}$$

Here $|dz|^2 := \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = du_1^2 + du_2^2$. Since (u_1, u_2) are isothermal parameters for M,

$$\langle \alpha, \alpha \rangle = 0, \tag{3-15}$$

and hence

$$g = 2\langle \alpha, \bar{\alpha}\rangle |dz|^2. \tag{3-16}$$

Noting that $\alpha_1 = \partial x_1/\partial z = \frac{1}{2}$, $\alpha_2 = \partial x_2/\partial z = \frac{1}{2}(a - b\sqrt{-1}) = \frac{1}{2}c$, (3-15) is equivalent to

$$\alpha_{2+m}^2 = \alpha_1^2 + \cdots + \alpha_{1+m}^2 = \frac{1+c^2}{4} + \alpha_3^2 + \cdots + \alpha_{1+m}^2. \tag{3-17}$$

Thus

$$\begin{aligned} \langle \alpha, \bar{\alpha}\rangle &= |\alpha_1|^2 + \cdots + |\alpha_{1+m}|^2 - |\alpha_{2+m}|^2 \\ &= \frac{1+|c|^2}{4} + |\alpha_3|^2 + \cdots + |\alpha_{1+m}|^2 - \Big|\frac{1+c^2}{4} + \alpha_3^2 + \cdots + \alpha_{1+m}^2\Big| \\ &\geq \frac{1+|c|^2 - |1+c^2|}{4}, \end{aligned}$$

and moreover,

$$g \geq \frac{1+|c|^2 - |1+c^2|}{2}|dz|^2. \tag{3-18}$$

Observing that $1 + |c|^2 - |1 + c^2| > 0$ is a direct corollary of $b > 0$, we get a conclusion as follows.

Corollary 3.2. *Let $M = \mathrm{graph}\, f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ be a spacelike stationary graph generated by $f : \mathbb{R}^2 \to \mathbb{R}^m_1$. Then the induced metric on M is complete.*

Remark. As shown in [Cheng and Yau 1976], if M is a spacelike hypersurface in $\mathbb{R}^{n+1}_1$ with constant mean curvature, so that M is a closed subset of $\mathbb{R}^{n+1}_1$ with respect to the Euclidean topology, then M is complete with respect to the induced

Lorentz metric. It is natural to raise the following problem. *Let M be an n-dimensional spacelike submanifold in $\mathbb{R}^{n+m}_1$ with parallel mean curvature, so that M is a closed subset of $\mathbb{R}^{n+m}_1$. Is M a complete Riemannian manifold?* Corollary 3.2 gives a partial positive answer to the above problem.

Equation (3-13) implies $dx_1 \wedge dx_2 = b\, du_1 \wedge du_2$, and hence

$$\begin{aligned} dM &= 2\langle \alpha, \bar{\alpha}\rangle\, du_1 \wedge du_2 \\ &= 2b^{-1}\langle \alpha, \bar{\alpha}\rangle\, dx_1 \wedge dx_2 \\ &= \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}\, dx_1 \wedge dx_2, \end{aligned}$$

with dM the area element of M. In other words,

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}. \tag{3-19}$$

4. On W-functions for entire stationary graphs in $\mathbb{R}^4_1$

Theorem 4.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2_1$ be a smooth function, such that $M = \operatorname{graph} f$ is a spacelike stationary graph. Then one and only one of the following three cases occurs*:

(i) *f is affine linear and $W \equiv r$, where r is an arbitrary positive constant.*

(ii) *$f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$ with h a nonlinear harmonic function on $\mathbb{R}^2$, $\boldsymbol{y}_0$ a nonzero lightlike vector in $\mathbb{R}^2_1$ and $\boldsymbol{y}_1$ a constant vector, and $W \equiv 1$.*

(iii) *W takes each value in $[r^{-1}, r]$ infinitely often, where r is an arbitrary number in $(1, \infty)$.*

Proof. Equation (3-15) is equivalent to

$$\alpha_3^2 - \alpha_4^2 = -(\alpha_1^2 + \alpha_2^2) = -\frac{1+c^2}{4}, \tag{4-1}$$

and (3-19) gives

$$W = \frac{1+|c|^2+4(|\alpha_3|^2-|\alpha_4|^2)}{2b}. \tag{4-2}$$

If α_3 is a constant function, then (4-1) shows α_4 is also constant, and

$$x_a(z) = \operatorname{Re}\int_0^z \alpha_a\, dz + x_a(0) \qquad \text{for all } \ a = 3, 4$$

is affine linear. Hence f is affine linear and $W \equiv r$, where r can be taken to be any value in $(0, \infty)$. This is case (i).

Now we assume α_3 is nonconstant. Then (4-1) implies α_4 is also nonconstant.

If $c=-\sqrt{-1}$, then (4-1) gives

$$0=\alpha_3^2-\alpha_4^2=(\alpha_3+\alpha_4)(\alpha_3-\alpha_4).$$

Noting that the zeros of a nonconstant holomorphic function have to be isolated, we get $\alpha_3+\alpha_4=0$ or $\alpha_3-\alpha_4=0$. Thus $|\alpha_3|=|\alpha_4|$ and then (4-2) shows $W\equiv 1$. Let β be the unique holomorphic function such that $\beta'=\alpha_3$ and $\beta(0)=0$. Then $\alpha_3\pm\alpha_4=0$ implies

$$\begin{aligned} f(x_1,x_2)&=(x_3(u_1,u_2),x_4(u_1,u_2))=(x_3(z),x_4(z))\\ &=\operatorname{Re}\int_0^z(\alpha_3,\alpha_4)\,dz+(x_3(0),x_4(0))\\ &=\operatorname{Re}\beta(z)(1,\mp 1)+f(0,0).\end{aligned}$$

Now we put $h:=\operatorname{Re}\beta(z)$, $\boldsymbol{y}_0:=(1,\mp 1)$ and $\boldsymbol{y}_1:=f(0,0)$. Then h is a nonlinear harmonic function, $\boldsymbol{y}_0$ is a lightlike vector and $f=h\boldsymbol{y}_0+\boldsymbol{y}_1$. This is case (ii).

Otherwise $c\neq-\sqrt{-1}$ and hence $-(1+c^2)/4\neq 0$. Let $\mu\neq 0$ such that $\mu^2=-(1+c^2)/4$, and h_1,h_2 be holomorphic functions such that $\alpha_3=\mu h_1$, $\alpha_4=\mu h_2$. Then $\mu^2(h_1^2-h_2^2)=\alpha_3^2-\alpha_4^2=\mu^2$ gives

$$1=h_1^2-h_2^2=(h_1+h_2)(h_1-h_2),$$

which implies h_1+h_2 is an entire function containing no zero. Hence there exists an entire function β, such that $h_1+h_2=e^{\beta}$, then $h_1-h_2=e^{-\beta}$ and hence

$$h_1=\cosh\beta,\qquad h_2=\sinh\beta. \tag{4-3}$$

By computing,

$$\begin{aligned}|h_1|^2-|h_2|^2&=|\cosh\beta|^2-|\sinh\beta|^2\\ &=\tfrac12(e^{\beta-\bar\beta}+e^{-\beta+\bar\beta})=\tfrac12\big(e^{2\operatorname{Im}\beta\sqrt{-1}}+e^{-2\operatorname{Im}\beta\sqrt{-1}}\big)\\ &=\cos(2\operatorname{Im}\beta),\end{aligned}$$

and hence

$$\begin{aligned} W&=\frac{1+|c|^2+4(|\alpha_3|^2-|\alpha_4|^2)}{2b}=\frac{1+|c|^2+4|\mu|^2(|h_1|^2-|h_2|^2)}{2b}\\ &=\frac{1+|c|^2+|1+c^2|\cos(2\operatorname{Im}\beta)}{2b}.\end{aligned} \tag{4-4}$$

Set

$$r_1:=\inf W=\frac{1+|c|^2-|1+c^2|}{2b}\quad\text{and}\quad r_2:=\sup W=\frac{1+|c|^2+|1+c^2|}{2b}.$$

Due to Picard's theorem, W takes each value in $[r_1,r_2]$ infinitely often. Noting that

$c = a - b\sqrt{-1}$, one computes

$$r_1 r_2 = \frac{(1+|c|^2)^2 - |1+c^2|^2}{4b^2} = \frac{1+2|c|^2+|c|^4-(1+c^2+\bar{c}^2+|c|^4)}{4b^2} = \frac{4b^2}{4b^2} = 1.$$

Hence $r_1 \in (0, 1)$ and $r_2 \in (1, \infty)$.

Now we take $b := 1$. Then $c = a - \sqrt{-1}$ and $r_2 = \frac{1}{2}(2 + a^2 + |a|\sqrt{a^2+4})$. Denote $\mu : t \in \mathbb{R}^+ \mapsto \mu(t) = \frac{1}{2}(2+t^2+|t|\sqrt{t^2+4})$. Then μ is a strictly increasing function and $\lim_{t\to 0} \mu(t) = 1$, $\lim_{t\to+\infty} \mu(t) = +\infty$. Hence for an arbitrary number $r \in (1, \infty)$, one can find $a \in \mathbb{R}^+$, such that $r_2 = r$ and then W takes each value in $[r^{-1}, r]$ infinitely often. This is case (iii). □

Corollary 4.2. *Let M be an entire spacelike stationary graph in $\mathbb{R}^4_1$ generated by a smooth function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2_1$. If $W \le 1$ (or $W \ge 1$), then f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, with h a nonlinear harmonic function, $\boldsymbol{y}_0$ a nonzero lightlike vector and $\boldsymbol{y}_1$ a constant vector. Moreover, $W > 1$ (or $W < 1$) forces f to be affine linear, representing an affine plane in $\mathbb{R}^4_1$.*

Remark. If $f_2 \equiv 0$, then $M = \operatorname{graph} f$ is a minimal entire graph in $\mathbb{R}^3$ and $W \ge 1$. By Corollary 4.2, f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, where h is a nonlinear harmonic function and $\boldsymbol{y}_0$ is a nonzero lightlike vector. But $f_2 \equiv 0$ precludes the latter case. Hence f is an affine linear function and so the classical Bernstein theorem [1915] can be derived from Corollary 4.2. Similarly, Corollary 4.2 implies any spacelike maximal entire graph in $\mathbb{R}^3_1$ has to be affine linear. This is Calabi's theorem [1970].

5. Bernstein-type theorems for entire stationary graphs in $\mathbb{R}^{2+m}_1$

It is natural to ask whether one can generalize the conclusion of Corollary 4.2 to higher-codimensional cases.

For the first statement, i.e., $W \le 1$, the answer is "yes":

Theorem 5.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^m_1$ be a smooth function, such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a spacelike stationary graph in $\mathbb{R}^{2+m}_1$. If the orthogonal projection $\mathcal{P}_0$ of M onto the coordinate plane $\mathbb{R}^2$ is area-increasing (i.e., $W \le 1$), then f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, with h a nonlinear harmonic function, $\boldsymbol{y}_0$ a nonzero lightlike vector and $\boldsymbol{y}_1$ a constant vector. Moreover, if $\mathcal{P}_0$ is strictly area-increasing (i.e., $W < 1$), then f has to be affine linear and M is an affine plane.*

Proof. We shall consider the problem in the following four cases.

Case I. $\alpha_3, \dots, \alpha_{2+m}$ are all constant functions. As in the proof of Theorem 4.1, one can show f is an affine linear function.

Case II. α_{2+m} is a constant function, but α_l is nonconstant for some $3 \le l \le 1+m$. By the classical Liouville Theorem, there exists a point in $\mathbb{C}$, such that

$$|\alpha_l|^2 \ge |\alpha_{2+m}|^2 + b - \tfrac{1}{4}(1+|c|^2)$$

at this point. Combing with (3-19) gives

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$\geq \frac{1+|c|^2+4(|\alpha_l|^2-|\alpha_{2+m}|^2)}{2b} \geq 2.$$

This gives a contradiction to the assumption that $W \leq 1$ everywhere. Hence this case cannot occur.

Case III. α_{2+m} is nonconstant and $c \neq -\sqrt{-1}$. Then $c \neq \sqrt{-1}$ implies

$$\frac{1+|c|^2}{2b} = \frac{1+b^2+a^2}{2b} > 1.$$

Denote $\delta := (1+|c|^2)/(2b) - 1$. Again the classical Liouville theorem implies the existence of a point such that $|\alpha_{2+m}|^2 < \frac{1}{2}b\delta$ at this point. Hence

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$\geq \frac{1+|c|^2-4|\alpha_{2+m}|^2}{2b} > 1+\delta - \frac{4\cdot\frac{1}{2}b\delta}{2b} = 1,$$

which causes a contradiction and therefore this case cannot happen.

Case IV. α_{2+m} is nonconstant and $c = -\sqrt{-1}$. Let $h_3, \ldots, h_{1+m}$ be meromorphic functions such that

$$\alpha_3^2 = h_3\alpha_{2+m}^2, \ldots, \alpha_{1+m}^2 = h_{1+m}\alpha_{2+m}^2.$$

Then (3-17) tells us

$$\alpha_{2+m}^2 = \frac{1+c^2}{4} + \alpha_3^2 + \cdots + \alpha_{1+m}^2 = \alpha_3^2 + \cdots + \alpha_{1+m}^2$$
$$= (h_3 + \cdots + h_{1+m})\alpha_{2+m}^2.$$

Since α_{2+m} is a nonconstant function, we have

$$h_3 + \cdots + h_{1+m} \equiv 1.$$

Due to the triangle inequality,

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$= 1 + 2(|\alpha_3^2| + \cdots + |\alpha_{1+m}^2| - |\alpha_{2+m}^2|)$$
$$= 1 + 2(|h_3| + \cdots + |h_{1+m}| - 1)|\alpha_{2+m}|^2 \geq 1,$$

and the equality holds if and only if the functions $h_3, \ldots, h_{1+m}$ all take values in $\mathbb{R}^+ \cup \{0, \infty\}$. Again using the Liouville Theorem, we know that $h_3, \ldots, h_{1+m}$

are all constant real functions. Therefore, there exist $v_3, \dots, v_{1+m} \in \mathbb{R}$, such that $v_3^2 + \cdots + v_{1+m}^2 = 1$ and

$$(\alpha_3, \dots, \alpha_{1+m}, \alpha_{2+m}) = (v_3, \dots, v_{1+m}, 1)\alpha_{2+m}.$$

Let β be the unique holomorphic function such that $\beta' = \alpha_{2+m}$ and $\beta(0) = 0$. Denote $h := \operatorname{Re}\beta$, $\boldsymbol{y}_0 := (v_3, \cdots, v_{1+m}, 1)$ and $\boldsymbol{y}_1 := f(0,0)$. Then h is a nonlinear harmonic function and $\boldsymbol{y}_0$ is a lightlike vector. We can proceed as in the proof of Theorem 4.1 to show $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$. Note that in this case $W \equiv 1$. □

But our answer is "no" for the second statement, i.e., $W \geq 1$. In fact, we have the following result:

Proposition 5.2. *For any real number $C \geq 1$ and $\varepsilon > 0$, there exists an entire spacelike stationary graph in $\mathbb{R}_1^{2+m}$ $(m \geq 3)$ generated by $f : \mathbb{R}^2 \to \mathbb{R}_1^m$ such that* $\inf W \cdot \sup W = C$ *and* $0 < \sup W - \inf W < \varepsilon$.

Proof. Now we put $c := -b\sqrt{-1}$ and let d be a real number to be chosen. Let μ be a complex number such that

$$\mu^2 = -\frac{1+c^2+d^2}{4} = -\frac{1-b^2+d^2}{4}.$$

Denote

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{c}{2} = -\frac{b}{2}\sqrt{-1}, \quad \alpha_3 = \cdots = \alpha_{m-1} = 0,$$
$$\alpha_m = \frac{d}{2}, \qquad \alpha_{1+m} = \mu \cosh z, \qquad \alpha_{2+m} = \mu \sinh z.$$

Since

$$\langle \alpha, \alpha \rangle = \alpha_1^2 + \alpha_2^2 + \alpha_m^2 + \alpha_{1+m}^2 - \alpha_{2+m}^2 = 0$$

and $\langle \alpha, \bar{\alpha} \rangle$ is positive, $z \mapsto \boldsymbol{x}(z) = \int_0^z \alpha(z)$ gives an entire spacelike stationary graph in $\mathbb{R}_1^{2+m}$.

As in the proof of Theorem 4.1, a similar calculation shows

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$= \frac{1+b^2+d^2+|1-b^2+d^2|\cos(2\operatorname{Im} z)}{2b}.$$

Denote $r_1 := \inf W$, $r_2 := \sup W$. Then $r_1 = (1+b^2+d^2-|1-b^2+d^2|)/(2b)$, $r_2 = (1+b^2+d^2+|1-b^2+d^2|)/(2b)$ and

$$r_1 r_2 = \frac{(1+b^2+d^2)^2-(1-b^2+d^2)^2}{4b^2} = 1+d^2, \quad r_2 - r_1 = \frac{|1-b^2+d^2|}{b}.$$

Now we put $d := \sqrt{C-1}$. Then $r_1 r_2 = C$, and one can choose b sufficiently close to $\sqrt{C}$, such that $r_2 - r_1 = |1-b^2+d^2|/b = |C-b^2|/b \in (0, \varepsilon)$. □

Remark. Calabi's theorem has been generalized to higher-dimensional cases. Namely, if f is a smooth real function on $\mathbb{R}^n$, so that $M = \text{graph } f := \{(x, f(x)) : x \in \mathbb{R}^n\}$ is an entire maximal hypersurface in $\mathbb{R}^{n+1}_1$, then f has to be affine linear. This is a well-known Bernstein-type result by Cheng and Yau [1976]. Observing that any maximal n-dimensional graph in $\mathbb{R}^{n+1}_1$ can be regarded as a stationary graph in $\mathbb{R}^{n+m}_1$ which satisfies $W \le 1$, we raise a conjecture:

Conjecture 5.3. Let $f : \mathbb{R}^n \to \mathbb{R}^m_1$ be a smooth function, such that $M = \text{graph } f := \{(x, f(x)) : x \in \mathbb{R}^n\}$ is a spacelike stationary graph in $\mathbb{R}^{n+m}_1$. If $W \le 1$, then M has to be a flat manifold. Moreover, $W < 1$ forces f to be affine linear and hence M has to be an affine n-plane.

6. Stationary graphs with finite total curvature

As demonstrated in [Ma et al. 2013], the Bernstein theorem can not be generalized directly to stationary graphs in $\mathbb{R}^4_1$, because one can easily construct complete stationary graphs in $\mathbb{R}^4_1$ which are not flat. Interestingly, these examples have infinite total curvature.

On the other hand, examples of complete stationary surfaces with finite total curvature are abundant, and there holds a generalized Jorge–Meeks formula about their total Gaussian curvature (and the total normal curvature) provided that they are algebraic [Ma et al. 2013]. Thus one is naturally interested to know whether there could be a stationary graph with finite total curvature. The answer to this question is the following Bernstein type theorem. (Note that here we do not need the algebraic assumption.)

Theorem 6.1. *Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2_1$ be a smooth function, such that $M = \text{graph } f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a spacelike stationary graph in $\mathbb{R}^4_1$ whose curvature integral $\int_M |K|\, dM$ converges absolutely. Then f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, with h a nonlinear harmonic function, $\boldsymbol{y}_0$ a nonzero lightlike vector and $\boldsymbol{y}_1$ a constant vector. In both cases, M is flat, i.e., $K \equiv 0$.*

Proof. Denote $z = u_1 + \sqrt{-1}u_2$ as before. As in the proof of Theorem 4.1, if M is not a flat surface as we claimed, then the holomorphic differential $\partial \boldsymbol{x}/\partial z$ can be expressed as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\tfrac{1}{2}, \tfrac{1}{2}c, \mu \cosh \beta, \mu \sinh \beta\right), \tag{6-1}$$

where $c = a - b\sqrt{-1}$ is a complex constant with $b > 0$, $\mu^2 = -\frac{1}{4}(1 + c^2)$, and $\beta = \beta(z)$ is a nonconstant holomorphic function defined on $\mathbb{C}$. We will derive a contradiction from this assumption.

By the Weierstrass representation formula given in [Ma et al. 2013], $\partial \boldsymbol{x}/\partial z$ can be expressed in terms of a pair of meromorphic functions ϕ, ψ (the *Gauss maps*) and a holomorphic differential $dh = h'(z)\,dz$ (the *height differential*) as below:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\phi + \psi, -\sqrt{-1}(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi)h'. \tag{6-2}$$

Comparing these two formulas, we obtain

$$h' = \frac{\mu}{2}e^{\beta}, \quad \phi = \frac{1+c\sqrt{-1}}{2\mu}e^{-\beta}, \quad \psi = \frac{1-c\sqrt{-1}}{2\mu}e^{-\beta}.$$

Note that $\dfrac{1+c\sqrt{-1}}{2\mu} \cdot \dfrac{1-c\sqrt{-1}}{2\mu} = -1$, and $b > 0$ implies

$$\left|\frac{1+c\sqrt{-1}}{2\mu}\right| > \left|\frac{1+\bar{c}\sqrt{-1}}{2\mu}\right|.$$

Denote $(1 + c\sqrt{-1})/(2\mu) := re^{\sqrt{-1}\theta}$ with $r > 1$ and $\theta \in \mathbb{R}$. Then

$$\frac{1 - c\sqrt{-1}}{2\mu} = -r^{-1}e^{-\sqrt{-1}\theta}.$$

In [Ma et al. 2013] the Gaussian curvature and the normal curvature of a stationary surface were unified in a single formula in terms of ϕ, ψ and the Laplacian with respect to the induced metric $g := e^{2\omega}|dz|^2$ as follows:

$$-K + \sqrt{-1}K^{\perp} = \Delta \ln(\phi - \bar{\psi}) = 4e^{-2\omega}\frac{\phi_z\bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2}. \tag{6-3}$$

Set $\beta := v_1 + \sqrt{-1}v_2$, where v_1, v_2 are both real functions on $\mathbb{C}$. Then

$$\begin{aligned}
|K|e^{2\omega} &= 4\left|\operatorname{Re}\frac{\phi_z\bar{\psi}_{\bar{z}}}{(\phi-\bar{\psi})^2}\right| = 4\left|\operatorname{Re}\frac{e^{2\sqrt{-1}\theta}e^{-\beta-\bar{\beta}}}{(re^{\sqrt{-1}\theta}e^{-\beta}+r^{-1}e^{\sqrt{-1}\theta}e^{-\bar{\beta}})^2}\right||\beta'(z)|^2 \\
&= 4\left|\operatorname{Re}\left(\frac{1}{(re^{(\bar{\beta}-\beta)/2}+r^{-1}e^{(\beta-\bar{\beta})/2})^2}\right)\right||\beta'(z)|^2 \\
&= \frac{4\bigl(2+(r^2+r^{-2})\cos 2v_2\bigr)}{|re^{-\sqrt{-1}v_2}+r^{-1}e^{\sqrt{-1}v_2}|^4}|\beta'(z)|^2 \geq \frac{4\bigl(2+(r^2+r^{-2})\cos 2v_2\bigr)}{|r+r^{-1}|^4}|\beta'(z)|^2.
\end{aligned} \tag{6-4}$$

Thus the assumption of finite total curvature is equivalent to saying that

$$\begin{aligned}
\infty > \int_M |K|\,dM &= \int_{\mathbb{C}} |K|e^{2\omega}\,du_1 \wedge du_2 \\
&\geq \int_{\mathbb{C}} \frac{4[2+(r^2+r^{-2})\cos(2v_2)]}{|r+r^{-1}|^4}|\beta'(z)|^2\,du_1 \wedge du_2 \\
&\geq \int_{\mathbb{C}} \frac{4[2+(r^2+r^{-2})\cos(2v_2)]}{|r+r^{-1}|^4}\,dv_1 \wedge dv_2,
\end{aligned} \tag{6-5}$$

where the final inequality follows from the assumption that β is a nonconstant entire function over $\mathbb{C}$, which takes almost every value of $\mathbb{C}$ at least one time. It is easily seen that the right-hand side of (6-5) is divergent, contradicting the finiteness of the total curvature. □

Remarks.

- Taking the imaginary part of (6-3), one can proceed as in (6-4)–(6-5) to get a contradiction when the condition "$\int_M |K|\,dM < \infty$" is replaced by "$\int_M |K^\perp|\,dM < \infty$". Therefore, if $M \subset \mathbb{R}^4_1$ is an entire spacelike stationary graph over $\mathbb{R}^2$, whose normal curvature integral converges absolutely, then M has to be a flat surface.
- Let M be a noncompact surface with a complete metric. If $\int_M |K|\,dM < \infty$, then there is a compact Riemann surface $\overline{M}$, such that M is conformally equivalent to $\overline{M} \setminus \{p_1, p_2, \dots, p_r\}$, with $p_1, \dots, p_r \in \overline{M}$. This is a purely intrinsic result, discovered by A. Huber [1957]. Moreover, if we additionally assume M to be a minimal surface in $\mathbb{R}^{2+m}$ (m is arbitrary), then the Gauss map of M is algebraic, and vice versa; see Theorem 1 of [Chern and Osserman 1967]. But this conclusion is no longer true for spacelike stationary surfaces in $\mathbb{R}^4_1$, due to the examples with finite total curvature and essential singularities; see [Ma et al. 2013]. Hence, unlike the $\mathbb{R}^4$ case [Osserman 1969], the conclusion of Theorem 6.1 cannot be deduced directly from (6-1).
- Combining Theorem 1 of [Chern and Osserman 1967] and §5 of [Osserman 1969], it is easy to conclude that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a minimal surface in $\mathbb{R}^4$ with finite total curvature if and only if $f = p(z)$ or $p(\bar{z})$, with p an arbitrary polynomial. Noting that any minimal graph in $\mathbb{R}^4$ over $\mathbb{R}^2$ can be regarded as a spacelike stationary graph in $\mathbb{R}^n_1$ ($n \geq 5$), the conclusion of Theorem 6.1 can not be generalized to spacelike stationary graphs in higher-dimensional Minkowski spaces.

References

[Barbosa 1979] J. L. M. Barbosa, "An extrinsic rigidity theorem for minimal immersions from S^2 into S^n", *J. Differential Geom.* **14**:3 (1979), 355–368. MR Zbl

[Bernstein 1915] S. Bernstein, "Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique", *Comm. Soc. Math. Kharkow* (2) **15**:1 (1915), 38–45. JFM

[Calabi 1970] E. Calabi, "Examples of Bernstein problems for some nonlinear equations", pp. 223–230 in *Global analysis* (Berkeley, CA, 1968), edited by S.-S. Chern and S. Smale, Proc. Sympos. Pure Math. **15**, American Mathematical Society, Providence, RI, 1970. MR Zbl

[Cheng and Yau 1976] S. Y. Cheng and S. T. Yau, "Maximal space-like hypersurfaces in the Lorentz–Minkowski spaces", *Ann. of Math.* (2) **104**:3 (1976), 407–419. MR Zbl

[Chern and Osserman 1967] S.-S. Chern and R. Osserman, "Complete minimal surfaces in Euclidean n-space", *J. Analyse Math.* **19** (1967), 15–34. MR Zbl

[Fischer-Colbrie 1980] D. Fischer-Colbrie, "Some rigidity theorems for minimal submanifolds of the sphere", *Acta Math.* **145**:1-2 (1980), 29–46. MR Zbl

[Huber 1957] A. Huber, "On subharmonic functions and differential geometry in the large", *Comment. Math. Helv.* **32** (1957), 13–72. MR Zbl

[Jost et al. 2014] J. Jost, Y. L. Xin, and L. Yang, “A spherical Bernstein theorem for minimal submanifolds of higher codimension”, preprint, 2014. arXiv

[Jost et al. 2015] J. Jost, Y. L. Xin, and L. Yang, “Curvature estimates for minimal submanifolds of higher codimension and small G-rank”, *Trans. Amer. Math. Soc.* **367**:12 (2015), 8301–8323. MR Zbl

[Lewy 1937] H. Lewy, “A priori limitations for solutions of Monge–Ampère equations, II”, *Trans. Amer. Math. Soc.* **41**:3 (1937), 365–374. MR Zbl

[Ma et al. 2013] X. Ma, C. Wang, and P. Wang, “Global geometry and topology of spacelike stationary surfaces in the 4-dimensional Lorentz space”, *Adv. Math.* **249** (2013), 311–347. MR Zbl

[Osserman 1969] R. Osserman, *A survey of minimal surfaces*, Van Nostrand, New York, 1969. Reprinted Dover, New York, 1986. MR

Received May 7, 2015. Revised February 2, 2016.

XIANG MA
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
CHINA
maxiang@math.pku.edu.cn

PENG WANG
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI, 200092
CHINA
netwangpeng@tongji.edu.cn

LING YANG
SCHOOL OF MATHEMATICAL SCIENCES
FUDAN UNIVERSITY
SHANGHAI, 200433
CHINA
yanglingfd@fudan.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.177

COMPARISON RESULTS FOR DERIVED DELIGNE–MUMFORD STACKS

MAURO PORTA

We establish a comparison between the notion of derived Deligne–Mumford stack in the sense of Toën and Vezzosi and the one introduced by Lurie. It is folklore that the two theories yield essentially the same objects, but it is difficult to locate in the literature a precise result, despite it sometimes being useful to be able to switch between the two frameworks.

Introduction

This short paper is devoted to establishing in a precise way the folklore equivalence between the theory of derived Deligne–Mumford stacks introduced by B. Toën and G. Vezzosi [2008] and the one defined by J. Lurie [2011b]. The main comparison result will be stated in the next section. See Theorem 1.7. Even though many of the results used to achieve the proof of the main theorem can be found scattered through the DAG series of J. Lurie, the precise form of Theorem 1.7 has not appeared anywhere in the literature, to the best of my knowledge.

As certain problems are easier to approach from the point of view of the functor of points, and others from the point of view of structured spaces, a precise comparison result can be useful. Moreover, this note can be helpful for someone who is trying to approach the subject of derived algebraic geometry for the first time. For this last reason, I preferred to be lengthy and to give thorough explanations even where perhaps they would not have been necessary.

Conventions. Throughout this paper we will work freely with the language of $(\infty, 1)$-categories. We will call them simply ∞-categories and our basic reference on the subject is [Lurie 2009]. Occasionally, it will be necessary to consider $(n, 1)$-categories. We will refer to such objects as n-categories, and we redirect the reader to [Lurie 2009, §2.3.4] for the definitions and the basic properties. There is no chance of confusion with the theory of (∞, n)-categories, since it plays no role in this note. The notation $\mathcal{S}$ will be reserved for the ∞-categories of spaces.

The paper was written when the author was a PhD student at the University of Paris Diderot and at the University of Florence.

MSC2010: 14A20.
Keywords: derived stack, Deligne–Mumford stack, spectral stack, HAG II, DAG V.

Whenever categorical constructions are used (such as limits, colimits, etc.), we mean the corresponding ∞-categorical notion. For the reader with a model categorical background, this means that we are always considering *homotopy* limits, *homotopy* colimits, etc. See [Lurie 2009, 4.2.4.1].

In [Lurie 2009] and more generally in the DAG series, whenever $\mathcal{C}$ is a 1-category the notation $\mathrm{N}(\mathcal{C})$ denotes $\mathcal{C}$ viewed (trivially) as an ∞-category. This notation stands for the nerve of the category $\mathcal{C}$ (and this is because an ∞-category in [Lurie 2009] is defined to be a quasicategory, that is a simplicial set with special lifting properties). In this note, we will systematically suppress this notation, and we encourage the reader to think of ∞-categories as model-independently as possible. For this reason, if k is a (discrete) commutative ring we chose to denote by CAlg_k the ∞-category underlying the category of simplicial commutative k-algebras and by $\mathrm{CAlg}_k^{\heartsuit}$ the 1-category of discrete k-algebras.

1. Statement of the comparison result

Let us begin by quickly reviewing the two theories.

HAG II framework. In [Toën and Vezzosi 2008], the authors work within the setting previously introduced in [Toën and Vezzosi 2005], where the theory of model topoi is introduced and extensively explored. In particular, model categories are used continuously throughout the whole paper. In order to compare their constructions with the ones of [Lurie 2011b], it will be convenient to rethink the paper in purely ∞-categorical language. This is essentially no more than an easy exercise, and we use this opportunity in this review to explain how it can be done.

Let k be a commutative ring (with unit). We will denote by sMod_k the category of simplicial k-modules. There is an adjunction

$$U : \mathrm{sMod}_k \rightleftarrows \mathrm{sSet} : F \qquad (F \dashv U)$$

which satisfies the hypothesis of the lifting principle (see [Schwede and Shipley 2000]) and therefore it allows us to lift the (Kan) model structure on sSet to a simplicial model structure on sMod_k. Moreover, with respect to this model structure, sMod_k becomes a monoidal model category (whose tensor product is computed objectwise). We set $\mathrm{sAlg}_k := \mathrm{Com}(\mathrm{sMod}_k)$. Using the fact that every object in sMod_k is fibrant, it is possible to establish that the adjunction

$$V : \mathrm{sAlg}_k \rightleftarrows \mathrm{sMod}_k : \mathrm{Sym}_k \qquad (\mathrm{Sym}_k \dashv V),$$

satisfies again the lifting principle (see [Schwede and Shipley 2000, §5]), and therefore the (simplicial) model structure on sMod_k induces a simplicial model structure on sAlg_k. We will simply denote by CAlg_k the ∞-category underlying sAlg_k, which can be explicitly thought as the coherent nerve [Lurie 2009, §1.1.5] of the category of fibrant cofibrant objects in sAlg_k. It is customary to denote the opposite of this ∞-category by dAff_k (the ∞-category of "affine derived schemes").

This ∞-category admits another description which is more useful for our purposes. Let $\mathcal{T}_{\mathrm{disc}}(k)$ the full subcategory of ordinary schemes over $\mathrm{Spec}(k)$ spanned by the relative finite-dimensional affine spaces $\mathbb{A}^n_k$. We can think of $\mathcal{T}_{\mathrm{disc}}(k)$ as a (one-sorted) Lawvere theory; equally, in the language of [Lurie 2011b], we can say that $\mathcal{T}_{\mathrm{disc}}(k)$ is a *discrete pregeometry*. The ∞-category of product-preserving functors with values in the ∞-category of spaces can be identified with the *sifted completion* of $\mathcal{T}_{\mathrm{disc}}(k)$ and we will denote it by $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$ (see [Lurie 2009, Definition 5.5.8.8]). This is a presentable ∞-category and therefore it admits a presentation by a model category [ibid., A.3.7.6], which can be easily obtained as follows: consider the category of simplicial presheaves on $\mathcal{T}_{\mathrm{disc}}(k)$ endowed with the global projective model structure. Then the underlying ∞-category of the Bousfield localization of this model category at the collection of maps $y(\mathbb{A}^n_k) \coprod y(\mathbb{A}^m_k) \to y(\mathbb{A}^{n+m}_k)$ (where y denotes the Yoneda embedding) precisely coincides with $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$. It is somehow remarkable that $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$ admits a much stricter presentation. Consider in fact the category of functors $\mathcal{T}_{\mathrm{disc}}(k) \to \mathrm{sSet}$ which *strictly* preserve products. It follows from a theorem of Quillen [ibid., 5.5.9.1] that this simplicial category admits a global projective model structure. Moreover, a theorem of J. Bergner [ibid., 5.5.9.2] shows that the underlying ∞-category coincides precisely with $\mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k))$. However, the category of product-preserving functors $\mathcal{T}_{\mathrm{disc}}(k) \to \mathrm{sSet}$ is precisely equivalent to sAlg_k, and the two model structures agree. Therefore, we have a categorical equivalence

$$\mathrm{CAlg}_k \simeq \mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k)).$$

The reader might want to consult also [Lurie 2011b, Remark 4.1.2] for another discussion of this equivalence.

