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C^1-UMBILICS WITH ARBITRARILY HIGH INDICES

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We show that C^1-umbilics with arbitrarily high indices exist. This implies that more than C^1-regularity is required to prove Loewner's conjecture.

1. Introduction

The *index* of an isolated umbilic on a given regular surface is the index of the curvature line flow of the surface at that point, which takes values in the set of half-integers. *Loewner's conjecture* asserts that any isolated umbilic on an immersed surface must have index at most 1. *Carathéodory's conjecture* asserts the existence of at least two umbilics on an immersed sphere in $\mathbb{R}^3$, which follows immediately from Loewner's conjecture. Although this problem was investigated mainly on real-analytic surfaces after Hamburger's work [1940; 1941a; 1941b], several geometers recently became interested in nonanalytic cases; see [Ando 2003; Bates 2001; Ghomi and Howard 2012; Gutierrez et al. 1996; Smyth and Xavier 1992]. In particular, Smyth and Xavier [1992] observed that Enneper's minimal surface is inverted to a branched sphere such that the index of the curvature line flow at the branch point is equal to two. Bates [2001] found that the graph of the function

$$B(x, y) := 2 + \frac{xy}{\sqrt{1+x^2}\sqrt{1+y^2}} \tag{1-1}$$

has no umbilics on $\mathbb{R}^2$ and inversion of it gives a genus zero surface without self-intersections, which is differentiable at the image of infinity under that inversion. Ghomi and Howard [2012] gave similar examples of genus zero surfaces using inversion. Moreover, they showed that Carathéodory's conjecture for closed convex surfaces can be reduced to the problem of existence of umbilics of certain entire graphs over $\mathbb{R}^2$. A brief history of Carathéodory's conjecture and recent developments are written also in [Ghomi and Howard 2012]. Recently, Guilfoyle and

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Klingenberg [2008; 2012] gave an approach to proving the Carathéodory and the Loewner conjectures in the smooth case.

Let $P : U \to \mathbb{R}^3$ be a C^1-immersion defined on an open subset U of $\mathbb{R}^2$ such that P is C^∞-differentiable on $U \setminus \{q\}$ and not C^2-differentiable at q. Then the point $q \in U$ is called a C^1-*umbilic* if the umbilics of P on $U \setminus \{q\}$ do not accumulate to q. At that point q, we can compute the index of the curvature line flow of P. In this paper, we prove the following assertion:

Theorem 1.1. *Let $U_1 \subset \mathbb{R}^2$ be the unit disk centered at the origin. For each positive integer m, there exists a C^1-function $f : U_1 \to \mathbb{R}$ satisfying the following properties*:

(1) *f is real-analytic on $U_1^* := U_1 \setminus \{(0,0)\}$*,

(2) $(0, 0, f(0,0))$ *is a C^1-umbilic of the graph of f with index* $1 + (m/2)$.

It should be remarked that the inversion of the graph of Bates' function $B(x, y)$ has a differentiable umbilic of index 2 although not of class C^1 (see Example 2.3). It was classically known that curvature line flows are closely related to the eigenflows of the Hessian matrices of functions (see Appendix A). As an application of the above result, we can show the following:

Corollary 1.2. *For each $m \geq 1$, there exists a C^1-function $\lambda : U_1 \to \mathbb{R}$ satisfying*

(1) *λ is real-analytic on U_1^*, and*

(2) *the eigenflow of the Hessian matrix of λ has an isolated singular point* $(0, 0)$ *with index* $1 + (m/2)$.

When we consider the eigenflow of the Hessian matrix of f, it is well known that the index of the flow at an isolated singular point is equal to half of the index of the vector field

$$d_f := 2 f_{xy} \frac{\partial}{\partial x} + (f_{yy} - f_{xx}) \frac{\partial}{\partial y}. \tag{1-2}$$

In addition, if $o := (0, 0)$ is an isolated singular point of the eigenflow of the Hessian matrix of f, then its index is equal to $1 + \mathrm{ind}_o(\delta_f)/2$ (see Appendix B), where $\mathrm{ind}_o(\delta_f)$ is the index of the vector field

$$\delta_f := 2(r f_{r\theta} - f_\theta) \frac{\partial}{\partial x} + (-r^2 f_{rr} + r f_r + f_{\theta\theta}) \frac{\partial}{\partial y} \tag{1-3}$$

at o, and $x = r\cos\theta,\ y = r\sin\theta$. In order to prove the above theorem, we introduce vector fields D_f and Δ_f analogous to d_f and δ_f, respectively (see Propositions 3.3 and 4.2), and prove the theorem by computing the index of Δ_f at infinity for each of the functions (see Section 5)

$$f = f_m(r, \theta) := 1 + \tanh(r^a \cos m\theta), \quad 0 < a < 1/4,\ m = 1, 2, \ldots. \tag{1-4}$$

We also give an alternative proof of Theorem 1.1 without use of inversion, by an explicit example of λ, see (6-1), satisfying (1) and (2) of Corollary 1.2 (see Section 6).

2. The regularity of the inversion

Let R be a positive number. Consider a function $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$, where

$$\Omega_R := \left\{(x, y) \in \mathbb{R}^2 \,;\, \sqrt{x^2+y^2} \le R\right\}. \tag{2-1}$$

Then $F = (x, y, f(x, y))$ gives a parametrization of the graph of f. The inversion of F is given by $F/(F \cdot F)$, where the dot denotes the inner product on $\mathbb{R}^3$. We consider the following coordinate change:

$$x = \frac{u}{u^2+v^2}, \qquad y = \frac{v}{u^2+v^2}. \tag{2-2}$$

Then

$$\Psi_f := \frac{1}{\rho^2 \hat{f}^2+1}(u, v, \rho^2 \hat{f}\,), \qquad \hat{f}(u, v) := f\left(\frac{u}{\rho^2}, \frac{v}{\rho^2}\right) \tag{2-3}$$

gives a parametrization of the inversion, where $\rho := \sqrt{u^2+v^2}$. The map Ψ_f is defined on the domain

$$U^*_{1/R} := U_{1/R} \setminus \{o\}, \quad \left(U_{1/R} := \left\{(u, v) \in \mathbb{R}^2 \,;\, \sqrt{u^2+v^2} < \frac{1}{R}\right\}\right), \tag{2-4}$$

where $o := (0, 0)$. If we set

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{2-5}$$

where $r > 0$, then (2-2) yields

$$\rho = \frac{1}{r}, \qquad u = \rho\cos\theta, \qquad v = \rho\sin\theta. \tag{2-6}$$

In particular, the angular parameter is common in the xy-plane and the uv-plane.

Proposition 2.1. *Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a C^∞-function such that f/r is bounded. Then the inversion $\Psi_f : U^*_{1/R} \to \mathbb{R}^3$ can be continuously extended to* $(0, 0)$, *and moreover, if*

$$\left|\frac{f^2 - 2rff_r}{r^2}\right| < 1, \qquad r > R, \tag{2-7}$$

then the image of $\Psi_f = (X, Y, Z)$ can be locally expressed as the graph of a function $Z = Z_f(X, Y)$ on a neighborhood of $(0, 0)$ *in the XY-plane. Under the assumption* (2-7), *the function $Z_f(X, Y)$ is differentiable if and only if*

$$\lim_{r\to\infty} \frac{f}{r} = 0.$$

Proof. We can write

$$\Psi_f(u, v) = \frac{1}{1+\varphi(u, v)^2}\left(u, v, \varphi(u, v)\sqrt{u^2+v^2}\right), \tag{2-8}$$

where

$$\varphi(u,v)=\sqrt{u^2+v^2}\,\hat{f}(u,v)=\frac{f(x,y)}{r}. \tag{2-9}$$

Since f/r is bounded, the function φ is bounded on $U^*_{1/R}$. Thus, using (2-8), we can prove $\lim_{\rho\to 0}\Psi_f=(0,0,0)$, i.e., $\Psi_f(u,v)$ can be continuously extended to $(0,0)$. We denote by $\Pi:\mathbb{R}^3\ni(x,y,z)\mapsto(x,y)\in\mathbb{R}^2$ the orthogonal projection. Setting

$$\psi(\rho,\theta):=\frac{\rho}{1+\varphi(\rho\cos\theta,\rho\sin\theta)^2},$$

it holds that

$$\Pi\circ\Psi_f(u,v)=\big(\psi(\rho,\theta)\cos\theta,\ \psi(\rho,\theta)\sin\theta\big). \tag{2-10}$$

Since $\hat{f}(\rho\cos\theta,\rho\sin\theta)=f(\cos\theta/\rho,\sin\theta/\rho)$, we have

$$\varphi_\rho=f-rf_r.$$

In particular, it holds that

$$\psi_\rho=\frac{1-(f^2-2rff_r)/r^2}{(1+f^2/r^2)^2}.$$

By (2-7), there exists $\varepsilon>0$ such that $\rho\mapsto\psi(\rho,\theta)$, $|\rho|\le\varepsilon$, is a monotone increasing function for each θ. Thus, by (2-10), we can conclude that $\Pi\circ\Psi_f:\overline{U}_\varepsilon\to\mathbb{R}^2$ is an injection. Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, the inverse map $G:\Omega\to U_\varepsilon$ of $\Pi\circ\Psi_f|_{U_\varepsilon}$ is continuous, where Ω is a neighborhood of the origin of the XY-plane in $\mathbb{R}^3$. Then the graph of

$$Z_f\left(=\frac{\rho\varphi}{1+\varphi^2}\right)=\frac{\varphi(G(X,Y))\rho(G(X,Y))}{1+\varphi(G(X,Y))^2} \tag{2-11}$$

coincides with the image of $\Psi_f=(X,Y,Z)$ around $(0,0,0)$. Then

$$X=\frac{u}{1+\varphi^2},\quad Y=\frac{v}{1+\varphi^2},\quad Z=\frac{\rho\varphi}{1+\varphi^2}.$$

Since $\rho\to 0$ as $(X,Y)\to(0,0)$, we obtain

$$\lim_{(X,Y)\to(0,0)}\frac{Z_f(X,Y)}{\sqrt{X^2+Y^2}}=\lim_{(X,Y)\to(0,0)}\frac{\varphi\rho}{\sqrt{u^2+v^2}}=\lim_{\rho\to 0}\varphi=\lim_{r\to\infty}\frac{f}{r}. \tag{2-12}$$ □

Corollary 2.2. *Suppose that $f:\mathbb{R}^2\setminus\Omega_R\to\mathbb{R}$ is a bounded C^∞-function satisfying*

$$\lim_{r\to\infty}\frac{f_r}{r}=0. \tag{2-13}$$

Then the inversion $\Psi_f:U^*_{1/R}\to\mathbb{R}^3$ can be continuously extended to $(0,0)$, and moreover, the image of Ψ_f is locally a graph which is differentiable at $(0,0)$.

Example 2.3. Bates' example mentioned in the introduction is differentiable. In fact, $B(x, y)$ in (1-1) is bounded and B_r/r converges to zero as $r \to \infty$. However, the inversion of $(x, y, B(x, y))$ is not C^1. In fact, the unit normal vector field of the graph of B is not continuously extended to the point at infinity. Since the inversion preserves the angle, the unit normal vector field of its inversion cannot be continuously extended to $(0, 0, 0)$.

Example 2.4. Ghomi and Howard [2012] gave an example:

$$f_{\mathrm{GH}} = 1 + \lambda \frac{1 + x + y^2}{\sqrt{1 + (x + y^2)^2}}, \qquad (\lambda > 0). \tag{2-14}$$

The graph of f_{GH} is umbilic-free (see Example 3.5 in Section 3). The function f_{GH} is bounded. In addition, since $(f_{\mathrm{GH}})_r$ is bounded, (2-13) is obvious. Therefore, as pointed out in [Ghomi and Howard 2012], the inversion of $(x, y, f_{\mathrm{GH}}(x, y))$ is differentiable. However, it is not a C^1-map. In fact, the limit of the unit normal vector field along $y = 0$ of the graph of f_{GH} is not equal to that along $x + y^2 = 0$ at the point at infinity.

Next, we give a condition for Ψ_f to be extendable as a C^1-map to $(0, 0)$.

Proposition 2.5. *Suppose that* $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ *is a bounded* C^∞*-function satisfying*

(a) $\lim_{r\to\infty} f_r = 0$,

(b) $\lim_{r\to\infty} f_\theta / r = 0$.

Then $\Psi_f = (X, Y, Z)$ *can be extended to* $(0, 0)$ *as a* C^1*-map. Moreover, the map* $(u, v) \mapsto (X(u, v), Y(u, v))$ *is a* C^1*-diffeomorphism from a neighborhood of the origin in the* uv*-plane onto a neighborhood of the origin in the* XY*-plane.*

To prove this, we prepare the following lemma.

Lemma 2.6. *The conditions* (a) *and* (b) *in Proposition 2.5 are equivalent to the following two conditions, respectively*:

(1) $\lim_{\rho\to 0} \rho^2 \hat{f}_\rho = 0$,

(2) $\lim_{\rho\to 0} \rho \hat{f}_\theta = 0$.

Proof. The equivalency of (2) and (b) is obvious. The equivalency of (1) and (a) follows from the identity $\hat{f}_\rho = -f_r/\rho^2$. □

Proof of Proposition 2.5. We see by Corollary 2.2 that Ψ_f can be extended to $(0, 0)$ as a differentiable map and that the map $(u, v) \mapsto (X(u, v), Y(u, v))$ is a homeomorphism from a neighborhood of $(0, 0)$ onto a neighborhood of $(0, 0)$. We set

$$h := \rho^2 \hat{f} (= \rho\varphi), \quad k := (\rho \hat{f})^2 (= \varphi^2). \tag{2-15}$$

By (2-3), we can write

$$\Psi_f = (X, Y, Z) = \frac{1}{k+1}(u, v, h). \tag{2-16}$$

To show that Ψ_f is a C^1-map at $(0, 0)$, it is sufficient to show that h and k are C^1-functions. Since h and k are C^∞-functions on $U^*_{1/R}$, they satisfy

$$\begin{aligned} h_u &= \rho\bigl((2\hat{f} + \rho\hat{f}_\rho)\cos\theta - \hat{f}_\theta \sin\theta\bigr),\\ h_v &= \rho\bigl((2\hat{f} + \rho\hat{f}_\rho)\sin\theta + \hat{f}_\theta \cos\theta\bigr), \end{aligned} \tag{2-17}$$

$$\begin{aligned} k_u &= 2\hat{f}\rho\bigl(\cos\theta(\hat{f} + \rho\hat{f}_\rho) - \hat{f}_\theta \sin\theta\bigr),\\ k_v &= 2\hat{f}\rho\bigl(\sin\theta(\hat{f} + \rho\hat{f}_\rho) + \hat{f}_\theta \cos\theta\bigr), \end{aligned} \tag{2-18}$$

on $U^*_{1/R}$. Using (1), (2) in Lemma 2.6, (2-17) and (2-18), one can easily see that

$$\lim_{\rho\to 0} h_u = \lim_{\rho\to 0} h_v = \lim_{\rho\to 0} k_u = \lim_{\rho\to 0} k_v = 0, \tag{2-19}$$

which shows that Ψ_f extends to $(0, 0)$ as a C^1-map. By (2-16) and (2-19), we have

$$X_u(0, 0) = 1, \quad X_v(0, 0) = 0, \quad Y_u(0, 0) = 0, \quad Y_v(0, 0) = 1.$$

Thus the second assertion follows from the inverse mapping theorem, because the Jacobi matrix of the map $(u, v) \mapsto (X(u, v), Y(u, v))$ is regular at $(0, 0)$. □

In Section 5, we need the following:

Proposition 2.7. *Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a bounded C^∞-function satisfying conditions* (a) *and* (b) *of Proposition 2.5. If there exists a constant $0 \le c < \frac{1}{2}$ such that*

$$r^{1-c/2} f_r, \quad r^{-c/2} f_\theta, \quad r^{2-c} f_{rr}, \quad r^{1-c} f_{r\theta}, \quad r^{-c} f_{\theta\theta}$$

are bounded on $\mathbb{R}^2 \setminus \Omega_R$, then the map $(u, v) \mapsto (X(u, v), Y(u, v))$ is a C^2-map at $(0, 0)$, where $\Psi_f = (X, Y, Z)$.

We prepare the following lemmas:

Lemma 2.8. *The boundedness of the five functions in Proposition 2.7 is equivalent to the boundedness of the functions*

$$\rho^{1+c/2} \hat{f}_\rho, \quad \rho^{c/2} \hat{f}_\theta, \quad \rho^{2+c} \hat{f}_{\rho\rho}, \quad \rho^{1+c} \hat{f}_{\rho\theta}, \quad \rho^c \hat{f}_{\theta\theta} \tag{2-20}$$

on $U \setminus \{(0, 0)\}$, where U is a sufficiently small neighborhood of $(0, 0)$.

Proof. Differentiating $\hat{f} = \hat{f}(\rho\cos\theta, \rho\sin\theta)$ by ρ, we get $\rho\hat{f}_\rho = -r f_r$ and $\rho^2 \hat{f}_{\rho\rho} = 2r f_r + r^2 f_{rr}$, which can be used to check the assertion. □

Lemma 2.9. *Suppose that the five functions in* (2-20) *are bounded on* $U \setminus \{(0,0)\}$. *Then* $\rho^{2c} k_{uu}$, $\rho^{2c} k_{uv}$ *and* $\rho^{2c} k_{vv}$ *are also bounded on* $U \setminus \{(0,0)\}$, *where* k *is the function given in* (2-15).

Proof. In fact, each of k_{uu}, k_{uv}, k_{vv} is written as a linear combination of

$$1, \quad \rho \hat{f}_\rho, \quad \hat{f}_\theta, \quad (\rho \hat{f}_\rho)^2, \quad \rho \hat{f}_\rho \hat{f}_\theta, \quad \hat{f}_\theta^2, \quad \rho^2 \hat{f}_{\rho\rho}, \quad \rho \hat{f}_{\rho\theta}, \quad \hat{f}_{\theta\theta},$$

with coefficients that are bounded functions. For example,

$$k_{uv} = \sin 2\theta \big(\rho^2 \hat{f}_\rho^2 + \hat{f}(\rho^2 \hat{f}_{\rho\rho} + 3\rho \hat{f}_\rho - \hat{f}_{\theta\theta}) - \hat{f}_\theta^2\big) + 2\cos 2\theta \big(\hat{f}_\theta(\rho \hat{f}_\rho + \hat{f}) + \rho \hat{f} \hat{f}_{\rho\theta}\big).$$

Thus, we get the assertion. □

Proof of Proposition 2.7. By Lemmas 2.8 and 2.9, the fact that $2c < 1$ yields that

$$\lim_{\rho\to 0} \rho k_{uu} = \lim_{\rho\to 0} \rho k_{uv} = \lim_{\rho\to 0} \rho k_{vv} = 0. \tag{2-21}$$

Since

$$X_{uu} = \frac{2uk_u^2 - 2(k+1)k_u - u(k+1)k_{uu}}{(k+1)^3},$$

$$X_{uv} = -\frac{k_v(-2uk_u + k + 1) + u(k+1)k_{uv}}{(k+1)^3},$$

$$X_{vv} = -\frac{u\big((k+1)k_{vv} - 2k_v^2\big)}{(k+1)^3},$$

we have that X_{uu}, X_{uv}, X_{vv} tend to 0 as $\rho \to 0$. This implies that X_u, X_v are C^1-functions. Similarly, Y_u, Y_v are also C^1-functions. □

3. The pair of identifiers for umbilics

Let U be a domain on $\mathbb{R}^2$. Consider a flow (i.e., a 1-dimensional foliation) $\mathcal{F}$ defined on $U \setminus \{p_1, \dots, p_n\}$, where $p_1, \dots, p_n$ are distinct points in U. We are interested in the case where $\mathcal{F}$ is

- the curvature line flow of an immersion $P : U \to \mathbb{R}^3$,
- the eigenflow of a matrix-valued function on U, or
- the flow induced by a vector field on U.

We fix a simple closed smooth curve $\gamma : T^1 \to U \setminus \{p_1, \dots, p_n\}$, where $T^1 := \mathbb{R}/2\pi\mathbb{Z}$. We set

$$\partial_x := \frac{\partial}{\partial x}, \qquad \partial_y := \frac{\partial}{\partial y}.$$

Then one can take a smooth vector field

$$V(t) := a(t)\partial_x + b(t)\partial_y$$

along the curve $\gamma(t)$ such that $V(t)$ is a nonzero tangent vector of $\mathbb{R}^2$ at $\gamma(t)$ which points in the direction of the flow $\mathcal{F}$. Then the map

$$\check{V} : T^1 \ni t \mapsto \frac{(a(t), b(t))}{\sqrt{a(t)^2 + b(t)^2}} \in S^1 := \{\boldsymbol{x} \in \mathbb{R}^2 \,;\ |\boldsymbol{x}| = 1\} \tag{3-1}$$

is called the *Gauss map* of $\mathcal{F}$ with respect to the curve γ. The mapping degree of the map $\check{V}$ is called the *rotation index* of $\mathcal{F}$ with respect to γ and denoted by $\operatorname{ind}(\mathcal{F}, \gamma)$, which is a half-integer, in general. If γ surrounds only p_j, then $\operatorname{ind}(\mathcal{F}, \gamma)$ is independent of the choice of such a curve γ. So we call it the (*rotation*) *index* of the flow $\mathcal{F}$ at p_j, and it is denoted by $\operatorname{ind}_{p_j}(\mathcal{F})$. If the flow $\mathcal{F}$ is generated by a vector field V defined on $U \setminus \{p_1, \dots, p_n\}$, then $\operatorname{ind}_{p_j}(\mathcal{F})$ is an integer, and we denote it by $\operatorname{ind}_{p_j}(V)$.

We denote by $S_2(\mathbb{R})$ the set of real symmetric 2-matrices. Let U be a domain in $\mathbb{R}^2$, and

$$A = \begin{pmatrix} a_{11}(x, y) & a_{12}(x, y) \\ a_{12}(x, y) & a_{22}(x, y) \end{pmatrix} : U \to S_2(\mathbb{R}),$$

a C^∞-map. A point $p \in U$ is called an *equidiagonal point* of A if $a_{11} = a_{22}$ and $a_{12} = 0$ at p. We now suppose that p is an isolated equidiagonal point. Without loss of generality, we may assume that A has no equidiagonal points on $U \setminus \{p\}$. Since two eigenflows of A are mutually orthogonal, the indices of the two eigenflows of the $S_2(\mathbb{R})$-valued function A are the same half-integer at p. We denote it by $\operatorname{ind}_p(A)$.

It is well known that for an $S_2(\mathbb{R})$-valued function A, the formula

$$\operatorname{ind}_p(A) = \tfrac{1}{2} \operatorname{ind}_p(\boldsymbol{v}_A) \tag{3-2}$$

holds, where $\boldsymbol{v}_A$ is the vector field on U given by

$$\boldsymbol{v}_A := (a_{11} - a_{22})\partial_x + a_{12}\partial_y. \tag{3-3}$$

We shall apply these facts to the computation of the indices of isolated umbilics on regular surfaces in $\mathbb{R}^3$ as follows. Let $f : U \to \mathbb{R}$ be a C^∞-function. The symmetric matrices associated with the first and the second fundamental forms of the graph of f are given by

$$I := \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}, \qquad II := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}. \tag{3-4}$$

We consider a $\mathrm{GL}(2, \mathbb{R})$-valued function

$$P := \begin{pmatrix} 0 & \sqrt{1 + f_x^2} \\ -\sqrt{(1 + f_x^2 + f_y^2)/(1 + f_x^2)} & f_x f_y / \sqrt{1 + f_x^2} \end{pmatrix}, \tag{3-5}$$

which satisfies the identity $PP^T = I$, where P^T is the transpose of P. Then

$$A_f := P^{-1}II(P^T)^{-1} = P^T(I^{-1}II)(P^T)^{-1}$$

is an $S_2(\mathbb{R})$-valued function. The umbilics of the graph of f correspond to the equidiagonal points of A_f. We show the following:

Proposition 3.1. *The symmetric matrix $A_f(p)$ is proportional to the identity matrix at $p \in U$ if and only if p gives an umbilic of the graph of f. Moreover, if p is an isolated umbilic, then* $\operatorname{ind}_p(A_f)$ *coincides with the index of the umbilic p.*

Proof. The first assertion follows from the definition of A_f. Without loss of generality, we may assume that p coincides with the origin $o := (0, 0)$, and the graph of f has no umbilics other than o on U. Take a sufficiently small positive number $\varepsilon > 0$ so that the circle

$$\gamma(t) = \varepsilon(\cos t, \sin t), \qquad 0 \le t \le 2\pi,$$

is null-homotopic in U.

We denote by $(a_1(t), b_1(t))^T$ and $(a_2(t), b_2(t))^T$, eigenvectors of $I^{-1}II$ and A_f at $\gamma(t)$, respectively. We may suppose

$$(a_1(t), b_1(t))P(\gamma(t)) = (a_2(t), b_2(t)), \qquad 0 \le t \le 2\pi.$$

We set

$$\boldsymbol{w}_i(t) := a_i(t)\partial_x + b_i(t)\partial_y, \qquad i = 1, 2.$$

Then $\boldsymbol{w}_1$ points in one of the principal directions of the graph of f. The matrix $P(\gamma(t))$ takes values in the set

$$\mathcal{T} := \left\{ \begin{pmatrix} 0 & x \\ -y & z \end{pmatrix} ;\ x, y > 0,\ z \in \mathbb{R} \right\}. \tag{3-6}$$

Since the set $\mathcal{T}$ is null-homotopic, the mapping degree of $\check{\boldsymbol{w}}_1(t)$ with respect to the origin is equal to that of $\check{\boldsymbol{w}}_2(t)$. Since the degree of $\check{\boldsymbol{w}}_2(t)$ with respect to o coincides with $\operatorname{ind}_o(A_f)$, we get the second assertion. □

By a straightforward calculation, one can get the following identity:

$$\tilde{A}_f := hk^3 A_f = \begin{pmatrix} f_x f_y(f_x f_y f_{xx} - 2h f_{xy}) + h^2 f_{yy} & lk \\ lk & k^2 f_{xx} \end{pmatrix},$$

where

$$h := 1 + f_x^2, \quad k := \sqrt{1 + f_x^2 + f_y^2}, \quad l := -h f_{xy} + f_x f_y f_{xx}.$$

Then the coefficients of the vector field

$$\boldsymbol{v}_{\tilde{A}_f} = v_1\partial_x + v_2\partial_y$$

defined as in (3-3) for $A = \tilde{A}_f$ are given by

$$v_1 = \tilde{a}_{11} - \tilde{a}_{22} = (-1 + f_x^2) f_y^2 f_{xx} - h f_{xx} - 2h f_x f_{xy} f_y + h^2 f_{yy},$$
$$v_2 = \tilde{a}_{12} = -k(h f_{xy} - f_x f_y f_{xx}),$$

where $\tilde{A}_f = (\tilde{a}_{ij})_{i,j=1,2}$. Hence, we get the following identity:

$$v_1 = \frac{2 f_x f_y}{k} v_2 + h\big(-f_{xx}(1 + f_y^2) + (1 + f_x^2) f_{yy}\big).$$

Consequently, we get the following fact (see [Ghomi and Howard 2012, (10)]):

Fact 3.2. *The graph of the function $z = f(x, y)$ defined on U has an umbilic at $p \in U$ if and only if the functions*

$$d_1(x, y) := (1+f_x^2) f_{xy} - f_x f_y f_{xx}, \quad \textit{and} \quad d_2(x, y) := (1+f_x^2) f_{yy} - f_{xx}(1+f_y^2)$$

both vanish at p.

We consider the vector field

$$D_f := d_1 \partial_x + d_2 \partial_y$$

defined on the domain U in the xy-plane. Suppose that p is a zero of D_f. The following assertion holds:

Proposition 3.3. *If p gives an isolated umbilic of the graph of f, then half of the index of the vector field D_f at p coincides with the index of the umbilic p.*

Proof. The half of the index of the vector field

$$X := -\boldsymbol{v}_{\hat{A}_f} = (2 f_x f_y d_1 - h d_2)\partial_x + k d_1 \partial_y$$

at p is equal to $\operatorname{ind}_p(\tilde{A}_f)$. We now set

$$X_s := (\partial_x, \partial_y) \begin{pmatrix} \dfrac{2s f_x f_y}{\sqrt{1 + s(f_x^2 + f_y^2)}} & -1 - s f_x^2 \\ & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \qquad 0 \le s \le 1.$$

Then $X = X_1$ and $X_0 = -d_2 \partial_x + d_1 \partial_y$, and the rotation index of X_s at p does not depend on $s \in [0, 1]$. Since the rotation index of $D_f = (d_1, d_2)$ at p coincides with that of X_0, we can conclude that X has the same rotation index as D_f at p. □

We call d_1, d_2 the *Cartesian umbilic identifiers* of the function f.

Example 3.4. For a function $f(x, y) := \operatorname{Re}(z^3) = x^3 - 3xy^2$ $(z = x + iy)$, the Cartesian umbilic identifiers are given by $d_1 = -6y\varphi_1$, $d_2 = -6x\varphi_2$, where

$$\varphi_1 := -9x^4 + 9y^4 + 1, \qquad \varphi_2 := 9x^4 + 18x^2y^2 + 9y^4 + 2.$$

Since $\varphi_i,\ i=1,2$, are positive at the origin $(0,0)$, the vector field D_f can be continuously deformed into the vector field $-y\partial_x - x\partial_y$ preserving the property that the origin is an isolated zero. Thus D_f is of index -1, and the graph of the function f has an isolated umbilic of index $-\frac{1}{2}$ at the origin.

Example 3.5. Bates' function $B(x,y)$ has no umbilics since $d_1 > 0$ on $\mathbb{R}^2$. On the other hand, the identifier d_1 with respect to Ghomi and Howard's function $f_{\mathrm{GH}}(x,y)$ in (2-14) vanishes if and only if $y=0$ or $x=-y^2$. Since d_2 never vanishes on these two sets, the graph of f_{GH} also has no umbilics on $\mathbb{R}^2$.

4. The pair of polar identifiers for umbilics

Let U be a domain in the xy-plane, and $f: U \to \mathbb{R}$ a C^∞-function. Let (r,θ) be the polar coordinate system associated to (x,y) as in (2-5). Then

$$F(r,\theta) := (r\cos\theta, r\sin\theta, f(r\cos\theta, r\sin\theta))$$

gives a parametrization of the graph of f with the unit normal vector

$$\nu := \frac{1}{\sqrt{f_\theta^2 + r^2(1+f_r^2)}}\big(f_\theta \sin\theta - r f_r\cos\theta, -r f_r \sin\theta - f_\theta\cos\theta, r\big).$$

Then

$$\widehat{I} := \begin{pmatrix} 1+f_r^2 & f_r f_\theta \\ f_r f_\theta & r^2+f_\theta^2 \end{pmatrix}$$

is the symmetric matrix consisting of the coefficients of the first fundamental form of F. If we set

$$Q = \begin{pmatrix} 0 & \sqrt{1+f_r^2} \\ -\sqrt{f_\theta^2 + r^2(1+f_r^2)}/\sqrt{1+f_r^2} & f_r f_\theta/\sqrt{1+f_r^2} \end{pmatrix},$$

then $QQ^T = \widehat{I}$. The symmetric matrix consisting of the coefficients of the second fundamental form is given by

$$\widehat{II} := \frac{1}{\sqrt{f_\theta^2 + r^2(1+f_r^2)}}\begin{pmatrix} r f_{rr} & r f_{r\theta} - f_\theta \\ r f_{r\theta} - f_\theta & r(f_{\theta\theta} + r f_r) \end{pmatrix}.$$

Then the symmetric matrix

$$B_f = Q^{-1}\widehat{II}(Q^{-1})^T = Q^T(\widehat{I}^{-1}\widehat{II})(Q^T)^{-1}$$

satisfies

$$\tilde{B}_f = \hat{h}\hat{k}^3 B_f = \begin{pmatrix} r f_r^2 f_\theta^2 f_{rr} + \hat{h} f_r\big(-2r f_\theta f_{r\theta} + 2 f_\theta^2 + r^2\hat{h}\big) + r\hat{h}^2 f_{\theta\theta} & \hat{l}\hat{k} \\ \hat{l}\hat{k} & r\hat{k}^2 f_{rr} \end{pmatrix},$$

where

$$\hat{h} := 1 + f_r^2, \quad \hat{k} := \sqrt{f_\theta^2 + r^2\bigl(1 + f_r^2\bigr)}, \quad \hat{l} := f_\theta(\hat{h} + r f_r f_{rr}) - r\hat{h} f_{r\theta}.$$

The following holds:

Proposition 4.1. *The symmetric matrix $\tilde{B}_f(p)$ is proportional to the identity matrix at $p \in U \setminus \{o\}$ if and only if p gives an umbilic of the graph of f. Moreover, if o is an isolated umbilic of the graph of f, then the index of the umbilic at o is equal to $1 + \mathrm{ind}_o(\tilde{B}_f)$.*

Proof. The first assertion follows from the above discussions. So we now prove the second assertion. Suppose o is an isolated umbilic. We take a simple closed smooth curve $\gamma(t)$ in the xy-plane, where $0 \le t \le 2\pi$, which surrounds the origin o anticlockwisely, and does not surround any other umbilics. Let $\boldsymbol{w}_1 : [0, 2\pi] \to \mathbb{R}^2$ be a vector field along γ such that $\boldsymbol{w}_1(t)$ is an eigenvector of the matrix $I^{-1}II$ at $\gamma(t)$ for each $t \in [0, 2\pi]$. Since

$$\partial_r = \cos\theta\,\partial_x + \sin\theta\,\partial_y,$$
$$\partial_\theta = -r\sin\theta\,\partial_x + r\cos\theta\,\partial_y,$$

we have that

$$(\partial_r, \partial_\theta) = (\partial_x, \partial_y)T_0, \qquad T_0 := \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Then, it holds that

$$\widehat{I}^{-1}\widehat{II} = (T_0)^{-1}(I^{-1}II)T_0.$$

In particular,

$$\boldsymbol{w}_2(t) := T_0(\gamma(t))^{-1}\boldsymbol{w}_1(t), \qquad 0 \le t \le 2\pi,$$

gives an eigenvector of the matrix $\widehat{I}^{-1}\widehat{II}$ at $\gamma(t)$. Let $T_s : U \to \mathrm{GL}(2, \mathbb{R}),\ 0 \le s \le 1$, be a map defined by

$$T_s := \begin{pmatrix} \cos\theta & -(r(1-s)+s)\sin\theta \\ \sin\theta & (r(1-s)+s)\cos\theta \end{pmatrix}, \qquad 0 \le s \le 1.$$

Then it gives a continuous deformation of T_0 to the rotation matrix T_1. Since the winding number of the curve $\gamma(t)$ with respect to the origin o is equal to 1, the difference between the rotation indices of $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ is equal to 1. Since the eigenflow of the symmetric matrix $\tilde{B}_f$ is associated with that of the matrix $\widehat{I}^{-1}\widehat{II}$ by Q, the fact that Q takes values in the set $\mathcal{T}$ in Section 3 yields that the index of the umbilic o is equal to $1 + \mathrm{ind}_o(\tilde{B}_f)$. □

We now set

$$\delta_1 := -\tilde{b}_{12}/\hat{k} = -f_\theta(1 + f_r^2 + r f_r f_{rr}) + r\bigl(1 + f_r^2\bigr) f_{r\theta},$$

where $\tilde{B}_f = (\tilde{b}_{ij})_{i,j=1,2}$. Then we have

$$\tilde{b}_{11} - \tilde{b}_{22} = -2 f_r f_\theta \delta_1 + r\bigl(1 + f_r^2\bigr)\delta_2,$$

where

$$\delta_2 := \bigl(1 + f_r^2\bigr)(r f_r + f_{\theta\theta}) - f_{rr}\bigl(r^2 + f_\theta^2\bigr).$$

Thus, as in the proof of Proposition 3.3, we get the following assertion:

Proposition 4.2. *Let U be a neighborhood of the origin $o := (0, 0)$. Let $f : U \to \mathbb{R}$ be a C^∞-function. Then the graph of f has an umbilic at $p \in U \setminus \{o\}$ if and only if the two functions $\delta_1(r, \theta)$, $\delta_2(r, \theta)$ both vanish at p, where $x = r\cos\theta$ and $y = r\sin\theta$. Further, if o is an isolated umbilic, then half of the index of the vector field*

$$\Delta_f := \delta_1 \partial_x + \delta_2 \partial_y$$

at o equals $-1 + I_f(o)$, where $I_f(o)$ is the index of the umbilic o.

We call δ_1, δ_2 the *polar umbilic identifiers* of the function f.

Example 4.3. Consider the function (where $z = x + iy$)

$$f(x, y) := \mathrm{Re}(z^2\bar{z}) = x^3 + xy^2 = r^3\cos\theta.$$

By straightforward calculations, we have

$$\delta_1 = -2r^3\sin\theta, \qquad \delta_2 = -2r^3(2 - 3r^4 - 6r^4\cos 2\theta)\cos\theta.$$

Since $2 - 3r^4 - 6r^4\cos 2\theta$ is positive for sufficiently small $r > 0$, the vector field Δ_f can be continuously deformed into the vector field $-\sin\theta\,\partial_r - \cos\theta\,\partial_\theta$ preserving the property that the origin is an isolated zero. Thus the rotation index of Δ_f at o is equal to -1, and $I_f(o) = 1 - \frac{1}{2} = \frac{1}{2}$.

We give a generalization of Proposition 4.2 for the computation of the index of the curvature line flow of a surface along an arbitrarily given simple closed curve surrounding several umbilics as follows. Let $z = f(x, y)$ be a C^∞-function defined on $\mathbb{R}^2$ admitting only isolated umbilics. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a C^∞-map satisfying $\gamma(t + 2\pi) = \gamma(t)$ which gives a simple closed curve in the xy-plane such that it surrounds a bounded domain containing the origin o anticlockwisely. Moreover, we assume that $\gamma(t)$ does not pass through any points corresponding to umbilics of the graph of f. We denote by $I_f(\gamma)$ (resp. $\mathrm{ind}_\gamma(\Delta_f)$) the rotation index of the curvature line flow (resp. of the vector field Δ_f) along the simple closed curve γ. Then the formula

$$I_f(\gamma) = 1 + \frac{\mathrm{ind}_\gamma(\Delta_f)}{2} \tag{4-1}$$

can be proved by modifying the proof of Proposition 4.2. Suppose that there exist at most finitely many points $t = t_1, \ldots, t_k \in [0, 2\pi]$ such that $\delta_1(\gamma(t))$ vanishes

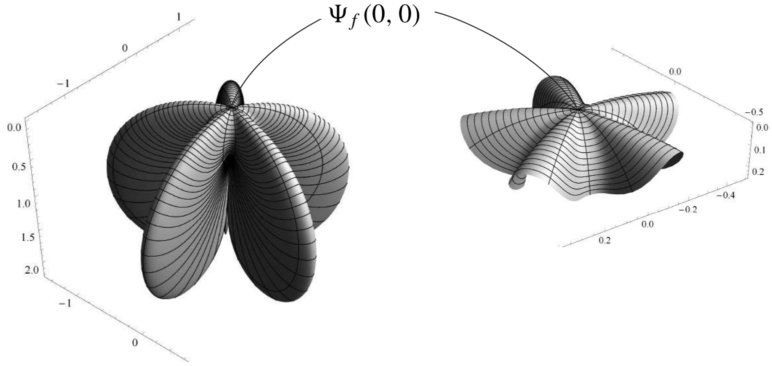

Figure 1. The inversion of the graph f_5 for $a = \frac{1}{5}$ (left) and its enlarged view (right). In these two figures, the z-axis points toward the downward direction.

at $t = t_j$. We now assume that $\delta_1'(\gamma(t)) := d\delta_1(\gamma(t))/dt$ does not vanish at $t = t_j$, for $j = 1, \ldots, k$. We set

$$\varepsilon(t_j) = \begin{cases} 0 & \text{for } \delta_2(\gamma(t_j)) < 0, \\ 1 & \text{for } \delta_1'(\gamma(t_j)) > 0 \text{ and } \delta_2(\gamma(t_j)) > 0, \\ -1 & \text{for } \delta_1'(\gamma(t_j)) < 0 \text{ and } \delta_2(\gamma(t_j)) > 0. \end{cases}$$

Then, it holds that

(4-2) $$\operatorname{ind}_\gamma(\Delta_f) = -\sum_{j=1}^{k} \varepsilon(t_j).$$

5. Proof of the main theorem

In this section, using the function $f = f_m$ ($m = 1, 2, 3, \ldots$) given in (1-4), we prove Theorem 1.1 and Corollary 1.2 in the introduction. More generally, we consider the function

(5-1) $$g = g_m(r, \theta) := 1 + F(r^a \cos m\theta), \qquad 0 < a < 1/4,\ m = 1, 2, 3, \ldots,$$

which is defined on $\{(r, \theta)\,;\ r > R\}$, where R is an arbitrarily fixed positive number, and $F : \mathbb{R} \to \mathbb{R}$ is a bounded C^∞-function satisfying the following conditions:

(i) $F(x)$ is an odd function, that is, it satisfies $F(-x) = -F(x)$,

(ii) the derivative $F'(x)$ of F is a positive-valued bounded function on $\mathbb{R}$,

(iii) the second derivative $F''(x)$ is a bounded function on $\mathbb{R}$ such that $F''(x) < 0$ for $x > 0$,

(iv) there exist three constants α, β and γ ($\beta \neq 0,\ \gamma > 0$) such that

$$\lim_{x\to\infty} e^{\gamma x} F'(x) = \alpha, \qquad \lim_{x\to\infty} e^{\gamma x} F''(x) = \beta.$$

One can easily construct a bounded C^∞-function $F(x)$ satisfying properties (i–iv). For example, one can construct an odd C^∞-function satisfying (ii) and (iii) so that

$$F(x) = 1 - e^{-x}, \qquad x \in [M, \infty),$$

for a positive number M. Then it satisfies (iv) also. However, to prove Theorem 1.1, we must choose the function $F(x)$ to be real-analytic, and

$$F(x) := \tanh x$$

satisfies all of the properties required. In this case, $g_m = f_m$ holds. From now on, we shall prove Theorem 1.1 and Corollary 1.2 using only the above four properties of $F(x)$.

The function g can be considered as a C^∞-function on $\mathbb{R}^2 \setminus \Omega_R$ in the xy-plane for any $R > 0$. The graph of g lies between two parallel planes orthogonal to the z-axis, and is symmetric under rotation by the angle $2\pi/m$ with respect to the z-axis (the entire figure of the inversion of the graph of f_5 is given in the left-hand side of Figure 1). The partial derivatives of the function g are given by

(5-2)
$$\begin{aligned}
g_r &= ar^{a-1} c_m F'(r^a c_m),\\
g_\theta &= -mr^a s_m F'(r^a c_m),\\
g_{rr} &= ar^{a-2} c_m \big(ar^a c_m F''(r^a c_m) + (a-1) F'(r^a c_m)\big),\\
g_{r\theta} &= -amr^{a-1} s_m \big(r^a c_m F''(r^a c_m) + F'(r^a c_m)\big),\\
g_{\theta\theta} &= m^2 r^a \big(r^a s_m^2 F''(r^a c_m) - c_m F'(r^a c_m)\big),
\end{aligned}$$

where

(5-3)
$$c_m := \cos m\theta, \qquad s_m := \sin m\theta.$$

Since $F(x)$ is a bounded function, g is bounded and satisfies (2-13), since $a < 2$. Therefore, the inversion Ψ_g can be expressed as a graph near $(0, 0, 0)$. Since $0 < a < 1$, the function g satisfies (a) and (b) of Proposition 2.5. Then $Z = Z_f(X, Y)$ as in (2-11), where $f := g$ is a C^1-function at $(0, 0)$. The graph of Z_g for $g = f_5$ near $(0, 0, 0)$ is indicated in the right-hand side of Figure 1. To prove Theorem 1.1, it is sufficient to show that $(0, 0, 0)$ is a C^1-umbilic of the graph of $Z_g(X, Y)$ with index $1 + (m/2)$. In the following discussions, we would like to show that there exists a positive number R such that the graph of g has no umbilics if $r > R$. We then compute the index $I_g(\Gamma)$ with respect to the circle

(5-4)
$$\Gamma(\theta) := (r\cos\theta, r\sin\theta), \qquad 0 \le \theta \le 2\pi,\ r > R,$$

using (4-1) and (4-2), which does not depend on the choice of $r > R$, as follows. We set

$$\check{\delta}_j(\theta) := \delta_j(\Gamma(\theta)), \qquad j = 1, 2. \tag{5-5}$$

The first polar identifier is given by

$$\delta_1 = -mr^a s_m\bigl(ar^a c_m F''(r^a c_m) + (a-1)F'(r^a c_m)\bigr). \tag{5-6}$$

Since $0 < a < 1$, condition (ii) yields that

$$(a-1)F'(r^a c_m) < 0. \tag{5-7}$$

On the other hand, by (i) and (iii), it holds that

$$xF''(x) \le 0, \qquad x := r^a c_m. \tag{5-8}$$

By (5-7) and (5-8), we can conclude that $\check{\delta}_1(\theta)$ changes sign only at the zeros of the function $\sin m\theta$. Since the function g is symmetric with respect to rotation by angle $2\pi/m$, to compute the rotation index of Δ_g along Γ, it is sufficient to check the sign changes of $\check{\delta}_i(\theta)$, $i = 1, 2$, for $\theta = 0$ and $\theta = \pi/m$. By (5-6), (5-7) and (5-8), we get the following:

$$\left.\frac{d\check{\delta}_1}{d\theta}\right|_{\theta=0} > 0, \qquad \left.\frac{d\check{\delta}_1}{d\theta}\right|_{\theta=\pi/m} < 0. \tag{5-9}$$

The second polar identifier δ_2 is given by

$$\begin{aligned} r^{2-3a}\delta_2 =& -r^{2-a}(a^2c_m^2 - m^2s_m^2)F''(c_mr^a) + ac_m(a^2c_m^2 - am^2 + m^2s_m^2)F'(c_mr^a)^3 \\ & - c_mr^{2-2a}(a^2 - 2a + m^2)F'(c_mr^a). \end{aligned}$$

We need the sign of $\check{\delta}_2(\theta)$ at $\theta \in (\pi/m)\mathbb{Z}$. In this case, $s_m = 0$ and $c_m = \pm 1$. Substituting these relations and using the fact that F' (resp. F'') is an even function (resp. an odd function), we have

$$r^{2-3a}\delta_2 = \mp r^{2-a}a^2F''(r^a) \pm a^2(a - m^2)F'(r^a)^3 \mp r^{2-2a}(a^2 - 2a + m^2)F'(r^a).$$

Since F' is bounded, the middle term is bounded. Hence, by (iv) and by the fact that $0 < a < 1$, there exists a positive number R such that the sign of δ_2 is determined by the sign of the first term $\mp r^{2-a}a^2F''(r^a)$ whenever $r > R$. Then, we have

$$-\check{\delta}_2(\pi/m) = \check{\delta}_2(0) > 0. \tag{5-10}$$

In particular, the image of the graph of g has no umbilics when $r > R$. By the $2\pi/m$-symmetry of g, (4-2), (5-9), and (5-10), the index $\operatorname{ind}_\Gamma(\Delta_g)$ is equal to $-m$. Then the index of the curvature line flow along Γ is equal to $I_g(\Gamma) = 1 - m/2$

by (4-1). Then after inversion, the Poincaré–Hopf index formula yields that the index I_0 of the umbilic of Ψ_g at the origin is

$$I_0 = 2 - I_g(\Gamma) = 1 + m/2.$$

If we choose $F(x) := \tanh x$, then the function $Z_g(X, Y)$ satisfies the properties of Theorem 1.1.

We next prove the corollary. We set

$$\lambda := \frac{Z\sqrt{1 + Z_X^2 + Z_Y^2}}{1 + \sqrt{1 + Z_X^2 + Z_Y^2}}, \tag{5-11}$$

where $Z := Z_g$ is the function given in (2-11). Suppose that λ and $\lambda\nu$ are a C^1-function and a C^1-vector field defined on a sufficiently small neighborhood of $(0, 0)$ in the XY-plane, respectively, where ν is a unit normal vector field of the graph of Z_g. Then the map

$$\Phi : (X, Y) \mapsto (\xi(X, Y), \eta(X, Y))$$

given by (A-4) for $f = Z_{f_m}$ is a local C^1-diffeomorphism, and is real-analytic on $U \setminus \{(0, 0)\}$. Then the proof of Fact A.1 in Appendix A is valid in our situation, and we can conclude that the eigenflow of the Hessian matrix of $\lambda(\xi, \eta)$ is equal to the curvature line flow of the map $P(\xi, \eta)$ given by (A-8). Since the image of $P(\xi, \eta)$ coincides with that of $\Psi_{f_m}(u, v)$, we get the proof of the corollary in the introduction.

Thus, it is sufficient to show that λ and $\lambda\nu$ are C^1 at $(X, Y) = (0, 0)$. By (5-11), we have the following expression

$$\lambda\nu = \frac{(ZZ_X, ZZ_Y, -Z)}{1 + \sqrt{1 + Z_X^2 + Z_Y^2}}. \tag{5-12}$$

By (5-11) and (5-12), we can say that $\lambda(X, Y)$ and $\lambda(X, Y)\nu(X, Y)$ are C^1 at $(0, 0)$ if

$$\lim_{(X,Y)\to(0,0)} ZZ_{XX} = \lim_{(X,Y)\to(0,0)} ZZ_{XY} = \lim_{(X,Y)\to(0,0)} ZZ_{YY} = 0 \tag{5-13}$$

hold. So to prove the corollary, it is sufficient to show (5-13). It can be easily seen that all of $r^{1-a}g_r$, $r^{-a}g_\theta$, $r^{2-2a}g_{rr}$, $r^{1-2a}g_{r\theta}$ and $r^{-2a}g_{\theta\theta}$ are bounded functions on $\mathbb{R}^2 \setminus \Omega_R$. Since $0 < a < \frac{1}{4}$, Proposition 2.7 yields that the map $(u, v) \mapsto (X, Y) = \Pi \circ \Psi_g(u, v)$ is a C^2-map. Then (5-13) is equivalent to

$$\lim_{(u,v)\to(0,0)} ZZ_{uu} = \lim_{(u,v)\to(0,0)} ZZ_{uv} = \lim_{(u,v)\to(0,0)} ZZ_{vv} = 0. \tag{5-14}$$

Since $Z = h/(k + 1)$, (5-14) follows from (2-19), (2-21) and the fact that

$$\lim_{\rho\to 0} \rho h_{uu} = \lim_{\rho\to 0} \rho h_{uv} = \lim_{\rho\to 0} \rho h_{vv} = 0.$$

6. An alternative proof of the main theorem

In the previous section, we have proved Corollary 1.2. However, it is natural to expect that one can give an explicit description of the function with the desired properties. The function λ given in (5-11) does not have a simple expression. On the other hand, we will see that functions

$$\Lambda = \Lambda_m := r^2 \tanh(r^{-a} \cos m\theta), \qquad m = 1, 2, 3, \dots, \tag{6-1}$$

satisfy (1) and (2) of Corollary 1.2 if $0 < a < 1$. We set

$$(\lambda :=)\, \lambda_m := r^2 F(r^{-a} \cos m\theta),$$

where $\xi = r \cos\theta$, $\eta = r \sin\theta$, and $F : \mathbb{R} \to \mathbb{R}$ is a function satisfying the properties (i–iv) given in the beginning of Section 5. Then Λ_m is a special case of λ_m for $F(x) := \tanh x$. It holds that

$$\begin{aligned}
\lambda_r &= r\bigl(2F(r^{-a}c_m) - ac_m r^{-a} F'(r^{-a}c_m)\bigr),\\
\lambda_\theta &= -mr^{2-a} s_m F'(r^{-a}c_m),\\
\lambda_{rr} &= 2F(r^{-a}c_m) + ar^{-2a}c_m\bigl((a-3)r^a F'(r^{-a}c_m) + ac_m F''(r^{-a}c_m)\bigr),\\
\lambda_{r\theta} &= ms_m r^{1-2a}\bigl((a-2)r^a F'(r^{-a}c_m) + ac_m F''(r^{-a}c_m)\bigr),\\
\lambda_{\theta\theta} &= -m^2 r^{2-2a}\bigl(r^a c_m F'(r^{-a}c_m) - s_m^2 F''(r^{-a}c_m)\bigr),
\end{aligned}$$

where c_m and s_m are defined in (5-3). We set

$$\zeta_1 := 2(r\lambda_{r\theta} - \lambda_\theta), \qquad \zeta_2 := -r^2\lambda_{rr} + r\lambda_r + \lambda_{\theta\theta}.$$

Then each component of the vector field $\delta_\lambda := \zeta_1 \partial_x + \zeta_2 \partial_y$ is an identifier for the eigenflow of the Hessian matrix of λ at the origin given in the introduction; see (1-3). By a direct calculation, we have

$$\begin{aligned}
\zeta_1 &= 2mr^{2-2a} s_m\bigl(ac_m F''(r^{-a}c_m) + (a-1)r^a F'(r^{-a}c_m)\bigr),\\
\zeta_2 &= -r^{2-2a}(a^2c_m^2 - m^2 s_m^2)F''(r^{-a}c_m) - (a^2 - 2a + m^2)r^{2-a}c_m F'(r^{-a}c_m).
\end{aligned}$$

By the property (ii) of F, $(a-1)r^a F'(r^{-a}c_m)$ is negative, and by (ii) and (iii), $c_m F''(r^{-a}c_m)$ is also negative. So ζ_1 is positively proportional to $-s_m (= -\sin m\theta)$. In particular, ζ_1 vanishes only when $s_m = 0$. Moreover, for fixed r, it holds that $d\zeta_1/d\theta < 0$ (resp. $d\zeta_1/d\theta > 0$) if $c_m = 1$ (resp. $c_m = -1$).

On the other hand, if $s_m = 0$ and r tends to zero, then $c_m = \pm 1$ and $F'(\pm r^{-a})$ and $F''(\pm r^{-a})$ tend to zero with exponential order (see condition (iv) for $F(x)$). Therefore, the leading term of ζ_2 for small r is $-r^{2-2a}(a^2c_m^2 - m^2s_m^2)F''(r^{-a}c_m)$. Hence, for a fixed sufficiently small r, the function ζ_2 is positive (resp. negative) if $c_m = 1$ (resp. $c_m = -1$). Summarizing these facts, one can easily show that the

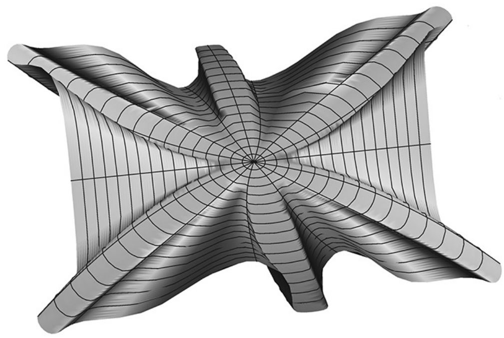

Figure 2. The image of P $\left(r \le \frac{1}{2}\right)$ for $m = 2$ and $a = \frac{1}{2}$.

index of the vector field δ_λ at $o := (0, 0)$ is equal to m. So the index of the eigenflow of the Hessian matrix of λ at o is equal to $1 + m/2$ (see Appendix B). One can easily check that λ is a C^1-function at o and the function λ satisfies (1) and (2) of Corollary 1.2. Since Λ is a special case of λ, we proved that Λ satisfies the desired properties.

