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SOME CLOSURE RESULTS FOR $\mathcal{C}$-APPROXIMABLE GROUPS

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We investigate closure results for $\mathcal{C}$-approximable groups, for certain classes $\mathcal{C}$, of groups with invariant length functions. In particular we prove, each time for certain (but not necessarily the same) classes $\mathcal{C}$ that: (i) the direct product of two $\mathcal{C}$-approximable groups is $\mathcal{C}$-approximable; (ii) the restricted standard wreath product $G \wr H$ is $\mathcal{C}$-approximable when G is $\mathcal{C}$-approximable and H is residually finite; and (iii) a group G with normal subgroup N is $\mathcal{C}$-approximable when N is $\mathcal{C}$-approximable and G/N is amenable. Our direct product result is valid for LEF, weakly sofic and hyperlinear groups, as well as for all groups that are approximable by finite groups equipped with commutator-contractive invariant length functions (considered by A. Thom). Our wreath product result is valid for weakly sofic groups, and we prove it separately for sofic groups. This last result has recently been generalised by Hayes and Sale, who proved that the restricted standard wreath product of any two sofic groups is sofic. Our result on extensions by amenable groups is valid for weakly sofic groups, and was proved by Elek and Szabó (2006) for sofic groups N.

1. Introduction

Our interest in $\mathcal{C}$-approximable groups stems from the fact that, by making an appropriate choice of the class $\mathcal{C}$, the definition of a $\mathcal{C}$-approximable group equates to that of one of a variety of classes of groups currently of interest, including sofic groups, hyperlinear groups, weakly sofic groups, linear sofic groups, and LEF groups. Hence techniques that apply to one such class can often be applied to another. In this article we develop some general techniques to establish some closure properties for many of these classes, specifically for direct products, for wreath products with residually finite groups, and for extensions by amenable groups. We shall refer to closure results in the literature, mostly for specific classes of $\mathcal{C}$-approximable groups; in some cases our proofs have been inspired by the proofs of those. We are grateful to the anonymous referee of the paper for a careful reading and several helpful comments and corrections.

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Our definition of a $\mathcal{C}$-approximable group is taken from [Thom 2012, Definition 1.6] and specialises to the definitions of sofic and hyperlinear groups in [Capraro and Lupini 2015]; we shall discuss some of the alternative definitions later on in this section. Our definition requires the concept of an *invariant length function* on a group K; that is, a map $\ell : K \to [0, 1]$ such that, for all $x, y \in K$:

$$\ell(x) = 0 \iff x = 1, \qquad \ell(x^{-1}) = \ell(x),$$
$$\ell(xy) \le \ell(x) + \ell(y), \quad \ell(xyx^{-1}) = \ell(y).$$

Every group admits the trivial length function ℓ_0 defined by $\ell_0(x) = 1$ if $x \ne 1$, $\ell_0(1) = 0$, and may admit many others. The Hamming norm, which computes the proportion of points moved by a permutation of a finite set, gives an invariant length function for finite symmetric groups.

In the following definition $\mathcal{C}$ is understood to be a set of pairs, each pair consisting of a group K together with an invariant length function ℓ_K on K; so the same group may occur in $\mathcal{C}$ with more than one length function. For a group K, the statement $K \in \mathcal{C}$ means that K is the group in at least one such pair.

Definition 1.1. (1) For a group G, a map $\delta : G \to \mathbb{R}$ (for which we write δ_g rather than $\delta(g)$) is a *weight function for* G if $\delta_1 = 0$ and $\delta_g > 0$ for all $1 \ne g \in G$.

(2) Let G be a group with weight function δ, let K be a group with invariant length function ℓ_K, let $\epsilon > 0$, and let F be a finite subset of G. Then the map $\phi : G \to K$ is an $(F, \epsilon, \delta, \ell_K)$-*quasihomomorphism* if

- $\phi(1) = 1$,
- $\forall g, h \in F$, $\ell_K(\phi(gh)\phi(h)^{-1}\phi(g)^{-1}) \le \epsilon$, and
- $\forall g \in F \setminus \{1\}$, $\ell_K(\phi(g)) \ge \delta_g$.

(3) Let $\mathcal{C}$ be a class of groups with associated invariant length functions. Then a group G is $\mathcal{C}$-*approximable* if it has a weight function δ, such that, for each $\epsilon > 0$ and for each finite subset F of G, there exists an $(F, \epsilon, \delta, \ell_K)$-quasihomomorphism $\phi : G \to K$ for some $(K, \ell_K) \in \mathcal{C}$.

Since these conditions cannot possibly be satisfied if $\delta_g > 1$ for some $g \in G$, we shall always assume that $\delta_g \le 1$.

In particular, sofic groups are precisely those groups that are $\mathcal{C}$-approximable with respect to the class $\mathcal{C}$ of finite symmetric groups with length function defined by the Hamming norms, and with weight functions of the form $\delta_g = c$ for all $1 \ne g \in G$, for some fixed constant $c > 0$; see [Pestov and Kwiatkowska 2009, Theorem 5.2].

The (normalised) Hilbert–Schmidt norm on the set of $n \times n$ complex matrices $A = (a_{ij})$ is defined by

$$\|(a_{ij})\|_{\mathrm{HS}_n} := \sqrt{\tfrac{1}{n}\textstyle\sum_{i,j}|a_{ij}|^2} = \sqrt{\tfrac{1}{n}\mathrm{Tr}(A^*A)}.$$

The hyperlinear groups are precisely those groups that are $\mathcal{C}$-approximable with respect to the class $\mathcal{C}$ of finite-dimensional unitary groups with length function defined by $\ell(g) = \frac{1}{2}\|g - I_n\|_{\mathrm{HS}_n}$, and with the same weight functions as for sofic groups; see [Pestov and Kwiatkowska 2009, Theorem 4.2]. Furthermore, weakly sofic groups, linear sofic groups and LEF groups can all be defined as $\mathcal{C}$-approximable groups, where the classes $\mathcal{C}$ are (respectively) the class $\mathcal{F}$ of all finite groups equipped with all associated invariant length functions, the groups $\mathrm{GL}_n(\mathbb{C})$ equipped with the norm $\ell(g) = \frac{1}{n}\mathrm{rk}(I_n - g)$ [Arzhantseva and Păunescu 2017], and the finite groups equipped with the trivial length function. We refer the reader to [Arzhantseva and Gal 2013; Ciobanu et al. 2014; Elek and Szabó 2006; 2011; Păunescu 2011; Stolz 2013] for a number of closure results involving various of these classes of groups.