The next step is to introduce the étale topology on the model category sAlg_k. As this notion only depends on the homotopy category of sAlg_k [Toën and Vezzosi 2005, Definition 4.3.1], it also defines a Grothendieck topology on the ∞-category CAlg_k [Lurie 2009, 6.2.2.3]. We briefly recall that a morphism $f : A \to B$ in sAlg_k is said to be étale if $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is étale and the canonical map

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$$

is an isomorphism (that is, the morphism is *strong*). Similarly, a morphism $f : A \to B$ is smooth if it is strong and $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is smooth. We denote by $\tau_{\text{ét}}$ the étale topology and by $\mathbf{P}_{\text{ét}}$ (resp. $\mathbf{P}_{\mathrm{sm}}$) the collection of étale (resp. smooth) morphisms. Using these data, one can form the model category of hypersheaves with respect to the étale topology. Recall that this is obtained in two steps:

(1) considering the global projective model structure on $\mathrm{Fun}(\mathrm{sAlg}_k, \mathrm{sSet})$;

(2) considering next the Bousfield localization of this model structure at the collection of hypercovers (see [Toën and Vezzosi 2005, §4.4 and §4.5] or [Lurie 2009, §6.5.3]).

The result is what is denoted in [Toën and Vezzosi 2008] by $\mathrm{dAff}^{\sim,\tau_{\text{ét}}}$. It follows from [Lurie 2009, 6.5.2.14, 6.5.2.15] that the underlying ∞-category of $\mathrm{dAff}^{\sim,\tau_{\text{ét}}}$ can be identified with the hypercompletion $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ (we refer the reader to [Lurie 2009, §6.5.2] for a detailed discussion of this notion). We usually refer to the objects in $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ as stacks (for the étale topology). The next step is to consider geometric stacks inside $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$. Since there are many references for this subject [Simpson 1996; Toën and Vezzosi 2008; Toën and Vaquié 2008; Porta and Yu 2016], we do not repeat the full definition here, but we limit ourselves to describing the general idea. Roughly speaking, geometric stacks are stacks X admitting a morphism $p : U \to X$ satisfying the following conditions:

(1) U is an affine derived scheme (seen as a stack via the ∞-categorical Yoneda embedding, see [Lurie 2009, §5.1.3] or [Lurie 2016a, §5.2.1]).

(2) $p : U \to X$ is an effective epimorphism (see [Lurie 2009, §6.2.3 and 7.2.1.14]).

(3) p is either an étale or a smooth morphism.

The notions of étale and smooth morphisms between geometric stacks must be defined with some care, proceeding by induction on the "geometric level" of the stack. See [Porta and Yu 2016, Definition 2.8] or [Toën and Vezzosi 2008, §1.3.3] for a complete review. When p can be chosen to be étale, we refer to X as a (higher) derived Deligne–Mumford stack; if instead p can only be chosen to be smooth, we refer to X as a (higher) derived Artin stack. We are mostly concerned with derived Deligne–Mumford stacks (see however Remark 1.9). We denote the full subcategory of $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$ spanned by derived Deligne–Mumford stacks by $\mathbf{DM}$. Let us complete the review of [Toën and Vezzosi 2008] with some additional remarks:

(1) Geometric stacks are stable under weak equivalences because only homotopy invariant categorical constructs are used in the definition (i.e., homotopy coproducts, homotopy geometric realizations, etc.). Therefore [Lurie 2009, 4.2.4.1] shows that the notion of geometric stack can be equally formulated at the level of the ∞-category $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$.

(2) The category $\mathbf{DM}$ is naturally filtered by the notion of geometric level: a stack is said to be (-1)-geometric if it is representable by an object in dAff_k. If $A \in \mathrm{CAlg}_k$, we choose to represent its functor of points by $\mathrm{Spec}(A) \in \mathbf{DM} \subset \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^\wedge$. Next, proceeding by induction, we say that a stack X is n-geometric if it admits an atlas $p : U \to X$ which is representable by $(n-1)$-geometric stacks in the following precise sense: for every representable stack $\mathrm{Spec}(A)$ and any map $\mathrm{Spec}(A) \to X$

the base change $\mathrm{Spec}(A) \times_X U$ is $(n-1)$-geometric. We say that a derived stack is geometric if it is n-geometric for some n.

(3) We denote by $\mathbf{DM}_n$ the full subcategory of $\mathbf{DM}$ spanned by geometric derived Deligne–Mumford stacks whose restriction to $\mathrm{CAlg}_k^{\heartsuit}$ is an n-truncated stack (i.e., it takes values in n-truncated spaces).

DAG V framework. The point of view taken in [Lurie 2011b] is quite different. We refer the reader to the introduction of [Porta 2015] for an expository account of the role of (pre)geometries (compare [Lurie 2011b, §1.2, 3.1]) in the construction of affine derived objects. Here, we content ourselves with a short review of the theory of $\mathcal{G}$-schemes for a given geometry $\mathcal{G}$ from the point of view of [Lurie 2011b]. Recall either from [Lurie 2011b, Definition 12.8] or from the introduction of [Porta 2015] that a geometry is an ∞-category $\mathcal{G}$ with finite limits and equipped with some extra structure, consisting of a collection of "admissible" morphisms and a Grothendieck topology τ on $\mathcal{G}$ generated by admissible morphisms. If $\mathcal{X}$ is an ∞-topos and $\mathcal{G}$ is a geometry, they define an ∞-category of $\mathcal{G}$-structures, denoted $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$. Recall that a $\mathcal{G}$-structure is a functor $\mathcal{G} \to \mathcal{X}$ which is left exact and takes τ-coverings to effective epimorphisms in $\mathcal{X}$.

Before moving on, it is important to discuss a very important special case. If $\mathcal{X}$ is the ∞-topos of $\mathcal{S}$-valued sheaves on some topological space X, we can think of a $\mathcal{G}$-structure on $\mathcal{X}$ as a sheaf on X with values in the ∞-category $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ having special behavior on the stalks, as the next key example shows:

Example 1.1. Let k be a fixed (discrete) commutative ring. We denote by $\mathcal{G}_{\text{ét}}(k)$ the category $((\mathrm{CRing}_k^{\heartsuit})^{\text{f.p.}})^{\mathrm{op}}$, the opposite of the category of discrete k-algebras of finite presentation. Moreover, we declare a morphism in $\mathcal{G}_{\text{ét}}(k)$ to be an admissible morphism if and only if it is étale, and we endow $\mathcal{G}_{\text{ét}}(k)$ with the usual étale topology. In this case, $\mathrm{Ind}(\mathcal{G}_{\text{ét}}(k)^{\mathrm{op}}) \simeq \mathrm{CAlg}_k^{\heartsuit}$, the category of discrete k-algebras of finite presentation. Then a $\mathcal{G}_{\text{ét}}(k)$-structure $\mathcal{O}$ on $\mathrm{Sh}(X)$ is a sheaf of discrete commutative rings on X whose stalks are strictly henselian local rings. The fact that $\mathcal{O}$ has to be discrete follows from its left-exactness (see [Lurie 2009, §5.5.6] for a general discussion of truncated objects in an ∞-category and more specifically [Lurie 2009, 5.5.6.16] for the needed property). The statement on stalks, instead, is due to the following fact: for every point $x \in X$ (formally seen as a geometric morphism $x^{-1} : \mathrm{Sh}(X) \rightleftarrows \mathcal{S} : x_*$), the stalk $\mathcal{O}_x := x^{-1}\mathcal{O}$ has to take étale coverings of k-algebras of finite presentation to epimorphisms in Set. Unraveling the definitions, this means that for every étale cover $\{A \to A_i\}$ in $\mathcal{G}_{\text{ét}}(k)$ and every solid diagram

$$\begin{tikzcd} & \coprod \mathrm{Spec}(A_i) \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_x) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(A), \end{tikzcd}$$

the lifting exists. This is a possible characterization of strictly henselian local rings (see [de Jong et al. 2005–, Tag 04GG, condition (8)]).

As in the case of locally ringed spaces, we are not really interested in all the transformations of $\mathcal{G}$-structures, but only in those that have good local behavior. This can be made precise by introducing the notion of *local transformation of* $\mathcal{G}$*-structures*. We recall that a morphism $f : \mathcal{O} \to \mathcal{O}'$ in $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ is said to be local if for every admissible morphism $f : U \to V$ in $\mathcal{G}$ the induced square

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}(V) \\ \downarrow & & \downarrow \\ \mathcal{O}'(U) & \longrightarrow & \mathcal{O}'(V) \end{array}$$

is a pullback in $\mathcal{X}$. In Example 1.1, morphisms satisfying the above condition simply become local morphisms of local rings.

Precisely as in the case of locally ringed spaces, we can use $\mathcal{G}$-structures and local morphisms of such to build an ∞-category of $\mathcal{G}$-structured topoi, denoted $\mathcal{T}\mathrm{op}(\mathcal{G})$. The actual construction is rather involved, and we refer to [Lurie 2011b, Definition 1.4.8] for the details. Here, we content ourselves with the following rougher idea: the ∞-category $\mathcal{T}\mathrm{op}(\mathcal{G})$ has as objects pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_{\mathcal{X}}$ is a $\mathcal{G}$-structure on $\mathcal{X}$, and as 1-morphisms pairs $(f, \alpha) : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, where f is a geometric morphism $f^{-1} : \mathcal{Y} \rightleftarrows \mathcal{X} : f_*$ and $\alpha : f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is a *local* transformation of $\mathcal{G}$-structures on $\mathcal{X}$.

The category $\mathcal{T}\mathrm{op}(\mathcal{G})$ is too huge to be of any practical interest. Therefore we are going to construct a full subcategory $\mathrm{Sch}(\mathcal{G})$ which intuitively corresponds to the subcategory of $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})^{\wedge}$ spanned by geometric stacks. We will see in discussing Theorem 1.7 that this is not quite a true statement, but until then it is a reasonable analogy. The idea is rather straightforward: as schemes are a full subcategory of locally ringed spaces spanned by those objects which are locally isomorphic to special ones constructed out of commutative rings, so objects $\mathrm{Sch}(\mathcal{G})$ are structured topoi locally equivalent to a collection of special models. As Example 1.1 suggests, it is possible to associate a $\mathcal{G}$-structured topos to every object of $\mathrm{Ind}(\mathcal{G})$. To keep the exposition as elementary as possible, we limit ourselves to considering the case of objects in $\mathcal{G}$, and we refer the reader to [Lurie 2011b, §2.2] for the general discussion.

Let $A \in \mathcal{G}^{\mathrm{op}}$. We will denote by A_{adm} the small admissible site of A. The underlying ∞-category of A_{adm} is the opposite of the full subcategory of $\mathcal{G}^{\mathrm{op}}_{A/}$ spanned by admissible morphisms $A \to B$. We then endow A_{adm} with the Grothendieck topology induced from the one on $\mathcal{G}$, which we still denote τ. Finally, we let $\mathcal{X}_A$ be the *nonhypercomplete* ∞-topos of ($\mathcal{S}$-valued) sheaves on A_{adm}. We next construct the $\mathcal{G}$-structure on $\mathcal{X}_A$. There is a forgetful functor $A_{\mathrm{adm}} \to \mathcal{G}$ which induces a composition

$$A_{\mathrm{adm}}^{\mathrm{op}} \times \mathcal{G} \to \mathcal{G}^{\mathrm{op}} \times \mathcal{G} \xrightarrow{y} \mathcal{S},$$

where y is the functor classifying the Yoneda embedding, see [Lurie 2016a, §5.2.1]. This corresponds to a functor

$$\mathcal{O}_A : \mathcal{G} \to \mathrm{PSh}(A_{\mathrm{adm}}) \xrightarrow{\mathrm{L}} \mathrm{Sh}(A_{\mathrm{adm}}, \tau),$$

where L is the sheafification functor. Note that if the Grothendieck topology on $\mathcal{G}$ were subcanonical, there would not be any need to apply L. Observe further that $\mathcal{O}_A$ is indeed left-exact by its very construction. We leave as an exercise to the reader to prove that $\mathcal{O}_A$ takes τ-coverings in effective epimorphisms (cf., [Lurie 2011b, Proposition 2.2.11]). Therefore the pair $(\mathcal{X}_A, \mathcal{O}_A)$ defines a $\mathcal{G}$-structured topos, which we denote as $\mathrm{Spec}^{\mathcal{G}}(A)$.

Remark 1.2. As often happens in the ∞-categorical world, the construction of the functoriality is the most subtle point in the definition of an ∞-functor. So, to build $\mathrm{Spec}^{\mathcal{G}}(-)$ as an ∞-functor $\mathcal{G} \simeq (\mathcal{G}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{T}\mathrm{op}(\mathcal{G})$, some additional effort is needed. The details are out of the scope of this review, but the rough idea is to prove that $\mathrm{Spec}^{\mathcal{G}}(A)$ enjoys a universal property, which makes $\mathrm{Spec}^{\mathcal{G}}(-)$ a right adjoint to the global section functor $\mathcal{T}\mathrm{op}(\mathcal{G}) \to \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$, informally defined by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto \mathrm{Map}_{\mathcal{X}}(\mathbf{1}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$. Observe that the latter becomes a finite-limit-preserving functor $\mathcal{G} \to \mathcal{S}$ and therefore can be identified with an element of $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. We refer the reader to [Lurie 2011b, §2.2] (and especially to [Lurie 2011b, Theorem 2.2.12]) for a detailed discussion.

With these preparations, it is now easy to define $\mathrm{Sch}(\mathcal{G})$ as a full subcategory of $\mathcal{T}\mathrm{op}(\mathcal{G})$. We will that a $\mathcal{G}$-structured topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a $\mathcal{G}$-*scheme* (resp. a $\mathcal{G}$-*scheme locally of finite presentation*) if there exists a collection of objects $U_i \in \mathcal{X}$ satisfying the following two conditions:

(1) The joint morphism $\coprod U_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism.

(2) For every index i, there exists an object $A_i \in \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ (resp. an object $A_i \in \mathcal{G}^{\mathrm{op}}$) and an equivalence of $\mathcal{G}$-structured topoi $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \simeq \mathrm{Spec}^{\mathcal{G}}(A_i)$.

We conclude this review with two important examples and some discussion about them.

Example 1.3. Let us go back to the geometry $\mathcal{G} := \mathcal{G}_{\text{ét}}(k)$ of Example 1.1. The category $\mathrm{Sch}(\mathcal{G})$ contains a very interesting full subcategory. To describe it, let us briefly recall that an ∞-topos $\mathcal{X}$ is said to be n-localic (for $n \geq -1$ an integer) if it can be thought of as the category of ($\mathcal{S}$-valued) sheaves on some Grothendieck site $(\mathcal{C}, \tau)$ with $\mathcal{G}$ being an n-category (see our conventions on the meaning of this). We refer the reader to [Lurie 2009, §6.4.5] for a more detailed account of this notion. Let $\mathrm{Sch}_{\leq 1}(\mathcal{G})$ be the full subcategory of $\mathrm{Sch}(\mathcal{G})$ spanned by $\mathcal{G}$-schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

such that $\mathcal{X}$ is 1-localic. Then [Lurie 2011b, Theorem 2.6.18] shows that $\mathrm{Sch}_{\leq 1}(\mathcal{G})$ is equivalent to the category of 1-geometric (underived) Deligne–Mumford stacks. More generally, Theorem 1.7 implies $\mathrm{Sch}_{\leq n}(\mathcal{G})$ is equivalent to the ∞-category of n-truncated (underived) Deligne–Mumford stacks.

Example 1.4. Let us define a new geometry $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$ as follows. We let the underlying ∞-category of $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$ to be the opposite of the full subcategory of CAlg_k spanned by compact objects. Observe that $\mathrm{CAlg}_k = \mathrm{Ind}(\mathcal{G}_{\text{ét}}^{\text{der}}(k)^{\text{op}})$. We say that a morphism in $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$ is admissible precisely when it is a (derived) étale morphism (see the previous section for the definition). We will further endow $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$ with the (derived) étale topology, which we will still denote $\tau_{\text{ét}}$ (observe that if $A \to B$ is an étale map in the derived sense and the source is discrete, then so is the target). In this special case, we write $\mathrm{Spec}^{\text{ét}}$ instead of $\mathrm{Spec}^{\mathcal{G}_{\text{ét}}^{\text{der}}(k)}$. Following [Lurie 2011b, Definition 4.3.20] (and using the important [Lurie 2011b, Proposition 4.3.15]), we say that a derived Deligne–Mumford stack (in the sense of [Lurie 2011b]) is a $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-scheme.

The following theorem summarizes several results of [Lurie 2011b]. We report them here because it clarifies the relation between the above two examples:

Theorem 1.5. (1) [Lurie 2011b, Proposition 4.3.15] *The natural inclusion*

$$\mathcal{T}_{\acute{e}t}(k) \to \mathcal{G}_{\acute{e}t}^{\text{der}}(k)$$

exhibits the latter as a geometric envelope of $\mathcal{T}_{\acute{e}t}(k)$.

(2) [Lurie 2011b, Remark 4.3.14 and Corollary 4.3.16] *The truncation functor* $\pi_0 : \mathcal{G}_{\text{ét}}^{\text{der}}(k) \to \mathcal{G}_{\acute{e}t}(k)$ *exhibits the latter as a* 0*-stub for* $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$*. In particular, the composition* $\mathcal{T}_{\acute{e}t}(k) \to \mathcal{G}_{\acute{e}t}^{\text{der}}(k) \to \mathcal{G}_{\acute{e}t}(k)$ *exhibits* $\mathcal{G}_{\acute{e}t}(k)$ *as a* 0*-truncated geometric envelope of* $\mathcal{T}_{\acute{e}t}(k)$.

(3) [Lurie 2011b, Proposition 4.3.21] *The category of* 1*-localic* $\mathcal{G}_{\acute{e}t}(k)$*-schemes is equivalent to the category of* $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$*-schemes which are* 1*-localic and* 0*-truncated.*

Remark 1.6. The derived Deligne–Mumford stacks of Example 1.4 are locally *connective*. There is a nonconnective variation of such objects, known as *spectral Deligne–Mumford stacks*. This plays a major role in a certain branch of algebraic topology known as *chromatic homotopy theory*. As we are not be concerned with such objects in this note, we invite the interested reader to consult [Lurie 2011c, §2, §8]. Then [Lurie 2011c, Corollary 9.28] completes the task of comparing the category of spectral Deligne–Mumford stacks with the one of Example 1.4. We would like to draw the attention of the reader to the fact that characteristic 0 is needed to have such a comparison. This is a complication that comes from the interaction with power operations in algebraic topology. In this note, no hypothesis on the characteristic is required.

The main theorem. Finally, we are ready to discuss the main comparison result. In order to avoid confusion, we will refer from this moment on to derived Deligne–Mumford stacks as the geometric stacks for the HAG context $(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathbf{P}_{\text{ét}})$ we discussed in Section 1, and to $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-schemes to the derived Deligne–Mumford stacks in the sense of [Lurie 2011b] we introduced in Example 1.4.

Taking inspiration from the comparison discussed in Example 1.3, we introduce the full subcategory $\mathrm{Sch}_{\leq n}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ of $\mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ spanned by $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ whose underlying ∞-topos $\mathcal{X}$ is n-localic. We further let $\mathrm{Sch}_{\mathrm{loc}}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ be the union of the ∞-categories $\mathrm{Sch}_{\leq n}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$ as n varies. The comparison result can therefore be stated as follows:

Theorem 1.7. *There exists an equivalence of ∞-categories*

$$\Phi : \mathrm{Sch}_{\mathrm{loc}}(\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k)) \rightleftarrows \mathbf{DM} : \Psi.$$

Moreover, for every $n \geq 1$, this restricts to an equivalence

$$\mathrm{Sch}_{\leq n}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)) \simeq \mathbf{DM}_n.$$

The next section is entirely devoted to the proof of this theorem.

Remark 1.8. The statement Theorem 1.7 is very similar to the one of [Porta 2015, Theorem 3.7]. However, the proof of Theorem 1.7 is somehow subtler. One of the key points is that if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived $\mathbb{C}$-analytic space (cf., [Lurie 2011a, Definition 12.3] or [Porta 2015, Definition 1.3]), then the ∞-topos $\mathcal{X}$ is always hypercomplete (see [Porta 2015, Lemma 3.2]). This is false in the algebraic setting, and the reason is that if $A \in \mathrm{CAlg}_k$, then usually $\mathcal{X}_A := \mathrm{Sh}(A_{\text{ét}})$ itself is not hypercomplete. As a consequence, there is no direct analogue in this setting of [Porta 2015, Corollary 3.4]: one needs to restrict oneself to the case of localic $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-schemes to prove the corresponding statement (see Proposition 2.3).

Another important point that marks the difference is that if $A \in \mathrm{CAlg}_k$ then $\mathcal{X}_A$ is 1-localic instead of 0-localic. Therefore the case of algebraic spaces needs to be dealt with separately and it cannot be uniformly included in an induction proof. This is done in Section 2.

Remark 1.9. Theorem 1.7 actually implies that the two ∞-categories of derived *Artin* stacks considered in [Toën and Vezzosi 2008] and in the DAG series are equivalent. Indeed, it is not possible to deal with Artin stacks from the point of view of structured topoi. Therefore, even in the DAG series and in J. Lurie's Ph.D. thesis [2004], derived Artin stacks are defined as geometric stacks with respect to the context of affine derived Deligne–Mumford stacks in $\mathbf{DM}_n$.

2. The proof of the comparison result

We start with the construction of the two functors Φ and Ψ. [Lurie 2011b, Theorem 2.4.1] provides us with a fully faithful embedding

$$\phi : \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k)) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k)^{\mathrm{op}}), \mathcal{S}) = \mathrm{Fun}(\mathrm{dAff}^{\mathrm{op}}, \mathcal{S}),$$

Unraveling the definition of ϕ, we see that for $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k))$, the functor $\phi(X)$

$$\phi(X) : \mathrm{CAlg}_k \to \mathcal{S}$$

is defined informally by

$$\phi(X)(A) = \mathrm{Map}_{\mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k))}(\mathrm{Spec}^{\mathrm{ét}}(A), X).$$

It follows from [Lurie 2011b, Lemma 2.4.13] that this functor factors through $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\mathrm{ét}})$.

To obtain the functor Φ of Theorem 1.7, we are left to show that the restriction of ϕ to $\mathrm{Sch}_{\mathrm{loc}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k))$ factors through $\mathbf{DM}$. Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k))$. More specifically, the proof of Theorem 1.7 breaks into the following independent steps:

(1) Let $n \geq 1$. If the underlying ∞-topos of X is n-localic, then $\phi(X)$ is hypercomplete.

(2) Let $n \geq 1$. If the underlying ∞-topos of X is n-localic, then $\phi(X)$ is geometric and n-truncated.

(3) The previous two points imply that ϕ factors through a fully faithful functor $\Phi : \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k)) \to \mathbf{DM}$. Therefore, to complete the proof, it will be sufficient to show that every object in $\mathbf{DM}$ arises is of the form $\phi(X)$ for $X \in \mathrm{Sch}_{\mathrm{loc}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{ét}}(k))$.

We deal with the first point in Section 2. In Section 2 we discuss the special case of derived algebraic spaces, which is then used as base for the proof by induction of the second point given in Section 2. Finally, we treat the third point in Section 2, where the proof of Theorem 1.7 will be achieved.

Hypercompleteness. Let us begin with a couple of preliminary lemmas.

Lemma 2.1. *Let $f : B \to A$ be a morphism in* CAlg_k *between finitely presented objects. The following conditions are equivalent*:

(1) *f is étale.*

(2) *The morphism* $\mathrm{Spec}^{\mathrm{ét}}(A) \to \mathrm{Spec}^{\mathrm{ét}}(B)$ *is étale in the sense of* [Lurie 2011b, Definition 2.3.1].

Proof. A proof of this lemma can be formally deduced from [Lurie 2011d, Theorem 1.2.1]. We will propose here a shorter proof that works fine in the connective situation. The implication (1) $\Rightarrow$ (2) is [Lurie 2011b, Example 2.3.8]. Let us show that (2) $\Rightarrow$ (1). Since both A and B are finitely presented, we see that $\pi_0(A) \to \pi_0(B)$ is finitely presented. If we show that $\mathbb{L}_{A/B} \simeq 0$, we will obtain that $B \to A$ is finitely presented (in virtue of [Lurie 2011a, Proposition 8.8][1]

Let

$$f^{-1} : \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}}) \to \mathrm{Sh}(B_{\text{ét}}, \tau_{\text{ét}})$$

be the inverse image functor. Consider the sheaf $\mathbb{L}_{\mathcal{O}_A/f^{-1}\mathcal{O}_B}$ on $A_{\text{ét}}$ defined by

$$C \mapsto \mathbb{L}_{\mathcal{O}_A(C)/f^{-1}\mathcal{O}_B(C)} = \mathbb{L}_{C/f^{-1}\mathcal{O}_B(C)}.$$

Since the morphism of $\mathcal{T}_{\text{ét}}(k)$-structured topoi $\mathrm{Spec}^{\text{ét}}(A) \to \mathrm{Spec}^{\text{ét}}(B)$ is étale in the sense of [Lurie 2011b, Definition 4.3.1], we see that $f^{-1}\mathcal{O}_B \simeq \mathcal{O}_A$. Therefore this sheaf is identically zero.

On the other side, if $\eta^{-1} : \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}}) \to \mathcal{S}$ is a geometric point, then

$$\eta^{-1}(\mathbb{L}_{\mathcal{O}_A/f^{-1}\mathcal{O}_B}) \simeq \mathbb{L}_{\eta^{-1}\mathcal{O}_A/\eta^{-1}f^{-1}\mathcal{O}_B}.$$

We can identify $\eta^{-1}f^{-1}\mathcal{O}_B$ with a strictly henselian B-algebra B'. Since the map $B \to B'$ is formally étale, we conclude that

$$\mathbb{L}_{\eta^{-1}\mathcal{O}_A/\eta^{-1}f^{-1}\mathcal{O}_B} \simeq \mathbb{L}_{\eta^{-1}\mathcal{O}_A/B}.$$

This is also the stalk of the sheaf on $A_{\text{ét}}$ defined by

$$C \mapsto \mathbb{L}_{C/B}.$$

Therefore, this sheaf vanishes as well. In particular, $\mathbb{L}_{A/B} \simeq 0$, completing the proof. □

Let us recall the following result from [Lurie 2011d]:

Lemma 2.2. *Let $\mathcal{T}\mathrm{op}_{\leq n}$ be the full subcategory of $\mathcal{R}\mathcal{T}\mathrm{op}$ spanned by n-localic ∞-topoi. Then $\mathcal{T}\mathrm{op}_{\leq n}$ is categorically equivalent to an $(n+1)$-category.*

Proof. This is a direct consequence of [Lurie 2011d, Lemma 1.3.5] and of [Lurie 2009, 2.3.4.18]. □

Proposition 2.3. *Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-scheme and suppose that $\mathcal{X}$ is n-localic, with $n \geq 1$. Then the functor $\phi(X) : \mathcal{C} \to \mathcal{S}$ is a hypercomplete sheaf.*

---

[1] We warn the reader that there is a small mistake in [Lurie 2011a, Example 8.4], when considering morphisms of finite presentation to order 0. Namely, it is not true that a discrete A-algebra B is finitely generated if the canonical map $\operatorname{colim} \mathrm{Hom}_A(B, C_\alpha) \to \mathrm{Hom}_A(B, \operatorname{colim} C_\alpha)$ is injective for every filtered diagram $\{C_\alpha\}$ of A-algebras, the easiest counterexample being $A = \mathbb{Z}$ and $B = \mathbb{Q}$. However, the converse is true, and this is precisely what is used afterwards. Therefore the subsequent results are not affected by this. This issue has been fixed in [Lurie 2016b].

Proof. Let $U^\bullet \to U$ be an étale hypercover in the category dAff_k. Let $\mathcal{T}\mathrm{op}_{\leq n}(\mathcal{G}_{\text{ét}}^{\mathrm{der}}(k))$ be the ∞-category of $\mathcal{G}_{\text{ét}}^{\mathrm{der}}(k)$-structured ∞-topoi which are m-localic for some $m \leq n$. We claim that the geometric realization of the simplicial object $\mathrm{Spec}^{\text{ét}}(U^\bullet)$ is $\mathcal{T}\mathrm{op}_{\leq n}(\mathcal{G}_{\text{ét}}^{\mathrm{der}}(k))$ is precisely $\mathrm{Spec}^{\text{ét}}(U)$. The claim directly implies the lemma, since

$$\begin{aligned}
\phi(X)(\mathrm{Spec}^{\text{ét}}(U)) &= \mathrm{Map}_{\mathrm{Sch}(\mathcal{T}_{\text{ét}}(k))}(\mathrm{Spec}^{\text{ét}}(U), X) \\
&= \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\leq n}(\mathcal{T}_{\text{ét}}(k))}(\mathrm{Spec}^{\text{ét}}(U), X) \\
&= \lim \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\leq n}(\mathcal{T}_{\text{ét}}(k))}(\mathrm{Spec}^{\text{ét}}(U^\bullet), X) \\
&= \lim \phi(X)(\mathrm{Spec}^{\text{ét}}(U^\bullet)).
\end{aligned}$$

We are therefore reduced to proving the claim. Let us denote by $\mathcal{X}_U$ the topos of (nonhypercomplete) sheaves on the small étale site of U. It follows from Lemma 2.1 that each face map

$$\mathrm{Spec}^{\text{ét}}(U^n) \to \mathrm{Spec}^{\text{ét}}(U^{n-1})$$

is étale. Thus, we can find objects $V^n \in \mathcal{X}_U$ and identifications $\mathcal{X}_{U^n} \simeq (\mathcal{X}_U)_{/V^n}$. The universal property of étale subtopoi (see [Lurie 2009, 6.3.5.6]), shows that we can arrange the V^n into a simplicial object in $\mathcal{X}_U$. Using statement (3′) in the proof of [Lurie 2011b, Proposition 2.3.5], we are reduced to prove that in $\mathcal{T}\mathrm{op}_{\leq n}$ one has an equivalence

$$\mathcal{X}_U \simeq \operatorname{colim} \mathcal{X}_{U^\bullet}.$$

Since $\mathcal{T}\mathrm{op}_{\leq n}$ is an n-category in virtue of Lemma 2.2, Proposition A.1 shows that a presheaf with values in $\mathcal{T}\mathrm{op}_{\leq n}$ has descent if and only if it has hyperdescent. We are therefore reduced to the case where $U^\bullet$ is the Čech nerve of the map $U^0 \to U$. In this case, the general descent theory for ∞-topoi (see [Lurie 2009, 6.1.3.9]) allows us to conclude. □

The case of algebraic spaces. Let $A \in \mathrm{CAlg}_k$. We denote by $A_{\mathrm{big,\,ét}}$ the big étale site of A: that is, its underlying ∞-category is the opposite of $(\mathrm{CAlg}_k)_{A/}$, and the Grothendieck topology is the (derived) étale one. There are continuous and cocontinuous morphisms of ∞-sites

$$(A_{\text{ét}}, \tau_{\text{ét}}) \xrightarrow{u} (A_{\mathrm{big,ét}}, \tau_{\text{ét}}) \xrightarrow{v} (\mathrm{dAff}_k, \tau_{\text{ét}}) \ .$$

Note that u commutes with finite limits. It follows from [Porta and Yu 2016, Lemma 2.14] that the induced adjunction

$$u_s : \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}}) \rightleftarrows \mathrm{Sh}(A_{\mathrm{big,\,ét}}, \tau_{\text{ét}}) : u^s$$

is a geometric morphism of ∞-topoi, in other words, u_s commutes with finite limits. Here u^s denotes the restriction functor along u and u_s is obtained via the left Kan extension along u. We refer the reader to [Porta and Yu 2016, §2.4] for a more detailed discussion of the chosen notations. In particular, we can use [Lurie 2009, 5.5.6.16] to conclude that u_s takes n-truncated objects to n-truncated objects.

This is not true for v, because it commutes only with weakly contractible limits. However, we still have an adjunction

$$v_s : \mathrm{Sh}(A_{\mathrm{big,\, \acute{e}t}}, \tau_{\acute{e}t}) \rightleftarrows \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\acute{e}t}) : v^s,$$

which can be identified with the canonical adjunction

$$v_s : \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\acute{e}t})_{/\,\mathrm{Spec}(A)} \rightleftarrows \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\acute{e}t}) : v^s,$$

where $\mathrm{Spec}(A)$ denotes the functor of points associated to A, accordingly to the notation introduced at the end of Section 1.

Definition 2.4. Let k be a commutative ring, A a commutative k-algebra and $X \in \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\acute{e}t})$ any sheaf equipped with a natural transformation $\alpha : X \to \mathrm{Spec}(A)$. We will say that α *exhibits X as an étale derived algebraic space over* $\mathrm{Spec}(A)$ if there exists a 0-truncated sheaf $F \in \mathrm{Sh}(A_{\acute{e}t}, \tau_{\acute{e}t})$ and an equivalence $X \simeq v_s(u_s(F))$ in $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\acute{e}t})_{/\,\mathrm{Spec}(A)}$.

Remark 2.5. The above definition is the analogue of [Lurie 2011b, Definition 2.6.4] in the derived setting. Indeed, let us replace the ∞-category CAlg_k with the 1-category $\mathrm{CAlg}_k^{\heartsuit}$. Keeping the same notations as above, we see that if $G \in \mathrm{Sh}(A_{\mathrm{big,\, \acute{e}t}}, \tau_{\acute{e}t})$ then

$$v_s(G) = \coprod_{\phi : A \to B} G(\phi).$$

If moreover F is an object in $\mathrm{Sh}(A_{\acute{e}t}, \tau_{\acute{e}t})$, then $(u_s F)(\phi) = \phi^{-1}(F)(B)$. In conclusion, we have

$$v_s(u_s(F))(B) = \{(\phi, \eta) \mid \phi \in \mathrm{Hom}_k(A, B),\, \eta \in (\phi^{-1} F)(B)\}.$$

This coincides precisely with the definition of $\widehat{F}$ given in [Lurie 2011b, Notation 2.6.2]. A similar description holds true in the derived setting. Indeed, there is a natural transformation $v_s(u_s(F)) \to \mathrm{Spec}(A)$. The fiber over a given map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ coincides precisely with the global sections of the discrete object $f^{-1}(F)$.

The following proposition is the analogue of [Lurie 2011b, 2.6.20]. The proof is essentially unchanged.