To give an alternative proof of Theorem 1.1, we consider the real analytic map $P : \mathbb{R}^2 \setminus \{o\} \to \mathbb{R}^3$ defined (see (A-8)) by

$$P(\xi, \eta) := (\xi, \eta, \Lambda(\xi, \eta)) - \Lambda(\xi, \eta)\nu(\xi, \eta),$$

where

$$\nu := \frac{1}{\Lambda_\xi^2 + \Lambda_\eta^2 + 1}(2\Lambda_\xi, 2\Lambda_\eta, \Lambda_\xi^2 + \Lambda_\eta^2 - 1). \tag{6-2}$$

One can easily verify that

$$\Lambda_\xi = r^{1-a}\big((ms_1 s_m - ac_1 c_m)\operatorname{sech}^2(r^{-a}c_m) + 2r^a c_1 \tanh(r^{-a}c_m)\big),$$
$$\Lambda_\eta = r^{1-a}\big(2r^a s_1 \tanh(r^{-a}c_m) - (as_1 c_m + mc_1 s_m)\operatorname{sech}^2(r^{-a}c_m)\big),$$

where $c_1 = \cos\theta$ and $s_1 = \sin\theta$. Using them, one can get the expressions

$$\Lambda_{\xi\xi} = \frac{1}{r^{2a}} h_1(r, \theta), \quad \Lambda_{\xi\eta} = \frac{1}{r^{2a}} h_2(r, \theta), \quad \Lambda_{\eta\eta} = \frac{1}{r^{2a}} h_3(r, \theta), \tag{6-3}$$

where $h_i(r, \theta)$, $i = 1, 2, 3$, are continuous functions defined on $\mathbb{R}^2$. Using (6-2), (6-3) and the fact $\lim_{r\to 0} \Lambda/r^{2a} = 0$, we have

$$\lim_{r\to 0} \Lambda\nu_\xi = \lim_{r\to 0} \frac{\Lambda}{r^{2a}}(r^{2a}\nu_\xi) = 0, \tag{6-4}$$

and also

$$\lim_{r\to 0} \Lambda\nu_\eta = 0. \tag{6-5}$$

Using (6-4), (6-5) and the fact

$$d(\Lambda\nu) = (d\Lambda)\nu + \Lambda d\nu,$$

we can conclude that $\Lambda\nu$ can be extended as a C^1-function at o. Thus $P(\xi, \eta)$ can also be extended as a C^1-differentiable map at o. One can also easily check that

$$P_\xi(0, 0) = (1, 0, 0), \qquad P_\eta(0, 0) = (0, 1, 0).$$

Hence P is an immersion at o, and

$$\Phi : (\xi, \eta) \mapsto (X(\xi, \eta), Y(\xi, \eta))$$

is a local C^1-diffeomorphism, where $P = (X, Y, Z)$. In particular,

$$Z_\Lambda := Z(\Phi^{-1}(X, Y))$$

gives a function defined on a neighborhood of $(X, Y) = (0, 0)$. By Fact A.1 in Appendix A, the index of the curvature line flow at $(0, 0)$ of the graph of Z_Λ is equal to the index of the eigenflow of the Hessian matrix of Λ, which implies Theorem 1.1. The image of P for $m = 3$ and $a = \frac{1}{2}$ is given in Figure 2.

7. The duality of indices

At the end of this paper, we consider the index at infinity for eigenflows of Hessian matrices. Let

$$f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}, \qquad g : U_{1/R} \setminus \{o\} \to \mathbb{R}$$

be C^2-functions, where Ω_R and $U_{1/R}$ are disks defined in Section 2. Let $\mathcal{H}_f$ (resp. $\mathcal{H}_g$) be the eigenflow of the Hessian matrix of f (resp. g). If the Hessian matrix of f has no equidiagonal points, then we can consider the index $\mathrm{ind}(\mathcal{H}_f, \Gamma)$ with respect to the circle Γ given in (5-4) and it is independent of the choice of $r > R$. So we denote it by $\mathrm{ind}_\infty(\mathcal{H}_f)$. Similarly, if the Hessian matrix of g has no equidiagonal points, then we can consider the index $\mathrm{ind}(\mathcal{H}_g, \Gamma')$ with respect to the circle $\Gamma'(\theta) := (\rho\cos\theta, \rho\sin\theta),\ 0 \le \theta \le 2\pi,\ \rho < 1/R$. Since it is independent of the choice of $\rho < 1/R$, we denote it by $\mathrm{ind}_o(\mathcal{H}_g)$. Consider the plane-inversion

$$\iota : \mathbb{R}^2 \in (u, v) \mapsto \frac{1}{u^2 + v^2}(u, v) \in \mathbb{R}^2.$$

Then the following assertion holds:

Proposition 7.1 (duality of indices). *Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a C^2-function whose Hessian matrix has no equidiagonal points. Then the function $g : \Omega_R \to \mathbb{R}$ defined by*

$$g(x, y) := (u^2 + v^2) f \circ \iota(u, v)$$

(*called the dual of* f) *satisfies*

$$\mathrm{ind}_o(\mathcal{H}_g) + \mathrm{ind}_\infty(\mathcal{H}_f) = 2.$$

Proof. Using the identification of (u, v) and $z = u + iv$, it holds that $u = (z + \bar{z})/2$ and $v = (z - \bar{z})/(2i)$. In particular, f can be considered as a function of variables z and $\bar{z}$, and can be denoted by $f = f(z, \bar{z})$. Since $\iota(z) = 1/\bar{z}$, we can write

$$g(z, \bar{z}) := z\bar{z} f(1/\bar{z}, 1/z).$$

Then

$$g_{zz}(z, \bar{z}) = \frac{\bar{z} f_{\bar{z}\bar{z}}(1/\bar{z}, 1/z)}{z^3}$$

holds, where

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Since $\Gamma(\theta) = re^{i\theta}$, we have that

$$g_{zz}(\Gamma(\theta)) = \frac{f_{\bar{z}\bar{z}}(\iota \circ \Gamma(\theta))}{r^2 e^{4i\theta}}.$$

Thus, it holds that

$$\mathrm{ind}_o(g_{zz}, \Gamma) = -4 + \mathrm{ind}_o(f_{\bar{z}\bar{z}}, \iota \circ \Gamma).$$

By (B-1), we have

$$\begin{aligned} \mathrm{ind}_o(g_{zz}, \Gamma) &= -2\,\mathrm{ind}_o(\mathcal{H}_g), \\ \mathrm{ind}_o(f_{\bar{z}\bar{z}}, \iota \circ \Gamma) &= -\,\mathrm{ind}_o(f_{zz}, \iota \circ \Gamma) = 2\,\mathrm{ind}_\infty(\mathcal{H}_f). \end{aligned}$$

Thus we get the assertion. □

Applying Proposition 7.1 for the function $g = \Lambda_m$, see (6-1), we get the following:

Corollary 7.2. *For each* $m \geq 1$, *there exists a* C^1*-function* $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ *satisfying*

(1) f *is real-analytic on* $\mathbb{R}^2 \setminus \Omega_R$,

(2) *the eigenflow of the Hessian matrix of* f *has no singular points, and*

(3) *the index at infinity of the eigenflow of* H_f *is equal to* $1 - m/2$.

The function Λ_m used in the second proof of Theorem 1.1 coincides with the dual of the function $f_m - 1$ given in (1-4).

Appendix A: The classical reduction

In this appendix we show the existence of a special coordinate system (ξ, η) of the graph of a function $f(x, y)$ which reduces the curvature line flow to the Hessian of a certain function, called Ribaucour's parametrization (Umehara learned this from Konrad Voss at the conference of Thessaloniki 1997). Although, the existence of such a coordinate system was classically known, and a proof is in the appendix of [Scherbel 1993], the authors will give the proof here for the sake of convenience. We set $P = (x, y, f(x, y))$, and suppose that $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Consider a sphere which is tangent to the graph of f at P and also tangent to the xy-plane at a point Q. Then, it holds that

$$Q + \lambda \boldsymbol{e}_3 = P + \lambda \nu, \tag{A-1}$$

where $\boldsymbol{e}_3 = (0, 0, 1)$ and $\nu = (f_x, f_y, -1)/\sqrt{1 + f_x^2 + f_y^2}$. Taking the third component of (A-1), we get

$$\lambda = \frac{f\sqrt{1 + f_x^2 + f_y^2}}{1 + \sqrt{1 + f_x^2 + f_y^2}}. \tag{A-2}$$

In particular, $\lambda(0, 0) = 0$. Since $f_x(0, 0) = f_y(0, 0) = 0$, we have that

$$d\lambda(0, 0) = df(0, 0) = 0. \tag{A-3}$$

Taking the exterior derivative of (A-1), and using (A-3) and $\lambda(0, 0) = 0$, we have $dP(0, 0) = dQ(0, 0)$. So, if we set $Q = (\xi(x, y), \eta(x, y), 0)$, then it holds that

$$\begin{aligned}(\xi_x(0, 0)dx + \xi_y(0, 0)dy,\ \eta_x(0, 0)dx + \eta_y(0, 0)dy, 0) &= dQ \\ = dP = (dx, dy, f_x(0, 0)dx + f_y(0, 0)dy) &= (dx, dy, 0),\end{aligned}$$

which implies that the Jacobi matrix of the map

$$\Phi : (x, y) \mapsto (\xi(x, y), \eta(x, y)) \tag{A-4}$$

is the identity matrix at $(0, 0)$. So we can take (ξ, η) as a new local coordinate system. Differentiating (A-1) by ξ and η, we get the following two identities:

$$Q_\xi + \lambda_\xi \boldsymbol{e}_3 = P_\xi + \lambda_\xi \nu + \lambda \nu_\xi, \quad Q_\eta + \lambda_\eta \boldsymbol{e}_3 = P_\eta + \lambda_\eta \nu + \lambda \nu_\eta.$$

Taking the inner products of them and ν, these two equations yield

$$Q_\xi \cdot \nu + \lambda_\xi \nu_3 = \lambda_\xi, \qquad Q_\eta \cdot \nu + \lambda_\eta \nu_3 = \lambda_\eta, \tag{A-5}$$

where we set $\nu = (\nu_1, \nu_2, \nu_3)$. Since $Q = (\xi, \eta, 0)$, we have that $Q_\xi = (1, 0, 0)$ and $Q_\eta = (0, 1, 0)$. So $Q_\xi \cdot \nu = \nu_1$ and $Q_\eta \cdot \nu = \nu_2$. Substituting this into (A-5), we have

$$\lambda_\xi = \frac{\nu_1}{1 - \nu_3}, \qquad \lambda_\eta = \frac{\nu_2}{1 - \nu_3}. \tag{A-6}$$

This implies that $(\lambda_\xi, \lambda_\eta)$ is the image of ν via the stereographic projection, and

$$\nu = \frac{1}{1+\lambda_\xi^2+\lambda_\eta^2}(2\lambda_\xi, 2\lambda_\eta, \lambda_\xi^2+\lambda_\eta^2-1). \tag{A-7}$$

By (A-1), we have

$$P = (\xi, \eta, 0) - \lambda\nu + (0, 0, \lambda). \tag{A-8}$$

We prove the following:

Fact A.1. *The curvature line flow of the graph $z = f(x, y)$ coincides with the eigenflow of the Hessian of the function $\lambda(\xi, \eta)$ given by* (A-2).

Proof. Noticing (A-8), we set

$$\Delta_{(\xi,\eta)} := \det\begin{pmatrix}\nu\\ dP\\ d\nu\end{pmatrix} = \det\begin{pmatrix}\nu\\ d\xi, d\eta, d\lambda\\ d\nu\end{pmatrix}.$$

Then this gives a map $\Delta_{(\xi,\eta)} : T_{(\xi,\eta)}\mathbb{R}^2 \to \mathbb{R}$ such that

$$\Delta_{(\xi,\eta)}\Big(a\frac{\partial}{\partial\xi} + b\frac{\partial}{\partial\eta}\Big) = \det\big(\nu, aP_\xi(\xi, \eta) + bP_\eta(\xi, \eta), a\nu_\xi(\xi, \eta) + b\nu_\eta(\xi, \eta)\big) \in \mathbb{R}.$$

It is well known that $\boldsymbol{w} \in T_{(\xi,\eta)}\mathbb{R}^2$ points in a principal direction of P at (ξ, η) if and only if $\Delta_{(\xi,\eta)}(\boldsymbol{w}) = 0$. Since $(\nu_1)^2 + (\nu_2)^2 + (\nu_3)^2 = 1$, (A-6) yields that

$$\lambda_\xi\nu_1 + \lambda_\eta\nu_2 = \frac{(\nu_1)^2 + (\nu_2)^2}{1-\nu_3} = \frac{1-(\nu_3)^2}{1-\nu_3} = 1 + \nu_3,$$

which implies $\nu_3 = \lambda_\xi\nu_1 + \lambda_\eta\nu_2 - 1$. We now set $\mu = 2/(1+\lambda_\xi^2+\lambda_\eta^2)$. Differentiating (A-7), we have

$$d\nu = \frac{d\mu}{\mu}\nu + \mu(d\lambda_\xi, d\lambda_\eta, \lambda_\xi d\lambda_\xi + \lambda_\eta d\lambda_\eta).$$

The first term of the right-hand side of the above equation is proportional to ν and does not affect the computation of $\Delta_{(\xi,\eta)}$. So we have that

$$\begin{aligned}\Delta_{(\xi,\eta)} &= \mu\begin{vmatrix}\nu_1 & \nu_2 & \lambda_\xi\nu_1 + \lambda_\eta\nu_2 - 1\\ d\xi & d\eta & \lambda_\xi d\xi + \lambda_\eta d\eta\\ d\lambda_\xi & d\lambda_\eta & \lambda_\xi d\lambda_\xi + \lambda_\eta d\lambda_\eta\end{vmatrix}\\ &= \mu\begin{vmatrix}\nu_1 & \nu_2 & -1\\ d\xi & d\eta & 0\\ d\lambda_\xi & d\lambda_\eta & 0\end{vmatrix} = -\mu\begin{vmatrix}d\xi & d\eta\\ d\lambda_\xi & d\lambda_\eta\end{vmatrix}\\ &= \mu\big((\lambda_{\xi\xi} - \lambda_{\eta\eta})d\xi d\eta - \lambda_{\xi\eta}(d\xi^2 - d\eta^2)\big).\end{aligned}$$

Fact A.1 follows from this representation of $\Delta_{(\xi,\eta)}$. □

Appendix B: Indices of eigenflows of Hessian matrices

Let $g : \Omega_R \setminus \{o\} \to \mathbb{R}$ be a C^2-function, where Ω_R is the closed disk of radius R centered at the origin $o := (0, 0)$; see (2-1). The Hessian matrix of g is given by

$$H_g := \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}.$$

We denote by $\mathcal{H}_g$ the eigenflow of H_g. A point $p \in \Omega_R \setminus \{o\}$ is called an *equidiagonal point* of $\mathcal{H}_g$ if $H_g(p)$ is proportional to the identity matrix. Consider the circle

$$\Gamma(\theta) := r(\cos\theta, \sin\theta), \qquad 0 \le \theta < 2\pi, r < R.$$

If there are no equidiagonal points on $\Omega_R \setminus \{o\}$, then we can define the index $\operatorname{ind}(\mathcal{H}_g, \Gamma)$ of the eigenflow $\mathcal{H}_g$ with respect to Γ, which does not depend on the choice of r. We call it the index of $\mathcal{H}_g$ at the origin and denote it by $\operatorname{ind}_o(\mathcal{H}_g)$. Consider the vector field

$$d_g := 2g_{xy}\frac{\partial}{\partial x} + (g_{yy} - g_{xx})\frac{\partial}{\partial y}.$$

It is well known that the mapping degree of the Gauss map, see (3-1),

$$\check{d}_g : T^1 := \mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto \frac{d_g(\Gamma(\theta))}{|d_g(\Gamma(\theta))|} \in S^1 := \{(x, y) \in \mathbb{R}^2\,;\ x^2 + y^2 = 1\}$$

is equal to $2\operatorname{ind}_o(\mathcal{H}_g)$. Using the correspondence $(x, y) \mapsto x + iy$, we identify $\mathbb{R}^2$ with $\mathbb{C}$, where $i = \sqrt{-1}$. Then

$$\begin{aligned} g_z &= \tfrac{1}{2}(g_x - ig_y), \\ g_{zz} &= \tfrac{1}{4}((g_{xx} - g_{yy}) - 2ig_{xy}), \end{aligned}$$

where $g_z := \partial g/\partial z,\ g_{zz} := \partial^2 g/\partial z^2$ and

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

Thus, d_g can be identified with the right-angle rotation of $\overline{g_{zz}}$. In particular,

$$\operatorname{ind}_o(\mathcal{H}_g) = -\tfrac{1}{2}\operatorname{ind}_o(g_{zz}). \tag{B-1}$$

Here g_{zz} is considered as a vector field and $\operatorname{ind}_o(g_{zz})$ is its index at the origin. Let (r, θ) be as in (2-5). Then $z = re^{i\theta}$ and

$$\begin{aligned} g_z &= \frac{e^{-i\theta}}{2r}(rg_r - ig_\theta), \\ g_{zz} &= \frac{e^{-2i\theta}}{4r^2}\big((r^2 g_{rr} - rg_r - g_{\theta\theta}) + 2i(g_\theta - rg_{r\theta})\big). \end{aligned}$$

We consider the vector field defined by

$$\delta_g := 2(rg_{r\theta} - g_\theta)\frac{\partial}{\partial x} + (-r^2 g_{rr} + rg_r + g_{\theta\theta})\frac{\partial}{\partial y}. \tag{B-2}$$

Since, from [Klotz 1959, (18)],

$$\mathrm{ind}_o(\overline{g_{zz}}) = 2 + \mathrm{ind}_o(\delta_g),$$

we obtain the following:

Lemma B.1. *The identity* $\mathrm{ind}_o(\mathcal{H}_g) = 1 + \mathrm{ind}_o(\delta_g)/2$ *holds.*

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References

[Ando 2003] N. Ando, "An umbilical point on a non-real-analytic surface", *Hiroshima Math. J.* **33**:1 (2003), 1–14. MR Zbl

[Bates 2001] L. Bates, "A weak counterexample to the Carathéodory conjecture", *Differential Geom. Appl.* **15**:1 (2001), 79–80. MR Zbl

[Ghomi and Howard 2012] M. Ghomi and R. Howard, "Normal curvatures of asymptotically constant graphs and Carathéodory's conjecture", *Proc. Amer. Math. Soc.* **140**:12 (2012), 4323–4335. MR Zbl

[Guilfoyle and Klingenberg 2008] B. Guilfoyle and W. Klingenberg, "Proof of the Carathéodory conjecture", preprint, 2008. arXiv

[Guilfoyle and Klingenberg 2012] B. Guilfoyle and W. Klingenberg, "From global to local: an index bound for umbilic points on smooth convex surfaces", preprint, 2012. arXiv

[Gutierrez et al. 1996] C. Gutierrez, F. Mercuri, and F. Sánchez-Bringas, "On a conjecture of Carathéodory: analyticity versus smoothness", *Experiment. Math.* **5**:1 (1996), 33–37. MR Zbl

[Hamburger 1940] H. Hamburger, "Beweis einer Caratheodoryschen Vermutung, I", *Ann. of Math.* (2) **41** (1940), 63–86. MR Zbl

[Hamburger 1941a] H. L. Hamburger, "Beweis einer Caratheodoryschen Vermutung, II", *Acta Math.* **73** (1941), 175–228. MR Zbl

[Hamburger 1941b] H. L. Hamburger, "Beweis einer Caratheodoryschen Vermutung, III", *Acta Math.* **73** (1941), 229–332. MR Zbl

[Klotz 1959] T. Klotz, "On G. Bol's proof of Carathéodory's conjecture", *Comm. Pure Appl. Math.* **12** (1959), 277–311. MR Zbl

[Scherbel 1993] H. Scherbel, *A new proof of Hamburger's index theorem on umbilical points*, Ph.D. thesis, Eidgenöessische Technische Hochschule Zürich, 1993, http://search.proquest.com/docview/304065217. MR

[Smyth and Xavier 1992] B. Smyth and F. Xavier, "A sharp geometric estimate for the index of an umbilic on a smooth surface", *Bull. London Math. Soc.* **24**:2 (1992), 176–180. MR Zbl

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WELL-POSEDNESS OF SECOND-ORDER DEGENERATE DIFFERENTIAL EQUATIONS WITH FINITE DELAY IN VECTOR-VALUED FUNCTION SPACES

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We give necessary and sufficient conditions of the L^p-well-posedness (respectively, $B^s_{p,q}$-well-posedness) for the second-order degenerate differential equation with finite delay: $(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t)$, ($t \in [0, 2\pi]$) with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, and F and G are bounded linear operators from $L^p([-2\pi, 0]; X)$ (respectively, $B^s_{p,q}([-2\pi, 0]; X)$) into X.

1. Introduction

The purpose of this paper is to study the well-posedness of the following second-order degenerate differential equations with finite delays:

$$(P_2)\qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t) \quad (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi), \end{cases}$$

where $\mathbb{T} := [0, 2\pi]$, A and M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, F and G are bounded linear operators from $L^p([-2\pi, 0]; X)$ (resp. $B^s_{p,q}([-2\pi, 0]; X)$) into X, u_t and u'_t are defined on $[-2\pi, 0]$ by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ when $t \in \mathbb{T}$.

Let $1 \le p < \infty$. We say that (P_2) is L^p-well-posed, if for all $f \in L^p(\mathbb{T}; X)$, there exists a unique $u \in W^{1,p}_{\text{per}}(\mathbb{T}; X) \cap L^p(\mathbb{T}; D(A))$, such that $u' \in L^p(\mathbb{T}; D(M))$, $Mu' \in W^{1,p}_{\text{per}}(\mathbb{T}; X)$, and (P_2) is satisfied a.e. on $\mathbb{T}$. Here $D(A)$ and $D(M)$ are equipped with their graph norms so that they become Banach spaces, and $W^{1,p}_{\text{per}}(\mathbb{T}; X)$ is the X-valued periodic Sobolev space of order 1. Our main result in this paper gives a necessary and sufficient condition for (P_2) to be L^p-well-posed. Precisely,

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we show that when the underlying Banach space X is a UMD Banach space and $1 < p < \infty$, if the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is Rademacher bounded, then (P_2) is L^p-well-posed if and only if $\rho_p(P_2) = \mathbb{Z}$, and the sets $\{k^2 M N_k : k \in \mathbb{Z}\}$, $\{k N_k : k \in \mathbb{Z}\}$ are Rademacher bounded, where

$$N_k = (k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}, \quad (k \in \mathbb{Z}), \tag{1-1}$$

$F_k, G_k \in \mathcal{L}(X)$ are defined by $F_k x = F(e_k x)$, $G_k x = G(e_k x)$ with $e_k(t) = e^{ikt}$ (see Theorem 2.4). We also study the well-posedness of (P_2) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$, and a necessary and sufficient condition for (P_2) to be $B^s_{p,q}$-well-posed is also given (see Theorem 3.3).

The main tools we will use are operator-valued Fourier multipliers on $L^p(\mathbb{T}; X)$ and $B^s_{p,q}(\mathbb{T}; X)$. Indeed, we will transform the well-posedness of (P_2) to an operator-valued Fourier multiplier problem in the corresponding vector-valued function spaces. Thus the operator-valued Fourier multipliers theorems obtained by Arendt and Bu [2002; 2004] on $L^p(\mathbb{T}; X)$ and $B^s_{p,q}(\mathbb{T}; X)$ are fundamental for us.

The results obtained in this paper recover the known results presented in Bu and Fang [2010] in the nondegenerate case when $M = I_X$ and $\alpha = 0$. Thus our results may be also regarded as generalizations of the previous known results when $M = I_X$ and $F = G = 0$ in the L^p-well-posedness and the $B^s_{p,q}$-well-posedness obtained in [Arendt and Bu 2002; 2004]. Our results also generalize the previous known results obtained by Bu [2013] in the simpler case when $F = G = 0$ and $\alpha = 0$.

A large number of partial differential equations arising in physics and applied sciences, such as in the flow of fluid through fissured rocks, thermodynamics and shear in second-order fluids or in the theory of control of dynamical systems, can be expressed by the model in the form of (P_2). See [Lizama 2006; Bu and Fang 2009; 2010; Lizama and Ponce 2011; 2013; Poblete and Pozo 2013; 2014] for the study of vector-valued degenerate equations with delays. See the monographs by Favini and Yagi [1999] and by Sviridyuk and Fedorov [2003] for detailed studies of abstract degenerate type differential equations.

At the end of this paper, we give concrete examples to which our abstract results may be applied. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $1 < p < \infty$ and m be a nonnegative bounded measurable function defined on Ω; let $X = H^{-1}(\Omega)$, $F, G : L^p([-2\pi, 0]; X) \to X$ be bounded linear operators. If M is the multiplication operator by m on $H^{-1}(\Omega)$ with domain of definition $D(M)$ and $A = \Delta$ is the Laplacian on X with Dirichlet boundary condition and we assume that $D(A) \subset D(M)$, then under suitable assumptions on F and G we obtain the L^p-well-posedness for the corresponding second-order degenerate differential equations with finite delays (see Example 4.1). Our abstract results can also be applied in the following situation: let H be a complex Hilbert space, $1 < p < \infty$ and $F, G \in \mathcal{L}(L^p([-2\pi, 0]; H), H)$ be delay operators, P be a densely defined positive selfadjoint operator on H with $P \geq \delta > 0$. If $M = P - \epsilon$ with $\epsilon < \delta$, and $A = \sum_{i=0}^{k} a_i P^i$ with $a_i \geq 0,\ a_k > 0$. If we assume that $0 \in \rho(M)$, then we obtain the L^p-well-posedness of the corresponding second-order degenerate differential equations with finite delays under suitable assumptions on F and G (see Example 4.2).

This work is organized as follows. In Section 2, we study the well-posedness of (P_2) in $L^p(\mathbb{T}; X)$. In Section 3, we consider the well-posedness of (P_2) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. In Section 4, we give examples of degenerate differential equations with finite delays to which our abstract results may be applied.

2. Well-posedness in Lebesgue–Bochner spaces

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y. If $X = Y$, we will denote it simply by $\mathcal{L}(X)$. Let $1 \leq p < \infty$. We denote by $L^p(\mathbb{T}; X)$ the space of all X-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\|f\|_{L^p} := \left(\int_0^{2\pi} \|f(t)\|^p \frac{dt}{2\pi}\right)^{1/p} < \infty.$$

If $f \in L^1(\mathbb{T}; X)$, we define

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t)\, dt,$$

the k-th Fourier coefficient of f, where $k \in \mathbb{Z}$ and $e_k(t) := e^{ikt}$ for $t \in \mathbb{T}$.

Definition. Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X, Y)$ is said to be Rademacher bounded (R-bounded, in short), if there exists $C > 0$ such that

$$\sum_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \epsilon_j T_j x_j \right\| \leq C \sum_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

It is clear from the definition that if $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R-bounded, then $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still R-bounded. It is also clear that each R-bounded set is norm bounded. It is known that each norm bounded subset of $\mathcal{L}(X)$ is R-bounded if and only if X is isomorphic to a Hilbert space [Arendt and Bu 2002, Proposition 1.13]. The main tool in the study of L^p-well-posedness of (P_2) is the operator-valued L^p-Fourier multipliers.

Definition. Let X, Y be Banach space and $1 \leq p < \infty$. We say $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique $u \in L^p(\mathbb{T}; Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

It follows easily from the closed graph theorem that when $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier, then there exists a unique $T \in \mathcal{L}(L^p(\mathbb{T}; X), L^p(\mathbb{T}; Y))$, such that $\widehat{Tf}(k) = M_k \hat{f}(k)$ when $f \in L^p(\mathbb{T}; X)$ and $k \in \mathbb{Z}$. The following results were established in [Arendt and Bu 2002]:

Proposition 2.1. *Let X, Y be Banach spaces and assume that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Theorem 2.2. *Let X, Y be* UMD *spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

In this section, we study the following second-order degenerate differential equation with finite delays:

$$(P_2) \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi), \end{cases}$$

where A, M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, and $F, G : L^p([-2\pi, 0]; X) \to X$ are fixed bounded linear operators. Moreover, for fixed $t \in \mathbb{T}$, u_t and u'_t are elements of $L^p([-2\pi, 0]; X)$ defined by $u_t(s) = u(t + s)$, $u'_t(s) = u'(t + s)$ for $-2\pi \le s \le 0$. Here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

To give the definition of the solution space for (P_2), we need to introduce vector-valued periodic Sobolev space of order 1. For $1 \le p < \infty$, we define the periodic "Sobolev" space of order 1 [Arendt and Bu 2002] by:

$$W^{1,p}_{\text{per}}(\mathbb{T}; X) := \{u \in L^p(\mathbb{T}; X) : \text{ there exists } v \in L^p(\mathbb{T}; X) \text{ such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Let $u \in L^p(\mathbb{T}; X)$. Then $u \in W^{1,p}_{\text{per}}(\mathbb{T}; X)$ if and only if u is differentiable a.e. on $\mathbb{T}$ and $u' \in L^p(\mathbb{T}; X)$; in this case, u is actually continuous and $u(0) = u(2\pi)$ [Arendt and Bu 2002, Lemma 2.1].

Let $1 \le p < \infty$. We define the solution space of the L^p-well-posedness for (P_2) by

$$S_p(A, M) := \{u \in L^p(\mathbb{T}; D(A)) \cap W^{1,p}_{\text{per}}(\mathbb{T}; X) : u' \in L^p(\mathbb{T}; D(M)),\ Mu' \in W^{1,p}_{\text{per}}(\mathbb{T}; X)\},$$

here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. When $u \in S_p(A, M)$, then $Fu_\bullet, Gu'_\bullet \in L^p(\mathbb{T}; X)$ as $\|Fu_t\| \le \|F\|\|u\|_p$ and $\|Fu'_t\| \le \|F\|\|u'\|_p$ when $t \in \mathbb{T}$. Thus all terms appearing in (P_2) belong to $L^p(\mathbb{T}; X)$. Moreover $S_p(A, M)$ is a Banach space with the norm

$$\|u\|_{S_p(A,M)} := \|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p}.$$

By [Arendt and Bu 2002, Lemma 2.1], if $u \in S_p(A, M)$, then u and Mu' are X-valued continuous on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$.

Definition. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}; X)$; $u \in S_p(A, M)$ is called a strong L^p-solution of (P_2) if (P_2) is satisfied a.e. on $\mathbb{T}$. We say that (P_2) is L^p-well-posed, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of (P_2).

If (P_2) is L^p-well-posed, there exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T}; X)$, if $u \in S_p(A, M)$ is the unique strong L^p-solution of (P_2), then

$$\|u\|_{S_p(A,M)} \le C\|f\|_{L^p}. \tag{2-1}$$

This is an easy consequence of the closed graph theorem by the closedness of A and M.

Let $F, G \in \mathcal{L}(L^p(-2\pi, 0); X), X)$ and $k \in \mathbb{Z}$. We define the linear operators F_k, G_k on X by

$$F_k x := F(e_k x) \quad \text{and} \quad G_k x := G(e_k x), \quad (x \in X). \tag{2-2}$$

It is clear that $F_k, G_k \in \mathcal{L}(X)$, $\|F_k\| \le \|F\|$ and $\|G_k\| \le \|G\|$ as $\|e_k\|_p = 1$. Moreover when $u \in L^p(\mathbb{T}; X)$,

$$\widehat{Fu_\bullet}(k) = F_k \hat{u}(k) \quad \text{and} \quad \widehat{Gu_\bullet}(k) = G_k \hat{u}(k), \quad (k \in \mathbb{Z}). \tag{2-3}$$

This implies that $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as

$$\|Fu_t\| \le \|F\|\|u_\bullet\|_p = \|F\|\|u\|_p, \quad (t \in \mathbb{T})$$

and thus $Fu_\bullet, Gu_\bullet \in L^p(\mathbb{T}; X)$. We define the resolvent set of (P_2) in the L^p-well-posedness setting by

$$\rho_p(P_2) := \{k \in \mathbb{Z} : k^2 M - i\alpha k + ikG_k + F_k + A \quad \text{is invertible from } D(A) \text{ onto } X \quad \text{and} \quad (k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} \in \mathcal{L}(X)\}.$$

If $k \in \rho_p(P_2)$, then $M(k^2M - i\alpha k + ikG_k + F_k + A)^{-1}$ and $A(k^2M - i\alpha k + ikG_k + F_k + A)^{-1}$ make sense as $D(A) \subset D(M)$ by assumption, and they belong to $\mathcal{L}(X)$ by the closed graph theorem. We need the following preparation.

Proposition 2.3. *Let A and M be closed linear operators defined on a* UMD *space X satisfying $D(A) \subset D(M)$, $1 < p < \infty$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$. Assume that $\rho_p(P_2) = \mathbb{Z}$ and that the sets $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ and $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (k^2M - i\alpha k + ikG_k + F_k + A)^{-1}$, F_k and G_k are defined by* (2-2) *when $k \in \mathbb{Z}$. Then $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$ and $(kMN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.*

Proof. Let $M_k = k^2MN_k$, $S_k = kN_k$ and $T_k = kMN_k$ when $k \in \mathbb{Z}$. The sets $\{G_k : k \in \mathbb{Z}\}$ and $\{F_k : k \in \mathbb{Z}\}$ are R-bounded by [Lizama 2006, Proposition 3.2]. It follows from the R-boundedness of the set $\{I_X/k : k \in \mathbb{Z} \setminus \{0\}\}$ that $\{N_k : k \in \mathbb{Z}\}$ is R-bounded, as the product of R-bounded sets is still R-bounded. Moreover, by the definition of N_k,

$$\begin{aligned} N_{k+1} - N_k &= N_{k+1}(N_k^{-1} - N_{k+1}^{-1})N_k \\ &= N_{k+1}[-(2k+1)M + i\alpha + ikG_k - i(k+1)G_{k+1} + F_k - F_{k+1}]N_k \\ &= -(2k+1)N_{k+1}MN_k + i\alpha N_{k+1}N_k - ikN_{k+1}(G_{k+1} - G_k)N_k \\ &\qquad - iN_{k+1}G_{k+1}N_k - N_{k+1}(F_{k+1} - F_k)N_k. \end{aligned} \tag{2-4}$$

It follows that

$$\begin{aligned} M_{k+1} - M_k &= (k+1)^2 MN_{k+1} - k^2 MN_k \\ &= k^2 M(N_{k+1} - N_k) + (2k+1)MN_{k+1} \\ &= -k^2(2k+1)MN_{k+1}MN_k + i\alpha k^2 MN_{k+1}N_k \\ &\qquad -ik^3 MN_{k+1}(G_{k+1} - G_k)N_k - ik^2 MN_{k+1}G_{k+1}N_k \\ &\qquad -k^2 MN_{k+1}(F_{k+1} - F_k)N_k + (2k+1)MN_{k+1}, \end{aligned} \tag{2-5}$$

$$\begin{aligned} S_{k+1} - S_k &= k(N_{k+1} - N_k)) + N_{k+1} \\ &= -k(2k+1)N_{k+1}MN_k + i\alpha kN_{k+1}N_k - ik^2 N_{k+1}(G_{k+1} - G_k)N_k \\ &\qquad -ikN_{k+1}G_{k+1}N_k - kN_{k+1}(F_{k+1} - F_k)N_k + N_{k+1}, \end{aligned} \tag{2-6}$$

and

$$\begin{aligned} T_{k+1} - T_k &= M(S_{k+1} - S_k) \\ &= -k(2k+1)MN_{k+1}MN_k + i\alpha kMN_{k+1}N_k - ik^2 MN_{k+1}(G_{k+1} - G_k)N_k \\ &\qquad - ikMN_{k+1}G_{k+1}N_k - kMN_{k+1}(F_{k+1} - F_k)N_k + MN_{k+1}. \end{aligned} \tag{2-7}$$

This implies that the sets $\{k(N_{k+1} - N_k) : k \in \mathbb{Z}\}$, $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$, $\{k(S_{k+1} - S_k) : k \in \mathbb{Z}\}$ and $\{k(T_{k+1} - T_k) : k \in \mathbb{Z}\}$ are R-bounded by the R-boundedness of the sets $\{k^2 MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$, $\{F_k : k \in \mathbb{Z}\}$ and $\{G_k : k \in \mathbb{Z}\}$. It follows that $(N_k)_{k\in\mathbb{Z}}$, $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by Theorem 2.2. This completes the proof. □

Our next result gives a necessary and sufficient condition for the L^p-well-posedness of (P_2) when X is a UMD space and $1 < p < \infty$.

Theorem 2.4. *Let X be a* UMD *space*, $1 < p < \infty$ *and let* A, M *be closed linear operators on X satisfying $D(A) \subset D(M)$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$ be such that the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is R-bounded. Then the following assertions are equivalent.*

(i) *(P_2) is L^p-well-posed.*

(ii) *$\rho_p(P_2) = \mathbb{Z}$ and the sets $\{k^2 M N_k : k \in \mathbb{Z}\}$ and $\{k N_k : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$.*

Proof. $(i) \Rightarrow (ii)$: Assume that (P_2) is L^p-well-posed. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt} y$ $(t \in \mathbb{T})$. Then $f \in L^p(\mathbb{T}; X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ for $n \neq k$. Since (P_2) is L^p-well-posed, there exists $u \in S_p(A, M)$ such that

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t) \quad \text{a.e. on } \mathbb{T}. \tag{2-8}$$

We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [Arendt and Bu 2002, Lemma 3.1] as $u \in L^p(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides of (2-8), we obtain

$$-(k^2 M - i\alpha k + ikG_k + F_k + A)\hat{u}(k) = y, \tag{2-9}$$

and $-(n^2 M - i\alpha n + inG_n + F_n + A)\hat{u}(n) = 0$ when $n \neq k$. This implies in particular that $k^2 M - i\alpha k + ikG_k + F_k + A$ is surjective. We are going to show that it is also injective. Let $x \in D(A)$ be such that

$$(k^2 M - i\alpha k + ikG_k + F_k + A)x = 0,$$

and let $u(t) = e^{ikt} x$ when $t \in \mathbb{T}$. Then $u \in S_p(A, M)$ and (P_2) holds a.e. on $\mathbb{T}$ when taking $f = 0$. Consequently u is a strong L^p-solution of (P_2) when $f = 0$. We obtain $u = 0$ by the uniqueness assumption and thus $x = 0$. We have shown that $k^2 M - i\alpha k + ikG_k + F_k + A$ is also injective. Therefore $k^2 M - i\alpha k + ikG_k + F_k + A$ is a bijection from $D(A)$ onto X.

Now we show the boundedness of $(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$. For $f(t) = e^{ikt} y$, we let $u \in S_p(A, M)$ be the strong L^p-solution of (P_2). Then

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ -(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} y, & n = k, \end{cases}$$

by (2-9). This means that $u(t) = -e^{ikt}(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} y$. By (2-1), there exists a constant $C > 0$ independent from y and k satisfying

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p} \leq C\|f\|_{L^p}.$$

In particular $\|u\|_{L^p} \leq C\|f\|_{L^p}$. This implies that $\|(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} y\| \leq C\|y\|$ for all $y \in X$. Thus

$$\|(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}\| \leq C.$$

We have shown that $k \in \rho_p(P_2)$. Hence $\rho_p(P_2) = \mathbb{Z}$.

Let $M_k = k^2 M(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$ and $S_k = ik(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$ when $k \in \mathbb{Z}$. We are going to show that $(M_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. Let $f \in L^p(\mathbb{T}; X)$ be fixed. Then there exists $u \in S_p(A, M)$

strong L^p-solution of (P_2) by assumption. Taking Fourier transforms on both sides of (P_2), we get that $\hat{u}(k) \in D(A)$ by [Arendt and Bu 2002, Lemma 3.1] and

$$-(k^2M - i\alpha k + ikG_k + F_k + A)\hat{u}(k) = \hat{f}(k)$$

when $k \in \mathbb{Z}$. Since $k^2M - i\alpha k + ikG_k + F_k + A$ is invertible, we have

$$\hat{u}(k) = -(k^2M - i\alpha k + ikG_k + F_k + A)^{-1}\hat{f}(k)$$

when $k \in \mathbb{Z}$. We have $\widehat{u'}(k) = ik\hat{u}(k)$ and $\widehat{(Mu')'}(k) = -k^2M\hat{u}(k)$ by [Arendt and Bu 2002, Lemma 3.1]. Consequently

$$\widehat{u'}(k) = -S_k\hat{f}(k), \quad \text{and} \quad \widehat{(Mu')'}(k) = -M_k\hat{f}(k)$$

when $k \in \mathbb{Z}$. We conclude that $(M_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as u', $(Mu')' \in L^p(\mathbb{T}; X)$ by assumption. It follows from Proposition 2.1 that the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{S_k : k \in \mathbb{Z}\}$ are R-bounded.

$(ii) \Rightarrow (i)$: Assume that $\rho_p(P_2) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded. Define $M_k = k^2MN_k$, $S_k = ikN_k$ and $T_k = ikMN_k$ when $k \in \mathbb{Z}$. It follows from Proposition 2.3 that $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. Then for all $f \in L^p(\mathbb{T}; X)$, there exists $u, v, w, g \in L^p(\mathbb{T}; X)$ satisfying

$$\begin{aligned} \hat{u}(k) &= -M_k\hat{f}(k), \quad \hat{v}(k) = S_k\hat{f}(k), \\ \hat{w}(k) &= N_k\hat{f}(k), \qquad \hat{g}(k) = T_k\hat{f}(k), \quad (k \in \mathbb{Z}). \end{aligned} \tag{2-10}$$

Consequently $\hat{v}(k) = ik\hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in W^{1,p}_{\text{per}}(\mathbb{T}; X)$ [Arendt and Bu 2002, Lemma 2.1] and $w' = v$. We note that $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by (2-3). Thus $(ikG_kN_k)_{k\in\mathbb{Z}}$ and $(F_kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as the product of L^p-Fourier multipliers is still an L^p-Fourier multiplier. We have

$$AN_k = I_X - M_k + i\alpha kN_k - ikG_kN_k - F_kN_k, \quad (k \in \mathbb{Z}).$$

It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also an L^p-Fourier multiplier as the sum of L^p-Fourier multipliers is still an L^p-Fourier multiplier. This together with the fact that $(N_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier implies that $N_k \in \mathcal{L}(X, D(A))$. Here we consider $D(A)$ as a Banach space equipped with its graph norm. We have shown that $w \in L^p(\mathbb{T}; D(A))$.

Noticing the facts that $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers, we have that $S_k \in \mathcal{L}(X, D(M))$. Since $\hat{v}(k) = S_k\hat{f}(k)$ when $k \in \mathbb{Z}$ by (2-10), we deduce that $v = w' \in L^p(\mathbb{T}; D(M))$. Again by (2-10),

$$\hat{u}(k) = -k^2MN_k\hat{f}(k) = ik\widehat{Mw'}(k)$$

when $k \in \mathbb{Z}$. Thus we have $Mw' \in W^{1,p}_{\text{per}}(\mathbb{T}; X)$ by [Arendt and Bu 2002, Lemma 2.1]. We have shown that $w \in S_p(A, M)$.

By (2-10), we have

$$\widehat{(Mw')'}(k) + i\alpha k\hat{w}(k) = A\hat{w}(k) + ikG_k\hat{w}(k) + F_k\hat{w}(k) + \hat{f}(k)$$

when $k \in \mathbb{Z}$. This together with the facts $\widehat{Fw_\bullet}(k) = F_k\hat{w}(k)$ and $\widehat{Gw'_\bullet}(k) = ikG_k\hat{w}(k)$ implies that

$$(Mw')'(t) + \alpha u'(t) = Aw(t) + Gw'_t + Fw_t + f(t) \quad \text{a.e. on } \mathbb{T}$$

by the uniqueness theorem [Arendt and Bu 2002, page 314]. Thus w is a strong L^p-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_p(A, M)$ satisfying

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t \quad \text{a.e. on } \mathbb{T}.$$

Taking the Fourier transforms on both sides, we have

$$(k^2M - i\alpha k + ikG_k + F_k + A)\hat{u}(k) = 0, \quad (k \in \mathbb{Z}).$$

Since $\rho_p(P_2) = \mathbb{Z}$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. So (P_2) is L^p-well-posed. This completes the proof. □

Theorem 2.4 recovers the known results presented in Bu and Fang [2010] in the nondegenerate case when $M = I_X$ and $\alpha = 0$. Thus it may be also regarded as generalizations of the previous known results when $M = I_X$, $\alpha = 0$ and $F = G = 0$ in the L^p-well-posedness obtained in [Arendt and Bu 2002]. Our results also generalize the previous known results obtained by Bu [2013] in the simpler case when $F = G = 0$ and $\alpha = 0$.

3. Well-posedness in periodic Besov spaces

In this section we study the $B^s_{p,q}$-well-posedness of (P_2). Firstly, we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [Arendt and Bu 2004]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = \sup_{x\in\mathbb{T}}|f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T}, X) := \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all continuous linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define periodic Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\},\ I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \quad \text{for} \quad x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$,

$$\sup_{x\in\mathbb{R},\, k\in\mathbb{N}_0} 2^{k\alpha} |\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \phi(\mathbb{R})$ be fixed. For $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}; X) := \left\{ f \in \mathcal{D}'(\mathbb{T}, X) : \|f\|_{B^s_{p,q}} := \left(\sum_{j\ge 0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \right\|_p^q \right)^{1/q} < \infty \right\}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T}; X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T}; X)$. Equipping $B^s_{p,q}(\mathbb{T}; X)$ with the norm $\|\cdot\|_{B^s_{p,q}}$ gives a Banach space. See [Arendt and Bu 2004, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T}; X)$. We know that if $s_2 \le s_1$, then $B^{s_1}_{p,q}(\mathbb{T}; X) \subset B^{s_2}_{p,q}(\mathbb{T}; X)$ and the embedding is continuous [Arendt and Bu 2004]. When $s > 0$, it is shown in the same work that $B^s_{p,q}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X)$, $f \in B^{s+1}_{p,q}(\mathbb{T}; X)$ if and only if f is differentiable a.e. on $\mathbb{T}$ and $f' \in B^s_{p,q}(\mathbb{T}; X)$. This implies that if $u \in B^s_{p,q}(\mathbb{T}; X)$ is such that there exists $v \in B^s_{p,q}(\mathbb{T}; X)$ satisfying $\hat{v}(k) = ik\hat{u}(k)$ when $k \in \mathbb{Z}$, then $u \in B^{s+1}_{p,q}(\mathbb{T}; X)$ and $u' = v$ [Arendt and Bu 2004, Lemma 2.1].

The main tool in the study of $B^s_{p,q}$-well-posedness of (P_2) is the operator-valued $B^s_{p,q}$-Fourier multiplier theory established in [Arendt and Bu 2004].

Definition. Let X, Y be Banach spaces, $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We say $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique $u \in B^s_{p,q}(\mathbb{T}; Y)$, such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

The following result, obtained in [Arendt and Bu 2004], gives a sufficient condition for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier.

Theorem 3.1. *Let X, Y be Banach spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We assume*

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty, \tag{3-1}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2} - 2M_{k+1} + M_k)\| < \infty. \tag{3-2}$$

Then for $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. If X is B-convex, then condition (3-1) *is already sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-Fourier multiplier.*

Recall that a Banach space X is B-convex if it does not contain l_1^n uniformly. This is equivalent to saying that X has Fourier type $1 < p \le 2$, i.e., the Fourier transform is a bounded linear operator from $L^p(\mathbb{R}; X)$ to $l^q(\mathbb{Z}; X)$, where $1/p + 1/q = 1$. It is well known that when $1 < p < \infty$, then $L^p(\mu)$ has Fourier type $\min\{p, p/(p-1)\}$.

Let $1 \le p,\ q \le \infty,\ s > 0$ be fixed. We consider the following second-order degenerate differential equation with finite delays:

$$(P_2) \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi), \end{cases}$$

where A, M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, and $F, G : B^s_{p,q}([-2\pi, 0]; X) \to X$ are bounded linear operators. Moreover, for fixed $t \in \mathbb{T}$, u_t and u'_t are elements of $B^s_{p,q}([-2\pi, 0]; X)$ defined by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ for $-2\pi \le s \le 0$. Here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

Let $F, G \in \mathcal{L}(B^s_{p,q}(-2\pi, 0); X), X)$ and $k \in \mathbb{Z}$. We define the linear operators $F_k, G_k \in \mathcal{L}(X)$ by $F_k x := F(e_k \otimes x)$, $G_k x := G(e_k \otimes x)$ for all $x \in X$. It is clear that there exists a constant $C > 0$ such that $\|e_k \otimes x\|_{B^s_{p,q}} \le C\|x\|$ for all $k \in \mathbb{Z}$. Thus

$$\|F_k\| \le C\|F\|, \quad \text{and} \quad \|G_k\| \le C\|G\|, \quad (k \in \mathbb{Z}). \tag{3-3}$$

It is easy to verify that when $u \in B^s_{p,q}(\mathbb{T}; X)$, then

$$\widehat{Fu_\bullet}(k) = F_k \hat{u}(k), \quad \text{and} \quad \widehat{Gu_\bullet}(k) = G_k \hat{u}(k), \quad (k \in \mathbb{Z}).$$

We define the resolvent set of (P_2) in the $B^s_{p,q}$-well-posedness setting by

$$\rho_{p,q,s}(P_2) := \{k \in \mathbb{Z} : k^2 M - ik\alpha + ikG_k + F_k + A \text{ is a bijection from } D(A) \text{ onto } X, \\ \text{and} \quad (k^2 M - ik\alpha + ikG_k + F_k + A)^{-1} \in \mathcal{L}(X)\}.$$

If $k \in \rho_{p,q,s}(P_2)$, then $M(k^2 M + ikG_k + F_k + A)^{-1}$, $A(k^2 M + ikG_k + F_k + A)^{-1}$ make sense as $D(A) \subset D(M)$ by assumption, and they are in $\mathcal{L}(X)$ by the closed graph theorem.

Let $1 \le p,\ q \le \infty,\ s > 0$. We notice that the functions $Fu_\bullet$ and $Gu'_\bullet$ are uniformly bounded on $\mathbb{T}$, but they are not necessarily in $B^s_{p,q}(\mathbb{T}; X)$. We define the solution space of the $B^s_{p,q}$-well-posedness for (P_2) by

$$S_{p,q,s}(A, M) := \{u \in B^s_{p,q}(\mathbb{T}; D(A)) \cap B^{1+s}_{p,q}(\mathbb{T}; X) : u' \in B^s_{p,q}(\mathbb{T}; D(M)), \\ Mu' \in B^{s+1}_{p,q}(\mathbb{T}; X) \quad \text{and} \quad Fu_\bullet, Gu'_\bullet \in B^s_{p,q}(\mathbb{T}; X)\}.$$

Here again we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S_{p,q,s}(A, M)$ is a Banach space with the norm

$$\|u\|_{S_{p,q,s}(A,M)} := \|u\|_{B^{1+s}_{p,q}} + \|Au\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Mu'\|_{B^{1+s}_{p,q}} + \|Fu_\bullet\|_{B^s_{p,q}} + \|Gu'_\bullet\|_{B^s_{p,q}}.$$

From [Arendt and Bu 2002, Lemma 2.1], if $u \in S_{p,q,s}(A, M)$, then u and Mu' are X-valued continuous on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$.

Definition. Let $1 \le p,\ q \le \infty,\ s > 0$ and $f \in B^s_{p,q}(\mathbb{T}; X)$. $u \in S_{p,q,s}(A, M)$ is called a strong $B^s_{p,q}$-solution of (P_2), if (P_2) is satisfied a.e. on $\mathbb{T}$. We say that (P_2) is $B^s_{p,q}$-well-posed, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique strong $B^s_{p,q}$-solution of (P_2).

If (P_2) is $B^s_{p,q}$-well-posed, there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T}; X)$, if $u \in S_{p,q,s}(A, M)$ is the unique strong $B^s_{p,q}$-solution of (P_2), then

$$\|u\|_{S_{p,q,s}(A,M)} \le C\|f\|_{B^s_{p,q}}. \tag{3-4}$$

This can be easily obtained by the closedness of the operators A and M and the closed graph theorem. We need the following preparation:

Proposition 3.2. *Let A and M be closed linear operators defined on a Banach space X satisfying $D(A) \subset D(M)$ and let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. Assume that $\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$, $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$, $\{k^2 M N_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (k^2 M - ik\alpha + ikG_k + F_k + A)^{-1}$ when $k \in \mathbb{Z}$. Then $(k^2 M N_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$, $(kMN_k)_{k\in\mathbb{Z}}$, $(F_k N_k)_{k\in\mathbb{Z}}$ and $(kG_k N_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers whenever $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$.*

Proof. Define $M_k = k^2 M N_k,\ S_k = kN_k,\ T_k = kMN_k,\ P_k = F_k N_k$ and $Q_k = kG_k N_k$ when $k \in \mathbb{Z}$. We know $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are norm bounded by (3-3). This implies that the sequences $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are norm bounded by assumption. Using the same argument used in the proof of Proposition 2.3, we obtain

$$\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty, \qquad \sup_{k\in\mathbb{Z}} \|k(N_{k+1} - N_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k(S_{k+1} - S_k)\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k(T_{k+1} - T_k)\| < \infty.$$

Moreover, it is easy to see that one has the stronger estimations

$$\sup_{k\in\mathbb{Z}} \|k^2(N_{k+1} - N_k)\| < \infty, \tag{3-5}$$

$$\sup_{k\in\mathbb{Z}} \|k^3 M(N_{k+1} - N_k)\| < \infty, \tag{3-6}$$

by using the norm boundedness of $\{k(G_{k+} - G_k) : k \in \mathbb{Z}\}$. For P_k and Q_k, when $k \in \mathbb{Z}$, we have

$$P_{k+1} - P_k = F_{k+1}(N_{k+1} - N_k) + (F_{k+1} - F_k)N_k, \tag{3-7}$$

$$Q_{k+1} - Q_k = G_{k+1}N_{k+1} + k(G_{k+1} - G_k)N_k + kG_k(N_{k+1} - N_k). \tag{3-8}$$

We deduce that

$$\sup_{k\in\mathbb{Z}} \|k(P_{k+1} - P_k)\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k(Q_{k+1} - Q_k)\| < \infty$$

by (3-5) and the boundedness of $(F_k)_{k\in\mathbb{Z}}$, $(G_k)_{k\in\mathbb{Z}}$ and $(k(G_{k+1} - G_k))_{k\in\mathbb{Z}}$.

By (2-3) we have

$$N_{k+1} - N_k = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} + I_k^{(5)},$$

where

$$\begin{aligned} I_k^{(1)} &:= -(2k+1)N_{k+1}MN_k, \\ I_k^{(2)} &:= i\alpha N_{k+1}N_k, \\ I_k^{(3)} &:= -ikN_{k+1}(G_{k+1} - G_k)N_k, \\ I_k^{(4)} &:= -iN_{k+1}G_{k+1}N_k, \\ I_k^{(5)} &:= -N_{k+1}(F_{k+1} - F_k)N_k. \end{aligned}$$

We have

$$\begin{aligned} I_{k+1}^{(1)} - I_k^{(1)} &= -(2k+3)N_{k+2}MN_{k+1} + (2k+1)N_{k+1}MN_k \\ &= -2N_{k+2}MN_{k+1} - (2k+1)(N_{k+2} - N_{k+1})MN_{k+1} \\ &\qquad - (2k+1)N_{k+1}M(N_{k+1} - N_k). \end{aligned} \tag{3-9}$$

This implies that

$$\sup_{k\in\mathbb{Z}} \|k^3(I_{k+1}^{(1)} - I_k^{(1)})\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k^4M(I_{k+1}^{(1)} - I_k^{(1)})\| < \infty$$

using (3-5) and (3-6). A similar argument shows that

$$\sup_{k\in\mathbb{Z}} \|k^3(I_{k+1}^{(i)} - I_k^{(i)})\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k^4M(I_{k+1}^{(i)} - I_k^{(i)})\| < \infty$$

when $i = 2, 3, 4, 5$ using inequalities (3-5), (3-6) and the norm boundedness of $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ and $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$. We have shown that

$$\sup_{k\in\mathbb{Z}} \|k^3(N_{k+2} - 2N_{k+1} + N_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^4M(N_{k+2} - 2N_{k+1} + N_k)\| < \infty. \tag{3-10}$$

In particular,

$$\sup_{k\in\mathbb{Z}} \|k^2(N_{k+2} - 2N_{k+1} + N_k)\| < \infty.$$

By (2-4), (2-5), (3-7), (3-8) and (3-10), and using similar argument used in the proof of (3-10), we show that

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2}-2M_{k+1}+M_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^2(S_{k+2}-2S_{k+1}+S_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k^2(T_{k+2}-2T_{k+1}+T_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^2(P_{k+2}-2P_{k+1}+P_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k^2(Q_{k+2}-2Q_{k+1}+Q_k)\| < \infty.$$

Thus $(N_k)_{k\in\mathbb{Z}}$, $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers by Theorem 3.1. □

Now we give a necessary and sufficient condition for (P_2) to be $B^s_{p,q}$-well-posed.

Theorem 3.3. *Let X be a Banach space, $1 \le p, q \le \infty$, $s > 0$ and let A and M be closed linear operators on X satisfying $D(A) \subset D(M)$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that the sets $\{k(F_{k+2}-2F_{k+1}+F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1}-G_k) : k \in \mathbb{Z}\}$ and $\{k^2(G_{k+2}-2G_{k+1}+G_k) : k \in \mathbb{Z}\}$ are norm bounded. Then the following assertions are equivalent*:

(i) *(P_2) is $B^s_{p,q}$-well-posed.*

(ii) *$\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$.*

Proof. $(i) \Rightarrow (ii)$: Assume that (P_2) is $B^s_{p,q}$-well-posed. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed, we define $f(t) = e^{ikt}y$ when $t \in \mathbb{T}$. Then $f \in B^s_{p,q}(\mathbb{T}; X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ for $n \ne k$. Since (P_2) is $B^s_{p,q}$-well-posed, there exists a unique $u \in S_{p,q,s}(A, M)$ satisfying

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), \quad \text{a.e. on } \mathbb{T}.$$

We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [Arendt and Bu 2002, Lemma 3.1] as $u \in B^s_{p,q}(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides, we obtain

$$-(k^2M - ik\alpha + ikG_k + F_k + A)\hat{u}(k) = y \tag{3-11}$$

and $-(k^2M + ikG_k + F_k + A)\hat{u}(n) = 0$ when $n \ne k$. This implies that the operator $k^2M - ik\alpha + ikG_k + F_k + A$ is surjective as the vector $y \in X$ is arbitrary. To show that $k^2M - ik\alpha + ikG_k + F_k + A$ is also injective, we let $x \in D(A)$ satisfying

$$(k^2M - ik\alpha + ikG_k + F_k + A)x = 0.$$

Let $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$. Then $u \in S_{p,q,s}(A, M)$ and (P_2) holds a.e. on $\mathbb{T}$ when $f = 0$. Thus u is a strong $B^s_{p,q}$-solution of (P_2) when $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that $k^2M - ik\alpha + ikG_k + F_k + A$ is injective. Thus it is bijective from $D(A)$ onto X.

