THE RICCI–BOURGUIGNON FLOW

GIOVANNI CATINO, LAURA CREMASCHI, ZINDINE DJADLI,
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We present some results on a family of geometric flows introduced by J. P. Bourguignon in 1981 that generalize the Ricci flow. For suitable values of the scalar parameter involved in these flows, we prove short time existence and provide curvature estimates. We also state some results on the associated solitons.

1. Introduction

In this paper we consider an n-dimensional, compact, smooth, Riemannian manifold M (without boundary) whose metric $g = g(t)$ is evolving according to the flow equation

$$(1\text{-}1) \qquad \frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\rho \mathrm{R} g = -2(\mathrm{Ric} - \rho \mathrm{R} g)$$

where Ric is the Ricci tensor of the manifold, R its scalar curvature and ρ is a real constant. This family of geometric flows contains, as a special case, the Ricci flow, setting $\rho = 0$. Moreover, by a suitable rescaling in time, when ρ is nonpositive, they can be seen as an interpolation between the Ricci flow and the Yamabe flow (see [Brendle 2005; Schwetlick and Struwe 2003; Ye 1994], for instance), obtained as a limit when $\rho \to -\infty$.

It should be noticed that for special values of the constant ρ the tensor $\mathrm{Ric} - \rho \mathrm{R} g$ appearing on the right-hand side of the evolution equation is of special interest. In particular,

- $\rho = 1/2$, the Einstein tensor $\mathrm{Ric} - \frac{\mathrm{R}}{2} g$,
- $\rho = 1/n$, the traceless Ricci tensor $\mathrm{Ric} - \frac{\mathrm{R}}{n} g$,
- $\rho = 1/2(n-1)$, the Schouten tensor $\mathrm{Ric} - \frac{\mathrm{R}}{2(n-1)} g$,
- $\rho = 0$, the Ricci tensor Ric.

In dimension two, the first three tensors are zero, hence the flow is static, and in higher dimension the values of ρ are strictly ordered as above, in descending order.

Apart from these special values of ρ, for which we will call the associated flows by the name of the corresponding tensor, in general we will refer to the evolution equation defined by the PDE system (1-1) as the Ricci–Bourguignon flow (or shortly RB flow).

The study of these flows was proposed by Jean-Pierre Bourguignon [1981, Question 3.24], building on some unpublished work of Lichnerowicz in the sixties and a paper of Aubin [1970]. In 2003, Fischer [2004] studied a conformal version of this problem where the scalar curvature is constrained along the flow. In 2011, Lu, Qing and Zheng [Lu et al. 2014] also proved some results on the conformal Ricci–Bourguignon flow. Some results concerning solitons of the Ricci–Bourguignon flow (called *gradient ρ-Einstein solitons*) can be found in [Catino and Mazzieri 2016; Catino et al. 2015b].

We will see in the next section that when ρ is larger than $1/2(n-1)$ the principal symbol of the operator in the right hand side of the second order quasilinear parabolic PDE (1-1) has negative eigenvalues, not allowing even a short time existence result for the flow for general initial data (manifold M and initial metric g_0). On the contrary, the main task of Section 2 will be to show that for any $\rho < 1/2(n-1)$, every initial compact Riemannian manifold (M, g_0) has a unique smooth solution $g(t)$ solving the flow equation (1-1), with $g(0) = g_0$, at least in a positive time interval.

However, the problem of knowing whether the "critical" *Schouten flow*

$$(1\text{-}2) \qquad \begin{cases} \dfrac{\partial}{\partial t} g = -2\mathrm{Ric} + \dfrac{\mathrm{R}}{n-1} g, \\ g(0) = g_0, \end{cases}$$

when $\rho = 1/2(n-1)$, admits or not a short time solution for general initial manifolds and metrics remains open, when $n \geq 3$.

We will see that if $\rho \leq 1/2(n-1)$, the principal symbol of the elliptic operator is nonnegative definite and it actually contains some zero eigenvalues due to the diffeomorphism invariance of the geometric flow. When $\rho < 1/2(n-1)$, these zero eigenvalues are the only ones, while all the others are actually positive, hence, they can be dealt with (as is customary by now) by means of the so-called DeTurck's trick [1983; 2003]. In the case of the Schouten flow $\rho = 1/2(n-1)$ instead, the principal symbol contains an extra zero eigenvalue besides the ones due to the diffeomorphism invariance, preventing this argument from being sufficient to conclude and to give a general short time existence result.

We mention that the presence of this extra zero eigenvalue should be expected, as the Cotton tensor, which is obtained from the Schouten tensor A by

$$\mathrm{C}_{ijk} = \nabla_k \mathrm{A}_{ij} - \nabla_j \mathrm{A}_{ik} = \nabla_k \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{ik} - \frac{1}{2(n-1)}(\nabla_k \mathrm{R} g_{ij} - \nabla_j \mathrm{R} g_{ik}),$$

satisfies the following invariance under the conformal change of metric $\widetilde{g} = e^{2u}g$,

$$e^{3u}\widetilde{C}_{ijk} = C_{ijk} + (n-2)W_{ijkl}\nabla^l u;$$

see [Catino et al. 2016, equation 3.35]. Recently, Delay [2014], following the work of Fischer and Marsden, gave some evidence on the fact that DeTurck's trick should fail for the Schouten tensor.

In Section 3, we will compute the evolution equations for the curvature.

In Section 4, by means of the maximum principle, we derive, from the evolution of the curvature, some conditions on the curvature which are preserved by the RB flow. In particular, we show that the Hamilton–Ivey estimate in dimension three holds.

In Section 5, we establish some *a priori* estimates on the Riemann tensor and prove that, if a compact solution of the flow exists up to a finite maximal time T, then the Riemann tensor is unbounded when approaching T.

Finally, in the last section we discuss the structure and the classification of the solitons of the RB flow.

1A. *Notation and preliminaries.*

1A. *Notation and preliminaries.* The Riemann curvature operator of a Riemannian manifold (M, g) of dimension n is defined as in [Gallot et al. 1990] by

$$\mathrm{Riem}(X, Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z,$$

and we will denote by $d\mu_g$ the canonical volume measure associated to the metric g.

In a local coordinate system, the components of the $(3, 1)$-Riemann curvature tensor are given by $R^l_{ijk}(\partial/\partial x^l) = \mathrm{Riem}(\partial/\partial x^i, \partial/\partial x^j)\partial/\partial x^k$, and we denote by $R_{ijkl} = g_{lm}R^m_{ijk}$ its $(4, 0)$-version.

With this choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v, w, v, w) = R_{ijkl}v^i w^j v^k w^l > 0$.

The Ricci tensor is obtained as the contraction $R_{ik} = g^{jl}R_{ijkl}$, and $R = g^{ik}R_{ik}$ will denote the scalar curvature.

The so-called Weyl tensor is then defined by the decomposition formula (see [Gallot et al. 1990, Chapter 3, Section K]) of the Riemann tensor in dimension $n \geq 3$,

$$(1\text{-}3)\quad W_{ijkl} = R_{ijkl} + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})$$
$$-\frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}).$$

The tensor W satisfies all the symmetries of the curvature tensor and all its traces with the metric are zero, as can be easily seen from the above formula.

In dimension three, W is identically zero for every Riemannian manifold (M, g), and it becomes relevant when $n \geq 4$ since it vanishes if and only if (M, g) is locally conformally flat. This latter condition means that around every point $p \in M$ there is a conformal deformation $\widetilde{g}_{ij} = e^f g_{ij}$ of the original metric g, such that the

new metric is flat, namely, the Riemann tensor associated to $\widetilde{g}$ is zero in U_p (here $f : U_p \to \mathbb{R}$ is a smooth function defined in a open neighborhood U_p of p).

2. Short time existence

Theorem 2.1. *Let $\rho < 1/2(n-1)$. Then, the evolution equation* (1-1) *has a unique solution for a positive time interval on any smooth, n-dimensional, compact Riemannian manifold M (without boundary) for any initial metric g_0.*

Proof. We first compute the linearized operator DL_{g_0} of the operator $L = -2(\mathrm{Ric} - \rho \mathrm{R}g)$ at a metric g_0. The Ricci tensor and the scalar curvature have the following linearizations (see [Besse 1987, Theorem 1.174] or [Topping 2006]), where we use the metric g_0 to lower and raise indices and to take traces:

$$D\mathrm{Ric}_{g_0}(h)_{ik} = \tfrac{1}{2}(-\Delta h_{ik} - \nabla_i \nabla_k \operatorname{tr}(h) + \nabla_i \nabla^t h_{tk} + \nabla_k \nabla^t h_{it}) + \mathrm{LOT},$$

$$D\mathrm{R}_{g_0}(h) = -\Delta(\operatorname{tr} h) + \nabla^s \nabla^t h_{st} + \mathrm{LOT}.$$

Here LOT stands for *lower order terms*.

Then, the linearization of L at g_0 is given by

$$
\begin{aligned}
DL_{g_0}(h)_{ik} &= -2(D\mathrm{Ric}_{g_0}(h)_{ik} - \rho D\mathrm{R}_{g_0}(h)(g_0)_{ik}) + 2\rho \mathrm{R}_{g_0} h_{ik} \\
&= \Delta h_{ik} + \nabla_i \nabla_k \operatorname{tr}(h) - \nabla_i \nabla^t h_{tk} - \nabla_k \nabla^t h_{it} \\
&\qquad\qquad - 2\rho(\Delta(\operatorname{tr} h) - \nabla^s \nabla^t h_{st})(g_0)_{ik} + \mathrm{LOT},
\end{aligned}
$$

for every bilinear form $h \in \Gamma(S^2 M)$. Now, we obtain the principal symbol of the linearized operator in the direction of an arbitrary cotangent vector ξ by replacing each covariant derivative ∇_α appearing in the higher order terms with the corresponding component ξ_α:

$$
\begin{aligned}
\sigma_\xi(DL_{g_0})(h)_{ik} &= \xi^t \xi_t h_{ik} + \xi_i \xi_k \operatorname{tr}_{g_0}(h) - \xi_i \xi^t h_{kt} - \xi_k \xi^t h_{it} \\
&\qquad\qquad - 2\rho \xi^t \xi_t \operatorname{tr}_{g_0}(h)(g_0)_{ik} + 2\rho \xi^t \xi^s h_{ts}(g_0)_{ik}.
\end{aligned}
$$

As usual, since the symbol is homogeneous we can assume that $|\xi|_{g_0} = 1$ and we perform all the computations in an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of $T_p M$ such that $\xi = g_0(e_1, \cdot)$, that is, $\xi_i = 0$ for $i \neq 1$.

Hence we obtain

$$\sigma_\xi(DL_{g_0})(h)_{ik} = h_{ik} + \delta_{i1}\delta_{k1} \operatorname{tr}_{g_0}(h) - \delta_{i1} h_{k1} - \delta_{k1} h_{i1} - 2\rho \operatorname{tr}_{g_0}(h)\delta_{ik} + 2\rho h_{11}\delta_{ik},$$

which can be represented in the coordinate system

$$(h_{11}, h_{22}, \dots, h_{nn}, h_{12}, \dots, h_{1n}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

for any $h \in \Gamma(S^2 M)$, by the following matrix

$$\sigma_\xi(DL_{g_0}) = \left(\begin{array}{ccc|c|c}
0 & 1-2\rho & \cdots & 1-2\rho & \\
\vdots & & A[n-1] & 0 & 0 \\
0 & & & & \\
\hline
& 0 & & 0 & 0 \\
\hline
& 0 & & 0 & \text{Id}_{(n-1)(n-2)/2}
\end{array} \right),$$

where $A[n-1]$ is the $(n-1) \times (n-1)$ matrix given by

$$A[n-1] = \begin{pmatrix} 1-2\rho & -2\rho & \cdots & -2\rho \\ -2\rho & 1-2\rho & \cdots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \cdots & 1-2\rho \end{pmatrix}.$$

We can see that there are at least n null eigenvalues, as would be expected by the diffeomorphism invariance of the operator L, and $(n-1)(n-2)/2$ eigenvalues equal to 1. The remaining $n-1$ eigenvalues can be computed by the following lemma which is easily proved by induction on the dimension of A.

Lemma 2.2. *Let $A[m]$ be the $m \times m$ matrix*

$$(2\text{-}1) \qquad A[m] = \begin{pmatrix} 1-2\rho & -2\rho & \cdots & -2\rho \\ -2\rho & 1-2\rho & \cdots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \cdots & 1-2\rho \end{pmatrix}.$$

Then we have

$$\det(A[m] - \lambda \text{Id}_m) = (1-\lambda)^{(m-1)}(1 - 2m\rho - \lambda).$$

Using this lemma, we conclude that the eigenvalues of the principal symbol of DL_{g_0} are 0 with multiplicity n, 1 with multiplicity $\frac{1}{2}(n+1)(n-2)$ and $1-2(n-1)\rho$ with multiplicity 1.