Following [Thom 2012] we say that an invariant length function $\ell : K \to [0, 1]$ is *commutator-contractive* if it satisfies the condition

$$\ell([x, y]) \le 4\ell(x)\ell(y) \quad \forall x, y \in K.$$

Note that the trivial length function is commutator-contractive. Let $\mathcal{F}_C$ be the class of all finite groups, each equipped with all commutator-contractive length functions. The main result of [Thom 2012] is that Higman's group [1951] is not $\mathcal{F}_C$-approximable. This group is widely seen as a candidate for a first example of a nonsofic group.

There are many variations in the literature of the definition of a $\mathcal{C}$-approximable group, not all of which are believed to be equivalent in general to our basic definition, although the paucity of known examples of groups that are not $\mathcal{C}$-approximable makes it difficult to prove their inequivalence.

Some definitions, such as [Glebsky 2015, Definition 2] and [Stolz 2013, §2] allow invariant length functions to take values in $[0, \infty)$ rather than in $[0, 1]$. This does not affect the classes of sofic, hyperlinear, linear sofic and LEF groups, since the length functions used in these classes all have range $[0, 1]$. It is also easily seen that the class of weakly sofic groups is not changed by this variant since, if a group is weakly sofic using length functions with range $[0, \infty)$, and ℓ_K is such a length function on a finite group K, then simply by replacing $\ell_K(g)$ by the new length function $\max(\ell_K(g), 1)$, we can show that G is weakly sofic using length functions with range $[0, 1]$. So this variation in the range of permissible length functions does not appear to us to be significant.

The more substantial variants involve the condition

$$\forall g \in F,\ \ \ell_K(\phi(g)) \ge \delta_g$$

in the definition of $\mathcal{C}$-approximability. These are discussed in Section 2 of [Stolz 2013]. The group G is said to have the *discrete $\mathcal{C}$-approximation property* if the weight function for G can be chosen to be constant on all nonidentity elements. It is said to have the *strong discrete $\mathcal{C}$-approximation property* if the condition above is replaced by

$$\forall g \in F,\ \ \ell_K(\phi(g)) \geq \operatorname{diam}(K) - \epsilon,$$

where $\operatorname{diam}(K)$ is defined to be $\sup\{\ell_K(x) : x \in K\}$, and ϵ is as in Definition 1.1(3). By choosing the weight function $\delta_g = \operatorname{diam}(G)/2$ for all $g \in G \setminus \{1\}$, we see immediately that the strong discrete $\mathcal{C}$-approximation property implies the discrete $\mathcal{C}$-approximation property, which clearly implies that G is $\mathcal{C}$-approximable using our definition. But the converse implications are not clear, and may not hold in general.

The definition given for sofic groups in [Elek and Szabó 2006] enforces the strong discrete approximation property. But it is shown in [Capraro and Lupini 2015, Exercise II.1.8] that, for this class, any $\mathcal{C}$-approximable group has the strong discrete $\mathcal{C}$-approximation property.

It is proved in [Arzhantseva and Păunescu 2017, Proposition 5.13] that linearly sofic groups have the discrete $\mathcal{C}$-approximation property, but it appears to be unknown whether they have the strong discrete $\mathcal{C}$-approximation property.

Hyperlinear groups do not have the strong $\mathcal{C}$-approximation property, and we are grateful to the referee for pointing this out to us. The diameter of the unitary group $\mathcal{U}(n)$ with length function defined as above by $\ell(g) = \frac{1}{2}\|g - I_n\|_{\mathrm{HS}_n}$ is 1. By using the identity

$$\|g-h\|_{\mathrm{HS}_n}^2 + \|g+h\|_{\mathrm{HS}_n}^2 = 4$$

for $g, h \in \mathcal{U}(n)$ and putting $h = I_n$, we see that, if $1 - \ell(g)$ is small, then g is close to $-I_n$ with respect to the Hilbert–Schmidt metric. So if $1 - \ell(g_1)$ and $1 - \ell(g_2)$ are both small, then $g_1 g_2$ is close to I_n and hence $\ell(g_1 g_2)$ is close to 0. It follows that a hyperlinear group with the strong discrete $\mathcal{C}$-approximation property must be finite with order at most 2.

For hyperlinear groups, it is true that, for any finite $F \subseteq G$ and $\epsilon > 0$, there exists an approximately multiplicative map $\phi : G \to \mathcal{U}(n)$ for which $|\mathrm{Tr}(\phi(g))/n| < \epsilon$ for all $g \in F \setminus \{1\}$. This was first proved in [Elek and Szabó 2005] using ideas introduced in [Rădulescu 2008].

It is not difficult to show that the classes of $\mathcal{F}$-approximable (i.e., weakly sofic) and $\mathcal{F}_C$-approximable groups both have the strong discrete $\mathcal{C}$-approximation property. For a finite subset F of a group G in one of these two classes, and $\epsilon > 0$, let $c = \min\{\delta_g : g \in F\}$, and let $\phi : G \to K$ be an $(F, c\epsilon, \delta, \ell_K)$-quasihomomorphism. Then, by replacing ℓ_K by the length function $\ell'_K(x) := \min(\ell_K(x)/c, 1)$, which is commutator-contractive if ℓ_K is, we see that ϕ is an $(F, \epsilon, \delta, \ell'_K)$-quasihomomorphism for which $\ell'_K(\phi(g)) = 1$ for all $g \in F$, so G has the strong discrete $\mathcal{C}$-approximation property.

We prove our closure results for direct products, wreath products, and extensions by amenable groups in Sections 2, 3 and 4, and 5, respectively. To prove the last of these, on extensions of $\mathcal{C}$-approximable groups N by amenable groups, we need to assume that the group N has the discrete $\mathcal{C}$-approximation property. For each of our closure results, it is straightforward to show that, if the groups that are assumed to be $\mathcal{C}$-approximable have the discrete or the strong discrete $\mathcal{C}$-approximation property, then so does the group G that is proved to be $\mathcal{C}$-approximable.

Concerning free products, we note that it is proved in [Elek and Szabó 2006, Theorem 1], [Stolz 2013, Theorem 5.6] and [Popa 1995; Voiculescu 1998], respectively, that the classes of sofic, linear sofic, and hyperlinear groups are closed under free products; further it is proved in [Brown et al. 2008] that free products of hyperlinear groups amalgamated over amenable subgroups are hyperlinear. We thank the referee for bringing to our attention the results for hyperlinear groups. We are unaware of any corresponding results for weakly sofic groups, and our efforts to prove such a result have so far been unsuccessful.