Next we show that $(k^2M - ik\alpha + ikG_k + F_k + A)^{-1} \in \mathcal{L}(X)$. For $y \in X$ and $f(t) = e^{ikt}y$, we let $u \in S_{p,q,s}(A, M)$ be the unique strong $B^s_{p,q}$-solution of (P_2). Then taking Fourier coefficients on both sides of (P_2), we obtain by (3-11)

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ -(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}y, & n = k. \end{cases}$$

Consequently, $u(t) = -e^{ikt}(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}y$ when $t \in \mathbb{T}$. By (3-4) there exists a constant $C > 0$ independent from y and k, such that

$$\|u\|_{B^{1+s}_{p,q}} + \|Au\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Mu'\|_{B^{1+s}_{p,q}} + \|Fu_\bullet\|_{B^s_{p,q}} + \|Gu'_\bullet\|_{B^s_{p,q}} \leq C\|f\|_{B^s_{p,q}}.$$

The estimation

$$\|u'\|_{B^s_{p,q}} \leq C\|f\|_{B^s_{p,q}}$$

implies that $\|k(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}y\| \leq C\|y\|$ for all $y \in X$. Therefore

$$\|k(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}\| \leq C.$$

We have shown that $k \in \rho_{p,q,s}(P_2)$ for all $k \in \mathbb{Z}$. Thus $\rho_{p,q,s}(P_2) = \mathbb{Z}$.

Next we show that $(M_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are norm bounded, where $M_k = k^2MN_k$ and $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$ when $k \in \mathbb{Z}$. For this it will suffice to show that $(M_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ define $B^s_{p,q}$-Fourier multipliers by [Arendt and Bu 2004]. Let $f \in B^s_{p,q}(\mathbb{T}; X)$. Then there exists $u \in S_{p,q,s}(A, M)$ which is a strong $B^s_{p,q}$-solution of (P_2) by assumption. Taking Fourier coefficients on both sides of (P_2), we get that $\hat{u}(k) \in D(A)$ and

$$-(k^2M - ik\alpha + ikG_k + F_k + A)\hat{u}(k) = \hat{f}(k),$$

or equivalently,

$$\hat{u}(k) = -(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}\hat{f}(k), \quad (k \in \mathbb{Z}).$$

It follows from $u \in S_{p,q,s}(A, M)$ that $\widehat{(Mu')'}(k) = -k^2M\hat{u}(k)$ and $\widehat{u'}(k) = ik\hat{u}(k)$. We obtain

$$\widehat{(Mu')'}(k) = -k^2M\hat{u}(k) = -M_k\hat{f}(k), \quad \text{and} \quad \widehat{u'}(k) = -ikN_k\hat{f}(k), \quad (k \in \mathbb{Z}).$$

We conclude that $(M_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ define $B^s_{p,q}$-Fourier multipliers as $(Mu')'$, $u' \in B^s_{p,q}(\mathbb{T}; X)$.

$(ii) \Rightarrow (i)$: Let $\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ be norm bounded, where $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$. Define $M_k = k^2MN_k$, $S_k = ikN_k$, $T_k = kMN_k$, $P_k = F_kN_k$ and $Q_k = ikG_kN_k$ when $k \in \mathbb{Z}$. It follows from Proposition 3.2 that $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$

are $B^s_{p,q}$-Fourier multipliers. Then for all $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists $u, v, w \in B^s_{p,q}(\mathbb{T}; X)$ satisfying

$$\hat{u}(k) = -k^2 M N_k \hat{f}(k), \quad \hat{v}(k) = ikN_k\hat{f}(k) \quad \text{and} \quad \hat{w}(k) = N_k\hat{f}(k), \tag{3-12}$$

when $k \in \mathbb{Z}$. We deduce from the facts that $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers that $Fw_\bullet, Gw'_\bullet \in B^s_{p,q}(\mathbb{T}; X)$ as

$$\widehat{Fw_\bullet}(k) = F_k\hat{w}(k) = F_kN_k\hat{f}(k) = P_k\hat{f}(k), \quad (k \in \mathbb{Z})$$

and

$$\widehat{Gw'_\bullet}(k) = G_k\widehat{w'}(k) = ikG_k\hat{w}(k) = ikG_kN_k\hat{f}(k) = Q_k\hat{f}(k), \quad (k \in \mathbb{Z}).$$

On the other hand, $\hat{v}(k) = ik\hat{w}(k)$ when $k \in \mathbb{Z}$ by (3-12). Therefore w is differentiable a.e. on $\mathbb{T}$ and $w' = v$. This implies that $w \in B^{1+s}_{p,q}(\mathbb{T}; X)$ as $v \in B^s_{p,q}(\mathbb{T}; X)$ [Arendt and Bu 2002, Lemma 2.1].

We note that

$$AN_k = M_k + \alpha S_k - P_k - Q_k + I_X, \quad (k \in \mathbb{Z}).$$

It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-Fourier multiplier. Therefore there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ satisfying

$$\hat{g}(k) = AN_k\hat{f}(k), \quad (k \in \mathbb{Z}). \tag{3-13}$$

Thus $\hat{g}(k) = A\hat{w}(k)$ when $k \in \mathbb{Z}$. This implies $w \in B^s_{p,q}(\mathbb{T}; D(A))$ by [Arendt and Bu 2002, Lemma 3.1].

Since $(T_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, there exists $h \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\hat{h}(k) = ikMN_k\hat{f}(k) = M\widehat{w'}(k), \quad (k \in \mathbb{Z}).$$

Thus $w' \in B^s_{p,q}(\mathbb{T}; D(M))$ by [Arendt and Bu 2002, Lemma 3.1]. In view of (3-12), we obtain

$$\hat{u}(k) = -k^2MN_k\hat{f}(k) = -k^2M\hat{w}(k) = ik\widehat{Mw'}(k), \quad (k \in \mathbb{Z})$$

which implies that $Mw' \in B^{s+1}_{p,q}(\mathbb{T}; X)$ by [Arendt and Bu 2002, Lemma 2.1]. We have shown that $u \in S_{p,q,s}(A, M)$.

By (3-12), we have

$$\widehat{(Mw')'}(k) + \alpha\widehat{w'}(k) = A\hat{w}(k) + ikG_k\hat{w}(k) + F_k\hat{w}(k) + \hat{f}(k), \quad (k \in \mathbb{Z}).$$

It follows that $(Mw')'(t) + \alpha w'(t) = Aw(t) + Gw'_t + Fw_t + f(t)$ a.e. on $\mathbb{T}$ by the uniqueness theorem [Arendt and Bu 2002, page 314]. Thus w is a strong $B^s_{p,q}$-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_{p,q,s}(A, M)$ satisfy

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t$$

a.e. on $\mathbb{T}$. Taking the Fourier coefficients on both sides, we have

$$-(k^2 M - \alpha S_k + ikG_k + F_k + A)\hat{u}(k) = 0$$

for all $k \in \mathbb{Z}$. Since $\rho_{p,q,s}(P_2) = \mathbb{Z}$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. So (P_2) is $B^s_{p,q}$-well-posed. This finishes the proof. □

By the proof of Theorem 2.4 and using Theorem 3.1, one can obtain the following result.

Theorem 3.4. *Let X be a B-convex Banach space, $1 \le p,\ q \le \infty,\ s > 0$ and let A, M be closed linear operators on X satisfying $D(A) \subset D(M)$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is norm bounded. Then the following assertions are equivalent*:

(i) *(P_2) is $B^s_{p,q}$-well-posed.*

(ii) *$\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k^2 MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (k^2 M - ik\alpha + ikG_k + F_k + A)^{-1}$.*

4. Applications

In the last section, we give some examples to which our abstract results (Theorem 2.4 and Theorem 3.3) may be applied.

Example 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and m be a nonnegative bounded measurable function defined on Ω. Let f be a given function on $[0, 2\pi] \times \Omega$ and $X = H^{-1}(\Omega)$. We consider the following periodic degenerate differential equations with finite delay:

$$(P)\quad \begin{cases} \frac{\partial^2}{\partial t^2}(m(x)u(t,x)) + \alpha\frac{\partial}{\partial t}u(t,x) + \Delta u = Fu_t + Gu'_t + f(t,x), & (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = 0, & (t,x) \in [0,2\pi] \times \partial\Omega, \\ u(0,x) = u(2\pi,x), & x \in \Omega, \\ \frac{\partial u(t,x)}{\partial t}|_{t=0} = \frac{\partial u(t,x)}{\partial t}|_{t=2\pi}, & x \in \Omega, \end{cases}$$

where $\alpha \in \mathbb{C}$ is fixed, $u_t(s, x) := u(t + s, x),\ u'_t(s, x) := u'(t + s, x)$ when $s \in [-2\pi, 0]$ and $x \in \Omega$, the delay operators $F, G : L^p([-2\pi, 0]; X) \to X$ are bounded linear operators for some fixed $1 < p < \infty$.

Let M be the multiplication operator by m on $H^{-1}(\Omega)$ with domain $D(M)$. Then it follows from [Favini and Yagi 1999, Section 3.7] that if we consider the Laplacian operator Δ on X with Dirichlet boundary condition, then there exists a constant $C > 0$ such that

$$\|M(zM-\Delta)^{-1}\| \le \frac{C}{1+|z|},$$

when $\mathrm{Re}(z) \ge -\beta(1+|\mathrm{Im}(z)|)$ for some positive constant β depending only on m, which implies that

$$\|M(k^2M-\Delta)^{-1}\| \le \frac{C}{1+|k|^2}, \quad (k \in \mathbb{Z}). \tag{4-1}$$

If we assume that m^{-1} is regular enough so that the multiplication operator by the function m^{-1} is bounded on $H^{-1}(\Omega)$, then there exists a constant C_1 such that

$$\|(k^2M-\Delta)^{-1}\| \le \frac{C_1}{1+|k|^2}, \quad (k \in \mathbb{Z}). \tag{4-2}$$

Assume that $D(\Delta) \subset D(M)$, that the set $\{k(G_{k+1}-G_k) : k \in \mathbb{Z}\}$ is norm bounded, and that $\rho_p(P) = \mathbb{Z}$, so that for all $k \in \mathbb{Z}$ the operator $-k^2M + i\alpha k + \Delta - F_k - ikG_k$ is a bijection from $D(\Delta)$ onto X, and $(-k^2M + i\alpha k + \Delta - F_k - ikG_k)^{-1} \in \mathcal{L}(X)$. We observe that

$$-k^2M+i\alpha k+\Delta-F_k-ikG_k = (I-(F_k+ikG_k-i\alpha k)(-k^2M+\Delta)^{-1})(-k^2M+\Delta)$$

for $k \in \mathbb{Z}$. From (4-2) we get $\lim_{k\to\infty} \|(F_k + ikG_k - i\alpha k)(-k^2M + \Delta)^{-1}\| = 0$ using the norm boundedness of $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$. This implies that the operator $I - (-k^2M + \Delta)^{-1}(F_k + ikG_k - i\alpha k)$ is invertible when $|k|$ is big enough. For such k we have

$$\begin{aligned}(-k^2M &+ i\alpha k + \Delta - F_k - ikG_k)^{-1} \\ &= (-k^2M + \Delta)^{-1}(I - (F_k + ikG_k - i\alpha k)(-k^2M + \Delta)^{-1})^{-1}.\end{aligned}$$

It follows from (4-1) and (4-2) that

$$\sup_{k\in\mathbb{Z}} \|k(-k^2M + i\alpha k + \Delta - F_k - ikG_k)^{-1}\| < \infty,$$

and

$$\sup_{k\in\mathbb{Z}} \|k^2M(-k^2M + i\alpha k + \Delta - F_k - ikG_k)^{-1}\| < \infty.$$

As a consequence, the sets $\{k(-k^2M + i\alpha k + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\}$ and $\{k^2M(-k^2M + i\alpha k + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\}$ are R-bounded. Here we used the fact that when the underlying Banach space X is a Hilbert space, then each norm bounded subset of $\mathcal{L}(X)$ is R-bounded [Arendt and Bu 2002, Proposition 1.13]. We deduce from Theorem 2.4 that (P) is L^p-well-posed when $X = H^{-1}(\Omega)$.

If we consider $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$, we may also apply Theorem 3.3 and Theorem 3.4 to obtain the $B^s_{p,q}$-well-posedness of (P) under suitable assumptions on F and G.

Example 4.2. Let H be a complex Hilbert space, let $1 < p < \infty$ and let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; H), H)$ be delay operators. Let P be a densely defined positive self-adjoint operator on H with $P \geq \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = \sum_{i=0}^{k} a_i P^i$ with $a_i \geq 0,\ a_k > 0$. Then there exists a constant $C > 0$, such that

$$\|M(zM + A)^{-1}\| \leq \frac{C}{1+|z|}$$

whenever $\operatorname{Re} z \geq -\beta(1 + |\operatorname{Im} z|)$ for some positive constant β depending only on A and M by [Favini and Yagi 1999, page 73]. This implies in particular that

$$\sup_{k\in\mathbb{Z}} \|k^2 M(k^2 M + A)^{-1}\| < \infty.$$

If we assume $0 \in \rho(M)$, then

$$\sup_{k\in\mathbb{Z}} \|k^2 (k^2 M + A)^{-1}\| < \infty.$$

Furthermore we assume that the set $\{k(G_{k+1} - G_k : k \in \mathbb{Z})\}$ is norm bounded. Then the argument used in the example on page 43 our first example shows that the degenerate differential equations with finite delay

$$(P') \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi) \end{cases}$$

is L^p-well-posed when $\rho_p(P') = \mathbb{Z}$. Under suitable assumptions on F, G, we may also apply Theorem 3.3 to (P') to obtain the $B^s_{p,q}$-well-posedness of (P') for all $1 \leq p,\ q \leq \infty,\ s > 0$.

We can also give a concrete example of (P'). We consider the following problem:

$$\begin{cases} \frac{\partial^2}{\partial t^2}\big(1 - \frac{\partial^2}{\partial x^2}\big)u(t,x) + \alpha\frac{\partial}{\partial t}u(t,x) = \frac{\partial^4}{\partial x^4}u(t,x) + Fu_t(\,\cdot\,,x) + G\big(\frac{\partial u}{\partial t}\big)_t(\,\cdot\,,x) + f(t,x), \\ u(t,0) = u(t,1) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,1) = 0, \\ u(0,x) = u(2\pi,x), \quad \big(1 - \frac{\partial^2}{\partial x^2}\big)u(0,x) = \big(1 - \frac{\partial^2}{\partial x^2}\big)u(2\pi,x), \\ \frac{\partial}{\partial t}\big(1 - \frac{\partial^2}{\partial x^2}\big)u(0,x) = \frac{\partial}{\partial t}\big(1 - \frac{\partial^2}{\partial x^2}\big)u(2\pi,x), \end{cases}$$

where $x \in \Omega,\ t \in (0, 2\pi)$ in the first equation, and $t \in [0, 2\pi]$ in the second equation. Here, $\Omega = (0, 1)$, $F, G \in \mathcal{L}(L^p([-2\pi, 0]; L^2(\Omega)), L^2(\Omega))$ and $u_t(s, x) := u(t+s, x)$ when $t \in [0, 2\pi]$ and $s \in [-2\pi, 0]$. Let $X = L^2(\Omega)$ and let $P = -\partial^2/\partial x^2$ with domain $D(P) = H^2(\Omega) \cap H^1_0(\Omega)$, i.e., P is the Laplacian on $L^2(\Omega)$ with Dirichlet boundary conditions. Then P is positive self adjoint on X. Let $M = P + I_X$ and $A = P^2$. It is clear that $-P$ generates an contraction semigroup on $L^2(\Omega)$ [Arendt et al. 2001, Example 3.4.7], hence $1 \in \rho(-P)$, or equivalently $M = I_X + P$ has a bounded inverse, i.e., $0 \in \rho(M)$. Then the abstract results obtained above for the problem (P') may be applied.

References

[Arendt and Bu 2002] W. Arendt and S. Bu, "The operator-valued Marcinkiewicz multiplier theorem and maximal regularity", *Math. Z.* **240**:2 (2002), 311–343. MR Zbl

[Arendt and Bu 2004] W. Arendt and S. Bu, "Operator-valued Fourier multipliers on periodic Besov spaces and applications", *Proc. Edinb. Math. Soc.* (2) **47**:1 (2004), 15–33. MR Zbl

[Arendt et al. 2001] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics **96**, Birkhäuser, Basel, 2001. MR Zbl

[Bu 2013] S. Bu, "Well-posedness of second order degenerate differential equations in vector-valued function spaces", *Studia Math.* **214**:1 (2013), 1–16. MR Zbl

[Bu and Fang 2009] S. Bu and Y. Fang, "Periodic solutions of delay equations in Besov spaces and Triebel–Lizorkin spaces", *Taiwanese J. Math.* **13**:3 (2009), 1063–1076. MR Zbl

[Bu and Fang 2010] S. Bu and Y. Fang, "Maximal regularity of second order delay equations in Banach spaces", *Sci. China Math.* **53**:1 (2010), 51–62. MR Zbl

[Favini and Yagi 1999] A. Favini and A. Yagi, *Degenerate differential equations in Banach spaces*, Monogr. Textbooks Pure Appl. Math. **215**, Dekker, New York, 1999. MR Zbl

[Lizama 2006] C. Lizama, "Fourier multipliers and periodic solutions of delay equations in Banach spaces", *J. Math. Anal. Appl.* **324**:2 (2006), 921–933. MR Zbl

[Lizama and Ponce 2011] C. Lizama and R. Ponce, "Periodic solutions of degenerate differential equations in vector-valued function spaces", *Studia Math.* **202**:1 (2011), 49–63. MR Zbl

[Lizama and Ponce 2013] C. Lizama and R. Ponce, "Maximal regularity for degenerate differential equations with infinite delay in periodic vector-valued function spaces", *Proc. Edinb. Math. Soc.* (2) **56**:3 (2013), 853–871. MR Zbl

[Poblete and Pozo 2013] V. Poblete and J. C. Pozo, "Periodic solutions of an abstract third-order differential equation", *Studia Math.* **215**:3 (2013), 195–219. MR Zbl

[Poblete and Pozo 2014] V. Poblete and J. C. Pozo, "Periodic solutions of a fractional neutral equation with finite delay", *J. Evol. Equ.* **14**:2 (2014), 417–444. MR Zbl

[Sviridyuk and Fedorov 2003] G. A. Sviridyuk and V. E. Fedorov, *Linear Sobolev type equations and degenerate semigroups of operators*, VSP, Utrecht, 2003. MR Zbl

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ON CUSP SOLUTIONS TO A PRESCRIBED MEAN CURVATURE EQUATION

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The nonexistence of "cusp solutions" of prescribed mean curvature boundary value problems in $\Omega \times \mathbb{R}$ when Ω is a domain in $\mathbb{R}^2$ is proven in certain cases and an application to radial limits at a corner is mentioned.

1. Introduction

Let Ω be a domain in $\mathbb{R}^2$ with locally Lipschitz boundary and $\mathcal{O} = (0, 0) \in \partial\Omega$ and $H \in C^{1,\beta}(\overline{\Omega} \times \mathbb{R})$, for some $\beta \in (0, 1)$. Let polar coordinates relative to $\mathcal{O}$ be denoted by r and θ and let $B_\delta(\mathcal{O})$ be the open ball in $\mathbb{R}^2$ of radius δ about $\mathcal{O}$. We shall assume there exist a $\delta^* > 0$ and $\alpha \in (0, \pi)$ such that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of two smooth arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached and for each $\theta \in (-\alpha, \alpha)$, there exists an $r(\theta) > 0$ such that $\{(r \cos\theta, r \sin\theta) : 0 < r < r(\theta)\} \subset \Omega$. Set $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$.

Consider a solution $f \in C^2(\Omega)$ of the prescribed mean curvature equation

$$\operatorname{div}(Tf)(x, y) = 2H(x, y, f(x, y)) \quad \text{for} \quad (x, y) \in \Omega^*, \tag{1}$$

which satisfies the conditions

$$\sup_{(x,y)\in\Omega^*} |f(x, y)| < \infty \quad \text{and} \quad \sup_{(x,y)\in\Omega^*} |H(x, y, f(x, y))| < \infty, \tag{2}$$

where $Tf = \nabla f/\sqrt{1 + |\nabla f|^2}$; examples of such functions might arise as solutions of a Dirichlet or contact angle boundary value problem for (1). We are interested in the radial limits of f:

$$Rf(\theta) \overset{\text{def}}{=} \lim_{r\downarrow 0} f(r \cos\theta, r \sin\theta), \quad -\alpha < \theta < \alpha. \tag{3}$$

When $\lim_{\partial^+\Omega^*\ni(x,y)\to\mathcal{O}} f(x, y)$ exists, we define $Rf(\alpha)$ to be this limit and when $\lim_{\partial^-\Omega^*\ni(x,y)\to\mathcal{O}} f(x, y)$ exists, we define $Rf(-\alpha)$ to be this limit.

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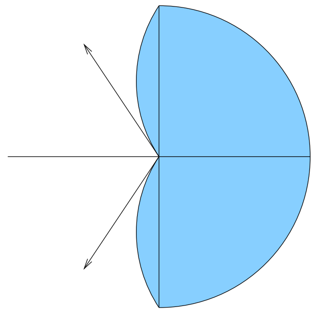

Figure 1. The domain Ω^*.

There are examples in which the radial limits do not exist for any $\theta \in (-\alpha, \alpha)$ [Lancaster 1989; Lancaster and Siegel 1996b]. For solutions of boundary value problems which satisfy appropriate conditions, $Rf(\theta)$ can be proven to exist for $\theta \in [-\alpha, \alpha] \setminus J$, where J is a countable subset of $(-\alpha, \alpha)$; see, e.g., [Entekhabi and Lancaster 2016; 2017; Lancaster 1988; 1991; 2012; Lancaster and Siegel 1996a; 1996b]. We know of no examples in which $J \neq \varnothing$ and we ask if $J = \varnothing$ always holds; this is related to the existence of *cusp solutions*.

A *cusp solution* for (1) is a domain $\Lambda \subset \mathbb{R}^2$ and a solution f of (1) in Λ such that $\partial\Lambda \setminus \{\mathcal{O}, A, B\} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $A, B, \mathcal{O}$ are distinct points on $\partial\Lambda$, and Γ_1, Γ_2 and Γ_3 are disjoint, smooth (open) arcs with endpoints $\{A, \mathcal{O}\}$, $\{B, \mathcal{O}\}$ and $\{A, B\}$, respectively; where Γ_1 and Γ_2 are tangent at $\mathcal{O}$ (so $\overline{\Lambda}$ has an "outward" cusp at $\mathcal{O}$, such as in Figure 2, which has a cusp at $(0, 0)$); and where $f(x, y) = c_j$ when $(x, y) \in \Gamma_j$ $(j = 1, 2)$, $c_1 < c_2$, and, for each $c \in (c_1, c_2)$, the level curves $\{(x, y) \in \Lambda : f(x, y) = c\}$ are tangent at $\mathcal{O}$; see, e.g., [Lancaster and Siegel 1996b, Section 5]. (Capillary surfaces in cusp regions were studied in [Aoki and Siegel 2012; Scholz 2004].) In cases where cusp solutions do not exist, we know $J = \varnothing$.

In [Lancaster and Siegel 1996a; 1996b], the nonexistence of cusp solutions is proven when (a) $H \in C^{1,\delta}(\overline{\Omega} \times \mathbb{R})$, $\delta \in (0, 1)$, and $H(x, y, z)$ is strictly increasing in z for each $(x, y) \in \overline{\Omega}$ or (b) H is real-analytic. The proof in [Lancaster and Siegel 1996b] for case (a) involves a "local" argument while that for (b) involves a "global" argument which shows (2) is violated. Using a "local" argument, we shall prove:

Theorem 1. *Suppose Ω is a domain in $\mathbb{R}^2$ with locally Lipschitz boundary*, $\mathcal{O} = (0, 0) \in \partial\Omega$ *and* $H \in C^{1,\beta}(\overline{\Omega}^* \times \mathbb{R})$ *for some* $\beta \in (0, 1)$. *Let* $f \in C^2(\Omega^*)$ *satisfy* (1) *and* (2). *Suppose* $H(x, y, z)$ *is weakly increasing in z for* (x, y) *in a neighborhood of* $(0, 0)$. *Then* f *cannot have a cusp solution* (*i.e., there is no "cusp region"* $\Lambda \subset \Omega$ *such that* (Λ, f) *is a cusp solution*).

We can exclude cusp solutions when H vanishes in the "cusp direction," which we may assume is the direction of the positive x-axis (see Figure 2).

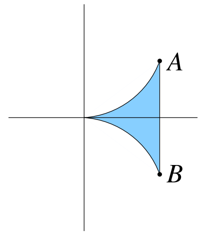

Figure 2. The cusp domain Λ.

Theorem 2. *Suppose Λ is a cusp domain in $\mathbb{R}^2$, $\partial\Lambda$ is tangent to $\vec{i}$ at $\mathcal{O}$, $H \in C^{1,\beta}(\overline{\Lambda} \times \mathbb{R})$ for some $\beta \in (0, 1)$, $f \in C^2(\Lambda)$ satisfies* (1) *and* (2) *and there exists a $\delta > 0$ such that*

$$H(x, 0, z) = 0 \quad \text{for} \quad (x, z) \in [0, \delta] \times [\liminf_{\Lambda \ni (x,y) \to \mathcal{O}} f(x, y), \limsup_{\Lambda \ni (x,y) \to \mathcal{O}} f(x, y)].$$

Then (Λ, f) cannot be a cusp solution.

What can we say when $H(x, y, z)$ is strictly decreasing in z? Unfortunately, as the following example illustrates, we cannot exclude cusp solutions in this case, even when H is real-analytic; a "global" argument (like in [Lancaster and Siegel 1996b, page 176]) is required to exclude cusp solutions when H is real-analytic. Thus, for example, the reasoning in [Aoki and Siegel 2012, 3B] cannot be used when $\kappa < 0$.

Example 3. Consider the cone $\mathcal{C} = \{X(\theta, t) : 0 \le \theta \le \frac{\pi}{2}, 0 < t < \infty\}$, where

$$X(\theta, t) = t(\cos\theta, \sin\theta - 1, 1).$$

Set $\Lambda = \{t(\cos\theta, \sin\theta - 1) : 0 < \theta < \frac{\pi}{2}, 1 < t < 2\}$ and $\mathcal{S} = \mathcal{C} \cap (\mathbb{R}^2 \times [1, 2])$. A straightforward computation shows that the mean curvature (with respect to the upward normal) is

$$H(\theta, t) = \frac{3 - 2\sin\theta}{2t(1 + (1 - \sin\theta)^2)^{3/2}};$$

that is, $H(x, y, z) = (z^2 - 2yz)/(2(y^2 + z^2)^{3/2})$. Now $y/z = \sin\theta - 1 \in [-1, 0]$ and $x = 0$ if and only if $\theta = \pi/2$; another calculation yields

$$2\frac{\partial H}{\partial z}(x, y, z) = -\frac{z^3}{(y^2 + z^2)^{5/2}}\left(1 - 4\left(\frac{y}{z}\right) - 2\left(\frac{y}{z}\right)^2 + 2\left(\frac{y}{z}\right)^3\right) < 0.$$

Finally observe that $\mathcal{S}$ is the graph of a cusp solution and satisfies (2) in Λ.

The hypotheses of [Entekhabi and Lancaster 2016] include the assumption that H satisfies one of the conditions which guarantees that cusp solutions do not exist; the following corollary is a consequence of Theorem 1 and that paper. (A second corollary, similar to Corollary 4, follows by applying Theorem 1 to [Entekhabi and Lancaster 2017, Theorems 1 and 2].)

Corollary 4 [Entekhabi and Lancaster 2016]. *Suppose* Ω, f *and* H *satisfy the hypotheses of Theorem 1 and either*

(i) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ *or*

(ii) $\alpha \in \left(0, \frac{\pi}{2}\right]$ *and one of* $Rf(\alpha)$ *or* $Rf(-\alpha)$ *exists.*

Then $Rf(\theta)$ *exists for each* $\theta \in (-\alpha, \alpha)$ *and* $Rf \in C^0\big((-\alpha, \alpha)\big)$. *If* $Rf(\alpha)$ *exists, then* $Rf \in C^0\big((-\alpha, \alpha]\big)$. *If* $Rf(-\alpha)$ *exists, then* $Rf \in C^0\big([-\alpha, \alpha)\big)$.

2. Proof of Theorem 1

Suppose (Λ, f) is a cusp solution and $\Lambda \subset \{(x, y) \in \mathbb{R}^2 : 0 < x < a,\ |y| < x\}$, $c_1 < c_2$ and the c-level curves of f in Λ are tangent to the positive x-axis at $\mathcal{O}$ for $c_1 \le c \le c_2$, for some $a > 0$ (see Figure 2). Since $H \in C^{1,\beta}(\overline{\Omega} \times \mathbb{R})$, the solution f is an element of $C^3(\Omega)$ and, as in [Lancaster and Siegel 1996a; 1996b], there exist an (open) rectangle $R_0 = (0, a) \times (c_1, c_2)$ and $g \in C^3(R)$, where $R = \bar{R}_0$, such that the graph of f over Λ, $\mathcal{G}$, is the set $\{(x, g(x, z), z) : (x, z) \in R_0\}$ (i.e., $z = f(x, y)$ if and only if $y = g(x, z)$ for $(x, z) \in R_0$ and $(x, y) \in \Lambda$) and $g(0, z) = \partial g(0, z)/\partial x = 0$ for $c_1 \le z \le c_2$. We may assume that $|\nabla g(x, z)| \le 1$ for $(x, z) \in R$.

The (upward) unit normal to the graph of f, $\mathcal{G}$, is

$$\vec{N}(x, y, z) = \frac{(-f_x(x, y), -f_y(x, y), 1)}{\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}}$$

and $\operatorname{div}(Tf)(x, y) = 2\vec{H}(x, y, z) \cdot \vec{N}(x, y, z)$ for $(x, y, z) \in \mathcal{G}$, where $2\vec{H}$ is the mean curvature vector of $\mathcal{G}$. Then

$$\operatorname{sgn}(g_z(x, z))\vec{N}(x, y, z) = \frac{(g_x(x, z), -1, g_z(x, z))}{\sqrt{1 + g_x^2(x, z) + g_z^2(x, z)}}.$$

Since $\operatorname{div}(Tg) = 2\vec{H} \cdot (-g_x, 1, -g_z)/\sqrt{1 + g_x^2 + g_z^2}$, we see that

$$\operatorname{div}(Tg)(x, z) = 2\vec{H}(x, y, z) \cdot \big(-\operatorname{sgn}(g_z(x, z))\big)\vec{N}(x, y, z) \quad \text{for} \quad (x, y, z) \in \mathcal{G}.$$

(Of course, if $g_z(x, z) = 0$ for some $(x, z) \in R$ with $x > 0$, then $\mathcal{G}$ has a horizontal unit normal at an interior point of Ω, which contradicts our hypothesis $f \in C^2(\Omega)$; hence $g_z(x, z) \ne 0$ when $(x, z) \in R$ with $x > 0$.)

Let us assume $\operatorname{sgn}(g_z(x, z)) = \operatorname{sgn}\big(f_y(x, g(x, z))\big) = +1$ for $(x, z) \in R$ with $x > 0$; the opposite choice will lead to the same (eventual) conclusion that cusp solutions do not exist. Then

$$Mg(x, z) = -2H(x, g(x, z), z),$$

where $Mg = \nabla \cdot Tg = \mathrm{div}(Tg)$. Suppose there exist a $\delta_1 > 0$ such that $H(x, y, z)$ is weakly increasing in z for each $(x, y) \in \Lambda$ and $z \in [c_1, c_2]$ when $x^2 + y^2 \le \delta_1^2$. We may assume $a \le \delta_1$.

Fix $\epsilon \in \left(0, \frac{1}{2}(c_2 - c_1)\right)$ and set $\tilde{c}_1 = c_1 + \epsilon$ and $\tilde{c}_2 = c_2 - \epsilon$; notice that $\tilde{c}_2 > \tilde{c}_1$. Set

$$g_j(x, z) := g(x, z + \tilde{c}_j) \quad \text{for} \quad 0 \le x \le a,\ -\epsilon \le z \le \epsilon,\ j = 1, 2, \tag{4}$$

and define $h = g_1 - g_2$.

If $h(x_0, z_0) = 0$ for some $(x_0, z_0) \in (0, a] \times [-\epsilon, \epsilon]$, then the graph of f fails the vertical line test since $(x_0, y_0, z_0 + \tilde{c}_1)$ and $(x_0, y_0, z_0 + \tilde{c}_2)$ are both points on the graph of f, where $y_0 = g_1(x_0, z_0) = g_2(x_0, z_0)$. Thus $h(x, z) \ne 0$ for all $0 < x \le a,\ -\epsilon \le z \le \epsilon$. Since $\mathrm{sgn}(g_z(x, z)) = +1$ when $(x, z) \in (0, a] \times [-\epsilon, \epsilon]$, we see that $h(x, z) < 0$ for all $(x, z) \in (0, a] \times [-\epsilon, \epsilon]$. (This is essentially the argument at the bottom of page 175 in [Lancaster and Siegel 1996b] since $h(0, z) > 0$ is the only option available there.)

Define

$$K(x, y) = 2H(x, y, \tilde{c}_1 + \epsilon), \quad 0 \le x \le a,\ (x, y) \in \Lambda,$$

and $d(x, z) = 2H(x, g(x, z), \tilde{c}_1 + \epsilon) - 2H(x, g(x, z), z)$. Notice that $d(x, z + \tilde{c}_1) \ge 0$ and $d(x, z + \tilde{c}_2) \le 0$ when $(x, z) \in [0, a] \times [-\epsilon, \epsilon]$. Now, for each $j = 1, 2,\ g_j$ is a solution of the Cauchy problem

$$\begin{aligned} Mg_j(x, z) &= -K(x, g_j(x, z)) + d(x, z + \tilde{c}_j) && \text{for } (x, z) \in [0, a] \times [-\epsilon, \epsilon], \\ g_j(0, z) &= \frac{\partial g_j}{\partial x}(0, z) = 0 && \text{for } z \in [-\epsilon, \epsilon]. \end{aligned}$$

Then, as in [Gilbarg and Trudinger 1983, pages 263–264], we have

$$\begin{aligned} 0 &= Mg_1(x, z) - Mg_2(x, z) + 2H(x, g_1(x, z), z + \tilde{c}_1) - 2H(x, g_2(x, z), z + \tilde{c}_2) \\ &= Lh(x, z) - d(x, z + \tilde{c}_1) + d(x, z + \tilde{c}_2), \end{aligned}$$

where, setting $D_1 := \partial/\partial x$ and $D_2 := \partial/\partial z$,

$$Lh = \sum_{i,j=1}^{2} a^{i,j} D_{ij} h + \sum_{i=1}^{2} b^i D_i h + ch; \tag{5}$$

here

$$a^{i,j}(x, z) = e^{i,j}(Dg_1(x, z)) \quad \text{for} \quad i, j = 1, 2, \tag{6}$$

with

$$\begin{aligned} e^{1,1}(p, q) &= (1 + q^2)W^{-3} \quad e^{1,2}(p, q) = e^{2,1}(p, q) = -pqW^{-3}, \\ e^{2,2}(p, q) &= (1 + p^2)W^{-3} \quad W = W(p, q) = \sqrt{1 + p^2 + q^2}, \end{aligned}$$

$$b^1(x,z)=\sum_{i,j=1}^{2} D_{ij}g_2(x,z)\frac{\partial e^{i,j}}{\partial p}(\xi_1,(g_1)_z(x,z)), \tag{7}$$

$$b^2(x,z)=\sum_{i,j=1}^{2} D_{ij}g_2(x,z)\frac{\partial e^{i,j}}{\partial q}((g_2)_x(x,z),\xi_2) \tag{8}$$

and $c(x,z)=\partial K(x,\xi)/\partial y=2\partial H(x,\xi,\tilde{c}_1+\epsilon)/\partial y$, for some ξ between $g_1(x,z)$ and $g_2(x,z)$, ξ_1 between $(g_1)_x(x,z)$ and $(g_2)_x(x,z)$ and ξ_2 between $(g_1)_z(x,z)$ and $(g_2)_z(x,z)$.

Notice that $a^{i,j}\in C^1(R)$ for $i,j\in\{1,2\}$, $b^i\in L^\infty(R)$ for $i\in\{1,2\}$ and $c\in L^\infty(R)$. Now $h(0,z)=\partial h(0,z)/\partial x=0$ for $|z|\le\epsilon$ and

$$Lh(x,z)=d(x,z+\tilde{c}_1)-d(x,z+\tilde{c}_2)\ge 0,\quad (x,z)\in[0,a]\times[-\epsilon,\epsilon]. \tag{9}$$

From (9) and the Hopf boundary point lemma (see, e.g., [Gilbarg and Trudinger 1983, Lemma 3.4]), we have

$$\frac{\partial h}{\partial x}(0,z)<0\quad\text{for each}\quad z\in(-\epsilon,\epsilon)$$

and this contradicts the fact that $h_x(0,z)=0$ if $z\in[-\epsilon,\epsilon]$. Thus we have proven Theorem 1. □

Remark 5. The assumption that H is weakly increasing in z is equivalent to one in the (weak) comparison principle (see, e.g., [Gilbarg and Trudinger 1983, Theorem 10.1] or [Finn 1986, Theorem 5.1]), which plays a critical role here.

3. Proof of Theorem 2

Suppose (Λ, f) is a cusp solution and $\Lambda\subset\{(x,y)\in\mathbb{R}^2 : 0<x<a,\ |y|<x\}$, $c_1<c_2$ and the c-level curves of f in Λ are tangent to the positive x-axis at $\mathcal{O}$ for $c_1\le c\le c_2$, for some $a>0$ (see Figure 2). As before, there exist an (open) rectangle $R_0=(0,a)\times(c_1,c_2)$ and $g\in C^3(R)$ such that the graph of f over Λ, $\mathcal{G}$, is the set $\{(x,g(x,z),z):(x,z)\in R_0\}$ and $g(0,z)=\partial g(0,z)/\partial x=0$ for $c_1\le z\le c_2$. We shall assume that $|\nabla g(x,z)|\le 1$ for $(x,z)\in R$.

Let us assume there exist $\delta\in(0,a]$ and $d_1,d_2\in[c_1,c_2]$ with $d_1<d_2$ such that $H(x,0,z)=0$ for $0\le x\le\delta$, $d_1\le z\le d_2$. Now $g_{xx}(0,z)=0$ for all $z\in[c_1,c_2]$ $\big($since $\triangle g(0,z)=Mg(0,z)=-2H(0,0,z)=0\big)$ and

$$H(x,g(x,z),z)=H(x,0,z)+\frac{\partial H}{\partial y}(x,\xi,z)g(x,z)=\frac{\partial H}{\partial y}(x,\xi,z)g(x,z)$$

for some ξ between 0 and $g(x,z)$. We may extend g as an even function in x by setting $g(x,z)=g(-x,z)$ for $-a\le x<0$, $c_1\le z\le c_2$, so that $g\in C^2(R\cup R^-)$, where $R^- = \{(-x, z) : (x, z) \in R\}$. Then

$$0 = Mg(x, z) + 2H(x, g(x, z), z) = \tilde{L}g(x, z),$$

where

$$a^{1,1}(x, z) = \frac{1 + g_z^2(x, z)}{W^3}, \qquad a^{1,2}(x, z) = -\frac{g_x(x, z)g_z(x, z)}{W^3},$$

$$a^{2,2}(x, z) = \frac{1 + g_x^2(x, z)}{W^3}, \qquad W(x, z) = \sqrt{1 + g_x^2(x, z) + g_z^2(x, z)},$$

$$a^{1,2} = a^{2,1}, \qquad \tilde{c}(x, z) = 2H_y(x, \xi, z)$$

and

$$\tilde{L}u = \sum_{i,j=1}^{2} a^{i,j} D_{ij}u + \tilde{c}u.$$

Since $|\nabla g(x, z)| \le 1$ for $(x, z) \in R$, $\tilde{L}$ is uniformly elliptic in R. Notice that $a^{i,j} \in C^1(R)$ for $i, j = 1, 2$ and $\tilde{c} \in C^0(R)$. Since $g \in C^2(R \cup R^-)$, Theorems 1* and 2* of [Hartman and Wintner 1953] imply that for each $z \in (d_1, d_2)$, there exist a natural number n and real constants e_1 and e_2, not both zero, such that

$$g_x(\rho\cos\theta, z + \rho\sin\theta) = \rho^n(e_1\cos(n\theta) + e_2\sin(n\theta)) + o(\rho^n)$$

and

$$g_z(\rho\cos\theta, z + \rho\sin\theta) = \rho^n(e_2\cos(n\theta) - e_1\sin(n\theta)) + o(\rho^n)$$

as $\rho \to 0$. Since $g_x(0, z) = 0$ and $g_z(0, z) = 0$ for $z \in [c_1, c_2]$, we see that

$$e_1\cos(n\pi/2) + e_2\sin(n\pi/2) = 0, \quad e_2\cos(n\pi/2) - e_1\sin(n\pi/2) = 0$$

and so $e_1 = e_2 = 0$. This contradicts the fact that at least one of e_1 or e_2 is nonzero. Thus we have proven Theorem 2. □

4. Radial limits

When radial limits for (1) exist, they behave in a different manner than do radial limits of, for example, Laplace's equation; see, e.g., [Bear and Hile 1983]. In particular, if f is a solution of (1) and the radial limits $Rf(\theta)$ exist for $\theta \in (-\alpha, \alpha)$, then they behave in one of the following ways:

(i) $Rf : (-\alpha, \alpha) \to \mathbb{R}$ is a constant function (i.e., f has a nontangential limit at $\mathcal{O}$).

(ii) There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on (α_1, α_2).

(iii) There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $(-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha)$ and is either strictly increasing on $(\alpha_1, \alpha_L]$ and strictly decreasing on $[\alpha_R, \alpha_2)$ or strictly decreasing on $(\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_R, \alpha_2)$.

References

[Aoki and Siegel 2012] Y. Aoki and D. Siegel, "Bounded and unbounded capillary surfaces in a cusp domain", *Pacific J. Math.* **257**:1 (2012), 143–165. MR Zbl

[Bear and Hile 1983] H. S. Bear and G. N. Hile, "Behavior of solutions of elliptic differential inequalities near a point of discontinuous boundary data", *Comm. Partial Differential Equations* **8**:11 (1983), 1175–1197. MR Zbl

[Entekhabi and Lancaster 2016] M. Entekhabi and K. Lancaster, "Radial limits of bounded nonparametric prescribed mean curvature surfaces", *Pacific J. Math.* **283**:2 (2016), 341–351. MR Zbl

[Entekhabi and Lancaster 2017] M. N. Entekhabi and K. E. Lancaster, "Radial limits of capillary surfaces at corners", *Pacific J. Math.* **288**:1 (2017), 55–67.

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren der Mathematischen Wissenschaften **284**, Springer, 1986. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **224**, Springer, 1983. MR Zbl

[Hartman and Wintner 1953] P. Hartman and A. Wintner, "On the local behavior of solutions of non-parabolic partial differential equations", *Amer. J. Math.* **75** (1953), 449–476. MR Zbl

[Lancaster 1988] K. E. Lancaster, "Nonparametric minimal surfaces in $\mathbb{R}^3$ whose boundaries have a jump discontinuity", *Internat. J. Math. Math. Sci.* **11**:4 (1988), 651–656. MR Zbl

[Lancaster 1989] K. E. Lancaster, "Existence and nonexistence of radial limits of minimal surfaces", *Proc. Amer. Math. Soc.* **106**:3 (1989), 757–762. MR Zbl

[Lancaster 1991] K. E. Lancaster, "Boundary behavior near re-entrant corners for solutions of certain elliptic equations", *Rend. Circ. Mat. Palermo* (2) **40**:2 (1991), 189–214. MR Zbl

[Lancaster 2012] K. E. Lancaster, "Remarks on the behavior of nonparametric capillary surfaces at corners", *Pacific J. Math.* **258**:2 (2012), 369–392. MR Zbl

[Lancaster and Siegel 1996a] K. E. Lancaster and D. Siegel, "Behavior of a bounded non-parametric H-surface near a re-entrant corner", *Z. Anal. Anwendungen* **15**:4 (1996), 819–850. MR Zbl

[Lancaster and Siegel 1996b] K. E. Lancaster and D. Siegel, "Existence and behavior of the radial limits of a bounded capillary surface at a corner", *Pacific J. Math.* **176**:1 (1996), 165–194. MR Zbl

[Scholz 2004] M. Scholz, "On the asymptotic behaviour of capillary surfaces in cusps", *Z. Angew. Math. Phys.* **55**:2 (2004), 216–234. MR Zbl

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RADIAL LIMITS OF CAPILLARY SURFACES AT CORNERS

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Dedicated to the memory of Amir Entekhabi

Consider a solution $f \in C^2(\Omega)$ of a prescribed mean curvature equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 2H(x, f) \quad \text{in } \Omega \subset \mathbb{R}^2,$$

where Ω is a domain whose boundary has a corner at $\mathcal{O} = (0, 0) \in \partial\Omega$ and the angular measure of this corner is 2α, for some $\alpha \in (0, \pi)$. Suppose $\sup_{x\in\Omega} |f(x)|$ and $\sup_{x\in\Omega} |H(x, f(x))|$ are both finite. If $\alpha > \frac{\pi}{2}$, then the (nontangential) radial limits of f at $\mathcal{O}$, namely

$$Rf(\theta) = \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

were recently proven by the authors to exist, independent of the boundary behavior of f on $\partial\Omega$, and to have a specific type of behavior.

Suppose $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, the contact angle $\gamma(\cdot)$ that the graph of f makes with one side of $\partial\Omega$ has a limit (denoted γ_2) at $\mathcal{O}$ and

$$\pi - 2\alpha < \gamma_2 < 2\alpha.$$

We prove that the (nontangential) radial limits of f at $\mathcal{O}$ exist and the radial limits have a specific type of behavior, independent of the boundary behavior of f on the other side of $\partial\Omega$. We also discuss the case $\alpha \in \left(0, \frac{\pi}{2}\right]$ and the displayed inequalities do not hold.

1. Introduction and statement of main theorems

Let Ω be a domain in $\mathbb{R}^2$ whose boundary has a corner at $\mathcal{O} \in \partial\Omega$. Suppose $H : \Omega \times \mathbb{R} \to \mathbb{R}$ and H satisfies one of the conditions which guarantees that "cusp solutions" (e.g., §5 of [Lancaster and Siegel 1996b]) do not exist; for example, $H(\boldsymbol{x}, t)$ is weakly increasing in t for each $\boldsymbol{x}$ [Echart and Lancaster 2017] or is real-analytic [Lancaster and Siegel 1996a]. We will assume $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball in $\mathbb{R}^2$ of radius δ^* about $\mathcal{O}$. Polar coordinates relative to $\mathcal{O}$ will be denoted by r and θ. We assume that $\partial\Omega$ is

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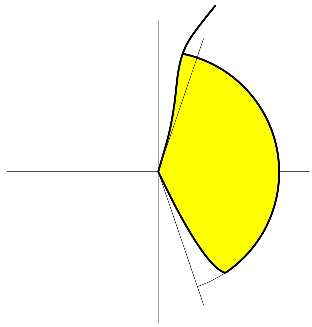

Figure 1. The domain Ω^*.

piecewise smooth and there exists $\alpha \in (0, \pi)$ such that $\partial\Omega \setminus \{\mathcal{O}\} \cap B_{\delta^*}(\mathcal{O})$ consists of two (open) C^1 arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached.

Suppose $\alpha > \frac{\pi}{2}$ and $f \in C^2(\Omega)$ satisfies the prescribed mean curvature equation

$$Nf(x) = 2H(x, f(x)), \quad \text{for } x \in \Omega, \tag{1}$$

where $Nf = \nabla \cdot Tf = \operatorname{div}(Tf)$, $Tf = \nabla f / \sqrt{1 + |\nabla f|^2}$, and

$$\sup_{x\in\Omega} |f(x)| < \infty \quad \text{and} \quad \sup_{x\in\Omega} |H(x, f(x))| < \infty. \tag{2}$$

In [Entekhabi and Lancaster 2016], the authors proved that the radial limits,

$$Rf(\theta) \stackrel{\text{def}}{=} \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

exist for all $\theta \in (-\alpha, \alpha)$, that $Rf(\,\cdot\,)$ is a continuous function on $(-\alpha, \alpha)$ and that these radial limits have similar behavior to that observed in Theorem 1 of [Lancaster and Siegel 1996b]. As illustrated in [Lancaster 1989] and in Theorem 3 of [Lancaster and Siegel 1996b], radial limits of nonparametric prescribed mean curvature surfaces do not necessarily exist.

Suppose $\alpha \le \frac{\pi}{2}$ (see Figure 1) and $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfies (1) and (2). In [Entekhabi and Lancaster 2016], it is shown that if

$$\lim_{\partial^-\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x}) \quad \text{exists,} \tag{3}$$

then the radial limits of f at $\mathcal{O}$ exist and behave as expected. In this paper, we consider the capillary problem as our model and suppose $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfies (1), (2) and the boundary condition

$$Tf(x) \cdot \nu(x) = \cos\gamma(x) \quad \text{for } x \in \partial^-\Omega^*, \tag{4}$$

where $\nu(x)$ is the exterior unit normal to Ω at $x \in \partial\Omega$ and $\gamma : \partial\Omega \to [0, \pi]$ is the contact angle between the graph of f and $\partial\Omega \times \mathbb{R}$, and

$$\lim_{\partial^-\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} \gamma(\boldsymbol{x}) = \gamma_2. \tag{5}$$

We shall prove

Theorem 1. *Let $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfy* (1) *and* (4) *and suppose* (2) *and* (5) *hold*, $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ *and*

$$\pi - 2\alpha < \gamma_2 < 2\alpha. \tag{6}$$

Then (3) *holds*, $Rf(\theta)$ *exists for all* $\theta \in (-\alpha, \alpha)$ *and* $Rf(\,\cdot\,)$ *is a continuous function on* $[-\alpha, \alpha)$, *where* $Rf(-\alpha)$ *equals the limit in* (3). *Further*, $Rf(\,\cdot\,)$ *behaves in one of the following ways*:

(i) $Rf : [-\alpha, \alpha) \to \mathbb{R}$ *is a constant function, hence* f *has a nontangential limit at* $\mathcal{O}$.

(ii) *There exist* α_1 *and* α_2 *so that* $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ *and* Rf *is constant on* $[-\alpha, \alpha_1]$ *and* $[\alpha_2, \alpha)$ *and strictly increasing or strictly decreasing on* $[\alpha_1, \alpha_2)$.

If $\alpha \in \left(0, \frac{\pi}{4}\right]$, then (6) cannot be satisfied. If $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ but $\gamma_2 \ge 2\alpha$ or $\gamma_2 \le \pi - 2\alpha$, then (6) is not satisfied. In both cases, Theorem 1 is not applicable. In these cases, we can prove the existence of $Rf(\,\cdot\,)$ if we add an assumption about the behavior of γ on $\partial^+\Omega^*$.

Theorem 2. *Let $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^* \cup \partial^+\Omega^*)$ satisfy* (1) *and* (4). *Suppose* (2) *and* (5) *hold*, $\alpha \in \left(0, \frac{\pi}{2}\right]$, *there exist* $\lambda_1, \lambda_2 \in [0, \pi]$ *with* $0 < \lambda_2 - \lambda_1 < 4\alpha$ *such that* $\lambda_1 \le \gamma(x) \le \lambda_2$ *for* $x \in \partial^+\Omega^*$ *and*

$$\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2. \tag{7}$$

Then the conclusions of Theorem 1 hold.

Remarks. (a) Theorem 2 only offers a new result when $\lambda_1 = 0$ or $\lambda_2 = \pi$; Figure 8 of [Shi 2006] illustrates one example in which $\lambda_1 = 0$ or $\lambda_2 = \pi$ occurs. If $0 < \lambda_1 < \lambda_2 < \pi$, then Theorem 2 is a consequence of [Lancaster and Siegel 1996b, Theorem 1]; in this case, the argument given in that reference (and here) implies that $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$.

(b) In [Concus and Finn 1996; Finn 1996] it was proved that, in a neighborhood $\mathcal{U}$ of $\mathcal{O}$ and assuming $\partial^+\Omega^*$ and $\partial^-\Omega^*$ are straight line segments, a solution of a constant mean curvature equation (i.e., H is constant in (1)) with constant contact angles γ_1 on $\mathcal{U} \cap \partial^+\Omega^*$ and γ_2 on $\mathcal{U} \cap \partial^-\Omega^*$ can exist only if $|\pi - \gamma_1 - \gamma_2| \le 2\alpha$. Using this, when $\gamma_1 = 0$, we would obtain a (local) upper bound for f in Theorem 1 when $\pi - 2\alpha < \gamma_2$ and, when $\gamma_1 = \pi$, a (local) lower bound for f when $\gamma_2 < 2\alpha$; these two inequalities are equivalent to (6).

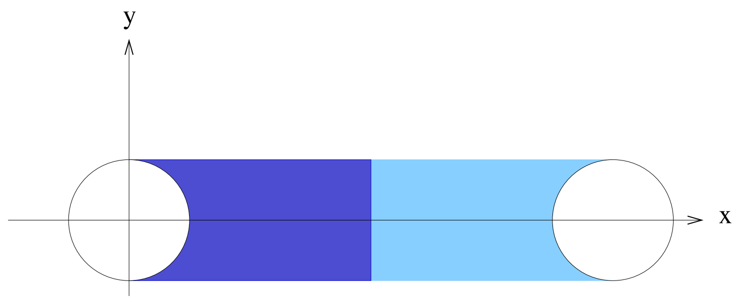

Figure 2. The regions Δ (dark blue) and Δ^R (light blue).

(c) As in [Lancaster and Siegel 1996b], conclusion (3) of Theorems 1 and 2 is a consequence of a general argument; establishing (3) is not a key step in the proof.

2. Preliminary remarks

Let $f \in C^2(\Omega)$ satisfy (1) and suppose (2) holds. Throughout the remainder of the article, let us assume that $M_1 \in (0, \infty)$, $M_2 \in [0, \infty)$,

$$\sup_{x\in\Omega} |f(x)| \le M_1 \quad \text{and} \quad \sup_{x\in\Omega} |H(x, f(x))| \le M_2. \tag{8}$$

2.1. ***A specific torus.*** We will use portions of tori and comparison function arguments as, for instance, in Examples 2 and 3 of [Lancaster and Siegel 1996b] and the Courant–Lebesgue lemma [Courant 1950, Lemma 3.1] to obtain upper and lower bounds on f near $\mathcal{O}$ in specific subsets of Ω and prove Theorems 1 and 2. Let us discuss the construction of a particular torus.

