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# A GAUSSIAN UPPER BOUND OF THE CONJUGATE HEAT EQUATION ALONG RICCI-HARMONIC FLOW

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**We mainly study the Ricci-harmonic flow. Using the monotonicity formulae of entropies, we show a uniform Sobolev inequality along Ricci-harmonic flow. Furthermore, we obtain a Gaussian upper bound for the fundamental solutions of the conjugate heat equation via Moser iteration and Sobolev inequality.**

## 1. Introduction

Let $M$ be a closed manifold of dimension $n$. List [2008] studied the following Ricci flow, coupled with a harmonic flow:

$$\begin{cases} \partial_t g(x,t) = -2\,\mathrm{Ric}_{g(x,t)} + 4\,d\phi(x,t)\otimes d\phi(x,t), \\ \partial_t \phi(x,t) = \Delta_{g(x,t)}\phi, \end{cases} \tag{1-1}$$

where $g(x,t)$ is a family of Riemannian metrics, and $\phi(x,t)$ is a scalar function on $M\times\mathbb{R}$. This flow is called Ricci-harmonic flow (see also [List 2008; Müller 2012; Zhu 2013]). If $\phi$ is a constant, the system (1-1) degenerates to Hamilton's Ricci flow, which has been discussed widely recently; see for example the book [Chow et al. 2006] and celebrated papers [Hamilton 1982; 1986; 1993; Li 2007; Ni 2006; Perelman 2002]. The stationary solutions of (1-1) satisfy the static Einstein vacuum system

$$\begin{cases} \mathrm{Ric} = 2\,d\phi\otimes d\phi, \\ \Delta\phi = 0. \end{cases}$$

Similarly to Ricci flow, corresponding theories for Ricci-harmonic flow have been established; see for instance [List 2008].

For the sake of convenience, we denote as in [List 2008] the symmetric tensor field $Sy \in \mathrm{Sym}_2(M)$ and its trace by

$$S_{ij} := R_{ij} - 2\partial_i\phi\partial_j\phi \quad \text{and} \quad S := R - 2|d\phi|^2,$$

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where $R$ denotes the scalar curvature of the Riemannian manifold $(M, g)$. Then the Ricci-harmonic flow can be written simply as

$$\begin{cases} \partial_t g = -2Sy, \\ \partial_t \phi = \Delta \phi. \end{cases}$$

It is well known that Sobolev inequality contains a host of analytical and geometric information (e.g., [Carrillo and Ni 2009; Chau et al. 2011; Hebey 1996; Saloff-Coste 2002]), including noncollapsing properties, isoperimetric inequalities and so on. Sobolev inequality is also an important tool in studying elliptic and parabolic differential equations on manifolds (see for example [Saloff-Coste 2002]). Via the monotonicity of Perelman's $W$ entropy, some uniform Sobolev inequalities were proven in Ricci flow, see [Carrillo and Ni 2009; Chau et al. 2011; Kuang and Zhang 2008; Zhang 2006; 2007; 2011]. Zhang [2007] showed a global upper bound for the fundamental solution of the heat equation along the backward Ricci flow

$$\begin{cases} \partial_t g = -2\,\mathrm{Ric}, \\ \Delta u + \partial_t u - Ru = 0, \end{cases}$$

providing Ricci curvature is nonnegative and the injective radius is bounded from below.

Along flow (1-1), we consider the conjugate heat equation

$$\partial_t u(x,t) + \Delta u(x,t) - S(x,t)u(x,t) = 0. \tag{1-2}$$

In [Zhu 2013], some pointwise gradient estimates for the positive solutions of (1-1) were proven, which can be viewed as Li–Yau estimates for the parabolic kernel of the Schrödinger operator in [Chau et al. 2011; Li and Yau 1986; Ni 2004; 2006].

The main goal of this paper is to establish certain Sobolev inequalities under system (1-1) and a global upper bound for the fundamental solutions of heat equation (1-2). Via the monotonicity of the entropies, we obtain the following Sobolev inequality.

**Theorem 1.1.** *Let $(M, g(x,t), \phi(x,t))$ be a solution of the system* (1-1) *for $t \in [0, T_0)$ with initial metric $g_0$, where $T_0 \le \infty$ is the life span of* (1-1). *Let $A_0$ and $B_0$ be positive numbers such that the following $L^2$ Sobolev inequality holds initially, i.e., for each $v \in W^{1,2}(M, g_0)$,*

$$\left(\int_M v^{2n/(n-2)}\,d\mu_{g_0}\right)^{(n-2)/n} \le A_0\int_M \left(|\nabla v|^2 + \tfrac{1}{4}Sv^2\right) d\mu_{g_0} + B_0\int_M v^2\,d\mu_{g_0}.$$

*Then for all $v \in W^{1,2}(M, g(t))$, we have*

$$\left(\int_M v^{2n/(n-2)}\,d\mu_{g(t)}\right)^{(n-2)/n} \le A(t)\int_M \left(|\nabla v|^2 + \tfrac{1}{4}Sv^2\right) d\mu_{g(t)} + B(t)\int_M v^2\,d\mu_{g(t)}, \tag{1-3}$$

*where $A(t)$ and $B(t)$ are positive constants depending on $A_0$, $(1+t)B_0$, $n$ and $S_0^-$. Here $S_0^- = \sup_{x\in M} S^-(x,0)$, and $S^-(x,0)$ denotes the negative part of $S(x,0)$.*

Via Sobolev inequality (1-3), combined with Morse iteration and Davies' heat kernel estimates, we prove the following Gaussian-type upper bound for the fundamental solutions of (1-2), with constants not depending on the lower bound of injective radius but on the first eigenvalue of the entropy, different from Zhang's result [2007]. More precisely:

**Theorem 1.2.** *Let $(M, g(x,t), \phi(x,t))$ be a smooth solution of system* (1-1) *in $M \times [0,T]$ and $G(x,t;y,T)$ be a fundamental solution of the following backward conjugate heat equation* (1-2); *that is,*

$$\begin{cases} \Delta_x G(x,t;y,T) + \partial_t G(x,t;y,T) - S(x,t)G(x,t;y,T) = 0 & \text{if } 0 \le t < T; \\ G(x,t;y,T) = \delta(x,y) & \text{if } t = T. \end{cases}$$

*Assume that $Sy \ge 0$ and the first eigenvalue $\lambda_0$ of $\inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + Sv^2)\, d\mu_{g_0}$ is positive. Then for each $t \in (0,T)$, and $x, y \in M$, we have the following estimates*:

$$G(x,t;y,T) \le \frac{c}{|B(y,\sqrt{T-t},T)|_T} \exp \frac{-c_1 d^2(x,y,T)}{T-t}, \tag{1-4}$$

*where $c_1$ is a constant depending only on the dimension $n$, and $c$ is a constant depending on $n$, $\lambda_0$ and the initial metric $g_0$. Here $d(x,y,T)$ denotes the distance between $x$ and $y$ with respect to metric $g(T)$, $B(y,\sqrt{T-t},T)$ denotes the geodesic ball centered at $y$ with radius $\sqrt{T-t}$, and $|B(y,\sqrt{T-t},T)|_T$ denotes the volume of the ball $B(y,\sqrt{T-t},T)$ with respect to the metric $g(T)$.*

The rest of the paper is organized as follows. We give the evolution equations of entropies under system (1-1) in Section 2. We prove Sobolev inequalities along Ricci-harmonic flow in Section 3. In Section 4, we prove Theorem 1.2.