2. The direct product result

In order to state and prove our closure result for direct products of $\mathcal{C}$-approximable groups, we must construct an appropriate invariant length function for the direct product of two groups in $\mathcal{C}$. Suppose that (J, ℓ_J), $(K,\ell_K) \in \mathcal{C}$. Then, for $p \in \mathbb{N} \cup \{\infty\}$, we define the functions $L^p_{\ell_J,\ell_K} : J \times K \to [0, 1]$ by

$$L^p_{\ell_J,\ell_K}(x, y) := \sqrt[p]{\tfrac{1}{2}(\ell_J(x)^p + \ell_K(y)^p)}, \quad p \in \mathbb{N},$$

and $L^\infty_{\ell_J,\ell_K}(x, y) := \max(\ell_J(x), \ell_K(y))$. We write just $L^p(x, y)$ when there is no ambiguity.

Note that $L^p(x, y) \le L^\infty(x, y) \le 1$ for all $p \ge 1$.

It follows immediately from Minkowski's inequality (basically the triangle inequality for the L^p norm) that L^p satisfies the rule

$$L^p(x_1x_2, y_1y_2) \le L^p(x_1, y_1) + L^p(x_2, y_2),$$

and hence is an invariant length function on $J \times K$. As we shall see below, we can use L^p (for some choice of p) to deduce the closure of $\mathcal{C}$-approximable groups under direct products provided that $(J \times K, L^p) \in \mathcal{C}$.

Theorem 2.1. *Let $\mathcal{C}$ be a class of groups with associated invariant length functions and suppose that, for some fixed $p \in \mathbb{N} \cup \{\infty\}$, and for any groups $J, K \in \mathcal{C}$,*

$$(J, \ell_J), (K, \ell_K) \in \mathcal{C} \Rightarrow (J \times K, L^p) \in \mathcal{C}.$$

Then the direct product $G \times H$ of two $\mathcal{C}$-approximable groups G and H is also $\mathcal{C}$-approximable.

Proof. Suppose that $\mathcal{C}$, p satisfy the conditions of the theorem.

Let G and H be $\mathcal{C}$-approximable with associated weight functions δ^G and δ^H. We define the weight function $\delta^{G\times H}$ by

$$\delta^{G\times H}(g,h) := \sqrt[p]{\tfrac{1}{2}(\delta^G(g)^p + \delta^H(h)^p)}.$$

Now suppose that $\epsilon > 0$ is given, and let F be a finite subset of $G \times H$. Then we can find finite subsets $F_G \subseteq G$, $F_H \subseteq H$ such that $F \subseteq F_G \times F_H$, pairs $(J, \ell_J), (K, \ell_K) \in \mathcal{C}$, an $(F_G, \epsilon, \delta^G, \ell_J)$-quasihomomorphism $\phi_G : G \to J$, and an $(F_H, \epsilon, \delta^H, \ell_K)$-quasihomomorphism $\phi_H : H \to K$.

We define $\phi : G \times H \to M := J \times K$ by $\phi(g,h) := (\phi_G(g), \phi_H(h))$ and $\ell_M(x,y) := L^p(x,y)$.

We verify easily that, for $(g_1,h_1), (g_2,h_2) \in F$, and hence $g_1, g_2 \in F_G$ and $g_2, h_2 \in F_H$,

$$\begin{aligned}&\ell_M(\phi(g_1g_2,h_1h_2)\phi(g_2,h_2)^{-1}\phi(g_1,h_1)^{-1})\\ &\qquad = L^p(\phi_G(g_1g_2)\phi_G(g_2)^{-1}\phi_G(g_1)^{-1}, \phi_H(h_1h_2)\phi_H(h_2)^{-1}\phi_H(h_1)^{-1}) \le \epsilon,\end{aligned}$$

and the other conditions are similarly verified. □

We can apply the result to deduce closure under direct products for the classes of weakly sofic groups, LEF groups, hyperlinear groups, linear sofic groups and Thom's class [2012] of $\mathcal{F}_C$-approximable groups.

For weakly sofic groups, the condition holds for any p, and for LEF groups it holds for $p = \infty$.

When ℓ_J, ℓ_K are Hilbert–Schmidt norms in the same dimension n, the function L^2 matches the Hilbert–Schmidt norm in dimension $2n$; observing that whenever G maps by a quasihomomorphism to a linear group in dimension m it also maps to a linear group in dimension rm, for any r, via a quasihomomorphism with the same parameters (the composite of the original quasihomomorphism and a diagonal map), we see that in essence the theorem applies with $p = 2$ to prove closure under direct products for the class of hyperlinear groups. Similarly it applies when $p = 1$ to prove closure under direct products for the class of linear sofic groups.

But for Hamming norms ℓ_J, ℓ_K, the function $L^p_{\ell_J,\ell_K}$ is not a Hamming norm, and hence we cannot deduce the closure of the class of sofic groups under direct products from this result.

Of course all of these specific closure results are already known, and the corresponding result for sofic groups is proved in [Elek and Szabó 2006].

The following lemma together with Theorem 2.1 shows that the class of $\mathcal{F}_C$-approximable groups is closed under direct products.

Lemma 2.2. *Suppose that the groups J, K have commutator-contractive length functions $\ell_J : J \to [0, 1]$, $\ell_K : K \to [0, 1]$. Then L^∞, as defined above, is a commutator-contractive length function for their direct product.*

Proof. Let $(g_1, h_1), (g_2, h_2) \in G \times H$. Then

$$\begin{aligned} L^\infty([(g_1, h_1), (g_2, h_2)]) &= L^\infty([g_1, g_2], [h_1, h_2]) \\ &= \max(l_J([g_1, g_2]), l_K([h_1, h_2])) \\ &\le \max(4l_J(g_1)l_J(g_2), 4l_K(h_1)l_K(h_2)) \\ &\le 4 \max(l_J(g_1), l_K(h_1)) \max(l_J(g_2), l_K(h_2)) \\ &= 4L^\infty(g_1, h_1)L^\infty(g_2, h_2). \end{aligned}$$ □

This result does not hold in general for L^p with $p \in [1, \infty)$.

3. The wreath product result

By definition, the restricted standard wreath product $W = G \wr H$ of two groups G, H is a semidirect product $H \ltimes B$. The *base group* B of W is the direct product of copies of G, one for each $h \in H$, and is viewed as the set of all functions $b : H \to G$ with finite support (that is, with $b(h)$ trivial for all but finitely many $h \in H$). Elements of B are multiplied componentwise; that is, $b_1b_2(h) = b_1(h)b_2(h)$ for $b_1, b_2 \in B$, $h \in H$. For $b \in B$, we denote by b^{-1} the function in B defined by $b^{-1}(h) = b(h)^{-1}$. The (right) action of H on B is defined by the rule $b^h(h') = b(h'h^{-1})$; we often abbreviate $(b^h)^{-1} = (b^{-1})^h$ as b^{-h}. So the elements of W have the form hb with $h \in H,\ b \in B$, and $(h_1b_1)(h_2b_2) = h_1h_2b_1^{h_2}b_2$, while $(h, b)^{-1} = (h^{-1}, b^{-h^{-1}})$.