Proposition 2.6. *Let $\alpha : Y \to \mathrm{Spec}(A)$ be a natural transformation of stacks. Write* $\mathrm{Spec}^{\acute{e}t}(A) = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. *The following conditions are equivalent*:

(1) *α exhibits Y as an étale derived algebraic space over* $\mathrm{Spec}(A)$.

(2) *Y is representable by a $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$-scheme $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and α induces an equivalence $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \simeq (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ for some discrete object $U \in \mathcal{X}$.*

(3) *The morphism α is 0-truncated and 0-representable by étale maps.*

Proof. We first prove the equivalence of (1) and (2). If α exhibits Y as an étale derived algebraic space over $\mathrm{Spec}(A)$, we can find a 0-truncated sheaf $U \in \mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}})$ and an equivalence $Y \simeq v_s(u_s(U))$ in $\mathrm{Sh}(\mathrm{dAff}, \tau_{\text{ét}})_{/\mathrm{Spec}(A)}$. Now, [Lurie 2011b, Remark 2.3.4] and Remark 2.5 show together that the functor represented by $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ coincides with Y. Conversely, if (2) is satisfied, then U defines an étale derived algebraic space $v_s(u_s(U))$ over $\mathrm{Spec}(A)$, and [Lurie 2011b, Remark 2.3.4] again allows us to identify it with Y.

Let us now prove the equivalence of (1) and (3) First, assume that (3) is satisfied. In this case, we can define a sheaf $U : A_{\text{ét}} \to \mathcal{S}$ by sending an étale map $f : A \to B$ to the fiber product

$$\begin{tikzcd} U(B) \arrow[r] \arrow[d] & Y(B) \arrow[d, "\alpha_B"] \\ \{*\} \arrow[r, "f"] & \mathrm{Map}(A, B). \end{tikzcd}$$

Since α is 0-truncated, we see that U takes values in Set. Since it is obviously a sheaf, it defines a 0-truncated object in $\mathrm{Sh}(A_{\text{ét}}, \tau_{\text{ét}})$. [Lurie 2011b, Remark 2.3.4] shows that $v_s(u_s(U))$ can be canonically identified with Y.

Finally, let us show that (1) implies (3). We already know that, in this situation, α is 0-truncated. Choosing sections $\eta_\alpha \in Y(A_\alpha)$ which generate Y, we obtain an effective epimorphism

$$\coprod \mathrm{Spec}(A_\alpha) \to v_s(u_s(Y))$$

in $\mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})$. Suppose that there exists a (-1)-truncated morphism

$$v_s(u_s(Y)) \to \mathrm{Spec}(B)$$

for some $B \in \mathrm{CAlg}_k$. In this case, we see that

$$\mathrm{Spec}(A_\alpha) \times_{v_s(u_s(Y))} \mathrm{Spec}(A_\beta) \simeq \mathrm{Spec}(A_\alpha) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A_\beta) \simeq \mathrm{Spec}(A_\alpha \otimes_B A_\beta).$$

In the general case, each fiber product $Y_{\alpha,\beta} := \mathrm{Spec}(A_\alpha) \times_{v_s(u_s(Y))} \mathrm{Spec}(A_\beta)$ is again a derived algebraic space étale over A. We claim moreover that the canonical morphism $Y_{\alpha,\beta} \to \mathrm{Spec}(A_\alpha \otimes_A A_\beta)$ is (-1)-truncated. Assuming the claim, it follows that $Y_{\alpha,\beta} \to \mathrm{Spec}(A)$ is (-1)-representable by étale maps, hence it would follow that the morphism $\mathrm{Spec}(A_\alpha) \to v_s(u_s(Y))$ is 0-representable. Finally, we see that it is representable by étale maps combining the equivalence between (1) and (2) with Lemma 2.1.

We are left to prove the claim. Fix $f_\alpha : A_\alpha \to B$, $f_\beta : A_\beta \to B$ together with a homotopy making the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_\alpha \arrow[d, "f_\alpha"] \\ A_\beta \arrow[r, "f_\beta"] & B \end{tikzcd}$$

commutative. We have pullback squares

$$\begin{tikzcd} Y_{\alpha,\beta} \arrow[r] \arrow[d] & v_s(u_s(Y)) \arrow[d] \\ \mathrm{Spec}(A_\alpha) \times \mathrm{Spec}(A_\beta) \arrow[r] & v_s(u_s(Y)) \times_{\mathrm{Spec}(A)} v_s(u_s(Y)), \end{tikzcd}$$

and since $\alpha : v_s(u_s(Y)) \to \mathrm{Spec}(A)$ is 0-truncated, the statement follows. □

$\phi(X)$ is geometric. We can now prove that if $X \in \mathrm{Sch}_{\leq n+1}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k))$, then $\phi(X)$ belongs to $\mathbf{DM}_n$. The proof goes by induction, and Proposition 2.6 serves as basis of the induction. Before doing that, however, it is convenient to prove the following lemma:

Lemma 2.7. *Let $n \geq 0$ be an integer. Fix $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{Sch}_{\leq n+1}(\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k))$ and let $V \in \mathcal{X}$ be an object such that $(\mathcal{X}_{/V}, \mathcal{O}_{\mathcal{X}}|_V) \simeq \mathrm{Spec}^{\acute{e}t}(A)$ for some $A \in \mathrm{CAlg}_k$. Then V is n-truncated.*

Proof. We start by replacing X with $\mathrm{t}_0(X) := (\mathcal{X}, \pi_0\mathcal{O}_X)$, which is a $\mathcal{G}_{\text{ét}}(k)$-scheme in virtue of [Lurie 2011b, Corollary 4.3.30]. We can therefore replace A by $\pi_0(A)$ (observe also that $\mathrm{Spec}^{\text{ét}}(\pi_0(A)) \simeq \mathrm{Spec}^{\mathcal{G}_{\text{ét}}(k)}(\pi_0(A))$).

Let us denote by $F_X : \mathrm{CAlg}_k^{\heartsuit} \to \mathcal{S}$ the (truncated) functor of points associated to X. Similarly, let $F_V : \mathrm{CAlg}_k^{\heartsuit} \to \mathcal{S}$ be the functor of points associated to $(\mathcal{X}_{/V}, \mathcal{O}_{\mathcal{X}}|_V)$. The hypothesis shows that F_V is nothing but the functor of points associated to $\pi_0(A)$ (with the notations of [Toën and Vezzosi 2008], this would be $\mathrm{t}_0(\mathrm{Spec}(\pi_0(A)))$). Reasoning as in the proof of [Lurie 2011b, Theorem 2.6.18], we see that to prove that V is n-truncated is equivalent to prove that for every (discrete) k-algebra B the fibers of $F_V(B) \to F_X(B)$ are n-truncated. [Lurie 2011b, Lemma 2.6.19] shows that $F(B)$ is $(n+1)$-truncated for every k-algebra B. On the other side, $F_V(B)$ is discrete by hypothesis. It follows from the long exact sequence of homotopy groups that the fibers of $F_V(B) \to F_X(B)$ are n-truncated, thus completing the proof. □

Proposition 2.8. *Let $X = (\mathcal{X}, \mathcal{O}_X) \in \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k))$ and suppose that $\mathcal{X}$ is n-localic for $n \geq 1$. Then the stack $\phi(X)$ is $(n+1)$-geometric and moreover its truncation $\mathrm{t}_0(\phi(X))$ is n-truncated.*

Proof. The fact that $\mathrm{t}_0(\phi(X))$ is n-truncated follows directly from [Lurie 2011b, Lemma 2.6.19].

Suppose now that $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an n-localic $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-scheme. By definition, we can find a collection of objects $V_i \in \mathcal{X}$ such that

(1) the morphism $\coprod V_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism, and
(2) the $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-schemes $(\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ are equivalent to $\mathrm{Spec}^{\text{ét}}(U_i)$ for $U_i \in \mathrm{dAff}_k$, and each U_i is of finite presentation.

Set $V := \coprod V_i$. By functoriality, we obtain a map

$$\coprod \phi(V_i) \to \phi(X).$$

We only need to show that this map is $(n-1)$-representable by étale morphisms and that it is an effective epimorphism. The second statement is an immediate consequence of [Lurie 2011b, Lemma 2.4.13].

Suppose first that $X \simeq \mathrm{Spec}^{\text{ét}}(A)$. In this case, the universal property of $\mathrm{Spec}^{\text{ét}}$ proved in [Lurie 2011b, §2.2] shows that $\phi(X) = \mathrm{Spec}(A)$, and therefore $\phi(X)$ is (-1)-geometric. Now suppose that X is a general n-localic $\mathcal{G}_{\text{ét}}^{\text{der}}(k)$-scheme. Since ϕ commutes with fiber products and is fully faithful, we see that for every map $\mathrm{Spec}(B) = \phi(\mathrm{Spec}^{\text{ét}}(B)) \to X$, one has

$$\mathrm{Spec}(B) \times_{\phi(X)} \phi(V_i) \simeq \phi(\mathrm{Spec}^{\text{ét}}(B) \times_{(\mathcal{X}, \mathcal{O}_X)} (\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})).$$

Let $(f_*, \varphi) : \mathrm{Spec}^{\text{ét}}(B) \to (\mathcal{X}, \mathcal{O}_X)$ be the given map. Then the fiber product $\mathrm{Spec}^{\text{ét}}(B) \times_{(\mathcal{X}, \mathcal{O}_X)} (\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ is the étale map to $\mathrm{Spec}^{\text{ét}}(B)$ classified by the object $f^{-1}(V_i) \in \mathcal{X}_A$, as it easily follows from [Lurie 2009, 6.3.5.8].

We complete the proof proving by induction on n that each morphism

$$\phi(\mathcal{X}_{/V_i}, \mathcal{O}_{\mathcal{X}}|_{V_i}) \to \phi(X)$$

is $(n-1)$-representable by étale maps. If $n = 1$, Lemma 2.7 shows that each object V_i is 0-truncated. It follows from Proposition 2.6 that the fiber product $\mathrm{Spec}(A) \times_{\phi(X)} \phi(V_i)$ is 1-geometric. Therefore, $\phi(X)$ is 2-geometric. Now suppose that $\mathcal{X}$ is n-localic for $n > 1$. Lemma 2.7 again shows that each V_i is $(n-1)$-truncated, and therefore [Lurie 2011b, Lemma 2.3.16] shows that the underlying ∞-topos of

$$\mathrm{Spec}^{\text{ét}}(A) \times_{(\mathcal{X}, \mathcal{O}_X)} (\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$$

is $(n-1)$-localic. The inductive hypothesis shows therefore that its image via the functor ϕ is n-geometric, and that the map to $\mathrm{Spec}(A)$ is étale. The proof is therefore complete. $\square$

Essential surjectivity. We finally prove that ϕ is essentially surjective. Let $X \in \mathbf{DM}$ be m-geometric and suppose that $\mathrm{t}_0(X)$ is n-truncated. It follows that the small étale site $(t_0(X))_{\text{ét}}$ is equivalent to an n-category. Recall that there is an equivalence of ∞-categories

$$X_{\text{ét}} \leftrightarrows (\mathrm{t}_0(X))_{\text{ét}}$$

(one can proceed as in [Porta 2015, Proposition 3.16] using as base of the induction [Toën and Vezzosi 2008, Corollary 2.2.2.10]). We conclude that $X_{\text{ét}}$ is an n-category. In particular, the ∞-topos $\mathcal{X} := \mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$ is n-localic. Define a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-structure on $\mathcal{X}$ as follows. Introduce the functor

$$\mathcal{G}^{\text{der}}_{\text{ét}}(k) \times (X_{\text{ét}})^{\text{op}} \to \mathcal{S},$$

defined as

$$(U, V) \mapsto \mathrm{Map}_{\mathrm{dAff}_k}(V, U).$$

Fix $U \in \mathcal{G}^{\text{der}}_{\text{ét}}(k)$. Since the Grothendieck topology on dAff_k is hypersubcanonical, we see that the resulting object of $\mathrm{Fun}((X_{\text{ét}})^{\text{op}}, \mathcal{S})$ is a hypersheaf. In particular, we obtain a well defined functor

$$\mathcal{O}_X : \mathcal{T}_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$$

that in fact factors through the hypercompletion of this category. In order to show that it is a $\mathcal{T}_{\text{ét}}$-structure, we only need to check the following statements:

(1) $\mathcal{O}_X$ is left-exact.

(2) $\mathcal{O}_X$ takes $\tau_{\text{ét}}$-coverings to effective epimorphisms.

Since limits in $\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$ are computed objectwise, the first statement follows directly from the definition of $\mathcal{O}_X$. We are left to show that $\mathcal{O}_X$ takes $\tau_{\text{ét}}$-coverings to effective epimorphisms. Let $\{U_i \to U\}$ be a $\tau_{\text{ét}}$-cover in $\mathcal{T}_{\text{ét}}(k)$. We have to show that the morphism

$$\coprod \mathcal{O}_X(U_i) \to \mathcal{O}_X(U)$$

is an effective epimorphism. In other words, we have to show that

$$\coprod \pi_0\mathcal{O}_X(U_i) \to \pi_0\mathcal{O}_X(U) \tag{2-1}$$

is an epimorphism of sheaves of sets.

Fix $V \in X_{\text{ét}}$ and let $\alpha \in (\pi_0\mathcal{O}_X(U))(V)$. By definition of the sheaf $\pi_0\mathcal{O}_X(U)$, this is equivalent to the given of an étale cover $\{V_j \to V\}$ plus morphisms $V_j \to U$. For every pair of indexes i and j, let

$$V_{ij} := U_i \times_U V_j.$$

Then the collection of morphisms $\{V_{ij} \to V_j\}_i$ for j fixed is an étale cover of V_j. Furthermore, the composition $V_{ij} \to V_j \to U$ can be seen as an element in $\alpha_{ij} \in (\pi_0 \mathcal{O}_X(U))(V_j)$, while the canonical map $V_{ij} \to U_i$ defines an element in $\beta_{ij} \in (\pi_0 \mathcal{O}_X(U_i))(V_{ij})$. The construction shows that the image of β_{ij} via the canonical map

$$(\pi_0 \mathcal{O}_X(U_i))(V_{ij}) \to \pi_0 \mathcal{O}_X(U)(V_{ij})$$

coincides with α_{ij}. Since the collection of maps $\{V_{ij} \to V\}_i$ is an étale cover, we have precisely proven that (2-1) is an epimorphism of sheaves of sets.

We therefore conclude that $\mathcal{O}_X$ is a hypercomplete $\mathcal{T}_{\text{ét}}(k)$-structure on $\mathcal{X}$. Since $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$ is a geometric envelope for $\mathcal{T}_{\text{ét}}(k)$, we can identify $\mathcal{O}_X$ with a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-structure on $\mathcal{X}$. The next step is to prove that the pair $(\mathcal{X}, \mathcal{O}_X)$ is a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-scheme. To do this, we need the following criterion for a morphism of Grothendieck sites to induce an equivalence between the associated hypercomplete ∞-topoi. It is the ∞-categorical analogue of [de Jong et al. 2005–, Tag 039Z], and we refer to [Porta and Yu 2016, Proposition 2.22] for a proof.

Lemma 2.9. *Let $(\mathcal{C}, \tau)$, $(\mathcal{D}, \sigma)$ be two ∞-sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(i) *u is continuous*;

(ii) *u is cocontinuous*;

(iii) *u is fully faithful*;

(iv) *for every object $V \in \mathcal{D}$ there exists a σ-covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i \in I}$*;

(v) *for every object $D \in \mathcal{D}$, the representable presheaf h_D is a hypercomplete sheaf.*

Then the induced adjunction $\mathrm{Sh}(\mathcal{C}, \tau)^\wedge \simeq \mathrm{Sh}(\mathcal{D}, \sigma)^\wedge$ is an equivalence of ∞-categories.

Proposition 2.10. *The pair $(\mathcal{X}, \mathcal{O}_X)$ is a $\mathcal{G}^{\text{der}}_{\text{ét}}(k)$-scheme.*

Proof. Choose an étale atlas $p : \coprod U_i \to X$ in the category **DM**. Since each morphism $p_i : U_i \to X$ is étale, we see each of them defines an element in the small étale site $(X_{\text{ét}}, \tau_{\text{ét}})$. Since this site is subcanonical, we can identify each U_i with objects $V_i \in \mathcal{X}$. Moreover, the étale subtopos $(\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ is canonically identified with $(\mathrm{Sh}((U_i)_{\text{ét}}, \tau_{\text{ét}}), \mathcal{O}_{U_i})$. The construction of the (absolute) spectrum functor of [Lurie 2011b, §2.2], shows that

$$\mathrm{Spec}^{\text{ét}}(U_i) \simeq (\mathrm{Sh}((U_i)_{\text{ét}}, \tau_{\text{ét}}), \mathcal{O}_{U_i}).$$

It will therefore be sufficient to show that the morphism $\coprod V_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism. In order to do this, it is convenient to replace the small étale site $X_{\text{ét}}$

with the site $((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})$ of étale maps $Y \to X$, where Y is a geometric stack such that $\mathrm{t}_0(Y)$ is n-truncated. We claim that the natural inclusion

$$(X_{\text{ét}}, \tau_{\text{ét}}) \to ((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}}) \tag{2-2}$$

is a Morita equivalence of sites. In other words, we claim that it induces an equivalence of ∞-topoi

$$\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}}) \simeq \mathrm{Sh}((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}}).$$

Lemma 2.9 implies that the morphism (2-2) induces an equivalence of hypercomplete ∞-topoi:

$$\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})^{\wedge} \simeq \mathrm{Sh}((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})^{\wedge}. \tag{2-3}$$

Observe now that the mapping spaces in $(\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}$ are n-truncated, hence [Lurie 2009, 2.3.4.18] implies $(\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}$ is (categorically equivalent to) an n-category. Therefore, the ∞-topos $\mathrm{Sh}((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})$ is n-localic. The same statement holds for $\mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})$, as we already discussed. Therefore, in order to check that the induced adjunction is an equivalence of ∞-categories, it is enough to check that the restriction to n-truncated object is an equivalence. This follows from equivalence (2-3), since we know from [Lurie 2009, 6.5.2.9] that n-truncated objects are hypercomplete.

In this way, we see that $\mathbf{1}_{\mathcal{X}}$ is the representable sheaf associated to the identity map $\mathrm{id}_X : X \to X$. We are therefore left to show that

$$\coprod \pi_0 \operatorname{Map}(-, U_i) \to \pi_0 \operatorname{Map}(-, X)$$

is an epimorphism of sheaves on $((\mathrm{Geom}^{\leq n}_{/X})_{\text{ét}}, \tau_{\text{ét}})$. This follows immediately from the fact that the maps $U_i \to X$ were an atlas for X. □

We are left to prove that $\phi(\mathcal{X}, \mathcal{O}_X) \simeq X$. We can proceed by induction on the geometric level m of X. If $m = -1$, the statement is obvious. Otherwise, let $U_i \to X$ be an étale atlas for X. Let $U := \coprod U_i$ and let $U^\bullet$ be the Čech nerve of $U \to X$. Combining the proof of Proposition 2.10, Proposition 2.8 and the induction hypothesis, we see that $U^\bullet$ is a groupoid presentation for both X and $\phi(\mathcal{X}, \mathcal{O}_X)$. We therefore proved that the essential image of the functor

$$\phi : \mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)) \to \mathrm{Sh}(\mathrm{dAff}_k, \tau_{\text{ét}})$$

contains all the Deligne–Mumford stacks in the sense of [Toën and Vezzosi 2008].

Appendix: Descent versus hyperdescent

Let $(\mathcal{C}, \tau)$ be an ∞-Grothendieck site. It is well known that for a presheaf on $\mathcal{C}$ with values in a truncated ∞-category, descent and hyperdescent are equivalent conditions. However, we could not locate a precise reference in the literature. For this reason, we decided to include a proof of this fact:

Proposition A.1. *Let* $(\mathcal{C}, \tau)$ *be an* ∞*-Grothendieck site and let* $\mathcal{D}$ *be an* $(n+1, 1)$*-category. Then A functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ *satisfies descent if and only if it satisfies hyperdescent.*

Proof. Let $D \in \mathcal{D}$ be any object and let $c_D : \mathcal{D} \to \mathcal{S}$ be the functor *corepresented* by D. Then F satisfies descent (resp. hyperdescent) if and only if $c_D \circ F$ does. Since $\mathcal{D}$ is an $(n+1, 1)$-category, we see that $c_D \circ F$ takes values in $\tau_{\leq n}\mathcal{S}$. Therefore, we may replace $\mathcal{D}$ with $\mathcal{S}$ and suppose that F takes values in the full subcategory of n-truncated objects. For every $U \in \mathcal{C}$, let us denote by h_U the sheafification of the presheaf associated to U. Since F is an n-truncated object, we see that

$$\mathrm{Map}_{\mathrm{Sh}_{\leq n}(\mathcal{C},\tau)}(\tau_{\leq n} h_U, F) \simeq \mathrm{Map}_{\mathrm{Sh}(\mathcal{C},\tau)}(h_U, F) \simeq F(U),$$

where the last equivalence is obtained combining the universal property of the sheafification with the Yoneda lemma. Therefore, it will be sufficient to show that for every hypercover $U^\bullet \to U$ in $\mathcal{C}$, the augmented simplicial diagram

$$\tau_{\leq n} h_{U^\bullet} \to \tau_{\leq n} h_U$$

is a colimit diagram in $\mathrm{Sh}_{\leq n}(\mathcal{C}, \tau)$. Since $\tau_{\leq n}$ is a left adjoint, we see that in $\mathrm{Sh}_{\leq n}(\mathcal{C}, \tau)$ the relation

$$|\tau_{\leq n} h_{U^\bullet}| \simeq \tau_{\leq n} |h_{U^\bullet}|$$

holds. Moreover, since $U^\bullet \to U$ is an hypercover, the morphism $|h_{U^\bullet}| \to h_U$ is ∞-connected in virtue of [Lurie 2009, 6.5.3.11]. Since $\tau_{\leq n}$ commutes with ∞-connected morphisms, we conclude that

$$\tau_{\leq n} |h_{U^\bullet}| \to \tau_{\leq n} h_U$$

is an ∞-connected morphism between n-truncated objects. Therefore it is an equivalence in $\mathrm{Sh}(\mathcal{C}, \tau)$. In conclusion, the morphism $|\tau_{\leq n} h_{U^\bullet}| \to \tau_{\leq n} h_U$ is an equivalence in $\mathrm{Sh}_{\leq n}(\mathcal{C}, \tau)$. The proof is now complete. □

Acknowledgments

I considered it necessary to look for a precise comparison result after a discussion I had with Marco Robalo. I use this opportunity to thank him for all the interesting conversations we had during this year. I am grateful as well to my advisor Gabriele Vezzosi for introducing me to such an interesting topic as derived geometry, in all its facets.

References

[de Jong et al. 2005–] A. J. de Jong et al., "The Stacks Project", electronic reference, 2005–, http://stacks.math.columbia.edu.

[Lurie 2004] J. Lurie, *Derived algebraic geometry*, Ph.D. thesis, Massachusetts Institute of Technology, 2004, http://search.proquest.com/docview/305095302. MR

[Lurie 2009] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies **170**, Princeton University Press, 2009. MR Zbl

[Lurie 2011a] J. Lurie, "Derived algebraic geometry, IX: Closed immersions", preprint, 2011, http://www.math.harvard.edu/~lurie/papers/DAG-IX.pdf.

[Lurie 2011b] J. Lurie, "Derived algebraic geometry, V: Structured spaces", preprint, 2011, http://www.math.harvard.edu/~lurie/papers/DAG-V.pdf.

[Lurie 2011c] J. Lurie, "Derived algebraic geometry, VII: Spectral schemes", preprint, 2011, http://www.math.harvard.edu/~lurie/papers/DAG-VII.pdf.

[Lurie 2011d] J. Lurie, "Derived algebraic geometry, VIII: Quasi-coherent sheaves and Tannaka duality theorems", preprint, 2011, http://www.math.harvard.edu/~lurie/papers/DAG-VIII.pdf.

[Lurie 2016a] J. Lurie, "Higher algebra", preprint, 2016, http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Lurie 2016b] J. Lurie, "Spectral algebraic geometry", book in progress, 2016, http://math.harvard.edu/~lurie/papers/SAG-rootfile.pdf.

[Porta 2015] M. Porta, "Derived complex analytic geometry, I: GAGA theorems", preprint, 2015. arXiv

[Porta and Yu 2016] M. Porta and T. Y. Yu, "Higher analytic stacks and GAGA theorems", *Adv. Math.* **302** (2016), 351–409. MR Zbl

[Schwede and Shipley 2000] S. Schwede and B. E. Shipley, "Algebras and modules in monoidal model categories", *Proc. London Math. Soc.* (3) **80**:2 (2000), 491–511. MR Zbl

[Simpson 1996] C. Simpson, "Algebraic (geometric) n-stacks", preprint, 1996. arXiv

[Toën and Vaquié 2008] B. Toën and M. Vaquié, "Algébrisation des variétés analytiques complexes et catégories dérivées", *Math. Ann.* **342**:4 (2008), 789–831. MR Zbl

[Toën and Vezzosi 2005] B. Toën and G. Vezzosi, "Homotopical algebraic geometry, I: Topos theory", *Adv. Math.* **193**:2 (2005), 257–372. MR Zbl

[Toën and Vezzosi 2008] B. Toën and G. Vezzosi, *Homotopical algebraic geometry, II: Geometric stacks and applications*, Mem. Amer. Math. Soc. **902**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

Received March 29, 2016. Revised July 29, 2016.

MAURO PORTA
LABORATORY DAVID RITTENHOUSE
UNIVERSITY OF PENNSYLVANIA
209 SOUTH 33RD STREET
PHILADELPHIA, PA 19104
UNITED STATES

maurop@math.upenn.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.199

ON LOCALLY COHERENT HEARTS

MANUEL SAORÍN

Let $\mathcal{G}$ be a locally coherent Grothendieck category. We show that, under particular conditions, if a t-structure τ in the unbounded derived category $\mathcal{D}(\mathcal{G})$ restricts to the bounded derived category $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ of its category of finitely presented (i.e, coherent) objects, then its heart $\mathcal{H}_\tau$ is a locally coherent Grothendieck category on which $\mathcal{H}_\tau \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ is the class of finitely presented objects. Those particular conditions are always satisfied when $\mathcal{G}$ is arbitrary and τ is the Happel–Reiten–Smalø t-structure in $\mathcal{D}(\mathcal{G})$ associated to a torsion pair in $\mathrm{fp}(\mathcal{G})$ or when $\mathcal{G} = \mathrm{Qcoh}(\mathbb{X})$ is the category of quasicoherent sheaves on a noetherian affine scheme $\mathbb{X}$ and τ is any compactly generated t-structure in $\mathcal{D}(\mathbb{X}) := \mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))$ which restricts to $\mathcal{D}^b(\mathbb{X}) := \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. In particular, the heart of any t-structure in $\mathcal{D}^b(\mathbb{X})$ is the category of finitely presented objects of a locally coherent Grothendieck category.

1. Introduction

Beilinson, Bernstein and Deligne [1982] introduced the notion of a t-structure in a triangulated category in their study of perverse sheaves on an algebraic or analytic variety. If $\mathcal{D}$ is such a triangulated category, a t-structure is a pair of full subcategories satisfying some axioms which guarantee that their intersection is an abelian category $\mathcal{H}$, called the heart of the t-structure. This category comes with a cohomological functor $\mathcal{D} \to \mathcal{H}$. Roughly speaking, a t-structure allows to develop an intrinsic (co)homology theory, where the homology "spaces" are again objects of $\mathcal{D}$ itself.

Nowadays, t-structures are used in several branches of mathematics, with special impact in algebraic geometry, homotopical algebra, and representation theory of groups and algebras. When dealing with t-structures, a natural question arises. It

This work is backed by research projects from the Ministerio de Economía y Competitividad of Spain (MTM201346837-P) and the Fundación "Séneca" of Murcia (19880/GERM/15), both partly supported by FEDER funds. The author thanks these institutions for their support. He also thanks Carlos Parra and the referee for their comments and careful reading of the paper.
MSC2010: 18E15, 18E30, 13DXX, 14AXX, 16EXX.
Keywords: locally coherent Grothendieck category, triangulated category, derived category, t-structure, heart of a t-structure.

asks under which conditions the heart of a given t-structure is a "nice" abelian category. Using a classical hierarchy for abelian categories introduced by Grothendieck, one may think of Grothendieck and module categories as the nicest possible abelian categories. It is therefore not surprising that the question of when the heart of a t-structure is a Grothendieck or module category received much attention in recent times (see, e.g., [Hoshino et al. 2002; Colpi et al. 2007; 2011; Colpi and Gregorio 2010; Mantese and Tonolo 2012; Parra and Saorín 2016b; 2015; Psaroudakis and Vitória 2015; Nicolás et al. 2015]).

Among Grothendieck categories, the most studied ones are those that have finiteness conditions (e.g., those which are locally coherent, locally noetherian or even locally finite). Module categories over noetherian or coherent rings or over Artin algebras, or the categories of quasicoherent sheaves over coherent or noetherian schemes provide examples of such categories. A natural subsequent question would ask when a given t-structure has a heart which is a Grothendieck category with good finiteness conditions. In this paper, we tackle the question for the locally coherent condition, assuming that the t-structure lives in the (unbounded) derived category $\mathcal{D}(\mathcal{G})$ of a Grothendieck category $\mathcal{G}$ which is itself locally coherent. Although to find a general answer seems to be hopeless, it is not so when the t-structure restricts to $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$, the bounded derived category of the category of finitely presented (i.e., coherent) objects. Our basic technical result in the paper, Proposition 4.5, gives a precise list of sufficient conditions on a t-structure in $\mathcal{D}(\mathcal{G})$ so that its heart $\mathcal{H}$ is a locally coherent Grothendieck category on which $\mathcal{H}\cap\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ is the class of its finitely presented objects. As an application, we give the main results of the paper, referring the reader to the next section for the notation and terminology used:

(1) (Theorem 5.2) Let $\mathcal{G}$ be a locally coherent Grothendieck category and $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$. The associated Happel–Reiten–Smalø t-structure in $\mathcal{D}(\mathcal{G})$ restricts to $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ and has a heart which is a locally coherent Grothendieck category if, and only if, $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$ and $\boldsymbol{t}$ restricts to $\mathrm{fp}(\mathcal{G})$.

(2) (Theorem 6.3) If R is a commutative noetherian ring, then any compactly generated t-structure in $\mathcal{D}(R)$ which restricts to $\mathcal{D}^b_{\mathrm{fg}}(R)\cong\mathcal{D}^b(R\text{-mod})$ has a heart $\mathcal{H}$ which is a locally coherent Grothendieck category on which $\mathcal{H}\cap\mathcal{D}^b_{\mathrm{fg}}(R)$ is the class of its finitely presented objects.

(3) (Corollary 6.4) If R is a commutative noetherian ring, then the heart of each t-structure in $\mathcal{D}^b_{\mathrm{fg}}(R)$ is equivalent to the category of finitely presented objects of some locally coherent Grothendieck category.

Of course, when taking the affine scheme $\mathbb{X} = \mathrm{Spec}\, R$ in (2) and (3), one obtains the geometric versions mentioned in the abstract (see also Corollary 6.5).

The organization of the paper is as follows. Section 2 introduces all the concepts and terminology used in the paper. In Section 3 we give some general results about locally coherent Grothendieck categories which are used later. Section 4 contains the technical Proposition 4.5, which is central to the paper, and a few auxiliary results needed for its proof. Section 5 is dedicated to the Happel–Reiten–Smalø t-structure and the proof of Theorem 5.2. The final Section 6 gives Theorem 6.3, of which Corollary 6.4 is a direct consequence, and two lemmas needed for its proof.

2. Preliminaries and terminology

All categories in this paper will be additive and all rings will be supposed to be associative with unit, unless otherwise specified. Whenever the term "module" is used over a noncommutative ring, it will mean "left module" and, for a given ring R, we will denote by R-Mod the category of all R-modules. Let $\mathcal{A}$ be an additive category in the rest of the paragraph. If $\mathcal{C}$ is any class of objects in $\mathcal{A}$, the symbol $\mathcal{C}^{\perp}$ (resp. ${}^{\perp}\mathcal{C}$) will denote the full subcategory of $\mathcal{A}$ whose objects are those $X \in \mathrm{Ob}(\mathcal{A})$ such that $\mathrm{Hom}_{\mathcal{A}}(C, X) = 0$ (resp. $\mathrm{Hom}_{\mathcal{A}}(X, C) = 0$), for all $C \in \mathcal{C}$. The expression "$\mathcal{A}$ has products (resp. coproducts)" will mean that $\mathcal{A}$ has arbitrary set-indexed products (coproducts). If $\mathcal{S}$ is a set of objects in $\mathcal{A}$, we denote by $\mathrm{sum}(\mathcal{S})$ the class of objects which are isomorphic to a finite coproduct of objects of $\mathcal{S}$, and by $\mathrm{add}(\mathcal{S})$ the class of objects isomorphic to a direct summand of a finite coproduct of objects of $\mathcal{S}$. When $\mathcal{A}$ has coproducts, we shall say that an object X is a *compact* (or *small*) *object* when the functor $\mathrm{Hom}_{\mathcal{A}}(X, ?) : \mathcal{A} \to \mathrm{Ab}$ preserves coproducts.

Two types of additive categories will get most of our interest in this paper: *abelian categories* (see [Popescu 1973]) and *triangulated categories* (see [Neeman 2001]). Diverting from the terminology in this latter reference, for a triangulated category $\mathcal{D}$, the shift or suspension functor will be denoted by $?[1]$, putting $?[k]$ for its k-th power, for each $k \in \mathbb{Z}$. We shall use the term *class* (resp. *set*) *of generators* with two different meanings, depending on whether we are in the abelian or the triangulated context. When $\mathcal{A}$ is an abelian category with coproducts, a class (resp. set) of generators $\mathcal{S}$ is a class (set) of objects such that each object in $\mathcal{A}$ is an epimorphic image of a coproduct of objects in $\mathcal{S}$. When $\mathcal{S}$ is a class (set) of objects in the triangulated category $\mathcal{D}$, we shall say that it is a class (set) of generators if an object X of $\mathcal{D}$ is zero exactly when $\mathrm{Hom}_{\mathcal{D}}(S[k], X) = 0$, for all $S \in \mathcal{S}$ and all $k \in \mathbb{Z}$.

Given a triangulated category $\mathcal{D}$, a subcategory $\mathcal{E}$ will be called a *triangulated subcategory* when it is closed under taking extensions and $\mathcal{E}[1] = \mathcal{E}$. If, in addition, it is closed under taking direct summands, we will say that $\mathcal{E}$ is a *thick subcategory* of $\mathcal{D}$. When $\mathcal{S}$ is a set of objects of $\mathcal{D}$, we shall denote by $\mathrm{tria}_{\mathcal{D}}(\mathcal{S})$ the smallest triangulated subcategory of $\mathcal{D}$ that contains $\mathcal{S}$, and by $\mathrm{thick}_{\mathcal{D}}(\mathcal{S}))$ the smallest thick subcategory of $\mathcal{D}$ that contains $\mathcal{S}$.