Now we apply the so-called *DeTurck's trick* [1983; 2003] to show that the RB flow is equivalent to a Cauchy problem for a strictly parabolic operator, modulo the action of the diffeomorphism group of M. Let $V : \Gamma(S^2 M) \to \Gamma(TM)$ be

"DeTurck's" vector field defined by

$$
\begin{aligned}
(2\text{-}2) \qquad V^j(g) &= -g_0^{jk} g^{pq} \nabla_p \big(\tfrac{1}{2} \operatorname{tr}_g(g_0) g_{qk} - (g_0)_{qk} \big) \\
&= -\tfrac{1}{2} g_0^{jk} g^{pq} \big(\nabla_k (g_0)_{pq} - \nabla_p (g_0)_{qk} - \nabla_q (g_0)_{pk} \big),
\end{aligned}
$$

where g_0 is a fixed Riemannian metric on M and g_0^{jk} are the components of the inverse matrix of g_0.

DeTurck's trick (see [DeTurck 1983, 2003] for details) states that in order to show the smooth existence part of the theorem, we only need to check that the operator $D(L - \mathscr{L}_V)_{g_0}$ is strongly elliptic, where $\mathscr{L}_V$ is the Lie derivative operator in the direction of V.

The principal symbol of this latter operator, with the same notation used above, is well known and is given by

$$
\sigma_\xi (D\mathscr{L}_V)_{g_0}(h)_{ik} = \delta_{i1}\delta_{k1} \operatorname{tr}_{g_0}(h) - \delta_{i1} h_{k1} - \delta_{k1} h_{i1}.
$$

Then we can easily see that the linearized DeTurck–Ricci–Bourguignon operator has principal symbol in the direction ξ, with respect to an orthonormal basis $\{\xi^\flat, e_2, \ldots, e_n\}$, given by

$$
\sigma_\xi((D(L - \mathscr{L}_V)_{g_0}) =
\left(
\begin{array}{c|c|c}
\begin{matrix} 1 & -2\rho & \cdots & -2\rho \\ \vdots & & A[n-1] & \\ 0 & & & \end{matrix} & 0 & 0 \\
\hline
0 & \mathrm{Id}_{(n-1)} & 0 \\
\hline
0 & 0 & \mathrm{Id}_{(n-1)(n-2)/2}
\end{array}
\right),
$$

expressed in the coordinate system

$$
(h_{11}, h_{22}, \ldots, h_{nn}, h_{12}, h_{13}, \ldots, h_{1n}, h_{23}, h_{24}, \ldots, h_{n-1,n})
$$

for any $h \in \Gamma(S^2 M)$.

Using Lemma 2.2 again, this matrix has $\tfrac{1}{2} n(n+1) - 1$ eigenvalues equal to 1 and 1 eigenvalue equal to $1 - 2(n-1)\rho$. Therefore, by DeTurck's trick, a sufficient condition for the existence of a solution is that $\rho < 1/(2(n-1))$.

The uniqueness part of the theorem is proven in the same way as for the Ricci flow (see [Hamilton 1995]). The RB flow is equivalent, via the one parameter group of diffeomorphisms generated by DeTurck's vector field, to the DeTurck–RB flow which is strictly parabolic. On the other hand, the one parameter group of diffeomorphisms satisfies the harmonic map flow introduced by Eells and Sampson

[1964], which is also parabolic. These two facts imply the uniqueness of the solution for the RB flow (see [Chow and Knopf 2004, Chapter 3, Section 4] for more details). □

3. Evolution of the curvature

3A. *The evolution of curvature.* As the metric tensor evolves by

$$\frac{\partial}{\partial t}g_{ij} = -2(\mathrm{R}_{ij} - \rho \mathrm{R}g_{ij}),$$

it is easy to see, differentiating the identity $g_{ij}g^{jl} = \delta_i^l$, that

(3-1) $$\frac{\partial}{\partial t}g^{jl} = 2(\mathrm{Ric}^{jl} - \rho \mathrm{R}g^{jl}).$$

It follows that the canonical volume measure μ satisfies

$$\frac{d\mu}{dt} = \frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\mathscr{L}^n = \frac{\sqrt{\det g_{ij}}\,g^{ij}\frac{\partial}{\partial t}g_{ij}}{2}\mathscr{L}^n = (n\rho - 1)\mathrm{R}\sqrt{\det g_{ij}}\,\mathscr{L}^n = (n\rho - 1)\mathrm{R}\mu.$$

Computing in a normal coordinate system, the evolution equation for the Christoffel symbols is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_{jk}^i &= \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}g_{kl}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t}g_{jl}\right) - \frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\
&\quad + \frac{1}{2}\frac{\partial}{\partial t}g^{il}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right\} \\
&= \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\
&= -g^{il}\{\nabla_j(\mathrm{R}_{kl} - \rho \mathrm{R}g_{kl}) + \nabla_k(\mathrm{R}_{jl} - \rho \mathrm{R}g_{jl}) - \nabla_l(\mathrm{R}_{jk} - \rho \mathrm{R}g_{jk})\} \\
&= -\nabla_j\mathrm{R}_k^i - \nabla_k\mathrm{R}_j^i - \nabla^i\mathrm{R}_{jk} + \rho(\nabla_j\mathrm{R}\delta_k^i + \nabla_k\mathrm{R}\delta_j^i + \nabla^i\mathrm{R}g_{jk}).
\end{aligned}$$

Proposition 3.1. *Along the RB flow on a n-dimensional Riemannian manifold* (M, g), *the curvature tensor, the Ricci tensor and the scalar curvature satisfy the following evolution equations*:

(3-2) $$\frac{\partial}{\partial t}\mathrm{R}_{ijkl} = \Delta\mathrm{R}_{ijkl} + 2(\mathrm{B}_{ijkl} - \mathrm{B}_{ijlk} - \mathrm{B}_{iljk} + \mathrm{B}_{ikjl})$$

$$-g^{pq}(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql})$$

$$-\rho(\nabla_i\nabla_k\mathrm{R}g_{jl} - \nabla_i\nabla_l\mathrm{R}g_{jk} - \nabla_j\nabla_k\mathrm{R}g_{il} + \nabla_j\nabla_l\mathrm{R}g_{ik}) + 2\rho\mathrm{R}\mathrm{R}_{ijkl},$$

where the tensor B *is defined as* $B_{ijkl} = g^{pq}g^{rs}R_{ipjr}R_{kqls}$,

$$(3\text{-}3) \qquad \frac{\partial}{\partial t}R_{ik} = \Delta R_{ik} + 2g^{pq}g^{rs}R_{pirk}R_{qs} - 2g^{pq}R_{pi}R_{qk}$$
$$- (n-2)\rho\nabla_i\nabla_k R - \rho\Delta R g_{ik},$$

$$(3\text{-}4) \qquad \frac{\partial}{\partial t}R = (1 - 2(n-1)\rho)\Delta R + 2|\text{Ric}|^2 - 2\rho R^2.$$

Proof. The following computation is analogous to the one for the Ricci flow performed by Hamilton [1982].

By the first variation formula for the $(4,0)$-Riemann tensor (see [Besse 1987, Theorem 1.174] or [Topping 2006]), we have in general

$$\frac{\partial}{\partial t}\text{Riem}(X,Y,W,Z) = \tfrac{1}{2}(h(\text{Riem}(X,Y)W,Z) - h(\text{Riem}(X,Y)Z,W))$$
$$- \tfrac{1}{2}(-\nabla^2_{Y,W}h(X,Z) - \nabla^2_{X,Z}h(Y,W) + \nabla^2_{X,W}h(Y,Z) + \nabla^2_{Y,Z}h(X,W)),$$

where $X, Y, W, Z \in \Gamma(TM)$ are vector fields and $h = (\partial/\partial t)g$. Along the RB flow $h = -2(\text{Ric} - \rho R g)$, and therefore

$$\frac{\partial}{\partial t}\text{Riem}(X, Y, W, Z)$$
$$= -\text{Ric}(\text{Riem}(X, Y)W, Z) + \text{Ric}(\text{Riem}(X, Y)Z, W)$$
$$- \nabla^2_{Y,W}\text{Ric}(X, Z) - \nabla^2_{X,Z}\text{Ric}(Y, W) + \nabla^2_{X,W}\text{Ric}(Y, Z) + \nabla^2_{Y,Z}\text{Ric}(X, W)$$
$$- \rho(-\nabla^2_{Y,W}R g(X, Z) - \nabla^2_{X,Z}R g(Y, W) + \nabla^2_{X,W}R g(Y, Z) + \nabla^2_{Y,Z}R g(X, W))$$
$$+ 2\rho R\text{Riem}(X, Y, W, Z).$$

Using the second Bianchi identity and the commutation formula for second covariant derivatives, we get the following equation for the Laplacian of the Riemann tensor:

$$\Delta\text{Riem}(X, Y, W, Z)$$
$$= -\nabla^2_{Y,W}\text{Ric}(X, Z) - \nabla^2_{X,Z}\text{Ric}(Y, W) + \nabla^2_{X,W}\text{Ric}(Y, Z) + \nabla^2_{Y,Z}\text{Ric}(X, W)$$
$$- \text{Ric}(\text{Riem}(W, Z)Y, X) + \text{Ric}(\text{Riem}(W, Z)X, Y) - 2(B(X, Y, W, Z)$$
$$- B(X, Y, Z, W) + B(X, W, Y, Z) - B(X, Z, Y, W)).$$

Plugging it into the evolution equation, we obtain

$$\frac{\partial}{\partial t}\text{Riem}(X, Y, W, Z)$$
$$= \Delta\text{Riem}(X, Y, W, Z) - \rho(\nabla^2 R \owedge g)(X, Y, W, Z)$$
$$+ 2(B(X, Y, W, Z) - B(X, Y, Z, W) + B(X, W, Y, Z) - B(X, Z, Y, W))$$
$$- \text{Ric}(\text{Riem}(X, Y)W, Z) + \text{Ric}(\text{Riem}(X, Y)Z, W) - \text{Ric}(\text{Riem}(W, Z)X, Y)$$
$$+ \text{Ric}(\text{Riem}(W, Z)Y, X) + 2\rho R\text{Riem}(X, Y, W, Z),$$

which is formula (3-2) once written in coordinates. Here the symbol $\oslash$ denotes the Kulkarni–Nomizu product of two symmetric bilinear forms p and q, defined by

$$(p \oslash q)(X, Y, Z, T)$$
$$= p(X, Z)q(Y, T) + p(Y, T)q(X, Z) - p(X, T)q(Y, Z) - p(Y, Z)q(X, T),$$

for every tangent vector fields $X, Y, Z, T \in \Gamma(TM)$.

Taking into account the evolution equation for the inverse of the metric (3-1), contracting equation (3-2) and using again the second Bianchi identity, formula (3-3) follows (see [Hamilton 1982] for details). Contracting again one gets the evolution equation (3-4) for the scalar curvature. $\qquad\square$

3B. *Uhlenbeck's trick and the evolution of the curvature operator.* In this subsection we want to study the evolution equation of the curvature operator, as was done for the Ricci flow by Hamilton [1986].

First of all, we simplify the expression of the reaction term in equation (3-2) by means of the so-called Uhlenbeck's trick [Hamilton 1986]. Briefly, we will relate the curvature tensor of the evolving metric to an evolving tensor of an abstract bundle with the same symmetries of the curvature (see Proposition 3.4) and a nicer evolution equation; afterwards we will find a suitable orthonormal moving frame of $(TM, g(t))$ and write the evolution equation of the coordinates of the Riemann tensor with respect to this frame. The result will be a system of *scalar* evolution equations and no more a tensorial equation (see [Chow and Knopf 2004] for more details on this method in the case of Ricci flow).

Let $(M, g(t))_{t \in [0,T)}$ be the solution of the RB flow with initial data g_0 and consider on the tangent bundle TM the family of endomorphisms $\{\varphi(t)\}_{t \in [0,T)}$ defined by the evolution equation

$$(3\text{-}5) \qquad \begin{cases} \frac{\partial}{\partial t}\varphi(t) = \mathrm{Ric}^{\#}_{g(t)} \circ \varphi(t) - \rho \mathrm{R}_{g(t)}\varphi(t), \\ \varphi(0) = \mathrm{Id}_{TM}, \end{cases}$$

where $\mathrm{Ric}^{\#}_{g(t)}$ is the endomorphism of the tangent bundle canonically associated to the Ricci tensor by raising an index.

For every point p of the manifold M, the evolution equation (3-5) represents a system of linear ODEs on the fiber $T_p M$; therefore a unique solution exists as long as the RB flow exists. Moreover, if we let $(h(t))_{t \in [0,T)}$ be the family of bundle metrics defined by $h(t) = \varphi(t)^*(g(t))$, where $\varphi(t)$ satisfies system (3-5), then $h(t) = g_0$ for every $t \in [0, T)$. As

$$\text{for all } t \in [0, T), \qquad \varphi(t) : (TM, g_0) \to (TM, g(t))$$

is a bundle isometry, the pullback via $\varphi(t)$ of the Levi-Civita connection associated to $g(t)$ is a connection on TM compatible with the metric g_0. In the following, we

will denote by (V, h) the vector bundle (TM, g_0) in order to stress the fact that we are not considering the Levi-Civita connection associated to g_0, but the family of time-dependent connections $D(t)$ defined via the bundle isometries $\varphi(t)$.