## 2. Entropies of Ricci-harmonic flow

In this section, we recall the definitions of entropies via corresponding conjugate heat equation, as Perelman's [2002] entropy in Ricci flow. Through direct computations, we obtain the monotonicity of the entropies. Although the monotonicity of the entropies were proven in [List 2008] via the entropies' invariance under diffeomorphism. But here for the completeness, we give a direct computation.

Let $u(x,t)$ be a positive solution to the conjugate heat equation (1-2):

$$H^* u = \Delta u - Su + \partial_t u = 0.$$

Note by (1-2) and equation (1-1) that

$$\frac{d}{dt}\int_M u(x,t)\,d\mu_{g(t)} = \int_M (\partial_t - S)u\,d\mu_{g(t)} = \int_M H^* u\,d\mu_{g(t)} = 0,$$

where we used the closure of $M$. Hereafter we always assume that $u(x,t)$ satisfies

$$\int_M u(x,t)\,d\mu_{g(t)} = 1 \tag{2-1}$$

for each $t \in [0, T]$.

Via the positive solution $u$, the entropies are defined (see, e.g., [List 2008]) as follows.

**Definition 2.1.** $F$ entropy is defined as the following integration:

$$F(t) = \int_M \Big(Su + \frac{|\nabla u|^2}{u}\Big)\,d\mu_{g(t)}, \tag{2-2}$$

and $W$ entropy is defined by

$$W(t) = \int_M \Big[\tau\Big(Su + \frac{|\nabla u|^2}{u}\Big) - u\ln u - \frac{n}{2}\ln(4\pi\tau)u - nu\Big]\,d\mu_{g(t)}, \tag{2-3}$$

where $d\tau/dt = -1$.

In order to simplify computations, we introduce a potential function $f(x,t)$ via

$$u(x,t) = \frac{e^{-f}}{(4\pi\tau)^{n/2}},$$

i.e.,

$$f = -\ln u - \frac{n}{2}(\ln 4\pi\tau). \tag{2-4}$$

With the above preparations, we now give a direct calculation of the following monotonicity formulae.

**Proposition 2.2** [List 2008, Theorem 6.1]. *Let $(M, g, \phi)$ be a solution of* (1-1) *and $u(x,t)$ be a positive solution of* (1-2)*. Then both $F$ entropy and $W$ entropy are nondecreasing in $t$. Moreover, we have*

$$\frac{d}{dt}F(t) = 2\int_M \big(|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\big)u\,d\mu_{g(t)} \geq 0, \tag{2-5}$$

*and*

$$\frac{d}{dt}W(t) = \int_M \Big(2\tau\Big|Sy + \nabla^2 f - \frac{g}{2\tau}\Big|^2 + 4\tau|\Delta\phi - d\phi(\nabla f)|^2\Big)u\,d\mu_{g(t)} \geq 0. \tag{2-6}$$

*Proof.* To start, we have by direct calculations that

$$H^*(u\ln u)=\frac{|\nabla u|^2}{u}+Su, \tag{2-7}$$

and

$$H^*\Big(\frac{|\nabla u|^2}{u}+Su\Big)=\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u} \\ +4\langle\nabla u,\nabla S\rangle+2u\Delta S+2(|Sy|^2+2|\Delta\phi|^2)u. \tag{2-8}$$

Here we used the well-known equation (see [List 2008, Lemma 3.2])

$$\partial_t S=\Delta S+2|Sy|^2+4|\Delta\phi|^2.$$

Note that

$$\frac{d}{dt}F=\frac{d}{dt}\int_M\Big(S+\frac{|\nabla u|^2}{u}\Big)\,d\mu=\int_M(\partial_t-S)\Big(Su+\frac{|\nabla u|^2}{u}\Big)\,d\mu=\int_M H^*\Big(Su+\frac{|\nabla u|^2}{u}\Big)\,d\mu,$$

and substituting (2-8) into the above equality we have

$$\frac{d}{dt}F=\int_M\Big[\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u} \\ +4\langle\nabla u,\nabla S\rangle+2u\Delta S+2\big(|Sy|^2+2|\Delta\phi|^2\big)u\Big]\,d\mu. \tag{2-9}$$

By integration by parts and the contracted second Bianchi identity, we see

$$\begin{aligned}\int_M\langle\nabla u,\nabla S\rangle\,d\mu&=\int_M\langle\nabla u,\nabla(R-2|d\phi|^2)\rangle\,d\mu\\&=\int_M(2u_i\nabla_jR_{ij}-4u_i\phi_j\phi_{ij})\,d\mu\\&=\int_M(-2u_{ij}R_{ij}+4u_{ij}\phi_j\phi_i+4u_i\phi_i\Delta\phi)\,d\mu\\&=\int_M(-2u_{ij}S_{ij}+4u_i\phi_i\Delta\phi)\,d\mu;\end{aligned} \tag{2-10}$$

Then substituting (2-10) into (2-9) yields

$$\begin{aligned}\frac{d}{dt}F&=\int_M\Big[\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u}\\&\qquad+2\langle\nabla u,\nabla S\rangle+2\big(|Sy|^2+2|\Delta\phi|^2\big)u\Big]\,d\mu\\&=\int_M\Big[\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u}\\&\qquad-4u_{ij}S_{ij}+8u_i\phi_i\Delta\phi+2\big(|Sy|^2+2|\Delta\phi|^2\big)u\Big]\,d\mu.\end{aligned}$$

Replacing $u$ with $f$ in the above equality yields

$$\begin{aligned}\frac{d}{dt}F &= \int_M \big[|f_{ij}|^2 + 2S_{ij}f_{ij} + |S_{ij}|^2 + 2|\Delta\phi|^2 + 2|2\,d\phi(\nabla f)|^2 - 4\Delta\phi(d\phi(\nabla f))\big]\,d\mu \\ &= 2\int_M \big(|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\big)u\,d\mu,\end{aligned}$$

proving formula (2-5).