For an additive category $\mathcal{A}$, we will denote by $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ the category of chain complexes of objects of $\mathcal{A}$ and the homotopy category of $\mathcal{A}$. Diverting from the classical notation, we will write superindices for chains, cycles and boundaries in ascending order. We will denote by $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^+(A)$, and $\mathcal{C}^b(\mathcal{A})$ the full subcategories of $\mathcal{C}(\mathcal{A})$ consisting of those objects isomorphic to upper bounded, lower bounded, and upper and lower bounded complexes, respectively, and similarly for $\mathcal{K}(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$, and $\mathcal{K}^b(\mathcal{A})$. Note that $\mathcal{K}(\mathcal{A})$ is always a triangulated category of which $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ are triangulated subcategories. When $\mathcal{A}$ is an abelian category, we will denote by $\mathcal{D}(\mathcal{A})$ its *derived category*, which is the one obtained from $\mathcal{C}(\mathcal{A})$ by keeping the same objects and formally inverting the quasi-isomorphisms (see [Verdier 1996] for the details). We shall denote by $\mathcal{D}^-(\mathcal{A})$, $\mathcal{D}^+(\mathcal{A})$, and $\mathcal{D}^b(\mathcal{A})$ the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of those complexes X such that $H^k(X) = 0$, for all $k \gg 0$, $k \ll 0$, and $|k| \gg 0$, respectively, where $H^k : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ denotes the k-th homology functor, for each $k \in \mathbb{Z}$. The objects of $\mathcal{D}^-(\mathcal{A})$ (resp. $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^b(\mathcal{A})$) will be called *homologically upper bounded* (resp. *homologically lower bounded*, *homologically bounded*) *complexes*. For integers $m \leq n$, we will denote by $\mathcal{D}^{[m,n]}(\mathcal{A})$ the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of the complexes X such that $H^k(X) = 0$ for integers k not in the closed interval $[m, n]$. We will also use $\mathcal{D}^{\leq n}(\mathcal{A})$, $\mathcal{D}^{<n}(\mathcal{A})$, $\mathcal{D}^{\geq n}(\mathcal{A})$, and $\mathcal{D}^{>n}(\mathcal{A})$ to denote the full subcategories consisting of the complexes X such that $H^i(X) = 0$, for all $i > n$, $i \geq n$, $i < n$, and $i \leq n$, respectively.

Strictly speaking, for a general abelian category $\mathcal{A}$, the category $\mathcal{D}(\mathcal{A})$ need not exist since the morphisms between two given objects could form a proper class and not just a set. However, this problem disappears when $\mathcal{A} = \mathcal{G}$ is a *Grothendieck category*. This is a cocomplete abelian category with a set of generators on which direct limits are exact. In a Grothendieck category $\mathcal{G}$ an object S is called *finitely presented* when $\mathrm{Hom}_{\mathcal{G}}(S, ?) : \mathcal{G} \to \mathrm{Ab}$ preserves direct limits. We say that $\mathcal{G}$ is *locally finitely presented* when it has a set of finitely presented generators. The reader is referred to [Crawley-Boevey 1994] for the corresponding more general concept of locally finitely presented additive categories with direct limits and is invited to check that, in the case of Grothendieck categories, it coincides with the one given here. Recall that an object in a Grothendieck category is called *noetherian* when it satisfies the ascending chain condition on subobjects. A *locally noetherian* Grothendieck category is a Grothendieck category which has a set of noetherian generators. When $\mathcal{G}$ is locally finitely presented and locally noetherian, an object N of $\mathcal{G}$ is noetherian if and only if it is finitely presented. (See [Krause 1997, Proposition A.11] for one direction, the reverse one being obvious since each noetherian object in such a category is an epimorphic image of a finitely presented one and the kernel of this epimorphism is again noetherian.)

Recall that if $\mathcal{D}$ and $\mathcal{A}$ are a triangulated and an abelian category, respectively, then an additive functor $H : \mathcal{D} \to \mathcal{A}$ is a *cohomological functor* when, given any triangle $X \to Y \to Z \xrightarrow{+}$, one gets an induced long exact sequence in $\mathcal{A}$:

$$\cdots \to H^{n-1}(Z) \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots,$$

where $H^n := H \circ (?[n])$, for each $n \in \mathbb{Z}$.

A *torsion pair* in the abelian category $\mathcal{A}$ is a pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories such that $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and each object X of $\mathcal{A}$ fits into an exact sequence $0 \to T_X \to X \to F_X \to 0$, where $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$. In this latter case the assignments $X \rightsquigarrow T_X$ and $X \rightsquigarrow F_X$ extend to endofunctors $t, (1 : t) : \mathcal{A} \to \mathcal{A}$. The functor t is usually called the *torsion radical* associated to $\boldsymbol{t}$. The torsion pair $\boldsymbol{t}$ will be called *hereditary* when $\mathcal{T}$ is closed under taking subobjects in $\mathcal{A}$.

Now let $\mathcal{D}$ be a triangulated category. A *t-structure* in $\mathcal{D}$ (see [Beilinson et al. 1982, Section 1]) is a pair $\tau = (\mathcal{U}, \mathcal{W})$ of full subcategories, closed under taking direct summands in $\mathcal{D}$, which satisfy the following properties:

(i) $\mathrm{Hom}_{\mathcal{D}}(U, W[-1]) = 0$, for all $U \in \mathcal{U}$ and $W \in \mathcal{W}$.

(ii) $\mathcal{U}[1] \subseteq \mathcal{U}$.

(iii) For each $X \in \mathrm{Ob}(\mathcal{D})$, there is a triangle $U \to X \to V \xrightarrow{+}$ in $\mathcal{D}$, where $U \in \mathcal{U}$ and $V \in \mathcal{W}[-1]$.

In this case $\mathcal{W} = \mathcal{U}^{\perp}[1]$ and $\mathcal{U} = {}^{\perp}(\mathcal{W}[-1]) = {}^{\perp}(\mathcal{U}^{\perp})$ and, for this reason, we will write a t-structure as $\tau = (\mathcal{U}, \mathcal{U}^{\perp}[1])$. We will call $\mathcal{U}$ and $\mathcal{U}^{\perp}$ the *aisle* and the *co-aisle* of the t-structure. The objects U and V in the above triangle are uniquely determined by X, up to isomorphism, and define functors $\tau_{\mathcal{U}} : \mathcal{D} \to \mathcal{U}$ and $\tau^{\mathcal{U}^{\perp}} : \mathcal{D} \to \mathcal{U}^{\perp}$ which are right and left adjoints to the respective inclusion functors. We call them the *left and right truncation functors* with respect to the given t-structure. The full subcategory $\mathcal{H} = \mathcal{U} \cap \mathcal{W} = \mathcal{U} \cap \mathcal{U}^{\perp}[1]$ is called the *heart* of the t-structure and it is an abelian category, where the short exact sequences "are" the triangles in $\mathcal{D}$ with its three terms in $\mathcal{H}$. Moreover, with the obvious abuse of notation, the assignments $X \rightsquigarrow (\tau_{\mathcal{U}} \circ \tau^{\mathcal{U}^{\perp}[1]})(X)$ and $X \to (\tau^{\mathcal{U}^{\perp}[1]} \circ \tau_{\mathcal{U}})(X)$ define naturally isomorphic functors $\mathcal{D} \to \mathcal{H}$ which are cohomological (see [Beilinson et al. 1982]). We will identify them and denote the corresponding functor by $\widetilde{H}$. When $\mathcal{D}$ has coproducts, the t-structure τ will be called *compactly generated* when there is a set $\mathcal{S} \subseteq \mathcal{U}$, formed by compact objects in $\mathcal{D}$, such that $\mathcal{W}[-1] = \mathcal{U}^{\perp}$ consists of the objects Y such that $\mathrm{Hom}_{\mathcal{D}}(S[k], Y) = 0$, for all $S \in \mathcal{S}$ and integers $k \geq 0$.

When $\mathcal{D}$ is a triangulated category with coproducts, we will use the term *Milnor colimit* of a sequence of morphisms $X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \cdots \xrightarrow{x_n} X_n \xrightarrow{x_{n+1}} \cdots$, which in [Neeman 2001] is called the homotopy colimit. It will be denoted $\mathrm{Mcolim}(X_n)$, without reference to the x_n.

3. Generalities about locally coherent Grothendieck categories

In this section we are interested in a particular case of locally finitely presented Grothendieck categories. Let us start with the following result which is folklore.

Lemma 3.1. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a full additive subcategory. The following assertions are equivalent*:

(1) *$\mathcal{B}$ is an abelian category such that the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ is exact.*

(2) *$\mathcal{B}$ is closed under taking finite* (*co*)*products*, *kernels and cokernels in $\mathcal{A}$.*

In this case we will say that $\mathcal{B}$ is an abelian exact subcategory *of $\mathcal{A}$.*

Note that if $\mathcal{G}$ is a locally finitely presented Grothendieck category, then the class $\mathrm{fp}(\mathcal{G})$ of finitely presented objects is skeletally small and is closed under taking cokernels and finite coproducts.

Definition. A Grothendieck category $\mathcal{G}$ is called *locally coherent* when it is locally finitely presented and the subcategory $\mathrm{fp}(\mathcal{G})$ is an abelian exact subcategory of $\mathcal{G}$ (equivalently, when $\mathrm{fp}(\mathcal{G})$ is closed under taking kernels).

Recall that a *pseudokernel* of a morphism $f : X \to Y$ in the additive category $\mathcal{A}$ is a morphism $u : Z \to X$ such that the sequence of contravariant functors $\mathrm{Hom}_{\mathcal{A}}(?, Z) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{A}}(?, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(?, Y)$ is exact, and similarly, a *pseudo-cokernel* of a morphism $f : X \to Y$ in the additive category $\mathcal{A}$ is a morphism $v : Y \to Z$ such that the sequence of covariant functors

$$\mathrm{Hom}_{\mathcal{A}}(Z, ?) \xrightarrow{v^*} \mathrm{Hom}_{\mathcal{A}}(Y, ?) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, ?)$$

is exact. We say that $\mathcal{A}$ *has pseudokernels* (resp. *pseudocokernels*) when each morphism in $\mathcal{A}$ has a pseudokernel (pseudocokernel).

Examples 3.2. Here are some locally coherent Grothendieck categories to which the results in this and next section apply. The first is well-known; for the others we provide a brief justification.

1. R-Mod, when R is a left coherent ring R (i.e., when each finitely generated left ideal of R is finitely presented).

2. The category $[\mathcal{C}, \mathrm{Ab}]$ (resp. $[\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}]$) of covariant (resp. contravariant) additive functors $\mathcal{C} \to \mathrm{Ab}$, where $\mathcal{C}$ is a skeletally small additive category with pseudo-cokernels (pseudokernels). In particular, when $\mathcal{C}$ is a skeletally small abelian or triangulated category, both $[\mathcal{C}, \mathrm{Ab}]$ and $[\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}]$ are locally coherent Grothendieck categories.

The covariant version follows from Propositions 1.3 and 2.1 of [Herzog 1997], taking into account that, in the second of these, the proof that each representable functor $(X, ?)$ is a coherent object only requires that each morphism $X \to Y$ has a pseudocokernel. The contravariant version follows by duality.

3. The category $\mathrm{Qcoh}(\mathbb{X})$ of quasicoherent sheaves, where $\mathbb{X}$ is a *coherent scheme*, i.e., a quasicompact and quasiseparated scheme admitting a covering $\mathbb{X} = \bigcup_{i \in I} U_i$ by affine open subschemes U_i such that $U_i = \operatorname{Spec} A_i$, for a commutative coherent ring A_i, for each $i \in I$.

For the proof, see [Garkusha 2009, Proposition 40], and also [Sitte 2014, Example 1.1.6.iv].

4. Any locally noetherian and locally finitely presented Grothendieck category.

This is clear, since $\mathrm{fp}(\mathcal{G})$ coincides with the class of noetherian objects in that case, and this latter class is always closed under taking kernels (even subobjects).

Lemma 3.3. *Let $\mathcal{G}$ be a locally coherent Grothendieck category, let $\mathcal{S}$ be a set of finitely presented generators of $\mathcal{G}$ and let M be any object in $\mathcal{D}(\mathcal{G})$. The following assertions hold*:

(1) *M is a homologically upper bounded complex whose homology objects are finitely presented if, and only if, M is isomorphic in $\mathcal{D}(\mathcal{G})$ to an upper bounded complex N of objects in* $\mathrm{sum}(\mathcal{S})$. *Moreover, N can be chosen such that* $\max\{i \in \mathbb{Z} : N^i \neq 0\} = \max\{i \in \mathbb{Z} : H^i(M) \neq 0\}$.

(2) *M is homologically bounded and its homology objects are finitely presented if, and only if, M is isomorphic in $\mathcal{D}(\mathcal{G})$ to a bounded complex*

$$\cdots\ 0 \to N^m \to N^{m+1} \to N^{n-1} \to N^n \to 0\ \cdots,$$

where the N^i are finitely presented objects (*and $N^i \in$* $\mathrm{sum}(\mathcal{S})$, *for $m < i \leq n$*).

If, moreover, the objects of $\mathcal{S}$ form a set of compact generators of $\mathcal{D}(\mathcal{G})$, then the following assertion also holds:

(3) *The compact objects of $\mathcal{D}(\mathcal{G})$ are those isomorphic to direct summands of bounded complexes of objects in* $\mathrm{add}(\mathcal{S})$.

Proof. We will frequently use the fact that if M is a complex whose homology objects are all finitely presented, then a given k-cycle object $Z^k = Z^k(M)$ is finitely presented if and only if so is the k-boundary object $B^k = B^k(M)$.

(1) The proof of this assertion is reminiscent of the dual of the proof of Lemma 4.6(3) in [Hartshorne 1966, Chapter I], with $\mathcal{A}' = \mathrm{fp}(\mathcal{G})$ and $\mathcal{A} = \mathcal{G}$, although the assumptions of that lemma do not hold in our situation. By truncating at the greatest integer i such that $H^i(M) \neq 0$ and shifting if necessary, we can assume without loss of generality that M is concentrated in degrees ≤ 0 and that $H^0(M) \neq 0$. We then inductively construct a sequence in $\mathcal{C}(\mathcal{G})$

$$\cdots\ M_n \xrightarrow{f_n} M_{n-1} \to \cdots \to M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M$$

satisfying the following properties:

(a) Each M_n is concentrated in degrees ≤ 0.

(b) The connecting chain maps $f_n : M_n \to M_{n-1}$ are quasi-isomorphisms, for all $n \in \mathbb{N}$ (with the convention that $M_{-1} = M$).

(c) Given $n \in \mathbb{N}$, one has $M_n^{-k} \in \operatorname{sum}(\mathcal{S})$ for $0 \leq k \leq n$.

(d) Given any $k \in \mathbb{N}$, the morphism $f_n^{-k} : M_n^{-k} \to M_{n-1}^{-k}$ is the identity map, for all $n > k$.

Once the sequence has been constructed, we clearly see that the inverse limit of the sequence, $X := \varprojlim_{\mathcal{C}(\mathcal{G})}(M_n)$, is a complex of objects in $\operatorname{sum}(\mathcal{S})$ concentrated in degrees ≤ 0 such that the induced chain map $X \to M$ is a quasi-isomorphism.

We now move on to construct the mentioned sequence. At the initial step, one easily gets a morphism $f : X^0 \to M^0$ such that $X^0 \in \operatorname{add}(\mathcal{S})$ and the composition $X^0 \xrightarrow{f} M^0 \xrightarrow{p} H^0(M)$ is an epimorphism, where p is the projection. Now, taking the pullback of f and the differential $M^{-1} \to M^0$, we easily get a quasi-isomorphism $f_0 : M_0 \to M$, where $f_0^{-k} : M_0^{-k} = M^{-k} \to M^{-k}$ is the identity map for all $k \geq 2$, and $f_0^0 : M_0^0 = X^0 \to M^0$ is f.

Assume now that $n > 0$ and that the quasi-isomorphisms

$$M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \to \cdots \to M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M$$

have already been constructed, satisfying the requirements. Note that $Z^{-k} := Z^{-k}(M_{n-1})$, and hence also $B^{-k} := B^{-k}(M_{n-1})$, are finitely presented objects for $k = 0, 1, \ldots, n-1$. Let us fix a direct system $(Y_i)_{i \in I}$ in $\operatorname{fp}(\mathcal{G})$ such that $\varinjlim Y_i \cong M_{n-1}^{-n}$. Replacing the directed set I by a cofinal subset if necessary, there is no loss of generality in assuming that the composition

$$Y_j \xrightarrow{u_j} \varinjlim Y_i \cong M_{n-1}^{-n} \xrightarrow{d^{-n}} B^{-n+1}$$

is an epimorphism, for all $j \in I$, where u_j is the canonical morphism to the direct limit. It is seen in a straightforward way that we have a direct system of exact sequences

$$0 \to u_i^{-1}(Z^{-n}) \to Y_i \xrightarrow{d^{-n} \circ u_i} B^{-n+1} \to 0 \qquad (i \in I)$$

whose direct limit is precisely the canonical exact sequence

$$0 \to Z^{-n} \to X^{-n} \xrightarrow{d^{-n}} B^{-n+1} \to 0.$$

Due to the fact that $H^{-n} := H^{-n}(M_{n-1})$ is finitely presented, there is some index $j \in I$ such that the composition $u_j^{-1}(Z^{-n}) \xrightarrow{u_j} Z^{-n} \xrightarrow{p} H^{-n}$ is an epimorphism. We fix such an index j and choose any epimorphism $\epsilon : X^{-n} \twoheadrightarrow Y_j$, with $X^{-n} \in \operatorname{sum}(\mathcal{S})$. Putting $M_n^{-n} := X^{-n}$, the composition $g : M_n^{-n} \xrightarrow{\epsilon} Y_j \xrightarrow{u_j} \varinjlim(Y_j) \cong M_{n-1}^{-n}$ is then a morphism which leads to the following commutative diagram, where all squares are bicartesian:

$$\begin{array}{ccccccc}
M_n^{-n-1} & \longrightarrow & \widetilde{B}^{-n} & \hookrightarrow & g^{-1}(Z^{-n}) & \hookrightarrow & M_n^{-n} \\
\downarrow{\scriptstyle g'} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\
M_{n-1}^{-n-1} & \xrightarrow{d^{-n-1}} & B^{-n} & \hookrightarrow & Z^{-n} & \hookrightarrow & M_{n-1}^{-n}
\end{array}$$

We derive a quasi-isomorphism $h : M_n \to M_{n-1}$, where $h^{-k} : M_n^{-k} = M_{n-1}^{-k} \to M_{n-1}^{-k}$ is the identity map, for $k \geq 0$ and $k \neq n, n+1$, and where $h^{-n-1} = g'$ and $h^{-n} = g$ are the morphisms from the last diagram.

(2) By assertion (1), we can assume that M is of the form

$$\cdots \to N^k \to N^{k+1} \to \cdots \to N^{n-1} \to N^n \to 0 \ \cdots,$$

where the N^i are in $\mathrm{sum}(\mathcal{S})$. Let us assume that $m = \min\{j \in \mathbb{Z} : H^j(M) \neq 0\}$. Then the intelligent truncation at m gives the complex

$$\tau^{\geq m} M : \ \cdots\ 0 \to B^m \hookrightarrow N^m \to N^{m+1} \to \cdots \to N^{n-1} \to N^n \to 0\ \cdots,$$

where B^m is an m-boundary object of M. But B^m is finitely presented because $Z^m = \mathrm{Ker}(N^m \to N^{m+1})$ is. We then take $N^m = B^m$ and the proof of the implication is complete because the canonical map $\tau^{\geq m} M \to M$ is an isomorphism in $\mathcal{D}(\mathcal{G})$.

In the rest of the proof, we assume that $\mathcal{S}$ is a set of compact generators of $\mathcal{D}(\mathcal{G})$.

(3) Note that each bounded complex of objects in $\mathrm{add}(\mathcal{S})$ is compact in $\mathcal{D}(\mathcal{G})$ since it is a finite iterated extension of stalks $X[k]$, with $X \in \mathrm{add}(\mathcal{S})$. Conversely, suppose that M is a compact object in $\mathcal{D}(\mathcal{G})$. It follows from [Keller 1994, Theorem 5.3] that it is a direct summand of a finite iterated extension of complexes of the form $S[k]$, with $S \in \mathcal{S}$ and $k \in \mathbb{Z}$. In particular M has bounded and finitely presented homology. If we fix now a quasi-isomorphism $f : P \to M$ such that P is a bounded above complex of objects in $\mathrm{add}(\mathcal{S})$, then we can assume without loss of generality that $P^0 \neq 0 = P^k$, for all $k > 0$. Note that then P is the Milnor colimit of the stupid truncations $\sigma^{\geq -n} P$. Since P is compact in $\mathcal{D}(\mathcal{G})$, an argument as in the proof of [Keller 1994, Theorem 5.3] shows that the identity map 1_P factors in the form $P \to \sigma^{\geq -n} P \to P$, for some $n \in \mathbb{N}$. It follows that $M \cong P$ is isomorphic in $\mathcal{D}(\mathcal{G})$ to a direct summand of a bounded complex of objects in $\mathrm{add}(\mathcal{S})$. □

When $\mathcal{G}$ is a locally coherent Grothendieck category, one easily gets from assertions (1) and (2) of the last lemma that $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ is equivalent, as a triangulated category, to the full subcategory $\mathcal{D}^b_{\mathrm{fp}}(\mathcal{G})$ of $\mathcal{D}(\mathcal{G})$ consisting of those complexes $M \in \mathcal{D}^b(\mathcal{G})$ such that $H^i(M) \in \mathrm{fp}(\mathcal{G})$, for all $i \in \mathbb{Z}$. In the sequel we will identify these equivalent triangulated categories, viewing $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ as a full triangulated subcategory of $\mathcal{D}(\mathcal{G})$.

Definition. Let $\mathcal{G}$ be a locally finitely presented Grothendieck category. An object Y of $\mathcal{G}$ will be called fp-*injective* when $\mathrm{Ext}^1_{\mathcal{G}}(?, Y)$ vanishes on finitely presented objects.

The following is an easy consequence of the proof of implication 1) $\Rightarrow$ 2) in [Šťovíček 2014, Proposition B.3], after a clear induction argument:

Lemma 3.4. *Let $\mathcal{G}$ be a locally coherent Grothendieck category. If Y is an* fp-*injective object of $\mathcal{G}$, then $\mathrm{Ext}^k_{\mathcal{G}}(?; Y)$ vanishes on finitely presented objects, for all $k > 0$.*

Recall that if $F : \mathcal{G} \to \mathrm{Ab}$ is any left exact functor, then an object Y of $\mathcal{G}$ is F*-acyclic* when the right derived functors $\boldsymbol{R}^k F : \mathcal{G} \to \mathrm{Ab}$ vanish on Y, for all $k > 0$. Recall also that, for each $X \in \mathrm{Ob}(\mathcal{G})$, one can calculate $\boldsymbol{R}^k F(X)$ by considering F-acyclic resolutions. That is, if one picks an exact sequence $0 \to X \to Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} \cdots Y^n \xrightarrow{d^n} \cdots$, where all the Y^k are F-acyclic, then $\boldsymbol{R}^k F(X)$ is the k-th homology group of the complex

$$\cdots\ 0 \to F(Y^0) \xrightarrow{F(d^0)} F(Y^1) \xrightarrow{F(d^1)} \cdots F(Y^n) \xrightarrow{F(d^n)} \cdots ,$$

for each integer $k \geq 0$. The following result seems to be well-known (see [Gillespie 2016, Introduction] or [Prest 2009, Chapter 11]), but we include a proof after not finding an explicit one in the literature.

Proposition 3.5. *Let $\mathcal{G}$ be a locally finitely presented Grothendieck category, let X be a finitely presented object, let $(M_i)_{i\in I}$ be a direct system in $\mathcal{G}$ and consider the canonical map $\mu_k : \varinjlim \mathrm{Ext}^k_{\mathcal{G}}(X, M_i) \to \mathrm{Ext}^k_{\mathcal{G}}(X, \varinjlim M_i)$, for each integer $k \geq 0$.*

(1) *μ_0 is an isomorphism and μ_1 is a monomorphism.*

(2) *When $\mathcal{G}$ is locally coherent, μ_k is an isomorphism, for all $k \geq 0$.*

Proof. (1) The case $k = 0$ follows from the definition of a finitely presented object. An element of $\varinjlim \mathrm{Ext}^1_{\mathcal{G}}(X, M_i)$ is represented by a direct system $(\epsilon_i)_{i\in I}$ of exact sequences

$$\epsilon_i : 0 \to M_i \to N_i \to X \to 0$$

whose "projection" on the first component is precisely the direct system $(M_i)_{i\in I}$ and where X is viewed as a constant direct system. The image of (ϵ_i) by the canonical map $\varinjlim \mathrm{Ext}^1_{\mathcal{G}}(X, M_i) \to \mathrm{Ext}_{\mathcal{G}}(X, \varinjlim M_i)$ is the induced exact sequence

$$0 \to \varinjlim M_i \to \varinjlim N_i \xrightarrow{\pi} X \to 0.$$

If this latter sequence splits and we fix a section $\mu : X \to \varinjlim N_i$ for π, then, since X is a finitely presented object, μ factors in the form $X \xrightarrow{\mu_j} N_j \xrightarrow{u_j} \varinjlim N_i$, for some $j \in I$, where u_j is the canonical morphism to the direct limit. This immediately implies that the j-th sequence $\epsilon_j : 0 \to M_j \to N_j \to X \to 0$ splits and, hence, that $(\epsilon_i)_{i\in I}$ is the zero element of $\varinjlim \operatorname{Ext}^1_{\mathcal{G}}(X, M_i)$.

(2) By [Adámek and Rosický 1994, Corollary 1.7 and subsequent remark], we can assume without loss of generality that $I = \lambda = \{\alpha \text{ ordinal} : \alpha < \lambda\}$ is an infinite limit ordinal and that, for each limit ordinal $\alpha < \lambda$, one has $M_\alpha = \varinjlim_{\beta<\alpha} M_\beta$. We now construct a direct system $(E_\alpha)_{\alpha<\lambda}$ in the category $\mathcal{C}(\mathcal{G})$ of complexes, satisfying the following properties:

(a) $E_\alpha : \cdots 0 \to E^0_\alpha \to E^1_\alpha \to \cdots \to E^n_\alpha \to \cdots$ is a complex concentrated in degrees ≥ 0 and $H^k(E_\alpha) = 0$, for all $\alpha < \lambda$ and all $k \neq 0$.

(b) E^n_α is an fp-injective object, for all $\alpha < \lambda$ and all integers $n \geq 0$.

(c) The direct system $(H^0(E_\alpha))_{\alpha<\lambda}$ in $\mathcal{G}$ is isomorphic to $(M_\alpha)_{\alpha<\lambda}$.

Once the direct system $(E_\alpha)_{\alpha<\lambda}$ is constructed, the exactness of the direct limit functor in $\mathcal{G}$ and the fact that the class of fp-injective objects is closed under taking direct limits (see [Šťovíček 2014, Proposition B.3]) will give that $E_\lambda := \varinjlim_{\mathcal{C}(\mathcal{G})} E_\alpha$ is a complex of fp-injective objects concentrated in degrees ≥ 0 whose only nonzero homology object is $H^0(E_\lambda) \cong \varinjlim_{\alpha<\lambda} M_\alpha$. That is, E_λ is a (deleted) fp-injective resolution of $M := \varinjlim_{\alpha<\lambda} M_\alpha$. By the previous lemma, we know that each fp-injective object is $\operatorname{Hom}_{\mathcal{G}}(X, ?)$-acyclic, whenever $X \in \operatorname{fp}(\mathcal{G})$. It follows that, for such an X, we have that $\operatorname{Ext}^k_{\mathcal{G}}(X, M)$ is the k-th homology abelian group of the complex $\operatorname{Hom}_{\mathcal{G}}(X, E_\lambda)$. But, by definition of E_λ and the fact that $\operatorname{Hom}_{\mathcal{G}}(X, ?)$ preserves direct limits, we have an isomorphism of complexes of abelian groups $\varinjlim_{\mathcal{C}(\mathrm{Ab})}(\operatorname{Hom}_{\mathcal{G}}(X, E_\alpha)) \cong \operatorname{Hom}_{\mathcal{G}}(X, E_\lambda)$. Then the k-th homology map will give the desired isomorphism

$$\varinjlim \operatorname{Ext}^k_{\mathcal{G}}(X, M_\alpha) \xrightarrow{\cong} \operatorname{Ext}^k_{\mathcal{G}}(X, M) = \operatorname{Ext}^k_{\mathcal{G}}(X, \varinjlim_{\alpha<\lambda} M_\alpha).$$

It remains to construct the direct system $(E_\alpha)_{\alpha<\lambda}$ in $\mathcal{C}(\mathcal{G})$. Let $u_\alpha : M_\alpha \to M_{\alpha+1}$ denote the morphism from the direct system $(M_\alpha)_{\alpha<\lambda}$. For a nonlimit ordinal α, E_α will be the (deleted) minimal injective resolution of M_α. If α is a limit ordinal and we already have defined the direct system $(E_\beta)_{\beta<\alpha}$, then $E_\alpha = \varinjlim_{\beta<\alpha} E_\beta$, where the direct limit is taken in $\mathcal{C}(\mathcal{G})$. Note that $H^0(E_\alpha) \cong \varinjlim_{\beta<\alpha} M_\beta = M_\alpha$. For the construction of $(E_\alpha)_{\alpha<\lambda}$ one just needs to define the connecting chain map $E_\alpha \to E_{\alpha+1}$, when $\alpha < \lambda$ is any ordinal for which E_α is already defined. This connecting chain map is defined by choosing a family $(f^n_\alpha : E^n_\alpha \to E^n_{\alpha+1})_{n\geq 0}$ of morphisms in $\mathcal{G}$ such that the following diagram is commutative and the induced map $\mathrm{Ker}(E^0_\alpha \to E^1_\alpha) \to \mathrm{Ker}(E^0_{\alpha+1} \to E^1_{\alpha+1})$ is the morphism $u_\alpha : M_\alpha \to M_{\alpha+1}$:

$$\begin{tikzcd} 0 \arrow[r] & E^0_\alpha \arrow[r] \arrow[d, "f^0_\alpha"] & E^1_\alpha \arrow[r] \arrow[d, "f^1_\alpha"] & \cdots \\ 0 \arrow[r] & E^0_{\alpha+1} \arrow[r] & E^1_{\alpha+1} \arrow[r] & \cdots \end{tikzcd}$$

The reader is invited to check that the direct system $(E_\alpha)_{\alpha<\lambda}$ satisfies all the requirements. □

4. Some sufficient conditions for the heart to be a locally coherent Grothendieck category

Definition. Let $\mathcal{D}'$ be a full triangulated subcategory of the triangulated category $\mathcal{D}$ and let $(\mathcal{U}, \mathcal{U}^\perp[1])$ be a t-structure in $\mathcal{D}$. We say that this t-structure *restricts to* $\mathcal{D}'$ when $(\mathcal{U} \cap \mathcal{D}', (\mathcal{U}^\perp \cap \mathcal{D}')[1])$ is a t-structure of $\mathcal{D}'$. This is equivalent to saying that, for each object X of $\mathcal{D}'$, the truncation triangle $\tau_{\mathcal{U}}(X) \to X \to \tau^{\mathcal{U}^\perp}(X) \xrightarrow{+}$ has its three vertices in $\mathcal{D}'$.

Lemma 4.1. *Let $\mathcal{D}'$ be a full triangulated subcategory of $\mathcal{D}$ and let $(\mathcal{U}, \mathcal{U}^\perp[1])$ be a t-structure in $\mathcal{D}$ whose heart is $\mathcal{H}$. If the t-structure restricts to $\mathcal{D}'$, then $\mathcal{H} \cap \mathcal{D}'$ is an abelian exact subcategory of $\mathcal{H}$.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{H} \cap \mathcal{D}'$ and complete it to a triangle, which is in $\mathcal{D}'$:

$$X \xrightarrow{f} Y \to Z \xrightarrow{+} .$$

Note that then $Z \in \mathcal{U} \cap \mathcal{U}^\perp[2]$ and hence $Z[-1] \in \mathcal{U}^\perp[1]$. According to [Parra and Saorín 2015, Lemma 3.1], we have $\widetilde{H}(Z) = \tau^{\mathcal{U}^\perp}(Z[-1])[1]$ and $\widetilde{H}(Z[-1]) = \tau_{\mathcal{U}}(Z[-1])$. Moreover, since the t-structure restricts to $\mathcal{D}'$ we get that both $\widetilde{H}(Z)$ and $\widetilde{H}(Z[-1])$ are in $\mathcal{H} \cap \mathcal{D}'$. But we then have a triangle

$$\widetilde{H}(Z[-1])[1] \to Z \to \widetilde{H}(Z) \xrightarrow{+} .$$

By [Beilinson et al. 1982], we have isomorphisms $\mathrm{Ker}_{\mathcal{H}}(f) \cong \widetilde{H}(Z[-1])$ and $\mathrm{Coker}_{\mathcal{H}}(f) \cong \widetilde{H}(Z)$ and, hence, $\mathcal{H} \cap \mathcal{D}'$ is closed under taking kernels and cokernels in $\mathcal{H}$. That it is also closed under taking finite coproducts is clear. □

Setting 4.2. In the rest of the section we assume that $\mathcal{G}$ is a locally coherent Grothendieck category and we fix a set $\mathcal{S}$ of finitely presented generators of $\mathcal{G}$. Recall that then $\mathcal{S}$ is also a set of generators of $\mathcal{D}(\mathcal{G})$ as a triangulated category (see [Nicolás et al. 2015, Lemma 9] or [Psaroudakis and Vitória 2015, Lemma 4.10]).

Lemma 4.3. *Let $X \in \mathcal{D}^{\leq 0}(\mathcal{G})$ have bounded finitely presented homology* (*i.e.*, *X is homologically bounded and $H^k(X) \in \mathrm{fp}(\mathcal{G})$, for all $k \in \mathbb{Z}$*) *and let n be a natural* *number. There is a complex* $P \in \mathcal{C}^b(\mathrm{sum}(\mathcal{S}))$ *together with a morphism* $g : P \to X$ *in* $\mathcal{D}(\mathcal{G})$ *such that the restriction of the natural transformation* $g^* : \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, ?) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(P, ?)$ *to* $\mathcal{D}^{[-n,0]}(\mathcal{G})$ *is a natural isomorphism.*

Proof. By Lemma 3.3, there is an isomorphism $p : Q \to X$ in $\mathcal{D}(\mathcal{G})$ such that Q is a complex of objects in $\mathrm{sum}(\mathcal{S})$ concentrated in degrees ≤ 0. We have that $p^* : \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, ?) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(Q, ?)$ is a natural isomorphism of functors $\mathcal{D}(\mathcal{G}) \to \mathrm{Ab}$. Stupid truncation at $-n-2$ gives a triangle in $\mathcal{K}(\mathcal{G})$

$$\sigma^{>-n-2}Q \xrightarrow{h} Q \to \sigma^{\leq -n-2}Q \xrightarrow{+},$$

where the left vertex is in $\mathcal{C}^b(\mathrm{sum}(\mathcal{S}))$. Since $\mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(\sigma^{\leq -n-2}Q[k], ?)$ vanishes on $\mathcal{D}^{[-n,0]}(\mathcal{G})$, for $k = -1, 0$, we get that the restriction of the natural transformation

$$h^* : \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(Q, ?) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(\sigma^{>-n-2}Q, ?)$$

to $\mathcal{D}^{[-n,0]}(\mathcal{G})$ is an isomorphism. Putting $P := \sigma^{>-n-2}Q$, the desired morphism g is the composition $P \xrightarrow{h} Q \xrightarrow{p} X$. □

Remark 4.4. Let $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ be a t-structure in any triangulated category $\mathcal{D}$ and suppose that it restricts to a full triangulated subcategory $\mathcal{D}'$. If $\widetilde{H} : \mathcal{D} \to \mathcal{H}$ is the associated cohomological functor, then $\widetilde{H}(M)$ is in $\mathcal{H} \cap \mathcal{D}'$, for all $M \in \mathcal{D}'$. This is because $\tau_{\mathcal{U}}(\mathcal{D}') \subseteq \mathcal{D}'$ and $\tau^{\mathcal{U}^{\perp}[1]}(\mathcal{D}') \subseteq \mathcal{D}'$.