The following lemma states some basic properties of these pullback connections:

Lemma 3.2 (see [Chow and Knopf 2004, Chapter 6, Section 2]). *Let $D(t) : \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$ be the pullback connection defined by*

$$D(t)_X \zeta = \varphi(t)^*(\nabla_X^{g(t)}(\varphi(t)(\zeta))),$$

for all $t \in [0, T)$, for all $X \in \Gamma(TM)$, for all $\zeta \in \Gamma(V)$, where $\nabla^{g(t)}$ is the Levi-Civita connection of $g(t)$.

Let again $D(t)$ be the canonical extension to the tensor powers of V and T be a covariant tensor on M. Then, for every $t \in [0, T)$ and $X \in \Gamma(TM)$ we have

$$D(t)_X(\varphi(t)^*(T)) = \varphi(t)^*(\nabla_X^{g(t)} T).$$

In particular, $D(t)_X h = \varphi^*(\nabla_X^{g(t)} g(t)) = 0$, so every connection of the family $D(t)$ is compatible with the bundle metric h on V.

Let $D^2 : \Gamma(TM) \times \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$ be the second covariant derivative defined by

$$D^2_{X,Y}(\zeta) = D_X(D_Y \zeta) - D_{\nabla_X^{g(t)} Y} \zeta, \qquad \text{for all } X, Y \in \Gamma(TM), \text{for all } \zeta \in \Gamma(V),$$

and the rough Laplacian defined by $\Delta_D = \operatorname{tr}_g(D^2)$. Then, for every covariant tensor T on M, we have

$$(3\text{-}6) \qquad D^2_{X,Y}(\varphi^*(T)) = \varphi^*(\nabla^2_{X,Y} T) \qquad \text{for all } X, Y \in \Gamma(TM),$$

$$(3\text{-}7) \qquad \Delta_D(\varphi^*(T)) = \varphi^*(\Delta_g T).$$

Remark 3.3. Let $\mathcal{R} \in \operatorname{End}(\Lambda^2 M)$ be the *Riemann curvature operator* defined by

$$(3\text{-}8) \qquad \langle \mathcal{R}(X \wedge Y), W \wedge Z \rangle = \operatorname{Riem}(X, Y, W, Z),$$

where $\langle \cdot, \cdot \rangle$ is the linear extension of g to the exterior powers of TM.

In the following, we use a convention on the Lie algebra structure of $\Lambda^2 M$ which is different from the original one chosen by Hamilton [1986]. More precisely, with his convention, the eigenvalues of the curvature operator are twice the sectional curvatures, whereas with our convention the curvature operator has the sectional curvatures as eigenvalues. In particular, every formula differs from the corresponding one in the usual theory of the Ricci flow by a factor of 2 (see also [Chow and Knopf 2004, Chapter 6, Section 3] for the details). We recall that $\mathcal{R}$ can be considered as an element of $\Gamma(S^2(\Lambda^2 M))$, and the following equations hold:

$$\mathrm{R} = 2 \sum_{i < k} \mathcal{R}^{(ik)}_{(ik)}, \quad (\mathcal{R}^2)_{ijkl} = \mathrm{B}_{ijkl} - \mathrm{B}_{ijlk}, \quad (\mathcal{R}\#\mathcal{R})_{ijkl} = \mathrm{B}_{ikjl} - \mathrm{B}_{iljk},$$

where B is defined as in Proposition 3.1. For more details on the structure of the curvature operator we refer the reader again to [Chow and Knopf 2004, Chapter 6, Section 3].

We now consider the pullback of the Riemann curvature tensor and the curvature operator.

Proposition 3.4. *Let* Piem *be the pullback of the Riemann curvature tensor via the family of bundle isometries* $\{\varphi(t)\}_{t\in[0,T)}$. *The following statements hold true*:

(1) Piem *has the same symmetry properties as* Riem, *i.e., it can be seen as an element of* $\Gamma(S^2(\Lambda^2 V))$ *and it satisfies the first Bianchi identity*;

(2) *For every* $p \in M$ *and* $t \in [0, T)$ *the* **algebraic curvature operator** $\mathcal{P}(p, t) \in$ End$(\Lambda^2 V_p)$ *(see Remark 3.7), defined by* $\varphi \circ \mathcal{P} = \mathcal{R} \circ \varphi$ *has the same eigenvalues as* $\mathcal{R}(p, t)$. *In particular,* $\mathcal{P}$ *is positive (nonnegative) definite if and only if* $\mathcal{R}$ *is positive (nonnegative) definite*;

(3) Pic$(t) = \mathrm{tr}_h(\mathrm{Piem}(t)) = \varphi(t)^*(\mathrm{Ric}_{g(t)})$;

(4) $\mathrm{P} = \mathrm{tr}_h(\mathrm{Pic}(t)) = \mathrm{R}_{g(t)}$;

(5) B(Piem) $= \varphi^*(\mathrm{B(Riem)})$, *where* B *is defined the same way as in Proposition 3.1 for a generic element of* $S^2(\Lambda^2 V^*)$.

Finally, we can compute the evolution of Piem and $\mathcal{P}$.

Proposition 3.5. *The tensors* Piem *and* $\mathcal{P}$ *satisfy respectively the following evolution equations*

$$(3\text{-}9) \quad \frac{\partial}{\partial t}(\mathrm{Piem})_{abcd} = \Delta_D(\mathrm{Piem})_{abcd} - \rho(\varphi^*(\nabla^2 \mathrm{R}) \oslash h)_{abcd}$$
$$+ 2(\mathrm{B(Piem)}_{abcd} - \mathrm{B(Piem)}_{abdc} + \mathrm{B(Piem)}_{acbd} - \mathrm{B(Piem)}_{adbc})$$
$$- 2\rho\mathrm{PPiem}_{abcd},$$

$$(3\text{-}10) \quad \frac{\partial}{\partial t}\mathcal{P} = \Delta_D\mathcal{P} - 2\rho\varphi^*(\nabla^2 \mathrm{tr}_h(\mathcal{P})) \oslash h + 2\mathcal{P}^2 + 2\mathcal{P}^{\#} - 4\rho\,\mathrm{tr}_h(\mathcal{P})\mathcal{P},$$

where $\mathrm{tr}_h(\mathcal{P}(t)) = \mathrm{tr}_{g(t)}(\mathcal{R}(t)) = \frac{1}{2}\mathrm{R}(t)$.

Remark 3.6. On the right-hand side of (3-9) the term $\varphi^*(\nabla^2 \mathrm{R})$ appears (i.e., the pullback of the Hessian of the scalar curvature, seen as a symmetric 2-form on the tangent bundle) and it cannot be expressed in terms of the connection $D(t)$.

Proof. Let $\zeta_1, \ldots, \zeta_4$ be sections of V; then combining the evolution equations of the Riemann tensor (3-2) and of the bundle isometry φ (3-5), we obtain

$$\frac{\partial}{\partial t}(\text{Piem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$$

$$= \varphi^*\Big(\frac{\partial}{\partial t}\text{Riem}\Big)(\zeta_1, \zeta_2, \zeta_3, \zeta_4) + \text{Riem}\Big(\frac{\partial \varphi}{\partial t}(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big)$$

$$\quad + \text{Riem}\Big(\varphi(\zeta_1), \frac{\partial \varphi}{\partial t}(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big) + \text{Riem}\Big(\varphi(\zeta_1), \varphi(\zeta_2), \frac{\partial \varphi}{\partial t}(\zeta_3), \varphi(\zeta_4)\Big)$$

$$\quad + \text{Riem}\Big(\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \frac{\partial \varphi}{\partial t}(\zeta_4)\Big)$$

$$= \varphi^*(\Delta_g \text{Riem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \rho \varphi^*(\nabla^2 \text{R} \oslash g)(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$$

$$\quad + 2\varphi^*(\text{B}(\text{Riem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \text{B}(\text{Riem})(\zeta_1, \zeta_2, \zeta_4, \zeta_3) - \text{B}(\text{Riem})(\zeta_1, \zeta_4, \zeta_2, \zeta_3)$$

$$\quad + \text{B}(\text{Riem})(\zeta_1, \zeta_3, \zeta_2, \zeta_4)) + 2\rho \text{R}\varphi^*(\text{Riem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$$

$$\quad - \text{Riem}(\text{Ric}^\#\circ\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)) - \text{Riem}(\varphi(\zeta_1), \text{Ric}^\#\circ\varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4))$$

$$\quad - \text{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), \text{Ric}^\#\circ\varphi(\zeta_3), \varphi(\zeta_4)) - \text{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \text{Ric}^\#\circ\varphi(\zeta_4))$$

$$\quad + \text{Riem}((\text{Ric}^\#\circ\varphi - \rho \text{R}\varphi)(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4))$$

$$\quad + \text{Riem}(\varphi(\zeta_1), (\text{Ric}^\#\circ\varphi - \rho \text{R}\varphi)(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4))$$

$$\quad + \text{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), (\text{Ric}^\#\circ\varphi - \rho \text{R}\varphi)(\zeta_3), \varphi(\zeta_4))$$

$$\quad + \text{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), (\text{Ric}^\#\circ\varphi - \rho \text{R}\varphi)(\zeta_4))$$

$$= \Delta_D(\text{Piem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \rho(\varphi^*(\nabla^2\text{R}) \oslash h)(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$$

$$\quad + 2\varphi^*(\text{B}(\text{Piem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \text{B}(\text{Piem})(\zeta_1, \zeta_2, \zeta_4, \zeta_3) - \text{B}(\text{Piem})(\zeta_1, \zeta_4, \zeta_2, \zeta_3)$$

$$\quad + \text{B}(\text{Piem})(\zeta_1, \zeta_3, \zeta_2, \zeta_4)) - 2\rho \text{PPiem}(\zeta_1, \zeta_2, \zeta_3, \zeta_4),$$

where we used several identities stated above. For $\zeta_1, \ldots, \zeta_4$ belonging to a local frame we get the desired equation (3-9).

Combining the evolution equation for Piem with the formulas in Remark 3.3, we find the evolution equation of $\mathcal{P}$.   □

Remark 3.7. It must be noticed that, even though for every $p \in M$ and $t \in [0, T)$, the tensor $\mathcal{P}(p, t)$ belongs to the set of algebraic curvature operators $\mathcal{C}_b(V_p)$, in general it does not coincide with the curvature operator of the pullback connection $D(t)$. In the present literature the pullback tensor is always denoted by Riem and this abuse of notation is somehow misleading, suggesting wrongly that $\text{Piem}(t) = \varphi(t)^*(\text{Riem}_{g(t)})$ is

equal to $\mathrm{Riem}_{\varphi(t)^*(g(t))} = \mathrm{Riem}_h$, but this is no longer true for general isomorphisms of the tangent bundle (however it is true for $\varphi \in \mathrm{Diff}(M)$).

By Uhlenbeck's trick, the evolution equation (3-10) for $\mathcal{P}$ allows a simpler use of the maximum principle for tensors as the reaction term is nicer and the metric on $S^2(\Lambda^2 V)$ is independent of time. Moreover, the subsets of $S^2(\Lambda^2 V)$ preserved by such PDE correspond to curvature conditions preserved under the RB flow.

4. Preserved curvature conditions

In this section we will use the maximum principle in various formulations in order to find curvature conditions which are preserved by the RB flow.

4A. *The scalar curvature.* We begin by considering the evolution equation for the scalar curvature (3-4), which behaves as under the Ricci flow.

Proposition 4.1. *Let $(M, g(t))_{t \in [0,T)}$ be a compact maximal solution of the RB flow (1-1). If $\rho \le 1/(2(n-1))$, the minimum of the scalar curvature is nondecreasing along the flow. In particular if $\mathrm{R}_{g(0)} \ge \alpha$, for some $\alpha \in \mathbb{R}$, then $\mathrm{R}_{g(t)} \ge \alpha$ for every $t \in [0, T)$. Moreover if $\alpha > 0$ then $T \le n/(2(1-n\rho)\alpha)$.*

Proof. As $\rho \le 1/(2(n-1)) \le 1/n$, for any $n > 1$, it follows that

$$\frac{\partial}{\partial t}\mathrm{R} = (1 - 2(n-1)\rho)\Delta\mathrm{R} + 2|\mathrm{Ric}|^2 - 2\rho\mathrm{R}^2$$
$$\ge (1 - 2(n-1)\rho)\Delta\mathrm{R} + 2\mathrm{R}^2/n - 2\rho\mathrm{R}^2$$
$$\ge (1 - 2(n-1)\rho)\Delta\mathrm{R},$$

hence, by the maximum principle, the minimum of the scalar curvature is nondecreasing along the RB flow on a compact manifold. In particular, for every $\alpha \in \mathbb{R}$, the condition $\mathrm{R} \ge \alpha$ is preserved.