From the definition of $W$ entropy, it follows that

$$\frac{d}{dt}W(t) = \int_M H^*\Big(\tau\Big(\frac{|\nabla u|^2}{u} + Su\Big)\Big) - H^*(u\ln u) - \frac{n}{2}H^*(u\ln\tau)\,d\mu.$$

Substituting (2-7) and (2-8) to the above equation yields

$$\frac{d}{dt}W = \tau\frac{d}{dt}F - 2F + \frac{n}{2\tau} = \tau\frac{d}{dt}F - 2\int_M \big(Su + |\nabla f|^2 u\big)\,d\mu(g(t)) + \frac{n}{2\tau}. \tag{2-11}$$

From the definition of $f$ and integrations by parts, we deduce

$$\int_M \Big(Su + \frac{|\nabla u|^2}{u}\Big)d\mu(g(t)) = \int_M (Su - \langle\nabla u, \nabla f\rangle)\,d\mu = \int_M (Su + \Delta f u)\,d\mu.$$

Substituting the above equality and equality (2-5) into (2-11), we have

$$\begin{aligned}\frac{d}{dt}W &= 2\tau\int_M \big(|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\big)u\,d\mu - 2\int_M (S + \Delta f)u\,d\mu + \frac{n}{2\tau} \\ &= \int_M 2\tau\Big(\Big|Sy + \mathrm{Hess}(f) - \frac{g}{2\tau}\Big|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\Big)u\,d\mu,\end{aligned}$$

completing the proof. □

Similarly to the Ricci flow, one can define a family of generalized $W$ entropy along the Ricci-harmonic flow by

$$\begin{aligned}W(a,t) &= \int_M \Big(\frac{a^2\tau}{2\pi}\Big(Su + \frac{|\nabla u|^2}{u}\Big) - u\ln u - \frac{n}{2}\ln(4\pi\tau)u - nu\Big)d\mu_{g(t)} \\ &= \int_M \Big(\frac{a^2\tau}{2\pi}(S + |\nabla f|^2) + f - n\Big)u\,d\mu_{g(t)}.\end{aligned} \tag{2-12}$$

Here the second equality is due to the relations between $u$ and $f$ given in (2-4). For applications of generalized entropy, we refer to the paper [Li 2007]. Using the calculations in [Kuang and Zhang 2008], one can easily show the following monotonicity formula of generalized $W$ entropy along Ricci-harmonic flow.

**Proposition 2.3.** *Let* $(M, g, \phi)$ *be a solution of* (1-1) *and* $u(x,t)$ *be a positive solution of* (1-2). *Then the generalized entropy* $W(a,t)$ *is nondecreasing in* $t$ *and* *we have*

$$\frac{d}{dt}W(a,t) \geq \frac{a^2\tau}{\pi}\int_M\Big(\Big|Sy+\mathrm{Hess}(f)-\frac{g}{2\tau}\Big|^2+2|\Delta\phi-d\phi(\nabla f)|^2\Big)u\,d\mu.$$

Since the proof is similar to that in Ricci flow (see for example [Kuang and Zhang 2008, Theorem 4.1]), we omit it.

## 3. Sobolev inequalities in Ricci-harmonic flow

In this section, we mainly use the monotonicity of $W$ entropy to derive a uniform Sobolev inequality along system (1-1), which will be useful in Section 4.

To prove Theorem 1.1, we need the following lemma, giving the equivalence of the logarithmic Sobolev inequality, the $W^{1,2}$ Sobolev inequality and the so-called ultracontractivity of the heat semigroup of the associated Schrödinger operator. The proof of this lemma is more or less standard.

**Lemma 3.1** [Zhang 2011, Theorem 4.2.1]. *Let $(M^n, g)$ be a closed Riemannian manifold $(n \geq 3)$. Then the following inequalities are equivalent up to constants.*

(I) *Sobolev inequality*: *there exists positive constants $A$ and $B$ such that for $v \in W^{1,2}(M)$*

$$\Big(\int_M v^{2n/(n-2)}\,d\mu\Big)^{(n-2)/n} \leq A\int_M|\nabla v|^2\,d\mu + B\int_M v^2\,d\mu;$$

(II) *Log-Sobolev inequality*: *for $v \in W^{1,2}(M)$ with $\|v\|_2 = 1$ and $\epsilon > 0$,*

$$\int_M v^2\ln v^2\,d\mu \leq \epsilon^2\int_M|\nabla v|^2\,d\mu - \frac{n}{2}\ln\epsilon^2 + BA^{-1}\epsilon^2 + \frac{n}{2}\ln\frac{nA}{2e};$$

(III) *Heat kernel upper bound*: *for $t > 0$,*

$$G(x,t;y) \leq \frac{(nA)^{n/2}}{t^{n/2}}e^{A^{-1}Bt}.$$

By Lemma 3.1, to prove Theorem 1.1 it suffices to show some log-Sobolev inequalities or heat kernel estimates for each $t \in [0, T_0)$. By the monotonicity of $W$ entropy, we obtain the following log-Sobolev inequality.

**Lemma 3.2** (log-Sobolev inequality). *Under the assumptions of Theorem 1.1, for each $t \in [0, T_0)$, $v \in W^{1,2}(M, g(t))$ with $\int_M v^2\,d\mu_{g(t)} = 1$ and $\epsilon > 0$, we have*

$$\int_M v^2\ln v^2\,d\mu_{g(t)} \leq \epsilon^2\int_M\big(4|\nabla v|^2+Sv^2\big)\,d\mu_{g(t)} - n\ln\epsilon + (t+\epsilon^2)B_0A_0^{-1} + \frac{n}{2}\ln\frac{nA_0}{2e}. \tag{3-1}$$

*Proof.* For $t_0 \in [0, T_0)$ and $\epsilon > 0$, we set

$$\tau(t) = \epsilon^2 + t_0 - t.$$

Recall that $W$ entropy is defined by

$$W(g, f, t) = \int_M \bigl(\tau(S + |\nabla f|^2) + f - n\bigr) u \, d\mu_{g(t)}.$$

Then from the monotonicity of $W$ entropy in Proposition 2.2, we deduce

$$\inf_{\int_M u \, d\mu_{g(t_0)}=1} W(g(t_0), f, \epsilon^2) \ge \inf_{\int_M u_0 \, d\mu_{g(0)}=1} W(g(0), f_0, t_0 + \epsilon^2). \tag{3-2}$$