To let us state and prove our closure result for wreath products of $\mathcal{C}$-approximable groups, we need to construct an appropriate invariant length function for the wreath product $J \wr X$ of a group $J \in \mathcal{C}$ by a finite group X.

Where B' is the base group of $J \wr X$, we define $\ell_J^X : J \wr X \to [0, 1]$ as follows. For $b' \in B'$, we put

$$\ell_L^X(b') = \max_{x \in X} \ell_J(b'(x)),$$

and then, for $x \neq 1$, put

$$\ell_J^X(xb') = 1.$$

It is straightforward to verify that ℓ_J^X is an invariant length function.

Theorem 3.1. *Let $\mathcal{C}$ be a class of groups with associated invariant length functions and suppose that, for all $(J, \ell_J) \in \mathcal{C}$ and all finite groups X, the wreath product $(J \wr X, \ell_J^X)$ is in $\mathcal{C}$. Suppose the group G is $\mathcal{C}$-approximable and the group H is residually finite. Then the restricted standard wreath product $G \wr H$ is $\mathcal{C}$-approximable.*

Proof. Suppose that G is $\mathcal{C}$-approximable with associated weight function δ, and that H is residually finite, and let $W = G \wr H$ be the restricted standard wreath product. Let B be the base group.

We define the weight function $\beta : W \to \mathbb{R}$ as follows:

$$\beta_{hb} = \begin{cases} 1 & \text{if } h \neq 1, \\ \max_{k \in H} \delta_{b(k)} & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$ be given, and let $F = \{h_i b_i : 1 \leq i \leq r\}$ be a finite subset of W. Our aim is to find $(K, \ell_K) \in \mathcal{C}$ and an $(F, \epsilon, \beta_W, \ell_K)$-quasihomomorphism $\psi : W \to K$.

Let E be a finite subset of H that contains

(i) h_i for $1 \leq i \leq r$;

(ii) all $h \in H$ with $b_j(h) \neq 1$ for some j with $1 \leq j \leq r$; and

(iii) all $h \in H$ with $b_j(hh_i^{-1}) \neq 1$ for some i, j with $1 \leq i \leq r,\ 1 \leq j \leq r$.

Choose $N \trianglelefteq H$ with H/N finite such that the images in H/N of the elements of E are all distinct and the images of $E \setminus \{1\}$ are nontrivial.

Let $D = \{b_j(h) : 1 \leq j \leq r,\ h \in H\}$. Then D is a finite subset of G so, by our definition of $\mathcal{C}$-approximability, for a given $\epsilon > 0$, there exists $(J, \ell_J) \in \mathcal{C}$, and a $(D, \epsilon, \delta, \ell_J)$-quasihomomorphism $\phi : G \to J$.

We will approximate W by $K := J \wr (H/N)$, and let ℓ_K be the length function $\ell_J^{H/N}$ defined above. Let B' be the base group of K, that is, the group of finitely supported functions from H/N to J.

We define $\psi : W \to K$ as follows. Suppose that $b \in B$, and $h, k \in H$. Note that our choice of N ensures that $E \cap kN$ is either empty or consists of a single element $k' \in kN$. We let $\psi(hb) := \bar{h}\hat{b}$, where we write $\bar{h}$ for hN and $\hat{b} : H/N \to J$ is defined by the rule

$$\hat{b}(kN) = \begin{cases} 1 & \text{when } E \cap kN = \varnothing, \\ \phi(b(k')) & \text{when } E \cap kN = \{k'\}. \end{cases}$$

We claim that ψ has the appropriate properties. Certainly $\psi(1) = 1$.

We first verify the required lower bound on $\ell_K(\psi(hb))$ for elements $hb \in F$. If $h \neq 1$ then our choice of N ensures that $\bar{h} \neq 1$, and so $\ell_K(\psi(hb)) = 1 = \beta_{hb}$.

If $h = 1$, then (where the maximum of an empty set of numbers in $[0, 1]$ is defined to be 0),

$$\begin{aligned} \ell_K(\psi(hb)) &= \ell_K(\psi(b)) = \ell_K(\hat{b}) \\ &= \max_{kN \in H/N\,:\,\{k'\} := kN \cap E \neq \varnothing} \ell_J(\phi(b(k'))) \\ &= \max_{k' \in E} \ell_J(\phi(b(k'))) \\ &\geq \max_{k' \in E} \delta_{b(k')} = \max_{k' \in H} \delta_{b(k')} = \beta_b. \end{aligned}$$

The equality of the two maxima in the final line follows from the definition of E, which ensures that $b(k)=1$ for any $k\in H\setminus E$ and hence that, for such k, $\delta_{b(k)}=0$.

It remains to show that, for $h_ib_i, h_jb_j\in F$,

$$l_K(\psi(h_ib_ih_jb_j)(\psi(h_ib_i)\psi(h_jb_j))^{-1})\le\epsilon.$$

We have

$$\psi(h_ib_ih_jb_j)=\psi(h_ih_jb_i^{h_j}b_j)=\overline{h_ih_j}\;\widehat{b_i^{h_j}b_j},$$

and

$$\psi(h_ib_i)\psi(h_jb_j)=(\bar{h}_i\hat{b}_i)(\bar{h}_j\hat{b}_j)=\bar{h}_i\bar{h}_j\hat{b}_i^{\bar{h}_j}\hat{b}_j.$$

Since l_K is invariant under conjugation, the length we need is that of the element

$$b':=\widehat{b_i^{h_j}b_j}\,\hat{b}_j^{-1}\big(\hat{b}_i^{\bar{h}_j}\big)^{-1}$$

of B'. By definition, $\ell_K(b')=\max_{kN\in H/N}\ell_J(b'(kN))$. So choose a coset kN. We want to bound $\ell_J(b'(kN))$ for each such choice. We have

$$\begin{aligned}
b'(kN)&=\widehat{b_i^{h_j}b_j}(kN)(\hat{b}_j(kN))^{-1}(\hat{b}_i^{\bar{h}_j}(kN))^{-1}\\
&=\widehat{b_i^{h_j}b_j}(kN)(\hat{b}_j(kN))^{-1}(\hat{b}_i(kh_j^{-1}N))^{-1}\\
&=\begin{cases}(\hat{b}_i(kh_j^{-1}N))^{-1} & \text{if } kN\cap E=\varnothing, \quad (1)\\ \phi(b_i(k'h_j^{-1})b_j(k'))(\phi(b_j(k')))^{-1}(\hat{b}_i(kh_j^{-1}N))^{-1} & \text{if } kN\cap E=\{k'\}, \quad (2)\end{cases}
\end{aligned}$$

since in case (1) we have $\widehat{b_i^{h_j}b_j}(kN)=\hat{b}_j(kN)=1$, and in case (2), we have $\widehat{b_i^{h_j}b_j}(kN)=\phi((b_i^{h_j}b_j)(k'))=\phi(b_i(k'h_j^{-1})b_j(k'))$, and $\hat{b}_j(kN)=\phi(b_j(k'))$.