Finally, integrating the inequality

$$\frac{\partial}{\partial t}\mathrm{R}_{\min} \ge 2\Big(\frac{1}{n} - \rho\Big)\mathrm{R}_{\min}^2,$$

that holds almost everywhere for $t \in [0, T)$ (by Hamilton's trick (see [Hamilton 1997], [Mantegazza 2011, Lemma 2.1.3])), we obtain

(4-1) $$\mathrm{R}_{\min}(t) \ge \frac{n\alpha}{n - 2(1-n\rho)\alpha t},$$

which, for $\alpha > 0$, gives the estimate on the maximal time of existence. $\square$

Remark 4.2. In the special case of the Schouten flow (when $\rho = 1/2(n-1)$), we actually have

$$\frac{\partial}{\partial t}\mathrm{R} \ge \frac{n-2}{n(n-1)}\mathrm{R}^2,$$

at every point of the manifold, which implies that the scalar curvature is pointwise nondecreasing and diverges in finite time.

Remark 4.3. By means of the strong maximum principle, it follows that if the initial manifold has nonnegative scalar curvature, then either the manifold is Einstein (Ric $= 0$) or the scalar curvature becomes positive for every positive time under any RB flow with $\rho \leq 1/(2(n-1))$.

Proposition 4.4. *Let* $(M, g(t))_{t \in (-\infty, 0]}$ *be a compact, n-dimensional, ancient solution of the RB flow* (1-1). *If* $\rho \leq 1/(2(n-1))$ *then, either* R > 0 *or* Ric $\equiv 0$ *on* $M \times (-\infty, 0]$.

Proof. As $g(t)$ is an ancient solution, for every $t_0 < t_1 \leq 0$, we can define $\widetilde{g}(s) = g(s + t_0)$, which is a solution of the RB flow for $s \in [0, t_1 - t_0]$. Then we have $\widetilde{R}_{\min}(0) = R_{\min}(t_0)$, hence, from formula (4-1)

$$\widetilde{R}_{\min}(s) \geq \frac{n}{n\widetilde{R}_{\min}^{-1}(0) - 2(1 - n\rho)s},$$

for every $s \in (0, t_1 - t_0]$. In particular, we have

$$R_{\min}(t_1) = \widetilde{R}_{\min}(t_1 - t_0) \geq \frac{n}{nR_{\min}^{-1}(t_0) - 2(1 - n\rho)(t_1 - t_0)}.$$

If $R_{\min}(t_0) \geq 0$, by Proposition 4.1, it follows that $R_{\min}(t_1) \geq 0$, so we can assume that $R_{\min}(t_0) < 0$, hence

$$R_{\min}(t_1) \geq \frac{n}{nR_{\min}^{-1}(t_0) - 2(1 - n\rho)(t_1 - t_0)} > -\frac{n}{2(1 - n\rho)(t_1 - t_0)},$$

for every $t_1 < t_0$, and sending t_0 to $-\infty$, we still conclude that $R_{\min}(t_1) \geq 0$. Since this holds for every $t_1 \leq 0$ the previous remark implies the result. $\qquad\square$

4B. *Maximum principle for uniformly elliptic operators.* Let M be a smooth compact manifold, $g(t)$, $t \in [0, T)$, a family of Riemannian metrics on M and $(E, h(t))$ $t \in [0, T)$, be a real vector bundle on M, endowed with a (possibly time-dependent) bundle metric. Let $D(t) : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ be a family of linear connections on E, compatible at each time with the bundle metric $h(t)$. We have already seen in Section 3B how to define the second covariant derivative, using also the Levi-Civita connections $\nabla_{g(t)}$ associated to the Riemannian metrics on M.

Definition 4.5. We consider a second order linear operator $\mathcal{L}$ on $\Gamma(E)$ that lacks a 0-th order term, and hence can be written in a local frame field $\{e_i\}_{i=1,\dots,n}$ of TM

$$(4\text{-}2) \qquad\qquad \mathcal{L} = a^{ij} D^2_{e_i e_j} - b^i D_{e_i}$$

where $a = a^{ij} e_i \otimes e_j \in \Gamma(S^2(TM))$ is a symmetric $(0, 2)$-tensor and $b = b^i e_i$ is a smooth vector field. We say $\mathcal{L}$ is *uniformly elliptic* if a is uniformly positive definite.

Remark 4.6. In the previous definition, both the coefficients and the connections are in general time-dependent and we say that $\mathcal{L}$ is uniformly elliptic if it is so for every $t \in [0, T)$ uniformly in time.

Weinberger [1975] proved the maximum principle for systems of solutions of a time-dependent heat equation in Euclidean space; Hamilton [1986] treated the general case of a vector bundle over an evolving Riemannian manifold. Here we present a slight generalization of Hamilton's theorem for parabolic equations with uniformly elliptic operator (Savas-Halilaj and Smoczyk [2014, Theorem 2.2] proved a "static" version). As before, $(M, g(t))$ is a smooth compact manifold equipped with a family of Riemannian metrics; we consider a real vector bundle E over M, equipped with a fixed bundle metric h and a family of time-dependent connections $D(t)$ compatible at every time with h.

Definition 4.7. Let $S \subset E$ be a subbundle and denote $S_p = S \cap E_p$ for every $p \in M$. We say that S is *invariant under parallel translation* with respect to D, if for every curve $\gamma : [0, l] \to M$ and vector $v \in S_{\gamma(0)}$, the unique parallel (with respect to D) section $v(s) \in E_{\gamma(s)}$ along $\gamma(s)$ with $v(0) = v$ is contained in S.

Theorem 4.8 (vectorial maximum principle). *Let $u : [0, T) \to \Gamma(E)$ be a smooth solution of the following parabolic equation*

$$(4\text{-}3) \qquad \frac{\partial}{\partial t} u = \mathcal{L}u + F(u, t),$$

where $\mathcal{L}$ is a uniformly elliptic operator as defined in (4-2) *and $F : E \times [0, T) \to E$ is a continuous map, locally Lipschitz in the E factor, which is also fiber-preserving, i.e., $F(v, t) \in E_p$ for every $p \in M$, $v \in E_p$, $t \in [0, T)$.*

Let $K \subset E$ be a closed subbundle (for the metric h), invariant under parallel translation with respect to $D(t)$, for every $t \in [0, T)$, and convex in the fibers, i.e., $K_p = K \cap E_p$ is convex for every $p \in M$.

Suppose that K is preserved by the ODE associated to (4-3), *i.e., for every $p \in M$ and $U_0 \in K_p$, the solution $U(t)$ of*

$$(4\text{-}4) \qquad \begin{cases} \frac{dU}{dt} & = F_p(U(t), t), \\ U(0) & = U_0. \end{cases}$$

remains in K_p. Then, if u is contained in K at time 0, u remains in K, i.e., $u(p, t) \in K_p$ for every $p \in M$, $t \in [0, T)$.

Proof. (Sketch) We can follow exactly the detailed proof written in [Chow et al. 2008, Chapter 10, Section 3], provided that we generalize their Lemma 10.34 to the analogue one for uniformly elliptic operator (see again [Savas-Halilaj and Smoczyk 2014, Lemma 2.2]): if $K \subset E$ satisfies all the hypotheses of Theorem 4.8 and

$u \in \Gamma(E)$ is a smooth section of E, then

$$u(p) \in K_p \quad \text{for all } p \in M \quad \Longrightarrow \quad \mathcal{L}(u)_p \in C_{u(p)}K_p \quad \text{for all } p \in M,$$

where $C_{u(p)}K_p$ is the tangent cone of the convex set K_p at $u(p)$.   □

There is a further generalization of this maximum principle which allows the subset K to be time-dependent.

Theorem 4.9 (vectorial maximum principle, time-dependent set). *Let $u : [0, T) \to \Gamma(E)$ be a smooth solution of the parabolic equation* (4-3), *with the notations of the previous theorem. For every $t \in [0, T)$, let $K(t) \subset E$ be a closed subbundle (for the metric h), invariant under parallel translation with respect to $D(t)$, convex in the fibers and such that the spacetime track*

$$\mathcal{T} = \{(v, t) \in E \times \mathbb{R} : v \in K(t), t \in [0, T)\}$$

is closed in $E \times [0, T)$. Suppose that, for every $t_0 \in [0, T)$, $K(t_0)$ is preserved by the ODE associated, i.e., for any $p \in M$, any solution $U(t)$ of the ODE that starts in $K(t_0)_p$ remains in $K(t)_p$, as long as it exists. Then, if $u(0)$ is contained in $K(0)$, $u(p, t) \in K(t)_p$ for ever $p \in M$, $t \in [0, T)$.

The proof of this theorem, when K depends continuously on time and F does not depend on time is due to Bohm and Wilking [2007, Theorem 1.1]. In the general case the proof can be found in [Chow et al. 2008, Chapter 10, Section 5], with the usual adaptation to the uniformly elliptic case.

As remarked before, the evolution equation (3-2) of the Riemann tensor has some mixed products of type Riem $*$ Ric which makes it difficult to understand the behavior of the reaction term. On the other hand, if we perform Uhlenbeck's trick, the evolution equation (3-9) becomes a little nicer and can be used to understand how the RB flow affects the geometry.

More precisely, we use the evolution equation (3-10) for the algebraic curvature operator $\mathcal{P} \in \Gamma(S^2(\Lambda^2 V^*))$ to prove that the cone of nonnegative curvature operators is preserved by the RB flow.

Proposition 4.10. *Let $(M, g(t))_{t \in [0,T)}$ be a compact solution of the RB flow* (1-1) *with $\rho < 1/(2(n-1))$ and such that the initial data g_0 has nonnegative curvature operator. Then $\mathcal{R}_{g(t)} \geq 0$ for every $t \in [0, T)$.*

Proof. We recall the evolution equation (3-10) for $\mathcal{P} = \varphi^{-1} \circ \mathcal{R} \circ \varphi$,

$$\frac{\partial}{\partial t}\mathcal{P} = \Delta_D \mathcal{P} - 2\rho\varphi^*(\nabla^2 \text{tr}_h(\mathcal{P})) \oslash h + 2\mathcal{P}^2 + 2\mathcal{P}^\# - 4\rho \, \text{tr}_h(\mathcal{P})\mathcal{P},$$

where $\text{tr}_h(\mathcal{P}(t)) = 1/2\text{R}(t)$ is half of the scalar curvature of the metric $g(t)$. By Proposition 3.4, it suffices to show that the nonnegativity of $\mathcal{P}$ is preserved by

equation (3-10). We want to apply the vectorial maximum principle Theorem 4.8, and therefore we must show that

$$\mathcal{L}(Q) = \Delta_D Q - 2\rho\varphi^*(\nabla^2 \operatorname{tr}_h(Q)) \oslash h$$

is a uniformly elliptic operator on the bundle $(\Gamma(S^2(\Lambda^2 V^*)), h, D(t))$.

As $\mathcal{L}$ is a linear second order operator, we compute as usual its principal symbol in the arbitrary direction ξ. In order to simplify the computations, we choose opportune frames at every point $p \in M$ and time $t \in [0, T)$. Then let $\{e_i\}_{i=1,\ldots,n}$ be an orthonormal basis of (V_p, h_p) such that $\xi = h_p(e_1, \cdot)$. According to Uhlenbeck's trick (Section 3B) and the convention on algebraic curvature operators (Section 3B) we have that $\{f_i = \varphi(t)_p(e_i)\}_{i_1,\ldots,n}$ is an orthonormal basis of $T_p M$ with respect to $g(t)_p$, the components of $\varphi(t)_p$ with these choices are $\varphi_i^a = \delta_i^a$, and $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis of $\Lambda^2 V_p$. Hence, the principal symbol of the operator $\mathcal{L}$ written in these frames is

$$\sigma_\xi(\mathcal{L}Q)_{(ij)(kl)} = \xi^p \xi_p Q_{(ij)(kl)} - 2\rho \delta_i^a \delta_j^b \delta_k^c \delta_l^d \operatorname{tr}_h(Q)(\xi \otimes \xi \oslash h)_{(ab)(cd)}$$

$$= |\xi|^2 Q_{(ij)(kl)} - 2\rho \operatorname{tr}_h(Q)(\xi \otimes \xi \oslash h)_{(ij)(kl)}$$

$$= Q_{(ij)(kl)} - 2\rho \left(\sum_{p<q} Q_{(pq)(pq)} \right) \delta_i^1 \delta_k^1 \delta_{jl},$$

where we used that $|\xi| = 1$, $i < j$ and $k < l$ in the last step. Now it is easy to see that the matrix representing the symbol has the following form:

$$\sigma_\xi(\mathcal{L}) = \begin{pmatrix} A[n-1] & \begin{matrix} -2\rho & \cdots & 2\rho \\ \vdots & \ddots & \vdots \\ -2\rho & \cdots & -2\rho \end{matrix} & 0 \\ 0 & \mathrm{Id}_{(n-1)(n-2)/2} & 0 \\ 0 & 0 & \mathrm{Id}_{N(N-1)/2} \end{pmatrix},$$

where we have ordered the components as follows: first the $n-1$ ones of the form $(1j)(1j)$ with $j > 1$, then the $(n-1)(n-2)/2$ ones of the form $(ij)(ij)$ with $1 < i < j$, and last the $N(N-1)/2$ "nondiagonal" ones, with $N = n(n-1)/2$ and A is the matrix defined in (2-1).