One can find a more detailed proof of this property in Section 3 of [Perelman 2002]. Here $f_0$ and $f$ are given via the formulae

$$u_0 = \frac{e^{-f_0}}{\bigl(4\pi(t_0+\epsilon^2)\bigr)^{n/2}} \quad \text{and} \quad u = \frac{e^{-f}}{(4\pi\epsilon^2)^{n/2}}.$$

Using this notation we rewrite (3-2) as

$$\begin{aligned}&\inf_{\int u \, d\mu_{g(t_0)}=1} \int_M \Bigl(\epsilon^2\bigl(S + |\nabla \ln u|^2\bigr) - \ln u - \frac{n}{2} \ln 4\pi\epsilon^2\Bigr) u \, d\mu_{g(t_0)} \\ &\quad \ge \inf_{\int u_0 \, d\mu_{g(0)}=1} \int_M \Bigl((\epsilon^2 + t_0)\bigl(S + |\nabla \ln u_0|^2\bigr) - \ln u_0 - \frac{n}{2} \ln 4\pi(t_0 + \epsilon^2)\Bigr) u_0 \, d\mu_{g(0)}.\end{aligned}$$

Let $v = \sqrt{u}$ and $v_0 = \sqrt{u_0}$, and the above inequality gives

$$\begin{aligned}&\inf_{\int v^2 \, d\mu_{g(t_0)}=1} \int_M \bigl[\epsilon^2(Sv^2 + 4|\nabla v|^2) - v^2 \ln v^2\bigr] \, d\mu_{g(t_0)} - \frac{n}{2} \ln \epsilon^2 \\ &\quad \ge \inf_{\int v_0^2 \, d\mu_{g(0)}=1} \int_M \bigl((\epsilon^2 + t_0)(Sv_0^2 + 4|\nabla v_0|^2) - v_0^2 \ln v_0^2\bigr) \, d\mu_{g(0)} - \frac{n}{2} \ln(t_0 + \epsilon^2)\end{aligned} \tag{3-3}$$

Since $\ln x$ is a concave function and $\int_M v_0^2 \, d\mu_{g(0)} = 1$, then applying Jensen's inequality we derive

$$\int_M v_0^2 \ln v_0^{q-2} \, d\mu_{g(0)} \le \ln \int v_0^{q-2} v_0^2 \, d\mu_{g(0)},$$

i.e.,

$$\int_M v_0^2 \ln v_0^2 d\mu_{g(0)} \le \tfrac{1}{2} n \ln \|v_0\|_q^2,$$

where $q = 2n/(n-2)$. By the assumption that the Sobolev inequality holds for the initial time $t = 0$, we have

$$\int_M v_0^2 \ln v_0^2 \, d\mu_{g(0)} \le \tfrac{1}{2} n \ln\Bigl(A_0 \int_M (4|\nabla v_0|^2 + S v_0^2) \, d\mu_{g(0)} + B_0\Bigr).$$

From the elementary inequality

$$\ln z \le yz - \ln y - 1,$$

we deduce that for any $y, z > 0$

$$\int_M v_0^2 \ln v_0^2 \, d\mu_{g(0)} \le \frac{n}{2} y \Big( A_0 \int_M \big(4|\nabla v_0|^2 + S v_0^2\big) \, d\mu_{g(0)} + B_0 \Big) - \frac{n}{2} \ln y - \frac{n}{2}.$$

Letting $y = 2(t_0 + \epsilon^2)/(nA_0)$ in the above inequality, we get

$$\int_M v_0^2 \ln v_0^2 \, d\mu_{g(0)} \le (t_0 + \epsilon^2) \int_M \big(4|\nabla v_0|^2 + S v_0^2\big) \, d\mu_{g(0)} + \frac{(t_0+\epsilon^2) B_0}{A_0} - \frac{n}{2} \ln \frac{2(t_0+\epsilon^2)}{nA_0} - \frac{n}{2}.$$

Substituting the above inequality to the right-hand side of (3-3), we arrive at

$$\int_M v^2 \ln v^2 \, d\mu_{g(t_0)} \le \epsilon^2 \int_M \big(4|\nabla v|^2 + S v^2\big) \, d\mu_{g(t_0)} - n \ln \epsilon + (t_0+\epsilon^2) B_0 A_0^{-1} + \frac{n}{2} \ln \frac{nA_0}{2e}.$$

Thus the log-Sobolev inequality (3-1) holds. □

*Proof of Theorem 1.1.* As the right-hand side of inequality (1-3) has an extra term $S$, we can not use Lemma 3.1 directly. Instead, we use Zhang's [2007] trick to obtain the estimates of the fundamental solutions of the heat equation, and then use Lemma 3.1 to derive the Sobolev inequality. More precisely, we consider the following heat equation:

$$\Delta_{g(t_0)} u(x,t) - \tfrac{1}{4} S(x,t_0) u(x,t) - S_0^- u(x,t) - u_t(x,t) = 0,$$

where $S_0^- = \sup_{x \in M} S^-(x,0)$ and the metric is fixed at $t_0$. Then following the same process as in [Zhang 2007]; we see the fundamental solution $p(x,T;y)$ is contractive and satisfies the estimates

$$p(x,T;y) \le \frac{C_1}{t^{n/2}} \quad \text{for} \quad t > 0,$$

where $C_1$ is a constant depending on $n$, $A_0$, $(1+t_0)B_0$ and $S_0^-$ Then from Lemma 3.1, we conclude that the Sobolev inequality (1-3) at $t = t_0$ holds with constants $A(t_0)$ and $B(t_0)$ (depending only on $n$, $A_0$, $(1+t_0)B_0$ and $S_0^-$). Thus the theorem is true by the arbitrariness of $t_0$. □

Since $(M, g_0)$ is a closed Riemannian manifold, the Sobolev inequality holds as described in Section 4.1 in [Zhang 2011]. That is, for any $v \in W^{1,2}(M)$, there exist positive constants $A_0$ and $B_0$ depending only on $n$ and the initial metric $g_0$ such that

$$\left( \int_M v^{2n/(n-2)} \, d\mu_{g_0} \right)^{(n-2)/n} \le A_0 \int_M |\nabla v|^2 \, d\mu_{g_0} + B_0 \int_M v^2 \, d\mu_{g_0}. \tag{3-4}$$

Recall that $\lambda_0$ is the first eigenvalue of $F$ entropy as characterized in (2-2), that is,

$$\lambda_0 = \inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + S v^2)\, d\mu_{g_0}. \tag{3-5}$$

This eigenvalue has been studied widely and is a very powerful tool for understanding Riemannian manifolds [Li 2007].