The following technical result is crucial for the main results of the paper.

Proposition 4.5. *Let* $\mathcal{G}$ *and* $\mathcal{S}$ *be as in* Setting 4.2, *let* $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ *be a t-structure in* $\mathcal{D}(\mathcal{G})$, *with heart* $\mathcal{H}$, *and let* $\widetilde{H} : \mathcal{D}(\mathcal{G}) \to \mathcal{H}$ *be the associated cohomological functor. Suppose that the following conditions hold*:

(1) $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ *restricts to* $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$.

(2) *There exist integers* $m \leq n$ *such that* $\mathcal{D}^{\leq m}(\mathcal{G}) \subseteq \mathcal{U} \subseteq \mathcal{D}^{\leq n}(\mathcal{G})$.

(3) $\mathcal{H} \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ *is a (skeletally small) class of generators of* $\mathcal{H}$.

(4) *For each direct system* $(M_i)_{i \in I}$ *in* $\mathcal{H}$, *for each* $S \in \mathcal{S}$ *and for each* $k \in \mathbb{Z}$, *the canonical map* $\eta_{S[k]} : \varinjlim \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(S[k], M_i) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(S[k], \varinjlim_{\mathcal{H}} M_i)$ *is an isomorphism.*

Then $\mathcal{H}$ *is a locally coherent Grothendieck category on which* $\mathcal{H} \cap \mathcal{D}^b(\mathcal{G})$ *is the class of its finitely presented objects.*

Proof. Take the cohomological functor $H' := \coprod_{S \in \mathcal{S}} \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(S, ?) : \mathcal{D}(\mathcal{G}) \to \mathrm{Ab}$. Using condition (4) and the fact that $\mathcal{S}$ is a set of generators of $\mathcal{D}(\mathcal{G})$, we see that, with the terminology of [Parra and Saorín 2015, Section 3], the pair $(H', +\infty)$ is a cohomological datum in $\mathcal{D}(\mathcal{G})$ for $\mathcal{H}$. Then [Parra and Saorín 2015, Proposition 3.4] says that $\mathcal{H}$ is an AB5 abelian category. But condition (3) says that it has a set of generators, so that $\mathcal{H}$ is a Grothendieck category.

Fix a direct system $(M_i)_{i\in I}$ in $\mathcal{H}$ in the sequel and consider the full subcategory $\mathcal{C}$ of $\mathcal{D}(\mathcal{G})$ consisting of those complexes X such that

$$\eta_{X[k]} : \varinjlim \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X[k], M_i) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X[k], \varinjlim_{\mathcal{H}} M_i)$$

is an isomorphism, for all $k \in \mathbb{Z}$. Using the Five Lemma, one readily sees that $\mathcal{C}$ is a thick subcategory of $\mathcal{D}(\mathcal{G})$ which, by condition (4), contains $\mathcal{S}$. We then have $\mathrm{thick}_{\mathcal{D}(\mathcal{G})}(\mathcal{S}) \subseteq \mathcal{C}$. In particular, if a complex $X \in \mathcal{C}^b(\mathrm{sum}(\mathcal{S}))$ is viewed as an object of $\mathcal{D}(\mathcal{G})$, then $X \in \mathcal{C}$.

We now claim that η_X is also an isomorphism, for each $X \in \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$. Indeed, condition (2) implies that $\mathcal{H} \subseteq \mathcal{D}^{[m,n]}(\mathcal{G})$. Let $X \in \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ be arbitrary. Replacing n by a larger integer if necessary, we can assume that $X \in \mathcal{D}^{\leq n}(\mathcal{G})$. Then the obvious generalization of Lemma 4.3 says that there exist a $P \in \mathcal{C}^b(\mathrm{sum}(\mathcal{S}))$ and a morphism $g : P \to X$ in $\mathcal{D}(\mathcal{G})$ such that the natural transformation

$$g^* : \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, ?) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(P, ?)$$

is an isomorphism when evaluated on objects of $\mathcal{D}^{[m,n]}(\mathcal{G})$. We then have the following commutative diagram

$$\begin{tikzcd}
\varinjlim \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, M_i) \arrow[r, "\eta_X"] \arrow[d, "g^*"] & \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, \varinjlim_{\mathcal{H}} M_i) \arrow[d, "g^*"] \\
\varinjlim \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(P, M_i) \arrow[r, "\eta_P"] & \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(P, \varinjlim_{\mathcal{H}} M_i)
\end{tikzcd}$$

where the vertical arrows are isomorphisms and, due to the previous paragraph, the lower horizontal arrow is an isomorphism also. This settles our claim. In particular, it implies that $\mathcal{H} \cap \mathcal{D}^b(\mathcal{G})$ is a class of finitely presented objects in $\mathcal{H}$ and, by conditions (1) and (3), it is a class of generators of $\mathcal{H}$ (see Remark 4.4). In particular $\mathcal{H}$ is locally finitely presented. Note also that, by condition (1) and Lemma 4.1, we know that $\mathcal{H} \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ is closed under taking cokernels (and kernels) in $\mathcal{H}$. It immediately follows that each finitely presented object of $\mathcal{H}$ is in $\mathcal{H} \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ since it is the cokernel of a morphism in this latter category. Then we have that $\mathcal{H} \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G})) = \mathrm{fp}(\mathcal{H})$, and this is an abelian exact subcategory of $\mathcal{H}$. Therefore $\mathcal{H}$ is locally coherent. □

Remark 4.6. Condition (1) of the last proposition is not necessary for the heart to be a locally coherent Grothendieck category. Indeed, by [Parra and Saorín 2014, Corollary 5.12] and using the terminology of that reference, if R is a commutative noetherian ring and $Z \subsetneq \mathrm{Spec}\, R$ is a perfect sp-subset, then $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure whose heart is equivalent to R_Z-Mod, where $\mathcal{U}$ consists of the complexes U such that $\mathrm{Supp}(H^j(U)) \subseteq Z$, for all $j > -1$. Then the heart is locally coherent since R_Z is a noetherian commutative ring. But the associated sp-filtration $\phi = \phi_{\mathcal{U}}$

of Spec R (see [Alonso et al. 2010, Section 2.8 and Theorem 3.11]) is given by $\phi(i) = \operatorname{Spec} R$, for $i \leq -1$, and $\phi(i) = Z$, for all $i > -1$. This sp-filtration does not satisfy in general the weak Cousin condition, in whose case $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ does not restrict to $\mathcal{D}^b(\mathrm{fp}(R\text{-Mod})) \cong \mathcal{D}^b_{\mathrm{fg}}(R)$ (see [Alonso et al. 2010, Corollary 4.5]). As an example of the last situation, consider $R = \mathbb{Z}$ and $Z = \operatorname{Spec} \mathbb{Z} \setminus \{0\}$, so that $R_Z = \mathbb{Q}$. We have a canonical triangle $\mathbb{Q}/\mathbb{Z}[-1] \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{+}$, where $\mathbb{Q}/\mathbb{Z}[-1] \in \mathcal{U}$ and $Q \in \mathcal{U}^{\perp}$.

5. The case of the Happel–Reiten–Smalø t-structure

Recall (see [Happel et al. 1996]) that if $\mathcal{A}$ is any abelian category and $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$, then $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1]) = (\mathcal{U}_{\boldsymbol{t}}, \mathcal{V}_{\boldsymbol{t}})$ is a t-structure in $\mathcal{D}(\mathcal{A})$, where

$$\mathcal{U}_{\boldsymbol{t}} = \{U \in \mathcal{D}^{\leq 0}(\mathcal{A}) : H^0(U) \in \mathcal{T}\} \quad \text{and} \quad \mathcal{V}_{\boldsymbol{t}} = \{V \in \mathcal{D}^{\geq -1}(\mathcal{A}) : H^{-1}(V) \in \mathcal{F}\}.$$

This t-structure will be called the *Happel–Reiten–Smalø* (or just *HRS*) *t-structure associated to* $\boldsymbol{t}$. In this paper we are only interested in the case when $\mathcal{A} = \mathcal{G}$ is a locally coherent Grothendieck category.

Therefore, all throughout this section, $\mathcal{G}$ will be a locally coherent Grothendieck category and $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ will be a torsion pair in $\mathcal{G}$. Recall that $\boldsymbol{t}$ is said to be of *finite type* when the torsion radical $t : \mathcal{G} \to \mathcal{T}$ preserves direct limits or, equivalently, when $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$ (see [Krause 1997, Section 2]). We shall say that $\boldsymbol{t}$ *restricts to* $\mathrm{fp}(\mathcal{G})$ when $t(X)$ is in $\mathrm{fp}(\mathcal{G})$, for each $X \in \mathrm{fp}(\mathcal{G})$. Note that this is equivalent to saying that $\boldsymbol{t}' = (\mathcal{T} \cap \mathrm{fp}(\mathcal{G}), \mathcal{F} \cap \mathrm{fp}(\mathcal{G}))$ is a torsion pair in $\mathrm{fp}(\mathcal{G})$.

Proposition 5.1. *Let* $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1])$ *be the HRS t-structure in* $\mathcal{D}(\mathcal{G})$ *associated to* $\boldsymbol{t}$. *The following assertions are equivalent*:

(1) *The t-structure* $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1])$ *restricts to* $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$.

(2) *The torsion pair* $\boldsymbol{t}$ *restricts to* $\mathrm{fp}(\mathcal{G})$.

In particular, if $\mathcal{G}$ *is locally noetherian then* $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1])$ *restricts to* $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$.

Proof. Given $M \in \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$, we have canonical triangles in $\mathcal{D}(\mathcal{G})$

$$\tau^{\leq -1}M \to M \to \tau^{\geq 0}M \xrightarrow{+},$$
$$t(H^0(M))[0] \to \tau^{\geq 0}M \to W \xrightarrow{+},$$

where $W \in \mathcal{D}^{\geq 0}(\mathcal{G})$, $H^0(W) \cong (H^0(M))/(t(H^0(M)))$ and $H^k(W) = H^k(M)$, for all $k > 0$. Then $W \in \mathcal{U}_{\boldsymbol{t}}^{\perp} = \mathcal{V}_{\boldsymbol{t}}[-1]$. Applying the octahedron axiom to the last two triangles, we obtain two new triangles

$$\tau^{\leq -1}M \to U \to t(H^0(M))[0] \xrightarrow{+},$$
$$U \to M \to W \xrightarrow{+}.$$

It follows from the first triangle that $U \in \mathcal{U}_{\boldsymbol{t}}$ since the outer vertices of the triangle are in $\mathcal{U}_{\boldsymbol{t}}$. We then conclude that the second triangle is precisely the truncation triangle of M with respect to $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1])$.

The last truncation triangle is in $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ if, and only if, $U \in \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$. But this happens exactly when $t(H^0(M))[0] \in \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$. That is, exactly when $t(H^0(M))$ is a finitely presented object. The equivalence of assertions (1) and (2) is now clear.

Noting that $\mathcal{G}$ is locally coherent all throughout this section, when $\mathcal{G}$ is also locally noetherian we have that $\mathrm{fp}(\mathcal{G})$ coincides with the class $\mathrm{noeth}(\mathcal{G})$ of noetherian objects, which is obviously closed under taking subobjects. Therefore $\boldsymbol{t}$ always restricts to $\mathrm{fp}(\mathcal{G})$. □

We are now ready to prove the first main result of the paper.

Theorem 5.2. *Let $\mathcal{G}$ be a locally coherent Grothendieck category, let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$, let $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1])$ be the associated t-structure in $\mathcal{D}(\mathcal{G})$ and let $\mathcal{H}_{\boldsymbol{t}}$ be its heart. The following assertions are equivalent:*

1) *$(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1])$ restricts to $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ and $\mathcal{H}_{\boldsymbol{t}}$ is a locally coherent Grothendieck category (with $\mathcal{H}_{\boldsymbol{t}} \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G})$ as the class of finitely presented objects).*
2) *$\boldsymbol{t}$ is of finite type and restricts to* $\mathrm{fp}(\mathcal{G})$.
3) *There exists a torsion pair $\boldsymbol{t}' = (\mathcal{T}', \mathcal{F}')$ in* $\mathrm{fp}(\mathcal{G})$ *such that $\boldsymbol{t} = (\varinjlim \mathcal{T}', \varinjlim \mathcal{F}')$.*

When in addition $\mathcal{G}$ is locally noetherian, these assertions are also equivalent to:

4) *$\boldsymbol{t}$ is of finite type.*

Proof. All throughout the proof, we fix a set $\mathcal{S}$ of finitely presented generators of $\mathcal{G}$.

1) ⇒ 2) By Proposition 5.1, we know that $\boldsymbol{t}$ restricts to $\mathrm{fp}(\mathcal{G})$ and, by [Parra and Saorín 2015, Theorem 4.8], we know that $\boldsymbol{t}$ is of finite type.

2) ⇒ 3) If we put $\mathcal{T}' = \mathcal{T} \cap \mathrm{fp}(\mathcal{G})$ and $\mathcal{F}' = \mathcal{F} \cap \mathrm{fp}(\mathcal{G})$, then $\boldsymbol{t}' = (\mathcal{T}', \mathcal{F}')$ is a torsion pair in $\mathrm{fp}(\mathcal{G})$ since $\boldsymbol{t}$ restricts to $\mathrm{fp}(\mathcal{G})$. By [Crawley-Boevey 1994, Lemma 4.4], we know that $(\varinjlim \mathcal{T}', \varinjlim \mathcal{F}')$ is a torsion pair in $\mathcal{G}$. But $\mathcal{T}$ and $\mathcal{F}$ are closed under taking direct limits in $\mathcal{G}$, which implies that $\varinjlim \mathcal{T}' \subseteq \mathcal{T}$ and $\varinjlim \mathcal{F}' \subseteq \mathcal{F}$. Since we always have $\mathcal{F} = \mathcal{T}^{\perp} \subseteq (\varinjlim \mathcal{T}')^{\perp} = \varinjlim \mathcal{F}'$ we conclude that $(\mathcal{T}, \mathcal{F}) = (\varinjlim \mathcal{T}', \varinjlim \mathcal{F}')$.

3) ⇒ 2) is clear.

2) ⇒ 1) The finite type condition of $\boldsymbol{t}$ implies that $\mathcal{H}_{\boldsymbol{t}}$ is a Grothendieck category (see [Parra and Saorín 2016a, Theorem 1.2]). Now, let $(M_i)_{i \in I}$ be a direct system in $\mathcal{H}_{\boldsymbol{t}}$. Bearing in mind that $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$ and using [Parra and Saorín 2015, Proposition 4.2], we get an exact sequence in $\mathcal{H}_{\boldsymbol{t}}$:

$$0 \to (\varinjlim H^{-1}(M_i))[1] \to \varinjlim_{\mathcal{H}_t} M_i \to (\varinjlim H^0(M_i))[0] \to 0.$$

To abbreviate, let us put $(X, Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, Y)$, for all $X, Y \in \mathcal{D}(\mathcal{G})$. Then, for each $S \in \mathcal{S}$ and each $k \in \mathbb{Z}$, we have a commutative diagram of abelian groups with exact columns, where the horizontal arrows are the canonical morphisms:

$$\begin{tikzcd}
(S[k], (\varinjlim H^0(M_i))[-1]) \arrow[d] & \varinjlim(S[k], H^0(M_i)[-1]) \arrow[l] \arrow[d] \\
(S[k], (\varinjlim H^{-1}(M_i))[1]) \arrow[d] & \varinjlim(S[k], H^{-1}(M_i)[1]) \arrow[l] \arrow[d] \\
(S[k], \varinjlim_{\mathcal{H}_t} M_i) \arrow[d] & \varinjlim(S[k], M_i) \arrow[l] \arrow[d] \\
(S[k], (\varinjlim H^0(M_i))[0]) \arrow[d] & \varinjlim(S[k], H^0(M_i)[0]) \arrow[l] \arrow[d] \\
(S[k], (\varinjlim H^{-1}(M_i))[2]) & \varinjlim(S[k], H^{-1}(M_i)[2]) \arrow[l]
\end{tikzcd}$$

By Proposition 3.5, we have that the two uppermost and the two lowermost horizontal arrows are isomorphisms, which implies the canonical map $\varinjlim(S[k], M_i) \to (S[k], \varinjlim_{\mathcal{H}_t} M_i)$ is also an isomorphism.

We will check now that all conditions (1)–(4) of Proposition 4.5 are satisfied by $(\mathcal{U}_t, \mathcal{U}_t[1])$. By Proposition 5.1, we know that $(\mathcal{U}_t, \mathcal{U}_t^{\perp}[1])$ restricts to $\mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$ and, by definition of the HRS t-structure, we know that $\mathcal{D}^{\leq -1}(\mathcal{G}) \subseteq \mathcal{U}_t \subseteq \mathcal{D}^{\leq 0}(\mathcal{G})$, so that conditions (1) and (2) of Proposition 4.5 hold. Moreover, the previous paragraph says that condition (4) also holds.

We will finally check that each object of $\mathcal{H}_t$ is an epimorphic image of a coproduct of objects of $\mathcal{H}_t \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$, which will give condition (3) of Proposition 5.1 and will end the proof. Let M be any object of $\mathcal{H}_t$ and let us write $H^0(M) = \varinjlim T_i$, for some direct system $(T_i)_{i \in I}$ in $\mathcal{T} \cap \mathrm{fp}(\mathcal{G})$. Note that this is possible since $\mathcal{T} = \varinjlim(\mathcal{T} \cap \mathrm{fp}(\mathcal{G}))$. Considering the canonical exact sequence

$$0 \to H^{-1}(M)[1] \to M \to H^0(M)[0] \to 0$$

and pulling it back, for each $i \in I$, along the obvious map $T_i[0] \to H^0(M)[0]$, we get a direct system of exact sequences in $\mathcal{H}_t$:

$$0 \to H^{-1}(M)[1] \to M_i \to T_i[0] \to 0.$$

Since $\mathcal{H}_t$ is a Grothendieck category it immediately follows that $M = \varinjlim_{\mathcal{H}_t} M_i$, so that M is an epimorphic image of $\coprod_{i \in I} M_i$. Replacing M by any of the M_i, we can and shall assume in the rest of the proof that $H^0(M) \in \mathcal{T} \cap \mathrm{fp}(\mathcal{G})$. We then write

M as a complex $\cdots 0 \to M^{-1} \to M^0 \to 0 \cdots$ concentrated in degrees -1 and 0. Note that if we put $M^0 = \varinjlim M_i^0$, where $(M_i^0)_{i\in I}$ is a direct system in $\mathrm{fp}(\mathcal{G})$, then some composition $M_j^0 \xrightarrow{\iota_j} \varinjlim M_i^0 = M^0 \overset{p}{\twoheadrightarrow} H^0(M)$ should be an epimorphism, because $H^0(M)$ is finitely presented. Replacing M^0 by M_j^0 if necessary, we can assume in the sequel that M^0 is also finitely presented.

Once we assume that $H^0(M)$ and M^0 are both finitely presented, we follow the lines of the proof of [Parra and Saorín 2015, Proposition 4.7] with an easy adaptation. The details are left to the reader. Since M^{-1} is a direct limit of finitely presented objects, we can fix an epimorphism $\coprod_{j\in J} X_j \twoheadrightarrow M^{-1}$ in $\mathcal{G}$, where $X_j \in \mathrm{fp}(\mathcal{G})$ for all $j \in J$. Now we construct a four-row commutative diagram as in the mentioned proof, where $G^{(J)}$ and $G^{(F)}$ are replaced in our case by $\coprod_{j\in J} X_j$ and $\coprod_{j\in F} X_j$, respectively. The key point now is that the appearing U_F and X_F are finitely presented objects. Since $\boldsymbol{t}$ restricts to $\mathrm{fp}(\mathcal{G})$, we also know that $t(X_F)$ and M_F^0 are finitely presented, for each finite subset $F \subseteq J$. If now $L = \widetilde{H}_{|\mathcal{U}_{\boldsymbol{t}}} : \mathcal{U}_{\boldsymbol{t}} \to \mathcal{H}_{\boldsymbol{t}}$ is the left adjoint to the inclusion functor (see [Parra and Saorín 2015, Lemma 3.1]), the mentioned proof shows that we have epimorphisms $\coprod_{F\subset J,\ F \text{ finite}} L(K_F) \twoheadrightarrow L(K_J)$ and $L(K_J) \twoheadrightarrow M$ in $\mathcal{H}_{\boldsymbol{t}}$, where $L(K_F)$ is the object of $\mathcal{H}_{\boldsymbol{t}}$ represented by the complex

$$\cdots\ 0 \to \frac{\coprod_{j\in F} X_j}{t(U_F)} \to M_F^0 \to 0\ \cdots,$$

concentrated in degrees -1 and 0. But $t(U_F)$ is finitely presented, because so is U_F. It follows that the latter complex is a complex of finitely presented objects, and hence $L(K_F) \in \mathcal{H}_{\boldsymbol{t}} \cap \mathcal{D}^b(\mathrm{fp}(\mathcal{G}))$.

4) $\Rightarrow$ 2) $=$ 3) If $\mathcal{G}$ is locally noetherian, each torsion pair restricts to its subcategory of noetherian objects, that is, to $\mathrm{fp}(\mathcal{G})$. □

6. The heart of a restricted t-structure in the derived category of a commutative noetherian ring

All throughout this section R is a commutative noetherian ring. To apply the results of earlier sections, we will consider $\mathcal{G} = R$-Mod the category of all R-modules, which is a locally noetherian Grothendieck category. Then we have that $\mathrm{fp}(\mathcal{G}) = R$-mod is the subcategory of finitely generated R-modules and, as usual (see comments on page 207), we identify $\mathcal{D}^b_{\mathrm{fg}}(R) := \mathcal{D}^b_{\mathrm{fp}}(R\text{-Mod})$ with $\mathcal{D}^b(R\text{-mod})$.

Recall that a *filtration by supports* or sp-*filtration* of Spec R is a decreasing map $\phi : \mathbb{Z} \to \mathcal{P}(\operatorname{Spec} R)$ such that $\phi(i) \subseteq \operatorname{Spec} R$ is a stable under specialization subset, for each $i \in \mathbb{Z}$. Filtrations by supports turn out to be in bijection with the compactly generated t-structures in $\mathcal{D}(R)$ (see [Alonso et al. 2010, Theorem 3.11]). Concretely, given an sp-filtration ϕ and putting

$$\mathcal{U}_\phi = \{U \in \mathcal{D}(R) : \operatorname{Supp}(H^i(U)) \subseteq \phi(i), \text{ for all } i \in \mathbb{Z}\},$$

we get a compactly generated t-structure $\tau_\phi = (\mathcal{U}_\phi, \mathcal{U}_\phi^{\perp}[1])$ and the assignment $\phi \rightsquigarrow \tau_\phi$ gives the mentioned bijection. All through this section, the reader is referred to [Alonso et al. 2010] for all nondefined terms that we might use.

Lemma 6.1. *Let $X \in \mathcal{D}^b_{\mathrm{fg}}(R)$ and $Y \in \mathcal{D}^+(R)$. For each $\boldsymbol{p} \in \operatorname{Spec} R$, the canonical map*

$$\operatorname{Hom}_{\mathcal{D}(R)}(X, Y)_{\boldsymbol{p}} \to \operatorname{Hom}_{\mathcal{D}(R_{\boldsymbol{p}})}(X_{\boldsymbol{p}}, Y_{\boldsymbol{p}})$$

is an isomorphism.

Proof. Let us fix $Y \in \mathcal{D}^+(R)$, which we consider to be a bounded below complex of injective R-modules. For each Z in $\mathcal{D}^b_{\mathrm{fg}}(R)$, we denote by η_Z the canonical map $\operatorname{Hom}_{\mathcal{D}(R)}(Z, Y)_{\boldsymbol{p}} \to \operatorname{Hom}_{\mathcal{D}(R_{\boldsymbol{p}})}(Z_{\boldsymbol{p}}, Y_{\boldsymbol{p}})$. We then consider the full subcategory $\mathcal{C}$ of $\mathcal{D}^b_{\mathrm{fg}}(R)$ consisting of those Z such that $\eta_{Z[k]}$ is an isomorphism, for all $k \in \mathbb{Z}$. It is clear that $\mathcal{C}$ is a thick subcategory of $\mathcal{D}^b_{\mathrm{fg}}(R)$.

We claim that $M[0] \in \mathcal{C}$, for each finitely generated R-module M. Once this is proved, the proof will be finished. Indeed, we will conclude that $\mathcal{C} = \mathcal{D}^b_{\mathrm{fg}}(R)$ since each $Z \in \mathcal{D}^b_{\mathrm{fg}}(R)$ is a finite iterated extension of the stalk complexes $H^{-k}(Z)[k]$, and each $H^{-k}(Z)$ is finitely generated. Recall that $\operatorname{Hom}_{\mathcal{D}(R)}(M[-k], Y)$ is the k-th homology module of the complex of R-modules $\operatorname{Hom}_R(M, Y)$. Similarly, $\operatorname{Hom}_{\mathcal{D}(R_{\boldsymbol{p}})}(M_{\boldsymbol{p}}[-k], Y_{\boldsymbol{p}})$ is the k-th homology module of the complex of $R_{\boldsymbol{p}}$-modules $\operatorname{Hom}_{R_{\boldsymbol{p}}}(M_{\boldsymbol{p}}, Y_{\boldsymbol{p}})$ since $Y_{\boldsymbol{p}}$ is a bounded below complex of injective $R_{\boldsymbol{p}}$-modules. The claim follows from the exactness of the localization at $\boldsymbol{p}$ and from the truth of the result when Y is a module (see, e.g., [Kunz 1985, Proposition IV.1.10]). □

Lemma 6.2. *Let R be connected, let $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ be a compactly generated t-structure in $\mathcal{D}(R)$ which restricts to $\mathcal{D}^b_{\mathrm{fg}}(R)$, let $\mathcal{H}$ be its heart and let $U \in \mathcal{D}^-(R) \cap \mathcal{U}$ be a complex with finitely generated homology modules. Then $\widetilde{H}(U)$ is in $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$.*

Proof. Let ϕ be the sp-filtration of $\operatorname{Spec} R$ associated to $(\mathcal{U}, \mathcal{U}^{\perp}[1])$. By Corollaries 4.5 and 4.8 of [Alonso et al. 2010], we know that there exists some $j_0 \in \mathbb{Z}$ such that $\phi(j_0) = \operatorname{Spec} R$. Without loss of generality, we assume that $j_0 = 0$. We then have

$$\mathcal{D}^{\leq 0}(R) \subseteq \mathcal{U} \quad \text{and} \quad \mathcal{H} \subseteq \mathcal{D}^{\geq 0}(R).$$

By considering now for the object U of the statement the canonical truncation triangle

$$\tau^{\leq 0}(U[-1]) \to U[-1] \xrightarrow{g} \tau^{>0}(U[-1]) \xrightarrow{+}$$

and applying the octahedron axiom, we see that

$$\tau^{\mathcal{U}^{\perp}}(g) : \tau^{\mathcal{U}^{\perp}}(U[-1]) \to \tau^{\mathcal{U}^{\perp}}(\tau^{>0}(U[-1]))$$

is an isomorphism. However, the codomain of this morphism is in $\mathcal{D}^b_{\mathrm{fg}}(R)$ since

$\tau^{>0}(U[-1]) \in \mathcal{D}^b_{\mathrm{fg}}(R)$ and the t-structure $(\mathcal{U}, \mathcal{U}^\perp[1])$ restricts to $\mathcal{D}^b_{\mathrm{fg}}(R)$. Then $\widetilde{H}(U) = \tau^{\mathcal{U}^\perp}(U[-1])[1]$ is in $\mathcal{D}^b_{\mathrm{fg}}(R)$ (see [Parra and Saorín 2015, Lemma 3.1]). □

We are now ready to prove the main result of the paper.

Theorem 6.3. *Let R be a commutative noetherian ring and let $(\mathcal{U}, \mathcal{U}^\perp[1])$ be a compactly generated t-structure in $\mathcal{D}(R)$ which restricts to $\mathcal{D}^b_{\mathrm{fg}}(R)$. The heart $\mathcal{H}$ of this t-structure is a locally coherent Grothendieck category where $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$ is the subcategory of its finitely presented objects.*

Proof. All throughout the proof, without loss of generality, we assume that R is connected. Remember that then the associated sp-filtration ϕ satisfies the weak Cousin condition and, hence, has the property that $\phi(i) = \operatorname{Spec} R$, for $i \ll 0$ (see [Alonso et al. 2010, Theorem 4.4 and Corollary 4.8]). This in turn implies that $\mathcal{H} = \mathcal{H}_\phi \subseteq \mathcal{D}^{\geq m}(R)$, for some $m \in \mathbb{Z}$. Moreover, by [Parra and Saorín 2014, Theorem 4.10], we know that $\mathcal{H} = \mathcal{H}_\phi$ is a Grothendieck category.

Step 1: *$\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$ is a (skeletally small) class of generators of $\mathcal{H}$*: Let $\mathcal{U}'$ denote the full subcategory of $\mathcal{U}$ consisting of complexes in $\mathcal{U} \cap \mathcal{D}^-(R)$ which have finitely generated homology modules. Each object of $\mathcal{U}'$ is isomorphic in $\mathcal{D}(R)$ to a bounded above complex of finitely generated R-modules. Let $L = \widetilde{H}_{|\mathcal{U}} : \mathcal{U} \to \mathcal{H}$ be the left adjoint to the inclusion functor $\mathcal{H} \hookrightarrow \mathcal{U}$. A slight modification of the proof of [Parra and Saorín 2014, Proposition 3.10] shows that $\mathcal{X} := L(\mathcal{U}')$ is a skeletally small class of generators of $\mathcal{H}$. By Lemma 6.2, we get that $\mathcal{X} \subseteq \mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$, which ends this first step.

Step 2: *The result is true when ϕ is eventually trivial* (i.e., when $\phi(i) = \varnothing$, for some $i \in \mathbb{Z}$): We shall check all conditions (1)–(4) of Proposition 4.5. Without loss of generality, we assume that the filtration is

$$\operatorname{Spec} R = \cdots \phi(-n-1) = \phi(-n) \supsetneq \phi(-n+1) \supseteq \cdots \supseteq \phi(0) \supsetneq \phi(1) = \phi(2) = \cdots = \varnothing,$$

in which case we have that $\mathcal{D}^{\leq -n}(R) \subseteq \mathcal{U} \subseteq \mathcal{D}^{\leq 0}(R)$ and $\mathcal{H} = \mathcal{H}_\phi \subseteq \mathcal{D}^{[-n,0]}(R)$ (see [Parra and Saorín 2014, Lemma 4.1]). Then condition (2) of Proposition 4.5 holds and condition (1) holds by hypothesis. Moreover, Step 1 of this proof gives condition (3) of that proposition. Finally, bearing in mind that we have a natural isomorphism $H^k \cong \operatorname{Hom}_{\mathcal{D}(R)}(R[-k], ?)$ of functors $\mathcal{D}(R) \to R$-Mod, by taking $\mathcal{S} = \{R\}$ and using [Parra and Saorín 2014, Theorem 4.9] we also get that condition (4) holds.

Step 3: *The general case.* The proof reduces to checking that $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R) \subseteq \operatorname{fp}(\mathcal{H})$. Indeed, if this is proved, then Step 1 implies that $\mathcal{H}$ is locally finitely presented and that each object in $\operatorname{fp}(\mathcal{H})$ is the cokernel of a morphism in $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$. It will follow from Lemma 4.1 that $\operatorname{fp}(\mathcal{H}) = \mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$ and that this is an abelian exact subcategory of $\mathcal{H}$. That is, $\mathcal{H}$ will be a locally coherent Grothendieck category with $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$ as its class of finitely presented objects.