By Lemma 2.2 the eigenvalues of the symbol are 1 with multiplicity $\frac{1}{2} N(N+1) - 1$ and $1 - 2(n-1)\rho$ with multiplicity 1, since $\rho < 1/2(n-1)$ the operator $\mathcal{L}$ is uniformly elliptic.

We next consider the reaction term $F(Q) = 2(Q^2 + Q^\# - 2\rho\,\mathrm{tr}_h(Q)Q)$. Clearly F is continuous, locally Lipschitz and fiber-preserving. Let $\Omega \subset \Gamma(S^2(\Lambda^2 V^*))$ be the set of nonnegative algebraic curvature operators, where we have identified $S^2(\Lambda^2 V^*) \simeq \mathrm{End}_{SA}(\Lambda^2 V)$ via the metric h. We observe that $\Omega = \{Q : \lambda_N(Q_p) \geq 0\}$, where $N = n(n-1)/2$ and λ_N is the least eigenvalue of Q_p. Hence Ω is clearly closed, by [Chow et al. 2008, Lemma 10.11] it is invariant under parallel translation with respect to every connection $D(t)$ and it is convex, provided that the function $Q \mapsto \lambda_N(Q_p)$ is concave. We can rewrite

$$\lambda_N(Q_p) = \inf_{\{v \in \Lambda^2 V_p : |v|_h = 1\}} h(Q_p(v), v),$$

so it is easy to conclude, by the bilinearity of the metric h and the concavity of inf, that the function defining Ω is actually concave and so its superlevels are convex. In order to finish the proof we have to show that the ODE $dQ/dt = F(Q)$ preserves Ω. Now, by standard facts in convex analysis, we only need to prove that $F_p(Q_p) \in T_{Q_p}\Omega_p$ for every $p \in M$ such that $Q_p \in \partial\Omega_p$, where $\partial\Omega_p$ is the set of $Q_p \in \Omega_p$ where there is $v \in \Lambda^2 V_p$ such that $Q_p(v, v) = 0$ and the tangent cone is

$$T_{Q_p}\Omega_p = \{S_p \in S^2(\Lambda^2 V_p^*) : S_p(v, v) \geq 0 \text{ for every } v \in \Lambda^2 V_p \text{ such that } Q_p(v, v) = 0\}$$

Let $v \in \Lambda^2 V_p$ and $\{\theta_\alpha\}$ be respectively a null eigenvector of Q_p and an orthonormal basis of $\Lambda^2 V_p$ that diagonalizes Q_p. Clearly $v = v^\alpha \theta_\alpha$ and $(Q_p)_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$. with $\lambda_\alpha \geq 0$. Then $(Q_p^2)_{\alpha\beta} = \lambda_\alpha^2 \delta_{\alpha\beta}$ and $(Q_p^\#)_{\alpha\beta} = \frac{1}{2}(c_\alpha^{\gamma\nu})^2 \lambda_\gamma \lambda_\nu \delta_{\alpha\beta}$ and

$$F_p(Q_p)(v, v) = \lambda_\alpha^2(v^\alpha)^2 + \tfrac{1}{2}(c_\alpha^{\gamma\nu})^2 \lambda_\gamma \lambda_\nu (v^\alpha)^2 \geq 0. \qquad \square$$

4C. *The evolution of the Weyl tensor.* By means of the evolution equations found for the curvatures, we are also able to write the equation satisfied by the Weyl tensor along the RB flow (1-1). In [Catino and Mantegazza 2011] the authors compute the evolution equation of the Weyl tensor during the Ricci flow (see [Catino et al. 2015a] for a significant application of this formula) and we use most of their computations.

Proposition 4.11. *During the RB flow of an n-dimensional Riemannian manifold (M, g) the Weyl tensor satisfies the following evolution equation:*

$$\frac{\partial}{\partial t}\mathrm{W}_{ijkl} = \Delta\mathrm{W}_{ijkl} + 2(\mathrm{B}(\mathrm{W})_{ijkl} - \mathrm{B}(\mathrm{W})_{ijlk} - \mathrm{B}(\mathrm{W})_{iljk} + \mathrm{B}(\mathrm{W})_{ikjl})$$
$$+ 2\rho\mathrm{RW}_{ijkl} - g^{pq}(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql})$$
$$+ \frac{2}{(n-2)^2}(\mathrm{Ric}^2 \oslash g)_{ijkl} + \frac{1}{(n-2)}(\mathrm{Ric} \oslash \mathrm{Ric})_{ijkl}$$
$$- \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{Ric} \oslash g)_{ijkl} + \frac{\mathrm{R}^2 - |\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g \oslash g)_{ijkl},$$

where $\mathrm{B}(\mathrm{W})_{ijkl} = g^{pq}g^{rs}\mathrm{W}_{ipjr}\mathrm{W}_{kqls}$.

Proof. By recalling the decomposition formula for the Weyl tensor (1-3) we have

$$\frac{\partial}{\partial t}W = \frac{\partial}{\partial t}\text{Riem} + \frac{1}{2(n-1)(n-2)}\left(\frac{\partial}{\partial t}Rg \oslash g + 2\frac{\partial}{\partial t}g \oslash g\right)$$
$$- \frac{1}{n-2}\left(\frac{\partial}{\partial t}\text{Ric} \oslash g + \text{Ric} \oslash \frac{\partial}{\partial t}g\right)$$

$$= \mathcal{L}_{II} + \mathcal{L}_0,$$

where $\mathcal{L}_{II}$ is the second order term in the curvatures and $\mathcal{L}$ the 0-th one. We deal first with the higher order term; plugging in the evolution equations of Riem, Ric and R (Proposition 3.1) we get

$$\mathcal{L}_{II} = \Delta\text{Riem} - \rho(\nabla^2 R \oslash g) + \frac{1-2(n-1)\rho}{2(n-1)(n-2)}\Delta Rg \oslash g$$
$$- \frac{1}{n-2}(\Delta\text{Ric} \oslash g - (n-2)\rho\nabla^2 R \oslash g - \rho\Delta Rg \oslash g)$$
$$= \Delta\text{Riem} + \frac{1-2(n-1)\rho + 2(n-1)\rho}{2(n-1)(n-2)}\Delta Rg \oslash g - \frac{1}{n-2}\Delta\text{Ric} \oslash g$$
$$= \Delta W.$$

Then we consider the lower order terms

$$(\mathcal{L}_0)_{ijkl} = 2(B(\text{Riem})_{ijkl} - B(\text{Riem})_{ijlk} - B(\text{Riem})_{iljk} + B(\text{Riem})_{ikjl})$$
$$- g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql})$$
$$+ 2\rho R\left(W - \frac{1}{2(n-1)(n-2)}Rg \oslash g + \frac{1}{n-2}\text{Ric} \oslash g\right)_{ijkl}$$
$$+ \frac{1}{2(n-1)(n-2)}(2|\text{Ric}|^2 g \oslash g - 2\rho R^2 g \oslash g - 4R\text{Ric} \oslash g + 4\rho R^2 g \oslash g)_{ijkl}$$
$$- \frac{1}{n-2}[2(\text{Riem} * \text{Ric}) \oslash g - 2\text{Ric}^2 \oslash g - 2\text{Ric} \oslash \text{Ric} + 2\rho R\text{Ric} \oslash g]_{ijkl}$$
$$= 2(B(\text{Riem})_{ijkl} - B(\text{Riem})_{ijlk} - B(\text{Riem})_{iljk} + B(\text{Riem})_{ikjl})$$
$$- g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) + 2\rho R W_{ijkl}$$
$$- \frac{2}{n-2}[(\text{Riem} * \text{Ric}) \oslash g - \text{Ric}^2 \oslash g - \text{Ric} \oslash \text{Ric}]_{ijkl}$$
$$- \frac{2R}{(n-1)(n-2)}(\text{Ric} \oslash g)_{ijkl} + \frac{|\text{Ric}|^2}{(n-1)(n-2)}(g \oslash g)_{ijkl},$$

where $(\text{Riem} * \text{Ric})_{ab} = R_{apbq}R_{st}g^{ps}g^{qt}$ and $(\text{Ric}^2)_{ab} = R_{ap}R_{bq}g^{pq}$.

Now we deal separately with every term containing the full curvature Riem, using its decomposition formula, expanding the Kulkarni–Nomizu products and then contracting again. We have that

$$[(g \oslash g) * \text{Ric}]_{ab} = 2[Rg - \text{Ric}]_{ab}, \quad [(\text{Ric} \oslash g) * \text{Ric}]_{ab} = [-2\text{Ric}^2 + R\text{Ric} + |\text{Ric}|^2 g]_{ab}.$$

Hence

$$(4\text{-}5) \quad (\mathrm{Riem} * \mathrm{Ric}) \oslash g = (\mathrm{W} * \mathrm{Ric}) \oslash g - \frac{2}{n-2}\mathrm{Ric}^2 \oslash g$$

$$+ \frac{n\mathrm{R}}{(n-1)(n-2)}\mathrm{Ric} \oslash g + \frac{(n-1)|\mathrm{Ric}|^2 - \mathrm{R}^2}{(n-1)(n-2)} g \oslash g.$$

Then

$$\mathrm{R}_{qi}\mathrm{R}_{pjkl}g^{pq} = \mathrm{R}_{qi}\left(\mathrm{W}_{pjkl} - \frac{\mathrm{R}}{(n-1)(n-2)}(g_{pk}g_{jl} - g_{pl}g_{jk})\right)g^{pq}$$

$$+ \frac{1}{n-2}\mathrm{R}_{qi}(\mathrm{R}_{pk}g_{jl} + \mathrm{R}_{jl}g_{pk} - \mathrm{R}_{pl}g_{jk} - \mathrm{R}_{jk}g_{pl})g^{pq}$$

$$= \mathrm{R}_{qi}\mathrm{W}_{pjkl}g^{pq} - \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk})$$

$$+ \frac{1}{n-2}(\mathrm{R}_{ik}^2 g_{jl} - \mathrm{R}_{il}^2 g_{jk} + \mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{il}\mathrm{R}_{jk}).$$

Interchanging the index and using the symmetry properties we get

$$(4\text{-}6) \quad g^{pq}(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql})$$

$$= g^{pq}(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql})$$

$$+ \frac{2}{n-2}(\mathrm{Ric}^2 \oslash g)_{ijkl} + \frac{2}{n-2}(\mathrm{Ric} \oslash \mathrm{Ric})_{ijkl} - \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{Ric} \oslash g)_{ijkl}.$$

Finally the "B"-terms:

$$B(\mathrm{Riem})_{abcd}$$

$$= \left(\mathrm{W} - \frac{\mathrm{R}}{2(n-1)(n-2)}g \oslash g + \frac{1}{n-2}\mathrm{Ric} \oslash g\right)_{apbq}$$

$$\left(\mathrm{W} - \frac{\mathrm{R}}{2(n-1)(n-2)}g \oslash g + \frac{1}{n-2}\mathrm{Ric} \oslash g\right)_{csdt} g^{ps}g^{qt}$$

$$(\mathrm{W}_{apbq}(g \oslash g)_{csdt} + (g \oslash g)_{apbq}\mathrm{W}_{csdt})g^{ps}g^{qt} = -2\mathrm{W}_{adbc} - 2\mathrm{W}_{cbda}$$

$$(\mathrm{W}_{apbq}(\mathrm{Ric} \oslash g)_{csdt} + (\mathrm{Ric} \oslash g)_{apbq}\mathrm{W}_{csdt})g^{ps}g^{qt}$$

$$= (\mathrm{W} * \mathrm{Ric})_{ab}g_{cd} + (\mathrm{W} * \mathrm{Ric})_{cd}g_{ab}$$

$$- (\mathrm{W}_{cbdp}\mathrm{R}_{aq} + \mathrm{W}_{cpda}\mathrm{R}_{bq} + \mathrm{W}_{adbp}\mathrm{R}_{cq} + \mathrm{W}_{apbd}\mathrm{R}_{dq})g^{pq}$$

$$(g \oslash g)_{apbd}(g \oslash g)_{csdt}g^{ps}g^{qt}$$

$$= 4((n-2)g_{ab}g_{cd} + g_{ac}g_{bd})((\mathrm{Ric}\oslash g)_{apbq}(g\oslash g)_{csdt} + (\mathrm{Ric}\oslash g)_{csdt}(g\oslash g)_{apbq})g^{ps}g^{qt}$$

$$= 2((n-4)\mathrm{R}_{ab}g_{cd} + (n-4)\mathrm{R}_{cd}g_{ab} + 2\mathrm{R}_{ac}g_{bd} + 2\mathrm{R}_{bd}g_{ac})$$

$$(\mathrm{Ric} \oslash g)_{abpq}(\mathrm{Ric} \oslash g)_{csdt}g^{ps}g^{qt}$$

$$= -2\mathrm{R}_{ab}^2 g_{cd} - 2\mathrm{R}_{cd}^2 g_{ab} + \mathrm{R}_{ac}^2 g_{bd} + \mathrm{R}_{bd}^2 g_{ac}$$

$$+ (n-4)\mathrm{R}_{ab}\mathrm{R}_{cd} + 2\mathrm{R}_{ac}\mathrm{R}_{bd} + \mathrm{R}(\mathrm{R}_{ab}g_{cd} + \mathrm{R}_{cd}g_{ab}) + |\mathrm{Ric}|^2 g_{ab}g_{cd}$$