When $E\cap kh_j^{-1}N=\varnothing$, we have $\hat{b}_i(kh_j^{-1}N)=1$. In that case, by the definition of E, we also have $b_i(k'h_j^{-1})=1$ and so, in both case (1) and case (2), we deduce that $b'(kN)=1$ and $\ell_J(b'(kN))=0$.

Otherwise $E\cap kh_j^{-1}N$ is nonempty, and its single element is equal to $k''h_j^{-1}$, for some $k''\in kN$.

Suppose first that $b_i(k''h_j^{-1})=1$, and hence again we have $\hat{b}_i(kh_j^{-1}N)=1$. If we are in case (2) then we must also have $b_i(k'h_j^{-1})=1$, since if $b_i(k'h_j^{-1})\ne 1$, then condition (ii) of the definition of E gives $k'h_j^{-1}\in E$, and so $k'=k''$, contradicting $b_i(k''h_j^{-1})=1$. Then, just as above, we see that in both cases (1) and (2) we again get $b'(kN)=1$ and $\ell_J(b'(kN))=0$.

Otherwise $b_i(k''h_j^{-1})\ne 1$ and condition (iii) of the definition of E gives $k''\in E$ and hence we are in case (2) with $k'=k''$. Then

$$b'(kN)=\phi(b_i(k'h_j^{-1})b_j(k'))\phi(b_j(k'))^{-1}\phi(b_i(k'h_j^{-1}))^{-1}.$$

Since ϕ was assumed to be a $(D,\epsilon,\delta,\ell_J)$-quasihomomorphism, $\ell_J(b'(kN))\le\epsilon$ and, since this is true for all $kN\in H/N$, we get $\ell_K(b')\le\epsilon$ as required. □

The conditions of the theorem clearly hold for the class $\mathcal{F}$, as well as for finite groups equipped with the trivial length function, and hence the classes of weakly sofic and LEF groups are both closed under restricted wreath products with residually finite groups. The following lemma together with Theorem 2.1 shows that the class of $\mathcal{F}_C$-approximable groups is also closed under restricted wreath products with residually finite groups.

Lemma 3.2. *Let J be a group equipped with an invariant function ℓ_J. If ℓ_J is commutator-contractive, then so is ℓ_J^X, for any finite group X.*

Proof. We consider the commutator of two elements x_1b_1 and x_2b_2 in J.

First suppose that x_1 and x_2 are both nontrivial. Then $\ell_J^X(x_1b_1) = \ell_J^X(x_2b_2) = 1$, and so the inequality holds trivially.

Now suppose that $x_1 = x_2 = 1$. Then

$$\begin{aligned}
\ell_J^X([b_1, b_2]) &= \max\nolimits_{x\in X} \ell_J([b_1, b_2](x)) \\
&= \max\nolimits_{x\in X} \ell_J([b_1(x), b_2(x)]) \\
&\le 4 \max\nolimits_{x\in X} \ell_J(b_1(x))\ell_J(b_2(x)) \\
&\le 4 \max\nolimits_{x\in X} \ell_J(b_1(x)) \max\nolimits_{y\in X} \ell_J(b_2(y)) \\
&= 4\ell_J^X(b_1)\ell_J^X(b_2).
\end{aligned}$$

Finally suppose that $x_1 = 1,\ x_2 \ne 1$ (the other case is very similar). Then

$$\begin{aligned}
\ell_J^X([b_1, x_2b_2]) &= \ell_J^X(b_1^{-1}b_2^{-1}x_2^{-1}b_1x_2b_2) \\
&= \ell_J^X(b_1^{-1}b_2^{-1}b_1^{x_2}b_2) \\
&= \max\nolimits_{x\in X} \ell_J(b_1(x)^{-1}b_2(x)^{-1}b_1^{x_2}(x)b_2(x)) \\
&= \max\nolimits_{x\in X} \ell_J(b_1(x)^{-1}b_2(x)^{-1}b_1(xx_2^{-1})b_2(x)) \\
&\le \max\nolimits_{x\in X}(\ell_J(b_1(x)^{-1}) + \ell_J(b_2(x)^{-1}b_1(xx_2^{-1})b_2(x))) \\
&= \max\nolimits_{x\in X}(\ell_J(b_1(x)^{-1}) + \ell_J(b_1(xx_2^{-1}))) \\
&\le \max\nolimits_{x\in X}(\ell_J(b_1(x)^{-1})) + \max\nolimits_{y\in X}(\ell_J(b_1(y))) \\
&\le 2 \max\nolimits_{x\in X}(\ell_J(b_1(x)^{-1}) = 2\ell_J^X(b_1).
\end{aligned}$$

□

4. The wreath product result for sofic groups

We prove now the corresponding result for sofic groups. For this, we are not free to choose our own norm function on the wreath product, but we must use the Hamming distance norm. The proof is nevertheless very similar in structure to that of Theorem 3.1. We use the definition of sofic groups given in [Elek and Szabó 2006] where, rather than having a weight function on the group G, we require that, for finite $F \subseteq G$, the proportion of moved points of elements of $F \setminus \{1\}$ in an (F, ϵ)-quasiaction of G on a finite set is at least $1-\epsilon$.

We note that this result has recently been generalised by Hayes and Sale [2016], who proved that the restricted standard wreath product of any two sofic groups is sofic.

Theorem 4.1. *The restricted standard wreath product $G \wr H$ of a sofic group G and a residually finite group H is sofic.*

Proof. Assume that G is sofic and H is residually finite, and let $W = G \wr H$ be the restricted standard wreath product. So, as in the proof of Theorem 3.1, W is the semidirect product of its base group B by H.