Note that

$$\int_M |\nabla v|^2 \, d\mu_{g_0} \le \int_M \Big(|\nabla v|^2 + \frac{S}{4} v^2\Big)\, d\mu_{g_0} + \frac{S_0^-}{4} \int_M v^2 \, d\mu_{g_0}.$$

Then if $\lambda_0 > 0$, we conclude from the inequality (3-4) that the assumption of Sobolev inequality in Theorem 1.1 holds initially as follows:

$$\Big(\int_M v^{2n/(n-2)} \, d\mu_{g_0}\Big)^{(n-2)/n} \le \Big[A_0 + \Big(\frac{S_0^-}{4} + B_0\Big)\frac{4}{\lambda_0}\Big] \int_M \big(|\nabla v|^2 + \tfrac{1}{4} S v^2\big)\, d\mu_{g_0}.$$

That is, the log-Sobolev inequality (3-1) in Lemma 3.2 holds with constant $B_0 = 0$. Therefore we conclude

**Corollary 3.3.** *Let* $(M, g, \phi)$ *be a solution of the system* (1-1). *Assume further that* $\lambda_0 > 0$. *Then for all* $v \in W^{1,2}(M, g(t))$, $t \in [0, T_0)$, *it holds that*

$$\Big(\int_M v^{2n/(n-2)} \, d\mu_{g(t)}\Big)^{(n-2)/n} \le \tilde{A}_0 \int_M \big(|\nabla v|^2 + \tfrac{1}{4} S v^2\big)\, d\mu_{g(t)},$$

*where* $\tilde{A}_0$ *depends on initial Sobolev constants* $A_0$ *and* $B_0$, *and* $\lambda_0$ *and* $S_0^-$ *are independent of* $t$.

## 4. Proof of Theorem 1.2

In this section, we prove a Gaussian-type upper bound for fundamental solutions of the conjugate heat equation. The Gaussian upper bound in Ricci flow was proven in [Zhang 2006] with the assumption on the lower bound of injectivity, via Sobolev inequality by Heybey [1996]. Here using the uniform Sobolev inequality in Corollary 3.3, we derive a similar Gaussian upper bound without the assumption on the lower bound of injectivity. To prove the theorem, we need the following interpolation theorem.

**Theorem 4.1.** *Let* $(M, g, \phi)$ *be a solution of* (1-1) *and* $u(x, t)$ *be a positive solution to heat equation*

$$\Delta u - \partial_t u = 0 \tag{4-1}$$

*for* $t \in [0, T]$. *Then it holds that*

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq \sqrt{\frac{1}{t}}\sqrt{\ln\frac{A}{u(x,t)}} \tag{4-2}$$

*for* $(x,t) \in M \times [0,T]$. *Here* $A = \sup_{M\times[0,T]} u$.

*Moreover, for each* $\delta > 0$, $x, y \in M$ *and* $0 < t < T$, *the following interpolation inequality holds*:

$$u(y,t) \leq A^{\delta/(1+\delta)} u^{1/(1+\delta)}(x,t) \exp\Big(\frac{d^2(x,y,t)}{4t\delta}\Big). \tag{4-3}$$

*Proof.* The proof is based on maximum principles, see also [Li and Yau 1986; Ni 2006; Zhu 2013]. Using (4-1), we compute

$$\begin{aligned}(\Delta - \partial_t)\Big(u\ln\frac{A}{u}\Big) &= \Delta u \ln\frac{A}{u} + u\Delta\Big(\ln\frac{A}{u}\Big) + 2\nabla u\nabla\ln\frac{A}{u} \\ &\qquad -\partial_t u\ln\frac{A}{u} - u\partial_t\Big(\ln\frac{A}{u}\Big) \\ &= \Delta u\ln\frac{A}{u} + u\Big(-\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}\Big) - 2\frac{|\nabla u|^2}{u} - \Delta u\ln\frac{A}{u} + \partial_t u \\ &= -\frac{|\nabla u|^2}{u},\end{aligned} \tag{4-4}$$

$$\begin{aligned}\Delta\Big(\frac{|\nabla u|^2}{u}\Big) &= \frac{\Delta|\nabla u|^2}{u} + \Delta\Big(\frac{1}{u}\Big)|\nabla u|^2 + 2\nabla|\nabla u|^2\nabla\Big(\frac{1}{u}\Big) \\ &= \frac{\Delta|\nabla u|^2}{u} + \Big(\frac{2|\nabla u|^2}{u^3} - \frac{\Delta u}{u^2}\Big)|\nabla u|^2 - 4\frac{u_i u_j u_{ij}}{u^2},\end{aligned} \tag{4-5}$$

and

$$\partial_t\Big(\frac{|\nabla u|^2}{u}\Big) = \frac{\partial_t|\nabla u|^2}{u} - \frac{|\nabla u|^2}{u^2}\partial_t u = \frac{2\langle\nabla u, \nabla\Delta u\rangle + 2S_{ij}u_i u_j}{u} - \frac{|\nabla u|^2}{u^2}\Delta u. \tag{4-6}$$

Putting (4-5) and (4-6) together, we get

$$\begin{aligned}(\Delta - \partial_t)\left(\frac{|\nabla u|^2}{u}\right) &= \frac{\Delta|\nabla u|^2}{u} + \frac{2|\nabla u|^4}{u^3} - 4\frac{u_i u_j u_{ij}}{u^2} - \frac{2\langle\nabla u,\nabla\Delta u\rangle + 2S_{ij}u_i u_j}{u} \\ &= \frac{2u_{ij}^2 + 4u_i u_j \phi_i \phi_j}{u} + \frac{2|\nabla u|^4}{u^3} - 4\frac{u_i u_j u_{ij}}{u^2} \\ &= \frac{4}{u}|d\phi(\nabla u)|^2 + \frac{2}{u}\Big|u_{ij} - \frac{u_i u_j}{u}\Big|^2.\end{aligned} \tag{4-7}$$

Combining (4-4) and (4-7), we have

$$\begin{aligned}(\Delta-\partial_t)\Big(\frac{t|\nabla u|^2}{u}-u\ln\frac{A}{u}\Big) &= -\frac{|\nabla u|^2}{u}+\frac{4}{u}|d\phi(\nabla u)|^2+\frac{2}{u}\Big|u_{ij}-\frac{u_iu_j}{u}\Big|^2+\frac{|\nabla u|^2}{u}\\ &= \frac{4}{u}|d\phi(\nabla u)|^2+\frac{2}{u}\Big|u_{ij}-\frac{u_iu_j}{u}\Big|^2\ge 0.\end{aligned}\tag{4-8}$$

By $A=\sup_{M\times[0,T]}u$, we know at $t=0$

$$\frac{t|\nabla u|^2}{u}-u\ln\frac{A}{u}=-u\ln\frac{A}{u}\le 0.$$

Then from (4-8), the maximum principle implies that

$$\frac{t|\nabla u|^2}{u}-u\ln\frac{A}{u}\le 0,$$

giving (4-2).