Now, adding the same type quantities for the different index permutations and using the symmetry properties of W we obtain

$$(4\text{-}7) \quad B(\text{Riem})_{ijkl} - B(\text{Riem})_{ijlk} - B(\text{Riem})_{iljk} + B(\text{Riem})_{ikjl}$$

$$= B(W)_{ijkl} - B(W)_{ijlk} - B(W)_{iljk} + B(W)_{ikjl}$$

$$+ \frac{1}{n-2}((W * \text{Ric}) \oslash g)_{ijkl} - \frac{1}{(n-2)^2}(\text{Ric}^2 \oslash g)_{ijkl} + \frac{1}{2(n-2)}(\text{Ric} \oslash \text{Ric})_{ijkl}$$

$$+ \frac{R}{(n-1)(n-2)^2}(\text{Ric} \oslash g)_{ijkl} + \Big(\frac{|\text{Ric}|^2}{2(n-2)^2} - \frac{R^2}{2(n-1)(n-2)^2} \Big)(g \oslash g)_{ijkl}.$$

We are ready to complete the computation of the 0-th order term in the evolution equation, using the previous formulas (4-5), (4-6), (4-7):

$$(\mathcal{L}_0)_{ijkl} = 2(B(W)_{ijkl} - B(W)_{ijlk} - B(W)_{iljk} + B(W)_{ikjl}) + 2\rho R W_{ijkl}$$

$$- g^{pq}(W_{pjkl}R_{qi} + W_{ipkl}R_{qj} + W_{ijpl}R_{qk} + W_{ijkp}R_{ql})$$

$$+ \frac{2}{(n-2)^2}(\text{Ric}^2 \oslash g)_{ijkl} + \frac{1}{(n-2)}(\text{Ric} \oslash \text{Ric})_{ijkl}$$

$$- \frac{2R}{(n-2)^2}(\text{Ric} \oslash g)_{ijkl} + \frac{R^2 - |\text{Ric}|^2}{(n-1)(n-2)^2}(g \oslash g)_{ijkl} \quad \square$$

4D. *Conditions preserved in dimension three.* In general dimension, it is very hard to find other curvature conditions preserved by the flow, and this is due principally to the complex structure of the reaction terms; for example in the evolution equation satisfied by the Ricci tensor (3-3), the reaction terms involve the full curvature tensor. Therefore it is easier to restrict our attention to the three dimensional case, in which the Weyl part of the Riemann tensor vanishes and all the geometric information is encoded in the Ricci tensor.

In the special case of dimension three, we can also use the evolution equation (3-10) of the pullback of the curvature operator to obtain more refined conditions preserved, because we can rewrite the ODE associated to the evolution of $\mathcal{P}$ as a system of ODEs in the eigenvalues of $\mathcal{P}$ that, by Proposition 3.4, are nothing but the sectional curvatures of $\mathcal{R}$. This point of view was introduced for the Ricci flow by Hamilton [1997] and can be easily generalized to the RB flow as follows:

Lemma 4.12. *If $n = 3$, then $\mathcal{P}_p$ has 3 eigenvalues, λ, μ, ν, and the ODE fiberwise associated to equation (3-10) can be written as the following system*:

$$(4\text{-}8) \quad \begin{cases} \frac{d\lambda}{dt} &= 2\lambda^2 + 2\mu\nu - 4\rho\lambda(\lambda + \mu + \nu), \\ \frac{d\mu}{dt} &= 2\mu^2 + 2\lambda\nu - 4\rho\mu(\lambda + \mu + \nu), \\ \frac{d\nu}{dt} &= 2\nu^2 + 2\lambda\mu - 4\rho\nu(\lambda + \mu + \nu). \end{cases}$$

In particular, if we assume $\lambda(0) \geq \mu(0) \geq \nu(0)$, then $\lambda(t) \geq \mu(t) \geq \nu(t)$ as long as the solution of the system exists.

Proof. We can pointwise identify V_p with an orthonormal frame of $\mathbb{R}^3$ with the standard basis. Then $\Lambda^2 V_p \simeq \mathfrak{so}(3)$ with the standard structure constants and if an algebraic operator Q_p is diagonal, both Q_p^2 and $Q_p^\#$ are diagonal with respect to the same basis (for the detailed computation of this fact, see [Chow and Knopf 2004, Chapter 6.4]). Hence the ODE $(d/dt)Q_p = F_p(Q_p)$, associated fiberwise to equation (3-10), preserves the eigenvalues of Q_p, that is, if $Q_p(0)$ is diagonal with respect to an orthonormal basis, $Q_p(t)$ stays diagonal with respect to the same basis and the ODE can be rewritten as the system (4-8) in the eigenvalues.

To prove the last statement, we observe that

$$\frac{d}{dt}(\lambda - \mu) = 2(\lambda - \mu)((1 - 2\rho)(\lambda + \mu) - (1 + 2\rho)\nu),$$

$$\frac{d}{dt}(\mu - \nu) = 2(\mu - \nu)((1 - 2\rho)(\mu + \nu) - (1 + 2\rho)\lambda). \qquad \square$$

Remark 4.13. We already proved that the differential operator in the evolution equation of $\mathcal{P}$ is uniformly elliptic if $\rho < 1/2(n-1)$, that is, $\rho < \frac{1}{4}$ in dimension three. Therefore any geometric condition expressed in terms of the eigenvalues is preserved along the RB flow if the cone identified by the condition is closed, convex and preserved by the system (4-8).

By using this method, we can prove:

Proposition 4.14. *Let* $(M, g(t))_{t \in [0, T)}$ *be a compact, three dimensional solution of the RB flow* (1-1). *If* $\rho < \frac{1}{4}$, *then*

(i) *nonnegative Ricci curvature is preserved along the flow*;

(ii) *nonnegative sectional curvature is preserved along the flow*;

(iii) *the pinching inequality* $\mathrm{Ric} \geq \varepsilon \mathrm{R} g$ *is preserved along the flow for any* $\varepsilon \leq \frac{1}{3}$.

Proof.

(i) If $\mathrm{Ric}(g(0)) \geq 0$, then $\mathrm{Ric}_{g(t)} \geq 0$. The eigenvalues of Ric are the pairwise sums of the sectional curvatures. Hence the condition is identified by the cone

$$K_p = \{Q_p : (\mu + \nu)(Q_p) \geq 0\}.$$

The closedness is obvious; in order to see that K_p is convex, we observe that the greatest eigenvalue can be characterized by $\lambda(Q_p) = \max\{Q_p(v, v) : v \in V_p |v|_h = 1\}$. Hence K_p is convex. Then the function $Q_p \mapsto \mu(Q_p) + \nu(Q_p) = \mathrm{tr}(Q_p) - \lambda(Q_p)$ is concave and this implies that its superlevels are convex. By system (4-8) we obtain

$$\frac{d}{dt}(\mu + \nu) = 2\mu^2 + 2\nu^2 + 2\lambda(\mu + \nu) - 4\rho(\mu + \nu)\mathrm{tr}(Q_p).$$

There is the stationary solution corresponding to $\mu(0) = 0 = \nu(0)$. Otherwise, whenever $\mu(t_0) + \nu(t_0) = 0$ with $\mu(t_0) \neq 0$ and $\nu(t_0) \neq 0$, $(d/dt)(\mu + \nu)(t_0) = 2(\mu^2 + \nu^2)(t_0) > 0$, then K is preserved.

(ii) If $\mathrm{Sec}(g(0)) \geq 0$, then $\mathrm{Sec}_{g(t)} \geq 0$. This condition is the nonnegativity of $\mathcal{P}$, already proved in general dimension in Proposition 4.10, identified by the cone $K_p = \{Q_p : \nu(Q_p) \geq 0\}$, which is convex as a superlevel of a concave function. We suppose that $\nu(t_0) = 0$. Then

$$\frac{d}{dt}\nu(t_0) = 2\lambda(t_0)\mu(t_0) \geq 0,$$

since the order between the eigenvalues is preserved and therefore $\lambda(t_0) \geq \mu(t_0) \geq 0$.

(iii) For every $\varepsilon \in \left(0, \frac{1}{3}\right]$, if $\mathrm{Ric}(g(0)) - \varepsilon \mathrm{R}(g(0))g(0) \geq 0$, then $\mathrm{Ric}_{g(t)} - \varepsilon \mathrm{R}_{g(t)}g(t) \geq 0$. Translating in terms of eigenvalues of $\mathcal{P}$, the condition means $\mu(Q_p) + \nu(Q_p) - 2\varepsilon \, \mathrm{tr}(Q_p) \geq 0$; that is, $\lambda(Q_p) \leq (1 - 2\varepsilon)/(2\varepsilon)(\mu(Q_p) + \nu(Q_p))$. Then the right cone is

$$K_p = \{Q_p : \lambda(Q_p) - C(\varepsilon)(\mu(Q_p) + \nu(Q_p)) \leq 0\},$$

where $C(\varepsilon) = (1 - 2\varepsilon)/(2\varepsilon) \in \left[\frac{1}{2}, +\infty\right)$. The defining function is the sum of two convex functions, hence its sublevels are convex. Now, for $C = \frac{1}{2}$, that corresponds to $\varepsilon = \frac{1}{3}$, and we have $\lambda(0) = \mu(0) = \nu(0)$ at each point of M; that is, the initial metric $g(0)$ has constant sectional curvature and this condition is preserved along the flow.

For $C > \frac{1}{2}$, we suppose $\lambda(t_0) = C(\mu(t_0) + \nu(t_0))$, then

$$\frac{d}{dt}(\lambda - C(\mu + \nu))(t_0)$$
$$= 2[\lambda^2 + \mu\nu - C(\mu^2 + \nu^2 + \lambda(\mu + \nu)) - 2\rho \, \mathrm{tr}(Q_p)(\lambda - C(\mu + \nu))](t_0)$$
$$= 2[C^2(\mu(t_0) + \nu(t_0))^2 + \mu(t_0)\nu(t_0) - C(\mu(t_0)^2 + \nu(t_0)^2) - C^2(\mu(t_0) + \nu(t_0))^2]$$
$$\leq (1 - 2C)(\mu(t_0)^2 + \nu(t_0)^2) \leq 0,$$

which completes the proof.   □

4E. *Hamilton–Ivey estimate.* A remarkable property of the three dimensional Ricci flow is the pinching estimate, independently proved by Hamilton [1995] and Ivey [1993], which says that positive sectional curvature dominates negative sectional curvature during the Ricci flow, that is, if the initial metric g_0 has a negative sectional curvature somewhere, the Ricci flow starting at g_0 evolves the scalar curvature towards the positive semiaxis in future times, which means that there will be a greater (in absolute value) positive sectional curvature.

We have generalized the pinching estimate and some consequences for positive values of the parameter ρ. In the same notation used before, let $\lambda \geq \mu \geq \nu$ be the ordered eigenvalues of the curvature operator.

Theorem 4.15 (Hamilton–Ivey estimate). *Let $(M, g(t))$ be a solution of the RB on a compact three-manifold such that the initial metric satisfies the normalizing assumption $\min_{p \in M} v_p(0) \geq -1$. If $\rho \in [0, \frac{1}{6})$, then at any point (p, t) where $v_p(t) < 0$, the scalar curvature satisfies*

$$(4\text{-}9) \qquad \mathrm{R} \geq |v|(\log |v| + \log(1 + 2(1 - 6\rho)t) - 3).$$

Proof. We would like to apply the maximum principle for time-dependent sets in Theorem 4.9. Hence we need to express condition (4-9) in terms of a family of closed, convex, invariant subsets of $S^2(\Lambda^2 V^*)$, where $(V, h(t), D(t))$ is the usual bundle isomorphism of the tangent bundle defined via Uhlenbeck's trick (Section 3B). Moreover, by [Chow et al. 2008, Lemma 10.11], we already know that, for any $t \in [0, T)$, the set

$$K_p(t) = \left\{ \begin{array}{l} Q_p : \mathrm{tr}(Q_p) \geq -\frac{3}{1+2(1-6\rho)t} \quad \text{and if} \quad v(Q_p) \leq -\frac{1}{1+2(1-6\rho)t}, \\ \text{then } \mathrm{tr}(Q_p) \geq |v(Q_p)|(\log |v(Q_p)| + \log(1 + 2(1 - 6\rho)t) - 3) \end{array} \right\}$$

defines a closed invariant subset of $S^2(\Lambda^2 V^*)$. Since, for $\rho \in [0, \frac{1}{6})$, $K(t)$ depends continuously on time, the spacetime track of $K(t)$ is closed in $S^2(\Lambda^2 V^*)$.