We then prove the inclusion $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R) \subseteq \mathrm{fp}(\mathcal{H})$. Let $(M_i)_{i\in I}$ be a direct system in $\mathcal{H}$ and let $X \in \mathcal{H} \cap \mathcal{D}^b_{\mathrm{fg}}(R)$ be any object. We consider the canonical morphism

$$\eta_X : \varinjlim \operatorname{Hom}_{\mathcal{D}(R)}(X, M_i) \to \operatorname{Hom}_{\mathcal{D}(R)}(X, \varinjlim_{\mathcal{H}} M_i),$$

which is a morphism in R-Mod. Localization at any prime ideal $\boldsymbol{p}$ preserves direct limits and, by [Parra and Saorín 2014, Proposition 3.11], we also have that $(\varinjlim_{\mathcal{H}} M_i)_{\boldsymbol{p}} \cong \varinjlim_{\mathcal{H}_{\boldsymbol{p}}} (M_i)_{\boldsymbol{p}}$. Here if $\mathcal{H} = \mathcal{H}_\phi$, then we put $\mathcal{H}_{\boldsymbol{p}} = \mathcal{H}_{\phi_{\boldsymbol{p}}}$, using the terminology of [Parra and Saorín 2014]. Therefore, using Lemma 6.1, we can identify $(\eta_X)_{\boldsymbol{p}} : (\varinjlim \operatorname{Hom}_{\mathcal{D}(R)}(X, M_i))_{\boldsymbol{p}} \to (\operatorname{Hom}_{\mathcal{D}(R)}(X, \varinjlim_{\mathcal{H}} M_i))_{\boldsymbol{p}}$ with the canonical morphism

$$\eta_{X_{\boldsymbol{p}}} : \varinjlim \operatorname{Hom}_{\mathcal{D}(R_{\boldsymbol{p}})}(X_{\boldsymbol{p}}, (M_i)_{\boldsymbol{p}}) \to \operatorname{Hom}_{\mathcal{D}(R_{\boldsymbol{p}})}(X_{\boldsymbol{p}}, \varinjlim_{\mathcal{H}_{\boldsymbol{p}}} (M_i)_{\boldsymbol{p}}).$$

But the sp-filtration $\phi_{\boldsymbol{p}}$ of Spec $R_{\boldsymbol{p}}$ also satisfies the weak Cousin condition and, since $R_{\boldsymbol{p}}$ has finite Krull dimension, we get that $\phi_{\boldsymbol{p}}$ is eventually trivial (see [Alonso et al. 2010, Corollary 4.8]). The truth of the theorem when the associated filtration is eventually trivial implies that $\eta_{X_{\boldsymbol{p}}}$ is an isomorphism, for all $\boldsymbol{p} \in \operatorname{Spec} R$, because $X_{\boldsymbol{p}} \in \mathrm{fp}(\mathcal{H}_{\boldsymbol{p}})$. Therefore the kernel and cokernel of η_X are R-modules with empty support. Then they are both zero, so that η_X is an isomorphism, and hence X is in $\mathrm{fp}(\mathcal{H})$ as desired. □

Corollary 6.4. *Let R be a commutative noetherian ring. The heart of any t-structure in $\mathcal{D}^b_{\mathrm{fg}}(R)$ is equivalent to the category of finitely presented objects of a locally coherent Grothendieck category.*

Proof. Each t-structure in $\mathcal{D}^b_{\mathrm{fg}}(R)$ is the restriction of the t-structure τ_ϕ in $\mathcal{D}(R)$ associated to an sp-filtration (see [Alonso et al. 2010, Corollary 3.12]). The result is then an immediate consequence of the last theorem, using [Alonso et al. 2010, Theorem 3.10]. □

As a final comment, we give the geometric translation of the last theorem and corollary:

Corollary 6.5. *Let $\mathbb{X}$ be an affine noetherian scheme and let $(\mathcal{U}, \mathcal{U}^\perp[1])$ be a t-structure in $\mathcal{D}(\mathbb{X}) := \mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))$ which restricts to $\mathcal{D}^b_{\mathrm{coh}}(\mathbb{X}) \cong D^b(\mathrm{coh}(\mathbb{X}))$. The heart $\mathcal{H}$ of the t-structure is a locally coherent Grothendieck category on which $\mathcal{H} \cap \mathcal{D}^b_{\mathrm{coh}}(\mathbb{X})$ is the class of finitely presented objects. In particular, the heart of each t-structure in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is equivalent to the category of finitely presented objects of a locally coherent Grothendieck category.*

References

[Adámek and Rosický 1994] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series **189**, Cambridge University Press, 1994. MR Zbl

[Alonso et al. 2010] L. Alonso Tarrío, A. Jeremías López, and M. Saorín, "Compactly generated t-structures on the derived category of a Noetherian ring", *J. Algebra* **324**:3 (2010), 313–346. MR Zbl

[Beilinson et al. 1982] A. A. Beilinson, J. Bernstein, and P. Deligne, "Faisceaux pervers", pp. 5–171 in *Analysis and topology on singular spaces, I* (Luminy, 1981), Astérisque **100**, Société Mathématique de France, Paris, 1982. MR Zbl

[Colpi and Gregorio 2010] R. Colpi and E. Gregorio, "The heart of cotilting theory pair is a Grothendieck category", preprint, 2010.

[Colpi et al. 2007] R. Colpi, E. Gregorio, and F. Mantese, "On the heart of a faithful torsion theory", *J. Algebra* **307**:2 (2007), 841–863. MR Zbl

[Colpi et al. 2011] R. Colpi, F. Mantese, and A. Tonolo, "When the heart of a faithful torsion pair is a module category", *J. Pure Appl. Algebra* **215**:12 (2011), 2923–2936. MR Zbl

[Crawley-Boevey 1994] W. Crawley-Boevey, "Locally finitely presented additive categories", *Comm. Algebra* **22**:5 (1994), 1641–1674. MR Zbl

[Garkusha 2009] G. A. Garkusha, "Classifying finite localizations of quasi-coherent sheaves", *Algebra i Analiz* **21**:3 (2009), 93–129. In Russian; translated in *St. Petersburg Math. J.* **21**:3 (2010), 433–458. MR Zbl

[Gillespie 2016] J. Gillespie, "Models for homotopy categories of injectives and Gorenstein injectives", *Comm. Algebra* (online publication October 2016).

[Happel et al. 1996] D. Happel, I. Reiten, and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **575**, Amer. Math. Soc., Providence, RI, 1996. MR Zbl

[Hartshorne 1966] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics **20**, Springer, 1966. MR Zbl

[Herzog 1997] I. Herzog, "The Ziegler spectrum of a locally coherent Grothendieck category", *Proc. London Math. Soc.* (3) **74**:3 (1997), 503–558. MR Zbl

[Hoshino et al. 2002] M. Hoshino, Y. Kato, and J.-I. Miyachi, "On t-structures and torsion theories induced by compact objects", *J. Pure Appl. Algebra* **167**:1 (2002), 15–35. MR Zbl

[Keller 1994] B. Keller, "Deriving DG categories", *Ann. Sci. École Norm. Sup.* (4) **27**:1 (1994), 63–102. MR Zbl

[Krause 1997] H. Krause, "The spectrum of a locally coherent category", *J. Pure Appl. Algebra* **114**:3 (1997), 259–271. MR Zbl

[Kunz 1985] E. Kunz, *Introduction to commutative algebra and algebraic geometry*, Birkhäuser, Boston, 1985. MR Zbl

[Mantese and Tonolo 2012] F. Mantese and A. Tonolo, "On the heart associated with a torsion pair", *Topology Appl.* **159**:9 (2012), 2483–2489. MR Zbl

[Neeman 2001] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, 2001. MR Zbl

[Nicolás et al. 2015] P. Nicolás, M. Saorín, and A. Zvonareva, "Silting theory in triangulated categories with coproducts", preprint, 2015. arXiv

[Parra and Saorín 2014] C. E. Parra and M. Saorín, "Hearts of t-structures in the derived category of a commutative Noetherian ring", preprint, 2014. To appear in *Trans. Amer. Math. Soc.* arXiv

[Parra and Saorín 2015] C. E. Parra and M. Saorín, "Direct limits in the heart of a t-structure: the case of a torsion pair", *J. Pure Appl. Algebra* **219**:9 (2015), 4117–4143. Addendum in **220**:6 (2016), 2467–2469. MR Zbl

[Parra and Saorín 2016a] C. E. Parra and M. Saorín, "Addendum to 'Direct limits in the heart of a t-structure: the case of a torsion pair"', *J. Pure Appl. Algebra* **220**:6 (2016), 2467–2469. MR Zbl

[Parra and Saorín 2016b] C. E. Parra and M. Saorín, "On hearts which are module categories", *J. Math. Soc. Japan* **68**:4 (2016), 1421–1460.

[Popescu 1973] N. Popescu, *Abelian categories with applications to rings and modules*, London Mathematical Society Monographs **3**, Academic Press, London, 1973. MR Zbl

[Prest 2009] M. Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications **121**, Cambridge University Press, 2009. MR Zbl

[Psaroudakis and Vitória 2015] C. Psaroudakis and J. Vitória, "Realisation functors in tilting theory", preprint, 2015. arXiv

[Sitte 2014] T. Sitte, *Local cohomology sheaves on algebraic stacks*, Ph.D. thesis, Universität Regensburg, 2014, available at http://epub.uni-regensburg.de/30539/1/thesis.pdf.

[Šťovíček 2014] J. Šťovíček, "On purity and applications to coderived and singularity categories", preprint, 2014. arXiv

[Verdier 1996] Jean-Louis. Verdier, *Des catégories dérivées des catégories abéliennes*, edited by G. Maltsiniotis, Astérisque **239**, Société Mathématique de France, Paris, 1996. MR Zbl

Received May 19, 2016. Revised September 20, 2016.

MANUEL SAORÍN
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE MURCIA
APTDO. 4021
30100 MURCIA
SPAIN

msaorinc@um.es

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.223

APPROXIMABILITY OF CONVEX BODIES AND VOLUME ENTROPY IN HILBERT GEOMETRY

CONSTANTIN VERNICOS

The approximability of a convex body is a number which measures the difficulty in approximating that convex body by polytopes. In the interior of a convex body one can define its Hilbert geometry. We prove on the one hand that the volume entropy is twice the approximability for a Hilbert geometry in dimension two or three, and on the other hand that in higher dimensions the approximability is a lower bound of the entropy. As a corollary we solve the volume entropy upper bound conjecture in dimension three and give a new proof in dimension two different from the one given in (*Pacific J. Math.* 245:2 (2010), 201–225). Moreover, our method allows us to prove the existence of Hilbert geometries with intermediate volume growth on the one hand, and that in general the volume entropy is not a limit on the other hand.

Introduction and statement of results

Hilbert geometries are all the metric spaces obtained by defining the so-called Hilbert distance on open bounded convex sets in $\mathbb{R}^n$. The definition of this distance uses cross ratios in the same way as in the Klein projective model of the hyperbolic geometry [Hilbert 1971]. These metric spaces are actually length spaces whose structure is defined by a Finsler metric which is Riemannian if and only if the underlying open bounded convex set is an ellipsoid [Kay 1967].

These geometries were introduced by D. Hilbert in a letter addressed to F. Klein and have attracted a lot of interest lately. The studies of the shape of spheres in [Busemann 1955, Chapter 18] and of perpendicularity in [Busemann and Kelly 1953, Chapter 28] seem to be among the first ones to appear. In the same period P. J. Kelly and E. Straus [1958], Y. Nasu [1961] and D. C. Kay [1967] were looking at characterisations of the hyperbolic geometry among them in terms of curvature, transitive actions and the ptolemaic inequality, respectively. After a break of twenty or so years, they started to be studied from the projective structure viewpoint by

The author acknowledges that this material is based upon works partially supported by the Science Foundation Ireland under a Stokes award.
MSC2010: primary 53C60; secondary 53C24, 58B20, 53A20.
Keywords: volume entropy, approximability, Hilbert geometries, Finsler metric, convex bodies.

W. Goldman [1990] and by I. Kim [2005], and from the perspective of the group acting on them by P. de la Harpe [1993]. At the start of the new millennium the quest for characterisation of the Hilbert geometries being hyperbolic in the sense of Gromov began with A. Karlsson and G. Noskov [2002] and more noticeably with an equivalence between the hyperbolicity and a property of the boundary called quasisymmetric convexity discovered by Y. Benoist [2003; 2008], who also studied dynamical aspects of these geometries and clarified the fractal shape of their boundary in dimension three. At the same time the infinite-dimensional ones were studied from a functional-analytical point of view; see, for instance, [Lins and Nussbaum 2008]. Lately, understanding the analogue of geometric finiteness in the setting of projective structures has been at the centre of the works of L. Marquis [2012], M. Crampon and Marquis [2014], and D. Cooper, D. Long and S. Tillman [Cooper et al. 2015]. Other aspects of interest can be found in the recent *Handbook of Hilbert geometry* [Papadopoulos and Troyanov 2014].

The present paper focuses on the volume growth of these geometries and more specifically on the volume entropy.

Let Ω be a bounded open convex set in $\mathbb{R}$ endowed with its Hilbert geometry. If we consider the *Busemann volume* Vol_Ω and denote by $B_\Omega(p,r)$ the metric ball of radius r centred at the point $p\in\Omega$, then the *lower* and the *upper volume entropies* of Ω will be defined respectively by

$$\underline{\mathrm{Ent}}\,\Omega=\liminf_{r\to+\infty}\frac{\ln(\mathrm{Vol}_\Omega B_\Omega(p,r))}{r}\quad\text{and}\quad\overline{\mathrm{Ent}}\,\Omega=\limsup_{r\to+\infty}\frac{\ln(\mathrm{Vol}_\Omega B_\Omega(p,r))}{r}. \tag{1}$$

When the two limits coincide we denote their common limit by $\mathrm{Ent}\,\Omega$ and call it the *volume entropy* of Ω.

Let us stress that in this definition the upper and lower volume entropy of Ω do not depend on the base point p and are actually projective invariants attached to Ω.

The question we address in this paper is twofold. On the one hand it is an investigation of the existence of an analogue, for all Hilbert geometries, of the relation between the volume entropy and the Hausdorff dimension of the radial limit set on the universal cover of a compact Riemannian manifold with nonpositive curvature. On the other hand we focus on the *volume entropy upper bound conjecture*, which states that if Ω is an open and bounded convex subset of $\mathbb{R}^n$, then $\overline{\mathrm{Ent}}\,\Omega\le n-1$. To put our work into perspective let us recall the main related results.

The first one is a complete answer to the conjecture in the two-dimensional case by G. Berck, A. Bernig and C. Vernicos in [Berck et al. 2010], where the authors actually obtained an upper bound as a function of d, the upper Minkowski dimension (or *ball-box* dimension) of the set of extreme points of Ω, namely

$$\overline{\mathrm{Ent}}\,\Omega\le\frac{2}{3-d}\le 1. \tag{2}$$

The second result is a more precise statement with respect to the asymptotic volume growth of balls. It involves another projective invariant introduced by Berck, Bernig and Vernicos in the introduction of [Berck et al. 2010], called the *centroprojective* area of Ω and defined by

$$\mathcal{A}_p(\Omega) := \int_{\partial\Omega} \frac{\sqrt{k(x)}}{\langle n(x), x-p\rangle^{\frac{1}{2}(n-1)}} \left(\frac{2\alpha(x)}{1+\alpha(x)}\right)^{\frac{1}{2}(n-1)} \mathrm{d}A(x), \tag{3}$$

where for any $x \in \partial\Omega$, $k(x)$ is the Gauss curvature, $n(x)$ is the outward normal and $\alpha(x) > 0$ is the function defined by $p - \alpha(x)(x-p) \in \partial\Omega$. Let us recall here that both k and n are defined almost everywhere as Alexandroff's theorem states [Alexandroff 1939].

Now, the second main theorem in [Berck et al. 2010] — which encompasses previous results given by B. Colbois and P. Verovic [2004] — asserts that if $\partial\Omega$ is $C^{1,1}$ we have

$$\lim_{r\to+\infty} \frac{\operatorname{Vol}_\Omega B_\Omega(p,r)}{\sinh^{n-1} r} = \frac{1}{n-1}\mathcal{A}_p(\Omega) \neq 0 \tag{4}$$

and $\operatorname{Ent}\Omega = n-1$ is a limit. Moreover, without any assumption on Ω we have $\underline{\operatorname{Ent}}\,\Omega \geq n-1$ whenever $\mathcal{A}_p(\Omega) \neq 0$.

The third one — which is also a rigidity result — requires stronger assumptions about Ω: it has to be divisible, meaning that it admits a compact quotient, and its Hilbert metric has to be hyperbolic in the sense of Gromov, which implies its boundary is C^1 and strictly convex by [Benoist 2003]. Let us stress that the Hilbert metric on such an Ω is the hyperbolic one if and only if Ω has a $C^{1,1}$ boundary, and that its volume entropy is positive since hyperbolicity implies the nonvanishing of the Cheeger constant (see Theorem 1.5 in [Colbois and Vernicos 2007]). A result by Crampon [2009] states that for a divisible open bounded convex set Ω in $\mathbb{R}^n$ whose boundary is C^1 we have $\operatorname{Ent}\Omega \leq n-1$ with equality if and only if Ω is an ellipsoid.

In the present paper we link the volume entropy to another invariant associated with a convex body, called the *approximability*. This name was introduced by R. Schneider and J. A. Wieacker [1981]. The approximability measures in some sense how well a convex set can be approximated by polytopes. More precisely, let $N(\varepsilon, \Omega)$ be the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than $\varepsilon > 0$. Then the lower and upper approximability of Ω are defined by

$$\underline{a}(\Omega) := \liminf_{\varepsilon\to 0} \frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon} \quad \text{and} \quad \bar{a}(\Omega) := \limsup_{\varepsilon\to 0} \frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon}. \tag{5}$$

The key inequality which is of interest in our work — obtained by Fejes Tóth [1948] in dimension two and by E. M. Bronshteyn and L. D. Ivanov [1975] in the general case — asserts that for any bounded convex set in $\mathbb{R}^n$ the following upper bound on the upper approximability holds: $\bar{a}(\Omega) \leq \frac{1}{2}(n-1)$.

Our main result is as follows.

Theorem 1 (main theorem). *Given an open bounded convex set Ω in $\mathbb{R}^n$, we have*

$$2\underline{a}(\Omega) \leq \underline{\mathrm{Ent}}\,\Omega \quad \text{and} \quad 2\bar{a}(\Omega) \leq \overline{\mathrm{Ent}}\,\Omega, \tag{6}$$

with equality for $n = 2$ or $n = 3$.

The equality case in (6), together with the upper bound for the upper approximability, implies the following corollary.

Corollary 2 (volume entropy upper bound conjecture). *For any open bounded convex set Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have $\overline{\mathrm{Ent}}\,\Omega \leq n-1$.*

The equality case in this main theorem heavily relies on the study of polytopal Hilbert geometries. As it happens we get an optimal control of the volume of metric balls in dimension two and three, for in those two cases the number of edges of a polytope is bounded from above by the number of its vertices up to a multiplicative and an additive constant. This does not hold in higher dimensions, following McMullen's upper bound theorem [McMullen 1971; McMullen and Shephard 1971].

The second important result concerns the two-dimensional case, where we can prove that there are Hilbert geometries with intermediate volume growth.

Theorem 3 (intermediate volume growth). *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function that satisfies*

$$\liminf_{r \to +\infty} \frac{e^r}{f(r)} > 0.$$

Then there exist an open bounded convex set Ω in $\mathbb{R}^2$ and a point o in Ω such that

$$\liminf_{r \to +\infty} \frac{\mathrm{Vol}_\Omega B_\Omega(o,r)}{f(r)} > 0 \quad \text{and} \quad \limsup_{r \to +\infty} \frac{\mathrm{Vol}_\Omega B_\Omega(o,r)}{f(r) r^2} < +\infty, \tag{7}$$

and

$$\underline{\mathrm{Ent}}\,\Omega = \liminf_{r \to +\infty} \frac{\ln f(r)}{r} \quad \text{and} \quad \overline{\mathrm{Ent}}\,\Omega = \limsup_{r \to +\infty} \frac{\ln f(r)}{r}. \tag{8}$$

In particular there are open bounded convex sets $\Omega \subset \mathbb{R}^2$ with

- *maximal volume entropy and zero centroprojective area,*
- *zero volume entropy which are not polytopes.*

This theorem is a consequence of our method for proving the equality in dimension two in the main theorem (see Section 2) and Schneider and Wieacker's results [1981] on the approximability in dimension two. The last statement follows from our work [Vernicos 2009], where we showed that polytopes have polynomial growth of order r^2 in dimension two.

The intermediate volume growth theorem allows us to settle in a quite definite way the question of whether the entropy is a limit or not.

Corollary 4. *The volume entropy is not a limit in general. More precisely, for any α and β with $0 \le \alpha \le \beta \le 1$ there exist an open bounded convex set Ω in $\mathbb{R}^2$ such that*

$$\underline{\mathrm{Ent}}\,\Omega = \alpha \quad \textit{and} \quad \overline{\mathrm{Ent}}\,\Omega = \beta.$$

The equalities and inequalities also imply the following new results:

Corollary 5. *Given an open bounded convex set Ω in $\mathbb{R}^n$, we have*

- $d_H \le \underline{\mathrm{Ent}}\,\Omega$, *where d_H is the Hausdorff dimension of the set of farthest points of Ω;*
- *if $n = 2$ or 3 then $\bar{a}(\Omega)$ is a projective invariant of Ω and $\overline{\mathrm{Ent}}\,\Omega = \overline{\mathrm{Ent}}\,\Omega^*$, where Ω^* is the polar dual of Ω;*
- *if $n = 2$, then $\bar{a}(\Omega) \le 1/(3-d)$.*

Section 1 presents the various lemmas and notions needed in Section 2 to prove the main theorem, and in Section 3 we present the proof of the intermediate volume growth theorem.

1. Preliminaries on Hilbert geometries and convex bodies

1.1. ***Notations and definitions.*** A *proper* open set in $\mathbb{R}^n$ is a set that does not contain a whole line. A nonempty proper open convex set in $\mathbb{R}^n$ will be called a *proper convex domain*. The closure of a bounded convex domain is usually called a *convex body*.

A Hilbert geometry (Ω, d_Ω) is a proper convex domain Ω in $\mathbb{R}^n$ endowed with its Hilbert distance d_Ω defined as follows: for any two distinct points p and q in Ω, the line passing through p and q meets the boundary $\partial\Omega$ of Ω at two points a and b such that a, p, q, b appear in that order on the line. We denote by $[a, p, q, b]$ the cross ratio of (a, p, q, b), i.e.,

$$[a, p, q, b] = \frac{qa}{pa} \times \frac{pb}{qb} > 1,$$

where for any two points x, y in $\mathbb{R}^n$, xy is their distance with respect to the standard Euclidean norm $\|\cdot\|$. Should a or b be at infinity, the corresponding ratio will be considered equal to 1. Then we define

$$d_\Omega(p, q) = \tfrac{1}{2} \ln[a, p, q, b].$$

Note that the invariance of the cross ratio by a projective map implies the invariance of d_Ω by such a map.

The proper convex domain Ω is also naturally endowed with the C^0 Finsler metric F_Ω defined as follows: given $p \in \Omega$ and $v \in T_p\Omega = \mathbb{R}^n$ with $v \neq 0$, the straight line passing through p with direction vector v meets $\partial\Omega$ at two points p_Ω^+ and p_Ω^- such that $p_\Omega^+ - p_\Omega^-$ and v have the same direction. Then let t^+ and t^- be the two positive numbers such that $p + t^+ v = p_\Omega^+$ and $p - t^- v = p_\Omega^-$ (in other words, these numbers correspond to the amount of time needed to reach the boundary of Ω when starting at p with the velocities v and $-v$, respectively). Then we define

$$F_\Omega(p, v) = \frac{1}{2}\left(\frac{1}{t^+} + \frac{1}{t^-}\right) \quad \text{and} \quad F_\Omega(p, 0) = 0.$$

Should p_Ω^+ or p_Ω^- be at infinity, then the corresponding ratio will be taken to be equal to 0.

The Hilbert distance d_Ω is the length distance associated to F_Ω. We shall denote by $B_\Omega(p, r)$ the metric ball of radius r centred at the point $p \in \Omega$ and by $S_\Omega(p, r)$ the corresponding metric sphere.

Thanks to that Finsler metric, we can make use of two important Borel measures on Ω. The first one, which coincides with the Hausdorff measure associated to the metric space (Ω, d_Ω) (see Example 5.5.13 in [Burago et al. 2001]), is the *Busemann* volume, denoted by Vol_Ω and defined as follows. Given any point p in Ω, let $\beta_\Omega(p) = \{v \in \mathbb{R}^n \mid F_\Omega(p, v) < 1\}$ be the open unit ball in $T_p\Omega = \mathbb{R}^n$ with respect to the norm $F_\Omega(p, \cdot)$ and let ω_n be the Euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^n$. Then given any Borel set A in Ω, its Busemann volume Vol_Ω is defined by

$$\mathrm{Vol}_\Omega A = \int_A \frac{\omega_n}{\lambda(\beta_\Omega(p))}\, \mathrm{d}\lambda(p),$$

where λ denotes the standard Lebesgue measure on $\mathbb{R}^n$.

The second one is the *Holmes–Thompson* volume on Ω, which we will denote by $\mu_{HT,\Omega}$. Given any Borel set A in Ω, its Holmes–Thompson volume is defined by

$$\mu_{HT,\Omega}(A) = \int_A \frac{\lambda(\beta_\Omega^*(p))}{\omega_n}\, \mathrm{d}\lambda(p),$$

where $\beta_\Omega^*(p)$ is the polar dual of $\beta_\Omega(p)$.

We can actually consider a whole family of measures as follows. Let $\mathcal{E}_n$ be the set of pointed proper open convex sets in $\mathbb{R}^n$. These are the pairs (ω, x) such that ω is a proper open convex set and x is a point in ω. We shall say that a function $f : \mathcal{E}_n \to \mathbb{R}$ is a *proper density* if it is positive and satisfies the three following properties:

- *Continuity* with respect to the Hausdorff pointed topology on $\mathcal{E}_n$.
- *Monotone decreasing* with respect to the inclusion; i.e., if $x \in \omega \subset \Omega$ then $f(\Omega, x) \leq f(\omega, x)$.
- *Chain rule compatibility*: for any projective transformation T one has

$$f\big(T(\omega), T(x)\big)\,\mathrm{Jac}(T, x) = f(\omega, x).$$

We will say that f is a *normalised proper density* if $f(\omega, x)\, d\lambda(x)$ is the Riemannian volume when ω is an ellipsoid. Let us denote by $\mathcal{PD}_n$ the set of proper densities over $\mathcal{E}_n$.

A result of Benzécri [1960] states that the action of the group of projective transformations on $\mathcal{E}_n$ is cocompact. Therefore, for any pair f, g in $\mathcal{PD}_n$, there exists a constant $C > 0$ ($C \geq 1$ for the normalised ones) such that for any $(\omega, x) \in \mathcal{E}_n$ one has

$$\frac{1}{C} \leq \frac{f(\omega, x)}{g(\omega, x)} \leq C. \tag{9}$$

Given a density f in $\mathcal{PD}_n$ there is a natural Borel measure associated to any open bounded convex set Ω, denoted by $\mu_{f,\Omega}$ and defined as follows: for any Borel subset A of Ω we let

$$\mu_{f,\Omega}(A) = \int_A f(\Omega, p)\, \mathrm{d}\lambda(p).$$

Integrating the inequalities (9) we obtain that for any two proper densities f, g in $\mathcal{PD}_n$, there exists a constant $C > 0$ such that for any Borel set $A \subset \Omega$ we have

$$\frac{1}{C}\mu_{g,\Omega}(A) \leq \mu_{f,\Omega}(A) \leq C\mu_{g,\Omega}(A). \tag{10}$$

We call the family of measures obtained in this way *proper measures with density*.

To a proper density $g \in \mathcal{PD}_{n-1}$ we can also associate an $(n-1)$-dimensional measure, denoted by $\mu_{\cdot,g,\Omega}$, on hypersurfaces in Ω as follows. Let Σ be a smooth hypersurface, and consider for a point p in the hypersurface Σ its tangent hyperplane $H(p)$. Then the measure will be given by

$$\frac{d\mu_{\Sigma,g,\Omega}}{d\sigma}(p) = \frac{d\mu_{g,\Omega\cap H(p)}}{d\sigma}(p), \tag{11}$$

where σ denotes the Hausdorff $(n-1)$-dimensional measure associated with the standard Euclidean distance. In Section 2 we will simply denote by $\mathrm{Vol}_{n-1,\Omega}$ and Area_Ω the $(n-1)$-dimensional measures associated with the Holmes–Thompson and the Busemann measures, respectively.

Let now $\mu_{f,\Omega}$ be a proper measure with density over Ω. Then the volume entropies of Ω are defined by

$$
\begin{aligned}
\underline{\operatorname{Ent}}\,\Omega &= \liminf_{r\to+\infty} \frac{\ln \mu_{f,\Omega}(B_\Omega(p,r))}{r},\\
\overline{\operatorname{Ent}}\,\Omega &= \limsup_{r\to+\infty} \frac{\ln \mu_{f,\Omega}(B_\Omega(p,r))}{r}.
\end{aligned} \tag{12}
$$

These numbers do not depend on either f nor p, and are equal to the spherical entropies (see Theorem 2.14 of [Berck et al. 2010]):

$$
\begin{aligned}
\underline{\operatorname{Ent}}\,\Omega &= \liminf_{r\to+\infty} \frac{\ln(\operatorname{Area}_\Omega S_\Omega(p,r))}{r},\\
\overline{\operatorname{Ent}}\,\Omega &= \limsup_{r\to+\infty} \frac{\ln(\operatorname{Area}_\Omega S_\Omega(p,r))}{r}.
\end{aligned} \tag{13}
$$

1.2. *Properties of the Holmes–Thompson and the Busemann measures.*

Lemma 6 (monotonicity of the Holmes–Thompson measure). *Let* (Ω, d_Ω) *be a Hilbert geometry in* $\mathbb{R}^n$. *The Holmes–Thompson area measure is monotonic on the set of convex bodies in* Ω; *that is, for any pair of convex bodies* K_1 *and* K_2 *in* Ω *such that* $K_1 \subset K_2$ *one has*

$$
\operatorname{Vol}_{n-1,\Omega} \partial K_1 \le \operatorname{Vol}_{n-1,\Omega} \partial K_2. \tag{14}
$$

Proof. If $\partial\Omega$ is C^2 with everywhere-positive Gaussian curvature then the tangent unit spheres of the Finsler metric are quadratically convex.

According to Álvarez Paiva and Fernandes [1998, Theorem 1.1 and Remark 2] there exists a Crofton formula for the Holmes–Thompson area, from which the inequality (14) follows.

Such smooth convex bodies are dense in the set of all convex bodies for the Hausdorff topology. By approximation, it follows that (14) is valid for any Ω. □

Lemma 6 associated with the Blaschke–Santaló inequality and the inequality (10) immediately implies the following result (see also [Berck et al. 2010, Lemma 2.12]).

Lemma 7 (rough monotonicity of the Busemann measure). *Let* (Ω, d_Ω) *be a Hilbert geometry*, *and let* p *be a point in* Ω. *There exists a monotonic function* f_Ω *and a constant* $C_n < 1$ *such that for all* $r > 0$

$$
C_n f_\Omega(r) \le \operatorname{Area}_\Omega S_\Omega(p,r) \le f_\Omega(r), \tag{15}
$$

where $f_\Omega(r)$ *is the Holmes–Thompson area of the sphere* $S_\Omega(p,r)$.

Let us finish by recalling one last statement also proved in [Berck et al. 2010, Lemma 2.13].

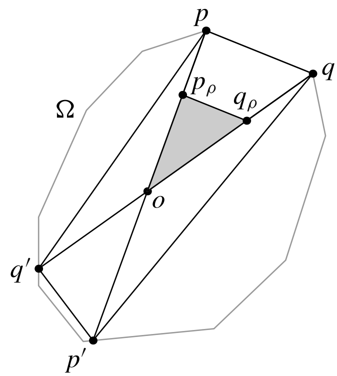


Figure 1. The area of the triangle (op_ρ, q_ρ) is bounded by $C\rho^2$.

Lemma 8 (coarea inequalities). *For all $r > 0$*

$$\frac{1}{2}\frac{\omega_n}{\omega_{n-1}}\operatorname{Area}_\Omega S_\Omega(p,r) \le \frac{\partial}{\partial r}\operatorname{Vol}_\Omega B_\Omega(p,r) \le \frac{n}{2}\frac{\omega_n}{\omega_{n-1}}\operatorname{Area}_\Omega S_\Omega(p,r).$$

1.3. ***Upper bound on the area of triangles.*** In this section we bound from above independently of the two-dimensional Hilbert geometries the area of affine triangles which are subset of a metric ball, when one of the vertices is the centre of that ball. We also give a lower bound on the length of some metric segments, when their vertices go to the boundary of the Hilbert geometry.

Lemma 9. *Let (Ω, d_Ω) be a two-dimensional Hilbert geometry. Then there exists a constant C independent of Ω such that for any point o in Ω and any pair of points p_ρ and q_ρ in the metric ball $B_\Omega(o,\rho)$, the area of the affine triangle $(op_\rho q_\rho)$ is less than $C\rho^2$.*

Proof. Given p_ρ and q_ρ in $B_\Omega(o,\rho)$, let p and q be the intersections of the boundary $\partial\Omega$ with the half-lines $[o, p_\rho)$ and $[o, q_\rho)$ respectively. Let p' and q' be, respectively, the intersections of the half-lines $[p_\rho, o)$ and $[q_\rho, o)$ with the boundary $\partial\Omega$. (See Figure 1.)

Then the volume of the triangle $(op_\rho q_\rho)$ with respect to the Hilbert geometry of Ω is less than or equal to its volume with respect to the Hilbert geometry of the quadrilateral $(pqp'q')$. However, the distances of p_ρ and q_ρ from o remain the same in both Hilbert geometries.

Up to a change of chart, we can suppose that this quadrilateral is actually a square. This allows us to use Theorem 1 from [Vernicos 2015], which states that the Hilbert geometry of the square is bi-Lipschitz to the product of the Hilbert geometries of its sides, using the identity as a map. In other words it is bi-Lipschitz to the Euclidean plane, with a Lipschitz constant equal to $C_0 > 1$, independent of our initial conditions.

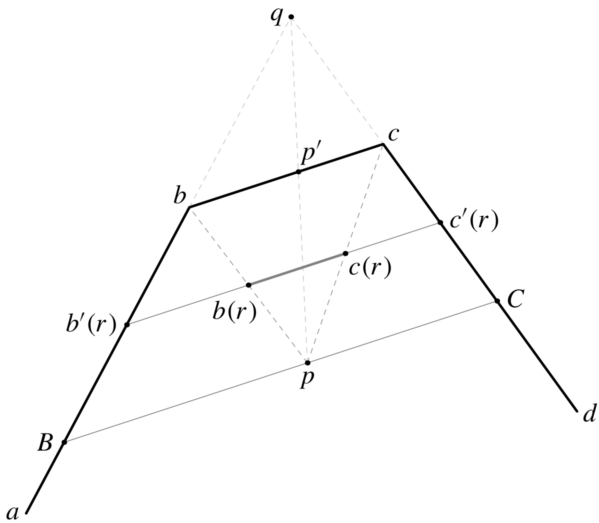


Figure 2. Distance estimate of Claim 10.

Thus our affine triangle is inside a Euclidean disc of radius $C_0\rho$, which implies that its area with respect to the Hilbert geometry of Ω is less than $C_0^4 \times \pi \times \rho^2$. $\square$

To prove that the volume entropy is bounded from below by the approximability we will need to bound from below the length of certain segments in a given Hilbert geometry Ω. To do so we will compare their length in the initial convex domain with their length in a convex domain projectively equivalent to a triangle, and containing the initial convex domain Ω.

Let us make this precise. Consider four points a, b, c and d in the Euclidean plane $(\mathbb{R}^2, \langle\cdot\rangle)$ such that $\mathcal{Q} = (abcd)$ is a convex quadrilateral. We assume that the scalar products $\langle\overrightarrow{ab}, \overrightarrow{bc}\rangle$ and $\langle\overrightarrow{bc}, \overrightarrow{cd}\rangle$ are positive and we let q be the intersection point between the straight lines (ab) and (cd).

Suppose that Ω is a convex domain such that the segments $[a, b]$, $[b, c]$ and $[c, d]$ belong to its boundary. Given p a point in the convex domain Ω we denote by p' the intersection between the straight line (pq) and the segment $[b, c]$, and we define $s = bp'/bc$.

We then denote by $[b(r), c(r)]$ the image of the segment $[b, c]$ under the dilation centred at p with ratio $0 < \tanh r < 1$. The image of the segment $[b, c]$ under the dilation centred at q sending p' to p will be denoted by $[B, C]$.