Set $\ell(x,t)=\ln(A/u(x,t))$. Then inequality (4-2) yields

$$|\nabla\sqrt{\ell(x,t)}|=\frac{1}{2}\Big|\frac{\nabla u}{u\sqrt{\ell}}\Big|\le\frac{1}{\sqrt{4t}}.$$

For each $x,y\in M$, integrating the above inequality along a minimizing geodesic joining $x$ and $y$ yields

$$\sqrt{\ln\frac{A}{u(x,t)}}\le\sqrt{\ln\frac{A}{u(y,t)}}+\frac{d(x,y,t)}{\sqrt{4t}}.$$

Then for any $\delta>0$ it follows

$$\begin{aligned}\ln\frac{A}{u(x,t)} &\le \ln\frac{A}{u(y,t)}+\frac{d^2(x,y,t)}{4t}+\sqrt{\ln\frac{A}{u(y,t)}}\frac{d(x,y,t)}{\sqrt{t}}\\ &\le \ln\frac{A}{u(y,t)}+\frac{d^2(x,y,t)}{4t}+\delta\ln\frac{A}{u(y,t)}+\frac{d^2(x,y,t)}{4t\delta},\end{aligned}$$

proving (4-3). □

Now we turn to proving Theorem 1.2. With the uniform Sobolev inequality in Corollary 3.3 and the interpolation theorem, we establish a mean value inequality via Moser iteration, and a weighted estimate in the spirit of Davies [1989], and then give the full proof of Theorem 1.2.

*Proof of Theorem 1.2.* We divide the proof into two steps.

**Step 1.** Using Morse iteration, we prove a mean value inequality for the positive solution $u$ of the conjugate equation (1-2).

For $p \geq 1$, it follows that

$$\Delta u^p - pSu^p + \partial_t u^p \geq 0. \tag{4-9}$$

Define

$$Q_{\sigma r} := \{(y, s) \mid y \in M, t \leq s \leq t + (\sigma r)^2, d(x, y, s) \leq \sigma r\},$$

with $r > 0$, $1 < \sigma \leq 2$. Let $\varphi(\rho) : [0, +\infty) \to [0, 1]$ be a smooth function satisfying:

$$|\varphi'| \leq \frac{2}{(\sigma - 1)r},$$

$\varphi' \leq 0$, $\varphi \geq 0$, $\varphi(\rho) = 1$ when $0 \leq \rho \leq r$, and $\varphi(\rho) = 0$ when $\rho \geq \sigma r$. Let $\eta(s) : [0, +\infty) \to [0, 1]$ be a smooth function satisfying:

$$|\eta'| \leq \frac{2}{(\sigma - 1)^2 r^2},$$

$\eta' \leq 0$, $\eta \geq 0$, $\eta(s) = 1$ when $s \leq t + r^2$, and $\eta(s) = 0$ when $t + (\sigma r)^2 \leq s \leq T$. Define a cutoff function $\psi(y, s)$ by

$$\psi(y, s) = \varphi(d(y, x, s))\eta(s).$$

Writing $\omega = u^p$, multiplying $\omega\psi^2$ to (4-9) and integrating by parts yield

$$\int_{Q_{\sigma r}} \nabla(\omega\psi^2)\nabla\omega \, dg(y, s) \, ds + p\int_{Q_{\sigma r}} S\omega^2\psi^2 \, dg(y, s) \, ds \leq \int_{Q_{\sigma r}} (\partial_s\omega)\omega\psi^2 \, dg(y, s) \, ds. \tag{4-10}$$

Integrating by parts, the right-hand side of (4-10) gives

$$\int_{Q_{\sigma r}} (\partial_s\omega)\omega\psi^2 \, dg(y, s) \, ds = -\int_{Q_{\sigma r}} \omega^2\psi\partial_s\psi \, dg(y, s) \, ds + \frac{1}{2}\int_{Q_{\sigma r}} (\psi\omega)^2 S \, dg(y, s) \, ds - \frac{1}{2}\int_{B_{\sigma r}(t)} (\psi\omega)^2 \, dg(y, t).$$

By the nonnegativity of $Sy$ and the identity (see [Chow et al. 2006; List 2008])

$$\partial_s d(x, y, s) = -\int_0^{d(x,y,s)} Sy(\gamma'(\tau), \gamma'(\tau)) \, d\tau \leq 0,$$

we have

$$\partial_s\psi = \eta(s)\varphi'(d(y, x, s))\partial_s d(x, y, s) + \varphi(d(y, x, s))\eta'(s) \geq \varphi(d(y, x, s))\eta'(s).$$

Hence

$$\int_{Q_{\sigma r}} (\partial_s \omega)\omega\psi^2 \, dg(y,s)\, ds \le \int_{Q_{\sigma r}} \omega^2 \psi \varphi(d(y,x,s))|\eta'(s)| \, dg(y,s)\, ds + \frac{1}{2}\int_{Q_{\sigma r}} (\psi\omega)^2 S \, dg(y,s)\, ds - \frac{1}{2}\int_{B_{\sigma r}(t)} (\psi\omega)^2 \, dg(y,t). \tag{4-11}$$

Also, note that

$$\int_{Q_{\sigma r}} \nabla(\omega\psi^2)\nabla\omega \, dg(y,s)\, ds = \int_{Q_{\sigma r}} |\nabla(\omega\psi)|^2 \, dg(y,s)\, ds - \int_{Q_{\sigma r}} |\nabla\psi|^2\omega^2 \, dg(y,s)\, ds. \tag{4-12}$$

Then from (4-10), (4-11) and (4-12), we deduce

$$\begin{aligned}\int_{Q_{\sigma r}} |\nabla(\omega\psi)|^2 \, dg(y,s)\, ds + \frac{1}{2}\int_{Q_{\sigma r}} S(\omega\psi)^2 \, dg(y,s)\, ds + \frac{1}{2}\int_{B_{\sigma r}(t)} (\psi\omega)^2 \, dg(y,t) \\ \le \int_{Q_{\sigma r}} \omega^2 \psi\varphi(d(y,x,s))|\eta'(s)| \, dg(y,s)\, ds + \int_{Q_{\sigma r}} |\nabla\psi|^2\omega^2 \, dg(y,s)\, ds \\ \le \frac{c}{(\sigma-1)^2 r^2}\int_{Q_{\sigma r}} \omega^2 \, dg(y,t).\end{aligned} \tag{4-13}$$

Using Hölder's inequality one finds

$$\int (\psi\omega)^{2(1+2/n)} \, dg \le \Big(\int (\psi\omega)^{2n/(n-2)} \, dg\Big)^{(n-2)/n}\Big(\int (\psi\omega)^2 \, dg\Big)^{2/n}, \tag{4-14}$$

and using Corollary 3.3, we see that for each $t \in (0, T)$

$$\Big(\int (\psi\omega)^{2n/(n-2)} \, dg(s)\Big)^{(n-2)/n} \le A_0 \int \big(|\nabla(\psi\omega)|^2 + S(\psi\omega)^2\big) \, dg(s), \tag{4-15}$$

where $A_0$ depends only on the dimension $n$, $\lambda_0$ and the initial metric $g_0$.