Let $F = \{h_i b_i : 1 \le i \le r\}$ be a finite subset of W. Then, for a given $\epsilon > 0$, we need to find an (F, ϵ)-quasiaction of W on some finite set Y.

We define the finite subset E of H, the normal subgroup N of H, and the finite subset D of G exactly as in the proof of Theorem 3.1. So, in particular, for any $k \in H$, $E \cap kN$ is either empty or consists of a single element $k' \in kN$. Let $m = |H/N|$.

Then, by [Elek and Szabó 2006, Lemma 2.1], for a given $\epsilon > 0$, there is a $(D, \epsilon/m)$-quasiaction $\phi : G \to \mathrm{Sym}(X)$ of G on some finite set X, and we may assume that $\phi(1) = 1$. Since we can choose both m and X to be arbitrarily large for given D and ϵ, we may assume that $|X|^{-m/2} < \epsilon$.

Let $Y = X^{H/N}$ be the set of functions $\delta : H/N \to X$. So $|Y| = |X|^m$. We define $\psi : W \to \mathrm{Sym}(Y)$ as follows. (The image of ψ is contained in the primitive wreath product of $\mathrm{Sym}(X)$ and H/N, as defined in [Dixon and Mortimer 1996, §2.6].)

For $b \in B,\ h, k \in H$, let $\delta^{\psi(hb)}(kN) := \delta(kh^{-1}N)^{\tau(b,k)}$, where

$$\tau(b,k) := \begin{cases} 1 & \text{when } E \cap kN = \varnothing, \\ \phi(b(k')) & \text{when } E \cap kN = \{k'\}. \end{cases}$$

We claim that ψ is an (F, ϵ)-quasiaction of W on Y. Observe first that $\psi(1) = 1$.

We check next that, for each $h_i b_i \in F \setminus \{1\}$, $\psi(h_i b_i)$ is $(1-\epsilon)$-different from 1. If $h_i \ne 1$ then, by assumption, $h_i \notin N$, so $kh_i^{-1}N \ne kN$ for all $kN \in H/N$. So, if $\delta \in Y$ is a fixed point of $\psi(h_i b_i)$, then the value of $\delta(kN)$ is uniquely determined by that of $\delta(kh_i^{-1}N)$ for each $kN \in H/N$, so the proportion of fixed points is at most $|X|^{m/2}/|X|^m = |X|^{-m/2}$, which we assumed to be less than ϵ.

If, on the other hand, $h_i = 1$ and $b_i \ne 1$, then there exists $h \in E$ with $b_i(h) \ne 1$. Now an element $\delta \in Y$ is fixed by $\psi(h_i b_i) = \psi(b_i)$ if and only if $\delta(kN)$ is fixed by $\tau(b, k)$ for all $kN \in H/N$. Hence, in particular, for a fixed point δ, we have $\delta(hN) = \delta(hN)^{\tau(b_i,h)}$, and so $\delta(hN)$ is a fixed point of $\tau(b_i, h) = \phi(b_i(h))$. Since the proportion of such points in X is, by assumption, at most ϵ, the same is true for $\psi(b_i)$.

Finally we need to verify that $\psi(h_i b_i)\psi(h_j b_j)$ is ϵ-similar to $\psi(h_i h_j b_i^{h_j} b_j)$ for each i, j with $1 \le i, j \le r$; that is, that the two permutations agree on at least a proportion $1-\epsilon$ of the points.

Now

$$\delta^{\psi(h_i b_i)\psi(h_j b_j)}(kN) = (\delta^{\psi(h_i b_i)}(kh_j^{-1}N))^{\tau(b_j,k)} = \delta(kh_j^{-1}h_i^{-1}N)^{\tau(b_i,kh_j^{-1})\tau(b_j,k)},$$

and

$$\delta^{\psi(h_i h_j b_i^{h_j} b_j)}(kN) = \delta(kh_j^{-1}h_i^{-1}N)^{\tau(b_i^{h_j} b_j,k)},$$

so we need to compare $\tau(b_i, kh_j^{-1})\tau(b_j, k)$ with $\tau(b_i^{h_j} b_j, k)$.

The argument is very similar to that in the analogous part of the proof of Theorem 3.1. We are in one of two cases. Either

(1) $E \cap kN = \varnothing$, in which case $\tau(b_j, k) = \tau(b_i^{h_j} b_j, k) = 1$, or

(2) $E \cap kN = \{k'\}$, for some $k' \in K$, and so $\tau(b_j, k) = \phi(b_j(k'))$, and $\tau(b_i^{h_j} b_j, k) = \phi((b_i^{h_j} b_j)(k')) = \phi(b_i(k'h_j^{-1})b_j(k'))$.

When $E \cap kh_j^{-1}N = \varnothing$, then $b_i(k'h_j^{-1}) = 1$ and, in both case (1) and case (2), $\tau(b_i, kh_j^{-1})\tau(b_j, k) = \tau(b_i^{h_j} b_j, k)$.

Otherwise, $E \cap kh_j^{-1}N = \{k''h_j^{-1}\}$ for some $k'' \in kN$.

Suppose first that $b_i(k''h_j^{-1}) = 1$. If we are in case (2) then $b_i(k'h_j^{-1}) = 1$, since otherwise, just as in the proof of Theorem 3.1, condition (ii) of the definition of E gives $k'h_j^{-1} \in E$, and so $k' = k''$, and we have a contradiction. Hence, in both case (1) and case (2) we again have $\tau(b_i, kh_j^{-1})\tau(b_j, k) = \tau(b_i^{h_j} b_j, k)$.

Otherwise $b_i(k''h_j^{-1}) \neq 1$, and then, again just as in the proof of Theorem 3.1, condition (iii) of the definition of E gives $k'' \in E$. Hence we are in case (2) and $k' = k''$. Then

$$\tau(b_i, gh_j^{-1})\tau(b_j, g) = \phi(b_i(k'h_j^{-1}))\phi(b_j(k'))$$

and

$$\tau(b_i^{h_j} b_j, g) = \phi(b_i(k'h_j^{-1})b_j(k')).$$

Since $b_i(k'h_j^{-1}),\ b_j(k') \in D$, the fact that ϕ is a $(D, \epsilon/m)$-quasiaction implies that the proportion of the points of X on which the permutations $\phi(b_i(k'h_j^{-1})b_j(k'))$ and $\phi(b_i(k'h_j^{-1}))\phi(b_j(k'))$ have the same image is at least $1 - \epsilon/m$.