Set

$$r_0 = \begin{cases} 1 & \text{if } M_2 = 0, \\ 1/M_2 + 1 - \sqrt{(1/M_2)^2 + 1} & \text{if } M_2 > 0. \end{cases}$$

Let

$$\Delta = \left\{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : |\boldsymbol{x}| \ge r_0, 0 \le x_1 \le 2, |x_2| \le r_0 \right\},$$

$$\Delta^R = \{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : (4 - x_1, x_2) \in \Delta \}, \text{ and}$$

$$\mathcal{T} = \Big\{ \big(2 + (2 + r_0 \cos v) \cos u, r_0 \sin v, (2 + r_0 \cos v) \sin u\big) : u \in [0, 2\pi], v \in \left[\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right] \Big\}.$$

$\mathcal{T}$ is the inner half of a torus of revolution with axis of symmetry $\{(2, y, 0) : y \in \mathbb{R}\}$, major radius $R_0 = 2$ and minor radius r_0; recall that the mean curvature of $\mathcal{T}$ (with respect to the exterior normal) at $\big(2+(2+r_0 \cos v) \cos u, r_0 \sin v, (2+r_0 \cos v) \sin u\big)$

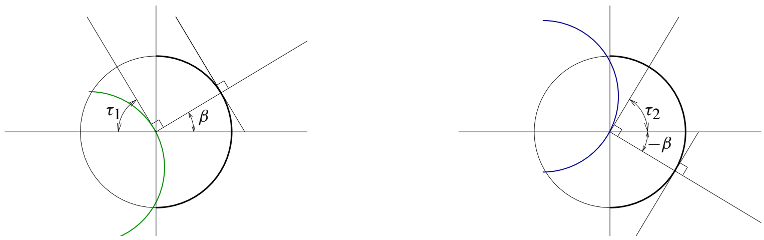

Figure 3. Left: $\beta+\tau_1=\frac{\pi}{2}$. Right: $-\beta+\tau_2=\frac{\pi}{2}$. In both cases, $\beta\geq 0$.

is given by

$$H_T=-\frac{2+2r_0\cos v}{2r_0(2+r_0\cos v)}.$$

A calculation shows that

$$-\Big(\frac{1}{r_0}+\frac{1}{2+r_0}\Big)\leq 2H_T\leq -\Big(\frac{1}{r_0}-\frac{1}{2-r_0}\Big)=-M_2. \tag{9}$$

Set

$$\mathcal{T}^+=\{(\boldsymbol{x},z)\in\mathcal{T}:\boldsymbol{x}\in\Delta,z\geq 0\}\quad\text{and}\quad \mathcal{T}^-=\{(\boldsymbol{x},z)\in\mathcal{T}:\boldsymbol{x}\in\Delta,z\leq 0\}.$$

Let $h^+,h^-:\Delta\to\mathbb{R}$ be functions whose graphs satisfy

$$\{(\boldsymbol{x},h^+(\boldsymbol{x})):\boldsymbol{x}\in\Delta\}=\mathcal{T}^+\quad\text{and}\quad \{(\boldsymbol{x},h^-(\boldsymbol{x})):\boldsymbol{x}\in\Delta\}=\mathcal{T}^-.$$

Then, from (9), we have

$$\operatorname{div}\frac{h^+}{\sqrt{1+|\nabla h^+|^2}}\geq M_2\quad\text{and}\quad \operatorname{div}\frac{h^-}{\sqrt{1+|\nabla h^-|^2}}\leq -M_2. \tag{10}$$

For each $\beta\in\big[-\frac{\pi}{2},\frac{\pi}{2}\big]$ let $\Delta_\beta=\mathcal{R}_\alpha\circ T_\beta(\Delta)$, where $\mathcal{R}_\alpha:\mathbb{R}^2\to\mathbb{R}^2$, given by

$$(x_1,x_2)\mapsto\big(\cos(\alpha)x_1+\sin(\alpha)x_2,-\sin(\alpha)x_1+\cos(\alpha)x_2\big),$$

is the rotation about $(0,0)$ through the angle $-\alpha$ and $T_\beta:\mathbb{R}^2\to\mathbb{R}^2$, given by

$$(x_1,x_2)\mapsto(x_1-r_0\cos\beta,x_2-r_0\sin\beta),$$

is the translation taking $(r_0\cos\beta,r_0\sin\beta)\in\partial\Delta$ to $(0,0)$. We will let τ_1 denote the angle that the upward tangent ray to $T_\beta(C)$ makes with the negative x_1–axis and let τ_2 denote the angle that the upward tangent ray to $T_{-\beta}(C)$ makes with the positive x_1–axis, where $C=\big\{\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}^2:|\boldsymbol{x}|=r_0,x_1\geq 0\big\}$. (Figure 3 illustrates this when $\beta>0$.) Let $h_\beta^\pm:\Delta_\beta\to\mathbb{R}$ be defined by $h_\beta^\pm=h^\pm\circ T_\beta^{-1}\circ\mathcal{R}_\alpha^{-1}$, see Figure 4.

Let q denote the be modulus of continuity of h^-, so that $|h^-_\beta(\boldsymbol{x}_1) - h^-_\beta(\boldsymbol{x}_2)| \le q(|\boldsymbol{x}_1 - \boldsymbol{x}_2|$. Notice that q is also the modulus of continuity of h^+, as well as for h^-_β and h^+_β for each $\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2.2. *Parametric framework.* Since $f \in C^0(\Omega)$, we may assume that f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| > \delta\}$ for each $\delta \in (0, \delta^*)$; if this is not true, we may replace Ω with a subset $U \subset \Omega$, such that $\partial\Omega \cap \partial U = \{\mathcal{O}\}$ and $\partial U \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+ U$ and $\partial^- U$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached. Set

$$S_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \Omega^*\} \quad \text{and} \quad \Gamma_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \partial\Omega^* \setminus \{\mathcal{O}\}\};$$

the points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled $A \in \partial^-\Omega^*$ and $B \in \partial^+\Omega^*$. From the calculation on page 170 of [Lancaster and Siegel 1996b], we see that the area of S_0^* is finite; let M_0 denote this area. For $\delta \in (0, 1)$, set

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(1/\delta)}}.$$

Let $E = \{(u, v) : u^2 + v^2 < 1\}$. As in [Elcrat and Lancaster 1986; Lancaster and Siegel 1996b], there is a parametric description of the surface S_0^*,

$$Y(u, v) = (a(u, v), b(u, v), c(u, v)) \in C^2(E : \mathbb{R}^3), \tag{11}$$

which has the following properties:

(a_1) Y is a diffeomorphism of E onto S_0^*.

(a_2) Set $G(u, v) = (a(u, v), b(u, v))$, $(u, v) \in E$. Then $G \in C^0(\bar{E} : \mathbb{R}^2)$.

(a_3) Let $\sigma = G^{-1}(\partial\Omega^* \setminus \{\mathcal{O}\})$; then σ is a connected arc of ∂E and Y maps σ strictly monotonically onto Γ_0^*. We may assume the endpoints of σ are $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ and there exist points $\boldsymbol{a}, \boldsymbol{b} \in \sigma$ such that $G(\boldsymbol{a}) = A$, $G(\boldsymbol{b}) = B$, G maps the (open) arc $\boldsymbol{o}_1\boldsymbol{b}$ onto $\partial^+\Omega$, and G maps the (open) arc $\boldsymbol{o}_2\boldsymbol{a}$ onto $\partial^-\Omega$. (Note that $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ are not assumed to be distinct at this point; Figures 4a and 4b of [Lancaster and Siegel 1997] illustrate this situation.)

(a_4) Y is conformal on E: $Y_u \cdot Y_v = 0$, $Y_u \cdot Y_u = Y_v \cdot Y_v$ on E.

(a_5) $\triangle Y := Y_{uu} + Y_{vv} = H(Y) Y_u \times Y_v$ on E.

Here by the (open) arcs $\boldsymbol{o}_1\boldsymbol{b}$ and $\boldsymbol{o}_2\boldsymbol{a}$ are meant the component of $\partial E \setminus \{\boldsymbol{o}_1, \boldsymbol{b}\}$ which does not contain $\boldsymbol{a}$ and the component of $\partial E \setminus \{\boldsymbol{o}_2, \boldsymbol{a}\}$ which does not contain $\boldsymbol{b}$, respectively. Let $\sigma_0 = \partial E \setminus \sigma$.

There are two cases we will need to consider during the proofs of Theorem 1 and Theorem 2:

$$\text{(A) } \boldsymbol{o}_1 = \boldsymbol{o}_2 \quad \text{or} \quad \text{(B) } \boldsymbol{o}_1 \ne \boldsymbol{o}_2.$$

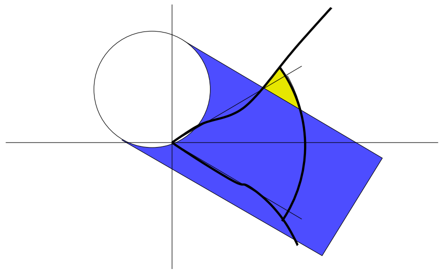

Figure 4. The domain (in blue) of a toroidal function $h_\beta^\pm$, $\alpha < \frac{\pi}{4}$.

These correspond to Cases 5 and 3 respectively in Step 1 of the proof of Theorem 1 of [Lancaster and Siegel 1996b].

3. Proof of Theorem 1

Since $\pi - 2\alpha < \gamma_2 < 2\alpha$, we can choose $\tau_1 \in (\pi - 2\alpha, \gamma_2)$ and $\tau_2 \in (\gamma_2, 2\alpha)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \frac{\pi}{2} - (\pi - \tau_2) = \tau_2 - \frac{\pi}{2}$. With these choices of β_1 and β_2, notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos\tau_1 > \cos\gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0,$$
$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos\tau_2 < \cos\gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0.$$

This implies that, for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough,

$$T(h_{\beta_1}^-)(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos\gamma(\boldsymbol{x}) \quad \text{and} \quad T(h_{\beta_2}^+)(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos\gamma(\boldsymbol{x}), \tag{12}$$

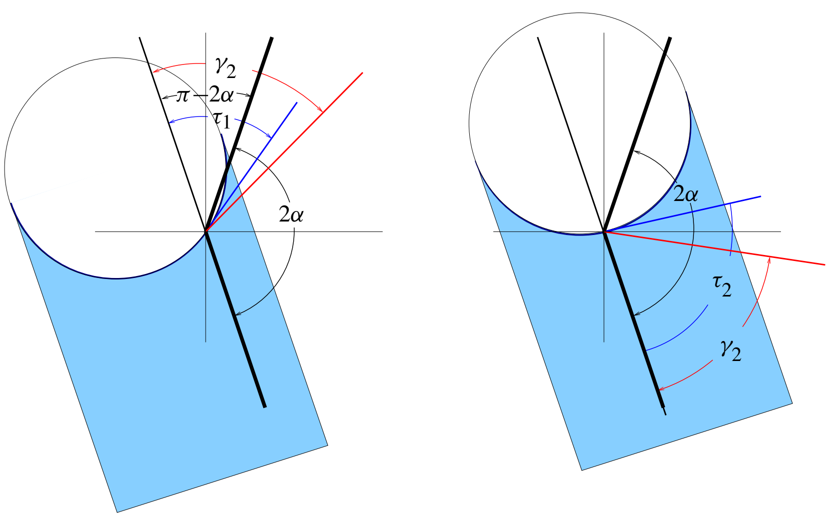

Figure 5. Left: Δ_{β_1}, the domain of $h_{\beta_1}^-$. Right: Δ_{β_2}, the domain of $h_{\beta_2}^+$.

for $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, where $\vec{\nu}(\boldsymbol{x})$ is the exterior unit normal to Ω at $\boldsymbol{x} \in \partial\Omega$. (See Figure 5.) (We may also assume $\nu(\boldsymbol{x}) \cdot (1, 1) < 0$, for $\boldsymbol{x} \in \partial^+\Omega$ with $|\boldsymbol{x}| < \delta_1$ and $\nu(\boldsymbol{x}) \cdot (1, -1) < 0$, for $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, since $\alpha > \frac{\pi}{4}$.)

Let $\mu \in (0, \min\{\gamma_2 - (\pi - 2\alpha), 2\alpha - \gamma_2\})$ and set $\tau_1(\mu) = \pi - 2\alpha + \mu$ and $\tau_2(\mu) = 2\alpha - \mu$, so that $\beta_1 = \beta_2$. Let us write $\delta_1(\mu)$ for $\delta_1(\beta_1, \beta_2)$, h^+_μ for $h^+_{\beta_2}$, h^-_μ for $h^-_{\beta_1}$ and Δ_μ for $\Delta_{\beta_1} = \Delta_{\beta_2}$. Since $\beta_1, \beta_2 \neq \pm\frac{\pi}{2}$, there exists a positive $R = R(\mu)$ such that $B(\mathcal{O}, R(\mu)) \cap \Omega^* \subset \Delta_\mu$ (where $B(\mathcal{O}, R) = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < R\}$).

Let us first assume that (A) holds and set $\boldsymbol{o} = \boldsymbol{o}_1 = \boldsymbol{o}_2$.

Claim. *f is uniformly continuous on Ω_0, where $\Omega_0 = \Omega^* \cap \Delta_\mu$.*

Proof. For $r > 0$, set $B_r = \{\boldsymbol{u} \in \bar{E} : |\boldsymbol{u} - \boldsymbol{o}| < r\}$, $C_r = \{\boldsymbol{u} \in \bar{E} : |\boldsymbol{u} - \boldsymbol{o}| = r\}$ and let l_r be the length of the image curve $Y(C_r)$; also let $C'_r = G(C_r)$ and $B'_r = G(B_r)$. From the Courant–Lebesgue lemma (e.g., Lemma 3.1 in [Courant 1950]), we see that for each $\delta \in (0, 1)$, there exists a $\rho = \rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength l_ρ of $Y(C_\rho)$ is less than $p(\delta)$. For $\delta > 0$, let $k(\delta) = \inf_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \inf_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$ and $m(\delta) = \sup_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \sup_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$; notice that $m(\delta) - k(\delta) \le l_\rho < p(\delta)$.

For each $\delta \in (0, 1)$ with $\sqrt{\delta} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$, there are two points in $C_{\rho(\delta)} \cap \partial E$; we denote these points as $\boldsymbol{e}_1(\delta) \in \boldsymbol{ob}$ and $\boldsymbol{e}_2(\delta) \in \boldsymbol{oa}$ and set $\boldsymbol{y}_1(\delta) = G(\boldsymbol{e}_1(\delta))$ and $\boldsymbol{y}_2(\delta) = G(\boldsymbol{e}_2(\delta))$. Notice that $C'_{\rho(\delta)}$ is a curve in $\bar{\Omega}$ which joins $\boldsymbol{y}_1 \in \partial^+\Omega^*$ and $\boldsymbol{y}_2 \in \partial^-\Omega^*$ and $\partial\Omega \cap C'_{\rho(\delta)} \setminus \{\boldsymbol{y}_1, \boldsymbol{y}_2\} = \varnothing$; therefore there exists $\eta = \eta(\delta) > 0$ such that $B_{\eta(\delta)}(\mathcal{O}) = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x}| < \eta(\delta)\} \subset B'_{\rho(\delta)}$ (see Figure 6).

Let $\epsilon > 0$. Choose $\delta > 0$ such that $\sqrt{\delta} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$, $p(\delta) < \delta_1(\mu)$, $p(\delta) < R(\mu)$, and $p(\delta) + q(p(\delta)) < \frac{1}{2}\epsilon$. Pick a point $\boldsymbol{w} \in C'_{\rho(\delta)}$ and define $b^\pm_j : \Delta_\mu \to \mathbb{R}$ by

$$b^\pm(\boldsymbol{x}) = f(\boldsymbol{w}) \pm p(\delta) \pm h^\mp_\mu(\boldsymbol{x}), \quad \boldsymbol{x} \in \Delta_\mu.$$

Recalling that $Tb^+ \cdot \eta_1 = 1$ on $C_1 = R_\alpha \circ T_{\beta_1}(C)$ and $Tb^- \cdot \eta_2 = -1$ on $C_2 = R_\alpha \circ T_{\beta_2}(C)$, where $\eta_j(\boldsymbol{x})$ is the interior unit normal to C_j at $\boldsymbol{x} \in C_j$ (and $C =$

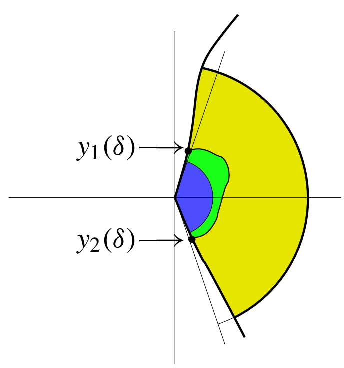

Figure 6. $B_{\eta(\delta)}(\mathcal{O})$ (blue region) and $B'_{\rho(\delta)}$ (blue and green regions).

$\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| = r_0, x_1 \geq 0\}$), it follows from (10), (12) and the general comparison principle (e.g., [Finn 1986, Theorem 5.1]) that

$$b^-(\boldsymbol{x}) < f(\boldsymbol{x}) < b^+(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in B'_{\rho(\delta)} \cap \Delta_\mu.$$

Thus if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ satisfy $|\boldsymbol{x}_1| < \eta(\delta)$, $|\boldsymbol{x}_2| < \eta(\delta)$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \eta(\delta)$, then

$$|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < 2p(\delta) + 2q(p(\delta)) < \epsilon. \tag{13}$$

Since f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| \geq \frac{1}{2}\eta(\delta)\}$, there exists a $\lambda > 0$ such that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega^*$ satisfy $|\boldsymbol{x}_1| \geq \frac{1}{2}\eta(\delta)$, $\boldsymbol{x}_2| \geq \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \lambda$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Now set $d = d(\epsilon) = \min\{\lambda, \frac{1}{2}\eta(\delta)\}$. If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \leq \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1| < \frac{1}{2}\eta(\delta)$, then $|\boldsymbol{x}_1| < \eta(\delta)$ and $|\boldsymbol{x}_2| < \eta(\delta)$; hence $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$ by (13). Next, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \leq \lambda$, $|\boldsymbol{x}_1| \geq \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_2| \geq \frac{1}{2}\eta(\delta)$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Therefore, for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ with $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon)$, we have $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. The claim is proven. □

Notice that if $\theta(\mu) = \alpha - \mu$ $(= \tau_2(\mu) - \alpha = \pi - \alpha - \tau_1(\mu))$, then

$$\{(r\cos\theta(\mu), r\sin\theta(\mu)) : r \geq 0\}$$

is the tangent ray to $\partial\Omega_0$ at $\mathcal{O}$ and it follows from the claim that $f \in C^0(\overline{\Omega_0})$; hence the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ exist for $\theta \in [-\alpha, \theta(\mu)]$ and the radial limits are identical (i.e., $Rf(\theta) = f(\mathcal{O})$ for all $\theta \in [-\alpha, \theta(\mu)]$, where $f(\mathcal{O})$ is the value at $\mathcal{O}$ of the restriction of f to $\overline{\Omega_0}$). Since $\lim_{\mu \downarrow 0} \theta(\mu) = \alpha$, Theorem 1 is proven in this case.

Let us next assume that (B) holds. This part of the proof is essentially the same as the proof of case (B) in Theorem 1 of [Entekhabi and Lancaster 2016]. As in that paper, and taking into account the hypothesis $\alpha \leq \frac{\pi}{2}$, we see that

(i) $c \in C^0(\bar{E} \setminus \{\boldsymbol{o}_1, \boldsymbol{o}_2\})$,

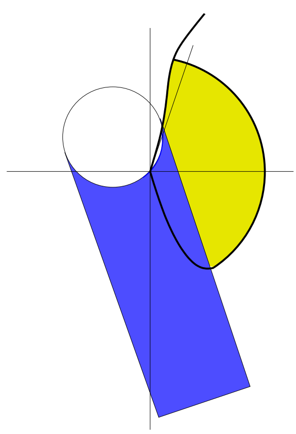

Figure 7. The domain (in blue) of the toroidal functions $h^\pm_\mu$, $\alpha > \frac{\pi}{4}$.

(ii) there exist $\alpha_1, \alpha_2 \in [-\alpha, \alpha]$ with $\alpha_1 < \alpha_2$ such that $Rf(\theta)$ exists when $\theta \in (\alpha_1, \alpha_2)$, and

(iii) Rf is strictly increasing or strictly decreasing on (α_1, α_2).

Taking hypothesis (5) into account and using cylinders as in case 3 of step 1 in the proof of Theorem 1 of [Lancaster and Siegel 1996b] (see Figure 2b in [Lancaster and Siegel 1997]) or using $h^{\pm}_{\mu}$ (see Figure 7), we see that in addition to (i)–(iii), we have

(iv) $c \in C^0(\bar{E} \setminus \{\boldsymbol{o}_1\})$ and

(v) $Rf(\theta)$ exists when $\theta \in [-\alpha, \alpha_2)$.

If $\alpha_2 = \alpha$, then Theorem 1 is proven. Otherwise, suppose $\alpha_2 < \alpha$ and fix $\delta_0 \in (0, \delta^*)$ and $\Omega_0 = \Omega^* \cap \Delta_\mu$ as before.

Claim. *Suppose $\alpha_2 < \alpha$. Then f is uniformly continuous on Ω_0^+, where*

$$\Omega_0^+ \stackrel{\text{def}}{=} \{(r\cos\theta, r\sin\theta) \in \Omega_0 : 0 < r < \delta^*, \alpha_2 < \theta < \pi\}.$$

Notice that the restriction of Y to $G^{-1}(\overline{\Omega_0^+})$ maps only one point, $\boldsymbol{o}_1$, to $\mathcal{O} \times \mathbb{R}$ and so the proof of this claim is the same as the proof of the previous claim. Thus $f \in C^0(\overline{\Omega_0^+})$; since $\lim_{\mu \downarrow 0} \theta(\mu) = \alpha$, we see that

$$Rf(\theta) = \lim_{\tau \uparrow \alpha_2} Rf(\tau) \quad \text{for all } \theta \in [\alpha_2, \alpha).$$

Thus Theorem 1 is proven. □

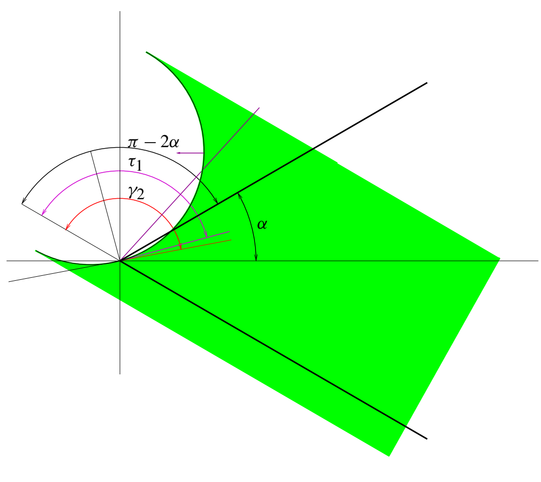

Figure 8. $\alpha = \frac{\pi}{6}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\pi}{2}$, $\gamma_2 = \frac{7\pi}{9}$, and $\tau_1 = \frac{27\pi}{36}$. The domain of $h^-_{\beta_1}$ is the green region.

4. Proof of Theorem 2

Suppose (6) does not hold. Since $\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2$, we can choose $\tau_1, \tau_2 \in (0, \pi)$ such that $\tau_1 \in (\pi - 2\alpha - \lambda_1, \gamma_2)$ and $\tau_2 \in (\gamma_2, \pi + 2\alpha - \lambda_2)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. (See Figures 8 and 9.) With these choices of β_1 and β_2, notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos\tau_1 > \cos\gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0,$$
$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos\tau_2 < \cos\gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0.$$

This implies that for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough,

$$(14) \qquad T(h^-_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos\gamma(\boldsymbol{x}) \quad \text{and} \quad T(h^+_{\beta_2})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos\gamma(\boldsymbol{x}),$$

for $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, where $\vec{\nu}(\boldsymbol{x})$ is the exterior unit normal to Ω at $\boldsymbol{x} \in \partial\Omega$. (See Figures 5, 8 and 9.)

Notice that the tangent plane at $(0, 0, 0)$ to the surface $\{(\boldsymbol{x}, h^-_{\beta_1}(\boldsymbol{x})) : \boldsymbol{x} \in \Delta_{\beta_1}\}$ is a vertical plane with (downward oriented) unit normal

$$\vec{n} = (-\sin(\tau_1 + \alpha), -\cos(\tau_1 + \alpha), 0)$$

and

$$\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{\nu}(\boldsymbol{x}) = (-\sin\alpha, \cos\alpha, 0).$$

Suppose $\tau_1 + 2\alpha \le \pi$. Then

$$\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{n} \cdot \vec{\nu}(\boldsymbol{x}) = -\cos(\tau_1 + 2\alpha) > -\cos(\pi - \lambda_1) = \cos\lambda_1,$$

since $\tau_1 + 2\alpha > \pi - \lambda_1$; since $\liminf_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \gamma(\boldsymbol{x}) \ge \lambda_1$, this implies that for some $\delta_2 > 0$ small enough,

$$(15) \qquad T(h^-_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos\gamma(\boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in \partial^+\Omega \text{ with } |\boldsymbol{x}| < \delta_2.$$

If $\tau_1 + 2\alpha > \pi$, then λ_1 doesn't matter and we argue as in the proof of Theorem 1; see Figure 8 for an illustration of this case.

Now the tangent plane at $(0, 0, 0)$ to the surface $\{(\boldsymbol{x}, h^+_{\beta_2}(\boldsymbol{x})) : \boldsymbol{x} \in \Delta_{\beta_2}\}$ is a vertical plane with (downward oriented) unit normal $\vec{m} = (\sin(\tau_2 - \alpha), -\cos(\tau_2 - \alpha), 0)$ and $\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{\nu}(\boldsymbol{x}) = (-\sin\alpha, \cos\alpha, 0)$.

Suppose $\tau_2 \ge 2\alpha$. Then

$$\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{m} \cdot \vec{\nu}(\boldsymbol{x}) = -\cos(\tau_2 - 2\alpha) < -\cos(\pi - \lambda_2) = \cos\lambda_2,$$

since $\tau_2 - 2\alpha < \pi - \lambda_2$; since $\limsup_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \gamma(\boldsymbol{x}) \le \lambda_2$, this implies that for some $\delta_3 > 0$ small enough,

$$(16) \qquad T(h^+_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos\gamma(\boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in \partial^+\Omega \text{ with } |\boldsymbol{x}| < \delta_3.$$

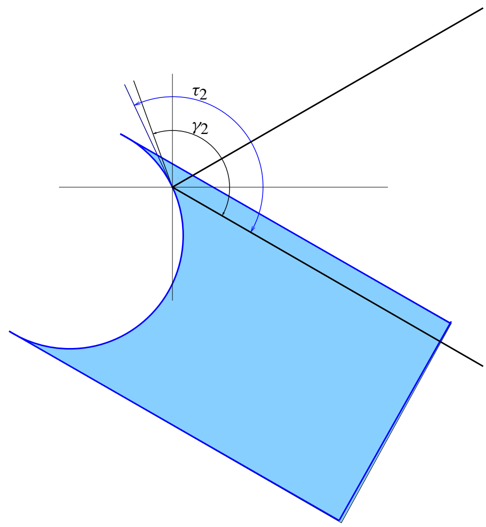

Figure 9. $\alpha = \frac{\pi}{6}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\pi}{2}$, $\gamma_2 = \frac{7\pi}{9}$, and $\tau_2 = \frac{29\pi}{36}$. The domain of $h^+_{\beta_2}$ is the blue region.

If $\tau_2 < 2\alpha$, then λ_2 doesn't matter and we argue as in the proof of Theorem 1.

Now set $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. The proof of Theorem 2 now follows essentially as in the proof of Theorem 1. □

References

[Concus and Finn 1996] P. Concus and R. Finn, "Capillary wedges revisited", *SIAM J. Math. Anal.* **27**:1 (1996), 56–69. MR Zbl

[Courant 1950] R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, New York, 1950. Reprinted by Springer, New York, 1977. MR Zbl

[Echart and Lancaster 2017] A. K. Echart and K. E. Lancaster, "On cusp solutions to a prescribed mean curvature equation", *Pacific J. Math.* **288** (2017), 47–54.

[Elcrat and Lancaster 1986] A. R. Elcrat and K. E. Lancaster, "Boundary behavior of a nonparametric surface of prescribed mean curvature near a reentrant corner", *Trans. Amer. Math. Soc.* **297**:2 (1986), 645–650. MR Zbl

[Entekhabi and Lancaster 2016] M. Entekhabi and K. Lancaster, "Radial limits of bounded nonparametric prescribed mean curvature surfaces", *Pacific J. Math.* **283**:2 (2016), 341–351. MR Zbl

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren der Mathematischen Wissenschaften **284**, Springer, New York, 1986. MR Zbl

[Finn 1996] R. Finn, "Local and global existence criteria for capillary surfaces in wedges", *Calc. Var. Partial Differential Equations* **4**:4 (1996), 305–322. MR Zbl

[Lancaster 1989] K. E. Lancaster, "Existence and nonexistence of radial limits of minimal surfaces", *Proc. Amer. Math. Soc.* **106**:3 (1989), 757–762. MR Zbl

[Lancaster and Siegel 1996a] K. E. Lancaster and D. Siegel, "Behavior of a bounded non-parametric H-surface near a reentrant corner", *Z. Anal. Anwendungen* **15**:4 (1996), 819–850. MR Zbl

[Lancaster and Siegel 1996b] K. E. Lancaster and D. Siegel, "Existence and behavior of the radial limits of a bounded capillary surface at a corner", *Pacific J. Math.* **176**:1 (1996), 165–194. MR Zbl

[Lancaster and Siegel 1997] K. E. Lancaster and D. Siegel, "Correction to: 'Existence and behavior of the radial limits of a bounded capillary surface at a corner' ", *Pacific J. Math.* **179**:2 (1997), 397–402. MR Zbl

[Shi 2006] D. Shi, "Capillary surfaces at a reentrant corner", *Pacific J. Math.* **224**:2 (2006), 321–353. MR Zbl

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A NEW BICOMMUTANT THEOREM

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We prove an analogue of Voiculescu's theorem: the relative bicommutant of a separable unital subalgebra A of an ultraproduct of simple unital C*-algebras is equal to A.

Ultrapowers[1] $A^{\mathcal{U}}$ of separable C*-algebras are, being subject to well-developed model-theoretic methods, reasonably well-understood; see, e.g., [Farah et al. 2016b, Theorem 1.2] and Section 2. Since the early 1970s and the influential work of McDuff and Connes, central sequence algebras $A' \cap A^{\mathcal{U}}$ play an even more important role than ultrapowers in the classification of II_1 factors and (more recently) C*-algebras. While they do not have a well-studied abstract analogue, in [Farah et al. 2016b, Theorem 1] it was shown that the central sequence algebra of a strongly self-absorbing algebra [Toms and Winter 2007] is isomorphic to its ultrapower if the continuum hypothesis holds. Relative commutants $B' \cap D^{\mathcal{U}}$ of separable subalgebras of ultrapowers of strongly self-absorbing C*-algebras play an increasingly important role in the classification program for separable C*-algebras [Bosa et al. 2016; Matui and Sato 2014, §3]; see also [Tikuisis et al. 2016; Winter 2016]. In the present note we make a step towards better understanding of these algebras.

A C*-algebra is *primitive* if it has a representation that is both faithful and irreducible. We prove an analogue of the well-known consequence of Voiculescu's theorem [1976, Corollary 1.9] and von Neumann's bicommutant theorem [Blackadar 2006, §I.9.1.2].

Theorem 1. *Assume $\prod_{\mathcal{U}} B_j$ is an ultraproduct of primitive* C**-algebras and A is a separable* C**-subalgebra. In addition, assume A is a unital subalgebra if $\prod_{\mathcal{U}} B_j$ is unital. With $\overline{A}^{\mathrm{WOT}}$ computed in the ultraproduct of faithful irreducible representations of B_js, we have*

$$A = \left(A' \cap \textstyle\prod_{\mathcal{U}} B_j\right)' = \overline{A}^{\mathrm{WOT}} \cap \textstyle\prod_{\mathcal{U}} B_j .$$

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[1]Throughout, $\mathcal{U}$ denotes a nonprincipal ultrafilter on $\mathbb{N}$.

A slightly weaker version of the following corollary to Theorem 1 (stated here with Aaron Tikuisis' kind permission) was originally proved using very different methods.

Corollary 2 (Farah and Tikuisis, 2015). *Assume* $\prod_{\mathcal{U}} B_j$ *is an ultraproduct of simple unital* C*-*algebras and* A *is a separable unital subalgebra. With* $Z(A)$ *denoting the center of* A, *we have*

$$Z\big(A' \cap \textstyle\prod_{\mathcal{U}} B_j\big) = Z(A). \qquad \square$$

At least two open problems are concerned with bicommutants of separable subalgebras of massive operator algebras. As is well-known, central sequence algebras $M' \cap M^{\mathcal{U}}$ of II_1 factors in tracial ultrapowers behave differently from the central sequence algebras of C*-algebras. For a II_1 factor M with separable predual, the central sequence algebra $M' \cap M^{\mathcal{U}}$ can be abelian or even trivial. Popa [2014, Conjecture 2.3.1] asked whether if P is a separable subalgebra of an ultraproduct of II_1 factors then $\big(P' \cap \prod_{\mathcal{U}} N_i\big)' = P$ implies P is amenable. In the domain of C*-algebras, G. K. Pedersen [1990, Remark 10.11] asked whether the following variant of Theorem 1 is true: if the corona $M(B)/B$ of a σ-unital C*-algebra B is simple and A is a separable unital subalgebra, is $(A' \cap M(B)/B)' = A$? (For the connection between ultraproducts and coronas, see the last paragraph of Section 3.)

The proof of Theorem 1 uses logic of metric structures [Ben Yaacov et al. 2008; Farah et al. 2014] and an analysis of the interplay between C*-algebra B and its second dual B^{**}.

1. Model theory of representations

We expand the language of C*-algebras introduced in [Farah et al. 2014, §2.3.1] to representations of C*-algebras. Readers' familiarity with, or at least easy access to, §2 of that paper is assumed. A structure in the expanded language $\mathcal{L}_{\mathrm{rep}}$ is a C*-algebra together with its representation on a Hilbert space. As in [Farah et al. 2014], the domains of quantification on a C*-algebra are D_n for $n \in \mathbb{N}$ and are interpreted as the n-balls. The domains of quantification on the Hilbert space are D_n^H for $n \in \mathbb{N}$ and are also interpreted as the n-balls. On all domains the metric is $d(x, y) = \|x - y\|$ (we denote both the operator norm on C*-algebras and the ℓ_2-norm on Hilbert spaces by $\|\cdot\|$). As in [Farah et al. 2014, §2.3.1], for every $\lambda \in \mathbb{C}$ we have a unary function symbol λ to be interpreted as multiplication by λ. We also have a binary function $+$ whose interpretation sends $D_m^H \times D_n^H$ to D_{m+n}^H. As the scalar product $(\cdot \mid \cdot)$ is definable from the norm via the polarization identity, we freely use it in our formulas, with the understanding that $(\xi \mid \eta)$ is an abbreviation for $\frac{1}{4}\sum_{j=0}^{3} i^j \|\xi + i^j \eta\|$. The language $\mathcal{L}_{\mathrm{rep}}$ also contains a binary function symbol π whose interpretation sends $D_n \times D_m^H$ to D_{mn}^H for all m and n. It is interpreted as an action of A on H.

Every variable is associated with a sort. In particular, variables x, y, z range over the C*-algebra and variables ξ, η, ζ range over the Hilbert space, all of them decorated with subscripts when needed.

We shall write $\bar{x}$ for a tuple $\bar{x} = (x_1, \dots, x_n)$ (with n either clear from the context or irrelevant). *Terms* come in two varieties. On the C*-algebra side, a term is a noncommutative *-polynomial in C*-variables. On the Hilbert space side, terms are linear combinations of Hilbert space variables and expressions of the form $\pi(\alpha(\bar{x}))\xi$, where $\alpha(\bar{x})$ is a term in the language of C*-algebras. *Formulas* are defined recursively. Atomic formulas are expressions of the form $\|t\|$ where t is a term.

The set of all formulas is the smallest set $\mathbb{F}$ containing all atomic formulas with the properties that

(i) for every n, all continuous $f : [0, \infty)^n \to [0, \infty)$ and all $\varphi_1, \dots, \varphi_n$ in $\mathbb{F}$, the expression $f(\varphi_1, \dots, \varphi_n)$ belongs to $\mathbb{F}$, and

(ii) if $\varphi \in \mathbb{F}$, and x and ξ are variable symbols, then each of $\sup_{\|\xi\|\leq m} \varphi$, $\inf_{\|\xi\|\leq m} \varphi$, $\sup_{\|x\|\leq m} \varphi$, and $\inf_{\|x\|\leq m} \varphi$ belongs to $\mathbb{F}$; see [Farah et al. 2014, §2.4] or [Farah et al. 2016a, Definition 2.1.1].

Suppose $\pi : A \to B(H)$ is a representation of a C*-algebra A on Hilbert space H. To (A, H, π) we associate the natural metric structure $\mathcal{M}(A, H, \pi)$ in the above language.

Suppose $\varphi(\bar{x}, \bar{\xi})$ is a formula whose free variables are included among $\bar{x}$ and $\bar{\xi}$. If $\pi : A \to B(H)$ is a representation of a C*-algebra on Hilbert space, $\bar{a}$ are elements of A and $\bar{\xi}$ are elements of H,[2] then the *interpretation* $\varphi(\bar{a}, \bar{\xi})^{\mathcal{M}(A,H,\pi)}$ is defined by recursion on the complexity of φ in the obvious way; see [Ben Yaacov et al. 2008, §3].

Proposition 1.1. *Triples (A, H, π) such that π is a representation of A on H form an axiomatizable class.*

Proof. As in [Farah et al. 2014, Definition 3.1], we need to define an $\mathcal{L}_{\mathrm{rep}}$-theory $\boldsymbol{T}_{\mathrm{rep}}$ such that the category of triples (A, H, π), where $\pi : A \to B(H)$ is a representation of a C*-algebra A, is equivalent to the category of metric structures that are models of $\boldsymbol{T}_{\mathrm{rep}}$, via the map

$$(A, H, \pi) \mapsto \mathcal{M}(A, H, \pi).$$

We use the axiomatization of C*-algebras from [Farah et al. 2014, §3.1]. In addition to the standard Hilbert space axioms, we need the following two axioms assuring

[2] Symbols $\xi, \eta, \zeta, \dots$ denote both Hilbert space variables and vectors in Hilbert space due to the font shortage; this shall not lead to a confusion.

that the interpretation of D_n^H equals the n-ball of the underlying Hilbert space for all n:

$$\sup_{\xi\in D_1} \|\xi\| \le n,$$

$$(*) \qquad \sup_{\xi\in D_n} \max\bigl((1 \dot- \|\xi\|), \inf_{\eta\in D_1} \|\xi-\eta\|\bigr),$$

where $s \dot- t := \max(s-t, 0)$. The standard axioms,

$$\pi(xy)\xi = \pi(x)\pi(y)\xi,$$
$$\pi(x+y)\xi = \pi(x)\xi + \pi(y)\xi,$$
$$(\pi(x)\xi \mid \eta) = (\xi \mid \pi(x^*)\eta)$$

are expressible as first-order sentences.[3] The axioms described here comprise theory $\boldsymbol{T}_{\rm rep}$.

One needs to check that the category of models of $\boldsymbol{T}_{\rm rep}$ is equivalent to the category of triples (A, H, π). Every triple (A, H, π) uniquely defines a model $\mathcal{M}(A, H, \pi)$. Conversely, assume $\mathcal{M}$ is a model of $\boldsymbol{T}_{\rm rep}$. The algebra $A_{\mathcal{M}}$ obtained from the first component of $\mathcal{M}$ is a C*-algebra by [Farah et al. 2014, Proposition 3.2]. Also, the linear space $H_{\mathcal{M}}$ obtained from the second component of $\mathcal{M}$ is a Hilbert space and the third component gives a representation $\pi_{\mathcal{M}}$ of A on H.

To see that this provides an equivalence of categories, we need to check that $\mathcal{M}(A_{\mathcal{M}}, H_{\mathcal{M}}, \pi_{\mathcal{M}}) \cong \mathcal{M}$ for every model $\mathcal{M}$ of $\boldsymbol{T}_{\rm rep}$. We need to show that the domains on $\mathcal{M}$ are determined by $A_{\mathcal{M}}$ and $H_{\mathcal{M}}$. The former was proved in the second paragraph of [Farah et al. 2014, Proposition 3.2], and the latter follows by (∗). □

Proposition 1.1 gives us full access to the model-theoretic toolbox, such as Łoś's theorem (see Section 2) and the Löwenheim–Skolem theorem [Farah et al. 2014, Theorem 4.6]. From now on, we shall identify triple (A, H, π) with the associated metric structure $\mathcal{M}(A, H, \pi)$ and stop using the latter notation. We shall also write $\sup_{\|\xi\|\le n}$ and $\inf_{\|\xi\|\le n}$ instead of $\sup_{\xi\in D_n}$ and $\inf_{\xi\in D_n}$, respectively.

Lemma 1.2. *The following properties of a representation π of A are axiomatizable*:

(1) *π is faithful.*

(2) *π is irreducible.*

Proof. We explicitly write the axioms for each of the properties of π. Fix a representation π. It is faithful if and only if it is isometric, which can be expressed as

$$\sup_{\|x\|\le 1} \inf_{\|\xi\|\le 1} \bigl| \|x\| - \|\pi(x)\xi\| \bigr| = 0.$$

---

[3]Our conventions are as described in [Farah et al. 2014, p. 485]. In particular $\alpha(x,\xi) = \beta(x,\xi)$ is an abbreviation for $\sup_{\xi\in D_n} \sup_{\xi} \|\alpha(x,\xi) - \beta(x,\xi)\| = 0$, for all n.

A representation π is irreducible if and only if for all vectors ξ and η in H such that $\|\eta\| \leq 1$ and $\|\xi\| = 1$, the expression $\|\eta - \pi(a)\xi\|$ can be made arbitrarily small when a ranges over the unit ball of A. In symbols,

$$\sup_{\|\xi\|\leq 1} \sup_{\|\eta\|\leq 1} \inf_{\|x\|\leq 1} \big| \|\xi\| \dot{-} 1 \big| \big\| \eta - \pi(x)\xi \big\| = 0.$$

The interpretation of this sentence in (A, H, π) is 0 if and only if the representation π is irreducible. □

A triple (D, θ, K) is an *elementary submodel* of (B, π, H), and (B, π, H) is an *elementary extension* of (D, θ, K), if $D \subseteq B$, $K \subseteq H$, $\theta(d) = \pi(d) \restriction H$ for all $d \in D$, and

$$\varphi(\bar{a})^{(D,\theta,K)} = \varphi(\bar{a})^{(B,\pi,H)}$$

for all formulas φ and all $\bar{a}$ in (D, θ, K) of the appropriate sort. Axiomatizable properties, such as being irreducible or faithful, transfer between elementary submodels and elementary extensions. Therefore the downward Löwenheim–Skolem theorem [Farah et al. 2014, Theorem 4.6] and Lemma 1.2 together imply, e.g., that if φ is a pure state of a nonseparable C*-algebra B then B is an inductive limit of separable subalgebras D such that the restriction of φ to D is pure. This fact was proved in [Akemann and Weaver 2004] and its slightly more precise version will be used in the proof of Lemma 3.2.

Some other properties of representations (such as not being faithful) are axiomatizable, but we shall concentrate on proving Theorem 1.

2. Saturation and representations

It has been known to logicians since the 1960s that the two defining properties of ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ in axiomatizable categories are Łoś's theorem [Farah et al. 2014, Proposition 4.3] and countable saturation [Farah et al. 2014, Proposition 4.11]. By the former, the diagonal embedding of a metric structure M into its ultrapower is elementary. More generally, if $\varphi(\bar{x})$ is a formula and $\bar{a}(j) \in M_j$ are of the appropriate sort then

$$\varphi(\bar{a})^{\prod_{\mathcal{U}} M_j} = \lim_{j\to\mathcal{U}} \varphi(\bar{a}(j))^{M_j}.$$

In order to define countable saturation, we recall the notion of a type from the logic of metric structures [Farah et al. 2014, §4.3]. A *closed condition* (or simply a *condition*; we shall not need any other conditions) is any expression of the form $\varphi \leq r$ for formula φ and $r \geq 0$ and a *type* is a set of conditions [Farah et al. 2014, §4.3]. As every expression of the form $\varphi = r$ is equivalent to the condition $\max(\varphi, r) \leq r$ and every expression of the form $\varphi \geq r$ is equivalent to the condition $\min(0, r - \varphi) \leq 0$, we shall freely refer to such expressions as conditions. For m

and n in $\mathbb{N}$ such that $m+n \geq 1$, an *(m, n)-type* is a type $\boldsymbol{t}$ such that all free variables occurring in conditions of $\boldsymbol{t}$ are among $\{x_1, \ldots, x_m\} \cup \{\xi_1, \ldots, \xi_n\}$.

Given a structure (A, H, π) and a subset X of $A \cup H$, we expand the language $\mathcal{L}_{\text{rep}}$ by adding constants for the elements of X (as in [Farah et al. 2014, §2.4.1]). The new language is denoted $(\mathcal{L}_{\text{rep}})_X$. C*-terms in $(\mathcal{L}_{\text{rep}})_X$ are *-polynomials in C*-variables and constants from $X \cap A$. Hilbert space terms are linear combinations of Hilbert space variables, constants in $X \cap H$, and expressions of the form $\pi(\alpha)\xi$, where α is a C*-term in the expanded language. The interpretation of an $(\mathcal{L}_{\text{rep}})_X$-formula is defined recursively in the natural way; see, e.g., the paragraph after Definition 2.1.1 in [Farah et al. 2016a].

A type *over X* is a type in $(\mathcal{L}_{\text{rep}})_X$. Such a type is *realized* in some elementary extension of (A, H, π) if the latter contains a tuple satisfying all conditions from the type. A type is *consistent* if it is realized in some ultrapower of (A, H, π), where the ultrafilter is taken over an arbitrary, not necessarily countable, set. This is equivalent to the type being realized in some elementary extension of (A, H, π).

By Łoś's theorem, a type $\boldsymbol{t}$ is consistent if and only every finite subset of $\boldsymbol{t}$ is ε-realized in (A, H, π) for every $\varepsilon > 0$ [Farah et al. 2014, Proposition 4.8].

A structure (A, H, π) is said to be *countably saturated* if every consistent type over a countable (or equivalently, norm-separable) set is realized in (A, H, π). Ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ are always countably saturated [Farah et al. 2014, Proposition 4.11]. A standard transfinite back-and-forth argument shows that a structure of density character $\aleph_1$ is countably saturated if and only if it is an ultraproduct. (The *density character* is the smallest cardinality of a dense subset.)

In the case when $A = B(H)$, we have

$$(B(H), H)^{\mathcal{U}} = (B(H)^{\mathcal{U}}, H^{\mathcal{U}});$$

in particular $B(H)^{\mathcal{U}}$ is identified with a subalgebra of $B(H^{\mathcal{U}})$. These two algebras are equal (still assuming $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$) if and only if H is finite-dimensional. As a matter of fact, no projection $p \in B(H^{\mathcal{U}})$ with a separable, infinite-dimensional range belongs to $B(H)^{\mathcal{U}}$ (this is proved by a standard argument, see, e.g., the last two paragraphs of the proof of Proposition 4.6 in [Farah et al. 2013]).

In the following, π will always be faithful and clear from the context and we shall identify A with $\pi(A)$ and suppress writing π. We shall therefore write (A, H) in place of (A, H, id).

The following two lemmas are standard (they were used in the proof of Corollary 2 on p. 344 of [Arveson 1977]) but we sketch the proofs for the reader's convenience.

Lemma 2.1. *Suppose A is a* C*-*algebra and φ is a functional on A. Then there are a representation $\pi : A \to B(K)$ and vectors ξ and η in K such that $\varphi(a) = (\pi(a)\,\xi \mid \eta)$ for all $a \in A$.*

Proof. Let $\bar{\varphi}$ be the unique extension of φ to a normal functional of the von Neumann algebra A^{**}. By Sakai's polar decomposition for normal linear functionals (see, for example, [Pedersen 1979, Proposition 3.6.7]) there exists a normal state ψ of A^{**} and a partial isometry v such that $\varphi(a) = \psi(av)$ for all $a \in A^{**}$. Let $\pi : A^{**} \to B(K)$ be the GNS representation corresponding to ψ. If η is the corresponding cyclic vector and $\xi = v\eta$, then the restriction of π to A is as required. □

Lemma 2.2. *Suppose A is a proper unital subalgebra of $C = \mathrm{C}^*(A, b)$. Then there exists a representation $\pi : C \to B(K)$ and a projection q in $\pi(A)' \cap B(K)$ such that $[q, b] \neq 0$.*

Proof. By the Hahn–Banach separation theorem, there exists a functional φ on C of norm 1 such that φ annihilates A and $\varphi(b) = \operatorname{dist}(A, b)$. Let $\pi : C \to B(K)$, and η and ξ be as guaranteed by Lemma 2.1. Let L be the norm-closure of $\pi(A)\xi$. Since A is unital, $L \neq \{0\}$. As $0 = \varphi(a) = (\pi(a)\,\xi \mid \eta)$ for all $a \in A$, η is orthogonal to L and therefore the projection p to L is nontrivial. Clearly $p \in \pi(A)' \cap B(K)$. Since $(\pi(b)\xi \mid \eta) = \varphi(b) \neq 0$, $\pi(b)$ does not commute with p and we therefore have $q \in \pi(A)' \cap B(K)$ such that $\|[\pi(b), q]\| > 0$. □

The proof of Theorem 1 would be much simpler if Lemma 2.2 provided an irreducible representation. This is impossible in general, as the following example shows. Let A be the unitization of the algebra of compact operators $\mathcal{K}(H)$ on an infinite-dimensional Hilbert space and let b be a projection in $B(H)$ which is Murray–von Neumann equivalent to $1-b$. Then $C = \mathrm{C}^*(A, b)$ has (up to equivalence) three irreducible representations. Two of those representations annihilate A and send b to a scalar, and the third representation is faithful and the image of b is in the weak operator closure of the image of A.

It is well-known that for a Banach space X, the second dual X^{**} can be embedded into an ultrapower of X [Heinrich 1980, Proposition 6.7]. In general, the second dual A^{**} of a C*-algebra A cannot be embedded into an ultrapower of A by a *-homomorphism for at least two reasons. First, A^{**} is a von Neumann algebra [Blackadar 2006, §III.5.2] and it therefore has real rank zero, while A may have no nontrivial projections at all. Since being projectionless is axiomatizable [Farah et al. 2016a, Theorem 2.5.1], if A is projectionless then Łoś's theorem implies that $A^{\mathcal{U}}$ is projectionless as well and A^{**} cannot be embedded into it. The referee pointed out another, much subtler, obstruction. In the context of Banach spaces, the embeddability of X^{**} into $X^{\mathcal{U}}$ is equivalent to a finitary statement, the so-called *local reflexivity* of Banach spaces, the C*-algebraic version of which does not hold for all C*-algebras [Effros and Haagerup 1985, §5]. In particular, for a large class of C*-algebras the diagonal embedding of A into $A^{\mathcal{U}}$ cannot be extended even to a unital completely positive map from A^{**} into $A^{\mathcal{U}}$. The referee also pointed out that a result of J. M. G. Fell is closely related to results of the present section. It is a standard fact that a representation of a discrete group is weakly contained in another representation of the same group if and only if it can be embedded into an ultrapower of the direct sum of infinitely many copies of the latter representation. In [Fell 1960, Theorem 1.2] it was essentially proved that this equivalence carries over to arbitrary C*-algebras.

All this said, Lemma 2.3 is a poor man's C*-algebraic variant of the fact that Banach space X^{**} embeds into $X^{\mathcal{U}}$. As in [Pedersen 1979, 3.3.6], we say that two representations π_1 and π_2 of A are *equivalent* if the identity map on A extends to an isomorphism between $\pi_1(A)''$ and $\pi_2(A)''$.

Lemma 2.3. *Assume* $\big(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\big)$ *is an ultraproduct of faithful irreducible representations of unital* C*-*algebras and* C *is a unital separable subalgebra of* $B^{\mathcal{U}}$.

(1) *If* $C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big) = \{0\}$ *then the induced representation of* C *on* $\prod_{\mathcal{U}} H_j$ *is equivalent to the universal representation of* C.

(2) *In general, if*

$$p = \bigvee \{q : q \text{ is a projection in } C \cap \mathcal{K}\big(\textstyle\prod_{\mathcal{U}} H_j\big)\}$$

then $p \in C' \cap B\big(\prod_{\mathcal{U}} H_j\big)$ *and* $c \mapsto (1-p)c$ *is equivalent to the universal representation of* $C/\big(C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)\big)$ *on* $(1-p)\prod_{\mathcal{U}} H_j$.

Proof. For a state ψ on C the $(0,1)$-type $\boldsymbol{t}_\psi(\xi)$ of a vector ξ implementing ψ consists of all conditions of the form $(a\xi \mid \xi) = \psi(a)$ for $a \in C$ and $\|\xi\| = 1$.

(1) Fix a state ψ on C. By Glimm's lemma [Davidson 1996, Lemma II.5.1], the type $\boldsymbol{t}_\psi$ is consistent with the theory of $\big(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\big)$. By the separability of C and countable saturation, there exists a unit vector $\eta \in \prod_{\mathcal{U}} H_j$ such that $\psi(c) = (c\eta \mid \eta)$ for all $c \in C$. Let L be the norm-closure of $C\eta$ in $\prod_{\mathcal{U}} H_j$. Then L is a reducing subspace for C and the induced representation of C on L is spatially isomorphic to the GNS representation of C corresponding to ψ. Since ψ was arbitrary, by [Pedersen 1979, Theorem 3.8.2] this completes the proof.

(2) For every $a \in C$ we have $pa \in C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)$ and therefore $pa(1-p) = 0$. Similarly $(1-p)ap = 0$, and therefore $p \in C' \cap B\big(\prod_{\mathcal{U}} H_j\big)$. Let p_n, for $n \in \mathbb{N}$, be a maximal family of orthogonal projections in $C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)$. It is countable by the separability of C and $p = \bigvee_n p_n$. Let ψ be a state of C that annihilates $C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)$. Let $\boldsymbol{t}^+_\psi(\xi)$ be the type obtained from $\boldsymbol{t}_\psi(\xi)$ by adding to it all conditions of the form $p_n\xi = 0$ for $n \in \mathbb{N}$. By Glimm's lemma (as stated in [Davidson 1996, Lemma II.5.1]) the type $\boldsymbol{t}^+_\psi(\xi)$ is consistent, and by the countable saturation we can find $\xi_1 \in \prod_{\mathcal{U}} H_j$ that realizes this type. Then $p\xi_1 = 0$ and therefore $\xi_1 \in (1-p)\prod_{\mathcal{U}} H_j$. Therefore every GNS representation of $C/\big(C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)\big)$ is spatially equivalent to a subrepresentation of $c \mapsto (1-p)c$, and by [Pedersen 1979, Theorem 3.8.2] this concludes the proof. □

3. Second dual and Day's trick

The natural embedding of a C*-algebra B into its second dual B^{**} is rarely elementary. For example, having real rank zero is axiomatizable [Farah et al. 2016a, Theorem 2.5.1] and B^{**}, being a von Neumann algebra, has real rank zero while B may have no nontrivial projections at all. However, we shall see that there is a restricted degree of elementarity between B and B^{**}, and it will suffice for our purposes.

We shall consider the language $(\mathcal{L}_{\mathrm{rep}})_B$ obtained by adding new constants for parameters in B; see Section 2. Term $\alpha(x)$ in the extended language is *linear* if it is of the form

$$\alpha(x) = xa + bx$$

for some parameters a and b.

A *restricted B-linear formula* is a formula of the form

(1) $\max_{j\leq m} \|\alpha_j(x) - b_j\| + \max_{j\leq n}(r_j \dot{-} \|\beta_j(x)\|)$,

where

(2) all b_j, for $1 \leq j \leq m$, are parameters in B,

(3) all r_j, for $1 \leq j \leq n$, are positive real numbers,

(4) all α_j, for $1 \leq j \leq m$, are linear terms with parameters in B, and

(5) all β_j, for $1 \leq j \leq n$, are linear terms with parameters in B.

The proof of the following is based on an application of the Hahn–Banach separation theorem first used by Day [1957]; see also [Elliott 1977, Section 2] for some uses of this method in the theory of C*-algebras.

Lemma 3.1. *Suppose B is a unital* C**-algebra and*

$$\gamma(x) = \max_{j\leq m} \|\alpha_j(x) - b_j\| + \max_{j\leq n}\bigl(r_j \dot{-} \|\beta_j(x)\|\bigr).$$

is a restricted B-linear formula. Then the following are equivalent:

(6) $\inf_{x\in B} \gamma(x) = 0$,

(7) $\inf_{x\in B^{**}} \gamma(x) = 0$.

Proof. Condition (6) implies (7) because B is isomorphic to a unital subalgebra of B^{**} and therefore $\inf_{x\in B^{**}} \gamma(x) \leq \inf_{x\in B} \gamma(x)$.

Assume (7) holds. Let a_j and c_j, for $j \leq n$, be such that $\alpha_j(x) = a_j x + x c_j$. For each j we identify α_j with its interpretation, a linear map from B to B. The second adjoint $\alpha_j^{**} : B^{**} \to B^{**}$ also satisfies $\alpha_j^{**}(x) = a_j x + x c_j$, hence $\alpha_j^{**}(x)$ is the interpretation of the term $\alpha_j(x)$ in B^{**}. The set

$$Z := \langle \alpha_j(x) : x \in B_{\leq 1} \rangle,$$

being an image of a convex set under a linear map, is a convex subset of B^m and by the Hahn–Banach theorem,

$$Z_1 := B^m \cap \langle \alpha_j(x) : x \in B^{**}_{\leq 1} \rangle$$

is included in the norm-closure of Z. By (7) we have $(b_1, \dots, b_m) \in Z_1$.