By (4-14) and (4-15), we obtain

$$\int_{B_{\sigma r}(s)} (\psi\omega)^{2(1+2/n)} \, dg(s) \le A_0 \Big(\int_{B_{\sigma r}(s)} \big(|\nabla(\psi\omega)|^2 + S(\psi\omega)^2\big) \, dg\Big)\Big(\int_{B_{\sigma r}(s)} (\psi\omega)^2 \, dg\Big)^{2/n}.$$

Setting $\theta = 1+2/n$, integrating the above inequality with respect to $s$ on $[t, t+(\sigma r)^2]$ and using (4-13), we reach

$$\int_{Q_{\sigma r}} (\psi\omega)^{2\theta} \, dg(y,s)\, ds \le A_0 \Big(\frac{1}{(\sigma-1)^2 r^2}\int_{Q_{\sigma r}(x,t)} \omega^2 \, dg(y,s)\, ds\Big)^{\theta},$$

which implies

$$\int_{Q_r} \omega^{2\theta} \, dg(y,s)\, ds \le A_0 \Big(\frac{1}{(\sigma-1)^2 r^2}\int_{Q_{\sigma r}(x,t)} \omega^2 \, dg(y,s)\, ds\Big)^{\theta}. \tag{4-16}$$

Now we choose the sequences of $\sigma_i$ and $p_i$ as $\sigma_0 = 2$, $\sigma_i = 2 - \sum_{j=1}^{i} 2^{-j}$, $p_i = \theta^i$. Then inequality (4-16) gives that

$$\|u^2\|_{L^{\theta^{i+1}}(\sigma_{i+1}r)} \le A_0^{1/\theta^{i+1}} \left( \frac{\sigma_{i+1}^2}{(\sigma_i - \sigma_{i+1})^2 r^2} \right)^{1/\theta^i} \|u^2\|_{L^{\theta^i}(\sigma_i r)},$$

which gives an $L^2$ mean value inequality

$$\sup_{Q_{r/2}(x,t)} u^2 \le \frac{c}{r^{2+n}} \int_{Q_r(x,t)} u^2 \, dg(y,s)\, ds, \tag{4-17}$$

where $c$ depends on the dimension $n$, $\lambda_0$ and the initial metric $g_0$. Then by a generic trick of Li and Schoen (see [Li 2012, Section 32]) we arrive at an $L^1$ mean value inequality: for $r > 0$,

$$\sup_{Q_{r/2}(x,t)} u \le \frac{c}{r^{2+n}} \int_{Q_r(x,t)} u \, dg(y,s)\, ds. \tag{4-18}$$

For $y \in M$ and $s > t$, applying (4-18) on $u = G(\cdot, \cdot : y, T)$ with $r = \sqrt{\frac{1}{2}(T-t)}$ and from the fact $\int_M u(z,\tau)\, dg(z,\tau)\, d\tau = 1$, we conclude

$$G(x,t;y,T) \le \frac{c}{(T-t)^{n/2}}. \tag{4-19}$$

**Step 2.** Using methods of the exponential weight due to Davies [1989], we prove the bound with the exponential term.

It is clear that we only have to deal with the case $d(x_0, y_0, T) \ge 2\sqrt{T-t}$. Otherwise, by (4-19) the Gaussian-type upper bound (1-4) holds obviously. Pick a point $x_0 \in M$, a number $\lambda < 0$ which is determined later and a function $f \in L^2(M, g(T))$. Consider the functions $u(x,t)$ and $H(x,t)$ defined by

$$\begin{aligned} u(x,t) &= \int_M G(x,t;y,T) e^{-\lambda d(y,x_0,T)} f(y)\, dg(y,T), \\ H(x,t) &= e^{\lambda d(x,x_0,t)} u(x,t). \end{aligned}$$

It is clear that $u$ is a solution of (1-2) with initial data

$$u(x,T) = e^{-\lambda d(x,x_0,T)} f(x).$$

Direct calculation shows

$$
\begin{aligned}
\partial_t \int_M H^2(x,t)\,dg(x,t) &= \partial_t \int_M e^{2\lambda d(x.x_0,t)} u^2(x,t)\,dg(x,t) \\
&= 2\lambda \int_M e^{2\lambda d(x.x_0,t)} \partial_t d(x,x_0,t) u^2(x,t)\,dg(x,t) \\
&\quad - \int_M e^{2\lambda d(x.x_0,t)} u^2(x,t) S(x,t)\,dg(x,t) \\
&\quad - 2 \int_M e^{2\lambda d(x.x_0,t)} u(x,t)(\Delta u - Su)\,dg(x,t) \\
&\geq -2 \int_M e^{2\lambda d(x.x_0,t)} u(x,t) \Delta u\,dg(x,t),
\end{aligned}
$$

where the last inequality holds due to $Sy \geq 0$, $\lambda < 0$, and $\partial_t\, d(x, x_0, t) \leq 0$.