It follows that the proportion of elements $\delta \in Y$ with

$$\delta^{\psi(h_i b_i)\psi(h_j b_j)}(kN) = \delta^{\psi(h_i h_j b_i^{h_j} b_j)}(kN)$$

is at least $1 - \epsilon/m$. But $\delta^{\psi(h_i b_i)\psi(h_j b_j)} = \delta^{\psi(h_i h_j b_i^{h_j} b_j)}$ if and only if they take the same values on all $kN \in H/N$, and the proportion of $\delta \in Y$ for which this is true is at least $1 - \epsilon$. □

5. Extensions by amenable groups

In Section 3 we defined the restricted standard wreath product $G \wr H$ of groups G, H. In this section, we shall need wreath products by permutation groups. For a group K and a finite set A, we define the permutation wreath product $W = K \wr \mathrm{Sym}(A)$ as $W = \mathrm{Sym}(A) \ltimes B$ where the base group is now the set of all functions $b : A \to K$. As before, we define $b_1 b_2(a) := b_1(a) b_2(a)$ for $b_1, b_2 \in B$, $a \in A$, and we define the action of $\mathrm{Sym}(A)$ on B by the rule $b^\alpha(a) = b(a^{\alpha^{-1}})$, for $\alpha \in \mathrm{Sym}(A)$, $a \in A$. Much as before, elements of the wreath product are represented as pairs (α, b) with $\alpha \in \mathrm{Sym}(A)$ and $b \in B$, multiplied according to the rule $(\alpha_1, b_1)(\alpha_2, b_2) = (\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$, and with $(\alpha, b)^{-1} = (\alpha^{-1}, b^{-\alpha^{-1}})$.

In general the length function for finite wreath products that we used in the proof of Theorem 3.1 is not suitable for the proof of Theorem 5.1 below. So we need to define a different one.

Given an invariant length function ℓ_K on K, we can define an invariant length function $\hat{\ell}_K^A$ on W by

$$\hat{\ell}_K^A(\alpha, b) = \frac{1}{|A|}\Bigg(\sum_{a \in A \,:\, a^\alpha = a} \ell_K(b(a)) + \sum_{a \in A \,:\, a^\alpha \neq a} 1 \Bigg).$$

Most of the conditions for $\hat{\ell}_K^A$ to be an invariant length function are straightforward consequences of the conditions on ℓ_K. The verification of

$$\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2) \le \hat{\ell}_K^A(\alpha_1, b_1) + \hat{\ell}_K^A(\alpha_2, b_2)$$

may require a little more thought. For this, we consider the terms corresponding to the various $a \in A$ in the three sums that make up $\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$, $\hat{\ell}_K^A(\alpha_1, b_1)$, and $\hat{\ell}_K^A(\alpha_2, b_2)$. We see that, for each $a \in A$ with $a^{\alpha_1} \neq a$ or $a^{\alpha_2} \neq a$, the term in $\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$ is at most $1/|A|$, but at least one of the two nonnegative terms in $\hat{\ell}_K^A(\alpha_1, b_1)$ and $\hat{\ell}_K^A(\alpha_2, b_2)$ is equal to $1/|A|$. On the other hand, for $a \in A$ with $a^{\alpha_1} = a$ and $a^{\alpha_2} = a$, the term corresponding to a in $\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$ is

$$\frac{1}{|A|}\ell_K(b_1^{\alpha_2}(a) b_2(a)) = \frac{1}{|A|}\ell_K(b_1(a) b_2(a)) \le \frac{1}{|A|}(\ell_K(b_1(a)) + \ell_K(b_2(a)),$$

which is the corresponding term in $\hat{\ell}_K^A(\alpha_1, b_1) + \hat{\ell}_K^A(\alpha_2, b_2)$.

Theorem 5.1. *Let $\mathcal{C}$ be a class of groups with associated invariant length functions and suppose that, for all $(K, \ell_K) \in \mathcal{C}$ and all finite sets A, the wreath product $(K \wr \mathrm{Sym}(A), \hat{\ell}_K^A)$ is in $\mathcal{C}$. Suppose that the group G has a normal subgroup N with the discrete $\mathcal{C}$-approximation property* (as defined in Section 1) *such that G/N is amenable. Then G has the discrete $\mathcal{C}$-approximation property.*

This result has already been proved for sofic groups [Elek and Szabó 2006, Theorem 1 (3)] and linear sofic groups [Stolz 2013, Theorem 5.3]. However, in order to avoid confusion we should comment that, while the above result considers extensions G of $\mathcal{C}$-approximable normal subgroups N with G/N amenable, by contrast, [Arzhantseva and Gal 2013, Theorem 7] considers extensions G of finitely generated residually finite normal subgroups N for which G/N is in a selected class $\mathcal{R}$ of groups (including groups that are residually amenable groups, LEF, LEA, sofic or surjunctive).

Proof. The proof is based on the corresponding proof in [Elek and Szabó 2006, Theorem 1 (3)] for sofic groups N.

By assumption, the normal subgroup N of G is $\mathcal{C}$-approximable using a weight function δ that takes a constant value c on all elements of $N \setminus \{1\}$. Since we can reduce the value of c without affecting the $\mathcal{C}$-approximability of N, we may assume that $c < 1$. If $N \neq \{1\}$ then we define the weight function β of G by $\beta_g = c$ for all $g \neq 1$, and if $N = \{1\}$, then we define β by $\beta_g = \frac{1}{2}$ for all $g \neq 1$.

For $g \in G$, let $\bar{g}$ be the homomorphic image of g in G/N and let $\sigma : G/N \to G$ be a section (so $\overline{\sigma(h)} = h$ for all $h \in G/N$), where $\sigma(\bar{1}) = 1$. We can lift σ to a map from G to G for which the image of $g \in G$ is $\sigma(\bar{g})$; we shall abuse notation and call that map σ as well.

To verify the $\mathcal{C}$-approximability condition on G, let F be a finite subset of G and let $\epsilon > 0$. We may assume that $\epsilon < \min\left(\frac{1}{2}, 1-c\right)$.

The amenability of G/N ensures the existence of a finite subset $\overline{A}$ of G/N containing the identity element such that $|\overline{A}\bar{g}\backslash\overline{A}| \leq \epsilon|\overline{A}|$ for all $g \in F \cup F^{-1} \cup F^2 \cup F^{-2}$. Let $A = \sigma(\overline{A})$; note that all points of A are fixed by the map $\sigma : G \to G$. We define a map $\phi : G \to \mathrm{Sym}(A)$ as follows:

$$\text{for } g \in G,\ a \in A, \quad a^{\phi(g)} := \begin{cases} \sigma(ag) & \text{if } \overline{ag} \in \overline{A}, \\ \text{any choice with } \phi(g) \in \mathrm{Sym}(A) & \text{otherwise.} \end{cases}$$

Let $E = N \cap (A \cdot F \cdot A^{-1})$. The $\mathcal{C}$-approximability of N ensures the existence of an $(E, \epsilon, \delta, \ell_K)$-quasihomomorphism $\psi : N \to K$ with $(K, \ell_K) \in \mathcal{C}$.