Fix $\varepsilon > 0$ and let

$$X_1 := \bigl\{ x \in B_{\leq 1} : \max_{j \leq m} \|\alpha_j(x) - b_j\| \leq \varepsilon \bigr\}.$$

By the above, this is a convex subset of the unit ball of B and (by using the Hahn–Banach separation theorem again) the weak*-closure of X_1 in B^{**} is equal to $\{x \in B^{**}_{\leq 1} : \max_{j \leq m} \|\alpha_j(x) - b_j\| \leq \varepsilon\}$.

Let $c \in B^{**}_{\leq 1}$ be such that $\gamma(c) < \varepsilon$. Then c belongs to the weak*-closure of X_1. For each $j \leq n$ we have $\|\beta_j(c)\| > r_j - \varepsilon$. Fix a norming functional $\varphi_j \in B^*$ such that $\|\varphi_j\| = 1$ and $\varphi_j(\beta_j(c)) > r_j - \varepsilon$. Then

$$U := \{x \in B^{**} : \varphi_j(\beta_j(x)) > r_j - \varepsilon \text{ for all } j\}$$

is a weak*-open neighborhood of c and, as c belongs to the weak*-closure of X_1, $U \cap X_1$ is a nonempty subset of $B_{\leq 1}$. Any $b \in U \cap X_1$ satisfies $\gamma(b) < \varepsilon$. As $\varepsilon > 0$ was arbitrary, this shows that (6) holds. □

In the following, $A \subseteq \prod_{\mathcal{U}} B_j$ is identified with a subalgebra of $B\bigl(\prod_{\mathcal{U}} H_j\bigr)$.

Lemma 3.2. *Suppose (B_j, H_j) is an irreducible representation of B_j on H_j for $j \in \mathbb{N}$ and A is a separable subalgebra of $\prod_{\mathcal{U}} B_j$.*

(1) *For every $b \in \prod_{\mathcal{U}} B_j$, we have that $b \in \bigl(A' \cap B\bigl(\prod_{\mathcal{U}} H_j\bigr)\bigr)'$ if and only if $b \in \bigl(A' \cap \prod_{\mathcal{U}} B_j\bigr)'$. Equivalently,*

$$\bigl(A' \cap B\bigl(\textstyle\prod_{\mathcal{U}} H_j\bigr)\bigr)' \cap \textstyle\prod_{\mathcal{U}} B_j = \bigl(A' \cap \textstyle\prod_{\mathcal{U}} B_j\bigr)' \cap \textstyle\prod_{\mathcal{U}} B_j.$$

(2) $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = \bigl(A' \cap \prod_{\mathcal{U}} B_j\bigr)'$.

Proof. (1) Since $\prod_{\mathcal{U}} B_j \subseteq B\bigl(\prod_{\mathcal{U}} H_j\bigr)$, we clearly have $\bigl(A' \cap B\bigl(\prod_{\mathcal{U}} H_j\bigr)\bigr)' \subseteq \bigl(A' \cap \prod_{\mathcal{U}} B_j\bigr)'$. In order to prove the converse inclusion, fix $b \in \prod_{\mathcal{U}} B_j$ and suppose that there exists $q \in A' \cap B\bigl(\prod_{\mathcal{U}} H_j\bigr)$ such that $\|[q, b]\| = r > 0$. We need to find $d \in A' \cap \prod_{\mathcal{U}} B_j$ satisfying $[d, b] \neq 0$.

Consider the $(1, 0)$-type $\boldsymbol{t}(x)$ consisting of all conditions of the form

$$\|[x, b]\| \geq r \quad \text{and} \quad [x, a] = 0$$

for $a \in A$. This type is satisfied in $B\bigl(\prod_{\mathcal{U}} H_j\bigr)$ by q. Since all formulas in $\boldsymbol{t}(x)$ are quantifier-free, their interpretation is unchanged when passing to a larger algebra.

Fix a finite subset of $\boldsymbol{t}(x)$ and let $F \subseteq A$ be the set of parameters occurring in this subset. Then

$$\gamma_F(x) := \inf_x \max_{a \in F} \|[x, a]\| + (r \mathbin{\dot{-}} \|[x, b]\|)$$

is a restricted $\prod_{\mathcal{U}} B_j$-linear formula. Since A is separable, we can find projection p in $\mathrm{C}^*(A, q)' \cap B(\prod_{\mathcal{U}} H_j)$ with separable range such that $q_1 := pq$ satisfies $\|[q_1, b]\| = r$. To find this p, take a separable elementary submodel (C, H_0) of

$$\big(B\big(\textstyle\prod_{\mathcal{U}} H_j\big), \prod_{\mathcal{U}} H_j\big)$$

such that $A \subseteq C$ and let p be the projection to H_0.

By the downward Löwenheim–Skolem theorem [Farah et al. 2014, Theorem 4.6] there exists a separable elementary submodel (D, K) of $\big(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\big)$ such that $\mathrm{C}^*(A, b) \subseteq D$ and the range of p is included in K. Part (2) of Lemma 1.2 and Łoś's theorem imply that $\overline{\prod_{\mathcal{U}} B_j}^{\mathrm{WOT}} = B\big(\prod_{\mathcal{U}} H_j\big)$ and $\overline{p_K D p_K}^{\mathrm{WOT}} = B\big(p_K \prod_{\mathcal{U}} H_j\big)$, where p_K denotes the projection to K. We can therefore identify p_K with a minimal central projection in D^{**}. Via this identification we have $q_1 \in D^{**}$. Since $\gamma_F(q_1) = 0$, Lemma 3.1 implies $\inf_{x \in D, \|x\| \le 1} \gamma_F(x) = 0$ and $\inf_{x \in \prod_{\mathcal{U}} B_j, \|x\| \le 1} \gamma_F(x) = 0$ (since γ_F is quantifier-free).

Since F was an arbitrary finite subset of A, the type $\boldsymbol{t}(x)$ is consistent with the theory of $\prod_{\mathcal{U}} B_j$. Since A is separable, by the countable saturation there exists $d \in A' \cap \prod_{\mathcal{U}} B_j$ satisfying $\|[d, b]\| \ge r$.

(2) By the von Neumann bicommutant theorem, $\overline{A}^{\mathrm{WOT}} = \big(A' \cap B\big(\prod_{\mathcal{U}} H_j\big)\big)'$ and therefore (1) implies $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$. □

4. Proof of Theorem 1

Suppose (B_j, H_j) is a faithful irreducible representation of B_j on H_j for $j \in \mathbb{N}$ and A is a separable subalgebra of $\prod_{\mathcal{U}} B_j$. By Lemma 1.2, $\big(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\big)$ is an irreducible faithful representation of $\prod_{\mathcal{U}} B_j$.

By (2) of Lemma 3.2, we have $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$. Then, since $A \subseteq \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$, it remains to prove $\big(A' \cap \prod_{\mathcal{U}} B_j\big)' \subseteq A$. Fix $b \in \prod_{\mathcal{U}} B_j$ such that $r := \mathrm{dist}(b, A) > 0$. By (1) of Lemma 3.2, it suffices to find $d \in A' \cap B\big(\prod_{\mathcal{U}} H_j\big)$ such that $[d, b] \neq 0$. Let

$$C := \mathrm{C}^*(A, b).$$

Lemma 4.1. *With A, b, C, r and $\prod_{\mathcal{U}} B_j$ as above, there exists a representation*

$$\pi : C/\big(C \cap \mathcal{K}\big(\textstyle\prod_{\mathcal{U}} H_j\big)\big) \to B(K)$$

and $q \in \pi(A)' \cap B(K)$ such that $[q, \pi(b)] \neq 0$.

Since the proof of Lemma 4.1 is on the long side, let us show how it completes the proof of Theorem 1. Lemma 2.3 implies that if

$$p = \bigvee \big\{q : q \text{ is a projection in } C \cap \mathcal{K}\big(\textstyle\prod_{\mathcal{U}} H_j\big)\big\}$$

then $p \in C' \cap B\big(\prod_{\mathcal{U}} H_j\big)$ and $c \mapsto (1 - p)c$ is equivalent to the universal representation of $C/\big(C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)\big)$ on $(1 - p) \prod_{\mathcal{U}} H_j$. Therefore q as in the conclusion of Lemma 4.1 can be found in $A' \cap B\big(\prod_{\mathcal{U}} H_j\big)$, implying $b \notin \big(A' \cap B\big(\prod_{\mathcal{U}} H_j\big)\big)'$. By Lemma 3.2 this implies $b \notin \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$, reducing the proof of Theorem 1 to the following.

Proof of Lemma 4.1. An easy special case is worth noting. If $C \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big) = \{0\}$ then Lemma 2.2 implies the existence of a representation $\pi : C \to B(K)$ and $q \in \pi(A)' \cap B(K)$ such that $[q, \pi(b)] \neq 0$.

In the general case, let q_n, for $n \in J$, be an enumeration of a maximal orthogonal set of minimal projections in $A \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)$. The index-set J is countable (and possibly finite or even empty) since A is separable. Let $p_n := \bigvee_{j \leq n} q_j$.

Suppose for a moment that there exists n such that $p_n b p_n \notin A$. Since the range of p_n is finite-dimensional, by von Neumann's bicommutant theorem [Blackadar 2006, §I.9.1.2] and the Kadison transitivity theorem [Blackadar 2006, Theorem II.6.1.13] there exists $d \in A' \cap B(p_n \prod_{\mathcal{U}} H_j)$ such that $[d, b] \neq 0$. Lemma 3.2 now implies $p_n b p_n \notin \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$ and $b \notin \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$.

We may therefore assume $p_n b p_n \in A$, for all n. Let $p = \bigvee_n p_n$. Lemma 2.3 (2) implies $p \in A' \cap B\big(\prod_{\mathcal{U}} H_j\big)$, and we may therefore assume $[b, p] = 0$. Since $C = \mathrm{C}^*(A, b)$ this implies $p \in C' \cap B\big(\prod_{\mathcal{U}} H_j\big)$. Since $p_n b p_n \in A$ for all n we have $A \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big) = pCp \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)$. If $c \in C$, then for every n, we have $p_n c(1-p) = 0$ and similarly $(1-p)cp_n = 0$. Since the sequence p_n, for $n \in \mathbb{N}$, is an approximate unit for $A \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big)$, the latter is an ideal of C. Let $\theta : C \to C/(A \cap \mathcal{K})$ be the quotient map. We claim that $\operatorname{dist}(\theta(b), \theta(A)) = \operatorname{dist}(b, A) > 0$. Fix $a \in A$. We need to show that $\|\theta(a-b)\| \geq r$.

Consider the (0, 1)-type $\boldsymbol{t}(\xi)$ consisting of all conditions of the form

$$\|\xi\| = 1, \quad \|(a-b)\xi\| \geq r, \quad p_n \xi = 0,$$

for $n \in J$. To see this type is consistent fix a finite $F \subseteq J$. Let $m \geq \max(F)$ and

$$a' := (1-p_m)a(1-p_m) + p_m b p_m.$$

As both summands belong to A, we have $a' \in A$ and therefore $\|a' - b\| \geq r$. Fix $\varepsilon > 0$. If $\xi \in \prod_{\mathcal{U}} H_j$ is a vector of norm ≤ 1 such that $\|(a'-b)\xi\| > r - \varepsilon$ then $\xi' = (1-p_m)\xi$ has the same property since $(a'-b)p_m = 0$. Since $\varepsilon > 0$ was arbitrary, $\boldsymbol{t}(\xi)$ is consistent. By the countable saturation there exists a unit vector $\xi \in \prod_{\mathcal{U}} H_j$ which realizes $\boldsymbol{t}(\xi)$. Since $p_n \xi = 0$ for all n, we have $p\xi = 0$ and therefore $\|\theta(a-b)\| \geq \big\|(1-p)(a-b)(1-p)\big\| \geq r$. Since $a \in A$ was arbitrary, we conclude that $\operatorname{dist}(\theta(b), \theta(A)) = r$.

Suppose for a moment that $(1-p)C(1-p) \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big) = \{0\}$. By (2) of Lemma 2.3 the representation

$$C \ni c \mapsto (1-p)c \in B\Big((1-p)\prod_{\mathcal{U}} H_j\Big)$$

is equivalent to the universal representation of C. Hence by Lemma 2.2 we can find $d \in (1-p)\big(A' \cap B\big(\prod_{\mathcal{U}} H_j\big)\big)$ that does not commute with b, and by the above, this concludes the proof in this case.

We may therefore assume that

$$(1-p)C(1-p) \cap \mathcal{K}\big(\prod_{\mathcal{U}} H_j\big) \neq \{0\}.$$

By the spectral theorem for self-adjoint compact operators and continuous functional calculus, there exists a nonzero projection $q \in (1-p)C(1-p)$ of finite rank. Fix $c \in C$ such that $(1-p)c(1-p) = q$.

By Lemma 3.2, it suffices to find $q \in A' \cap (1-p)B\big(\prod_{\mathcal{U}} H_j\big)(1-p)$ such that $[q, c] \neq 0$. Suppose otherwise, so that $c \in \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$. Lemma 3.2 (2) implies that $c \in \overline{A}^{\mathrm{WOT}}$. By the Kaplansky density theorem [Blackadar 2006, Theorem I.9.1.3] there is a net of positive contractions in A converging to c in the weak operator topology. By the continuous functional calculus and the Kadison transitivity theorem [Blackadar 2006, Theorem II.6.1.13], we may choose this net among the members of

$$Z := \{a \in A_+ : \|a\| = 1, qaq = q\}.$$

Consider the (0, 1)-type $\boldsymbol{t}_1(\xi)$ consisting of all conditions of the form

$$\|\xi\| = 1, \qquad a\xi = \xi,$$
$$q\xi = 0, \quad p_n\xi = 0,$$

for $n \in \mathbb{N}$ and $a \in Z$.

We claim that $\boldsymbol{t}_1(\xi)$ is consistent. Fix $\varepsilon > 0$ and $a_1, a_2, \dots, a_n$ in Z. Let

$$a := a_1 a_2 \cdots a_{n-1} a_n a_{n-1} \cdots a_2 a_1.$$

Then $a \in Z$ and $q \leq a$. By the choice of p the operator $(1-p)(a-s)_+$ is not compact for any $s < 1$. Therefore there exists a unit vector ξ_0 in $(1-p-q)\prod_{\mathcal{U}} H_j$ such that $\|\xi_0 - a\xi_0\|$ is arbitrarily small. By the countable saturation there exists a unit vector $\xi_1 \in (1-(p+q))\prod_{\mathcal{U}} H_j$ such that $a\xi_1 = \xi_1$. As each a_j is a positive contraction, we have $a_j\xi_1 = \xi_1$ for $1 \leq j \leq n$. Since $a_1, \dots, a_n$ was an arbitrary subset of Z, this shows that $\boldsymbol{t}_1(\xi)$ is consistent.

Since Z is separable, by the countable saturation there exists $\xi \in \prod_{\mathcal{U}} H_j$ realizing $\boldsymbol{t}_1(\xi)$. Then ξ is a unit vector in $(1-(p+q))\prod_{\mathcal{U}} H_j$ such that $a\xi = \xi$ for all $a \in Z$. As $c\xi = 0$, this contradicts c being in the weak operator topology closure of Z.

Therefore there exists

$$q \in A' \cap (1-p)B\big(\prod_{\mathcal{U}} H_j\big)(1-p)$$

such that $[q, c] \neq 0$. Since $c \in C = \mathrm{C}^*(A, b)$ we have $[q, b] \neq 0$, and this concludes the proof. □

5. Concluding remarks

In the following infinitary form of the Kadison transitivity theorem, p_K denotes projection to a closed subspace K of $\prod_{\mathcal{U}} H_j$.

Proposition 5.1. *Assume $\big(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\big)$ is an ultraproduct of faithful and irreducible representations of unital* C*-algebras. Also assume K is a separable closed subspace of $\prod_{\mathcal{U}} H_j$ and $T \in B(K)$.*

(1) *There exists $b \in \prod_{\mathcal{U}} B_j$ such that $\|b\| = \|T\|$ and $p_K b p_K = T$.*

(2) *If T is self-adjoint, positive, or unitary in $B(K)$, then b can be chosen to be self-adjoint, positive, or unitary, respectively, in $B\big(\prod_{\mathcal{U}} H_j\big)$.*

Proof. (1) This is a consequence of the Kadison transitivity theorem and countable saturation of the structure $\big(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\big)$. Let p_n, for $n \in \mathbb{N}$, be an increasing sequence of finite-dimensional projections converging to p_K in the strong operator topology and let a_n, for $n \in \mathbb{N}$, be a dense subset of A. We need to check that the type $\boldsymbol{t}(x)$ consisting of all conditions of the form

$$\|p_n(x - T)p_n\| = 0, \qquad \|x\| = \|T\|,$$

for $n \in \mathbb{N}$ is consistent. Since the representation of $\prod_{\mathcal{U}} B_j$ on $\prod_{\mathcal{U}} H_j$ is irreducible by Lemma 1.2, every finite subset of $\boldsymbol{t}(x)$ is consistent by the Kadison transitivity theorem. We can therefore find $b \in \prod_{\mathcal{U}} B_j$ that satisfies $\boldsymbol{t}(x)$ and thus $p_K b p_K = T$ and $\|b\| = \|T\|$.

(2) If T is self-adjoint, add the condition $x = x^*$ to $\boldsymbol{t}(x)$. By [Pedersen 1979, Theorem 2.7.5] the corresponding type is consistent, and the assertion again follows by countable saturation. The case when T is unitary uses the same theorem. □

An important consequence of Voiculescu's theorem is that any two unital representations $\pi_j : A \to B(H)$ of a separable unital C*-algebra A on H such that $\ker(\pi_1) = \ker(\pi_2)$ and $\pi_1(A) \cap \mathcal{K}(H) = \pi_2(A) \cap \mathcal{K}(H) = \{0\}$ are approximately unitarily equivalent [Voiculescu 1976, Corollary 1.4]. The analogous statement is in general false for the ultraproducts. Let $B_n = M_n(\mathbb{C})$ for $n \in \mathbb{N}$ and let $A = \mathbb{C}^2$. The group $K_0\big(\prod_{\mathcal{U}} M_n(\mathbb{C})\big)$ is isomorphic to $\mathbb{Z}^{\mathbb{N}}$ with the natural ordering and the identity function id as the order-unit. Every unital representation of A corresponds to an element of this group that lies between 0 and id, and there are $2^{\aleph_0}$ inequivalent representations. Also, $K_0\big(\prod_{\mathcal{U}} M_n(\mathbb{C})\big)$ is isomorphic to the ultraproduct $\prod_{\mathcal{U}} \mathbb{Z}$ and $2^{\aleph_0}$ of these extensions remain inequivalent even after passing to the ultraproduct.

We return to Pedersen's question [1990, Remark 10.11], whether a bicommutant theorem $(A' \cap M(B)/B)' = A$ is true for a separable unital subalgebra A of a corona $M(B)/B$ of a σ-unital C*-algebra B? A simple and unital C*-algebra C is *purely infinite* if for every nonzero $a \in C$ there are x and y such that $xay = 1$.

Question 5.2. Suppose C is unital, simple, purely infinite, and separable and A is a unital subalgebra of C. Is $(A' \cap C^{**})' \cap C = A$?

Let us prove that a positive answer to Question 5.2 would imply a positive answer to Pedersen's question. If A is a separable and unital subalgebra of $M(B)/B$ and $b \in (M(B)/B) \setminus A$, then there exists a separable elementary submodel C of $M(B)/B$ containing b. By [Lin 2004], $M(B)/B$ is simple if and only if it is purely infinite, and since being simple and purely infinite is axiomatizable [Farah et al. 2016a, Theorem 2.5.1], C is simple and purely infinite. If $(A' \cap C^{**})' \cap C = A$ then Proposition 5.3 implies that there exists $d \in A' \cap M(B)/B$ such that $[d, b] \neq 0$.

Proposition 5.3. *Suppose B is a* C*-*algebra*, A *is a separable subalgebra of* B, $b \in B$ *and* $r \geq 0$. *If B is an ultraproduct or a corona of a σ-unital, nonunital* C*-*algebra then*

$$\sup_{d \in (A' \cap B)_+, \|d\| \leq 1} \|[d, b]\| = \sup_{d \in (A' \cap B^{**})_+, \|d\| \leq 1} \|[d, b]\|.$$

Proof. The only property of B used in this proof is that of being countably degree-1 saturated [Farah and Hart 2013, Theorem 1]. Since $B \subseteq B^{**}$, it suffices to prove "$\geq$" in the above inequality. Suppose $b \in B$ and $d \in (A' \cap B^{**})_+$ are such that $\|d\| = 1$ and $r \dot{-} \|[b, d]\|$. Consider the type $\boldsymbol{t}(x)$ consisting of conditions $\|x\| = 1$, $x \geq 0$, $\|xb - bx\| \geq r$, and $\|[x, a]\| = 0$ for a in a countable dense subset of A. This is a countable degree-1 type. If $\varphi_j = 0$, for $j < n$, is a finite subset of $\boldsymbol{t}(x)$ then $\gamma(x) := \max_{j<n} \varphi_j(x)$ is a restricted B-linear formula and Lemma 2.3 implies that it is approximately satisfied in B. By the countable degree-1 saturation of B [Farah and Hart 2013, Theorem 1] we can find a realization d' of $\boldsymbol{t}(x)$ in B. Clearly $d' \in (A' \cap B)_+$, $\|d'\| = 1$, and $\|[d', b]\| \geq r$, completing the proof. □

Some information on a special case of Pedersen's conjecture can also be found in [Elliott and Kucerovsky 2007].

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References

[Akemann and Weaver 2004] C. Akemann and N. Weaver, "Consistency of a counterexample to Naimark's problem", *Proc. Natl. Acad. Sci. USA* **101**:20 (2004), 7522–7525. MR Zbl

[Arveson 1977] W. Arveson, "Notes on extensions of C^*-algebras", *Duke Math. J.* **44**:2 (1977), 329–355. MR Zbl

[Ben Yaacov et al. 2008] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, "Model theory for metric structures", pp. 315–427 in *Model theory with applications to algebra and analysis*, vol. 2, edited by Z. Chatzidakis et al., London Math. Soc. Lecture Note Ser. **350**, Cambridge Univ. Press, 2008. MR Zbl

[Blackadar 2006] B. Blackadar, *Operator algebras: theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences **122**, Springer, 2006. MR Zbl

[Bosa et al. 2016] J. Bosa, N. P. Brown, Y. Sato, A. P. Tikuisis, S. White, and W. Winter, "Covering dimension of C^*-algebras and 2-coloured classification", preprint, 2016. To appear in *Mem. Amer. Math. Soc.* arXiv

[Davidson 1996] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs **6**, American Mathematical Society, Providence, RI, 1996. MR Zbl

[Day 1957] M. M. Day, "Amenable semigroups", *Illinois J. Math.* **1** (1957), 509–544. MR Zbl

[Effros and Haagerup 1985] E. G. Effros and U. Haagerup, "Lifting problems and local reflexivity for C^*-algebras", *Duke Math. J.* **52**:1 (1985), 103–128. MR Zbl

[Elliott 1977] G. A. Elliott, "Some C^*-algebras with outer derivations, III", *Ann. of Math.* (2) **106**:1 (1977), 121–143. MR Zbl

[Elliott and Kucerovsky 2007] G. A. Elliott and D. Kucerovsky, "A relative double commutant theorem for hereditary sub-C^*-algebras", *C. R. Math. Acad. Sci. Soc. R. Can.* **29**:1 (2007), 22–27. MR Zbl

[Farah and Hart 2013] I. Farah and B. Hart, "Countable saturation of corona algebras", *C. R. Math. Acad. Sci. Soc. R. Can.* **35**:2 (2013), 35–56. MR Zbl

[Farah et al. 2013] I. Farah, B. Hart, and D. Sherman, "Model theory of operator algebras, I: stability", *Bull. Lond. Math. Soc.* **45**:4 (2013), 825–838. MR Zbl

[Farah et al. 2014] I. Farah, B. Hart, and D. Sherman, "Model theory of operator algebras, II: model theory", *Israel J. Math.* **201**:1 (2014), 477–505. MR Zbl

[Farah et al. 2016a] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter, "Model theory of C^*-algebras", preprint, 2016. arXiv

[Farah et al. 2016b] I. Farah, B. Hart, M. Rørdam, and A. Tikuisis, "Relative commutants of strongly self-absorbing C^*-algebras", *Sel. Math. New Ser.* (online publication April 2016).

[Fell 1960] J. M. G. Fell, "The dual spaces of C^*-algebras", *Trans. Amer. Math. Soc.* **94** (1960), 365–403. MR Zbl

[Hadwin 1978] D. W. Hadwin, "An asymptotic double commutant theorem for C^*-algebras", *Trans. Amer. Math. Soc.* **244** (1978), 273–297. MR Zbl

[Hadwin 2011] D. Hadwin, "Approximate double commutants in von Neumann algebras", preprint, 2011. Zbl arXiv

[Hadwin and Shen 2014] D. Hadwin and J. Shen, "Approximate double commutants and distance formulas", *Oper. Matrices* **8**:2 (2014), 529–553. MR Zbl

[Heinrich 1980] S. Heinrich, "Ultraproducts in Banach space theory", *J. Reine Angew. Math.* **313** (1980), 72–104. MR Zbl

[Lin 2004] H. Lin, "Simple corona C^*-algebras", *Proc. Amer. Math. Soc.* **132**:11 (2004), 3215–3224. MR Zbl

[Matui and Sato 2014] H. Matui and Y. Sato, "Decomposition rank of UHF-absorbing C^*-algebras", *Duke Math. J.* **163**:14 (2014), 2687–2708. MR Zbl

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic Press, London, 1979. MR Zbl

[Pedersen 1990] G. K. Pedersen, "The corona construction", pp. 49–92 in *Operator theory* (Indianapolis, IN, 1988), edited by J. B. Conway and B. B. Morrel, Pitman Res. Notes Math. Ser. **225**, Longman Sci. Tech., Harlow, 1990. MR Zbl

[Popa 2014] S. Popa, "Independence properties in subalgebras of ultraproduct II_1 factors", *J. Funct. Anal.* **266**:9 (2014), 5818–5846. MR Zbl

[Tikuisis et al. 2016] A. Tikuisis, S. White, and W. Winter, "Quasidiagonality of nuclear C^*-algebras", preprint, 2016. To appear in *Ann. Math.* arXiv

[Toms and Winter 2007] A. S. Toms and W. Winter, "Strongly self-absorbing C^*-algebras", *Trans. Amer. Math. Soc.* **359**:8 (2007), 3999–4029. MR Zbl

[Voiculescu 1976] D. Voiculescu, "A non-commutative Weyl–von Neumann theorem", *Rev. Roumaine Math. Pures Appl.* **21**:1 (1976), 97–113. MR Zbl

[Winter 2016] W. Winter, "QDQ vs. UCT", pp. 321–342 in *Operator algebras and applications: the Abel symposium 2015*, edited by T. M. Carlsen et al., Abel Symposia **12**, Springer, 2016.

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NONCOMPACT MANIFOLDS THAT ARE INWARD TAME

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We continue our study of ends of noncompact manifolds, with a focus on the inward tameness condition. For manifolds with compact boundary, inward tameness, by itself, has significant implications. For example, such manifolds have stable homology at infinity in all dimensions. Here we show that these manifolds have "almost perfectly semistable" fundamental group at each of their ends. That observation leads to further analysis of the group-theoretic conditions at infinity, and to the notion of a "near pseudocollar" structure. We obtain a complete characterization of n-manifolds ($n \geq 6$) admitting such a structure, thereby generalizing our previous work (*Geom. Topol.* 10 (2006), 541–556). We also construct examples illustrating the necessity and usefulness of the new conditions introduced here. Variations on the notion of a perfect group, with corresponding versions of the Quillen plus construction, form an underlying theme of this work.

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1. Introduction

In [Guilbault 2000; Guilbault and Tinsley 2003; 2006] we carried out a program to generalize L. C. Siebenmann's famous manifold collaring theorem [1965] in ways applicable to manifolds with nonstable fundamental group at infinity. Motivated by some important examples of finite-dimensional manifolds and a seminal paper by

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T. A. Chapman and Siebenmann [1976] on Hilbert cube manifolds, we chose the following definitions.

A manifold N^n with compact boundary is called a *homotopy collar* if $\partial N^n \hookrightarrow N^n$ is a homotopy equivalence. If N^n contains arbitrarily small homotopy collar neighborhoods of infinity, we call N^n a *pseudocollar*. Clearly, an actual open collar N^n, i.e., $N^n \approx \partial N^n \times [0, \infty)$, is a special case of a pseudocollar. Fundamental to [Siebenmann 1965; Chapman and Siebenmann 1976] and our earlier work is the notion of inward tameness.

A manifold M^n is *inward tame* if each of its clean neighborhoods of infinity is finitely dominated; it is *absolutely inward tame* if those neighborhoods all have finite homotopy type. An alternative formulation of this definition (see p. 95) justifies the adjective "inward" — a term that helps distinguish this version of tameness from a similar, but inequivalent, version found elsewhere in the literature.

In [Guilbault and Tinsley 2006] a classification of pseudocollarable n-manifolds for $6 \leq n < \infty$ was obtained. In simplified form, it says:

Theorem 1.1 (pseudocollarability characterization — simple version). *A* 1*-ended* *n-manifold* M^n ($n \geq 6$) *with compact boundary is pseudocollarable if and only if*

(a) M^n *is absolutely inward tame*, *and*

(b) *the fundamental group at infinity is* $\mathcal{P}$*-semistable.*

A "$\mathcal{P}$-semistable (or perfectly semistable) fundamental group at infinity" indicates that an inverse sequence of fundamental groups of neighborhoods of infinity can be arranged so that bonding homomorphisms are surjective with perfect kernels.

By way of comparison, the simple version of Siebenmann's collaring theorem is obtained by replacing (b) with the stronger condition of π_1-stability, while Chapman and Siebenmann's pseudocollarability characterization for Hilbert cube manifolds is obtained by omitting (b) entirely. Thus, the differences among these three results lie entirely in the fundamental group at infinity.

In this paper we take a close look at n-manifolds satisfying only the inward tameness hypothesis. By necessity, our attention turns to the group theory at the ends of those spaces. Unlike the case of infinite-dimensional manifolds, CW complexes, or even n-manifolds with noncompact boundary, inward tameness has major implications for the fundamental group at the ends of n-manifolds with compact boundary. Unfortunately, inward tameness (ordinary or absolute) does not imply $\mathcal{P}$-semistability — an example from [Guilbault and Tinsley 2003] attests to that — but it comes remarkably close. One of the main results of this paper is the following.

Theorem 1.2. *Let* M^n *be an inward tame n-manifold with compact boundary. Then* M^n *has an* $\mathcal{AP}$*-semistable* (*almost perfectly semistable*) *fundamental group at each of its finitely many ends.*

The initial goals of this paper are developing the appropriate group theory (including the definition of $\mathcal{AP}$-semistable) and proving the above theorem. After that is accomplished, we apply those investigations by proving a structure theorem for manifolds that are inward tame, but not necessarily pseudocollarable.

Theorem 1.3 (near pseudocollarability characterization — simple version). *A* 1*-ended n-manifold* M^n $(n \geq 6)$ *with compact boundary is nearly pseudocollarable if and only if*

(a) M^n *is absolutely inward tame*, *and*

(b) *the fundamental group at infinity is* $\mathcal{SAP}$*-semistable.*

The notion of near pseudocollarability will be defined and explored in Section 4. For now, we note that nearly pseudocollarable manifolds admit arbitrarily small clean neighborhoods of infinity N, containing compact codimension 0 submanifolds A for which $A \hookrightarrow N$ is a homotopy equivalence. Obtaining a near pseudocollar structure requires a slight strengthening of $\mathcal{AP}$-semistability to $\mathcal{SAP}$-semistability (strong almost perfect semistability). The essential nature of this stronger condition is verified by a final result, in which our group-theoretic explorations come together in a concrete set of examples.

Theorem 1.4. *For all* $n \geq 6$, *there exist* 1*-ended open n-manifolds that are absolutely inward tame but do not have* $\mathcal{SAP}$*-semistable fundamental group at infinity*, *and thus*, *are not nearly pseudocollarable.*

In Section 7, we close with a discussion of some open questions.

Remark 1.5. Throughout this paper attention is restricted to noncompact manifolds with compact boundaries. When a boundary is noncompact, its end topology gets entangled with that of the ambient manifold, leading to very different issues. In the study of noncompact manifolds, a focus on those with compact boundaries is analogous to a focus on closed manifolds in the study of compact manifolds. An investigation of manifolds with noncompact boundaries is planned for [Guilbault and Gu $\geq$ 2017].

2. Definitions and background

Variations on the notion of a perfect group. In this subsection we review the definition of perfect group and discuss some variations.

Given elements a and b of a group K, the *commutator* $a^{-1}b^{-1}ab$ will be denoted by $[a, b]$. The *commutator subgroup* of K, denoted by $[K, K]$, is the subgroup generated by all commutators. It is a standard fact that $[K, K]$ is normal in K and is the smallest such subgroup with an abelian quotient. We call K *perfect* if $K = [K, K]$.

Now suppose K and J are normal subgroups of G. Define $[K, J]$ to be the subgroup of G generated by the set of commutators

$$[k, j] = \{k^{-1} j^{-1} k j \mid k \in K \text{ and } j \in J\}.$$

The following is standard and easy to verify.

Lemma 2.1. *For normal subgroups K and J of a group G,*

(1) $[K, J] \trianglelefteq G$,

(2) $[K, J] \trianglelefteq K$ *and* $[K, J] \trianglelefteq J$, *and*

(3) $[K, J] = [J, K]$.

Given the above setup, we say that K is *J-perfect* if $K \subseteq [J, J]$, and that K is *strongly J-perfect* if $K \subseteq [K, J]$. By Lemma 2.1, both of these conditions imply that $K \trianglelefteq J$; so we customarily begin with that as an assumption.

The following two lemmas are immediate. We state them explicitly for the purpose of comparison.

Lemma 2.2. *Let $K \trianglelefteq J$ be normal subgroups of G.*

(1) *K is perfect if and only if each element of K can be expressed as $\prod_{i=1}^{k}[a_i, b_i]$, where $a_i, b_i \in K$ for all i.*

(2) *K is J-perfect if and only if each element of K can be expressed as $\prod_{i=1}^{k}[a_i, b_i]$, where $a_i, b_i \in J$ for all i.*

(3) *K is strongly J-perfect if and only if each element of K can be expressed as $\prod_{i=1}^{k}[a_i, b_i]$, where $a_i \in K$ and $b_i \in J$ for all i.*

Lemma 2.3. *Let $K \trianglelefteq J \trianglelefteq L$ be normal subgroups of G.*

(1) *If K is* [*strongly*] *J-perfect, then K is* [*strongly*] *L-perfect for every normal subgroup L containing J.*

(2) *K is* [*strongly*] *K-perfect if and only if K is a perfect group.*

Remark 2.4. Lemma 2.3 suggests a key theme: the smaller the group L for which K is [strongly] L-perfect, the closer K is to being a genuine perfect group.

The various levels of perfectness can be nicely characterized using *group homology*. The $\mathbb{Z}$-homology of a group G may be defined as the $\mathbb{Z}$-homology of a $K(G, 1)$ space K_G. If $\lambda : G \to H$ is a homomorphism, there is a map $f_\lambda : K_G \to K_H$, unique up to basepoint-preserving homotopy, inducing λ on fundamental groups. Define $\lambda_* : H_*(G; \mathbb{Z}) \to H_*(H; \mathbb{Z})$ to be the homomorphism induced by f_λ.

Lemma 2.5. *Let $K \trianglelefteq J$, $i : K \hookrightarrow J$ be inclusion, and $q : J \to J/K$ be projection.*

(1) *K is perfect if and only if $H_1(K; \mathbb{Z}) = 0$.*

(2) *K is J-perfect if and only if* $i_* : H_1(K;\mathbb{Z}) \xrightarrow{0} H_1(J;\mathbb{Z})$ *if and only if* $q_* : H_1(J;\mathbb{Z}) \xrightarrow{\cong} H_1(J/K;\mathbb{Z})$.

(3) *K is strongly J-perfect if and only if K is J-perfect and* $q_* : H_2(J;\mathbb{Z}) \to H_2(J/K;\mathbb{Z})$ *is surjective.*

Proof. Claim (1) is clear from the standard fact that $H_1(K) \cong K/[K, K]$. Claim (2) can be verified with elementary group theory. Claim (3) follows from a well-known 5-term exact sequence due to Stallings [1965] and Stammbach [1966]. Due to its importance in this paper, we state it as a separate lemma. □

Lemma 2.6 (5-term exact sequence for group homology). *Given a normal subgroup K of a group J, there is a natural exact sequence*:

$$H_2(J;\mathbb{Z}) \to H_2(J/K;\mathbb{Z}) \to K/[K, J] \to H_1(J;\mathbb{Z}) \to H_1(J/K;\mathbb{Z}) \to 0.$$

The following elementary facts about group homology will be useful.

Lemma 2.7. *Let $f : X \to Y$ be a map between connected CW complexes and $\lambda : \pi_1(X) \to \pi_1(Y)$ the induced homomorphism. Then*

(1) $H_1(X;\mathbb{Z}) \cong H_1(\pi_1(X, *);\mathbb{Z})$;

(2) $f_* : H_1(X;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ *realizes* $\lambda_* : H_1(\pi_1(X);\mathbb{Z}) \to H_1(\pi_1(Y);\mathbb{Z})$; *and*

(3) *if* $f_* : H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ *is surjective, then* $\lambda_* : H_2(\pi_1(X);\mathbb{Z}) \to H_2(\pi_1(Y);\mathbb{Z})$ *is also surjective.*

Proof. Build a $K(\pi_1(X), 1)$ complex X' by attaching cells of dimension ≥ 3 to X and a $K(\pi_1(Y), 1)$ complex Y' by attaching cells of dimension ≥ 3 to Y. Both $X \overset{i}{\hookrightarrow} X'$ and $Y \overset{j}{\hookrightarrow} Y'$ induce isomorphisms on π_1 and H_1, so (1) follows. Use the asphericity of Y' to extend f to $f' : X' \to Y'$, also inducing λ on π_1. Clearly $i_* : H_2(X;\mathbb{Z}) \to H_2(X';\mathbb{Z})$ and $j_* : H_2(Y;\mathbb{Z}) \to H_2(Y';\mathbb{Z})$ are surjective.

This gives a commutative diagram

$$\begin{array}{ccc} H_2(X;\mathbb{Z}) & \xrightarrow{f_*} \!\!\!\!\!\twoheadrightarrow & H_2(Y;\mathbb{Z}) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ H_2(\pi_1(X);\mathbb{Z}) & \xrightarrow{f'_*} & H_2(\pi_1(Y);\mathbb{Z}) \end{array}$$

Since the other maps are all surjective, so is f'_*. □

Lastly we offer a topological characterization of the various levels of perfectness. For the purposes of this paper, these are possibly the most useful.

Let S_g denote a compact orientable surface of genus g with a single boundary component. A collection of oriented simple closed curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ on S_g with the property that each α_i intersects β_i transversely at a single point, and each of $\alpha_i \cap \alpha_j$, $\beta_i \cap \beta_j$, and $\alpha_i \cap \beta_j$ is empty when $i \neq j$, is called a *complete*

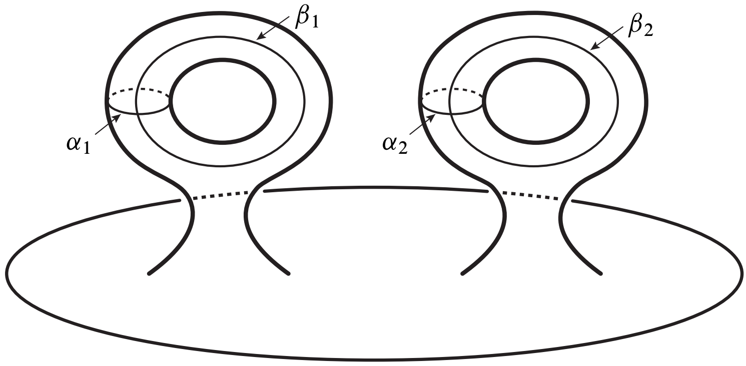


Figure 1. Complete set of handle curves ($g = 2$ case).

set of handle curves for S_g. A complete set of handle curves on S_g is not unique; however, given any such set, there exists a homeomorphism of S_g to the "disk with g handles" pictured in Figure 1 taking each α_i and β_i to the corresponding curves in the diagram.

Given a (not necessarily embedded) loop γ in a topological space X, we say that γ *bounds a compact orientable surface in* X if, for some g, there exists a map $f : S_g \to X$ such that $f|_{\partial S_g} = \gamma$. Notice that we do not require that f be an embedding. We often abuse terminology slightly by saying that γ bounds the surface S_g in X. Similarly, we often do not distinguish between a set of handle curves on S_g and their images in X.

Lemma 2.8. *Let X be a space with $\pi_1(X, x_0) \cong G$ and let $K \trianglelefteq J$ be normal subgroups of G. Then*:

(1) *K is perfect if and only if each loop γ in X representing an element of K bounds a surface S_g in X containing a complete set of handle curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ with each α_i and β_i belonging to K.*

(2) *K is J-perfect if and only if each loop γ in X representing an element of K bounds a surface S_g in X containing a complete set of handle curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ with each α_i and β_i belonging to J.*

(3) *K is strongly J-perfect if and only if each loop γ in X representing an element of K bounds a surface S_g in X containing a complete set of handle curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ with each α_i belonging to K and each β_i belonging to J.*

Remark 2.9. We are being informal in the statement of Lemma 2.8. Since the handle curves are not based, we should also choose, for each pair (α_i, β_i), an arc τ_i in S_g from x_0 to $p_i = \alpha_i \cap \beta_i$. The element of $\pi_1(X, x_0)$ represented by α_i is then $\tau_i * \alpha_i * \tau_i^{-1}$, and similarly for β_i. Notice that, by normality, the question of whether one of these loops belongs to K or J is independent of the choice of τ_i.

Algebra of inverse sequences. Understanding the fundamental group at infinity requires the language of inverse sequences. We briefly review the necessary definitions and terminology.

Throughout this subsection all arrows denote homomorphisms, while those of type $\twoheadrightarrow$ or $\twoheadleftarrow$ specify surjections. The symbol $\cong$ denotes isomorphisms.

Let

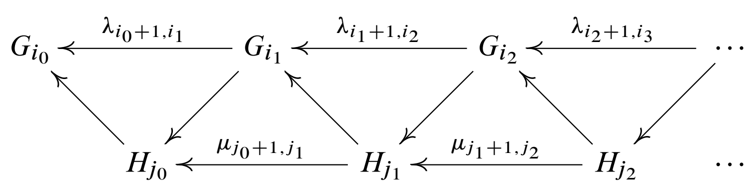
$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence of groups and homomorphisms. A *subsequence* is an inverse sequence of the form

$$G_{i_0} \xleftarrow{\lambda_{i_0+1}\circ\cdots\circ\lambda_{i_1}} G_{i_1} \xleftarrow{\lambda_{i_1+1}\circ\cdots\circ\lambda_{i_2}} G_{i_2} \xleftarrow{\lambda_{i_2+1}\circ\cdots\circ\lambda_{i_3}} \cdots .$$

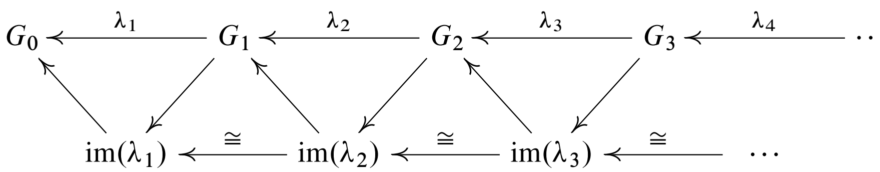
In the future we denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \le j)$ by $\lambda_{i,j}$.

Sequences $\{G_i, \lambda_i\}$ and $\{H_i, \mu_i\}$ are *pro-isomorphic* if, after passing to subsequences, there exists a commuting diagram:



Clearly an inverse sequence is pro-isomorphic to each of its subsequences. To avoid tedious notation, we often do not distinguish $\{G_i, \lambda_i\}$ from its subsequences. Instead we assume $\{G_i, \lambda_i\}$ has the properties of a preferred subsequence — prefaced by the words "after passing to a subsequence and relabeling".

An inverse sequence $\{G_i, \lambda_i\}$ is *stable* if it is pro-isomorphic to an $\{H_i, \mu_i\}$ for which each μ_i is an isomorphism. A more usable formulation is that $\{G_i, \lambda_i\}$ is stable if, after passing to a subsequence and relabeling, there is a commutative diagram of the form

(∗) 

where all unlabeled maps are obtained by restriction. If $\{H_i, \mu_i\}$ can be chosen so that each μ_i is an epimorphism, we call our sequence *semistable* (or *Mittag-Leffler*, or *pro-epimorphic*). In that case, it can be arranged that the maps in the bottom row of (∗) are epimorphisms. Similarly, if $\{H_i, \mu_i\}$ can be chosen so that each μ_i is a monomorphism, we call our sequence *pro-monomorphic*; it can then be arranged that the restriction maps in the bottom row of $(*)$ are monomorphisms. It is easy to show that an inverse sequence that is semistable and pro-monomorphic is stable.

An inverse sequence is *perfectly semistable* if it is pro-isomorphic to an inverse sequence

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

of finitely presentable groups and surjections, where each $\ker(\lambda_i)$ is perfect. A straightforward argument [Guilbault 2000, Corollary 1] shows that sequences of this type behave well under passage to subsequences.

Augmented inverse sequences and almost perfect semistability. An *augmentation* of an inverse sequence $\{G_i, \lambda_i\}$ is a sequence $\{L_i\}$, where $L_i \trianglelefteq G_i$ and $\lambda_i(L_i) \leq L_{i-1}$ for each i. The corresponding *augmentation sequence* is the sequence $\{L_i, \lambda|_{L_i}\}$.

The *minimal augmentation* (or the *unaugmented case*) occurs when $L_i = \{1\}$; the *maximal augmentation* is the case where $L_i = G_i$; and the *standard augmentation* occurs when $L_i = \ker \lambda_i$ for each i. Any augmentation where $L_i \leq \ker \lambda_i$ for each i is called a *small augmentation*. For each subsequence $\{G_{k_i}\}$ of a sequence $\{G_i, \lambda_i\}$ augmented by $\{L_i\}$, there is a corresponding augmentation $\{L_{k_i}\}$.

We say that $\{G_i, \lambda_i\}$ satisfies the *$\{L_i\}$-perfectness property* if, for each i, $\ker \lambda_i$ is $\lambda_i^{-1}(L_{i-1})$-perfect; it satisfies the *strong $\{L_i\}$-perfectness property* if each $\ker \lambda_i$ is strongly $\lambda_i^{-1}(L_{i-1})$-perfect. More concisely, if $K_i = \ker \lambda_i$ and $J_i = \lambda_i^{-1}(L_{i-1})$, these conditions require that each K_i be [strongly] J_i-perfect.

Employing the above terminology, we can restate the definition of perfect semistability (abbreviated *$\mathcal{P}$-semistable*) by requiring that the sequence be pro-isomorphic to an inverse sequence of finitely presented groups and surjections satisfying the $\{L_i\}$-perfectness property for the minimal augmentation $\{L_i\} = \{1\}$. More generally, we call an inverse sequence of groups

- *$\mathcal{AP}$-semistable* (for almost perfectly semistable) if it is pro-isomorphic to an inverse sequence $\{G_i, \lambda_i\}$ of finitely presentable groups and surjections, satisfying the $\{L_i\}$-perfectness property for some small augmentation $\{L_i\}$, and
- *$\mathcal{SAP}$-semistable* (for strongly almost perfectly semistable) if it is pro-isomorphic to an inverse sequence $\{G_i, \lambda_i\}$ of finitely presentable groups and surjections satisfying the strong $\{L_i\}$-perfectness property for some small augmentation $\{L_i\}$.

Remark 2.10. Note that an inverse sequence satisfies the [strong] $\{L_i\}$-perfectness property for some small augmentation $\{L_i\}$ if and only if it satisfies that property for the standard augmentation.

When applying sequences of the above types to geometric constructions, it is frequently desirable to pass to subsequences without losing the defining property of the sequence. For that reason, the following observation is crucial.

Proposition 2.11. *If an inverse sequence* $\{G_i, \lambda_i\}$ *of surjections augmented by* $\{L_i\}$ *satisfies the* [*strong*] $\{L_i\}$*-perfectness property, then any subsequence* $\{G_{k_i}\}$ *satisfies the corresponding* [*strong*] $\{L_{k_i}\}$*-perfectness property.*

Proof. Since the proofs for perfectness and strong perfectness are similar, we prove only the latter. Assume $\{G_i, \lambda_i\}$ augmented by $\{L_i\}$ satisfies strong $\{L_i\}$-perfectness. Simplifying notation, a portion of the given subsequence becomes

$$G_a \xleftarrow{\lambda_{a+1,b}} G_b \xleftarrow{\lambda_{b+1,c}} G_c,$$

where $-1 \leq a < b < c$. We must show that

$$\ker(\lambda_{b+1,c}) \subseteq [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)].$$

Suppose the proposition holds for $j < c$. If $c = b + 1$, then $\lambda_{b+1,c} = \lambda_c$, and the result follows by hypothesis. Now, assume $c \geq b + 2$ and write

$$\lambda_{b+1,c} = \lambda_{b+1,c-1} \circ \lambda_c : G_c \to G_{c-1} \to G_b.$$

Let $\omega \in \ker(\lambda_{b+1,c})$; then $\lambda_c(\omega) \in \ker(\lambda_{b+1,c-1})$. By induction, $\ker(\lambda_{b+1,c-1}) \subseteq [\ker(\lambda_{b+1,c-1}), \lambda_{b+1,c-1}^{-1}(L_b)]$; so, $\lambda_c(\omega)$ is a product of commutators $[\alpha_m, \beta_m]$, where $\beta_m \in \lambda_{b+1,c-1}^{-1}(L_b)$ and $\alpha_m \in \ker(\lambda_{b+1,c-1})$. Since λ_c is surjective over G_{c-1} we identify for each m a pair of elements $\alpha'_m, \beta'_m \in G_c$ that map to α_m and β_m, respectively. Thus, $\beta'_m \in \lambda_{b+1,c}^{-1}(L_b)$, $\alpha'_m \in \ker(\lambda_{b+1,c})$, and $[\alpha'_m, \beta'_m] \in [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)]$.

Now, let ν be the product of the commutators with $[\alpha'_m, \beta'_m]$ replacing $[\alpha_m, \beta_m]$. By construction, $\lambda_c(\omega) = \lambda_c(\nu)$ and $\nu \in [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)]$. Thus,

$$\omega\nu^{-1} \in \ker(\lambda_c) \subseteq [\ker(\lambda_c), \lambda_c^{-1}(L_{c-1})] \subseteq [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)].$$

Consequently, $\omega \in [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)]$ as well. □

Topology of ends of manifolds. Next we supply some topological definitions and background. Throughout the paper, $\approx$ represents homeomorphism and $\simeq$ indicates homotopic maps or homotopy equivalent spaces. The word *manifold* means *manifold with* (*possibly empty*) *boundary*. A manifold is *open* if it is noncompact and has no boundary. As noted earlier, we restrict our attention to manifolds with compact boundaries.

For convenience, all manifolds are assumed to be PL; analogous results may be obtained for smooth or topological manifolds in the usual ways. Our standard resource for PL topology is [Rourke and Sanderson 1972]. Some of the results presented here are valid in all dimensions. Others are valid in dimensions ≥ 4 or ≥ 5, but require the purely topological 4-dimensional techniques found in [Freedman and Quinn 1990] for the 4- and/or 5-dimensional cases; there the conclusions are only topological. The main focus of this paper is on dimensions ≥ 6.

Let M^n be a manifold with compact (possibly empty) boundary. A set $N \subseteq M^n$ is a *neighborhood of infinity* if $\overline{M^n - N}$ is compact. A neighborhood of infinity N is *clean* if

- N is a closed subset of M^n,
- $N \cap \partial M^n = \varnothing$, and
- N is a codimension 0 submanifold of M^n with bicollared boundary.

It is easy to see that each neighborhood of infinity contains a clean neighborhood of infinity.

We say that M^n *has k ends* if it contains a compactum C such that, for every compactum D with $C \subseteq D$, $M^n - D$ has exactly k unbounded components, i.e., k components with noncompact closures. When k exists, it is uniquely determined; if k does not exist, we say M^n *has infinitely many ends*. If M^n is k-ended, then it contains a clean neighborhood of infinity N consisting of k connected components, each of which is a 1-ended manifold with compact boundary. Thus, when studying manifolds with finitely many ends, it suffices to understand the 1-ended situation. That is the case in this paper, where our standard hypotheses ensure finitely many ends. (See Theorem 3.1.)

A connected clean neighborhood of infinity with connected boundary is called a *0-neighborhood of infinity*. A 0-neighborhood of infinity N for which $\partial N \hookrightarrow N$ induces a π_1-isomorphism is called a *generalized 1-neighborhood of infinity*. If, in addition, $\pi_j(N, \partial N) = 0$ for $j \leq k$, then N is a *generalized k-neighborhood of infinity*.

A nested sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of neighborhoods of infinity is *cofinal* if $\bigcap_{i=0}^{\infty} N_i = \varnothing$. We will refer to any cofinal sequence $\{N_i\}$ of closed neighborhoods of infinity with $N_{i+1} \subseteq \operatorname{int} N_i$, for all i, as an *end structure* for M^n. Descriptors will be added to indicate end structures with additional properties. For example, if each N_i is clean we call $\{N_i\}$ a *clean end structure*; if each N_i is clean and connected we call $\{N_i\}$ a *clean connected end structure*; and if each N_i is a generalized k-neighborhood of infinity, we call $\{N_i\}$ a *generalized k-neighborhood end structure*.

Remark 2.12. The word "generalized" in the above definitions is in deference to the terminology in [Siebenmann 1965], where the ambient manifold M^n is assumed to have stable fundamental group at infinity. There a (nongeneralized) k-neighborhood of infinity N is also required to satisfy $\pi_1(\varepsilon(M^n)) \xrightarrow{\cong} \pi_1(N)$.

Building upon the above terminology, the primary goal of this paper is to identify, construct, and detect the existence of various end structures for manifolds. A central example: the *pseudocollar* can be described as an end structure $\{N_i\}$ where each N_i is a homotopy collar.

We say M^n is *inward tame* if, for arbitrarily small neighborhoods of infinity N, there exist homotopies $H : N \times [0,1] \to N$ such that $H_0 = \mathrm{id}_N$ and $\overline{H_1(N)}$ is compact. Thus inward tameness means each neighborhood of infinity can be pulled into a compact subset of itself. In this case we refer to H as a *taming homotopy*.

In [Guilbault 2000], the existence of generalized $(n-3)$-neighborhood end structures is shown for all inward tame M^n $(n \geq 5)$.

Recall that a space X is *finitely dominated* if there exists a finite complex K and maps $u : X \to K$ and $d : K \to X$ such that $d \circ u \simeq \mathrm{id}_X$. The following lemma uses this notion to offer equivalent formulations of inward tameness.

Lemma 2.13 [Guilbault and Tinsley 2003, Lemma 2.4]. *For a manifold M^n, the following are equivalent.*

(1) *M^n is inward tame.*

(2) *Each clean neighborhood of infinity in M^n is finitely dominated.*

(3) *For each clean end structure $\{N_i\}$, the inverse sequence*

$$N_0 \xleftarrow{j_1} N_1 \xleftarrow{j_2} N_2 \xleftarrow{j_3} \cdots$$

is pro-homotopy equivalent to an inverse sequence of finite polyhedra.

Given a clean connected end structure $\{N_i\}_{i=0}^{\infty}$, basepoints $p_i \in N_i$, and paths $\alpha_i \subseteq N_i$ connecting p_i to p_{i+1}, we obtain an inverse sequence:

$$\pi_1(N_0, p_0) \xleftarrow{\lambda_1} \pi_1(N_1, p_1) \xleftarrow{\lambda_2} \pi_1(N_2, p_2) \xleftarrow{\lambda_3} \cdots .$$

Here, each $\lambda_{i+1} : \pi_1(N_{i+1}, p_{i+1}) \to \pi_1(N_i, p_i)$ is the homomorphism induced by inclusion followed by the change-of-basepoint isomorphism determined by α_i. The singular ray obtained by piecing together the α_i is called the *base ray* for the inverse sequence. Provided the sequence is semistable, its pro-isomorphism class does not depend on any of the choices made above (see [Guilbault 2016] or [Geoghegan 2008, §16.2]). In the absence of semistability, the pro-isomorphism class of the inverse sequence depends on the base ray; hence, the ray becomes part of the data. The same procedure may be used to define $\pi_k(\varepsilon(M^n))$ for all $k \geq 1$. Similarly, but without need for a base ray or connectedness, we may define $H_k(\varepsilon(M^n))$.