By integration by parts, we obtain

$$
\begin{aligned}
\partial_t \int_M H^2(x,t)\,dg(x,t) \geq 4\lambda \int_M e^{2\lambda d(x.x_0,t)} u(x,t) \langle \nabla u, \nabla d(x,x_0,t)\rangle\,dg(x,t) \\
+ 2 \int_M e^{2\lambda d(x.x_0,t)} |\nabla u|^2\,dg(x,t),
\end{aligned}
$$

and also

$$
\begin{aligned}
&\int_M |\nabla H(x,t)|^2\,dg(x,t) \\
&= \int_M |\nabla (u(x,t) e^{\lambda d(x.x_0,t)})|^2\,dg(x,t) \\
&= \int_M e^{2\lambda d(x.x_0,t)} |\nabla u|^2\,dg(x,t) + 2\lambda \int_M e^{2\lambda d(x.x_0,t)} u(x,t) \langle \nabla u, \nabla d(x,x_0,t)\rangle\,dg(x,t) \\
&\qquad + \lambda^2 \int_M e^{2\lambda d(x.x_0,t)} |\nabla d|^2 u^2\,dg(x,t).
\end{aligned}
$$

Combining the above two expressions, we conclude

$$
\partial_t \int_M H^2(x,t)\,dg(x,t) \geq 2 \int_M |\nabla H(x,t)|^2\,dg(x,t) - 2\lambda^2 \int_M e^{2\lambda d(x.x_0,t)} u^2\,dg(x,t),
$$

which implies

$$
\partial_t \int_M H^2(x,t)\,dg(x,t) \geq -2\lambda^2 \int_M H^2(x,t)\,dg(x,t).
$$

Integrating on $[t, T]$, we arrive at the $L^2$ estimate

$$\text{(4-20)}\qquad \int_M H^2(x,t)\,dg(x,t) \le e^{2\lambda^2(T-t)} \int_M H^2(x,T)\,dg(x,T) = e^{2\lambda^2(T-t)} \int_M f^2(x)\,dg(x,T).$$

Therefore, by the mean value inequality (4-17) with $r = \sqrt{\frac{1}{2}(T-t)}$, it holds that

$$u^2(x,t) \le \frac{c}{(T-t)^{1+n/2}} \int_t^{(T+t)/2} \int_{B(x,\sqrt{(T-t)/2},\tau)} u^2(z,\tau)\,dg(z,\tau)\,d\tau \le \frac{c}{(T-t)^{1+n/2}} \int_t^{(T+t)/2} \int_{B(x,\sqrt{(T-t)/2},\tau)} e^{-2\lambda d(z,x_0,\tau)} H^2(z,\tau)\,dg(z,\tau)\,d\tau.$$

Particularly, at $x = x_0$, we get

$$u^2(x_0,t) \le \frac{ce^{-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{1+n/2}} \int_t^{(T+t)/2} \int_{B(x_0,\sqrt{(T-t)/2},\tau)} H^2(z,\tau)\,dg(z,\tau)\,d\tau.$$

From (4-20), it follows that

$$u^2(x_0,t) \le \frac{ce^{2\lambda^2(T-t)-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{n/2}} \int_M f^2(y)\,dg(y,T),$$

i.e.,

$$\text{(4-21)}\qquad \left(\int_M G(x_0,t;z,T)e^{-\lambda d(z,x_0,T)} f(z)\,dg(z,T)\right)^2 \le \frac{ce^{2\lambda^2(T-t)-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{n/2}} \int_M f^2(y)\,dg(y,T).$$

Now we fix $y_0$ such that $d(y_0,x_0,T)^2 \ge 4(T-t)$. Then it follows from the triangle inequality that

$$-\lambda d(z,x_0,T) \ge -\tfrac{1}{2}\lambda d(x_0,y_0,T),$$

provided by $d(z,y_0,T) \le \sqrt{T-t}$. Then (4-21) implies

$$\left(\int_{B(y_0,\sqrt{T-t},T)} G(x_0,t;z,T) f(z)\,dg(T)\right)^2 \le \frac{ce^{\lambda d(x_0,y_0,T)+2\lambda^2(T-t)-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{n/2}} \int_M f^2(y)\,dg(T).$$

Note that by the Cauchy–Schwartz inequality

$$2\lambda^2(T-t) - 2\lambda\sqrt{\tfrac{1}{2}(T-t)} \le 3\lambda^2(T-t) + \tfrac{1}{2},$$

and letting $\lambda = -d(x_0, y_0, T)/(b(T-t))$, we obtain

$$\left(\int_{B(y_0,\sqrt{T-t},T)} G(x_0, t; z, T) f(z)\, dg(z, T)\right)^2 \le \frac{ce^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}}}{(T-t)^{n/2}} \int_M f^2(y)\, dg(y, T)$$

with $b > 0$ sufficiently large, and $c_1$ is an absolute constant. Then by the arbitrariness of $f$, we derive

$$\int_{B(y_0,\sqrt{T-t},T)} G^2(x_0, t; z, T)\, dg(z, T) \le \frac{ce^{-c_1\frac{d^2(x_0,y_0,T)}{T-t}}}{(T-t)^{n/2}}.$$

Hence, there exists $z_0 \in B(y_0, \sqrt{T-t}, T)$ such that

$$G^2(x_0, t; z_0, T) \le \frac{ce^{-c_1\frac{d^2(x_0,y_0,T)}{T-t}}}{(T-t)^{n/2}|B(y_0, \sqrt{T-t}, T)|_T}. \tag{4-22}$$

Let us recall that in [Guenther 2002] the adjoint property of the $G(x_0, t : \cdot, \cdot)$ is obtained, thus

$$\Delta_z G(x, t; z, \tau) - \partial_\tau G(x, t; z, \tau) = 0$$

along Ricci-harmonic flow (1-1).

Choosing $\delta = 1$ in Theorem 4.1 and $z_0 \in B(y_0, \sqrt{T-t}, T)$, it then follows that

$$\begin{aligned} G(x_0, t; y_0, T) &\le \sqrt{G(x_0, t; z_0, T)}\sqrt{A}e^{d^2(y_0,z_0,T)/4(T-t)} \\ &\le e^{1/4}\sqrt{G(x_0, t; z_0, T)}\sqrt{A}, \end{aligned} \tag{4-23}$$

where $A = \sup_{M\times[(t+T)/2,T]} G(x_0, t; \cdot, \cdot)$.

Since (4-19) implies

$$A \le \frac{c}{(T-t)^{n/2}},$$

then combining with (4-22) and (4-23) we have

$$G(x_0, t; y_0, T)^2 \le \frac{c}{(T-t)^{n/2}} \frac{1}{(T-t)^{n/4}\sqrt{|B(y_0, \sqrt{T-t}, T)|_T}} e^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}}.$$

Therefore by the Cauchy–Schwartz inequality, we get

$$\begin{aligned} G(x_0, t; y_0, T) &\le c\left(\frac{1}{(T-t)^{n/2}} + \frac{1}{|B(y_0, \sqrt{T-t}, T)|_T}\right)e^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}} \\ &\le \frac{c}{|B(y_0, \sqrt{T-t}, T)|_T} e^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}}, \end{aligned}$$

where $c$ depends on the dimension $n$, $\lambda_0$ and the initial metric $g_0$, and $c_1$ depends only on dimension $n$. In the last inequality, we used the volume comparison theorem with the nonnegative Ricci curvature. By the arbitrariness of $x_0$ and $y_0$, we complete the proof. □

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