Now we show that $K_p(t)$ is convex for every $p \in M$ and $t \in [0, T)$. Following [Chow and Knopf 2004, Lemma 9.5], we consider the map

$$\Phi : S^2(\Lambda^2 V_p^*) \to \mathbb{R}^2, \qquad \Phi(Q_p) = (|v(Q_p)|, \mathrm{tr}(Q_p))$$

Clearly, we have that $Q_p \in K_p(t)$ if and only if $\Phi(Q_p) \in A(t)$, where

$$A(t) = \left\{ \begin{array}{l} (x, y) \in \mathbb{R}^2 : y \geq -\frac{3}{1+2(1-6\rho)t}; \quad y \geq -3x; \\ \text{if } x \geq \frac{1}{1+2(1-6\rho)t}, \text{ then } y \geq x(\log x + \log(1 + 2(1 - 6\rho)t) - 3) \end{array} \right\}$$

is a convex subset of $\mathbb{R}^2$. Then in order to show that $K_p(t)$ is convex it is sufficient to show that the segment between any two algebraic operators in $K_p(t)$ is sent by the map Φ into $A(t)$. Therefore let $Q_p, Q_p' \in K_p(t)$, $s \in [0, 1]$ and $Q_p(s) = s Q_p + (1 - s) Q_p'$. About the first defining condition for $A(t)$, the trace is a linear functional, hence it is obviously fulfilled by $Q_p(s)$, while the second condition is satisfied by any algebraic operator.

The third condition is a bit tricky. If $v(Q_p), v(Q_p') > -1/(1 + (1 - 6\rho)t)$, then the condition is empty for every point of the segment because v is a concave function. By continuity we can assume that $v(Q_p(s)) \leq -1/(1 + (1 - 6\rho)t)$ without loss of generality for every $s \in [0, 1]$, and hence $x(Q_p(s)) = -v(Q_p(s))$ is a convex function and $x(Q_p(s)) \leq s x(Q_p) + (1 - s) x(Q_p')$. On the other hand the second condition implies that $x(Q_p(s)) \geq -y(Q_p(s))/3 = -\frac{1}{3}(s y(Q_p) + (1 - s) y(Q_p'))$. Then $\Phi(Q_p(s))$ belongs to the trapezium of vertices

$$\Phi(Q_p), \left(-\tfrac{1}{3} y(Q_p), y(Q_p)\right), \Phi(Q_p'), \left(-\tfrac{1}{3} y(Q_p'), y(Q_p')\right),$$

which is contained in $A(t)$, as its vertices are, and $A(t)$ is convex.

Now we prove that $K(t)$ is preserved by the system (4-8). By taking the sum of the three equations in the system (see also Remark 4.13) we get

$$\frac{d}{dt}\operatorname{tr}(Q_p) \geq \frac{4}{3}(1-3\rho)\operatorname{tr}(Q_p)^2.$$

By hypothesis, $\nu(Q_p)(0) \geq -1$, hence $\operatorname{tr}(Q_p)(0) \geq -3$ for every $p \in M$ and by integrating the previous inequality,

$$\operatorname{tr}(Q_p)(t) \geq -\frac{3}{1+4(1-3\rho)t} \geq -\frac{3}{1+2(1-6\rho)t},$$

which holds for any $\rho \in \left[0, \frac{1}{6}\right)$.

In order to prove that the second inequality is preserved, too, we consider, for every $p \in M$ such that $\nu(Q_p)(0) < 0$, the function

$$(4\text{-}10) \qquad f(t) = \frac{\operatorname{tr}(Q_p)}{-\nu(Q_p)} - \log(-\nu(Q_p)) - \log(1 + 2(1-6\rho)t),$$

and we compute its derivative along the flow:

$$\begin{aligned}
\frac{d}{dt}f &= \frac{1}{\nu^2}[(-2\nu)(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \lambda\nu + \mu\nu - 2\rho(\lambda+\mu+\nu)^2) \\
&\quad + 2(\lambda+\mu+\nu)(\nu^2 + \lambda\mu - 2\rho\nu(\lambda+\mu+\nu))] \\
&\quad - \frac{2}{\nu}(\nu^2 + \lambda\mu - 2\rho\nu(\lambda+\mu+\nu)) - \frac{2(1-6\rho)}{1+2(1-6\rho)t} \\
&= \frac{2}{\nu^2}[-\nu(\lambda^2+\mu^2+\lambda\mu) + \lambda\mu(\lambda+\mu) - \nu^3 + 2\rho\nu^2(\lambda+\mu+\nu)] - \frac{2(1-6\rho)}{1+2(1-6\rho)t}.
\end{aligned}$$

As in the case of the Ricci flow, it is easy to see that the quantity $-\nu(\lambda^2+\mu^2+\lambda\mu) + \lambda\mu(\lambda+\mu)$ is always nonnegative if $\nu < 0$. In fact, if $\mu > 0$ it is obvious, whereas if $\mu \leq 0$ one has

$$-\nu(\lambda^2 + \mu^2 + \lambda\mu) + \lambda\mu(\lambda+\mu) = (\mu-\nu)(\lambda^2 + \mu^2 + \lambda\mu) - \mu^3 \geq 0.$$

Hence we get

$$(4\text{-}11) \qquad \frac{d}{dt}f(t) \geq -2\nu + 4\rho(\lambda+\mu+\nu) - \frac{2(1-6\rho)}{1+2(1-6\rho)t}.$$

If $\rho \geq 0$, since $\lambda + \mu + \nu \geq 3\nu$, we obtain

$$\frac{d}{dt}f \geq -2(1-6\rho)\left(\nu + \frac{1}{1+2(1-6\rho)t}\right) \geq 0$$

whenever $\nu \leq -1/(1 + 2(1-6\rho)t)$ and $\rho \leq \frac{1}{6}$.

Hence, if (λ, μ, ν) is a solution of system (4-8) in $[0, T)$ with $(\lambda(0), \mu(0), \nu(0)) \in K_p(0)$, we suppose that there is $t_1 > 0$ such that $\nu(t_1) < -1/(1 + 2(1-6\rho)t_1)$. Then

either $v(t) < -1/(1+2(1-6\rho)t)$ for any $t \in [0, t_1]$, or there exists $t_0 < t_1$ such that $v(t_0) = -1/(1+2(1-6\rho)t_0)$ and $v(t) < -1/(1+2(1-6\rho)t)$ for any $t \in (t_0, t_1]$. In the first case, by hypothesis we obtain $f(0) \geq -3$ and $(d/dt)f(t) \geq 0$ for any $t \in [0, t_1]$, therefore $f(t_1) \geq -3$; in the second case $f(t_0) = (\lambda + \mu + v)(t_0)/-v(t_0) \geq -3$ and $(d/dt)f(t) \geq 0$ for any $t \in [t_0, t_1]$, therefore again $f(t_1) \geq -3$, which is equivalent to the second inequality. $\square$

Remark 4.16. The extra term $4\rho(\lambda + \mu + v)$ on the key equation (4-11) requires strong assumptions on the parameter ρ since we have no information on the sign of the trace. However, combining equation (4-11) with Proposition 4.4, we can enlarge the range of ρ to $[0, \frac{1}{4})$, simply by dropping the extra term, nonnegative for ancient solutions and therefore conclude that an ancient solution to the RB flow on a compact three-manifold with bounded scalar curvature has nonnegative sectional curvature for any value of $\rho \in [0, \frac{1}{4})$ (see [Chow and Knopf 2004, Corollary 9.8]).

Proposition 4.17. *Let $(M, g(t))_{t \in (-\infty, 0]}$ be a compact, three dimensional, ancient solution of the RB flow (1-1) with uniformly bounded scalar curvature. If $\rho \in [0, \frac{1}{4})$ then the sectional curvature is nonnegative.*

5. Curvature estimates

5A. *Technical lemmas.* Before proving the curvature estimates for the RB flow, we need some technical results, the first being the following proposition:

Proposition 5.1. *Let $k \in \mathbb{N}$, $p \in [1, +\infty]$ and $q \in [1, +\infty)$. There exists a constant $C(n, k, p, q)$ such that for all $0 \leq j \leq k$ and all tensors T*

$$\|\nabla^j T\|_{r_j} \leq C\|T\|_p^{1-j/k}\|\nabla^k T\|_q^{j/k},$$

where $1/r_j = (1 - j/k)/p + j/k/q$.

To prove this proposition, we need several lemmas.

Lemma 5.2. *Let $p \in [1, +\infty]$, $q \in [1, +\infty)$ and $r \in [2, +\infty)$ such that $2/r = 1/p + 1/q$. There exists a constant $C(n, r)$ such that for all tensors T,*

$$\|\nabla T\|_r^2 \leq C\|T\|_p\|\nabla^2 T\|_q.$$

Proof.

$$\|\nabla T\|_r^r = \int_M \langle \nabla T, |\nabla T|^{r-2} \nabla T \rangle \, d\mu_g$$

$$= - \int_M \langle T, \nabla \left(|\nabla T|^{r-2} \nabla T \right) \rangle \, d\mu_g$$

$$= - \int_M \langle T, (r-2) \nabla^2 T |\nabla T|^{r-3} \nabla T \rangle \, d\mu_g - \int_M \langle T, |\nabla T|^{r-2} \nabla^2 T \rangle \, d\mu_g$$

$$\leq C \int_M |T| |\nabla^2 T| |\nabla T|^{r-2} \, d\mu_g$$

$$\leq C \|T\|_p \|\nabla^2 T\|_q \|\nabla T\|_r^{r-2},$$

using Hölder's inequality with $(r-2)/r + 1/p + 1/q = 1$. This ends the proof of this lemma. $\qquad\square$

Lemma 5.3 [Hamilton 1982, Corollary 12.5]. *Let $k \in \mathbb{N}$. If $f : \{0, \dots, k\} \to \mathbb{R}$ satisfies for all $0 < j < k$*

$$f(j) \leq C f(j-1)^{\frac{1}{2}} f(j+1)^{\frac{1}{2}},$$

where C is a positive constant, then for all $0 \leq j \leq k$,

$$f(j) \leq C^{j(k-j)} f(0)^{1-j/k} f(k)^{j/k}.$$

Proof of Proposition 5.1. We apply Lemma 5.3 with $f(j) = \|\nabla^j T\|_{r_j}$. Since $2/r_j = 1/r_{j-1} + 1/r_{j+1}$, Lemma 5.2 shows that there exists $C(n, k, p, q)$ such that

$$f(j) \leq C f(j-1)^{\frac{1}{2}} f(j+1)^{\frac{1}{2}},$$

and then Lemma 5.3 gives Proposition 5.1, since $r_0 = p$ a $r_k = q$. $\qquad\square$

Lemma 5.4. *For all tensors of the form $S * T$, there exists C depending on the dimension and the coefficients in the expression such that*

$$|S * T| \leq C |S| |T|.$$

Proof. By the Cauchy–Schwarz inequality, $(\operatorname{tr}_g T)^2 = (g^{\alpha\beta} T_{\alpha\beta})^2 \leq n T_{\alpha\beta} T^{\alpha\beta} = n |T|^2$. Then

$$|S * T| \leq C(n) |S \otimes T \otimes g^{\otimes j} \otimes (g^{-1})^{\otimes k} \leq C(n) n^{\frac{j+k}{2}} |S| |T|. \qquad\square$$

Let $k \in \mathbb{N}$, and set, for a tensor T, $F_g(T) = \sum_{j+l=k;\, j,l \geq 0} \nabla^j T * \nabla^l T * \nabla^k T$.