Wall [1965] showed that each finitely dominated connected space X determines a well-defined $\sigma(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1 X])$ (the reduced projective class group) that vanishes if and only if X has the homotopy type of a finite complex. Given a clean connected end structure $\{N_i\}_{i=0}^{\infty}$ for an inward tame M^n, we have a Wall finiteness obstruction $\sigma(N_i)$ for each i. These may be combined into a single obstruction

$$\sigma_\infty(M^n) = (-1)^n(\sigma(N_0), \sigma(N_1), \sigma(N_2), \dots) \\ \in \widetilde{K}_0(\pi_1(\varepsilon(M^n))) \equiv \varprojlim \widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$$

that is well defined and which vanishes if and only if each clean neighborhood of infinity in M^n has finite homotopy type. See [Chapman and Siebenmann 1976] or [Guilbault 2000] for details.

We now state the full version of the main theorem of [Guilbault and Tinsley 2006].

Theorem 2.14 (pseudocollarability characterization — complete version). *A* 1*-ended n-manifold* M^n ($n \geq 6$) *with compact boundary is pseudocollarable if and only if*

(1) M^n *is inward tame*,

(2) $\pi_1(\varepsilon(M^n))$ *is* $\mathcal{P}$*-semistable*, *and*

(3) $\sigma_\infty(M^n) = 0 \in \widetilde{K}_0(\pi_1(\varepsilon(M^n)))$.

3. Some consequences of inward tameness

In this section we show that, for manifolds with compact boundary, the inward tameness condition, by itself, has significant implications. The main goal is a proof of Theorem 1.2 — that every inward tame manifold with compact boundary has $\mathcal{AP}$-semistable fundamental group at each of its finitely many ends. Results in this section are valid in all (finite) dimensions and build upon an earlier theorem.

Theorem 3.1 [Guilbault and Tinsley 2003]. *If an n-manifold with compact* (*possibly empty*) *boundary is inward tame*, *then it has finitely many ends*, *each of which has semistable fundamental group and stable homology in all dimensions.*

Remark 3.2. Note that *none* of the above conclusions is valid for Hilbert cube manifolds, polyhedra, or manifolds with noncompact boundary. See, for example, [Guilbault 2016, §4.5].

As preparation for the proof of Theorem 1.2, we look at an easier result that follows directly from Theorem 3.1.

Let M^n be an inward tame n-manifold with compact boundary. Since M^n is finite-ended, there is no loss of generality in assuming that M^n is 1-ended. By taking a product with $\mathbb{S}^k$ ($k \geq 2$) if necessary, we may arrange that $n \geq 6$, without changing the fundamental group at infinity. So, by the semistability conclusion of Theorem 3.1 combined with the generalized 1-neighborhood theorem [Guilbault 2000, Theorem 4], we may choose a generalized 1-neighborhood end structure

$\{N_i\}$ for which each bonding map in the inverse sequence

$$\pi_1(N_0, p_0) \xleftarrow{\lambda_1} \pi_1(N_1, p_1) \xleftarrow{\lambda_2} \pi_1(N_2, p_2) \xleftarrow{\lambda_3} \cdots \tag{3-1}$$

is surjective. Abelianization gives an inverse sequence

$$H_1(N_0) \xleftarrow{\lambda_{1*}} H_1(N_1) \xleftarrow{\lambda_{2*}} H_1(N_2) \xleftarrow{\lambda_{3*}} \cdots, \tag{3-2}$$

which, by Theorem 3.1, is stable. It follows that all but finitely many of the epimorphisms in (3-2) are isomorphisms, so by omitting finitely many terms (then relabeling), we may assume all bonds in (3-2) are isomorphisms. A term-by-term application of Lemma 2.5 gives the following.

Proposition 3.3. *Every* 1*-ended inward tame manifold* M^n *with compact boundary admits a generalized* 1*-neighborhood end structure* $\{N_i\}$ *for which all bonding maps in the sequence* $\{\pi_1(N_i, p_i), \lambda_i\}$ *are surjective and each* $\ker \lambda_i$ *is* $\pi_1(N_i, p_i)$*-perfect*; *in other words*, *if* $\{L_i = \pi_1(N_i, p_i)\}$ *is the maximal augmentation*, *then* $\{\pi_1(N_i, p_i), \lambda_i\}$ *satisfies the* $\{L_i\}$*-perfectness property.*

Theorem 1.2 is a stronger version of Proposition 3.3. For clarity, we restate it in a similar form.

Proposition 3.4. *Every* 1*-ended inward tame manifold* M^n *with compact boundary admits a generalized* 1*-neighborhood end structure* $\{N_i\}$ *for which all bonding maps in the sequence* $\{\pi_1(N_i, p_i), \lambda_i\}$ *are surjective and*, *if we let* $K_i = \ker \lambda_i$ *for each* $i \geq 1$ (*the standard augmentation*), *then* K_i *is* $\lambda_i^{-1}(K_{i-1})$*-perfect for all* $i \geq 2$. *In other words*, $\{\pi_1(N_i, p_i), \lambda_i\}$ *satisfies the* $\{K_i\}$*-perfectness property*; *so* M^n *has* $\mathcal{AP}$*-semistable fundamental group at infinity.*

Proof. Assume the sequence $\{N_i\}$ was chosen so that, for each i, N_{i+1} is sufficiently small that a taming homotopy H^i pulls N_i into $A_i = N_i - \operatorname{int} N_{i+1}$, i.e., $\overline{H_1^i(N_i)} \subseteq A_i$, and N_{i+3} is sufficiently small that $H^i(\partial N_{i+2} \times [0,1]) \cap N_{i+3} = \varnothing$. By compactness of $\overline{H_1^i(N_i)}$ and $H^i(\partial N_{i+2} \times [0,1])$ those choices can be made.

Now let $i \geq 2$ be fixed and $q_{i-2} : \widetilde{N}_{i-2} \to N_{i-2}$ be the universal covering projection. Let $\widetilde{A}_{i-2} = q_{i-2}^{-1}(A_{i-2})$ and for $j > i-2$, $\widehat{N}_j = q_{i-2}^{-1}(N_j)$ and $\widehat{A}_j = p_{i-2}^{q-1}(A_j)$. Then

$$\widetilde{N}_{i-2} \supset \widehat{N}_{i-1} \supset \widehat{N}_i \supset \widehat{N}_{i+1};$$

and H^{i-2} lifts to a proper homotopy $\widetilde{H}^{i-2}$ that pulls $\widetilde{N}_{i-2}$ into $\widetilde{A}_{i-2}$ and for which $\widetilde{H}^i(\partial \widehat{N}_i \times [0,1])$ misses $\widehat{N}_{i+1}$.

We may associate $\lambda_i^{-1}(K_{i-1})$ with $\pi_1(\widehat{N}_i)$ and K_i with $\ker(\pi_1(\widehat{N}_i) \to \pi_1(\widehat{N}_{i-1}))$. Thus, an arbitrary element of K_i may be viewed as a loop α in $\partial \widehat{N}_i$ that bounds a disk D in $\widehat{A}_{i-1}$. To prove the proposition, it suffices to show that α bounds an orientable surface in $\widehat{N}_i$. By π_1-surjectivity and the fact that the N_j are generalized 1-neighborhoods, α may be homotoped within $\widehat{A}_i$ to a loop α_0 in $\partial \widehat{N}_{i+1}$. Let E be the cylinder in $\widehat{A}_i$ between α and α_0 traced out by that homotopy. Then the disk $D \cup E$ may be viewed as an element $[\beta] \in H_2(\widehat{A}_i \cup \widehat{A}_{i-1}, \partial\widehat{N}_{i+1})$. Let

$$\widehat{f} : \partial\widehat{N}_i \times [0,1] \cup_{\partial\widehat{N}_i \times \{0\}} \widehat{A}_i \to \widetilde{A}_{i-2} \cup \widehat{A}_{i-1} \cup \widehat{A}_i$$

be the identity on $\widehat{A}_i$ and $\widetilde{H}^{i-2}|$ on $\partial\widehat{N}_i \times [0,1]$. By PL transversality theory (see [Rourke and Sanderson 1968] or [Buoncristiano et al. 1976, §II.4]), we may — after a small proper adjustment that does not alter $\widehat{f}$ on $(\partial\widehat{N}_i \times \{0,1\}) \cup \widehat{A}_i$ — assume that $\widehat{f}^{-1}(\widehat{A}_{i-1} \cup \widehat{A}_i)$ is a manifold with boundary that is a homeomorphism over a collar neighborhood of $\partial\widehat{N}_{i+1}$. Let $\widehat{C}$ be the component of $\widehat{f}^{-1}(\widehat{A}_{i-1} \cup \widehat{A}_i)$ containing that neighborhood. Then, by local characterization of degree, $\widehat{f}| : \widehat{C} \to \widehat{A}_{i-1} \cup \widehat{A}_i$ is a proper degree 1 map, and $\widehat{f}|^{-1}(\partial\widehat{N}_{i+1}) = \partial\widehat{N}_{i+1}$. Thus we have a surjection

$$\widehat{f}|_* : H_2(\widehat{C}, \partial\widehat{N}_{i+1}) \to H_2(\widehat{A}_i \cup \widehat{A}_{i-1}, \partial\widehat{N}_{i+1}).$$

Let $[\beta']$ be a preimage of $[\beta]$. We may assume that β' is an orientable surface with boundary in $\widehat{C}$. Since $\widehat{f}$ is the identity on $\partial\widehat{N}_{i+1}$, $\partial\beta'$ is homologous in $\partial\widehat{N}_{i+1}$ to $\partial\beta = \alpha_0$. Without loss of generality, we may assume that $\partial\beta' = \alpha_0$. Since $\widehat{C}$ lies in $\partial\widehat{N}_i \times [0,1] \cup_{\partial\widehat{N}_i \times \{0\}} \widehat{A}_i$, we may push β', rel boundary, into $\widehat{A}_i$. This provides an orientable surface in $\widehat{A}_i$ with boundary α_0. Gluing the cylinder E to that surface along α_0 produces the bounding surface for α that we desire. □

Early attempts to prove $\mathcal{P}$-semistability (hence pseudocollarability) with only an assumption of absolute inward tameness were brought to a halt by the discovery of a key example presented in [Guilbault and Tinsley 2003]. Ideas contained in that example play an important role here, so we provide a quick review.

An easy way to denote normal subgroups will be helpful. Let G be a group and $S \subseteq G$. The *normal closure of S in G* is the smallest normal subgroup of G containing S. We denote it by $\operatorname{ncl}(S, G)$.

Example 3.5 (main example from [Guilbault and Tinsley 2003]). For all $n \geq 6$, there exist 1-ended absolutely inward tame open n-manifolds with fundamental group system

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots,$$

where

$$G_i = \langle a_0, a_1, \ldots, a_i \mid a_1 = [a_1, a_0],\ a_2 = [a_2, a_1],\ \ldots,\ a_i = [a_i, a_{i-1}] \rangle$$

and λ_i sends a_j to a_j for $0 \leq j \leq i-1$ and a_i to 1.

By a largely algebraic argument, it was shown that these examples do not have $\mathcal{P}$-semistable fundamental group at infinity, and thus are not pseudocollarable. Notice, however, that each $K_i = \ker \lambda_i$ is the normal closure of a_i and $a_i = [a_i, a_{i-1}]$ in G_i; so $K_i \trianglelefteq [K_i, \lambda_i^{-1}(K_{i-1})]$. In other words, $\{G_i, \lambda_i\}$ satisfies the strong $\{K_i\}$-perfectness property, and is therefore $\mathcal{SAP}$-semistable.

In addition to the above algebra, these examples have nice topological properties. Although they do not contain small homotopy collar neighborhoods of infinity, they do contain arbitrarily small generalized 1-neighborhoods of infinity N for which $\partial N \hookrightarrow N$ is $\mathbb{Z}$-homology equivalence. In fact, they contain a sequence $\{N_i\}$ of generalized 1-neighborhoods of infinity with $\pi_1(N_i) \cong G_i$ and $\partial N_i \hookrightarrow N_i$ a $\mathbb{Z}[G_{i-1}]$-homology equivalence.

The observation in Example 3.5 provides much of the motivation for the remainder of this paper.

4. Generalizing one-sided h-cobordisms, homotopy collars and pseudocollars

We begin developing ideas for placing Example 3.5 into a general context. We will see that end structures like those found in that example are possible only when kernels satisfy a strong relative perfectness condition. Conversely, we will show that whenever such a group-theoretic condition is present, a corresponding "near pseudocollar" structure is attainable.

We have already defined a pseudocollar structure on the end of a manifold M^n to be an end structure $\{N_i\}$ for which each N_i is a homotopy collar, i.e., each $\partial N_i \hookrightarrow N_i$ is a homotopy equivalence. The existence of such a structure allows us to express each N_i as a union

$$N_i = W_i \cup W_{i+1} \cup W_{i+2} \cup \cdots ,$$

where $W_i = N_i - \operatorname{int} N_{i+1}$, and each triple $(W_i, \partial N_i, \partial N_{i+1})$ is a compact *one-sided h-cobordism* in the sense that $\partial N_i \hookrightarrow W_i$ is a homotopy equivalence (and $\partial N_{i+1} \hookrightarrow W_i$ is probably not). One-sided cobordisms play an important role in manifold topology in general, and the topology of ends in particular. See [Guilbault 2000, §4] for details. For later use, we review a few key properties of one-sided h-cobordisms. See, for example, [Guilbault and Tinsley 2003, Theorem 2.5].

Theorem 4.1. *Let (W, P, Q) be a compact cobordism between closed manifolds with $P \hookrightarrow W$ a homotopy equivalence. Then*

(1) *$P \hookrightarrow W$ and $Q \hookrightarrow W$ are $\mathbb{Z}[\pi_1(W)]$-homology equivalences, i.e.,*

$$H_*(W, P; \mathbb{Z}[\pi_1(W)]) = 0 = H_*(W, Q; \mathbb{Z}[\pi_1(W)]);$$

(2) *$\pi_1(Q) \to \pi_1(W)$ is surjective; and*

(3) *$K = \ker(\pi_1(Q) \to \pi_1(W))$ is perfect.*

Moving forward, we require generalizations of the fundamental concepts of homotopy equivalence, homotopy collar, one-sided h-cobordism and pseudocollar:

- Let (X, A) be a CW pair for which $i : A \hookrightarrow X$ induces a π_1-isomorphism and let $L \trianglelefteq \pi_1(A)$. Call i a (mod L)-*homotopy equivalence* if $H_*(X, A; \mathbb{Z}[\pi_1(A)/L])$

is zero for all $*$. Extension to arbitrary maps is accomplished by use of mapping cylinders.

- A manifold N with compact boundary is a $(\operatorname{mod} L)$-*homotopy collar* if $L \trianglelefteq \pi_1(\partial N)$ and $\partial N \hookrightarrow N$ is a $(\operatorname{mod} L)$-homotopy equivalence.
- Let (W, P, Q) be a compact cobordism between closed manifolds and $L \trianglelefteq \pi_1(W)$. We call (W, P, Q) a $(\operatorname{mod} L)$-*one-sided h-cobordism* if $i : P \hookrightarrow W$ is a $(\operatorname{mod} L)$-homotopy equivalence and $j : Q \hookrightarrow W$ induces a surjection on fundamental groups.
- Let $\{N_i\}$ be a generalized 1-neighborhood end structure on a manifold M^n, chosen so that the bonding maps in

$$\pi_1(N_0) \xleftarrow{\lambda_1} \pi_1(N_1) \xleftarrow{\lambda_2} \pi_1(N_2) \xleftarrow{\lambda_3} \cdots .$$

are surjective, and let $\{L_i\}$ be an augmentation of this sequence. Call $\{N_i\}$ a $\operatorname{mod}(\{L_i\})$ *pseudocollar structure* if each $\partial N_i \hookrightarrow N_i$ is a $(\operatorname{mod} L_i)$-homotopy equivalence.

Remark 4.2. (i) Each of the above definitions reduces to its traditional counterpart when the subgroup(s) involved are trivial.

(ii) In the generalization of one-sided h-cobordism, we *require* $j_\# : \pi_1(Q) \to \pi_1(W)$ to be surjective — a condition that is automatic when $L = \{1\}$, but not in general. Analogs of the other two assertions of Theorem 4.1 will be shown to follow.

(iii) For the maximal augmentation, the generalization of pseudocollar requires only that each $\partial N_i \hookrightarrow N_i$ be a $\mathbb{Z}$-homology equivalence, whereas, for the trivial augmentation, we have a genuine pseudocollar. The key dividing line between those extremes occurs when $\{L_i\}$ is a small augmentation ($L_i \leq \ker \lambda_i$ for all i). In those cases, we call $\{N_i\}$ a *near pseudocollar* structure, and say that a 1-ended M^n with compact boundary is *nearly pseudocollarable* if it admits such a structure. The geometric significance of the small augmentation requirement will become clear in the proof of Theorem 5.1. Further discussion of that topic is contained in Section 7.

The following lemma adds topological meaning to the definition of $(\operatorname{mod} L)$-homotopy equivalence.

Lemma 4.3. *Let (X, A) be a CW pair for which $i : A \hookrightarrow X$ induces a π_1-isomorphism, $L \trianglelefteq \pi_1(A)$, and $S \subseteq L$ for which $\operatorname{ncl}(S, \pi_1(A)) = L$. Obtain A' from A by attaching a 2-disk D_s along each $s \in S$; let $X' = X \cup \left(\bigcup_{s \in S} D_s\right)$, and $i' : A' \hookrightarrow X'$. Then i is a $(\operatorname{mod} L)$-homotopy equivalence if and only if i' is a homotopy equivalence.*

Proof. Let $p : \widehat{X} \to X$ be the covering projection corresponding to L. Then $\widehat{A} = p^{-1}(A)$ is the cover of A corresponding to L. Viewing S as a collection of loops in A and $\widehat{S}$ the set of all lifts of those loops, then attaching 2-disks to $\widehat{A}$ (and simultaneously $\widehat{X}$) along $\widehat{S}$ produces universal covers $\widetilde{A}'$ of A' and $\widetilde{X}'$ of X'.

Assume now that $i : A \hookrightarrow X$ is a $(\mathrm{mod}\, L)$-homotopy equivalence. Then by Shapiro's lemma [Davis and Kirk 2001, p. 100], $H_*(\widehat{X}, \widehat{A}; \mathbb{Z}) = 0$, so by excision $H_*(\widetilde{X}', \widetilde{A}'; \mathbb{Z}) = 0$. Because both spaces are simply connected, the relative Hurewicz theorem implies that $\pi_*(\widetilde{X}', \widetilde{A}') = 0$; therefore $\pi_*(X', A') = 0$. By Whitehead's theorem i' is a homotopy equivalence.

Conversely, if i' is a homotopy equivalence, then its lift $\widetilde{A}' \hookrightarrow \widetilde{X}'$ is a homotopy equivalence. Therefore $H_*(\widetilde{X}', \widetilde{A}'; \mathbb{Z}) = 0$, so by excision $H_*(\widehat{X}, \widehat{A}; \mathbb{Z}) = 0$, and by Shapiro's lemma $H_*(X, A; \mathbb{Z}[\pi_1(A)/L]) = 0$. □

The following is a useful corollary.

Lemma 4.4. *Let (X, A) be a CW pair for which $i : A \hookrightarrow X$ induces a π_1-isomorphism and suppose $L \trianglelefteq \pi_1(A)$. If $H_*(X, A; \mathbb{Z}[\pi_1(A)/L]) = 0$, then $H_*(X, A; \mathbb{Z}[\pi_1(A)/J]) = 0$ for any J with $L < J \trianglelefteq \pi_1(A)$. In particular, $H_*(X, A; \mathbb{Z}) = 0$.*

The next observation is a direct analog of Theorem 4.1.

Theorem 4.5. *Let (W, P, Q) be a compact $(\mathrm{mod}\, L)$-one-sided h-cobordism between closed manifolds with $L \trianglelefteq \pi_1(W)$. Let $j : Q \hookrightarrow W$ and $L' = j_\#^{-1}(L)$. Then*

(1) *both $P \hookrightarrow W$ and $Q \hookrightarrow W$ are $\mathbb{Z}[\pi_1(W)/L]$-homology equivalences, i.e.,*

$$H_*(W, P; \mathbb{Z}[\pi_1(W)/L]) = 0 = H_*(W, Q; \mathbb{Z}[\pi_1(W)/L]);$$

and

(2) *$K = \ker j_\# \trianglelefteq \pi_1(Q)$ is strongly L'-perfect.*

Proof. First note that by the surjectivity of $j_\# : \pi_1(Q) \to \pi_1(W)$, there is a canonical isomorphism $\pi_1(Q)/L' \xrightarrow{\cong} \pi_1(W)/L$ that is assumed throughout. Let $p : \widehat{W}_L \to W$ be the covering projection corresponding to L, $\widehat{P} = p^{-1}(P)$ and $\widehat{Q} = p^{-1}(Q)$. Then both $\widehat{P}$ and $\widehat{Q}$ are connected, and their projections onto P and Q are the coverings corresponding to L and L'.

The assertion that $H_*(W, P; \mathbb{Z}[\pi_1(W)/L]) = 0$ is part of the hypothesis, and (by Shapiro's lemma [Davis and Kirk 2001, p. 100]) equivalent to the assumption that $H_*(\widehat{W}_L, \widehat{P}; \mathbb{Z}) = 0$. To show that $H_*(W, Q; \mathbb{Z}[\pi_1(W)/L])$ vanishes in all dimensions, it suffices to show that $H_*(\widehat{W}_L, \widehat{Q}; \mathbb{Z}) = 0$. This will follow from Poincaré duality for noncompact manifolds if we can verify:

Claim. $H_f^*(\widehat{W}_L, \widehat{P}; \mathbb{Z}) = 0$, *where the f indicates cellular cohomology based on finite cochains.* (See [Geoghegan 2008, Chapter 12].)

Applying Lemma 4.3, attach 2-cells to W along a collection S of loops in P to kill L, obtaining spaces P' and W', and a homotopy equivalence $P' \hookrightarrow W'$.

Since W is compact, any strong deformation retraction of W' onto P' is proper, and hence, lifts to a proper strong deformation retraction of universal covers $\widetilde{W}'$ onto $\widetilde{P}'$ [Geoghegan 2008, §10.1]. It follows that $H_f^*(\widehat{W}', \partial\widehat{N}'_{i-1}; \mathbb{Z}) = 0$. Both universal covers are obtained by attaching disks along the collection $\widehat{S}$ of lifts to $\widehat{P}$ and $\widehat{W}$ of the loops in S. By excising the interiors of those disks, we conclude that $H_f^*(\widehat{W}, \partial\widehat{N}; \mathbb{Z}) = 0$.

To verify assertion (2), consider the short exact sequence

$$1 \to K \to L' \xrightarrow{q} L'/K \to 1,$$

where L'/K may be identified with L. Lemma 2.6 provides the 5-term exact sequence

$$H_2(L'; \mathbb{Z}) \xrightarrow{q_{*2}} H_2(L'/K; \mathbb{Z}) \to K/[K, L'] \to H_1(L'; \mathbb{Z}) \xrightarrow{q_{*1}} H_1(L'/K; \mathbb{Z}) \to 0,$$

from which the L'-perfectness of K can be deduced by showing that q_{*2} is an epimorphism and q_{*1} an isomorphism.

Since $\widehat{Q} \hookrightarrow \widehat{W}_L$ induces $q : L' \to L$ and since $H_2(\widehat{W}_L, \widehat{Q}; \mathbb{Z}) = 0$, the long exact sequence for that pair ensures that $H_1(L'; \mathbb{Z}) \xrightarrow{\cong} H_1(L; \mathbb{Z})$. In addition, the surjectivity of $H_2(\widehat{Q}; \mathbb{Z}) \to H_2(\widehat{W}_L; \mathbb{Z})$ combines with Lemma 2.7 to imply the surjectivity of $H_2(L'; \mathbb{Z}) \to H_2(L; \mathbb{Z})$. □

5. Structure of inward tame ends

With all necessary definitions in place, we are ready to prove the second main theorem described in the introduction. We begin by stating a strong form of the theorem, written in the style of earlier characterization theorems from [Siebenmann 1965; Guilbault and Tinsley 2006].

Theorem 5.1 (near pseudocollarability characterization). *A 1-ended n-manifold M^n ($n \geq 6$) with compact boundary is nearly pseudocollarable if and only if*

(1) *M^n is inward tame,*

(2) *the fundamental group at infinity is $\mathcal{SAP}$-semistable, and*

(3) *$\sigma_\infty(M^n) = 0 \in \widetilde{K}_0(\pi_1(\varepsilon(M^n)))$.*

Recall that condition (2) calls for the existence of a representation of $\pi_1(\varepsilon(M^n)$ of the form

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots \tag{5-1}$$

with a small augmentation $\{L_i\}$ ($L_i \trianglelefteq K_i = \ker\lambda_i$ for all i) so that each K_i is strongly J_i-perfect, where $J_i = \lambda_i^{-1}(L_{i-1})$.

Proof. First we verify that a nearly pseudocollarable 1-ended manifold with compact boundary must satisfy conditions (1)–(3).

The hypothesis provides a generalized 1-neighborhood end structure $\{N_i\}$ on M^n with group data

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots \tag{5-2}$$

($G_i = \pi_1(N_i)$) and a small augmentation $\{L_i\}$ ($L_i \trianglelefteq K_i = \ker \lambda_i$) such that each N_i is a mod(L_i)-homotopy collar.

To simultaneously verify (1) and (3), it suffices to exhibit a cofinal sequence of clean neighborhoods of infinity, each having finite homotopy type. Lemma 4.4 ensures that each N_i is a mod(K_i)-homotopy collar, and since each λ_i is a surjection between finitely presented groups, each K_i is finitely generated as a normal subgroup of G_i. Let i be fixed and $A = \{\alpha_j\}$ be a finite collection of loops in ∂N_i that normally generates K_i in G_i. By Lemma 4.3, if we abstractly attach a 2-disk Δ_j^2 along each α_j, we obtain a homotopy equivalence

$$\partial N_i \cup \left(\bigcup \Delta_j^2\right) \hookrightarrow N_i \cup \left(\bigcup \Delta_j^2\right).$$

In particular, $N_i \cup \left(\bigcup \Delta_j^2\right)$ has the homotopy type of a finite complex. But, since each α_j represents an element of $\ker \lambda_i$, we may assume that each Δ_j^2 is properly embedded in $N_{i-1} - \operatorname{int} N_i$. By thickening these 2-disks to 2-handles, we obtain a clean neighborhood of infinity N_i^* with finite homotopy type, lying in N_{i-1}.

This leaves only $\mathcal{SAP}$-semistability to be checked. We will show that (5-2) satisfies the strong $\{L_i\}$-perfectness property; in other words, each K_i is strongly J_i-perfect, where $J_i = \lambda_i^{-1}(K_{i-1})$.

For each $i > 0$, let $W_{i-1} = N_{i-1} - \operatorname{int} N_i$.

Claim. $(W_{i-1}, \partial N_{i-1}, \partial N_i)$ *is a* (mod L_{i-1})*-one-sided h-cobordism.*

Fix i and let $p : \widehat{N}_{i-1} \to N_{i-1}$ be the covering corresponding to $L_{i-1} \trianglelefteq G_{i-1} = \pi_1(N_{i-1}) \cong \pi_1(W_{i-1})$. Let $\widehat{W}_{i-1}$ denote $p^{-1}(W_{i-1})$ and let $\widehat{N}_i$ denote $p^{-1}(N_i)$. Then $\widehat{W}_{i-1}$ is the cover of W_{i-1} corresponding to J_{i-1}, and $\widehat{N}_i$ is the cover of N_i corresponding to $J_i \trianglelefteq G_i = \pi_1(N_i)$. By Lemma 4.4 and Shapiro's lemma

$$0 = H_*(N_i, \partial N_i; \mathbb{Z}[G_i/J_i]) \cong H_*(\widehat{N}_i, \partial \widehat{N}_i; \mathbb{Z}),$$

and from the long exact homology sequence for the triple $(\widehat{N}_{i-1}, \widehat{W}_{i-1}, \partial \widehat{N}_{i-1})$, excision and Shapiro's lemma

$$H_*(\widehat{W}_{i-1}, \partial \widehat{N}_{i-1}; \mathbb{Z}) \cong H_*(W_{i-1}, \partial N_{i-1}; \mathbb{Z}[G_{i-1}/L_{i-1}]) = 0.$$

The claim follows.

Finally, since the bonding map $G_{i-1} \overset{\lambda_i}{\twoheadleftarrow} G_i$ is represented by the inclusion $W_{i-1} \hookleftarrow \partial N_i$, K_i is strongly J_i-perfect by Theorem 4.5.

For the converse, we must show that conditions (1)–(3) imply the existence of a near pseudocollar structure on M^n. Though the proof is rather complicated, it follows the same outline as that in [Guilbault 2000], which followed the original proof in [Siebenmann 1965]. For a full understanding, the reader should be familiar with [Guilbault 2000]. The new argument presented here generalizes the final portions of that proof. A concise review of [Guilbault 2000] can be found in [Guilbault and Tinsley 2006, §4].

In [Guilbault 2000; Guilbault and Tinsley 2006] the goal was to improve arbitrarily small neighborhoods of infinity to homotopy collars. That is impossible with our weaker hypotheses; instead, the goal is to improve neighborhoods of infinity to homotopy collars modulo certain subgroups of their fundamental groups.

By condition (2) the pro-isomorphism class of $\pi_1(\varepsilon(M^n))$ may be represented by a sequence

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots \tag{5-3}$$

of finitely presented groups, along with a small augmentation $\{L_i\}$ ($L_i \trianglelefteq K_i = \ker \lambda_i$ for all i) so that each K_i is strongly J_i-perfect, where $J_i = \lambda_i^{-1}(L_{i-1})$.

By [Guilbault 2000, Lemma 8] there is a sequence $\{N_i\}$ of generalized 1-neighborhoods of infinity whose inverse sequence of fundamental groups is isomorphic to a subsequence of $\{G_i\}$.

$$\begin{array}{ccccccccc}
G_{i_0} & \overset{\lambda_{i_0+1,i_1}}{\twoheadleftarrow} & G_{i_1} & \overset{\lambda_{i_1+1,i_2}}{\twoheadleftarrow} & G_{i_2} & \overset{\lambda_{i_2+1,i_3}}{\twoheadleftarrow} & G_{i_3} & \overset{\lambda_{i_3+1,i_4}}{\twoheadleftarrow} & \cdots \\
\updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \\
\pi_1(N_0, p_0) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \pi_1(N_1, p_1) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \pi_1(N_2, p_2) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \pi_1(N_3, p_3) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \cdots
\end{array}$$

This diagram and Proposition 2.11 ensure that, for each j, $\ker(\lambda_{i_{j-1}+1,i_j})$ is strongly $\lambda^{-1}_{i_{j-1}+1,i_j}(L_{i_{j-1}})$-perfect. So by passing to this subsequence and relabeling, we may assume that sequence (5-1) and the corresponding subgroup data match the fundamental group data of $\{N_i\}$. Note here that the J-groups (which are not viewed as part of the original data) are not the same as the previous J-groups; they are now preimages of *compositions* of the original bonding maps.

Next we inductively improve the sequence $\{N_j\}$ to generalized k-neighborhoods of infinity for increasing values of k, up to $k = n-3$. We must frequently pass to subsequences; however, each improvement of a given N_j leaves its fundamental group and that of ∂N_i intact. So at each stage, the "new" fundamental group data will be a subsequence of the original (5-1), along with the subsequence augmentation. The J-groups will change as per their definition, but, by Proposition 2.11, we always maintain the appropriate strong relative perfectness condition.

This neighborhood improvement process uses only the hypothesis that M^n is inward tame; it is identical to that used in [Guilbault 2000, Theorem 5] and outlined in [Guilbault and Tinsley 2006, Theorem 3.2]. To save on notation we relabel the neighborhood sequences and their corresponding groups at each stage, designating the resulting cofinal sequence of generalized $(n-3)$-neighborhoods of infinity by $\{N_i\}$, with $G_i = \pi_1(N_i)$, $\lambda_i : G_i \to G_{i-1}$ the corresponding homomorphism, $L_i \trianglelefteq K_i = \ker \lambda_i$, and $J_i = \lambda_i^{-1}(L_{i-1})$.

For each i, let $R_i = N_i - \overset{\circ}{N}_{i+1}$ and consider the collection of cobordisms $\{(R_i, \partial N_i, \partial N_{i+1})\}$. The following summary comprises the contents of Lemmas 11 and 12 of [Guilbault 2000], along with new hypotheses regarding kernels.

(i) Each N_i is a generalized $(n-3)$-neighborhood of infinity.

(ii) Each induced bonding map $\pi_1(N_i) \twoheadleftarrow \pi_1(N_{i+1})$ is surjective.

(iii) Each inclusion $\partial N_i \hookrightarrow R_i \hookrightarrow N_i$ induces a π_1-isomorphism.

(iv) Each $\partial N_{i+1} \hookrightarrow R_i$ induces a π_1-epimorphism with kernel strongly J_i-perfect.

(v) $\pi_k(R_i, \partial N_i) = 0$ for all $k < n-3$ and all i.

(vi) Each $(R_i, \partial N_i, \partial N_{i+1})$ admits a handle decomposition based on ∂N_i containing handles only of index $n-3$ and $n-2$.

(vii) Each N_i admits an infinite handle decomposition with handles only of index $n-3$ and $n-2$.

(viii) Each $(N_i, \partial N_i)$ has the homotopy type of a relative CW pair $(K_i, \partial N_i)$ with $\dim(K_i - \partial N_i) \le n-2$.

The obvious next goal is attempting to improve the N_i to generalized $(n-2)$-neighborhoods of infinity, which by item (viii) would necessarily be homotopy collars. In previous work [Siebenmann 1965; Guilbault 2000; Guilbault and Tinsley 2006], that is the final (and most difficult and interesting) step. The same is true here, where the weakened hypotheses create greater difficulties and the strategy and end goal must eventually be altered. For now, we continue with the earlier strategies by turning our attention to $\pi_{n-2}(N_i, \partial N_i) \cong H_{n-2}(\widetilde{N}_i, \partial \widetilde{N}_i)$, which may be viewed as a $\mathbb{Z}[\pi_1 N_i]$-module $H_{n-2}(N_i, \partial N_i; \mathbb{Z}[\pi_1 N_i])$. The content of [Guilbault 2000, Lemma 13] is given by the next two items.

(ix) $H_{n-2}(\widetilde{N}_i, \partial \widetilde{N}_i)$ is a finitely generated projective $\mathbb{Z}[\pi_1 N_i]$-module.

(x) As an element of $\widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$, $[H_{n-2}(\widetilde{N}_i, \partial \widetilde{N}_i)] = (-1)^n \sigma(N_i)$, where $\sigma(N_i)$ is the Wall finiteness obstruction for N_i.

Together, these elements of $\widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$ determine the obstruction $\sigma_\infty(\varepsilon(M^n))$ found in condition (3). From now on we assume that $\sigma_\infty(M^n)$ vanishes. This is equivalent to assuming that each $\sigma(N_i)$ is the trivial element of $\widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$, in other words, each $H_{n-2}(\widetilde{N}_i, \partial \widetilde{N}_i)$ is a stably free $\mathbb{Z}[\pi_1 N_i]$-module. Therefore:

(xi) By carving out finitely many trivial $(n-3)$-handles from each N_i, we can arrange that $H_{n-2}(\widetilde{N}_i, \partial \widetilde{N}_i)$ is a finitely generated free $\mathbb{Z}[\pi_1 N_i]$-module.

Item (xi) can be done so that these sets remain a generalized $(n-3)$-neighborhood of infinity, and so that their fundamental groups and those of their boundaries are unchanged. Again, to save on notation, we denote the improved collection by $\{N_i\}$. See [Guilbault 2000, Lemma 14] for details.

The finite generation of $H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ allows us to, after again passing to a subsequence and relabeling, assume that

(xii) $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) \twoheadrightarrow H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is surjective for each i.

The long exact sequence for the triple $(\widetilde{N}_i, \widetilde{R}_i, \partial\widetilde{N}_i)$ from there shows that

(xiii) $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) \xrightarrow{\cong} H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is an isomorphism for each i (and hence, $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i)$ is a finitely generated free $\mathbb{Z}[\pi_1 R_i]$-module).

As above, we may choose handle decompositions for the R_i based on ∂N_i having handles only of index $n-3$ and $n-2$.

From now on, let i be fixed. After introducing some trivial $(n-3, n-2)$-handle pairs, an algebraic lemma and some handle slides allow us to obtain a handle decomposition of R_i based on ∂N_i with $(n-2)$-handles $h_1^{n-2}, h_2^{n-2}, \dots, h_r^{n-2}$ and an integer $s \le r$, such that the subcollection $\{h_1^{n-2}, h_2^{n-2}, \dots, h_s^{n-2}\}$ is a free $\mathbb{Z}[\pi_1 R_i]$-basis for $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i)$. So we have:

(xiv) The $\mathbb{Z}[\pi_1 R_i]$-cellular chain complex for $(R_i, \partial N_i)$ may be expressed as

$$0 \to \langle h_1^{n-2}, \dots, h_s^{n-2}\rangle \oplus \langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle \xrightarrow{\partial} \langle h_1^{n-3}, \dots, h_t^{n-3}\rangle \to 0, \tag{5-4}$$

where

- $\langle h_1^{n-2}, \dots, h_s^{n-2}\rangle$ and $\langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle$ represent free $\mathbb{Z}[\pi_1 R_i]$-submodules of $\widetilde{C}_{n-2}$ generated by the corresponding handles;
- $\langle h_1^{n-3}, \dots, h_t^{n-3}\rangle = \widetilde{C}_{n-3}$ is the free $\mathbb{Z}[\pi_1 R_i]$-module generated by the $(n-3)$-handles in R_i;
- $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) = \ker\partial = \langle h_1^{n-2}, \dots, h_s^{n-2}\rangle \oplus \{0\}$; and
- ∂ takes $\{0\} \oplus \langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle$ injectively into $\langle h_1^{n-3}, \dots, h_t^{n-3}\rangle$.

Item (xiv) and the preceding paragraph are the content of Lemma 15 in [Guilbault 2000].

To this point, we have only used the hypotheses of inward tameness and triviality of the Wall obstruction to build the structure described by items (i)–(xiv). All arguments used thus far appear in [Guilbault 2000; Guilbault and Tinsley 2006], with simpler analogs in [Siebenmann 1965].

Under the π_1-stability hypothesis of [Siebenmann 1965], $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i)$ can now be killed by sliding the offending $(n-2)$-handles $\{h_1^{n-2}, \dots, h_s^{n-2}\}$ off the $(n-3)$-handles and carving out their interiors. Under the weaker $\mathcal{P}$-semistability hypothesis of [Guilbault and Tinsley 2006], a similar strategy works, but only after

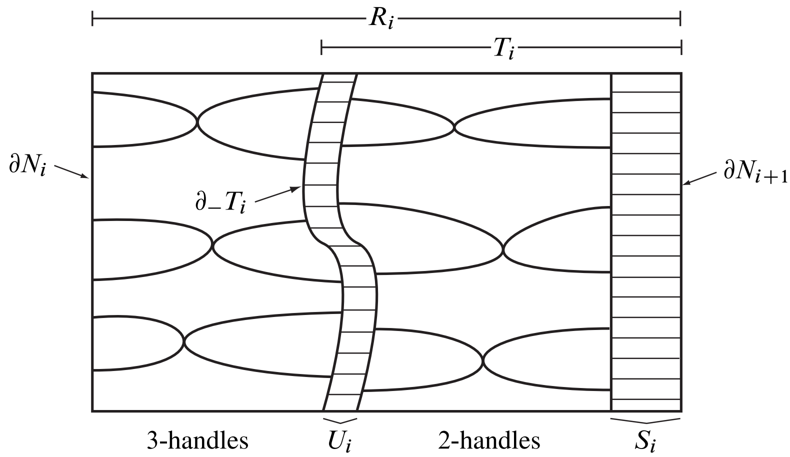


Figure 2. Schematic of R_i.

a significant preparatory step, made possible by perfect kernels. In [Guilbault 2000] an alternate strategy was employed. Instead of killing $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) = \ker\partial$ by removing its generating handles $\{h_1^{n-2}, \dots, h_s^{n-2}\}$, the task was accomplished by introducing new $(n-3)$-handles, which became images of the $\{h_1^{n-2}, \dots, h_s^{n-2}\}$ under the resulting boundary map, thereby trivializing the kernel. Complete discussions of these approaches can be found in [Guilbault and Tinsley 2006, §3] and [Guilbault 2000, §8]; the strategy employed here is based on the latter.

It is helpful to change our perspective by switching to the dual handle decomposition of R_i. Let S_i be a closed collar neighborhood of ∂N_{i+1} in R_i, and for each $(n-2)$-handle h_k^{n-2} identified earlier, let $\bar{h}_k^2$ be its dual, attached to S_i. Similarly, for each $(n-3)$-handle h_k^{n-3}, let $\bar{h}_k^3$ be its dual. As is standard, the attaching and belt spheres of a given handle switch roles in its dual.

Let $T_i = S_i \cup (\bar{h}_1^2 \cup \dots \cup \bar{h}_s^2 \cup \bar{h}_{s+1}^2 \cup \dots \cup \bar{h}_r^2)$, $\partial_- T_i = \partial T_i - \partial N_{i+1}$, and U_i be a closed collar on $\partial_- T_i$ in T_i. Observe that $R_i = T_i \cup (\bar{h}_1^3 \cup \dots \cup \bar{h}_t^3)$. See Figure 2.

A simplified view of the next step is that we will find a collection of 3-handles $\{\bar{k}_1^3, \dots, \bar{k}_s^3\}$ attached to the left-hand boundary of R_i and lying in R_{i-1} so that the collection $\{\Gamma_j^2\}_{j=1}^s$ of attaching spheres of those 3-handles is algebraically dual to the belt spheres of $\{\bar{h}_1^2, \dots, \bar{h}_s^2\}$ and has trivial algebraic intersection with the belt spheres of $\{\bar{h}_{s+1}^2, \dots, \bar{h}_r^2\}$. Adding those 3-handles to the mix, then inverting the handle decomposition again, results in a cobordism with chain complex

$$(5\text{-}5)\quad 0 \to \langle h_1^{n-2}, \dots, h_s^{n-2}\rangle \oplus \langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle \xrightarrow{\partial} \langle k_1^{n-3}, \dots, h_s^{n-3}\rangle \oplus \langle h_1^{n-3}, \dots, h_t^{n-3}\rangle \to 0$$

in which $\ker\partial = 0$ as desired — but with a caveat. Although addition of the 3-handles does not change the fundamental group of the cobordism, the arranged algebraic intersections between the attaching spheres of $\{\bar{k}_1^3, \dots, \bar{k}_s^3\}$ and the belt spheres of the existing 2-handles are $\mathbb{Z}[\pi_1(R_i)/L_i]$-intersection numbers; this is the best the hypotheses will allow. Then, to arrive at the desired conclusion — that we have effectively killed the relative second homology — it is necessary to switch the coefficient ring to $\mathbb{Z}[\pi_1(R_i)/L_i]$ (in other words, mod out by L_i), and reinterpret (5-5) as a $\mathbb{Z}[\pi_1(R_i)/L_i]$-complex. Then, letting $V_i = N_i \cup (\bar{k}_1^3 \cup \cdots \cup \bar{k}_s^3)$, it follows that

$$\pi_1(V_i) \cong \pi_1(R_i) \cong \pi_1(N_i),$$

$\partial V_i \hookrightarrow V_i$ induces a π_1-isomorphism, and $H_*(V_i, \partial V_i; \mathbb{Z}[\pi_1(R_i)/L_i]) = 0$. In other words, V_i is a mod(L_i)-homotopy collar.

In order to carry out the above program, we first identify a collection $\{\Gamma_j^2\}_{j=1}^s$ of pairwise disjoint 2-spheres in $\partial_- T_i$ algebraically dual over $\mathbb{Z}[\pi_1(R_i)/L_i]$ to the collection $\{\beta_j^{n-3}\}_{j=1}^s$ of belt spheres of the 2-handles $\{\bar{h}_1^2, \dots, \bar{h}_s^2\}$ and having trivial $\mathbb{Z}[\pi_1(R_i)/L_i]$-intersections with the belt spheres $\{\beta_j^{n-3}\}_{j=s+1}^r$ of the remaining 2-handles $\{\bar{h}_{s+1}^2, \dots, \bar{h}_r^2\}$. Keeping in mind that $\pi_1(R_i)/L_i$ is canonically isomorphic to $\pi_1(R_{i+1})/J_{i+1}$, and using the hypothesis that K_{i+1} is strongly J_{i+1}-perfect, such a collection $\{\Gamma_j^2\}_{j=1}^s$ exists, as is shown in [Guilbault and Tinsley 2013, §5]. By general position, the collection can be made disjoint from the attaching tubes of the 3-handles $\{\bar{h}_1^3, \dots, \bar{h}_t^3\}$, so they may be viewed as lying in ∂N_i. If the collection $\{\Gamma_j^2\}_{j=1}^s$ bounds a pairwise disjoint collection of embedded 3-disks in R_{i-1}, regular neighborhoods of those disks would provide the desired 3-handles, and the proof is complete. (The argument from [Guilbault 2000, §8] provides details.)

For $n \geq 7$, the issue is just whether the 2-spheres $\{\Gamma_j^2\}_{j=1}^s$ contract in R_{i-1}. (In dimension 6, a special argument is needed to get pairwise disjoint embeddings.) Contractibility is not guaranteed; but with additional work it can be arranged. The additional work involves the *spherical alteration* of 2-handles developed in [Guilbault and Tinsley 2013]. The idea is to alter the 2-handles $\{\bar{h}_1^2, \dots, \bar{h}_s^2\}$ in a planned manner so that the correspondingly altered $\{\Gamma_j^2\}_{j=1}^s$ contract in the new R_{i-1}. Along the way it will be necessary to reconstruct the 3-handles $\{\bar{h}_1^3, \dots, \bar{h}_t^3\}$ as well; for later use, let $\{\Theta_j^2\}_{j=1}^t$ denote the attaching spheres of those handles.

All of the details were carefully laid out in [Guilbault and Tinsley 2013], with this application in mind. The tailor-made lemma, stated in the final section of that paper, is repeated here.

Lemma 5.2 [Guilbault and Tinsley 2013, Lemma 6.1]. *Let $R' \subseteq R$ be a pair of n-manifolds ($n \geq 6$) with a common boundary component B, and suppose there is a subgroup L' of* $\ker(\pi_1(B) \to \pi_1(R))$ *for which* $K = \ker(\pi_1(B) \to \pi_1(R'))$ *is strongly L'-perfect. Suppose further that there is a clean submanifold $T \subseteq R'$ consisting of a finite collection $\mathcal{H}^2$ of 2-handles in R' attached to a collar neighborhood S of B with $T \hookrightarrow R'$ inducing a π_1-isomorphism (the 2-handles precisely kill* *the group K) and a finite collection $\{\Theta_t^2\}$ of pairwise disjoint embedded 2-spheres in $\partial T - B$, each of which contracts in R'.*

Then on any subcollection $\{h_j^2\}_{j=1}^k \subseteq \mathcal{H}^2$, one may perform spherical alterations to obtain 2-handles $\{\dot{h}_j^2\}_{j=1}^k$ in R' so that in $\partial\dot{T}-B$ (where $\dot{T}$ is the correspondingly altered version of T) there is a collection of 2-spheres $\{\dot{\Gamma}_j^2\}_{j=1}^k$ algebraically dual over $\mathbb{Z}[\pi_1(B)/L']$ to the belt spheres $\{\beta_j^{n-3}\}_{j=1}^k$ common to $\{h_j^2\}_{j=1}^k$ and $\{\dot{h}_j^2\}_{j=1}^k$ with the property that each $\dot{\Gamma}_j^2$ contracts in R.

Furthermore, each correspondingly altered 2-sphere $\dot{\Theta}_t^2$ (now lying in $\partial\dot{T} - B$) has the same $\mathbb{Z}[\pi_1(B)/L']$-intersection number with those belt spheres and with any other oriented $(n-3)$-manifold lying in both $\partial T - B$ and $\partial\dot{T} - B$ as did Θ_t^2. Whereas the 2-spheres $\{\Theta_t^2\}$ each contracted in R', the $\dot{\Theta}_t^2$ each contract in R.

We apply Lemma 5.2 to the current setup, with the following substitutions:

Lemma 5.2	**Current situation**
R'	R_i
R	$R_i \cup R_{i-1}$
B	∂N_{i+1}
$\mathcal{H}^2$	$\{\bar{h}_1^2, \ldots, \bar{h}_s^2, \bar{h}_{s+1}^2, \ldots, \bar{h}_r^2\}$
L'	$J_{i+1} = \lambda_{i+1}^{-1}(L_i)$
T	$T_i = S_i \cup (\bar{h}_1^2 \cup \cdots \cup \bar{h}_s^2 \cup \bar{h}_{s+1}^2 \cup \cdots \cup \bar{h}_r^2)$
$k \in \mathbb{Z}$	$s \in \mathbb{Z}$
$\{h_j^2\}_{j=1}^k$	$\{\bar{h}_j^2\}_{j=1}^s$
$\{\Gamma_j^2\}_{j=1}^k$	$\{\Gamma_j^2\}_{j=1}^s$
$\{\Theta_t^2\}$	$\{\Theta_j^2\}_{j=1}^t$

After applying this lemma, the collection $\{\bar{h}_j^2\}_{j=1}^s$ is replaced by altered versions $\{\dot{\bar{h}}_j^2\}_{j=1}^s$ and the original collection $\{\bar{h}_j^2\}_{j=s+1}^r$ is retained. Let

$$\dot{T}_i = S_i \cup \left(\dot{\bar{h}}_1^2 \cup \cdots \cup \dot{\bar{h}}_s^2 \cup \bar{h}_{s+1}^2 \cup \cdots \cup \bar{h}_r^2\right)$$

and $\partial_-\dot{T}_i = \partial\dot{T}_i - \partial N_{i+1}$. The collections $\{\Gamma_j^2\}_{j=1}^s$ and $\{\Theta_j^2\}_{j=1}^t$ are replaced by altered versions $\{\dot{\Gamma}_j^2\}_{j=1}^s$ and $\{\dot{\Theta}_j^2\}_{j=1}^t$ which lie in $\partial_-\dot{T}_i$ and contract in

$$\overline{R_i \cup R_{i-1} - \dot{T}}.$$

The original 3-handles $\{\bar{h}_j^3\}_{j=1}^t$ must be discarded since their attaching tubes have been disrupted; replacements will be constructed shortly. When $n \geq 7$, use general position to choose a pairwise disjoint collection of properly embedded 3-disks in

$$\overline{R_i \cup R_{i-1} - \dot{T}}$$

with boundaries corresponding to the 2-spheres $\{\dot{\Gamma}_j^2\}_{j=1}^k \cup \{\dot{\Theta}_t^2\}$. Those 3-disks may be thickened to 3-handles by taking regular neighborhoods. With all of these handles finally in place, the argument described earlier completes the proof. When $n = 6$, the same is true, but the π-π argument used in [Guilbault and Tinsley 2013, Theorems 4.2 and 5.3] is needed in order to find pairwise disjoint embedded 3-disks. □

Remark 5.3. In reality, we have shown a stronger result than what is stated in Theorem 5.1. Specifically, the near pseudocollar structures obtained are as close to actual pseudocollars as the augmentation is to the trivial augmentation. For example, if $\{L_i\}$ is the trivial augmentation, the above argument contains an alternative proof of the main result of [Guilbault and Tinsley 2006] (stated here as Theorem 2.14). More generally, if $\{L_i\}$ lies somewhere between the trivial augmentation and the standard augmentation, then a near pseudocollar structure on M^n can be chosen to reflect that augmentation.

6. The examples: proof of Theorem 1.4

Introduction to the examples. The main examples of [Guilbault and Tinsley 2003], described here in Example 3.5, proved the existence of (absolutely) inward tame open manifolds that are not pseudocollarable. In this section we construct open manifolds that are absolutely inward tame but not nearly pseudocollarable. Since the examples from that paper are nearly pseudocollarable, the new examples fill a gap in the spectrum of known end structures.

The examples of [Guilbault and Tinsley 2003] began with algebra. The main theorems of that paper showed that all inward tame open manifolds have pro-finitely generated, semistable fundamental group, and stable $\mathbb{Z}$-homology, at infinity. The missing ingredient for detecting a pseudocollar structure was $\mathcal{P}$-semistability. With that knowledge, an inverse sequence of groups satisfying the necessary properties, but failing $\mathcal{P}$-semistability, became the blueprint for an example. A nontrivial handle-theoretic strategy was needed to realize the examples, but the heart of the matter was the group theory.

A similar story plays out here. We will begin with an inverse sequence of finitely presented groups with surjective bonding maps that become isomorphisms upon abelianization; but this time, in light of Theorems 1.2 and 1.3, we want an $\mathcal{AP}$-semistable sequence that is not $\mathcal{SAP}$-semistable. The first step is to identify such a sequence.

Let $\mathbb{F}_3 = \langle a_1, a_2, a_3 \mid \rangle$, the free group on the three generators; $r_{1,1} = [a_2, a_3]$, $r_{1,2} = [a_1, a_3]$, and $r_{1,3} = [a_1, a_2]$; $\mathbb{A}_1 = \mathrm{ncl}(\{r_{1,1}, r_{1,2}, r_{1,3}\}, \mathbb{F}_3)$; and $G_1 = \mathbb{F}_3/\mathbb{A}_1$. Notice that $\mathbb{A}_1$ is precisely the commutator subgroup $[\mathbb{F}_3, \mathbb{F}_3]$, so $G_1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Suppose $r_{2,1} = [r_{1,2,}, r_{1,3}]$, $r_{2,2} = [r_{1,1}, r_{1,3}]$, and $r_{1,3} = [r_{1,1}, r_{1,2}]$; $\mathbb{A}_2 = \mathrm{ncl}(\{r_{2,1}, r_{2,2}, r_{2,3}\}, \mathbb{F}_3)$; and $G_2 = \mathbb{F}_3/\mathbb{A}_2$. Since $\mathbb{A}_2 \leq \mathbb{A}_1$, there is an induced epimorphism

$$G_1 \xleftarrow{\lambda_2} G_2$$

which abelianizes to the identity map on $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Continue inductively, letting $r_{i+1,1} = [r_{i,2}, r_{i,3}]$, $r_{i+1,2} = [r_{i,1}, r_{i,3}]$, and $r_{i+1,3} = [r_{i,1}, r_{i,2}]$; $\mathbb{A}_{i+1} = \operatorname{ncl}(\{r_{i+1,1}, r_{i+1,2}, r_{i+1,3}\}, \mathbb{F}_3)$; and $G_{i+1} = \mathbb{F}_3/\mathbb{A}_{i+1}$. The result is a nested sequence of normal subgroups of $\mathbb{F}_3$, $\mathbb{A}_1 \geq \mathbb{A}_2 \geq \mathbb{A}_3 \geq \cdots$, and a corresponding inverse sequence of quotient groups

$$G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} G_3 \overset{\lambda_4}{\twoheadleftarrow} \cdots \tag{6-1}$$

which abelianizes to the constant inverse sequence

$$\mathbb{Z}^3 \xleftarrow{\text{id}} \mathbb{Z}^3 \xleftarrow{\text{id}} \mathbb{Z}^3 \xleftarrow{\text{id}} \cdots .$$

A more delicate motivation for our choices is the following: For each $i > 1$, $\ker \lambda_i = \operatorname{ncl}(\{r_{i-1,1}, r_{i-1,2}, r_{i-1,3}\}, G_i)$; similarly, for each $i > 2$,

$$\ker(\lambda_{i-1}\lambda_i) = \operatorname{ncl}(\{r_{i-2,1}, r_{i-2,2}, r_{i-2,3}\}, G_i).$$

Moreover, since the elements of $\{r_{i-1,1}, r_{i-1,2}, r_{i-1,3}\}$ are precisely the commutators of the elements of $\{r_{i-2,1}, r_{i-2,2}, r_{i-2,3}\}$,

$$\ker(\lambda_i) \leq [\ker(\lambda_{i-1}\lambda_i), \ker(\lambda_{i-1}\lambda_i)].$$

So, for the standard augmentation, $L_i = \ker \lambda_i$, (6-1) is $\{L_i\}$-perfect, hence, $\mathcal{AP}$-semistable.

Two tasks remain:

- prove that (6-1) is not $\mathcal{SAP}$-semistable, and
- construct 1-ended absolutely inward tame open manifolds with fundamental groups at infinity representable by (6-1).

Since these tasks are independent, the ordering of the following two subsections is arbitrary.