Claim 10. *Under the above assumption*,

$$d_\Omega\big(b(r), c(r)\big) \geq \tag{16}$$
$$\frac{1}{2}\ln\!\left(\frac{bc}{s\cdot BC}\,\frac{\tanh r}{1-\tanh r}+1\right)+\frac{1}{2}\ln\!\left(\frac{bc}{(1-s)\cdot BC}\,\frac{\tanh r}{1-\tanh r}+1\right).$$

Proof. Straightforward computation, using the fact that the convex domain Ω is inside the convex domain Q obtained as the intersection of the half-planes defined by the lines (ab), (bc) and (cd), and therefore

$$d_\Omega\big(b(r), c(r)\big) \geq d_Q\big(b(r), c(r)\big).$$

Let $b'(r)$ be the intersection of the lines (ab) and $\big(b(r)c(r)\big)$, and let $c'(r)$ be the intersection of the lines (cd) and $\big(b(r)c(r)\big)$. (See Figure 2.) Then we have

$$d_Q\big(b(r), c(r)\big) = \tfrac{1}{2}\ln\left(\frac{b(r)c'(r)}{c(r)c'(r)} \cdot \frac{c(r)b'(r)}{b(r)b'(r)}\right).$$

Let us focus on the first ratio. On the one hand $b(r)c'(r) = b(r)c(r) + c(r)c'(r)$, and on the other hand following Thales' theorem

$$\begin{aligned} b(r)c(r) &= \tanh(r)bc, \\ c(r)c'(r) &= (1-\tanh r)pC. \end{aligned} \tag{17}$$

But $pC = BC \cdot (p'c/bc) = (1-s)BC$, and therefore we obtain

$$\ln\left(\frac{b(r)c'(r)}{c(r)c'(r)}\right) = \ln\left(\frac{bc}{(1-s)\cdot BC}\,\frac{\tanh r}{1-\tanh r} + 1\right).$$

The second ratio is treated in the same way. □

1.4. ***Intrinsic and extrinsic Hausdorff topologies of Hilbert geometries.*** We describe the link between the Hausdorff topology induced by a Euclidean metric with the Hausdorff topology induced by the Hilbert metric on a compact subset of an open convex set.

We recall that the Löwner ellipsoid of a compact set is the ellipsoid with least volume containing that set. In this section we will suppose, without loss of generality, that Ω is a bounded open convex set whose Löwner ellipsoid $\mathcal{E}$ is the Euclidean unit ball with centre o. It is a standard result that $(1/n)\mathcal{E}$ is then contained in Ω; i.e.,

$$\frac{1}{n}\mathcal{E} \subset \Omega \subset \mathcal{E}. \tag{18}$$

Definition 11 (asymptotic ball and sphere). The *asymptotic ball* of radius R centred at o is the image of Ω by the dilation of ratio $\tanh R$ centred at o, and we denote it by $\mathrm{AsB}(o, R)$. The image of the boundary $\partial\Omega$ by the same dilation will be called the *asymptotic sphere* of radius R centred at o and denoted by $\mathrm{AsS}(o, R)$.

Recall that the Hausdorff distance is the distance between nonempty compact subsets in a metric space. We shall use both the Euclidean and Hilbert distance and we will use the terminology *Hausdorff–Euclidean* and *Hausdorff–Hilbert* to distinguish both cases.

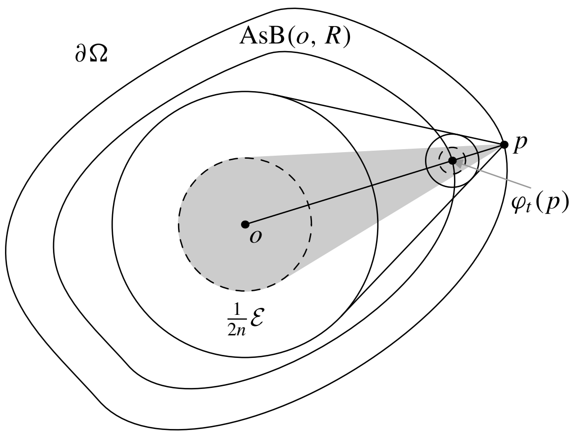


Figure 3. Illustration of Proposition 12's proof.

We would like to relate the Hausdorff–Hilbert neighbourhoods of the asymptotic ball $\mathrm{AsB}(o, R)$ with its Hausdorff–Euclidean neighbourhoods.

Proposition 12. *Let Ω be a convex domain and let o be the centre of its Löwner ellipsoid, which we assume to be the unit Euclidean ball.*

(1) *The $(1-\tanh R)/(2n)$-Hausdorff–Euclidean neighbourhood of the asymptotic ball $\mathrm{AsB}(o, R)$ is contained in its $\left(\frac{1}{2}\ln 3\right)$-Hausdorff–Hilbert neighbourhood.*

(2) *For any $K > 0$, the K-Hausdorff–Hilbert neighbourhood of the asymptotic ball $\mathrm{AsB}(o, R)$ is contained in its $(1-\tanh R)$-Hausdorff–Euclidean neighbourhood.*

Proof. For any point $p \in \partial\Omega$ on the boundary of Ω and for $0 < t < 1$ let $\varphi_t(p) = o + t \cdot \overrightarrow{op}$. This map sends $\partial\Omega$ bijectively to the asymptotic sphere $\mathrm{AsS}(o, \operatorname{arctanh} t)$ centred at o with radius $\operatorname{arctanh} t$. (See Figure 3.)

Proof of part (1). Any point of a compact set in the $(1-\tanh R)/(2n)$-Hausdorff–Euclidean neighbourhood of $\mathrm{AsB}(o, R)$, either lies inside $\mathrm{AsB}(o, R)$ or is contained in a Euclidean ball of radius $(1-\tanh R)/(2n)$ centred on a point of $\mathrm{AsB}(o, R)$.

We recall that the ball of radius $1/n$ is a subset of Ω, and thus so is the ball of radius $1/(2n)$; that is,

$$\frac{1}{2n}\mathcal{E} \subset \frac{1}{n}\mathcal{E} \subset \Omega.$$

Let $p \in \partial\Omega$ be a point on the boundary. By convexity, the interior of $K(p)$, the convex hull of p and $(1/n)\mathcal{E}$, is a subset of Ω — it is the projection of a cone of basis $(1/n)\mathcal{E}$. Hence $\mathcal{E}_{p,\alpha}$, the image of $(1/n)\mathcal{E}$ by the dilation of ratio $0 < \alpha < 1$ centred at p, lies in the "cone" $K(p)$. The set $\mathcal{E}_{p,\alpha}$ is therefore a Euclidean ball of radius α/n centred at $\varphi_{1-\alpha}(p)$, and it is a subset of Ω.

A point in the Euclidean ball of radius $\alpha/(2n)$ centred at $\varphi_{1-\alpha}(p)$ is at a distance less than or equal to $\frac{1}{2}\ln 3$ from $\varphi_{1-\alpha}(p)$ with respect to the Hilbert distance of $\mathcal{E}_{p,\alpha}$.

Now a standard comparison argument states that for any two points x and y in $\mathcal{E}_{p,\alpha} \subset \Omega$,

$$d_\Omega(x, y) \le d_{\mathcal{E}_{p,\alpha}}(x, y).$$

From this inequality it follows that any point in the Euclidean ball of radius $\alpha/(2n)$ centred at $\varphi_{1-\alpha}(p)$ is in the Hilbert metric ball centred at $\varphi_{1-\alpha}(p)$ of radius $\frac{1}{2}\ln 3$.

Now for any $1 \ge \alpha > 1 - \tanh R$, the Euclidean ball of radius $\alpha/(2n)$ contains the Euclidean ball of radius $(1 - \tanh R)/(2n)$.

This implies that for any point x in the asymptotic ball $\mathrm{AsB}(o, R)$, the Euclidean ball of radius $(1 - \tanh R)/(2n)$ centred at x is contained in the Hilbert ball of radius $\frac{1}{2}\ln 3$ centred at x, which allows us to obtain the first part of our claim.

Proof of part (2). This follows from the fact that under our assumptions, Ω itself is in the $(1-\tanh R)$-Hausdorff–Euclidean neighbourhood of the asymptotic ball $\mathrm{AsB}(o, R)$. □

Corollary 13. *Let Ω be a convex domain and let o be the centre of its Löwner ellipsoid, which we assume to be the unit Euclidean ball.*

(1) *The* $(1-\tanh(R+\ln 2))/(2n)$*-Hausdorff–Euclidean neighbourhood of* $B(o, R)$ *is contained in its* $\ln(3(n+1))$*-Hausdorff–Hilbert neighbourhood.*

(2) *For any* $K > 0$, *the K-Hausdorff–Hilbert neighbourhood of* $B(o, R)$ *is contained in its* $(1-\tanh(R+K-\ln(n+1)))$*-Hausdorff–Euclidean neighbourhood.*

The proof of this corollary is a straightforward consequence of the following lemma applied to the conclusion of the Proposition 12.

Lemma 14. *Let Ω be a convex domain, and suppose that o is a point in the interior of Ω such that the unit Euclidean open ball centred at o contains Ω, and Ω contains the Euclidean closed ball centred at o of radius* $1/(2n)$. *Then we have*

$$B(o, R) \subset \mathrm{AsB}(o, R + \ln 2) \quad \text{and} \quad \mathrm{AsB}(o, R) \subset B(o, R + \ln(n + 1)). \tag{19}$$

This lemma is a refinement of a result of [Colbois and Verovic 2004] in our case.

Proof of Lemma 14. Let x be a point on the boundary $\partial\Omega$ of Ω, and let x^* be the second intersection of the straight line (ox) with $\partial\Omega$. Then our assumption implies

$$\frac{1}{2n} < xo \le 1 \quad \text{and} \quad \frac{1}{2n} < ox^* \le 1. \tag{20}$$

Actually the first inclusion is always true. Indeed suppose y is on the half-line $[ox)$ such that $d_\Omega(o, y) \le R$, which in other words implies that we have

$$\frac{ox}{yx}\frac{yx^*}{ox^*} \le e^{2R};$$

therefore

$$ox \le e^{2R}\frac{ox^*}{yx^*}(ox - oy) \le e^{2R}(ox - oy),$$

which implies in turn that

$$oy \le \frac{e^{2R}-1}{e^{2R}}ox \le (1-e^{-2R})ox \le \tanh(R+\ln 2)ox.$$

Now regarding the second inclusion: consider a point y on the half-line $[ox)$ such that $oy \le \tanh(R)ox$. On the one hand we have

$$\frac{ox}{yx} = \frac{ox}{ox-oy} \le \frac{1}{1-\tanh R} = \frac{e^{2R}+1}{2},$$

and, on the other hand, thanks to the inequalities (20) we get

$$\frac{yx^*}{ox^*} \le \frac{ox+ox^*}{ox^*} \le 1 + \frac{ox}{ox^*} \le 1+2n, \tag{21}$$

which implies that

$$\frac{ox}{yx}\frac{yx^*}{ox^*} \le \frac{e^{2R}+1}{2}(1+2n) \le (1+2n)e^{2R} \le (1+n)^2 e^{2R}. \tag{22}$$

The conclusion follows. □

1.5. ***Distance function to a sphere in a Hilbert geometry.*** This section is an adaptation in the realm of Hilbert geometries of a result concerning the spheres in a Minkowski space provided to the author by A. Thompson [2012].

Let us first start by recalling the following important fact regarding the distance of a point to a geodesic in a Hilbert geometry (see [Busemann 1955, Chapter II, Section 18, page 109]):

Proposition 15. *Let (Ω, d_Ω) be a Hilbert geometry. The distance function of a straight geodesic (that is, given by an affine line) to a point is a peakless function; i.e., if $\gamma : [t_1, t_2] \to \Omega$ is a geodesic segment, then for any $x \in \Omega$ and $t_1 \le s \le t_2$ one has*

$$d_\Omega\big(x, \gamma(s)\big) \le \max\big\{d_\Omega\big(x, \gamma(t_1)\big), d_\Omega\big(x, \gamma(t_2)\big)\big\}.$$

Let us now turn our attention to metric spheres in a two-dimensional Hilbert geometry.

Proposition 16. *Let (Ω, d_Ω) be a two-dimensional Hilbert geometry. Suppose o is a point of Ω, and p and q are two points on the intersection of the metric sphere $S(o, R)$ centred at o and of radius R with a line passing by o. If C denotes one of the arcs of the sphere $S(o, R)$ from p to q, then for any point p' on the half-line $[o, p)$, the function $\varphi(x) = d_\Omega(p', x)$ is monotonic on C.*

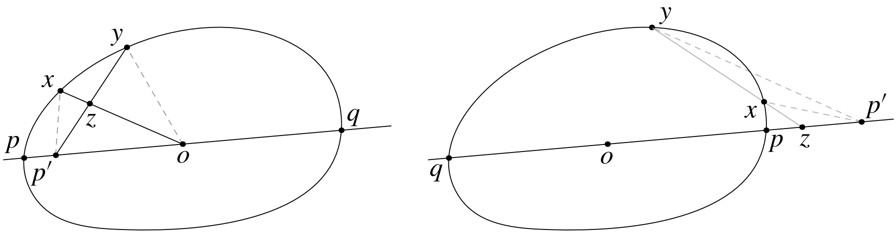


Figure 4. Monotonicity of the distance of a point to a sphere.

Proof. Let p, x, y, q be points in that order on C. We have to show that

$$d_\Omega(p', x) \leq d_\Omega(p', y).$$

Suppose first that the line segments $[o, x]$ and $[p', y]$ intersect at a point z. (See Figure 4.) Hence we have

$$\begin{aligned} d_\Omega(o, x) + d_\Omega(p', y) &= \big(d_\Omega(o, z) + d_\Omega(z, x)\big) + \big(d_\Omega(p', z) + d_\Omega(z, y)\big) \\ &= \big(d_\Omega(p', z) + d_\Omega(z, x)\big) + \big(d_\Omega(o, z) + d_\Omega(z, y)\big) \\ &\geq d_\Omega(p', x) + d_\Omega(o, y). \end{aligned}$$

Now, as $d_\Omega(o, y) = d_\Omega(o, x) = R$, the result follows.

Suppose now that $[o, x]$ and $[p', y]$ do not intersect, which implies that p' is outside the ball $B(o, R)$. Then the line (yx) intersects (op) at z. Because x and y lie on the sphere of radius R, we have $d_\Omega(o, z) > R$. Also, as p is one of the nearest points to p' on C, we have $d_\Omega(p', z) \leq d_\Omega(p', p) \leq d_\Omega(p', y)$. Hence if we apply Proposition 15 to the segment $[z, y]$ and p', as $x \in [z, y]$ we get

$$d_\Omega(p', x) \leq \max\{d_\Omega(p', z), d_\Omega(p', y)\} = d_\Omega(p', y). \qquad \square$$

2. Volume entropy and approximability

This section is devoted to the proof of the main theorem. This is done in two steps. The first step consists in bounding the entropy from above in dimension two and three by the approximability thanks to the study of the volume growth in polytopes. The second step is to bound the entropy from below. This is done by exhibiting a separated subset of the Hilbert geometry whose growth is bigger than the approximability. We conclude this section with the various corollaries implied.

Theorem 17. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. The doubles of the approximabilities of Ω are bigger than the volume entropies*; *i.e.*,

$$\underline{\mathrm{Ent}}\,\Omega \leq 2\underline{a}(\Omega) \quad \textit{and} \quad \overline{\mathrm{Ent}}\,\Omega \leq 2\overline{a}(\Omega).$$

The proof of this theorem relies on the following stronger statement which is a sort of uniform bound on the volume of metric balls and metric spheres in a polytopal Hilbert geometry. The key fact is that this bound depends, in a coarse sense, linearly on the number of vertices of the polytope.

Theorem 18. *Let $n = 2$ or $n = 3$. There are affine maps a_n, b_n from $\mathbb{R}$ to $\mathbb{R}$ and polynomials q_n, p_{n-1} of degree n and $n-1$ such that for any open convex polytope $\mathcal{P}_N$ with N vertices inside the unit Euclidean ball of $\mathbb{R}^n$ and containing the ball of radius $1/(2n)$, one has*

$$\begin{aligned} \mathrm{Vol}_{n-1,\mathcal{P}_N} S_{\mathcal{P}_N}(o, R) &\leq a_n(N) p_{n-1}(R), \\ \mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, R) &\leq b_n(N) q_n(R). \end{aligned} \tag{23}$$

The same result holds for the asymptotic balls.

Let us stress that our method also yields a control in terms of the vertices in higher dimensions as well, using the so-called upper bound conjecture proved by McMullen [McMullen 1971; McMullen and Shephard 1971], but alas a polynomial of degree strictly bigger than 1 replaces the affine functions a_n and b_n. This is why we can't state the equality in the main theorem in higher dimensions.

Notice that this theorem is still valid if we replace the Hausdorff measures by any measures defined by a pair of proper densities $f \in \mathcal{PD}_n$ and $g \in \mathcal{PD}_{n-1}$. The change of measures will only impact the values of the constants.

Proof of Theorem 18. We will have to deal with dimension two and dimension three separately, even though both cases follow the same main steps.

The first step of our proof consists in proving the first inequality of (23) for the Holmes–Thompson measure and for an asymptotic sphere. The uniform inclusion of metric balls into asymptotic balls (19) then implies the result for the metric spheres thanks to the monotonicity of the Holmes–Thompson measure (Lemma 6).

The second step is an integration using the coarea inequality (25), which allows us to get the second inequality of (23) for metric balls with respect to the Busemann measure.

Let us now make all this more precise. We fix a polytope $\mathcal{P}_N$ with N vertices and for any real $R > 0$ we let P_R be the asymptotic ball of radius R centred at o, and let ∂P_R be the associated asymptotic sphere. We also introduce the constant $c_n = \ln(n+1)$.

Two-dimensional case. The idea is to find an upper bound on the length of each edge of the asymptotic sphere ∂P_R, depending only on R.

To do so, we can use the fact that each edge belongs to the triangle defined by joining its extremities to the point o. Hence, thanks to the triangle inequality its length is less than the sum of these two other segments. However, using the second inclusion (19) of Lemma 14, we know that the asymptotic ball P_R is inside the

Hilbert ball of radius $R + c_2$ centred at o of the convex polygon $\mathcal{P}_N$. Hence the length of each edge is less than $2 \cdot (R + c_2)$. Therefore the length of the polygon ∂P_R is less than $N \cdot 2 \cdot (R + c_2)$.

Following the first inclusion (19) of Lemma 14, the metric ball of radius r centred at o is a subset of the asymptotic ball of radius $r + \ln 2$ centred at o. Therefore, we can use the monotonicity of the Holmes–Thompson length (see Lemma 6) to get for all $r > 0$,

$$\text{Length}_{\mathcal{P}_N} S_{\mathcal{P}_N}(o, r) \leq \text{Length}_{\mathcal{P}_N} \partial P_{r+\ln 2} \leq N \cdot 2(r + \ln 2 + c_2). \tag{24}$$

Now using the coarea inequality of Lemma 8, taking into account that the Busemann length is equal to the Holmes–Thompson length one gets

$$\frac{\partial}{\partial r} \text{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, r) \leq \frac{\pi}{4} \cdot N \cdot 2(r + \ln 2 + c_2). \tag{25}$$

Hence, integrating the inequality (25) over the interval $[0, R]$, we finally obtain the following inequality for the ball of radius $R > 0$:

$$\text{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, R) \leq \frac{\pi}{4} \cdot N \cdot (R^2 + 2(\ln 2 + c_2)R). \tag{26}$$

The inequalities (24) and (26) are the expected results in dimension two.

Three-dimensional case. Once again the idea is to find an upper bound on the area of faces of the asymptotic sphere ∂P_R. Alas, contrary to the two-dimensional case, there is not a unique type of faces, and it is therefore pointless to look for an upper bound depending only on the radius R.

However, each face can be seen as the basis of a pyramid with apex the point o. All other faces are then triangles, whose areas can be bounded thanks to Lemma 9. An analogue of the triangle inequality is available in the form of the minimality of the Holmes–Thompson area (see Berck [2009]). In other words, the Holmes–Thompson area of each face of ∂P_R is less than the sum of the Holmes–Thompson areas of the triangles obtained as the convex hull of o and an edge of the given face of ∂P_R. Let us call $\mathcal{T}_o$ such a triangle (the subscript o is to stress the fact that the point o is one of its vertices).

To bound the area of the triangle $\mathcal{T}_o$ it suffices to focus on the intersection of the polytope $\mathcal{P}_N$ with the affine plane containing the triangle $\mathcal{T}_o$. This is a polygon $\widetilde{P}$, to which we can apply Lemma 9, which bounds from above the area of a two-dimensional triangle inside a metric ball centred on one of its vertices. This is exactly the situation of our triangle $\mathcal{T}_o$ with respect to the Hilbert geometry associated to the polygon $\widetilde{P}$. Indeed it is included in the asymptotic ball of radius R, and again thanks to Lemma 14 we know that it is inside the metric ball of radius $R + c_3$ with respect to the Hilbert geometry of $\mathcal{P}_N \in \mathbb{R}^3$. As $\widetilde{P}$ is a plane section of $\mathcal{P}_N \in \mathbb{R}^3$, this still holds for $\mathcal{T}_o$ seen as a subset of $\widetilde{P}$. Hence Lemma 9 implies that the area of the triangle $\mathcal{T}_o$ is less than $C(R+c_3)^2$, for some constant $C>1$ independent of R.

Therefore, if $e(N)$ is the number of edges of $\mathcal{P}_N$, the area of the asymptotic sphere ∂P_R is less than $2e(N)C(R+c_3)^2$.

Let $f(N)$ be the number of faces of $\mathcal{P}_N$ and let us recall Euler's formula:

$$N - e(N) + f(N) = 2.$$

Each face being surrounded by at least three edges and each edge belonging to two faces, one has the classical inequality (where equality is obtained in a simplex)

$$3f(N) \leq 2e(N).$$

Combining the previous two inequalities we get a linear upper bound on the number of edges by the number of vertices:

$$2 \leq N - \tfrac{1}{3}e(N) \quad \Rightarrow \quad e(N) \leq 3N-6.$$

Hence the area of the asymptotic sphere ∂P_R is less than $(3N-6)\cdot 2C\cdot(R+c_3)^2$.

We can now conclude almost as in the two-dimensional case. Following the first inclusion (19) of Lemma 14, the metric ball of radius r centred at o is a subset of the asymptotic ball of radius $r+\ln 2$ centred at o. Therefore, we can use the monotonicity of the Holmes–Thompson area measure (see Lemma 6) to get for all $r>0$,

$$\text{(27)} \qquad \mathrm{Vol}_{2,\mathcal{P}_N} S_{\mathcal{P}_N}(o,r) \leq \mathrm{Vol}_{2,\mathcal{P}_N} \partial P_{r+\ln 2} \leq (3N-6)\cdot 2C\cdot(r+\ln 2+c_3)^2.$$

Notice that this inequality (27) corresponds to the first part of the inequality (23).

The rough monotonicity of the Busemann measure (see the right-hand side of the inequality (15) in Lemma 7) states that the Busemann area is smaller than the Holmes–Thompson one, hence combined with the inequality (27) above, we get that for all $r>0$

$$\text{(28)} \qquad \mathrm{Area}_{\mathcal{P}_N} S_{\mathcal{P}_N}(o,r) \leq (3N-6)\cdot 2C\cdot(r+\ln 2+c_3)^2.$$

Taking into account the coarea inequality (see Lemma 8) in conjunction with the inequality (28) leads to the differential inequality

$$\text{(29)} \qquad \frac{\partial}{\partial r}\mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o,r) \leq 2\cdot(3N-6)\cdot 2C\cdot(r+\ln 2+c_3)^2,$$

which we can integrate over the interval $[0,\, R]$ to finally obtain that for all $R>0$

$$\text{(30)} \qquad \mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o,\, R) \leq 2\cdot(N-2)\cdot 2C\cdot\big((r+\ln 2+c_3)^3 - c_3^3\big).$$

This concludes our proof in the three-dimensional case. $\square$

Let us remark that if we link this to our study of the asymptotic volume of the Hilbert geometry of polytopes [Vernicos 2013] we obtain the following corollary:

Corollary 19. *Let $\mathcal{P}_N$ be an open convex polytope with N vertices in $\mathbb{R}^n$, for $n = 2$ or 3. Then there are three constants α_n, β_n and γ_n such that for any point $p \in \mathcal{P}_N$ one has*

$$\alpha_n \cdot N \leq \liminf_{R\to+\infty} \frac{\mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(p, R)}{R^n} \leq \beta_n \cdot N + \gamma_n.$$

Now let us come back to our initial problem and see how Theorem 18 implies Theorem 17.

Proof of Theorem 17. We remind the reader that $\mathrm{Vol}_{n-1,\Omega}$ stands for the $(n-1)$-dimensional Holmes–Thompson measure. Let o be the centre of the Löwner ellipsoid of Ω, which we assume to be the unit Euclidean ball. We consider R large enough in order to have the Euclidean ball of radius $1/(2n)$ inside all the asymptotic balls involved in the sequel.

The idea of the proof consists in replacing for all R large enough the convex set Ω by a convex polytope $\mathcal{P}_R$ such that

- $\mathcal{P}_R$ is a subset of Ω;
- the asymptotic ball P_R of the polytope $\mathcal{P}_R$ is inside the $(1-\tanh R)/(2n)$-Euclidean neighbourhood of the corresponding asymptotic ball $\mathrm{AsB}_\Omega(o, R)$ of Ω;
- the exponential volume growth, with respect to the geometry of Ω, of the two families of asymptotic balls $(P_R)_{R\in\mathbb{R}}$ and $(\mathrm{AsB}_\Omega(o, R))_{R\in\mathbb{R}}$ is the same.

Let us insist on the fact that the convex polytope $\mathcal{P}_R$ depends on R.

Then using Theorem 18 we will bound from above the area in dimension three or the perimeter in dimension two of the convex polytope P_R by a function depending linearly on the number of vertices of P_R and polynomially on R. This will allow us to conclude.

Fix R. Among all polytopes included in both the asymptotic ball $\mathrm{AsB}_\Omega(o, R)$ and its $(1-\tanh R)/(2n)$-Hausdorff–Euclidean neighbourhood pick a polytope P_R with the minimal number of vertices $N(R)$. Notice that we have

$$N(R) = N\left(\frac{1-\tanh R}{2n\tanh R}, \Omega\right). \tag{31}$$

Claim. *There exists a constant $C > 0$ such that for all R,*

$$\mathrm{AsB}_\Omega(o, R-C) \subset P_R \subset \mathrm{AsB}_\Omega(o, R). \tag{32}$$

To prove this claim, on the one hand we deduce from the first inclusion of Lemma 14 that

$$B_\Omega(o, R - \ln 2) \subset \mathrm{AsB}_\Omega(o, R).$$

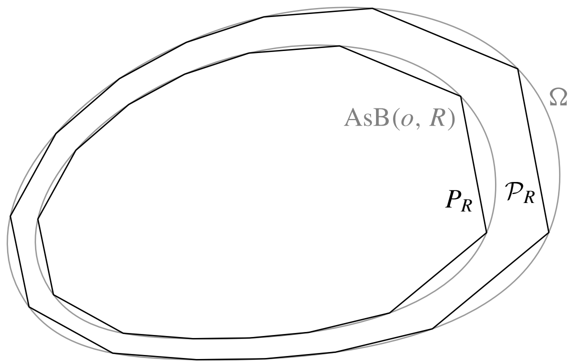


Figure 5. The asymptotic ball and an approximating polytope.

On the other hand the comparison of both Hausdorff–Hilbert and Hausdorff–Euclidean neighbourhoods, as stated in Proposition 12, implies that the convex polytope P_R lies in the $\left(\frac{1}{2}\ln 3\right)$-Hausdorff–Hilbert neighbourhood of the asymptotic ball $\mathrm{AsB}_\Omega(o, R)$. From these we deduce the inclusion

$$B_\Omega(o, R-\ln 6) \subset P_R \subset \mathrm{AsB}_\Omega(o, R). \tag{33}$$

Taking into account the second inclusion of Lemma 14 we get

$$\mathrm{AsB}_\Omega(o, R-\ln 6-\ln(n+1)) \subset P_R \subset \mathrm{AsB}_\Omega(o, R), \tag{34}$$

which proves our claim with $C = \ln 6 + \ln(n+1)$.

Thanks to the monotonicity of the Holmes–Thompson measure (see Lemma 6) we know that the area of the boundary ∂P_R is less than the area of the asymptotic sphere $\mathrm{AsS}_\Omega(o, R)$, but larger than the area of the asymptotic sphere of radius $R-C$; that is,

$$\mathrm{Vol}_{n-1,\Omega}\,\mathrm{AsS}_\Omega(o, R-C) \le \mathrm{Vol}_{n-1,\Omega}\,\partial P_R \le \mathrm{Vol}_{n-1,\Omega}\,\mathrm{AsS}_\Omega(o, R). \tag{35}$$

From (35) we deduce that the logarithms of the areas of ∂P_R and $\mathrm{AsS}_\Omega(o, R)$ are asymptotically the same in the following sense:

$$\lim_{R\to+\infty} \frac{\ln(\mathrm{Vol}_{n-1,\Omega}\,\mathrm{AsS}_\Omega(o, R))}{\ln(\mathrm{Vol}_{n-1,\Omega}\,\partial P_R)} = 1. \tag{36}$$

Let us denote by $\mathcal{P}_R$ the image of P_R by the dilation of ratio $1/\tanh R$. This is the dilation sending $\mathrm{AsB}_\Omega(o, R)$ to Ω. (See Figure 5.) Hence, by construction, $\mathcal{P}_R \subset \Omega$ and therefore we have

$$\mathrm{Vol}_{n-1,\Omega}\,\partial P_R \le \mathrm{Vol}_{n-1,\mathcal{P}_R}\,\partial P_R. \tag{37}$$

Now thanks to Theorem 18, for $n=2$ or $n=3$ and $R>0$ such that $\tanh R > \frac{3}{4}$, there are two constants a_n, b_n and a polynomial Q_n of degree n such that

$$\operatorname{Vol}_{n-1,\Omega} \partial P_R \leq (a_n N(R) + b_n) Q_n(R). \tag{38}$$

To conclude we remark that

$$\liminf_{R\to+\infty} \frac{\ln N(R)}{R} = 2\underline{a}(\Omega) \quad \text{and} \quad \limsup_{R\to+\infty} \frac{\ln N(R)}{R} = 2\overline{a}(\Omega),$$

and use it with the inequality (38) to get for instance

$$\limsup_{R\to+\infty} \frac{\ln(\operatorname{Vol}_{n-1,\Omega} \partial P_R)}{R} \leq 2\overline{a}(\Omega).$$

Finally the limit (36) implies that

$$\limsup_{R\to+\infty} \frac{\ln(\operatorname{Vol}_{n-1,\Omega} \operatorname{AsS}_\Omega(o, R))}{R} \leq 2\overline{a}(\Omega).$$

The left-hand side of this last inequality is easily seen to be the spherical entropy (see (13)), which ends our proof. □

The next corollary follows from a result of Bronshteyn and Ivanov (Theorem 31) which states that $2\overline{a} \leq n-1$.

Corollary 20. *Let Ω be an open bounded convex set in $\mathbb{R}^n$ for $n=2$ or 3. Then*

$$\overline{\operatorname{Ent}}\,\Omega \leq n-1.$$

We are now going to study the reverse inequality.

Theorem 21. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. The volume entropies of Ω are greater than or equal to twice the approximabilities of Ω; i.e.,*

$$2\underline{a}(\Omega) \leq \underline{\operatorname{Ent}}\,\Omega \quad \textit{and} \quad 2\overline{a}(\Omega) \leq \overline{\operatorname{Ent}}\,\Omega.$$

Proof. Without loss of generality we suppose that the Euclidean unit ball is the Löwner ellipsoid of Ω and that o is the centre of that ball.

The idea of the proof is the following:

- We will show that for a good positive δ and any positive real number R there exists a δ-separated set $\mathcal{S}_R$ in the metric ball of radius $B(o, R+2\delta)$ such that the convex closure P_R of that set contains the ball $B(o, R)$.
- We will then use the fact that the cardinality of this δ-separated set will be larger than the cardinality of the set of vertices of a vertex-minimising convex polytope included in the annulus $B(o, R+2\delta) \setminus B(o, R)$.

In other words, the number of points in the δ-separated set will be bounded from below by the number $N(\varepsilon(R), \Omega)$ from the introduction. Here ε will be a function of R.

- To conclude we will take into account that the union of the open metric balls of radius $\frac{1}{2}\delta$ centred at the point of the δ-separated set $\mathcal{S}_R$ is disjoint and is in the ball $B(o, R+3\delta)$. Thus we get a lower bound on the volume of the ball $B(o, R+3\delta)$ in terms of $N(\varepsilon(R), \Omega)$ times a constant depending on the dimension.

Let us now start the proof. Consider the $\left(\frac{1}{2}\ln 3\right)$-Hilbert neighbourhood of the metric ball $B(o, R)$, that is,

$$V(R) = B\left(o, R + \tfrac{1}{2}\ln 3\right),$$

and take a maximal ($\delta = \frac{1}{4}\ln 3$)-separated set $\mathcal{S}_R$ on its boundary. This set contains $\#\mathcal{S}_R$ points. Now let us take the convex hull $\mathcal{C}_R$ of these points. This is a polytope with $N_2(R) \leq \#\mathcal{S}_R$ vertices.

Claim 22. *The polytope $\mathcal{C}_R$ is included in the 2δ-Hilbert neighbourhood of $B(o, R)$ and contains $B(o, R)$.*

Notice that if the claim holds, then for some real constant c independent of R (see Corollary 13 once again), we have

$$\#\mathcal{S}_R \geq N_2(R) \geq \widetilde{N}(R-c) := N\left(\tfrac{1}{4}(1-\tanh(R-c)), \mathrm{AsB}(o, R-c)\right). \tag{39}$$

Proof of Claim 22. First notice that $V(R)$ is a convex set (see Busemann [1955, Chapter II, Section 18, page 105]). Therefore the convex hull is inside the 2δ-Hilbert neighbourhood of $B(o, R)$, that is, $V(R)$.

Now let us suppose by contradiction that $\mathcal{C}_R$ does not contain $B(o, R)$. Hence there exists some point q in $B(o, R)$ which is not in $\mathcal{C}_R$. We will show that we can find a point on the sphere $S(o, R+2\delta)$ which is at a distance bigger than δ from all points of $\mathcal{S}_R$, which will contradict its maximality.

Under our assumption, the Hahn–Banach separation theorem asserts that there exists a linear form a, some constant c and a hyperplane $H = \{x \mid a(x) = c\}$ which separates q and $\mathcal{C}_R$, i.e., $a(q) > c$ and $a(x) < c$ for all $x \in \mathcal{C}_R$. Consider then $H_q = \{x \mid a(x) = a(q)\}$, the hyperplane parallel to H containing q. Let us say that a point x such that $a(x) \geq a(q)$ is *above* the hyperplane H_q.

Then let us define by $V_o' = \{x \in \partial V(R) \mid a(x) \geq a(q)\}$ the part of the boundary of $V(R)$ which is above H_q. Now we want to metrically project each point of V_o' onto H_q, that is to say that to each point of V_o' we associate its closest point on H_q. However if Ω is not strictly convex, the projection might not be unique (see the Appendix); that is why we are going to distinguish two cases.

First case: the convex set Ω is strictly convex. Then the metric projection is a map from V_o' to H_q and it is continuous; furthermore the points on $H_q \cap V_o'$ are fixed and by convexity $H_q \cap V_o'$ is homeomorphic to an $(n-2)$-dimensional sphere. Therefore by the Borsuk–Ulam theorem (or a version of it known as the *antipodal map theorem*), there is a point p on V_o' whose metric projection is q.