Lemma 5.5. *Let $k \in \mathbb{N}$. Let $p \in [2, +\infty]$ and $q \in [2, +\infty)$ such that $1/p + 2/q = 1$. There exists $C(n, k, p, q, F)$ such that for all tensors T,*

$$\int_M |F_g(T)| \, d\mu_g \leq C \|T\|_p \|\nabla^k T\|_q^2.$$

Proof. Let us consider one term in $F_g(T)$ that can be written $\nabla^j T * \nabla^l T * \nabla^k T$, $j, l \geq 0$. We set

$$\frac{1}{r_j} = \frac{1 - \frac{j}{k}}{p} + \frac{\frac{j}{k}}{q} \quad \text{and} \quad \frac{1}{r_l} = \frac{1 - \frac{l}{k}}{p} + \frac{\frac{l}{k}}{q}.$$

Since $1/r_j + 1/r_j + 1/q = 1$, by Lemma 5.4 and Hölder's inequality we have

$$\int_M |\nabla^j T * \nabla^l T * \nabla^k T| \, d\mu_g \leq C' \int_M |\nabla^j T| |\nabla^l T| |\nabla^k T| \, d\mu_g$$
$$\leq C' \|\nabla^j T\|_{r_j} \|\nabla^l T\|_{r_l} \|\nabla^k T\|_q,$$

Then, by applying Proposition 5.1 to the first two factors, we get

$$\int_M |\nabla^j T * \nabla^l T * \nabla^k T| \, d\mu_g \leq C \|T\|_p \|\nabla^k T\|_q^2.$$

The result follows since $F_g(T)$ is a linear combination of such terms.   □

5B. *Curvature estimates.*

Theorem 5.6. *Assume $\rho < 1/(2(n-1))$. If $g(t)$ is a compact solution of the RB flow for $t \in [0, T)$ such that*

$$\sup_{(x,t) \in M \times [0,T)} |\mathrm{Riem}(x, t)| \leq K,$$

then for all $k \in \mathbb{N}$ there exists a constant $C(n, \rho, k, K, T)$ such that for all $t \in (0, T]$

$$\|\nabla^k \mathrm{Riem}_{g(t)}\|_2^2 \leq \frac{C}{t^k} \sup_{t \in [0,T)} \|\mathrm{Riem}_{g(t)}\|_2^2.$$

Proof. A direct computation gives

$$\frac{\partial}{\partial t} |\mathrm{Riem}|^2 = \Delta(|\mathrm{Riem}|^2) - 2|\nabla \mathrm{Riem}|^2 - 8\rho R_{ij} \nabla^i \nabla^j R + \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem}$$

$$\frac{\partial}{\partial t} R^2 = (1 - 2(n-1)\rho)\Delta(R^2) - 2(1 - 2(n-1)\rho)|\nabla R|^2 + 4R|\mathrm{Ric}|^2 - 4\rho R^3.$$

It follows that

$$\frac{d}{dt} \int_M |\mathrm{Riem}|^2 \, d\mu_g = -2 \int_M |\nabla \mathrm{Riem}|^2 \, d\mu_g - 8\rho \int_M R_{ij} \nabla^i \nabla^j R \, d\mu_g$$
$$+ \int_M \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \, d\mu_g$$

$$\frac{d}{dt} \int_M R^2 d\mu_g = -2(1 - 2(n-1)\rho) \int_M |\nabla R|^2 d\mu_g + \int_M \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \, d\mu_g.$$

Now we want to compute $\int_M \mathrm{R}_{ij} \nabla^i \nabla^j \mathrm{R} \, d\mu_g$. Using the Bianchi identity we have

$$\int_M \mathrm{R}_{ij} \nabla^i \nabla^j \mathrm{R} \, d\mu_g = -\frac{1}{2} \int_M |\nabla \mathrm{R}|^2 \, d\mu_g \, .$$

We conclude that

$$\frac{d}{dt} \int_M |\mathrm{Riem}|^2 \, d\mu_g = -2 \int_M |\nabla \mathrm{Riem}|^2 \, d\mu_g + 4\rho \int_M |\nabla \mathrm{R}|^2 \, d\mu_g$$
$$+ \int_M \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \, d\mu_g$$

and

$$\frac{d}{dt} \int_M \mathrm{R}^2 \, d\mu_g = -2(1 - 2(n-1)\rho) \int_M |\nabla \mathrm{R}|^2 \, d\mu_g + \int_M \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \, d\mu_g.$$

As we did before, a straightforward computation gives:

$$\frac{d}{dt} \int_M |\nabla^k \mathrm{Riem}|^2 \, d\mu_g = -2 \int_M |\nabla^{k+1} \mathrm{Riem}|^2 \, d\mu_g + 4\rho \int_M |\nabla^{k+1} \mathrm{R}|^2 \, d\mu_g$$
$$+ \sum_{j+l=k;\, j,l \geq 0} \int_M \nabla^j \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^k \mathrm{Riem} \, d\mu_g$$

$$\frac{d}{dt} \int_M |\nabla^k \mathrm{R}|^2 \, d\mu_g = -2(1 - 2(n-1)\rho) \int_M |\nabla^{k+1} \mathrm{R}|^2 \, d\mu_g$$
$$+ \sum_{j+l=k;\, j,l \geq 0} \int_M \nabla^j \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^k \mathrm{Riem} \, d\mu_g.$$

Consider

$$\mathcal{A}_k := \int_M |\nabla^k \mathrm{Riem}|^2 \, d\mu_g + \frac{4|\rho|}{(1 - 2(n-1)\rho)} \int_M |\nabla^k \mathrm{R}|^2 \, d\mu_g,$$

and set $f_k(t) := \sum_{j=0}^k (\beta^j t^j / j!) \mathcal{A}_j$, where $\beta := \min(1, 1 - 2(n-1)\rho)$. We have

$$(5\text{-}1) \qquad f_k'(t) = \sum_{j=0}^{k-1} \frac{\beta^j t^j}{j!} (\mathcal{A}_j' + \beta \mathcal{A}_{j+1}) + \frac{\beta^k t^k}{k!} \mathcal{A}_k'.$$

We have by a direct computation, for any j:

$$\mathcal{A}_j' + \beta \mathcal{A}_{j+1} = (-2 + \beta)\|\nabla^{j+1} \mathrm{Riem}\|_2^2 + \left(4\rho - 8|\rho| + \frac{4\beta|\rho|}{1 - 2(n-1)\rho}\right)\|\nabla^{j+1} \mathrm{R}\|_2^2$$
$$+ \sum_{i+l=j;\, i,l \geq 0} \int_M \nabla^i \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^j \mathrm{Riem} \, d\mu_g.$$

We need to estimate

$$\sum_{i+l=j,\,i,l\geq 0} \int_M \nabla^i \text{Riem} * \nabla^l \text{Riem} * \nabla^j \text{Riem}\, d\mu_g.$$

For this we use Lemma 5.5 with $p = +\infty$ and $q = 2$:

$$\sum_{i+l=j,\,i,l\geq 0} \int_M \nabla^i \text{Riem} * \nabla^l \text{Riem} * \nabla^j \text{Riem}\, d\mu_g \leq C\|\text{Riem}\|_\infty \|\nabla^j \text{Riem}\|_2^2.$$

Using Proposition 5.1, with $k = j + 1$ we get

$$\sum_{i+l=k} \int_M \nabla^i \text{Riem} * \nabla^l \text{Riem} * \nabla^j \text{Riem}\, d\mu_g \leq C\|\text{Riem}\|_\infty (\|\text{Riem}\|_2^2)^{\frac{1}{j+1}} (\|\nabla^{j+1}\text{Riem}\|_2^2)^{\frac{j}{j+1}},$$

where $i, l \geq 0$. Now we apply Young's inequality $ab \leq a^p/p + b^q/q$, where

$$a = C\|\text{Riem}\|_\infty (\|\text{Riem}\|_2^2)^{\frac{1}{j+1}}, \quad b = (\|\nabla^{j+1}\text{Riem}\|_2^2)^{\frac{j}{j+1}}$$

and $p = j + 1$, $q = (j + 1)/j$. We use the hypothesis on the boundedness of $\|\text{Riem}\|_\infty$ and we obtain

$$\sum_{i+l=j} \int_M \nabla^i \text{Riem} * \nabla^l \text{Riem} * \nabla^j \text{Riem}\, d\mu_g \leq C'(n, \rho, j, K)\|\text{Riem}\|_2^2 + \|\nabla^{j+1}\text{Riem}\|_2^2,$$

where $i, l \geq 0$. Putting this last inequality in the previous computation, we obtain

$$\begin{aligned}
\mathcal{A}'_j + \beta \mathcal{A}_{j+1} &\leq (-1 + \beta)\|\nabla^{j+1}\text{Riem}\|_2^2 + \left(4\rho - 8|\rho| + \frac{4\beta|\rho|}{1 - 2(n-1)\rho}\right)\|\nabla^{j+1}\text{R}\|_2^2 \\
&\quad + C'(n, \rho, j, K)\|\text{Riem}\|_2^2 \\
&\leq C'(n, \rho, j, K)\|\text{Riem}\|_2^2,
\end{aligned}$$

where we use the facts that $-1 + \beta \leq 0$ and $4\rho - 8|\rho| + 4|\rho|\beta/(1 - 2(n-1)\rho) \leq 0$. The same estimates holds for the last term in equation (5-1), since

$$\mathcal{A}'_k \leq \mathcal{A}'_k + \beta \mathcal{A}_{k+1} \leq C'(n, \rho, k, K)\|\text{Riem}\|_2^2$$

Therefore

$$\begin{aligned}
f'_k(t) &\leq \sum_{j=0}^{k} \frac{\beta^j t^j}{j!} C'(n, \rho, j, K)\|\text{Riem}\|_2^2 \\
&\leq \overline{C}(n, \rho, k, K)\|\text{Riem}\|_2^2 (e^{\beta t} - 1) \leq \widetilde{C}(n, \rho, k, K, T)\|\text{Riem}\|_2^2.
\end{aligned}$$

Since $f_k(0) = \mathcal{A}_0 \le C(\rho, n)\|\text{Riem}\|_2^2$, by integrating the previous inequality we finally get

$$\|\nabla^k\text{Riem}\|_2^2 \le \mathcal{A}_k \le \frac{k!}{\beta^k t^k} f_k(t) \le \frac{\widehat{C}}{t^k}\big[f_k(0) + \widetilde{C}t\|\text{Riem}\|_2^2\big]$$

$$\le \frac{\widehat{C}[C(\rho, n) + \widetilde{C}t]}{t^k}\|\text{Riem}\|_2^2 \le \frac{C}{t^k}\|\text{Riem}\|_2^2,$$

which concludes the proof of the theorem. $\qquad\square$

5C. *Long time existence.* In this section we will prove the following result.

Theorem 5.7. *Assume $\rho < 1/(2(n-1))$. If $g(t)$ is a compact solution of the RB flow on a maximal time interval $[0, T)$, $T < +\infty$, then*

$$\limsup_{t \to T} \max_M |\text{Riem}(\,\cdot\,, t)| = +\infty.$$

Proof. This proof follows exactly the one given by Hamilton for the Ricci flow (see [Hamilton 1982, Section 14]). First of all we observe that, if the Riemann tensor is uniformly bounded as $t \to T$ and $T < +\infty$, then also its L^2-norm is uniformly bounded, because from the previous computations, for $\mathcal{A}_0 = \|\text{Riem}\|_2^2 + 4|\rho|/(1 - 2(n-1)\rho)\|\text{R}\|_2^2$, so we have $\mathcal{A}_0' \le C\mathcal{A}_0$.

Then, by Theorem 5.6, we get, for any $j \in \mathbb{N}$

$$\|\nabla^j\text{Riem}\|_2^2 \le C_j.$$

Now, by using the interpolation inequalities in Proposition 5.1 with $p = \infty$, $q = 2$, we immediately get the estimates

$$\|\nabla^j\text{Riem}\|_{\frac{2k}{j}} \le C_{j,k},$$

for all $j \in \mathbb{N}$ and $k \ge j$. Therefore, by interpolation the same result holds for a generic exponent r, with a constant that depends on j and r.

Now, let $E_j := |\nabla^j\text{Riem}|^2$. Then, for all $r < +\infty$ we have

$$\int_M (|E_j|^r + |\nabla E_j|^r)\,d\mu_g \le C_{j,r}'.$$

Thus, by the Sobolev inequality, if $r > j$, one has

$$\max_M |E_j|^r \le C_t \int_M (|E_j|^r + |\nabla E_j|^r)\,d\mu_g.$$

Notice that the constant C_t depends on the metric $g(t)$, but it does not depend on the derivatives of $g(t)$. Moreover, from [Hamilton 1982, Lemma 14.2], it follows that the metrics are all equivalent. Hence, the constant C_t is uniformly bounded as $t \to T$

and, from the previous estimates, it follows that, if $|\text{Riem}| \leq C$ on $M \times [0, T)]$, for every $j \in \mathbb{N}$ one has

$$\max_M |\nabla^j \text{Riem}| \leq C_j,$$

where the constant C_j depends only on the initial value of the metric and the constant C.

Arguing now as in [Hamilton 1982, Section 14], it follows that the metrics $g(t)$ converge to some limit metric $g(T)$ in the C^∞ topology (with all their time/space ordinary partial derivatives, once written in local coordinates), hence, we can restart the flow with this initial metric $g(T)$, obtaining a smooth flow in some larger time interval $[0, T + \delta)$, in contradiction with the fact that T was the maximal time of smooth existence. This completes the proof of Theorem 5.7. □

Acknowledgements

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GIOVANNI CATINO
DIPARTIMENTO DI MATEMATICA
POLITECNICO DI MILANO
VIA BONARDI 9
I-20133 MILANO
ITALY
giovanni.catino@polimi.it

LAURA CREMASCHI
SCUOLA NORMALE SUPERIORE DI PISA
PIAZZA DEI CAVALIERI 7
I-56126 PISA
ITALY
laura.cremaschi@sns.it

ZINDINE DJADLI
INSTITUT FOURIER
UNIVERSITÉ GRENOBLE
100 RUE DES MATHS
38402 ST MARTIN D'HERES
FRANCE
and
LABORATORIO FIBONACCI
PIAZZA DEI CAVALIERI 7
I-56126 PISA
ITALY
Zindine.Djadli@ujf-grenoble.fr

CARLO MANTEGAZZA
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI
UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II
VIA CINTIA
MONTE S. ANGELO
I-80126 NAPOLI
ITALY
carlo.mantegazza@sns.it

LORENZO MAZZIERI
DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DEGLI STUDI DI TRENTO
VIA SOMMARIVE 14
I-38123 POVO (TN)
ITALY
lorenzo.mazzieri@unitn.it

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