The sequence* (6-1) *is not $\mathcal{SAP}$-semistable. Let $\mathbb{F}_n = \langle a_1, \ldots, a_n \mid \ \rangle$, the free group on n generators. We will exploit two standard constructions from group theory. The *derived series of* $\mathbb{F}_n$ is defined by

$$\mathbb{F}_n^{(0)} = \mathbb{F}_n \qquad \text{and} \qquad \mathbb{F}_n^{(k+1)} = [\mathbb{F}_n^{(k)}, \mathbb{F}_n^{(k)}] \quad \text{for } k \geq 0.$$

The *lower central series of* $\mathbb{F}_n$ is given by $(\mathbb{F}_n)_1 = \mathbb{F}_n$ and then $(\mathbb{F}_n)_{k+1} = [(\mathbb{F}_n)_k, \mathbb{F}_n]$ for $k \geq 0$. By inspection

$$\mathbb{F}_n^{(k+1)} \leq \mathbb{F}_n^{(k)}, \quad (\mathbb{F}_n)_{k+1} \leq (\mathbb{F}_n)_k, \quad \mathbb{F}_n^{(k)} \leq (\mathbb{F}_n)_{k+1} \qquad \text{for all } k.$$

A well-known fact, similar in spirit to our goal in this subsection, is that

$$\bigcap_{k=0}^{\infty} \mathbb{F}_n^{(k)} = \{1\} = \bigcap_{k=1}^{\infty} (\mathbb{F}_n)_k.$$

The following representation of $\mathbb{F}_n$ was discovered by Magnus; our general reference is [Lyndon and Schupp 1977].

Proposition 6.1 [Lyndon and Schupp 1977, Proposition 10.1]. *Let $\mathcal{P}_n$ be the non-commuting power series ring in indeterminates $\{x_1, x_2, \ldots, x_n\}$ with $x_j^2 = 0$ for $j = 1, 2, \ldots, n$. Then the function $\beta(a_j) = 1 + x_j$ $(j = 1, 2, \ldots, n)$ induces a faithful representation of $\mathbb{F}_n$ into $\mathcal{P}_n^*$, the multiplicative group of units of $\mathcal{P}_n$.*

In $\mathcal{P}_n$, the *fundamental ideal* Δ is the kernel of the homomorphism $\rho : \mathcal{P}_n \to \mathbb{Z}$ that takes each x_j to 0. The elements of Δ are all sums of the form $\sum_{\nu=1}^{\infty} \pi_\nu$ where each π_ν is a homogeneous polynomial of degree at least one. Consequently, for any positive integer k the ideal Δ^k is made of all sums of the form $\sum_{\nu=1}^{\infty} \pi_\nu$ where each π_ν is a homogeneous polynomial of degree at least k.

The next proposition and lemma are useful for monitoring the location of commutators in a group.

Proposition 6.2 [Lyndon and Schupp 1977, Proposition 10.2]. *Let $\beta : \mathbb{F}_n \to \mathcal{P}^*$ be the representation given above. If $w_1, w_2 \in \mathbb{F}_n$ such that $\beta(w_1) - 1 \in \Delta^r$ and $\beta(w_2) - 1 \in \Delta^s$, then $\beta([w_1, w_2]) - 1 \in \Delta^{r+s}$.*

By applying Proposition 6.2 inductively, we obtain the following useful facts.

Lemma 6.3. *For all integers $n, i \geq 1$,*

(1) $\{\beta(w) - 1 \mid w \in \mathbb{F}_n^{(i)}\} \subseteq \Delta^{2^i}$,

(2) $\{\beta(w) - 1 \mid w \in (\mathbb{F}_n)_i\} \subseteq \Delta^i$,

(3) $\bigcap_{k=1}^{\infty} \Delta^k = 0$, *and*

(4) $\bigcap_{k=1}^{\infty} \mathbb{F}_n^{(k)} = \{1\} = \bigcap_{k=1}^{\infty} (\mathbb{F}_n)_k$.

We now focus our attention on $\mathbb{F}_3$ and its subgroups $\mathbb{A}_i = \mathrm{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, \mathbb{F}_3)$, as defined earlier.

Lemma 6.4. *For each $k \geq 1$ and $j \in \{1, 2, 3\}$,*

(1) *$r_{k,j}$ is a member of at least one free basis for $\mathbb{F}_3^{(k)}$, and*

(2) $r_{k,j} \in \mathbb{F}_3^{(k)} - \mathbb{F}_3^{(k+1)}$.

Proof. Assertion (1) can be obtained from an inductive argument using Schreier systems. A model argument can be found in [Massey 1967, Example 8.1].

Assertion (2) follows from (1), since the quotient map $\mathbb{F}_3^k \to \mathbb{F}_3^k / \mathbb{F}_3^{k+1}$ is the abelianization of $\mathbb{F}_3^k$. □

Since $\mathbb{A}_i \leq \mathbb{F}_3^{(i)}$, the following is an easy consequence of Lemmas 6.3 and 6.4.

Lemma 6.5. *For each $i \geq 1$ and $j \in \{1,2,3\}$,*

(1) $\beta(r_{i,j}) - 1 \neq 0$, *and*

(2) $\{\beta(h) - 1 \mid h \in \mathbb{A}_i\} \subseteq \Delta^{2^i}$.

The definitions of derived and lower central series are clearly applicable to arbitrary groups. To expand those notions further, the following definition is useful. For $H \trianglelefteq G$, let $\Omega_1(H,G) = H$ and $\Omega_k(H,G) = [\Omega_{k-1}(H,G), G]$ for $k > 1$. By normality, $H = \Omega_1(H,G) \geq \Omega_2(H,G) \geq \Omega_3(H,G) \geq \cdots$. When H is strongly G-perfect, $\Omega_k(H,G) = H$ for all k.

Proposition 6.6. *For each $i \geq 1$, there exists $p_i > 0$ and $q_i \geq p_i$ such that*

(1) *for each $j \in \{1,2,3\}$, $\beta(r_{i,j}) - 1 \notin \Delta^{2^i + p_i}$, and*

(2) $\{\beta(w) - 1 \mid w \in \Omega_{q_i}(\mathbb{A}_i, \mathbb{F}_3)\} \subseteq \Delta^{2^i + p_i}$.

Proof. Let i be fixed. Existence of p_i follows from item (3) of Lemma 6.3. Existence of q_i may be obtained from an inductive application of Proposition 6.2. □

We shift focus one more time, from $\mathbb{F}_3$ and its subgroups to the quotient groups $G_i = \mathbb{F}_3/\mathbb{A}_i$ and their subgroups. In doing so, we will allow a word in the generators of $\mathbb{F}_3$ to represent both an element of $\mathbb{F}_3$ and the corresponding element of a G_i. For example, recalling that $\lambda_{i+1,j} = \lambda_{i+1} \circ \cdots \circ \lambda_j : G_j \to G_i$, we say $\ker(\lambda_{i+1,j}) = \operatorname{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, G_j)$.

The following result is simple but useful.

Lemma 6.7. *Suppose $\lambda : G \to G'$ is a surjective homomorphism, $H \trianglelefteq G$, and $q \geq 0$. Then $\lambda(\Omega_q(H,G)) = \Omega_q(\lambda(H), G')$.*

Lemma 6.7 ensures that, for each $i < k$ and all $q \geq 0$, the quotient maps $\mathbb{F}_3 \twoheadrightarrow G_k$ restrict to epimorphisms

$$\Omega_q(\mathbb{A}_i, \mathbb{F}_3) \twoheadrightarrow \Omega_q(\operatorname{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, G_k), G_k). \tag{6-2}$$

Proposition 6.8. *For p_i and q_i as chosen in Proposition 6.6, and each $j \in \{1,2,3\}$, $r_{i,j} \notin \Omega_{q_i}(\ker(\lambda_{i+1,k}), G_k)$ whenever $2^k \geq 2^i + p_i$.*

Proof. Suppose $r_{i,j} \in \Omega_{q_i}(\ker(\lambda_{i+1,k}), G_k) = \Omega_{q_i}(\operatorname{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, G_k), G_k)$. Surjection (6-2) provides a $w \in \Omega_{q_i}(\mathbb{A}_i, \mathbb{F}_3)$ with cosets $\mathbb{A}_k \cdot r_{i,j} = \mathbb{A}_k \cdot w$. Consequently, there is an $h \in \mathbb{A}_k$ with $r_{i,j} = hw$ in $\mathbb{F}_3$. Then

$$\begin{aligned} \beta(r_{i,j}) - 1 &= \beta(h)\beta(w) - 1 \\ &= \beta(h)\beta(w) - \beta(h) + \beta(h) - 1 \\ &= \beta(h)(\beta(w) - 1) + (\beta(h) - 1). \end{aligned}$$

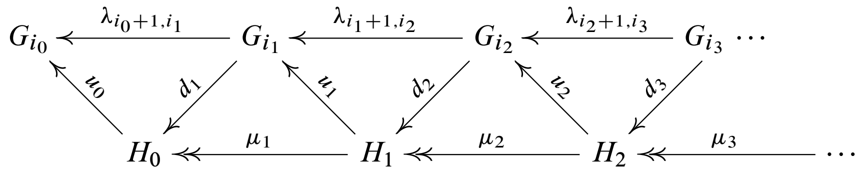
Since $\beta(w) - 1 \in \Delta^{2^i + p_i}$ and $\beta(h) - 1 \in \Delta^{2^k} \subseteq \Delta^{2^i + p_i}$, then $\beta(r_{i,j}) - 1 \in \Delta^{2^i + p_i}$, violating the choice of p_i. □

We are now ready for the main result of this subsection.

Theorem 6.9. *The inverse sequence* $\{G_i, \lambda_i\}_{i=0}^{\infty}$ *is not* $\mathcal{SAP}$*-semistable. In fact,* $\{G_i, \lambda_i\}_{i=0}^{\infty}$ *is not pro-isomorphic to any inverse sequence* $\{H_i, \mu_i\}$ *of surjections that satisfies the strong* $\{H_i\}$*-perfectness property.*

Proof. We proceed directly to the stronger assertion. Suppose $\{G_i, \lambda_i\}$ is pro-isomorphic to an inverse sequence $\{H_i, \mu_i\}$ of surjections that is strongly $\{H_i\}$-perfect; in other words, $\ker \mu_i = [\ker \mu_i, H_i]$ for all i.

By Proposition 2.11, each subsequence of $\{H_i, \mu_i\}$ satisfies the same essential property, so by our assumption, $\{G_i, \lambda_i\}$ contains a subsequence that fits into a commutative diagram of the following form:



Passing to a further subsequence if necessary, we may assume $2^{i_n} \geq 2^{i_{n-1}} + p_{i_{n-1}}$ for all n.

By Lemma 6.4, $1 \neq r_{i_1,j} \in \ker(\lambda_{i_1+1,i_2}) \leq G_{i_2}$. Choose $\alpha' \in H_2$ with $u_2(\alpha') = r_{i_1,j}$. Then, $\alpha' \in \ker(\mu_{1,2})$, and consequently $\alpha' \in [\ker(\mu_{1,2}), H_2]$, since $\ker(\mu_{1,2})$ is strongly H_2-perfect (again using Proposition 2.11). Thus $\alpha' \in \Omega_q(\ker(\mu_{1,2}), H_2)$ for all q. Moreover, since $u_2(\ker(\mu_{1,2})) \subseteq \ker(\lambda_{i_0+1,i_2})$,

$$r_{i_1,j} = u_2(\alpha') \in \Omega_q(u_2(\ker(\mu_{1,2})), G_{i_2}) \subseteq \Omega_q(\ker(\lambda_{i_0+1,i_2}), G_{i_2})$$

for all q, thereby contradicting Proposition 6.8. $\square$

Construction of the examples. The goal of this subsection is to construct, for each $n \geq 6$, a 1-ended open manifold M^n that is absolutely inward tame and has fundamental group at infinity represented by the inverse sequence (6-1). By Theorem 1.3 or Theorem 5.1, such an example fails to be nearly pseudocollarable, thus completing the proof of Theorem 1.4.

Overview. We will construct M^n as a countable union of codimension 0 submanifolds

$$M^n = C_1 \cup A_1 \cup A_2 \cup A_3 \cup \cdots,$$

where C_1 is a compact "core" and $\{(A_i, \Gamma_i, \Gamma_{i+1})\}$ is a sequence of compact cobordisms between closed connected $(n-1)$-manifolds with $A_i \cap A_{i+1} = \Gamma_{i+1}$ for each $i \geq 1$, and $\partial C_1 = \Gamma_1$. Letting

$$N_i = A_i \cup A_{i+1} \cup A_{i+2} \cup \cdots$$

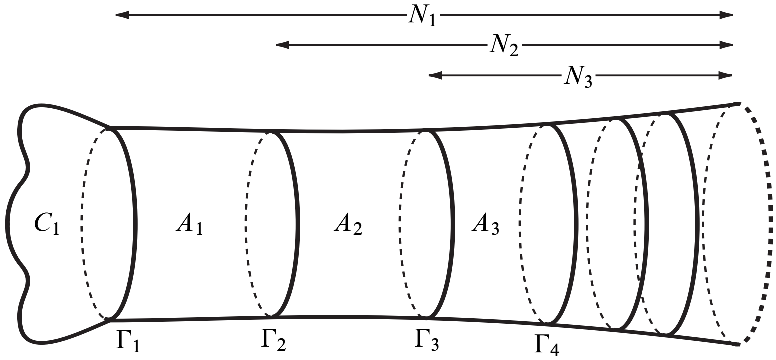


Figure 3. $M^n = C_1 \cup A_1 \cup A_2 \cup A_3 \cup \cdots$.

gives a preferred end structure $\{N_i\}$ with $\partial N_i = \Gamma_i$ for each i. See Figure 3.

So that pro-$\pi_1(\varepsilon(M^n))$ is represented by (6-1), the A_i will be constructed to satisfy:

(a) For all $i \geq 1$, $\pi_1(\Gamma_i, p_i) \cong G_i$ and $\Gamma_i \hookrightarrow A_i$ induces a π_1-isomorphism.

(b) The isomorphism between $\pi_1(\Gamma_i, p_i)$ and G_i may be chosen so that

$$\begin{array}{ccccc} G_i & & \xleftarrow{\quad\lambda_{i+1}\quad} & & G_{i+1} \\ \downarrow \cong & & & & \downarrow \cong \\ \pi_1(\Gamma_i, p_i) & \xleftarrow{\cong} & \pi_1(A_i, p_i) & \xleftarrow{\psi_{i+1}} & \pi_1(\Gamma_{i+1}, p_{i+1}) \end{array}$$

commutes. Here ψ_{i+1} is the composition

$$\pi_1(A_i, p_i) \xleftarrow{\widehat{\rho}_i} \pi_1(A_i, p_{i+1}) \xleftarrow{\iota_{i+1}} \pi_1(\Gamma_{i+1}, p_{i+1}),$$

where ι_{i+1} is induced by inclusion and $\widehat{\rho}_i$ is a change-of-basepoint isomorphism with respect to a path ρ_i in A_i between p_i and p_{i+1}.

From there it follows from Van Kampen's theorem that each $\Gamma_i = \partial N_i \hookrightarrow N_i$ induces a π_1-isomorphism, so by repeated application of (a) and (b), the inverse sequence

$$\pi_1(N_1, p_1) \xleftarrow{\mu_2} \pi_1(N_2, p_2) \xleftarrow{\mu_3} \pi_1(N_3, p_3) \xleftarrow{\mu_4}$$

is isomorphic to (6-1).

It will also be shown that each N_i has finite homotopy type; so M^n is absolutely inward tame. That argument requires specific details of the construction; it will be presented later.

Details of the construction. Recall that a p-handle h^p attached to an n-manifold P^n and a $(p+1)$-handle h^{p+1} attached to $P^n \cup h^p$ form a *complementary pair* if the attaching sphere of h^{p+1} intersects the belt sphere of h^p transversely in a single point. In that case $P^n \cup h^p \cup h^{p+1} \approx P^n$; moreover, we may arrange (by an isotopy of the attaching sphere of h^{p+1}) that $P^n \cap (h^p \cup h^{p+1})$ is an $(n-1)$-ball in ∂P^n. Conversely, for any ball $B^{n-1} \subseteq \partial P^n$, one may introduce a pair of complementary handles $P^n \cup h^p \cup h^{p+1}$ so that $P^n \cap (h^p \cup h^{p+1}) = B^{n-1}$. We call (h^p, h^{p+1}) a *trivial handle pair*. Note that the difference between a complementary pair and trivial pair is just a matter of perspective. In general, we say that h^p is *attached trivially* to P^n if it is possible to attach an h^{p+1} so that (h^p, h^{p+1}) is a complementary pair.

After a preliminary step where we construct the core manifold C_1, our proof proceeds inductively. At the i-th stage we construct the cobordism $(A_i, \Gamma_i, \Gamma_{i+1})$, along with a compact manifold C_{i+1} with $\partial C_{i+1} = \Gamma_{i+1}$, to be used in the following stage. Throughout the construction, we abuse notation slightly by letting $\partial C_i \times [0, \varepsilon]$ denote a small regular neighborhood of ∂C_i in C_i and $\Gamma_i \times [0, \varepsilon]$ to denote a small regular neighborhood of Γ_i in A_i.

Step 0 (preliminaries). Let C_0 be the n-manifold obtained by attaching three orientable 1-handles $\{h^1_{0,j}\}_{j=1}^3$ to the n-ball B^n. Choose a basepoint $p_0 \in \partial C_0$ and let a_1, a_2, and a_3 be embedded loops in ∂C_0, one through each 1-handle, intersecting only at p_0. Abuse notation slightly by writing

$$\pi_1(\partial C_0) = \pi_1(C_0) = \langle a_1, a_2, a_3 \mid \rangle.$$

A convenient way to arrange that the 1-handles are orientable is by attaching three trivial $(1,2)$-handle pairs $\{h^1_{0,j}, h^2_{0,j}\}_{j=1}^3$, then discarding the 2-handles.

Recall that

$$G_1 = \langle a_1, a_2, a_3 \mid r_{1,1}, r_{1,2}, r_{1,3} \rangle,$$

where $r_{1,1} = [a_2, a_3]$, $r_{1,2} = [a_1, a_3]$, and $r_{1,3} = [a_1, a_2]$. Attach a trio of 2-handles $\{h^2_{1,j}\}_{j=1}^3$ to C_0, where $h^2_{1,j}$ has attaching circle $r_{1,j}$. Choose the framings of these handles so that, if the 2-handles $\{h^2_{0,j}\}_{j=1}^3$ were added back in, then $\{h^2_{1,j}\}_{j=1}^3$ would be trivially attached (to an n-ball). Let

$$C_1 = C_0 \cup h^2_{1,1} \cup h^2_{1,2} \cup h^2_{1,3}$$

and note that $\pi_1(C_1) \cong \pi_1(\partial C_1) \cong G_1$.

Step 1 (constructing A_1 and C_2). Attach three trivial $(2,3)$-handle pairs to C_1, disjoint from the existing handles, then perform handle slides on each of the trivial 2-handles (over the handles $\{h^2_{1,j}\}_{j=1}^3$) so the resulting 2-handles $h^2_{2,1}$, $h^2_{2,2}$ and $h^2_{2,3}$ have attaching circles spelling out the words $r_{2,1}$, $r_{2,2}$ and $r_{2,3}$, respectively. This is possible since each $r_{2,k}$ can be viewed as a product of the loops $\{r_{1,j}\}_{j=1}^3$ and their inverses, which are the attaching circles of $\{h^2_{1,j}\}_{j=1}^3$. Sliding a 2-handle over $h^2_{1,j}$ inserts the loop $r^{\pm 1}_{1,j}$ into the new attaching circle of that 2-handle (with ± 1 depending on the orientation chosen).

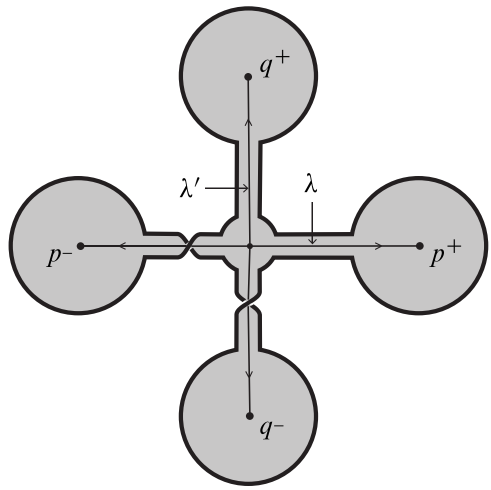


Figure 4. Attaching a $(2,3)$-handle pair.

By keeping track of the attaching 2-spheres of the trivial 3-handles after the handle slides, it is possible to attach 3-handles $h^3_{2,1}$, $h^3_{2,2}$, and $h^3_{2,3}$ to $C_1 \cup h^2_{2,1} \cup h^2_{2,2} \cup h^2_{2,3}$ that are complementary to $h^2_{2,1}$, $h^2_{2,2}$, and $h^2_{2,3}$, respectively. Then

$$C_1 \cup \left(\bigcup_{j=1}^{3} h^2_{2,j}\right) \cup \left(\bigcup_{j=1}^{3} h^3_{2,j}\right) \approx C_1.$$

For later purposes, it is useful to have a schematic image of the attaching circles of $\{h^2_{1,j}\}_{j=1}^3$ and the attaching 2-spheres of the complementary handles $\{h^2_{1,j}\}_{j=1}^3$. Figure 4 provides such an image for one complementary pair. The outer loop represents the attaching circle for an $h^2_{2,j}$ and the shaded region represents the "lower hemisphere" of the attaching 2-sphere of $h^3_{2,j}$; the "upper hemisphere", which is not shown, is a parallel copy of the core of $h^2_{2,j}$. Within the lower hemisphere, the small central disk represents the lower hemisphere of the 2-sphere before handle slides. The arms are narrow strips whose centerlines are the paths along which the handle slides were performed; diametrically opposite paths lead to the same 2-handle, and are chosen to be parallel to a fixed path. We have indicated this by labeling one pair of centerlines λ and the other λ'. The four outer disks are parallel to the cores of the 2-handles over which the slides were made. A twist in the strip leading to an outer disk is used to reverse the orientation of the boundary of that disk. Thus, diametrically opposite outer disks are parallel to each other, but with opposite orientations. Center points of the outer disks represent transverse intersections with belt spheres of those handles; thus, p^+ and p^- are nearby intersections with the same belt sphere, and similarly for q^+ and q^-.

By rewriting

$$C_1 \cup \left(\bigcup_{j=1}^{3} h^2_{2,j}\right) \cup \left(\bigcup_{j=1}^{3} h^3_{2,j}\right)$$

as

$$C_0 \cup \left(\bigcup_{j=1}^3 h_{1,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{2,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{2,j}^3\right),$$

we may reorder the handles so that $h_{2,1}^2$, $h_{2,2}^2$, and $h_{2,3}^2$ are attached first. Define

$$C_2 = C_0 \cup \left(\bigcup_{j=1}^3 h_{2,j}^2\right)$$

and note that $\pi_1(C_2) \approx \pi_1(\partial C_2) \approx G_2$. Furthermore,

$$C_2 \cup \left(\bigcup_{j=1}^3 h_{1,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{2,j}^3\right) \approx C_1.$$

So, if we let

$$A_1 = (\partial C_2 \times [0, \varepsilon]) \cup \left(\bigcup_{j=1}^3 h_{1,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{2,j}^3\right),$$

(the result of excising the interior of a slightly shrunken copy of C_2), then $\partial A_1 \approx \partial C_2 \sqcup \partial C_1$. By letting $\Gamma_1 = \partial C_1$ and $\Gamma_2 = \partial C_2$ we obtain the first cobordism of the construction $(A_1, \Gamma_1, \Gamma_2)$. By avoiding the basepoint $p_0 \in \partial C_0$ in all of the above handle additions, we may let the arc $\rho_1 \subseteq A_1$ be the product line $p_0 \times [0, \varepsilon]$, with p_1 and p_2 its endpoints. Conditions (a) and (b) are then clear.

Inductive step (constructing A_i and C_{i+1}). Assume the existence of a cobordism $(A_{i-1}, \Gamma_{i-1}, \Gamma_i)$ satisfying (a) and (b) along with a compact manifold $C_i = C_0 \cup \left(\bigcup_{j=1}^3 h_{i,j}^2\right)$, with the attaching circle of each $h_{i,j}^2$ representing the relator $r_{i,j}$ in the presentation of G_i, and $\partial C_i = \Gamma_i$. Attach three trivial $(2,3)$-handle pairs to C_i, then perform handle slides on each of the trivial 2-handles (over the handles $\{h_{i,j}^2\}_{j=1}^3$) so that the resulting 2-handles $h_{i+1,1}^2$, $h_{i+1,2}^2$ and $h_{i+1,3}^2$ have attaching circles spelling out the words $r_{i+1,1}$, $r_{i+1,2}$ and $r_{i1,3}$, respectively. This is possible since each $r_{i+1,k}$ can be viewed as a product of the loops $\{r_{i,j}\}_{j=1}^3$ and their inverses, which are the attaching circles of $\{h_{i,j}^2\}_{j=1}^3$.

By keeping track of the attaching 2-spheres of the trivial 3-handles under the above handle slides, it is possible to attach 3-handles $h_{i+1,1}^3$, $h_{i+1,2}^3$, and $h_{i+1,3}^3$ to

$$C_i \cup h_{i+1,1}^2 \cup h_{i+1,2}^2 \cup h_{i+1,3}^2$$

that are complementary to $h_{i+1,1}^2$, $h_{i+1,2}^2$, and $h_{i+1,3}^2$, respectively. Then

$$C_i \cup \left(\bigcup_{j=1}^3 h_{i+1,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{i+1,j}^3\right) \approx C_i.$$

A picture like Figure 4, but with different indices, describes the current situation.

Rewrite

$$C_i \cup \left(\bigcup_{j=1}^3 h_{i+1,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{i+1,j}^3\right)$$

as

$$C_0 \cup \left(\bigcup_{j=1}^3 h_{i,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{i+1,j}^2\right) \cup \left(\bigcup_{j=1}^3 h_{i+1,j}^3\right),$$

then reorder the handles so that $h^2_{i+1,1}$, $h^2_{i+1,2}$, and $h^2_{i+1,3}$ are attached first. Define

$$C_{i+1} = C_0 \cup \left(\bigcup_{j=1}^3 h^2_{i+1,j}\right)$$

and note that $\pi_1(C_{i+1}) \approx \pi_1(\partial C_{i+1}) \approx G_{i+1}$.

Furthermore,

$$C_{i+1} \cup \left(\bigcup_{j=1}^3 h^2_{i,j}\right) \cup \left(\bigcup_{j=1}^3 h^3_{i+1,j}\right) \approx C_i.$$

Excising the interior of a slightly shrunken copy of C_{i+1} gives

$$A_{i+1} = (\partial C_{i+1} \times [0,\varepsilon]) \cup \left(\bigcup_{j=1}^3 h^2_{i,j}\right) \cup \left(\bigcup_{j=1}^3 h^3_{i+1,j}\right);$$

then $\partial A_{i+1} \approx \partial C_{i+1} \sqcup \partial C_i$. Noting that $\Gamma_i = \partial C_i$ and letting $\Gamma_{i+1} = \partial C_{i+1}$, we obtain $(A_i, \Gamma_i, \Gamma_{i+1})$. By avoiding $p_i \in \partial C_i$ in all of the handle additions, letting $\rho_i \subseteq A_i$ be the product line $p_i \times [0,\varepsilon]$, and p_{i+1} the new endpoint, conditions (a) and (b) are clear.

Assembling the pieces in the manner described in Figure 3 completes the construction. In particular, we obtain a 1-ended open manifold

$$M^n = C_1 \cup A_1 \cup A_2 \cup A_3 \cup \cdots$$

whose fundamental group at infinity is represented by the inverse sequence (6-1).

Remark 6.10. In the construction of $(A_i, \Gamma_i, \Gamma_{i+1})$, we have written Γ_i on the left and Γ_{i+1} on the right to match the blueprint laid out in Figure 3. In that case, the handle decomposition of A_i implicit in the construction goes from right to left, with handles being attached to a collar neighborhood $\Gamma_{i+1} \times [0,\varepsilon]$ of Γ_{i+1}. Later, when our perspective becomes reversed, we will pass to the dual decomposition

$$A_i = (\Gamma_i \times [0,\varepsilon]) \cup \left(\bigcup_{j=1}^3 \bar{h}^{n-3}_{1,j}\right) \cup \left(\bigcup_{j=1}^3 \bar{h}^{n-2}_{2,j}\right),$$

where each $\bar{h}^{n-p}$ is the dual of an original h^p and $\Gamma_i \times [0,\varepsilon]$ is a thin collar neighborhood of Γ_i.

Absolute inward tameness of M^n. The following proposition will complete the proof of Theorem 1.4.

Proposition 6.11. *For the manifolds M^n constructed above, each clean neighborhood of infinity*

$$N_i = A_i \cup A_{i+1} \cup A_{i+2} \cup \cdots$$

has finite homotopy type. Thus, M^n is absolutely inward tame.

We will prove this by examining $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$ (equivalently, $H_*(\widetilde{N}_i, \widetilde{\Gamma}_i; \mathbb{Z})$ viewed as a $\mathbb{Z}G_i$-module), where $G_i = \pi_1(N_i) = \pi_1(\Gamma_i)$. In particular, we will prove:

Claim 6.12. *For each i, $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$ is trivial in all dimensions except for $* = n-2$, where it is isomorphic to the free module $(\mathbb{Z}G_i)^3 = \mathbb{Z}G_i \oplus \mathbb{Z}G_i \oplus \mathbb{Z}G_i$.*

Once this claim is established, Proposition 6.11 follows from [Siebenmann 1965, Lemma 6.2]. In Remark 6.13 at the conclusion of this section, we explain why this final observation is elementary, requiring no discussion of finite dominations or finiteness obstructions.

Proof. It is useful to consider compact subsets of the form

$$A_{i,k} = A_i \cup A_{i+1} \cup \cdots \cup A_k.$$

By repeated application of Remark 6.10, there is a handle decomposition of $A_{i,k}$ based on $\Gamma_i \times [0, \varepsilon]$ with handles only of indices $n-3$ and $n-2$. By reordering the handles, $(A_{i,k}, \Gamma_i)$ is seen to be homotopy equivalent to a finite relative CW complex $(K_{i,k}, \Gamma_i)$, where $K_{i,k}$ consists of Γ_i with an $(n{-}3)$-cell attached for each $(n{-}3)$-handle of $A_{i,k}$ followed by an $(n{-}2)$-cell for each $(n{-}2)$-handle. In the usual way, the $\mathbb{Z}G_i$-incidence number of an $(n{-}2)$-cell with an $(n{-}3)$-cell is equal to the $\mathbb{Z}G_i$-intersection number between the belt sphere of the corresponding $(n{-}3)$-handle and the attaching sphere of the corresponding $(n{-}2)$-handle. This process produces a sequence

$$K_{i,i} \subseteq K_{i,i+1} \subseteq K_{i,i+2} \subseteq \cdots$$

of relative CW complexes with direct limit a relative CW pair $(K_{i,\infty}, \Gamma_i)$ homotopy equivalent to (N_i, Γ_i). So we can determine $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$ by calculating $H_*(A_{i,k}, \Gamma_i; \mathbb{Z}G_i)$ and taking the direct limit as $k \to \infty$.

The $\mathbb{Z}G_i$-handle chain complex for $(A_{i,k}, \Gamma_i)$ (equivalently, the $\mathbb{Z}G_i$-cellular chain complex for $(K_{i,k}, \Gamma_i)$) looks like

$$0 \longrightarrow \mathcal{C}_{n-2} \xrightarrow{\partial} \mathcal{C}_{n-3} \longrightarrow 0,$$

where $\mathcal{C}_{n-2}$ and $\mathcal{C}_{n-3}$ are finitely generated free $\mathbb{Z}G_i$-modules generated by the handles of $A_{i,k}$, and the boundary map is determined by $\mathbb{Z}G_i$-intersection numbers between the belt spheres of $(n{-}3)$-handles and attaching spheres of the $(n{-}2)$-handles. These intersection numbers will be determined by returning to the construction.

Beginning with the compact manifold

$$C_i = C_0 \cup \left(\bigcup_{j=1}^3 h_{i,j}^2\right),$$

attach three trivial $(2,3)$-handle pairs, then perform handle slides on the 2-handles (over the handles $\{h_{i,j}^2\}_{j=1}^3$) to obtain $h_{i+1,1}^2$, $h_{i+1,2}^2$ and $h_{i+1,3}^2$ with attaching circles $r_{i+1,1}$, $r_{i+1,2}$ and $r_{i+1,3}$, respectively. Having kept track of the attaching

2-spheres of the trivial 3-handles under the handle slides, attach 3-handles $h^3_{i+1,1}$, $h^3_{i+1,2}$, and $h^3_{i+1,3}$ to

$$C_i \cup h^2_{i+1,1} \cup h^2_{i+1,2} \cup h^2_{i+1,3}$$

that are complementary to $h^2_{i+1,1}$, $h^2_{i+1,2}$, and $h^2_{i+1,3}$, respectively (all as described in inductive step above). This can all be done so that $h^3_{i+1,1}$, $h^3_{i+1,2}$, and $h^3_{i+1,3}$ do not touch the earlier 2-handles $h^2_{i,1}$, $h^2_{i,2}$ and $h^2_{i,3}$. Next attach a second trio of trivial $(2,3)$-handle pairs, taking care that they are disjoint from the existing handles, and slide the trivial 2-handles over the 2-handles $\{h^2_{i+1,j}\}_{j=1}^3$ so that the resulting 2-handles $\{h^2_{i+2,j}\}_{j=1}^3$ have attaching circles $r_{i+2,1}$, $r_{i+2,2}$ and $r_{i+2,3}$. Again, having kept track of the attaching 2-spheres of the trivial 3-handles under the handle slides, attach 3-handles $h^3_{i+2,1}$, $h^3_{i+2,2}$, and $h^3_{i+2,3}$ to

$$C_i \cup \left(\bigcup_{j=1}^3 h^2_{i+1,j}\right) \cup \left(\bigcup_{j=1}^3 h^3_{i+1,j}\right) \cup \left(\bigcup_{j=1}^3 h^2_{i+2,j}\right)$$

that are complementary to $h^2_{i+2,1}$, $h^2_{i+2,2}$, and $h^2_{i+2,3}$, respectively, while taking care that these new 3-handles are completely disjoint from all 2- and 3-handles of lower index. Continue this process $k-i$ times, at each stage attaching three trivial $(2,3)$-handle pairs disjoint from the existing handles; sliding the trivial 2-handles over the 2-handles created in the previous step, in the manner prescribed above; and then attaching 3-handles complementary to these new 2-handles (and disjoint from earlier 2- and 3-handles) along the images of the attaching 2-spheres of the trivial 3-handles after the handle slides.

Since all of the 2- and 3-handles mentioned above, except for the original 2-handles $h^2_{i,1}$, $h^2_{i,2}$ and $h^2_{i,3}$, occur in complementary pairs, the manifold we just created is just a thickened copy of C_i; let us call it C_i'. By the standard reordering lemma, we may arrange that the 2-handles are pairwise disjoint, and all are attached before any of the 3-handles — which are also are attached in a pairwise disjoint manner. Then

$$\begin{aligned}
C_i' &= C_i \cup \left(\bigcup_{s=1}^k \left(\bigcup_{j=1}^3 h^2_{i+s,j}\right)\right) \cup \left(\bigcup_{s=1}^k \left(\bigcup_{j=1}^3 h^3_{i+s,j}\right)\right) \\
&= C_0 \cup \left(\bigcup_{j=1}^3 h^2_{i,j}\right) \cup \left(\bigcup_{s=1}^k \left(\bigcup_{j=1}^3 h^2_{i+s,j}\right)\right) \cup \left(\bigcup_{s=1}^k \left(\bigcup_{j=1}^3 h^3_{i+s,j}\right)\right) \\
&= C_0 \cup \left(\bigcup_{j=1}^3 h^2_{i+k,j}\right) \cup \left(\bigcup_{j=1}^3 h^2_{i,j}\right) \cup \left(\bigcup_{s=1}^{k-1} \left(\bigcup_{j=1}^3 h^2_{i+s,j}\right)\right) \\
&\qquad\qquad \cup \left(\bigcup_{s=1}^k \left(\bigcup_{j=1}^3 h^3_{i+s,j}\right)\right) \\
&= C_k \cup \left(\bigcup_{j=1}^3 h^2_{i,j}\right) \cup \left(\bigcup_{s=1}^{k-1} \left(\bigcup_{j=1}^3 h^2_{i+s,j}\right)\right) \cup \left(\bigcup_{s=1}^k \left(\bigcup_{j=1}^3 h^3_{i+s,j}\right)\right),
\end{aligned}$$

where, going from the first to the second line, we apply the definition of C_i; going from the second to the third, we bring the last triple of 2-handles forward to the beginning; and in going from the third to the fourth, we apply the definition of C_k.

Excising a slightly shrunken copy of the interior of C_k from C_i' results in a cobordism between $\partial C_k = \Gamma_k$ and $\partial C_i' \approx \Gamma_i$, which has a handle decomposition

$$(\Gamma_k \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^{3} h_{i,j}^2\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} h_{i+s,j}^2\big)\big) \cup \big(\bigcup_{s=1}^{k}\big(\bigcup_{j=1}^{3} h_{i+s,j}^3\big)\big).$$

Comparing this handle decomposition to our earlier construction reveals that this cobordism is precisely $A_i \cup A_{i+1} \cup \cdots \cup A_k = A_{i,k}$. In order to match the orientation of Figure 3, view Γ_k as the right-hand boundary and Γ_i as the left-hand boundary, with 2- and 3-handles being attached from right to left. Before switching to the dual handle decomposition, we analyze the $\mathbb{Z}G_i$-intersection numbers between the attaching spheres of the 3-handles and the belt spheres of the 2-handles. All should be viewed as submanifolds of the left-hand boundary of

$$(\Gamma_k \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^{3} h_{i,j}^2\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} h_{i+s,j}^2\big)\big),$$

which has fundamental group G_i.

For each $1 \le s \le k$ and $j \in \{1,2,3\}$ let $\alpha_{i+s,j}^2$ denote the attaching 2-sphere of $h_{i+s,j}^3$; and for each $0 \le s' \le k-1$ and $j' \in \{1,2,3\}$ let $\beta_{i+s',j'}^{n-3}$ denote the belt $(n-3)$-sphere of $h_{i+s',j'}^2$ There are three cases to consider.

Case 1: $s = s'$. Then for each j, the pair $(h_{i+s,j}^2, h_{i+s,j}^3)$ is complementary; in other words $\alpha_{i+s,j}^2$ intersects $\beta_{i+s,j}^{n-3}$ transversely in a single point. Adjusting base paths, if necessary, and being indifferent to orientation (since it will not affect our computations), we have

$$\varepsilon_{\mathbb{Z}G_i}(\alpha_{i+s,j}^2, \beta_{i+s,j}^{n-3}) = \pm 1.$$

If $j \ne j'$, then $h_{i+s,j}^3$ does not intersect $h_{i+s,j'}^2$, so

$$\varepsilon_{\mathbb{Z}G_i}(\alpha_{i+s,j}^2, \beta_{i+s,j'}^{n-3}) = 0.$$

Case 2: $s = s'+1$. For each j, $\alpha_{i+s,j}^2$ can be split into a pair of disks. The "upper hemisphere" lies in the 2-handle $h_{i+s,j}^2$ and it intersects $\beta_{i+s,j}^{n-3}$ transversely in a single point; that point of intersection was accounted for in Case 1. The "lower hemisphere" is analogous to the one pictured in Figure 4. If $\{u,v\} = \{1,2,3\} - \{j\}$, then one pair of the diametrically opposite disks has boundaries labelled $r_{i+s-1,u}$ and $r_{i+s-1,u}^{-1}$ and the disks are parallel to the core of $h_{i+s-1,u}^2$, so each intersects $\beta_{i+s-1,u}^{n-3}$ transversely in points p_u^+ and p_u^-. Due to the flipped orientation of one of the disks, these points of intersection, between $\alpha_{i+s,j}^2$ and $\beta_{i+s-1,u}^{n-3}$, have opposite sign. Connecting p_u^+ and p_u^- by a path homotopic to $\lambda^{-1} * \lambda$ in $\alpha_{i+s,j}^2$ and a short path μ connecting p_u^+ and p_u^- in $\beta_{i+s-1,u}^{n-3}$ yields a loop that is contractible in the left-hand boundary of

$$(\Gamma_k \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^{3} h_{i,j}^2\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} h_{i+s,j}^2\big)\big).$$

So together p_u^+ and p_u^- contribute 0 to the $\mathbb{Z}G_i$-intersection number of $\alpha^2_{i+s,j}$ and $\beta^{n-3}_{i+s-1,u}$; hence,

$$\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s-1,u}) = 0.$$

Similarly

$$\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s-1,v}) = 0.$$

Finally, $\alpha^2_{i+s,j}$ and $\beta^{n-3}_{i+s-1,j}$ do not intersect, so

$$\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s-1,j}) = 0$$

as well.

Case 3: $s \notin \{s', s'+1\}$. In this case, the handles $h^3_{i+s,j}$ and $h^2_{i+s',u}$ are disjoint, so $\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s,j'}) = 0$.

Now invert the above handle decomposition to obtain a handle decomposition of the cobordism $(A_{i,k}, \Gamma_i, \Gamma_k)$, based on Γ_i, containing only $(n-3)$- and $(n-2)$-handles. Specifically, we have

$$(\Gamma_i \times [0,\varepsilon]) \cup \big(\bigcup_{s=1}^{k}\big(\bigcup_{j=1}^{3} \bar{h}^{n-3}_{i+s,j}\big)\big) \cup \big(\bigcup_{j=1}^{3} \bar{h}^{n-2}_{i,j}\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} \bar{h}^{n-2}_{i+s,j}\big)\big).$$

Since the belt sphere of each $\bar{h}^{n-3}$ is the attaching sphere of its dual h^3 and the attaching sphere of each $\bar{h}^{n-2}$ is the belt sphere of its dual h^2, the incidence numbers between these handles of this handle decomposition are determined (up to sign) by the earlier calculations. So the cellular $\mathbb{Z}G_i$-chain complex for the $(A_{i,k}, \Gamma_i)$ is isomorphic to

$$0 \to \bigoplus_{s=0}^{k-1}(\mathbb{Z}G_i)^3 \xrightarrow{\partial} \bigoplus_{s=1}^{k}(\mathbb{Z}G_i)^3 \to 0,$$

where the $(\mathbb{Z}G_i)^3$ summands on the left are generated by the handles $\{\bar{h}^{n-2}_{i+s,j}\}_{j=1}^3$ and those on the right by $\{\bar{h}^{n-3}_{i+s,j}\}_{j=1}^3$. Since $\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s,j}) = \pm 1$ for all $1 \le s \le k-1$ and all other intersection numbers are 0, the boundary map is trivial on the 0-th copy of $(\mathbb{Z}G_i)^3$; misses the k-th copy of $(\mathbb{Z}G_i)^3$ in the range; and restricts to an isomorphism $\bigoplus_{s=1}^{k-1}(\mathbb{Z}G_i)^3 \xrightarrow{\cong} \bigoplus_{s=1}^{k-1}(\mathbb{Z}G_i)^3$ elsewhere. Thus

$$H_{n-2}(A_{i,k}, \Gamma_i; \mathbb{Z}G_i) = \ker \partial \cong (\mathbb{Z}G_i)^3, \text{ and}$$

$$H_{n-3}(A_{i,k}, \Gamma_i; \mathbb{Z}G_i) = \operatorname{coker} \partial \cong (\mathbb{Z}G_i)^3,$$

where $H_{n-2}(K_{i,k}, \Gamma_i)$ is generated by the $s = 0$ summand and $H_{n-3}(K_{i,k}, \Gamma_i)$ is generated by the $s = k$ summand.

Now consider the inclusion $A_{i,k} \hookrightarrow A_{i,k+1}$ and the corresponding inclusion of $\mathbb{Z}G_i$-chain complexes. The chain complex of $A_{i,k+1}$ will contain an extra $(\mathbb{Z}G_i)^3$ summand in each dimension, generated by $\{\bar{h}^{n-2}_{i+k,j}\}_{j=1}^3$ and $\{\bar{h}^{n-3}_{i+k+1,j}\}_{j=1}^3$, respectively. The boundary map takes the new summand in the domain onto the previous cokernel, thereby killing $H_{n-3}(A_{i,k}, \Gamma_i; \mathbb{Z}G_i)$, and replacing it with a cokernel generated by $\{\bar{h}^{n-3}_{i+k+1,j}\}^3_{j=1}$. Said differently, the inclusion induced map

$$i_* : H_{n-3}(K_{i,k}, \Gamma_i; \mathbb{Z}G_i) \xrightarrow{0} H_{n-3}(K_{i,k+1}, \Gamma_i; \mathbb{Z}G_i)$$

is trivial. On the other hand, the expansion from $K_{i,k}$ to $K_{i,k+1}$ does not change $\ker \partial$, which is still generated by the handles $\{\bar{h}^{n-2}_{i,j}\}^3_{j=1}$. In other words, the inclusion induced map

$$i_* : H_{n-2}(K_{i,k}, \Gamma_i; \mathbb{Z}G_i) \xrightarrow{\cong} H_{n-2}(K_{i,k+1}, \Gamma_i; \mathbb{Z}G_i)$$

is an isomorphism.

Taking direct limits, we have

$$H_*(N_i, \Gamma_i; \mathbb{Z}G_i) \cong \begin{cases} (\mathbb{Z}G_i)^3 & \text{if } * = n-2, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Remark 6.13. The appeal to [Siebenmann 1965, Lemma 6.2] may give the impression that obtaining Proposition 6.11 from Claim 6.12 is complicated — that is not the case. The conclusion can be obtained directly as follows: If $\{e^{n-2}_{i,j}\}^3_{j=1}$ represents the cores of the $(n-2)$-handles $\{\bar{h}^{n-2}_{i,j}\}$, which generate $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$, abstractly attach $(n-2)$-disks $\{f^{n-2}_{i,j}\}^3_{j=1}$ to Γ_i along their boundaries. This does not affect fundamental groups, so by excision, the pair

$$\big(N_i \cup \{f^{n-2}_{i,j}\}^3_{j=1}, \Gamma_i \cup \{f^{n-2}_{i,j}\}^3_{j=1}\big)$$

has the same $\mathbb{Z}G_i$ -homology as (N_i, Γ_i), with the same generating set. Now attach an $(n-1)$-cell g^{n-1}_j along each sphere $e^{n-2}_{i,j} \cup f^{n-2}_{i,j}$ to obtain a pair

$$\big(N_i \cup \{f^{n-2}_{i,j}\}^3_{j=1} \cup \{g^{n-2}_{i,j}\}^3_{j=1}, \Gamma_i \cup \{f^{n-2}_{i,j}\}^3_{j=1}\big)$$

with trivial $\mathbb{Z}G_i$-homology in all dimensions. It follows that

$$\Gamma_i \cup \{f^{n-2}_{i,j}\}^3_{j=1} \hookrightarrow N_i \cup \{f^{n-2}_{i,j}\}^3_{j=1} \cup \{g^{n-1}_{i,j}\}^3_{j=1}$$

is a homotopy equivalence. But notice that each $g^{n-1}_{i,j}$ has a free face $f^{n-2}_{i,j}$, so

$$N_i \cup \{f^{n-2}_{i,j}\}^3_{j=1} \cup \{g^{n-1}_{i,j}\}^3_{j=1}$$

collapses onto N_i. Therefore, N_i is homotopy equivalent to $\Gamma_i \cup \{f^{n-2}_{i,j}\}^3_{j=1}$.

7. Remaining questions

In the introduction we commented that nearly pseudocollarable manifolds admit arbitrarily small clean neighborhoods of infinity N containing compact codimension 0 submanifolds A for which $A \hookrightarrow N$ is a homotopy equivalence. Call such a pair (N, A) a *wide homotopy collar*. The difference, of course, between a wide homotopy collar and a homotopy collar is that, in the latter, the subspace is required to be the (codimension 1) boundary of N. The fact that nearly pseudocollarable manifolds contain arbitrarily small wide homotopy collars is immediate from the following easy lemma.

Lemma 7.1. *Suppose N' is a* $(\mathrm{mod}\ J)$*-homotopy collar neighborhood of infinity in a manifold M^n ($n \geq 5$), where J is a normally finitely generated subgroup of* $\ker(\pi_1(N') \to \pi_1(M^n))$*. Then M^n contains a wide homotopy collar neighborhood of infinity (N, A), where $N' \subseteq N \subseteq M^n$.*

Proof. Choose a finite collection of pairwise disjoint properly embedded 2-disks $\{D_i^2\}_{i=1}^k$ in $\overline{M^n - N'}$, with boundaries comprising a normal generating set for $\ker(\pi_1(N') \to \pi_1(M^n))$. Then let (N, A) be a regular neighborhood pair for

$$\big(N' \cup \big(\textstyle\bigcup_{i=1}^k D_i^2\big), \partial N' \cup \big(\textstyle\bigcup_{i=1}^k D_i^2\big)\big)$$

and apply Lemma 4.3. □

The following seem likely but, thus far, we have been unable to find proofs.

Questions. Must a manifold with compact boundary that contains arbitrarily small wide homotopy collar neighborhoods of infinity be nearly pseudocollarable? More specifically, can it be shown that the nonpseudocollarable examples in Section 6 do not contain arbitrarily small wide homotopy collar neighborhoods of infinity?

References

[Buoncristiano et al. 1976] S. Buoncristiano, C. P. Rourke, and B. J. Sanderson, *A geometric approach to homology theory*, London Mathematical Society Lecture Note Series **18**, Cambridge University Press, 1976. MR Zbl

[Chapman and Siebenmann 1976] T. A. Chapman and L. C. Siebenmann, "Finding a boundary for a Hilbert cube manifold", *Acta Math.* **137**:3-4 (1976), 171–208. MR Zbl

[Davis and Kirk 2001] J. F. Davis and P. Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics **35**, American Mathematical Society, Providence, RI, 2001. MR Zbl

[Freedman and Quinn 1990] M. H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series **39**, Princeton University Press, 1990. MR Zbl

[Geoghegan 2008] R. Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics **243**, Springer, 2008. MR Zbl

[Guilbault 2000] C. R. Guilbault, "Manifolds with non-stable fundamental groups at infinity", *Geom. Topol.* **4** (2000), 537–579. MR Zbl

[Guilbault 2016] C. R. Guilbault, "Ends, shapes, and boundaries in manifold topology and geometric group theory", pp. 45–125 in *Topology and geometric group theory* (Columbus, OH, 2010–2011), edited by M. W. Davis et al., Proc. Math. Stat. **184**, Springer, Cham, 2016. Zbl

[Guilbault and Gu ≥ 2017] C. R. Guilbault and S. Gu, "Compactifications of manifolds with boundary", in progress.

[Guilbault and Tinsley 2003] C. R. Guilbault and F. C. Tinsley, "Manifolds with non-stable fundamental groups at infinity, II", *Geom. Topol.* **7** (2003), 255–286. MR Zbl

[Guilbault and Tinsley 2006] C. R. Guilbault and F. C. Tinsley, "Manifolds with non-stable fundamental groups at infinity, III", *Geom. Topol.* **10** (2006), 541–556. MR Zbl

[Guilbault and Tinsley 2013] C. R. Guilbault and F. C. Tinsley, "Spherical alterations of handles: embedding the manifold plus construction", *Algebr. Geom. Topol.* **13**:1 (2013), 35–60. MR Zbl

[Lyndon and Schupp 1977] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik **89**, Springer, Berlin, 1977. Reprinted in the series Classics in Mathematics **89**, Springer, Berlin, 2001. MR Zbl

[Massey 1967] W. S. Massey, *Algebraic topology: an introduction*, Harcourt, New York, 1967. MR Zbl

[Rourke and Sanderson 1968] C. P. Rourke and B. J. Sanderson, "Block bundles, II: Transversality", *Ann. of Math.* (2) **87** (1968), 256–278. MR Zbl

[Rourke and Sanderson 1972] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Ergebnisse der Mathematik **69**, Springer, Berlin, 1972. Reprinted in the series Springer Study Edition **69**, Springer, Berlin, 1982. MR Zbl

[Siebenmann 1965] L. C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Ph.D. thesis, Princeton University, 1965, available at http://search.proquest.com/docview/302173863. MR

[Stallings 1965] J. Stallings, "Homology and central series of groups", *J. Algebra* **2** (1965), 170–181. MR Zbl

[Stammbach 1966] U. Stammbach, "Anwendungen der Homologietheorie der Gruppen auf Zentralreihen und auf Invarianten von Präsentierungen", *Math. Z.* **94** (1966), 157–177. MR Zbl

[Wall 1965] C. T. C. Wall, "Finiteness conditions for CW-complexes", *Ann. of Math.* (2) **81** (1965), 56–69. MR Zbl

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p-ADIC VARIATION OF UNIT ROOT L-FUNCTIONS

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Dwork's conjecture, now proven by Wan, states that unit root L-functions "coming from geometry" are p-adic meromorphic. In this paper we study the p-adic variation of a family of unit root L-functions coming from a suitable family of toric exponential sums. In this setting, we find that the unit root L-functions each have a unique p-adic unit root. We then study the variation of this unit root over the family of unit root L-functions. Surprisingly, we find that this unit root behaves similarly to the classical case of families of exponential sums, as studied by Adolphson and Sperber (2012). That is, the unit root is essentially a ratio of $\mathcal{A}$-hypergeometric functions.

1. Introduction

Dwork [1973] conjectured that certain L-functions, constructed as Euler products of p-adic unit roots coming from the fibers of an algebraic family of L-functions, are p-adic meromorphic. He proved this in a few cases using the idea of an excellent lifting of Frobenius, but was unable to prove it in general, mainly because excellent lifting in its original form does not always exist. Wan [1999; 2000b; 2000a] proved Dwork's conjecture using a new technique which avoided excellent lifting. In the present paper, we extend Wan's techniques, as established in [Haessig 2014], by constructing a dual theory in which to study the p-adic variation of unit root L-functions.

Let Ψ be a nontrivial additive character on the finite field $\mathbb{F}_q$. Additionally, let $f \in \mathbb{F}_q[\lambda_1^{\pm}, \dots, \lambda_s^{\pm}, x_1^{\pm}, \dots, x_n^{\pm}]$ be a Laurent polynomial, and consider for each $\bar{\lambda} \in (\bar{\mathbb{F}}_q^{\times})^s$ and $m \geq 1$ the exponential sum

$$S_m(f, \bar{\lambda}) := \sum_{\bar{x} \in (\mathbb{F}_{q^{m \cdot \deg(\bar{\lambda})}}^{\times})^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{m \cdot \deg(\bar{\lambda})}} / \mathbb{F}_q}(f(\bar{\lambda}, \bar{x}))$$

where $\deg(\bar{\lambda}) := [\mathbb{F}_q(\bar{\lambda}) : \mathbb{F}_q]$. Define the associated L-function by $L(f, \bar{\lambda}, T) := \exp\left(\sum_{m \geq 1} S_m(f, \bar{\lambda}) T^m / m\right)$. It is known that $L(f, \bar{\lambda}, T)^{(-1)^{n+1}}$ is a rational function

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with a unique p-adic unit root, say $\pi_0(\bar{\lambda})$, which is also a 1-unit. The unit root L-function of this family is defined by

$$L_{\mathrm{unit}}(\kappa, T) := \prod_{\bar{\lambda} \in |\mathbb{G}_m^s/\mathbb{F}_q|} \frac{1}{1 - \pi_0(\bar{\lambda})^{\kappa} T^{\deg(\bar{\lambda})}},$$

where κ takes on values in the p-adic integers $\mathbb{Z}_p$. This is a p-adic meromorphic function in T and a p-adic continuous function in κ. As shown in a remark below, $L_{\mathrm{unit}}(\kappa, T)^{(-1)^{s+1}}$ will have a unique p-adic unit root. We conjecture that the unit root will have the following description.

Writing $f(\lambda, x) = \sum a_{\gamma,u} \lambda^{\gamma} x^u$ with $a_{\gamma,u} \in \mathbb{F}_q$, let $\hat{a}_{\gamma,u}$ be the Teichmüller lift of $a_{\gamma,u}$ and write $\hat{a} = (\hat{a}_{\gamma,u})_{(\gamma,u)\in \mathrm{supp}(f)}$. Let $\pi \in \overline{\mathbb{Q}}_p$ be such that $\pi^{p-1} = -p$. Define a new polynomial $\tilde{f}$ by replacing the coefficients of f by new variables $\Lambda_{\gamma,u}$ for each monomial $\lambda^{\gamma} x^u$, that is, define $\tilde{f}(\Lambda, \lambda, x) := \sum \Lambda_{\gamma,u} \lambda^{\gamma} x^u$. Writing $\exp \pi \tilde{f}(\Lambda, \lambda, x) = \sum g_{\gamma,u}(\Lambda) \lambda^{\gamma} x^u$, Adolphson and Sperber [2012] have shown that $\mathcal{G}(\Lambda) := g_{0,0}(\Lambda)/g_{0,0}(\Lambda^p)$ converges on the closed polydisk $|\Lambda_{\gamma,u}|_p \le 1$. Thus, it makes sense to evaluate $\mathcal{G}(\hat{a}) := \mathcal{G}(\Lambda)|_{\Lambda=\hat{a}}$. We conjecture that the unit root of $L_{\mathrm{unit}}(\kappa, T)^{(-1)^{s+1}}$ is of the form $(\mathcal{G}(\hat{a})\mathcal{G}(\hat{a}^p) \cdots \mathcal{G}(\hat{a}^{p^{a-1}}))^{\kappa}$ where $q = p^a$.