Now as p is on the boundary of $V(R)$, that is, the sphere $B(o, R+2\delta)$, and q is in $B(o, R)$ we necessarily have

$$d_\Omega(p, q) \geq \tfrac{1}{2} \ln 3.$$

Hence for all points x in $H_q \cap V_o'$, we have

$$d_\Omega(p, x) \geq d_\Omega(p, q) \geq \tfrac{1}{2} \ln 3.$$

Second case: the convex set Ω is not strictly convex. Then let us approximate it by a smooth and strictly convex set Ω' such that $\Omega \subset \Omega'$, and for all pairs of points $x, y \in V(R)$,

$$\tfrac{2}{3} \times d_{\Omega'}(x, y) \geq d_\Omega(x, y) \geq d_{\Omega'}(x, y). \tag{40}$$

Then metrically project V_o' onto H_q with respect to Ω'. By the same argument as in the first case, we obtain a point p such that for all x in $H_q \cap V_o'$ we have

$$d_{\Omega'}(p, x) \geq d_{\Omega'}(p, q) \geq \tfrac{3}{2} d_\Omega(p, q) \geq \tfrac{3}{4} \ln 3,$$

which also implies by the inequalities (40) that for all x in $H_q \cap V_o'$ we have

$$d_\Omega(p, x) \geq \tfrac{3}{4} \ln 3.$$

In either case, using Proposition 16 of Section 1.5, we deduce that all points on ∂V_R at distance less than or equal to $\frac{1}{4} \ln 3$ from p are above H_q and are therefore contained in V_o'. We then infer that there are no points of $\mathcal{S}_R$ at distance less than or equal to $\frac{1}{4} \ln 3$ from p, which contradicts the maximality of the set $\mathcal{S}_R$. □

Now consider the union of the balls of radius $\frac{1}{2}\delta$ centred at the points of $\mathcal{S}_R$. This union is a subset of the ball $B(o, R+3\delta)$ and the balls are mutually disjoint. Now following our paper [Vernicos 2013], there exists a constant a_n such that for any open proper convex set Ω and $x \in \Omega$, the volume of the ball of radius r centred at x is at least $a_n r^n$. Hence from this fact and the inequality (39) we get that for all $R > 0$,

$$\begin{aligned} \operatorname{Vol}_\Omega B(o, R+3\delta) &\geq \#\mathcal{S}_R \cdot a_n \delta^n \\ &\geq N\bigl(\tfrac{1}{4}(1-\tanh(R-c)), \operatorname{AsB}(o, R-c)\bigr) \cdot a_n \delta^n. \end{aligned} \tag{41}$$

Now if we take the logarithm of the previous inequalities, divide by R and take either the lim inf or the lim sup we conclude the proof of Theorem 21. □

The proof of the main theorem (Theorem 1) is now complete, and we turn to its corollaries.

A point x of a convex body K is called a *farthest point* of K if and only if, for some point $y \in \mathbb{R}^n$, x is farthest from y among the points of K. The set of farthest points of K, which are special exposed points, will be denoted by $\exp^* K$. Thus a point $x \in K$ belongs to $\exp^* K$ if and only if there exists a ball which circumscribes K and contains x in its boundary.

In dimension two we get the following corollary:

Corollary 23. *Let Ω be a plane Hilbert geometry, and let d_M be the Minkowski dimension of extremal points and d_H the Hausdorff dimension of the set $\exp^* \Omega$ of farthest points. Then we have*

$$d_H \leq \underline{\mathrm{Ent}}\,\Omega \leq \overline{\mathrm{Ent}}\,\Omega \leq \frac{2}{3-d_M}. \tag{42}$$

The inequality on the left remains valid for higher-dimensional Hilbert geometries.

Proof. The inequality on the left of (42) comes from [Schneider and Wieacker 1981], whereas the one on the right is the first main theorem in [Berck et al. 2010]. □

Remark 24. Inequality (42) induces a new result concerning the approximability in dimension two, as it implies that

$$\bar{a}(\Omega) \leq \frac{1}{3-d}.$$

Lastly we are also able to prove the following result, which relates the entropy of a convex set and the entropy of its polar body.

Corollary 25. *Let Ω be a Hilbert geometry of dimension two or three. Then*

$$\underline{\mathrm{Ent}}\,\Omega = \underline{\mathrm{Ent}}\,\Omega^* \quad \textit{and} \quad \overline{\mathrm{Ent}}\,\Omega = \overline{\mathrm{Ent}}\,\Omega^*.$$

Proof. It suffices to prove that the approximability of a convex body Ω containing the origin and its polar Ω^* are equal. Without loss of generality we can assume that the unit ball is Ω's John ellipsoid. Hence Ω is contained in the ball of radius the dimension and its polar contains the ball of radius the inverse of the dimension and is included in the unit ball. Now, notice that for ε small enough, if P_k is a polytope with k vertices inside the ε-Hausdorff neighbourhood of Ω, then its polar P_k^* is a polytope with k faces containing Ω^* and contained in its $(\varepsilon \cdot C)$-Hausdorff neighbourhood for some constant C depending only on the dimension. A known fact (see Gruber [2007, Section 11.2]) states that the approximability can be computed by minimising either the vertices or the faces. Hence $\underline{a}(\Omega) = \underline{a}(\Omega^*)$ and $\bar{a}(\Omega) = \bar{a}(\Omega^*)$. The statement therefore follows from the main theorem. □

3. Intermediate growth

In this section we focus on the two-dimensional case. The intermediate volume growth will follow from Theorem 18 and the following proposition, which allows us to control both the length of sphere and its volume in dimension two from below, thanks to the number of vertices of an ad hoc approximating polytope, in the fashion of Theorem 18, except that here the lower bounds depend on Ω.

Proposition 26. *Let Ω be an open bounded convex set in $\mathbb{R}^2$ whose Löwner ellipsoid is the Euclidean unit ball centred at $o \in \Omega$. Let $N(\varepsilon, \Omega)$ be the minimal number of vertices of a polygon containing Ω at Hausdorff–Euclidean distance less than ε from Ω, and for any positive real number R let*

$$N(R) := N\left(\frac{1-\tanh R}{4\tanh R}, \Omega\right).$$

Then there exist three constants R_2, K_2 and C_2 independent of Ω such that for all real numbers $R > R_2$ we have

$$\begin{aligned} \text{Length}_\Omega\, S_\Omega(o, R) &\geq \big(N\big(R - \tfrac{3}{2}\ln 3\big) - 2\big)K_2, \\ \text{Vol}_\Omega\, B_\Omega\big(o, R + \tfrac{1}{2}K_2\big) &\geq \big(N\big(R - \tfrac{3}{2}\ln 3\big) - 2\big)C_2(K_2)^2. \end{aligned} \tag{43}$$

The same result holds for the asymptotic balls with $R > R_2 + \ln 2$.

We want to stress once again that there is actually no loss in generality in supposing the Euclidean unit ball to be the Löwner ellipsoid of Ω.

Proof. For any positive real number R let $\varepsilon(R) = \frac{1}{4}(1 - \tanh R)$. The idea is to build a convex polygon in the $\varepsilon(R)$-neighbourhood of an asymptotic ball of radius R in such a way that we can control uniformly from below the length of the edges. More precisely we have the following.

Claim 27. *There exists a convex polygon $\mathcal{P}_R$ such that*

- *$\mathcal{P}_R$ contains the asymptotic ball* $\mathrm{AsB}(o, R)$ *and is in its $\varepsilon(R)$-Hausdorff–Euclidean neighbourhood*;
- *all the edges of $\mathcal{P}_R$ but one are tangent to* $\mathrm{AsB}(o, R)$ *and all its vertices belong to the boundary* $\partial_R\,\mathrm{AsB}$ *of the $\varepsilon(R)$-Hausdorff neighbourhood of the asymptotic ball* $\mathrm{AsB}_\Omega(o, R)$.

This claim is a consequence of the following algorithm:

Step 1 Draw one tangent to $\mathrm{AsB}_\Omega(o, R)$. It will meet the boundary $\partial_R\,\mathrm{AsB}$ of its $\varepsilon(R)$-Hausdorff neighbourhood at two points x_1 and x_2, where $\overrightarrow{ox_1}$ and $\overrightarrow{ox_2}$ are positively oriented.

Step 2 We start from x_2 and draw the second tangent to $\mathrm{AsB}_\Omega(0, R)$ passing by x_2. This second tangent will meet the boundary $\partial_R\,\mathrm{AsB}$ at a second point x_3.

Step 3 For $k > 2$, if the second tangent t_{k+1} to $\mathrm{AsB}_\Omega(0, R)$ passing by x_k has its second intersection with $\partial_R \mathrm{AsB}$ on the arc from x_1 to x_k (in the orientation of the construction), we stop and consider for $\mathcal{P}_R$ the convex hull of $x_1, \dots, x_k$; otherwise we take for x_{k+1} that second intersection of the tangent t_{k+1} with $\partial_R \mathrm{AsB}$ and start that step again.

This algorithm will necessarily finish, because by convexity the arclength of $x_i x_{i+1}$ on $\partial_R \mathrm{AsB}$ built this way is bigger than $2\varepsilon(R)$. At the end of this algorithm we obtain, by minimality, a polygon which has at least $N(R) = N\big(\varepsilon(R), \mathrm{AsB}_\Omega(o, R)\big) = N\big(\varepsilon(R)/\tanh R, \Omega\big)$ edges.

Recall that Proposition 12 guarantees us that the $\varepsilon(R)$-Euclidean neighbourhood of the asymptotic ball $\mathrm{AsB}_\Omega(o, R)$ is included in its $\big(\frac{1}{2}\ln 3\big)$-Hausdorff–Hilbert neighbourhood and therefore, taking into account the inclusions (19), we obtain

$$B_\Omega(o, R - \ln 2) \subset \mathrm{AsB}_\Omega(o, R) \subset \mathcal{P}_R \subset B_\Omega\big(o, R + \tfrac{3}{2}\ln 3\big).$$

Moreover, the length coincides with the Holmes–Thompson one-dimensional measure. Therefore, the monotonicity of the latter, as seen in Lemma 6, implies the following inequalities:

$$\begin{aligned} \mathrm{Length}_\Omega S_\Omega(o, R - \ln 2) &\le \mathrm{Length}_\Omega \partial \mathrm{AsB}_\Omega(o, R) \\ &\le \mathrm{Length}_\Omega \partial \mathcal{P}_R \\ &\le \mathrm{Length}_\Omega S_\Omega\big(o, R + \tfrac{3}{2}\ln 3\big). \end{aligned} \tag{44}$$

Now let $\mathfrak{P}_R$ be the image of $\mathcal{P}_R$ under the dilation of ratio $1/\tanh R$ centred at o. By construction $\mathfrak{P}_R$ contains Ω, which implies

$$\mathrm{Length}_{\mathfrak{P}_R} \partial \mathcal{P}_R \le \mathrm{Length}_\Omega \partial \mathcal{P}_R.$$

Therefore it suffices to prove the following claim:

Claim 28. *Let $I(R) \in \partial_R \mathrm{AsB}$ be a vertex of $\mathcal{P}_R$ such that the two edges containing $I(R)$ are tangent to $\mathrm{AsB}_\Omega(o, R)$ at $b(R)$ and $c(R)$. Then for any $R > \tanh^{-1}\big(\frac{1}{2}\big) = R_2$,*

$$d_\Omega\big(b(R), c(R)\big) \ge d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \ge \ln \tfrac{6}{5} = K_2.$$

Indeed, let us assume that Claim 28 is true, and for $R > r_2$ consider a vertex v of $\mathcal{P}_R$ whose incident edges are tangent to $\mathrm{AsB}(o, R)$. Let b and c be the two points of tangency. Then by the triangle inequality,

$$d_\Omega(b, v) + d_\Omega(c, v) \ge d_\Omega(b, c) \ge K_2.$$

Therefore the length of $\mathcal{P}_R$ is bigger than $(\widetilde{N}(R) - 2)K_2$, where $\widetilde{N}(R)$ is the number of edges of $\mathcal{P}_R$ (because of the possible exception at x_1 and the last point of the construction above). Hence taking $R_2 = r_2 + \frac{3}{2}\ln 3$, thanks to (44), we get for $R > R_2$

$$\text{Length}_\Omega\, S_\Omega(o, R) \geq \bigl(\widetilde{N}\bigl(R - \tfrac{3}{2}\ln 3\bigr) - 2\bigr)K_2, \tag{45}$$

and as $\widetilde{N}\bigl(R - \frac{3}{2}\ln 3\bigr) \geq N\bigl(R - \frac{3}{2}\ln 3\bigr)$ the first inequality in (43) is proved.

Now concerning the volume of the ball, Claim 28 and Proposition 16 imply that the contact points of the edges of $\mathcal{P}_R$ with $\text{AsB}_\Omega(o, R)$ form a K_2-separated set. Hence we can conclude in the same way as we did during the proof of Theorem 21; i.e., the balls of radius $\frac{1}{2}K_2$ centred at those points are disjoint and included in the metric ball $B_\Omega\bigl(o, R + \frac{3}{2}\ln 3 + \frac{1}{2}K_2\bigr)$. Now following [Vernicos 2013], there exists a constant C depending only on the dimension such that the volume of the ball of radius r is at least $C \cdot r^2$. Hence we obtain that

$$\text{Vol}_\Omega\, B_\Omega\bigl(o, R + \tfrac{3}{2}\ln 3 + \tfrac{1}{2}K_2\bigr) \geq \bigl(\widetilde{N}(R) - 2\bigr) \cdot C \cdot \bigl(\tfrac{1}{2}K_2\bigr)^2, \tag{46}$$

and the last inequality (43) follows once again from the inequality $\widetilde{N}(R) \geq N(R)$.

Proof of Claim 28. Let $a(R)$ (respectively $d(R)$) be the vertex opposite $I(R)$ on the edge containing $b(R)$ (respectively $c(R)$).

Now let us consider the images I, a, b, c and d of the five points $I(R)$, $a(R)$, $b(R)$, $c(R)$ and $d(R)$ by the dilation of ratio $1/\tanh R$ centred at o. Then we are in the same configuration as in Claim 10, with $\mathfrak{P}_R$ instead of Ω. Let

$$u(R) = \frac{bc}{BC}\frac{\tanh R}{1 - \tanh R};$$

then following (16) we have

$$d_{\mathfrak{P}_R}\bigl(b(R), c(R)\bigr) \geq \tfrac{1}{2}\ln\left(1 + \frac{u(R) + u(R)^2}{s(1-s)}\right).$$

Therefore we need to obtain a lower bound for $u(R)$. To do this, let p be the intersection of the line oI with the lines (bc). Then thanks to Thales' theorem we have

$$\frac{BC}{bc} = \frac{oI}{pI} = \frac{op + pI}{pI} = 1 + \frac{op}{pI}.$$

Concerning the distance op, recall that the unit ball centred at o is the Löwner ellipsoid of Ω and therefore we get $op \leq 1/\tanh R$, because by convexity p is in Ω. Regarding the distance pI, as $I(R)$ is on the boundary of the $\frac{1}{4}(1 - \tanh R)$-Euclidean neighbourhood of $\text{AsB}(o, R)$, we have that I is on the boundary of the $(1 - \tanh R)/(4\tanh R)$-neighbourhood of Ω. Hence we obtain

$$pI \geq \frac{1 - \tanh R}{4\tanh R},$$

because the segment $[p, I]$ intersects Ω. In this way we obtain

$$\frac{BC}{bc} \leq 1 + \frac{4}{1-\tanh R},$$

which in turn implies that

$$1 \leq \frac{5-\tanh R}{1-\tanh R}\,\frac{bc}{BC} \leq \frac{5}{1-\tanh R}\,\frac{bc}{BC}.$$

Hence

$$\tfrac{1}{5}\tanh R \leq u(R). \tag{47}$$

Therefore if $\tanh R_2 = \frac{1}{2}$ then for all $R > R_2$ we get $10u(R) > 1$.

Finally, using the fact that $s(1-s) \leq \frac{1}{4}$ and taking $R > R_2$ we get

$$d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \geq \tfrac{1}{2}\ln\big(1 + \tfrac{2}{5} + \tfrac{1}{25}\big) = \ln\tfrac{6}{5} > 0.18. \qquad \square$$

Proof of Theorem 3 (*intermediate volume growth theorem*). Following Theorem 4 of [Schneider and Wieacker 1981, page 154] and its proof, for any increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\liminf_{r\to+\infty} \frac{e^r}{f(r)} > 0$$

there exists a convex set Ω_f such that

$$0 < \liminf_{r\to+\infty} \frac{N(1-\tanh r, \Omega_f)}{f(r)} \leq \limsup_{r\to+\infty} \frac{N(1-\tanh r, \Omega_f)}{f(r)} < +\infty. \tag{48}$$

In the sequel we will write $N(r) = N(1-\tanh r, \Omega_f)$ and drop the subscript Ω_f in the notation of metric and asymptotic balls.

Now let o be the centre of the Löwner ellipsoid of Ω_f. Following Proposition 26 for $K_2 = \ln\frac{6}{5}$ and $r > 0$ satisfying

$$\tanh\big(r - \tfrac{3}{2}\ln 3 - \tfrac{1}{2}K_2\big) \geq \tfrac{1}{2},$$

we have that

$$\mathrm{Vol}_{\Omega_f} B(o, r) \geq \big(N\big(r - \tfrac{3}{2}\ln 3 - \tfrac{1}{2}K_2\big) - 2\big)C(K_2)^2. \tag{49}$$

This inequality implies that

$$\liminf_{r\to+\infty} \frac{\mathrm{Vol}_{\Omega_f} B(o, r)}{f(r)} \geq C(K_2)^2 \liminf_{r\to+\infty} \frac{N\big(r - \frac{3}{2}\ln 3 - \frac{1}{2}K_2\big) - 2}{f(r)}. \tag{50}$$

Now using inequalities (35) to (38) from the proof of Theorem 17 we get the existence of three constants a, b and c such that if $K = \ln 18$ and $r > 0$ is a real number satisfying $\tanh(r - C) > \frac{3}{4}$ then

$$\mathrm{Vol}_{\Omega_f} \mathrm{AsB}(o, r - C) \leq N\left(\frac{1-\tanh r}{4\tanh r}, \Omega_f\right)(ar^2 + br + c). \tag{51}$$

The inclusion $B(o, r-\ln 2-C) \subset \operatorname{AsB}(o, r-C)$ given by (19) in Corollary 13's proof allows us to obtain

$$\operatorname{Vol}_{\Omega_f} B(o, r-C-\ln 2) \leq N\left(\frac{1-\tanh r}{4\tanh r}, \Omega_f\right)(ar^2+br+c), \tag{52}$$

which in turn implies that

$$\limsup_{r\to+\infty} \frac{\operatorname{Vol}_{\Omega_f} B(o,r)}{r^2 f(r)} \leq a \times \limsup_{r\to+\infty} \frac{N\left(\frac{1-\tanh r}{4\tanh r}, \Omega_f\right)}{f(r)}. \tag{53}$$

Combining inequalities (49) and (51) and using the asymptotic comparison (48) we finally conclude that

$$\liminf_{r\to+\infty} \frac{\ln(\operatorname{Vol}_{\Omega_f} B(o,r))}{r} = \liminf_{r\to+\infty} \frac{\ln f(r)}{r}.$$

In the above proofs we can replace lim inf by lim sup.

To obtain the penultimate statement consider $f(r) = e^r/r^3$, and apply our result to get a convex set Ω_f whose entropy is 1. However, by the definition of the centroprojective area and our result in the two-dimensional case [Berck et al. 2010] we have

$$\begin{aligned}\mathcal{A}_o(\Omega_f) = \lim \frac{\operatorname{Vol}_{\Omega_f} B(o,r)}{\sinh r} &= \limsup \frac{\operatorname{Vol}_{\Omega_f} B(o,r)}{\sinh r}\\ &= \limsup \frac{\operatorname{Vol}_{\Omega_f} B(o,r)}{e^r r^{-1}} \times \frac{e^r}{r \sinh r} = 0.\end{aligned} \tag{54}$$

For the last statement take $f(r) = r^3$ and apply our result to get a convex set Ω_f such that

$$\limsup \frac{\operatorname{Vol}_{\Omega_f} B(o,r)}{r^2} = \limsup \frac{r \operatorname{Vol}_{\Omega_f} B(o,r)}{r^3} = +\infty;$$

hence, following our paper [Vernicos 2013], Ω_f is not a polytope. Furthermore the entropy of such a convex set is zero as we have $\limsup_\infty \ln(r^3)/r = 0$. □

To conclude this section let us show how Corollary 4, related to the values attained by the lower and upper volume entropies, easily follows: Suppose first that $0 < \alpha \leq \beta \leq 1$, and start by considering a sequence $(U_n)_{n\in\mathbb{N}}$ defined for some $x > 0$ by $U_0 = e^{bx}$, and for all $k \geq 0$ by

$$U_{2k+1} = e^{\alpha U_{2k}} \quad \text{and} \quad U_{2k+2} = e^{\beta U_{2k+1}}.$$

Then take an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $r \in \mathbb{R}$,

$$e^{\alpha r} \leq f(r) \leq e^{\beta r},$$

and $f(U_n) = U_{n+1}$ for all $n \geq 0$. We can define such a function piecewise linearly.

If $\alpha = 0$, replace $r \mapsto e^{\alpha r}$ by $r \mapsto 2r$ above and take $U_{2k+1} = 2U_{2k}$ for all $k \geq 0$.

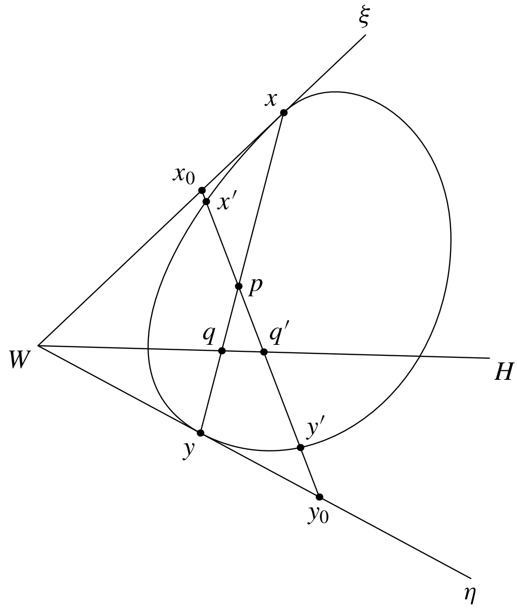


Figure 6. Metric projection of p on H.

Appendix: Metric projection in a Hilbert geometry

The following is a reformulation and a detailed proof of a statement found in [Busemann and Kelly 1953, Sections 21 and 28] in any dimension.

Proposition 29. *Let* (Ω, d_Ω) *be a Hilbert geometry in* $\mathbb{R}^n$*. Let* p *be a point of* Ω *and* H *a hyperplane intersecting* Ω*. Then* $q \in H \cap \Omega$ *is a metric projection of* p *onto* H*, i.e.,*

$$d_\Omega(p, H) = d_\Omega(p, q),$$

if and only if $\partial\Omega$ *has, at its intersection with the straight line* (pq)*, supporting hyperplanes concurrent with* H *(the intersection of these three hyperplanes is an* $(n-2)$*-dimensional affine space).*

Proof. Let us suppose first that such concurrent support hyperplanes exist. Let x and y be the intersections of the line (pq) with $\partial\Omega$. Assume that ξ and η are supporting hyperplanes of $\partial\Omega$ respectively at x and y whose intersection with H is the $(n-2)$-dimensional affine space W. (See Figure 6.) Let us show that for any $p' \in (pq)$ and any $q' \in H$ we have

$$d_\Omega(p', q') \geq d_\Omega(p', q). \tag{55}$$

Let us suppose that x is on the half-line $[qp')$ and y on the half-line $[p'q)$ and denote by x' and y' the intersections of $\partial\Omega$ with the half-lines $[q'p')$ and $[p'q')$

respectively. Then let x_0 be the intersection of ξ with the line $(p'q')$ and y_0 the intersection of $(p'q')$ with η. By Thales' theorem, the cross ratio of $[x_0, p', q', y_0]$ is equal to the cross ratio of $[x, p', q, y]$ and standard computation shows that

$$[x_0, p', q', y_0] \leq [x', p', q', y'],$$

with equality if and only if $x_0 = x'$ and $y' = y_0$. Hence the inequality (55) holds, and if the convex set is strictly convex, this inequality is always strict, for $q' \neq q$.

Reciprocally: recall that when a point q' of Ω goes to the boundary, its distance to p goes to infinity. Hence by continuity of the distance and compactness there exists a point q on $H \cap \Omega$ such that $d_\Omega(p, H) = d_\Omega(p, q)$. Now consider the Hilbert ball $B_\Omega(p, r)$ of radius $r = d_\Omega(p, H)$ centred at p. Let once more x, y, ξ and η be defined as before, and let H' be the hyperplane passing by q and $\xi \cap \eta = W$. Then this hyperplane has to be tangent to the ball $B_\Omega(p, r)$; otherwise one can find a point q' on H' inside the open ball (i.e., $d(p, q') < r$). However, by the reasoning done in our first step we would conclude that this point is at a distance greater than or equal to r, which would be a contradiction. By minimality of the point q, H is also a supporting hyperplane of $B_\Omega(p, r)$ at q. Hence we have to distinguish between two cases. If Ω is C^1, then by the uniqueness of the tangent hyperplanes at every point, $H = H'$. Otherwise, Ω is not C^1 at x or y. In that case it is possible to replace one of the hyperplanes, say ξ, with ξ' passing by x and $H \cap \eta$ (which might be at infinity, which would mean that we consider parallel hyperplanes). □

Notice that there is no uniqueness of the metric projections (also called "foot" by Busemann). However, if the convex set is strictly convex, then we will have a unique projection, and if furthermore the convex set is C^1, this projection will be given by a unique pair of supporting hyperplanes.

A.1. ***Approximability of convex bodies seen as a dimension.*** In this section we relate our definition of approximability with the definition given in [Schneider and Wieacker 1981].

Recall that for a convex body Ω and $\varepsilon > 0$, $N(\varepsilon, \Omega)$ denotes the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than ε.

Theorem 30 [Schneider and Wieacker 1981]. *Let* $\underline{a}_s := \liminf_{\varepsilon \to 0^+} N(\varepsilon, \Omega)\varepsilon^s$. *Then* $s \to \underline{a}_s$ *admits a critical value* $\underline{a}(\Omega)$, *called the* approximability number *of* Ω, *such that if* $s > \underline{a}(\Omega)$ *then* $\underline{a}_s(\Omega) = 0$, *and if* $s < \underline{a}(\Omega)$ *then* $\underline{a}_s(\Omega) = \infty$.

In the same way, we can introduce the *upper approximability number* of Ω, denoted by $\overline{a}(\Omega)$, as the critical value of $s \mapsto \overline{a}_s(\Omega)$, where

$$\overline{a}_s(\Omega) := \limsup_{\varepsilon \to 0^+} N(\varepsilon, \Omega)\varepsilon^s.$$

The reader familiar with the definition of the ball-box dimension (also known as the Minkowski dimension) will have no difficulty seeing that this definition coincides with the one given in the Introduction.

Now the main result in [Bronshteyn and Ivanov 1975] asserts that for any convex set Ω inscribed in the unit Euclidean ball, there are no more than $c(n)\varepsilon^{(1-n)/2}$ points whose convex hull is no more than ε away from Ω in the Hausdorff topology, which gives the next result.

Theorem 31 [Bronshteyn and Ivanov 1975]. *Let Ω be a convex body in $\mathbb{R}^n$. Then*

$$\bar{a}(\Omega) \leq \tfrac{1}{2}(n-1).$$

References

[Alexandroff 1939] A. D. Alexandroff, "Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it", *Uchenye Zapiski Leningrad. Gos. Univ. Ser. Mat. Nauk.* **6** (1939), 3–35. In Russian. Reprinted in his Избранные труды, И: Геометрия и приложения, edited by O. A. Ladizhenskaya et al., Nauka, Novosibirsk, 2006, pp. 171–207. MR Zbl

[Álvarez Paiva and Fernandes 1998] J. C. Álvarez Paiva and E. Fernandes, "Crofton formulas in projective Finsler spaces", *Electron. Res. Announc. Amer. Math. Soc.* **4** (1998), 91–100. MR Zbl

[Benoist 2003] Y. Benoist, "Convexes hyperboliques et fonctions quasisymétriques", *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), 181–237. MR Zbl

[Benoist 2008] Y. Benoist, "A survey on divisible convex sets", pp. 1–18 in *Geometry, analysis and topology of discrete groups*, edited by L. Ji et al., Adv. Lect. Math. **6**, International Press, Somerville, MA, 2008. MR Zbl

[Benzécri 1960] J.-P. Benzécri, "Sur les variétés localement affines et localement projectives", *Bull. Soc. Math. France* **88** (1960), 229–332. MR Zbl

[Berck 2009] G. Berck, "Minimality of totally geodesic submanifolds in Finsler geometry", *Math. Ann.* **343**:4 (2009), 955–973. MR Zbl

[Berck et al. 2010] G. Berck, A. Bernig, and C. Vernicos, "Volume entropy of Hilbert geometries", *Pacific J. Math.* **245**:2 (2010), 201–225. MR Zbl

[Bronshteyn and Ivanov 1975] E. M. Bronshteyn and L. D. Ivanov, "The approximation of convex sets by polyhedra", *Sibirsk. Mat. Zh.* **16**:5 (1975), 1110–1112. In Russian; translated in *Siberian Math. J.* **16**:5 (1975), 852-853. MR Zbl

[Burago et al. 2001] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics **33**, American Mathematical Society, Providence, RI, 2001. MR Zbl

[Busemann 1955] H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955. MR Zbl

[Busemann and Kelly 1953] H. Busemann and P. J. Kelly, *Projective geometry and projective metrics*, Academic Press, New York, 1953. MR Zbl

[Colbois and Vernicos 2007] B. Colbois and C. Vernicos, "Les géométries de Hilbert sont à géométrie locale bornée", *Ann. Inst. Fourier* (*Grenoble*) **57**:4 (2007), 1359–1375. MR Zbl

[Colbois and Verovic 2004] B. Colbois and P. Verovic, "Hilbert geometry for strictly convex domains", *Geom. Dedicata* **105** (2004), 29–42. MR Zbl

[Cooper et al. 2015] D. Cooper, D. D. Long, and S. Tillmann, "On convex projective manifolds and cusps", *Adv. Math.* **277** (2015), 181–251. MR Zbl

[Crampon 2009] M. Crampon, "Entropies of strictly convex projective manifolds", *J. Mod. Dyn.* **3**:4 (2009), 511–547. MR Zbl

[Crampon and Marquis 2014] M. Crampon and L. Marquis, "Le flot géodésique des quotients géométriquement finis des géométries de Hilbert", *Pacific J. Math.* **268**:2 (2014), 313–369. MR Zbl

[Fejes Tóth 1948] L. Fejes Tóth, "Approximation by polygons and polyhedra", *Bull. Amer. Math. Soc.* **54** (1948), 431–438. MR Zbl

[Goldman 1990] W. M. Goldman, "Convex real projective structures on compact surfaces", *J. Differential Geom.* **31**:3 (1990), 791–845. MR Zbl

[Gruber 2007] P. M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften **336**, Springer, 2007. MR Zbl

[de la Harpe 1993] P. de la Harpe, "On Hilbert's metric for simplices", pp. 97–119 in *Geometric group theory* (Sussex, 1991), vol. 1, edited by G. A. Niblo and M. A. Roller, London Math. Soc. Lecture Note Ser. **181**, Cambridge Univ. Press, 1993. MR Zbl

[Hilbert 1971] D. Hilbert, *Les fondements de la géométrie*, Dunod, Paris, 1971. Zbl

[Karlsson and Noskov 2002] A. Karlsson and G. A. Noskov, "The Hilbert metric and Gromov hyperbolicity", *Enseign. Math.* (2) **48**:1–2 (2002), 73–89. MR Zbl

[Kay 1967] D. C. Kay, "The ptolemaic inequality in Hilbert geometries", *Pacific J. Math.* **21** (1967), 293–301. MR Zbl

[Kelly and Straus 1958] P. Kelly and E. Straus, "Curvature in Hilbert geometries", *Pacific J. Math.* **8** (1958), 119–125. MR Zbl

[Kim 2005] I. Kim, "Compactification of strictly convex real projective structures", *Geom. Dedicata* **113** (2005), 185–195. MR Zbl

[Lins and Nussbaum 2008] B. Lins and R. Nussbaum, "Denjoy–Wolff theorems, Hilbert metric nonexpansive maps and reproduction-decimation operators", *J. Funct. Anal.* **254**:9 (2008), 2365–2386. MR Zbl

[Marquis 2012] L. Marquis, "Surface projective convexe de volume fini", *Ann. Inst. Fourier* (*Grenoble*) **62**:1 (2012), 325–392. MR Zbl

[McMullen 1971] P. McMullen, "On the upper-bound conjecture for convex polytopes", *J. Combinatorial Theory Ser. B* **10** (1971), 187–200. MR Zbl

[McMullen and Shephard 1971] P. McMullen and G. C. Shephard, *Convex polytopes and the upper bound conjecture*, London Mathematical Society Lecture Note Series **3**, Cambridge University Press, 1971. MR Zbl

[Nasu 1961] Y. Nasu, "On Hilbert geometry", *Math. J. Okayama Univ.* **10**:2 (1961), 101–112. Zbl

[Papadopoulos and Troyanov 2014] A. Papadopoulos and M. Troyanov (editors), *Handbook of Hilbert geometry*, IRMA Lectures in Mathematics and Theoretical Physics **22**, European Mathematical Society, Zürich, 2014. MR Zbl

[Schneider and Wieacker 1981] R. Schneider and J. A. Wieacker, "Approximation of convex bodies by polytopes", *Bull. London Math. Soc.* **13**:2 (1981), 149–156. MR Zbl

[Thompson 2012] A. Thompson, "Essay 17: Lengths of curves", in a book in progress, 2012.

[Vernicos 2009] C. Vernicos, "Spectral radius and amenability in Hilbert geometries", *Houston J. Math.* **35**:4 (2009), 1143–1169. MR Zbl

[Vernicos 2013] C. Vernicos, "Asymptotic volume in Hilbert geometries", *Indiana Univ. Math. J.* **62**:5 (2013), 1431–1441. MR Zbl

[Vernicos 2015] C. Vernicos, "On the Hilbert geometry of products", *Proc. Amer. Math. Soc.* **143**:7 (2015), 3111–3121. MR Zbl

Received June 10, 2015. Revised July 11, 2016.

CONSTANTIN VERNICOS
INSTITUT MONTPELLIÉRAIN ALEXANDER GROTHENDIECK
UNIVERSITÉ DE MONTPELLIER
CASE COURRIER 051
PLACE EUGÈNE BATAILLON
F-34095 MONTPELLIER CEDEX
FRANCE

constantin.vernicos@umontpellier.fr

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 287 No. 1 March 2017