Now we let $W = K \wr \mathrm{Sym}(A) = \mathrm{Sym}(A) \ltimes B$ and define $\Phi : G \to W$ by $\Phi(g) = (\phi(g), b)$ where, for $a \in A$, $b(a) := \psi(\sigma(ag^{-1})ga^{-1})$.

We show first that $\hat{\ell}_K^A(\Phi(g)) \geq \beta_g$ for $g \in F$. If $g \notin N$ then, since $\phi(g)$ moves all points $a \in A$ for which $\overline{ag} \in \overline{A}$, we have

$$\hat{\ell}_K^A(\Phi(g)) \geq 1 - \epsilon > \tfrac{1}{2} = \delta_g.$$

If $g \in N \setminus \{1\}$ then $\overline{ag^{-1}} = \bar{a}$, so $\sigma(ag^{-1}) = a$ for all $a \in A$, and $\hat{\ell}_K^A(\Phi(g))$ is the average over $a \in A$ of $\ell_K(\psi(aga^{-1}))$. But since each $aga^{-1} \in E \setminus \{1\}$, these all exceed δ_g.

Now let $g, h \in F$. We aim to show that

$$\hat{\ell}_K^A(\Phi(gh)\Phi(h)^{-1}\Phi(g)^{-1}) \leq 5\epsilon.$$

For $a \in A$, we have

$$\Phi(g) = (\phi(g), b), \qquad \text{where } b(a) = \psi(\sigma(ag^{-1})ga^{-1}),$$

$$\Phi(h) = (\phi(h), c), \qquad \text{where } c(a) = \psi(\sigma(ah^{-1})ha^{-1}),$$

$$\Phi(gh) = (\phi(gh), d), \qquad \text{where } d(a) = \psi(\sigma(ah^{-1}g^{-1})gha^{-1}),$$

$$\begin{aligned}\Phi(g)\Phi(h) = (\phi(g)\phi(h), b^{\phi(h)}c),&\\ \text{where } (b^{\phi(h)}c)(a) = b^{\phi(h)}(a)c(a) &= b(a^{\phi(h)^{-1}})c(a)\\ &= \psi(\sigma(a^{\phi(h)^{-1}}g^{-1})ga^{-\phi(h)^{-1}})\psi(\sigma(ah^{-1})ha^{-1})\end{aligned}$$

(where, for $a, k \in G$, we write a^{-k} as shorthand for $(a^{-1})^k = (a^k)^{-1}$). Then

$$\begin{aligned}\Phi(gh)(\Phi(g)\Phi(h))^{-1} &= (\phi(gh), d)(\phi(g)\phi(h), b^{\phi}(h)c)^{-1}\\ &= (\phi(gh), d)((\phi(g)\phi(h))^{-1}, (b^{\phi}(h)c)^{-(\phi(g)\phi(h))^{-1}})\\ &= (\phi(gh)(\phi(g)\phi(h))^{-1}, (d(b^{\phi}(h)c)^{-1})^{(\phi(g)\phi(h))^{-1}}).\end{aligned}$$

Now, for a proportion of at least $1 - 2\epsilon$ of the points $a \in A$, we have both $\overline{ah^{-1}} \in \overline{A}$ and $\overline{ah^{-1}g^{-1}} \in \overline{A}$. For those points a, we have $a^{\phi(h)^{-1}} = \sigma(ah^{-1})$ and so the final expression for $(b^{\phi(h)}c)(a)$ above becomes

$$\psi(\sigma(ah^{-1}g^{-1})g\sigma(ah^{-1})^{-1}) \times \psi(\sigma(ah^{-1})ha^{-1}),$$

and we see that the image of a under the second component of $\Phi(gh)(\Phi(g)\Phi(h))^{-1}$ is equal to a conjugate of

$$\psi(xy)\psi(y)^{-1}\psi(x)^{-1},$$

where $x = \sigma(ah^{-1}g^{-1})g\sigma(ah^{-1})^{-1}$ and $y = \sigma(ah^{-1})ha^{-1}$. The elements x, y are both in the finite subset E of G, and hence, since ψ is a quasihomomorphism, $\ell_K(\psi(xy)\psi(y)^{-1}\psi(x)^{-1}) < \epsilon$, and we deduce that

$$\ell_K((d(b^{\phi(h)}c)^{-1})^{(\phi(g)\phi(h))^{-1}})(a)) < \epsilon,$$

for at least a proportion $1 - 2\epsilon$ of the points of A.

Our choice of A ensures also that $\phi(gh)(\phi(g)\phi(h))^{-1}(a) = a$ for at least a proportion $1 - 2\epsilon$ of the points a of A.

Now, for at least a proportion $1 - 4\epsilon$ of the points of A, the conditions of both of the last two paragraphs hold, and so we can deduce

$$\hat{\ell}_K^A(\Phi(gh)\Phi(h)^{-1}\Phi(g)^{-1}) < \epsilon(1 - 4\epsilon) + 4\epsilon < 5\epsilon. \qquad \square$$

In particular, by taking $\mathcal{C} = \mathcal{F}$ with each $K \in \mathcal{F}$ associated with all possible length functions, we see that the class of weakly sofic groups is closed under extension by amenable groups.

In general, ℓ_K commutator-contractive does not imply that $\hat{\ell}_K^A$ is commutator-contractive. But if, instead, we define ℓ_K^A as we did in Section 3 (that is, for $b \in B$, $\ell_K^A(b) = \max_{a \in A} \ell_K(b(a))$, and $\ell_K^A(\alpha b) = 1$ when $1 \neq \alpha \in \mathrm{Sym}(A)$) then, as we proved in Lemma 3.2, ℓ_K^A is commutator-contractive.

Our proof of Theorem 5.1 does not always work with this commutator-contractive norm, but it does work if $\phi : G/N \to A$ is a homomorphism. In particular, when $G/N \cong (\mathbb{Z}, +)$, we can choose A to be $\{x \in \mathbb{Z} : -m \leq x \leq m\}$ for some m and define ϕ to be addition modulo $2m+1$. So, by applying this repeatedly, we have:

Proposition 5.2. *The class of $\mathcal{F}_c$-approximable groups is closed under extension by polycyclic groups.*

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