Our first main result will be to prove this conjecture when $f(\lambda, x)$ satisfies a lower deformation hypothesis stated below. Our second main result, which explains the paper's length, is the development of a dual theory for L-functions of infinite symmetric powers $L^{(0)}(\kappa, \bar{t}, T)$, defined on page 137. These seem to have a theory similar to that of classical L-functions of exponential sums over finite fields. For example, they display the same type of δ-structure (10) as well as having an attached p-adic cohomology theory (see, e.g., [Haessig 2016]). There is some slight evidence that these may be related to p-adic automorphic forms.

As mentioned above, in this paper we study the p-adic variation of unit root L-functions such as these. The following setup is similar to that of the above family, but more technical for the following reason. As unit root L-functions come from families, and we wish to study a family of unit root L-functions, we need to consider a family of families. The role of the variables in the following is: x denotes the space variables, λ denotes the parameters of the family, and t denotes the parameters defining the family of families.

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^n$. We define the Newton polyhedron of $\mathcal{A}$ at ∞, denoted $\Delta_\infty(\mathcal{A})$, to be the convex closure of $\mathcal{A} \cup 0$ in $\mathbb{R}^n$. We make the simplifying hypothesis that every element $u \in \mathcal{A}$ lies on the Newton boundary at ∞ of $\Delta_\infty(\mathcal{A})$, that is, the union of all faces of $\Delta_\infty(\mathcal{A})$ which do not contain the origin. In other language this is the same as the hypothesis that $w(u) = 1$ for all $u \in \mathcal{A}$, where w is the usual polyhedral weight defined by $\Delta_\infty(\mathcal{A})$ (see the next section for definition). The generic polynomial f, with x-support equal to $\mathcal{A}$, is given by $f(t, x) = \sum t_u x^u \in \mathbb{F}_q[\{t_u\}_{u\in\mathcal{A}}, x_1^{\pm}, \dots, x_n^{\pm}]$, where u runs over $\mathcal{A}$ and $\{t_u\}_{u\in\mathcal{A}}$ are new variables. Let $\Delta_\infty(f)(=\Delta_\infty(\mathcal{A}))$ be the Newton polyhedron at infinity of f. Let $P(\lambda, x) \in \mathbb{F}_q[\lambda_1^\pm, \dots, \lambda_s^\pm, x_1^\pm, \dots, x_n^\pm]$ be such that the monomials $\lambda^\gamma x^v$ in the support of $P(\lambda, x)$ all satisfy $0 < w(v) < 1$. Such deformations were studied in [Haessig and Sperber 2014]. It is convenient to assume the origin is not in the set $\mathcal{A}$ and if $\lambda^\gamma x^v$ is in the support of P, then $v \neq 0$ so that neither f nor P have a constant term (with respect to the x-variables). This assumption will be made throughout this work. Let $G(t, \lambda, x) := f(t, x) + P(\lambda, x)$.

We construct a family of L-functions as follows. Let $\bar{t} \in (\overline{\mathbb{F}}_q^*)^{|\mathcal{A}|}$, and denote by $\deg(\bar{t}) = [\mathbb{F}_q(\bar{t}) : \mathbb{F}_q]$ the degree of $\bar{t}$, where $\mathbb{F}_q(\bar{t})$ means we adjoin every coordinate of $\bar{t}$ to $\mathbb{F}_q$. We will often write $d(\bar{t})$ for $\deg(\bar{t})$. For convenience, write $q_{\bar{t}} := q^{d(\bar{t})}$ so that $\mathbb{F}_{q_{\bar{t}}} = \mathbb{F}_q(\bar{t})$. Next, let $\bar{\lambda} \in (\overline{\mathbb{F}}_q^*)^s$. Denote by $\deg_{\bar{t}}(\bar{\lambda})$ or $d_{\bar{t}}(\bar{\lambda})$ the degree $[\mathbb{F}_{q_{\bar{t}}}(\bar{\lambda}) : \mathbb{F}_{q_{\bar{t}}}]$; set $q_{\bar{t},\bar{\lambda}} := q_{\bar{t}}^{d_{\bar{t}}(\bar{\lambda})}$ and $\mathbb{F}_{q_{\bar{t},\bar{\lambda}}} = \mathbb{F}_{q_{\bar{t}}}(\bar{\lambda})$. For each $m \geq 1$, define the exponential sum

$$S_m(\bar{t}, \bar{\lambda}) := \sum_{\bar{x} \in (\mathbb{F}^*_{q^m_{\bar{t},\bar{\lambda}}})^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m_{\bar{t},\bar{\lambda}}}/\mathbb{F}_q}(G(\bar{t}, \bar{\lambda}, \bar{x}))$$

and its associated L-function

$$L(\bar{t}, \bar{\lambda}, T) := \exp\biggl(\sum_{m=1}^\infty S_m(\bar{t}, \bar{\lambda}) \frac{T^m}{m}\biggr).$$

It is well-known [Adolphson and Sperber 2012] that $L(\bar{t}, \bar{\lambda}, T)^{(-1)^{n+1}}$ has a unique reciprocal p-adic unit root $\pi_0(\bar{t}, \bar{\lambda})$, which is a 1-unit. Let $\kappa \in \mathbb{Z}_p$ be a p-adic integer. For each $\bar{t}$, the unit root L-function is defined by

$$L_{\mathrm{unit}}(\kappa, \bar{t}, T) := \prod_{\bar{\lambda} \in |\mathbb{G}_m^s/\mathbb{F}_{q_{\bar{t}}}|} \frac{1}{1 - \pi_0(\bar{t}, \bar{\lambda})^\kappa T^{d_{\bar{t}}(\bar{\lambda})}},$$

where κ takes values in the p-adic integers $\mathbb{Z}_p$. Wan's theorem tells us that this L-function is p-adic meromorphic and so may be written as a quotient of p-adic entire functions:

$$L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}} = \frac{\prod_{i=1}^\infty (1 - \alpha_i(\kappa, \bar{t})T)}{\prod_{j=1}^\infty (1 - \beta_j(\kappa, \bar{t})T)}, \quad \alpha_i, \beta_j \to 0 \text{ as } i, j \to \infty.$$

Little is known about the zeros and poles of unit root L-functions. In Theorem 1.1 below we show, for each $\bar{t}$ and κ, that $L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ itself has a unique unit zero (and no unit poles), which is a 1-unit. We then study the variation of this unit root as a function of $\bar{t}$ and κ. We note that the variation of the unit root L-function with respect to the parameter κ has been studied before in Wan's proof of Dwork's conjecture, and is connected to the Gouvêa–Mazur conjecture [1992]. On the other hand, as far as we know, the study of the p-adic analytic variation of the unit root

L-function with respect to $\bar{t}$ is new. To state the main result, first denote by $\pi \in \overline{\mathbb{Q}}_p$ an element satisfying $\pi^{p-1} = -p$. Next, writing

$$\begin{aligned} G(t, \lambda, x) &= f(t, x) + P(\lambda, x) \\ &= \sum t_u x^u + \sum A(\gamma, v)\lambda^\gamma x^v \\ &\in \mathbb{F}_q[x_1^{\pm}, \ldots, x_n^{\pm}, \lambda_1^{\pm}, \ldots, \lambda_s^{\pm}, \{t_u\}_{u \in \mathrm{supp}(f)}], \end{aligned}$$

let $\hat{A}(\gamma, v)$ be the Teichmüller lift of $A(\gamma, v)$ in $\mathbb{Q}_q$ for each $(\gamma, v) \in \mathrm{supp}(P)$. We now replace every coefficient $A(\gamma, v)$ of $P(\lambda, x)$ with a new variable Λ: set $\mathcal{P}(\Lambda, \lambda, x) := \sum_{(\gamma,v)\in\mathrm{supp}(P)} \Lambda_{\gamma,v}\lambda^\gamma x^v$ and define

$$H(t, \Lambda, \lambda, x) := f(t, x) + \mathcal{P}(\Lambda, \lambda, x).$$

Note that the series

$$\exp(\pi H(t, \Lambda, \lambda, x)) = \sum_{\substack{\gamma\in\mathbb{Z}^s \\ u\in\mathbb{Z}^n}} K_{\gamma,u}(t, \Lambda)\lambda^\gamma x^u$$

is well-defined, and its coefficients $K_{\gamma,u}(t, \Lambda)$ are themselves elements in the power-series ring $\mathbb{Z}_p[\zeta_p][[\{t_u\}_{u\in\mathcal{A}}, \{\Lambda_{\gamma,v}\}_{(\gamma,v)\in\mathrm{supp}(P)}]]$, and so converge in the open polydisk $D(0, 1^-)^{|\mathcal{A}|+|\,\mathrm{supp}(P)|}$ which is defined by the inequalities $|t_u| < 1$ for all $u \in \mathcal{A}$ and $|\Lambda_{\gamma,v}| < 1$ for all $(\gamma, v) \in \mathrm{supp}(P)$. Of particular interest is $K_{0,0}(t)$, a principal p-adic unit for all t and Λ in the polydisk. Define $\mathcal{F}(t, \Lambda) := K_{0,0}(t, \Lambda)/K_{0,0}(t^p, \Lambda^p)$ and set $\mathcal{F}_m(t, \Lambda) := \prod_{i=0}^{m-1} \mathcal{F}(t^{p^i}, \Lambda^{p^i})$. By Adolphson and Sperber [2012], $\mathcal{F}(t, \Lambda)$ analytically continues to the closed polydisc $D(0,1^+)^{|\mathcal{A}|+|\mathrm{supp}(P)|}$ defined by $|t_u| \le 1$, $u \in \mathcal{A}$ and $|\Lambda_{\gamma,v}| \le 1$, $(\gamma, v) \in \mathrm{supp}(P)$.

Theorem 1.1. *Let $\hat{t}$ be the Teichmüller lift of $\bar{t}$. Then*

$$\mathcal{F}_{ad(\bar{t})}(\hat{t}, \hat{A})^\kappa = \prod_{i=0}^{ad(\bar{t})} \mathcal{F}(\hat{t}^{p^i}, \hat{A}^{p^i})^\kappa$$

is the unique unit root of $L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ at each fiber $\bar{t}$ and $\kappa \in \mathbb{Z}_p$, where $\mathcal{F}_{ad(\bar{t})}(\hat{t}, \hat{A})$ means setting each $t_u = \hat{t}_u$ and $\Lambda_{\gamma,v} = \hat{A}(\gamma, v)$.

Remark. It is worthwhile to compare this result to the result in [Adolphson and Sperber 2012]. To that end, consider the (total) family $H(t, \Lambda, \lambda, x)$ above. For each $\bar{t} \in (\overline{\mathbb{F}}_q^\times)^{|\mathcal{A}|}$ and $m \ge 1$, define the exponential sum

$$S_m(H, \bar{t}) := \sum_{(\bar{\lambda},\bar{x})\in(\mathbb{F}^\times_{q^{m\cdot\deg(\bar{t})}})^s\times(\mathbb{F}^\times_{q^{m\cdot\deg(\bar{t})}})^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{m\cdot\deg(\bar{t})}}/\mathbb{F}_q}(H(\bar{t}, A, \bar{\lambda}, \bar{x})).$$

Define by $L(H, \bar{t}, T) := \exp\left(\sum_{m\ge1} S_m(H, \bar{t})T^m/m\right)$ the associated L-function, a rational function over $\mathbb{Q}(\zeta_p)$. By [Adolphson and Sperber 2012], $L(H, \bar{t}, T)^{(-1)^{s+n+1}}$

has a unique p-adic unit root given by $\mathcal{F}_{ad(\bar{t})}(\hat{t}, \hat{A})$. As mentioned above, this relation should conjecturally hold in greater generality.

Remark. The existence of a unique p-adic unit root is a general result for unit root L-functions defined over the torus $\mathbb{G}_m^s$. This includes the classical case of L-functions of exponential sums defined over the torus; see [Haessig 2014, Section 3] for details.

To give an indication of the proof, we use the language of σ-modules. See [Haessig 2014] as a reference for the following notation. Let K be a finite extension field of $\mathbb{Q}_p$ with uniformizer π, ring of integers R, and residue field $\mathbb{F}_q$. Let (M, ϕ) be a $c \cdot \log$-convergent, nuclear σ-module over R, ordinary at slope zero of rank one ($h_0 = 1$) with basis $\{e_i\}_{i \geq 0}$. Assume further the normalization condition $\phi e_0 \equiv e_0 \bmod(\pi)$ and $\phi e_i \equiv 0 \bmod(\pi)$ for all $i \geq 1$. With this setup, it follows that the associated unit root L-function $L_{\text{unit}}(\kappa, \phi, T)^{(-1)^{s+1}}$ has a unique p-adic unit root (and no unit poles). To see this we first note that, by [Haessig 2014], $L_{\text{unit}}(\kappa, \phi, T)^{(-1)^{s+1}} \equiv \det(1 - F_{B^{[\kappa]}}T) \bmod \pi$. Next, it follows from the normalization condition that the matrix $B^{[\kappa]}$ takes the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \bmod \pi$, and thus $\det(1 - F_{B^{[\kappa]}}T) \equiv 1 - T \bmod \pi$. Hence, the Fredholm determinant $\det(1 - F_{B^{[\kappa]}}T)$ has a unique p-adic unit root proving the result.

2. Lower deformation family

Let $f \in \mathbb{F}_q[\{t_u\}_{u \in \operatorname{supp}(f)}, x_1^{\pm}, \dots, x_n^{\pm}]$ be of the form $f(t, x) = \sum t_u x^u$. In particular, the coefficient of every monomial x^u in f is a new variable t_u. Denote by $\Delta_\infty(f)$ the Newton polytope at infinity of f, defined as the convex closure of $\operatorname{supp}(f) \cup \{0\}$ in $\mathbb{R}^n$. Let $\operatorname{Cone}(f)$ be the union of all rays emanating from the origin and passing through $\Delta_\infty(f)$, and set $M := M(f) := \operatorname{Cone}(f) \cap \mathbb{Z}^n$. We define a weight function w on M as follows. For $u \in M$, let $w(u)$ be the smallest nonnegative rational number such that $u \in w(u)\Delta(f)$. It is convenient to assume $w(u) = 1$ for all u in the x-support of f. In particular this implies that f has no constant term. Let D denote the smallest positive integer such that $w(M) \subset (1/D)\mathbb{Z}_{\geq 0}$. The weight function w satisfies the following norm-like properties:

(1) $w(u) = 0$ if and only if $u = 0$.

(2) $w(cu) = cw(u)$ for every $c \geq 0$.

(3) $w(u + v) \leq w(u) + w(v)$ for every $u, v \in M$, with equality holding if and only if u and v are cofacial.

It is convenient to assume the lower-order deformation $P \in \mathbb{F}_q[\lambda_1^{\pm}, \dots, \lambda_s^{\pm}, x_1^{\pm}, \dots, x_n^{\pm}]$ has no constant term so the origin in $\mathbb{R}^n$ is not in the x-support of P. In fact, if we write $P(\lambda, x) = \sum_{u \in M} P_u(\lambda) x^u$, then $0 < w(u) < 1$. Our lower deformation family then is defined by $G(t,\lambda,x) := f(t,x) + P(\lambda,x)$. Set

$$U := \left\{ \left(\tfrac{1}{1-w(u)}\right)\gamma \in \mathbb{Q}^s \;\middle|\; (\gamma,u) \in \operatorname{supp}(P) \right\}, \tag{1}$$

and let $\Gamma := \Delta_\infty(U) \subset \mathbb{R}^s$. Similarly, define $M(\Gamma) := \operatorname{Cone}(\Gamma) \cap \mathbb{Z}^s$ with associated polyhedral weight function w_Γ. The polyhedral weight makes sense as well on points in $\operatorname{Cone}(\Gamma)$ having real coordinates. Since $0 < w(u) < 1$ for $(\gamma,u) \in \operatorname{supp}(P)$, it follows that $w_\Gamma(\delta) < 1$ for any $\delta = \gamma/(1-w(u)) \in U$. Equivalently, $w_\Gamma(\gamma) < 1$ for any $(\gamma,u) \in \operatorname{supp}(P)$. We call Γ the *relative polytope* of the family $G(x,t)$.

Rings of p-adic analytic functions. Let ζ_p be a primitive p-th root of unity, $\mathbb{Q}_q$ be the unramified extension of $\mathbb{Q}_p$ of degree $a := [\mathbb{F}_q : \mathbb{F}_p]$, and denote by $\mathbb{Z}_q$ its ring of integers. Then $\mathbb{Z}_q[\zeta_p]$ and $\mathbb{Z}_p[\zeta_p]$ are the rings of integers of $\mathbb{Q}_q(\zeta_p)$ and $\mathbb{Q}_p(\zeta_p)$, respectively. Let $\pi \in \overline{\mathbb{Q}}_p$ satisfy $\pi^{p-1} = -p$, and let $\tilde{\pi}$ be an element which satisfies $\operatorname{ord}_p(\tilde{\pi}) = (p-1)/p^2$. We may have occasion to work over a purely ramified extension $\Omega_0 = \mathbb{Q}_p(\hat{\pi})$ of $\mathbb{Q}_p$ with uniformizer $\hat{\pi}$ which contains $\mathbb{Q}_p(\zeta_p, \tilde{\pi})$ and for which $\tilde{\pi}$ is an integral power of $\hat{\pi}$. Let $\Omega = \mathbb{Q}_q(\hat{\pi})$. Denote by R the ring of integers of Ω, and R_0 the ring of integers of Ω_0. Set

$$\mathcal{O}_0 := \left\{ \sum_{\gamma \in M(\Gamma)} C(\gamma) \tilde{\pi}^{w_\Gamma(\gamma)} \lambda^\gamma \;\middle|\; C(\gamma) \in R,\, C(\gamma) \to 0 \text{ as } \gamma \to \infty \right\}.$$

(We note that the fractional powers of $\tilde{\pi}$ are to be understood as integral powers of a uniformizer of R.) Then $\mathcal{O}_0$ is a ring with a discrete valuation given by

$$\left| \sum_{\gamma \in M(\Gamma)} C(\gamma) \lambda^\gamma \tilde{\pi}^{w_\Gamma(\gamma)} \right| := \sup_{\gamma \in M(\Gamma)} |C(\gamma)|.$$

Define

$$\mathcal{C}_0(\mathcal{O}_0) := \left\{ \xi = \sum_{\mu \in M(\bar{f})} \xi(\mu) \tilde{\pi}^{w(\mu)} x^\mu \;\middle|\; \xi(\mu) \in \mathcal{O}_0,\, \xi(\mu) \to 0 \text{ as } \mu \to \infty \right\},$$

an $\mathcal{O}_0$-algebra.

In the following, $q = p^a$ is an arbitrary power of p (including the case $a = 0$), so we can handle the cases of t^q, t^p, and t at the same time. Define

$$\mathcal{O}_{0,q} := \left\{ \sum_{\gamma \in M(\Gamma)} C(\gamma) \lambda^\gamma \tilde{\pi}^{w_{q\Gamma}(\gamma)} \;\middle|\; C(\gamma) \in R,\, C(\gamma) \to 0 \text{ as } \gamma \to \infty \right\}. \tag{2}$$

This ring is the same as $\mathcal{O}_0$ except using a weight function defined by the dilation $q\Gamma$ (that is, $w_{q\Gamma}(\gamma) = w_\Gamma(\gamma)/q$). We note that here $\mathcal{O}_{0,1} = \mathcal{O}_0$. A discrete valuation may be defined as follows. If $\xi = \sum_{\gamma \in M(\Gamma)} C(\gamma) \tilde{\pi}^{w_{q\Gamma}(\gamma)} \lambda^\gamma \in \mathcal{O}_{0,q}$ then the valuation on $\mathcal{O}_{0,q}$ is given by

$$|\xi| := \sup_{\gamma \in M(\Gamma)} |C(\gamma)|.$$

We may also define the space

$$\mathcal{C}_0(\mathcal{O}_{0,q}) := \left\{ \sum_{u \in M(f)} \xi_u x^u \tilde{\pi}^{w(u)} \,\middle|\, \xi_u \in \mathcal{O}_{0,q}, \xi_u \to 0 \text{ as } u \to \infty \right\}. \tag{3}$$

For $\eta = \sum_{u \in M(\bar{f})} \xi_u \tilde{\pi}^{w(u)} x^u \in \mathcal{C}_0(\mathcal{O}_{0,q})$, we set

$$|\eta| = \sup_{u \in M(f)} |\xi_u|.$$

Frobenius. At present, we fix $\bar{t} \in (\bar{\mathbb{F}}_q)^{|A|}$, returning to variation in $\bar{t}$ in the last section. Recall the notation $\deg(\bar{t}) = d(\bar{t}) = [\mathbb{F}_q(\bar{t}) : \mathbb{F}_q]$ and $q_{\bar{t}} = q^{d(\bar{t})}$. Now let $\bar{\lambda} \in (\bar{\mathbb{F}}_q)^s$. Similarly, denote by $\deg(\bar{\lambda})$ or $d(\bar{\lambda})$ the degree $[\mathbb{F}_q(\bar{\lambda}, \bar{t}) : \mathbb{F}_q(\bar{t})]$, and $q_{\bar{t},\bar{\lambda}} = q^{d(\bar{t})d(\bar{\lambda})}$.

Dwork defines a splitting function by $\theta(T) := \exp \pi(T - T^p) = \sum_{i=0}^{\infty} \theta_i T^i$. It is well-known that $\mathrm{ord}_p(\theta_i) \geq i(p-1)/p^2$ for all $i \geq 0$. Writing

$$G(\bar{t}, \lambda, x) = f(\bar{t}, x) + P(\lambda, x) = \sum \bar{t}_u x^u + \sum \bar{A}(\gamma, v) \lambda^\gamma x^v$$

in $\mathbb{F}_{q_{\bar{t}}}[x_1^{\pm}, \dots, x_n^{\pm}, \lambda_1^{\pm}, \dots, \lambda_s^{\pm}]$, we let

$$\hat{G}(\hat{t}, \lambda, x) := \sum \hat{t}_u x^u + \sum \hat{A}(\gamma, v) \lambda^\gamma x^v \in R[x_1^{\pm}, \dots, x_n^{\pm}, \lambda_1^{\pm}, \dots, \lambda_s^{\pm}]$$

be the lifting of G by lifting the coefficients $\bar{A}(\gamma, u)$ and $\bar{t}$ by Teichmüller units. Set

$$F(\hat{t}, \lambda, x) := \prod_{u \in \mathrm{supp}(f)} \theta(\hat{t}_u x^u) \cdot \prod_{(\gamma, v) \in \mathrm{supp}(P)} \theta(\hat{A}(\gamma, v) \lambda^\gamma x^v) \tag{4}$$

and for any $m \geq 1$,

$$F_m(\hat{t}, \lambda, x) := \prod_{i=0}^{m-1} F^{\sigma^i}(\hat{t}, \lambda^{p^i}, x^{p^i}), \tag{5}$$

where σ is the extension of the usual Frobenius generator of $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ to Ω with $\sigma(\hat{\pi}) = \hat{\pi}$. Then, σ acts on a series with coefficients in Ω by acting on these coefficients. Note that if we set

$$F_m(\hat{t}, \lambda, x) = \sum_{u \in M(f)} \mathcal{B}^m(u) x^u = \sum_{\substack{\gamma \in M(\Gamma) \\ u \in M(f)}} \mathcal{B}^m(\gamma, u) \lambda^\gamma x^u,$$

then

$$\mathrm{ord}_p(\mathcal{B}^m(\gamma, u)) \geq \frac{w_\Gamma(\gamma) + w(u)}{p^{m-1}} \cdot \frac{p-1}{p^2}.$$

Define ψ_x by $\sum C(u)x^u \mapsto \sum C(pu)x^u$. Set

$$\alpha_1 := \sigma^{-1} \circ \psi_x \circ F(\hat{t}, \lambda, x).$$

A similar argument to that in [Haessig and Sperber 2014] demonstrates that α_1 is a σ^{-1}-semilinear map of $\mathcal{C}_0(\mathcal{O}_0)$ into $\mathcal{C}_0(\mathcal{O}_{0,p})$. Similarly, for $m \geq 1$, if we define

$$\alpha_m := \sigma^{-m} \circ \psi_x^m \circ F_m(\hat{t}, \lambda, x),$$

then α_m maps $\mathcal{C}_0(\mathcal{O}_0)$ into $\mathcal{C}_0(\mathcal{O}_{0,p^m})$. In particular,

$$\alpha_m(\tilde{\pi}^{w(v)} x^v) = \sum_{u \in M(f)} \tilde{\pi}^{w(v)-w(u)} \mathcal{B}^m(p^m u - v) \tilde{\pi}^{w(u)} x^u,$$

with $\operatorname{ord}_p(\tilde{\pi}^{w(v)-w(u)} \mathcal{B}^m(p^m u - v)) \geq \big((p-1)w(u) + (1 - 1/p^{m-1})w(v)\big) \operatorname{ord}_p(\tilde{\pi})$. Summarizing, in $\mathcal{C}_0(\mathcal{O}_{0,p^m})$ we have $|\alpha_m(\tilde{\pi}^{w(v)} x^v)| \leq |\tilde{\pi}|^{w(v)(p^{m-1}-1)/p^{m-1}}$.

Fibers. Define

$$\alpha_{\bar{t},\bar{\lambda}} := \psi_x^{ad(\bar{t})d(\bar{\lambda})} \circ F_{ad(\bar{t})d(\bar{\lambda})}(\hat{t}, \hat{\lambda}, x),$$

where $\hat{t}$ and $\hat{\lambda}$ are the Teichmüller representatives of $\bar{t}$ and $\bar{\lambda}$, respectively. Notice that $\alpha_{\bar{t},\bar{\lambda}}$ is an endomorphism of $\mathcal{C}_0(\hat{\lambda})$, where $\mathcal{C}_0(\hat{\lambda})$ denotes the space obtained from $\mathcal{C}_0(\mathcal{O}_0)$ by applying the map on $\mathcal{O}_0$ which sends λ to $\hat{\lambda}$.

To relate the L-function $L(\bar{t}, \bar{\lambda}, T)$ to the operator $\alpha_{\bar{t},\bar{\lambda}}$ it is convenient to introduce the following operation: for any function $g(T)$, define $g(T)^{\delta_q} := g(T)/g(qT)$. Set $q_{\bar{t},\bar{\lambda}} := q^{d(\bar{t})d(\bar{\lambda})}$. Dwork's trace formula states

$$(q_{\bar{t},\bar{\lambda}}^m - 1)^n \operatorname{Tr}(\alpha_{\bar{t},\bar{\lambda}}^m \mid \mathcal{C}_0(\hat{\lambda})) = \sum_{\bar{x} \in (\mathbb{F}^*_{q^m_{\bar{t},\bar{\lambda}}})^n} \Psi \circ \operatorname{Tr}_{\mathbb{F}_{q^m_{\bar{t},\bar{\lambda}}}/\mathbb{F}_q}(G(\bar{t}, \bar{\lambda}, \bar{x}))$$

Equivalently,

$$L(\bar{t}, \bar{\lambda}, T)^{(-1)^{n+1}} = \det\big(1 - \alpha_{\bar{t},\bar{\lambda}} T \mid \mathcal{C}_0(\hat{\lambda})\big)^{\delta^n_{q_{\bar{t},\bar{\lambda}}}}.$$

This is a rational function, and it is well-known that $L(\bar{t}, \bar{\lambda}, T)^{(-1)^{n+1}}$ has a unique unit (reciprocal) root $\pi_0(\bar{t}, \bar{\lambda})$ (see [Adolphson and Sperber 2012], for example). This unit root is a 1-unit, so it makes sense to define, for any p-adic integer κ, the unit root L-function at the fiber $\bar{t}$:

$$L_{\text{unit}}(\kappa, \bar{t}, T) := \prod_{\bar{\lambda} \in |\mathbb{G}_m^s/\mathbb{F}_q(\bar{t})|} \frac{1}{1 - \pi_0(\bar{t}, \bar{\lambda})^{\kappa} T^{\deg(\bar{\lambda})}}. \tag{6}$$

Denote the roots of $\det(1-\alpha_{\bar{t},\bar{\lambda}}T \mid \mathcal{C}_0(\hat{\lambda}))$ by $\pi_i(\bar{t},\bar{\lambda})$, and order them such that $\mathrm{ord}_p\,\pi_i(\bar{t},\bar{\lambda}) \le \mathrm{ord}_p\,\pi_{i+1}(\bar{t},\bar{\lambda})$ for $i \ge 0$. For each $m \ge 0$, define

$$L^{(m)}(\kappa,\bar{t},T) := \prod_{\bar{\lambda}\in|\mathbb{G}_m^s/\mathbb{F}_{q_{\bar{t}}}|}\prod\bigl(1-\pi_0(\bar{t},\bar{\lambda})^{\kappa-r-m}\pi_{i_1}(\bar{t},\bar{\lambda})\cdots\pi_{i_r}(\bar{t},\bar{\lambda})\pi_{j_1}(\bar{t},\bar{\lambda})\cdots\pi_{j_m}(\bar{t},\bar{\lambda})T^{\deg(\bar{\lambda})}\bigr)^{-1},$$

where the inner product runs over all $r \ge 0$, $1 \le i_1 \le i_2 \le \cdots$, and $0 \le j_1 \le j_2 < \cdots < j_m$. Note that the factors indexed by i_k are allowed to repeat, whereas the factors indexed by j_l are distinct. Intuitively, the inner product is $\det(1-\mathrm{Sym}^{\kappa-m}\,\alpha_{\bar{t},\bar{\lambda}} \otimes \wedge^m \alpha_{\bar{t},\bar{\lambda}}T)$. From [Haessig 2014, Lemma 2.1],

$$L_{\mathrm{unit}}(\kappa,\bar{t},T) = \prod_{i=0}^{\infty} L^{(i)}(\kappa,\bar{t},T)^{(-1)^{i-1}(i-1)} = L^{(0)}(\kappa,\bar{t},T)\prod_{i\ge 2} L^{(i)}(\kappa,\bar{t},T)^{(-1)^{i-1}(i-1)}. \tag{7}$$

In the next section, we will show each $L^{(i)}$ with $i \ge 1$ has no unit root or pole, whereas $L^{(0)}$ will. This will show $L_{\mathrm{unit}}(\kappa,\bar{t},T)^{(-1)^{s+1}}$ has a unique unit root.

3. Infinite symmetric powers

Denote by $\mathcal{S}(\hat{\lambda}) := R[\hat{\lambda}][\![\{e_u\}_{u\in M\setminus\{0\}}]\!]$ the formal power series ring over $R[\hat{\lambda}]$ in the variables $\{e_u\}_{u\in M\setminus\{0\}}$ which are formal symbols indexed by $M \setminus \{0\}$. We equip this ring with the sup-norm on coefficients (in $R[\hat{\lambda}]$). This ring will play the role of the formal infinite symmetric power of $\mathcal{C}_0(\hat{\lambda})$ over $R[\hat{\lambda}]$ in a way we describe below. It is convenient to write the monomials of degree r in the variables $\{e_u\}$ using the notation $e_{\boldsymbol{u}} := e_{u_1}\cdots e_{u_r}$, where $u_1,\dots,u_r \in M(f)\setminus\{0\}$ for $r \ge 0$. To fix ideas, it helps to assume we have a linear order on $M(f)\setminus\{0\}$ with the property that if $w(u) \le w(v)$ for $u, v \in M(f)\setminus\{0\}$, then $u \le v$. We may extend this to all of $M(f)$ by taking 0 as the least element. We then agree that in this notation we have $0 < u_1 \le u_2 \le \cdots \le u_r$ (equality indicating repeated variables). When $r = 0$ we understand there is only the monomial 1 of degree 0. We extend the weight function w to such monomials by defining, for $e_{\boldsymbol{u}} := e_{u_1}\cdots e_{u_r}$, the weight $w(\boldsymbol{u}) := w(u_1)+\cdots+w(u_r)$. Denote by $\mathcal{S}(M)$ the set of all indices $\boldsymbol{u}$ corresponding to monomials $e_{\boldsymbol{u}}$. We emphasize that we will often equate elements $\boldsymbol{u} \in \mathcal{S}(M)$ with the monomials $e_{\boldsymbol{u}}$; it should be clear from the context which meaning is desired. We may assume $\mathcal{S}(M)$ has a linear order defined on it such that the weight $w(\boldsymbol{u})$ is nondecreasing and such that the restriction of this linear order to $M(f)$ is our earlier linear order.

We may identify $\mathcal{C}_0(\hat{\lambda})$ as an $R[\hat{\lambda}]$-submodule of $\mathcal{S}(\hat{\lambda})$ by defining an $R[\hat{\lambda}]$-linear map

$$\Upsilon : \mathcal{C}_0(\hat{\lambda}) \to \mathcal{S}(\hat{\lambda}) \quad \text{via} \quad \sum_{u \in M(f)} \xi_u \tilde{\pi}^{w(u)} x^u \longmapsto \xi_0 + \sum_{u \in M(f) \setminus \{0\}} \xi_u e_u.$$

That is, the image $\Upsilon(\mathcal{C}_0(\hat{\lambda}))$ consists of the powers series with support in the monomials of $\mathcal{S}(\hat{\lambda})$ of degree ≤ 1 with coefficients $\{\xi_u\}_{u \in M(f)} \subset R[\hat{\lambda}]$ satisfying $\xi_u \to 0$ as $u \to \infty$. Note that $\Upsilon(\tilde{\pi}^{w(u)} x^u) = e_u$ for $u \in M \setminus \{0\}$, and $\Upsilon(1) := 1$. Define the $R[\hat{\lambda}]$-subalgebra of $\mathcal{S}(\hat{\lambda})$

$$\mathcal{S}_0(\hat{\lambda}) := \left\{ \xi = \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi(\boldsymbol{u}) e_{\boldsymbol{u}} \;\middle|\; \xi(\boldsymbol{u}) \in R[\hat{\lambda}], \xi(\boldsymbol{u}) \to 0 \text{ as } w(\boldsymbol{u}) \to \infty \right\}.$$

Hence, $\Upsilon(\mathcal{C}_0(\hat{\lambda})) \subset \mathcal{S}_0(\hat{\lambda})$. Note that we may write $\alpha_{\bar{t},\bar{\lambda}}(1) = 1 + \eta(x)$ for some element $\eta \in \mathcal{C}_0(\hat{\lambda})$ satisfying $|\eta| < 1$ with support of η in $M(f) \setminus \{0\}$. For $\xi = \sum \xi(\boldsymbol{u}) e_{\boldsymbol{u}} \in \mathcal{S}_0(\hat{\lambda})$, define $|\xi| := \sum_{\boldsymbol{u} \in \mathcal{S}(M)} |\xi(\boldsymbol{u})|$, which makes $\mathcal{S}_0(\hat{\lambda})$ a p-adic Banach algebra over $R[\hat{\lambda}]$. Then for any $\zeta \in \mathcal{C}_0(\hat{\lambda})$, $|\Upsilon(\zeta)| = |\zeta|$. It follows that $(\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(1))^\tau$ is defined and belongs to $\mathcal{S}_0(\hat{\lambda})$ for any $\tau \in \mathbb{Z}_p$. Define $[\alpha_{\bar{t},\bar{\lambda}}]_\kappa : \mathcal{S}_0(\hat{\lambda}) \to \mathcal{S}_0(\hat{\lambda})$ by extending linearly over $R[\hat{\lambda}]$ the action on monomials of degree r:

$$[\alpha_{\bar{t},\bar{\lambda}}]_\kappa(e_{u_1} \cdots e_{u_r}) := (\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(1))^{\kappa - r} (\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(\tilde{\pi}^{w(u_1)} x^{u_1})) \cdots (\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(\tilde{\pi}^{w(u_r)} x^{u_r})).$$

By a similar argument to [Haessig 2014, Corollary 2.4, part 2],

$$\det\bigl(1 - [\alpha_{\bar{t},\bar{\lambda}}]_\kappa T \mid \mathcal{S}_0(\hat{\lambda})\bigr) = \prod_{r=0}^{\infty} \prod \bigl(1 - \pi_0(\bar{t},\bar{\lambda})^{\kappa - r} \pi_{i_1}(\bar{t},\bar{\lambda}) \cdots \pi_{i_r}(\bar{t},\bar{\lambda}) T\bigr),$$

where the inner product runs over all multisets $\{i_1, \ldots, i_r\}$ of positive integers of cardinality r satisfying $1 \leq i_1 \leq i_2 \leq \cdots$.

Infinite symmetric power on the family. Denote by $\mathcal{S}(\mathcal{O}_0) := \mathcal{O}_0[\![\{e_u\}_{u \in M \setminus \{0\}}]\!]$, the formal power series ring supported by the monomials $\mathcal{S}(M)$, with coefficients in the ring $\mathcal{O}_0$. As in the constant fiber case above, this ring is equipped with the sup-norm on coefficients. Define the p-adic Banach algebra over $\mathcal{O}_0$,

$$\begin{aligned}\mathcal{S}_0(\mathcal{O}_0) &:= \left\{ \xi = \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi(\boldsymbol{u}) e_{\boldsymbol{u}} \;\middle|\; \xi(\boldsymbol{u}) \in \mathcal{O}_0, \xi(\boldsymbol{u}) \to 0 \text{ as } w(\boldsymbol{u}) \to \infty \right\} \\ &= \left\{ \xi = \sum_{\substack{\gamma \in M(\Gamma) \\ \boldsymbol{u} \in \mathcal{S}(M)}} C(\gamma, \boldsymbol{u}) \tilde{\pi}^{w_\Gamma(\gamma)} \lambda^\gamma e_{\boldsymbol{u}} \;\middle|\; C(\gamma, \boldsymbol{u}) \in R, C(\gamma, \boldsymbol{u}) \to 0 \text{ as } w_\Gamma(\gamma) + w(\boldsymbol{u}) \to \infty \right\},\end{aligned}$$

and similarly, for any $q = p^a$ an arbitrary power of p (including the case when $a = 0$),

$$\mathcal{S}_0(\mathcal{O}_{0,q}) := \left\{ \sum_{\boldsymbol{u}\in\mathcal{S}(M)} \xi(\boldsymbol{u})e_{\boldsymbol{u}} \,\middle|\, \xi(\boldsymbol{u}) \in \mathcal{O}_{0,q},\, \xi(\boldsymbol{u}) \to 0 \text{ as } w(\boldsymbol{u}) \to \infty \right\}.$$

Note that $\mathcal{S}_0(\mathcal{O}_{0,q})$ is a p-adic Banach algebra over $\mathcal{O}_{0,q}$ with $\mathcal{S}(M)$ an orthonormal basis. We embed $\mathcal{C}_0(\mathcal{O}_{0,q}) \hookrightarrow \mathcal{S}_0(\mathcal{O}_{0,q})$ via a map Υ defined in the same way as on the fibers. Again, $(\Upsilon \circ \alpha_m(1))^\tau \in \mathcal{S}_0(\mathcal{O}_{0,p^m})$ for any $\tau \in \mathbb{Z}_p$. We define a map $[\alpha_m]_\kappa : \mathcal{S}_0(\mathcal{O}_0) \to \mathcal{S}_0(\mathcal{O}_{0,p^m})$ as follows. On a basis element $e_{\boldsymbol{u}} = e_{u_1} \cdots e_{u_r}$ with $r > 0$ and $0 < u_1 \le \cdots \le u_r$,

$$\begin{aligned}[\alpha_m](e_{\boldsymbol{u}}) &:= [\alpha_m]_\kappa(e_{u_1} \cdots e_{u_r}) \\ &:= (\Upsilon \circ \alpha_m(1))^{\kappa - r}(\Upsilon \circ \alpha_m(\tilde{\pi}^{w(u_1)}x^{u_1})) \cdots (\Upsilon \circ \alpha_m(\tilde{\pi}^{w(u_r)}x^{u_r})).\end{aligned}$$

If $r = 0$,

$$[\alpha_m]_\kappa(1) := \Upsilon(\alpha_m(1))^\kappa.$$

We may calculate an estimate for $\alpha_m(\tilde{\pi}^{w(u)}x^u)$, where we recall that $\alpha_m := \sigma^{-m} \circ \psi_x^m \circ F_m(\bar{t}, \lambda, x)$. As noted earlier, we may write

$$F_m(\hat{t}, \lambda, x) = \sum_{\gamma\in M(\Gamma), v\in M(f)} B(\gamma, v)\tilde{\pi}^{(w_\Gamma(\gamma)+w(v))/p^{m-1}}\lambda^\gamma x^v, \tag{8}$$

with $\operatorname{ord}_p B(\gamma, r) \ge 0$, and set $\mathcal{B}^m(\gamma, v) = B(\gamma, v)\tilde{\pi}^{(w_\Gamma(\gamma)+w(v))/p^{m-1}}$. So

$$\begin{aligned}\alpha_m(\tilde{\pi}^{w(u)}x^u) &= \psi_x^m(F_m(\hat{t}, \lambda, x) \cdot \tilde{\pi}^{w(u)}x^u) \\ &= \sum \big(\tilde{\pi}^{(w_\Gamma(\gamma)+w(p^m v-u))/p^{m-1}+w(u)-w_\Gamma(\gamma)/p^{m-1}-w(v)} \\ &\qquad\qquad \times B(\gamma, p^m v - u) \cdot \tilde{\pi}^{w_\Gamma(\gamma)/p^{m-1}}\lambda^\gamma \cdot \tilde{\pi}^{w(v)}x^v\big).\end{aligned}$$

We note that

$$\begin{aligned}\tfrac{w(p^m v-u)}{p^{m-1}} + w(u) - w(v) &\ge pw(v) - \tfrac{w(u)}{p^{m-1}} + w(u) - w(v) \\ &\ge (p-1)w(v) + \tfrac{p^{m-1}-1}{p^{m-1}}w(u).\end{aligned}$$

Hence,

$$|\Upsilon(\alpha_m(\tilde{\pi}^{w(u)}x^u))| \le |\tilde{\pi}|^{w(u)(p^{m-1}-1)/p^{m-1}}. \tag{9}$$

The R-linear map $\psi_\lambda : \mathcal{S}_0(\mathcal{O}_{0,p}) \to \mathcal{S}_0(\mathcal{O}_0)$ is defined by

$$\psi_\lambda : \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} A(\gamma, \boldsymbol{u})\lambda^\gamma e_{\boldsymbol{u}} \longmapsto \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} A(p\gamma, \boldsymbol{u})\lambda^\gamma e_{\boldsymbol{u}}.$$

We may in the usual manner view $\mathcal{S}_0(\mathcal{O}_0)$ as a p-adic Banach space over R with orthonormal basis $\{\tilde{\pi}^{w_\Gamma(\gamma)}\lambda^\gamma e_{\boldsymbol{u}} \mid \gamma \in M(\Gamma), \boldsymbol{u} \in \mathcal{S}(M)\}$. Then

$$\beta_{\kappa,\bar{t}} := \psi_\lambda^{ad(\bar{t})} \circ [\alpha_{ad(\bar{t})}]_\kappa : \mathcal{S}_0(\mathcal{O}_0) \to \mathcal{S}_0(\mathcal{O}_0)$$

is a completely continuous operator (over R). Set $\mathcal{B} := \{e_{\boldsymbol{u}} \mid \boldsymbol{u} \in \mathcal{S}(M)\}$. Let $B_{\bar{t}}^{[\kappa]}(\lambda)$ be the matrix of $[\alpha_{ad(\bar{t})}]_\kappa$ with respect to $\mathcal{B}$, the basis of $\mathcal{S}_0(\mathcal{O}_0)$ over $\mathcal{O}_0$ (as well as $\mathcal{S}_0(\mathcal{O}_{0,p^m})$ over $\mathcal{O}_{0,p^m}$). The entries of $B_{\bar{t}}^{[\kappa]}(\lambda)$ are series with support in $\mathcal{B}$ and coefficients in $\mathcal{O}_{0,p^m}$ (which tend to 0 as $w(\boldsymbol{u}) \to \infty$). We may write $B_{\bar{t}}^{[\kappa]}(\lambda) = \sum_{\gamma \in M(\Gamma)} b_\gamma^{[\kappa]}\lambda^\gamma$, where $b_\gamma^{[\kappa]}$ is a matrix with rows and columns indexed by $M(\Gamma)$ and entries in R. We define the matrix $F_{B_{\bar{t}}^{[\kappa]}} := (b^{[\kappa]}_{q_{\bar{t}}\gamma-\mu})_{(\gamma,\mu)}$ indexed by $\gamma, \mu \in M(\Gamma)$, and we set $b^{[\kappa]}_{q_{\bar{t}}\gamma-\mu} := 0$ if $q_{\bar{t}}\gamma - \mu \notin M(\Gamma)$. Note that $F_{B_{\bar{t}}^{[\kappa]}}$ is a matrix with entries in R whose (γ, μ) entry is again a matrix in R with rows and columns indexed by $M(\Gamma)$. As we showed in [Haessig and Sperber 2014, §2.3], $F_{B_{\bar{t}}^{[\kappa]}}$ is the matrix of the completely continuous operator $\beta_{\kappa,\bar{t}}$, and as such it has a well-defined Fredholm determinant. In particular, the Dwork trace formula gives

$$\begin{aligned}(q_{\bar{t}}^m - 1)^s \operatorname{Tr}(\beta_{\kappa,\bar{t}}^m) &= (q_{\bar{t}}^m - 1)^s \operatorname{Tr}(F^m_{B_{\bar{t}}^{[\kappa]}}) \\ &= \sum_{\lambda^{q_{\bar{t}}^m-1}=1} \operatorname{Tr}\big(B_{\bar{t}}^{[\kappa]}(\hat{\lambda}^{q_{\bar{t}}^{m-1}}) \cdots B_{\bar{t}}^{[\kappa]}(\hat{\lambda}^{q_{\bar{t}}}) B^{[\kappa]}\bar{t}(\hat{\lambda})\big) \\ &= \sum_{\substack{\bar{\lambda} \in (\mathbb{F}^*_{q_{\bar{t}}^m})^s \\ \hat{\lambda} = \mathrm{Teich}(\bar{\lambda})}} \operatorname{Tr}\big([\alpha_{\bar{t},\bar{\lambda}}]_\kappa^m \mid \mathcal{S}_0(\hat{\lambda})\big).\end{aligned}$$

Using an argument similar to that succeeding [Haessig 2014, (8)], it follows that

$$L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}} = \det(1 - \beta_{\kappa,\bar{t}} T)^{\delta^s_{q_{\bar{t}}}}. \tag{10}$$

Since the Fredholm determinant $\det(1-\beta_{\kappa,\bar{t}}T)$ is p-adically entire, this demonstrates the meromorphic continuation of $L^{(0)}(\kappa, \bar{t}, T)$. Since the matrix of $\beta_{\kappa,\bar{t}}$ shows that $\det(1-\beta_{\kappa,\bar{t}}T)$ has a unique unit root, it follows that $L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ has a unique unit root equal in fact to the unique unit root of $\det(1 - \beta_{\kappa,\bar{t}}T)$.

In a similar way, define on the space $\mathcal{S}_0(\mathcal{O}_0) \otimes \wedge^m \mathcal{C}_0(\mathcal{O}_0)$, the operator $\beta_{\kappa,\bar{t}}^{(m)} := \psi_\lambda^{ad(\bar{t})} \circ \big([\alpha_{ad(\bar{t})}]_{\kappa-m} \otimes \wedge^m \alpha_{ad(\bar{t})}\big)$. Then

$$L^{(m)}(\kappa, \bar{t}, T)^{(-1)^{s+1}} = \det(1 - \beta_{\kappa,\bar{t}}^{(m)} T)^{\delta^s_{q_{\bar{t}}}}.$$

In particular, for $m \geq 2$, due to the wedge product, $L^{(m)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ has no zeros or poles on the closed unit disk. Hence, by (7), we have:

Theorem 3.1. *$L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ has a unique p-adic unit root which in fact is the unique unit root of $L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$.*

4. Dual theory

In this section, we define a dual theory for the operator $\beta_{\kappa,\bar{t}}$ acting on $\mathcal{S}_0(\mathcal{O}_0)$. We begin by defining a dual map to $\alpha_{ad(\bar{t})}$. For $q = p^a$ an arbitrary power of p (including the case $a = 0$) define the $\mathcal{O}_{0,q}$-module

$$\mathcal{C}_0^*(\mathcal{O}_{0,q}) := \left\{ \sum_{u \in M(f)} \xi(u)\tilde{\pi}^{-w(u)} x^{-u} \,\middle|\, \xi(u) \in \mathcal{O}_{0,q} \right\},$$

equipped with the sup-norm on the set of coefficients $\{\xi(u)\}_{u\in M(f)}$. Define the projection (or truncation) map

$$\mathrm{pr}_{M(f)} : \quad \sum_{u\in\mathbb{Z}^n} A(u)x^{-u} \longmapsto \sum_{u\in M(f)} A(u)x^{-u}.$$

For each $m \geq 1$, define

$$\alpha_m^* := \mathrm{pr}_{M(f)} \circ F_m(\hat{t}, \lambda, x) \circ \Phi_x^m \circ \sigma^m,$$

where $\sigma \in \mathrm{Gal}(\Omega/\Omega_0)$ acts on coefficients (as mentioned above), and Φ_x acts on monomials by $\Phi_x(x^u) := x^{pu}$.

Lemma 4.1. *$\alpha_m^* : \mathcal{C}_0^*(\mathcal{O}_{0,p^m}) \to \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ is a linear map over $\mathcal{O}_{0,p^m}$. Furthermore, writing*

$$\alpha_m^*(\tilde{\pi}^{-w(v)}x^{-v}) = \sum_{z\in M(f)} C_v(z)\tilde{\pi}^{-w(z)}x^{-z},$$

with $C_v(z) \in \mathcal{O}_{0,p^m}$, then $C_v(z) \to 0$ in $\mathcal{O}_{0,p^m}$ as $w(v) \to \infty$. In addition, we may write $\alpha_m^*(1) = 1 + \eta_m^*(\lambda, x)$, with $\eta_m^*(\lambda, x) \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ having $|\eta_m^*| \leq |\tilde{\pi}|$.

Proof. We consider $\alpha_m^*(\tilde{\pi}^{-w(v)}x^{-v})$ with $v \in M(f)$. Using (8), we may write this as

$$\alpha_m^*(\tilde{\pi}^{-w(v)}x^{-v}) = \sum_{\substack{z\in M(f)\\ \gamma\in M(\Gamma)}} \Big(B(\gamma, -z + p^m v)\tilde{\pi}^{w_\Gamma(\gamma)/p^{m-1}}\lambda^\gamma \times \tilde{\pi}^{-w(v)+w(z)+(w(-z+p^m v)/p^{m-1})}\tilde{\pi}^{-w(z)}x^{-z}\Big).$$

Since

$$-w(v) + w(z) + \tfrac{1}{p^{m-1}}w(-z + p^m v) \geq \tfrac{p^{m-1}-1}{p^{m-1}}w(z) + (p-1)w(v),$$

we see that

$$\alpha_m^*(\tilde{\pi}^{-w(v)}x^{-v}) = \tilde{\pi}^{(p-1)w(v)}\zeta_v^*(\lambda, x), \tag{11}$$

where $\zeta_v^*(\lambda, x) \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$.

If $\xi^* \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ with $\xi^* = \sum_{v\in M(f)} A_v(\lambda)\tilde{\pi}^{-w(v)}x^{-v}$, then

$$\alpha_m^*(\xi^*) = \sum_{v\in M(f)} \tilde{\pi}^{(p-1)w(v)}A_v(\lambda)\zeta_v^*(\lambda, x) \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m}).$$

Finally, note that by the above,

$$\alpha_m^*(1) = 1 + \sum_{\gamma \in M(\Gamma)\setminus\{0\}} B(\gamma, 0)\tilde{\pi}^{w(\gamma)/p^{m-1}} \lambda^\gamma + \sum_{\substack{z \in M(f)\setminus\{0\} \\ \gamma \in M(\Gamma)}} B(\gamma, -z)\tilde{\pi}^{w(z)+(w(-z)/p^{m-1})} (\tilde{\pi}^{w(\gamma)/p^{m-1}} \lambda^\gamma)(\tilde{\pi}^{w(-z)} x^{-z}).$$

This proves the lemma. □

Define

$$\mathcal{A}_0 := \left\{ \sum_{\gamma \in M(\Gamma)} A(\gamma)\lambda^\gamma \,\middle|\, A(\gamma) \in R \text{ and } A(\gamma) \to 0 \text{ as } w(\gamma) \to \infty \right\}.$$

For q_1 and q_2 any two powers of the prime p, define a pairing

$$(\cdot\,,\cdot) : \mathcal{C}_0(\mathcal{O}_{0,q_1}) \times \mathcal{C}_0^*(\mathcal{O}_{0,q_2}) \to \mathcal{A}_0$$

by

$$(\xi, \xi^*) := \text{ the constant term with respect to } x \text{ of the product } \xi \cdot \xi^*.$$

This product is well-defined since if $\{\eta_1(v)\}_{v \in M(\Gamma)} \subset \mathcal{O}_{0,q_1}$ with $\eta_1(v) \to 0$ as $w(v) \to \infty$, and $\{\eta_2(v)\}_{v \in M(\Gamma)} \subset \mathcal{O}_{0,q_2}$, then $\sum_{v \in M(\Gamma)} \eta_1(v)\eta_2(v) \in \mathcal{A}_0$. Next let $\xi \in \mathcal{C}_0(\mathcal{O}_0)$, $\xi^* \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$. Writing F_m for $F_m(\hat{t}, \lambda, x)$, observe that

(12) $$((\psi_x^m \circ F_m)\xi, \xi^*) = (F_m \xi, \Phi_x^m \xi^*) = (\xi, (\mathrm{pr}_{M(f)} \circ F_m \circ \Phi_x^m)(\xi^*)).$$

Symmetric powers. We construct in a now familiar manner formal k-th symmetric powers of $\mathcal{C}_0(\mathcal{O}_0)$ and $\mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ over $\mathcal{O}_0$. Similar to the construction used above, we consider a linear order on $\{u \in M(f)\}$ under which the weight is nondecreasing, say $0 = u_0 \le u_1 \le \cdots$. We will for convenience of notation write the "basis" as $\{E_u := \tilde{\pi}^{w(u)} x^u \mid u \in M(f)\}$, and the k-th symmetric power of the basis as

$$E_{\boldsymbol{u}} := E_{u_{j_1}} E_{u_{j_2}} \cdots E_{u_{j_k}} \quad (0 \le j_1 \le j_2 \le \cdots \le j_k),$$

where $\boldsymbol{u}$ runs over multisets of indices of cardinality k, say

$$\{\boldsymbol{u} = (u_{j_1}, u_{j_2}, \ldots, u_{j_k}) \mid 0 \le u_{j_1} \le u_{j_2} \le \cdots \le u_{j_k}\}.$$

Defining

$$\mathrm{Sym}^k_{\mathcal{O}_0} \mathcal{C}_0(\mathcal{O}_0) := \left\{ \xi = \sum_{|\boldsymbol{u}|=k} \xi_{\boldsymbol{u}}(\lambda) E_{\boldsymbol{u}} \,\middle|\, \xi_{\boldsymbol{u}}(\lambda) \in \mathcal{O}_0, \xi_{\boldsymbol{u}}(\lambda) \to 0 \text{ as } w(\boldsymbol{u}) \to +\infty \right\},$$

we then define the map

$$\mathrm{Sym}^k \alpha_m : \mathrm{Sym}^k_{\mathcal{O}_0} \mathcal{C}_0(\mathcal{O}_0) \to \mathrm{Sym}^k_{\mathcal{O}_{0,p^m}} \mathcal{C}_0(\mathcal{O}_{0,p^m})$$

as follows. Let

$$\begin{aligned}\alpha_m(\tilde{\pi}^{w(u)}x^u) &= \sum_{v\in M(f)} \mathcal{A}^m_{v,u}(\lambda)\tilde{\pi}^{w(v)}x^v \\ &= \sum_{v\in M(f)} \mathcal{A}^m_{v,u}(\lambda)E_v.\end{aligned}$$

We know from Section 2 that

$$\mathcal{A}^m_{u,v} = \sum_{\gamma\in M(\Gamma), v\in M(f)} \tilde{\pi}^{w(u)-w(v)}\mathcal{B}^m(\gamma, p^m v - u)\lambda^\gamma.$$

Then

$$\mathrm{Sym}^k \alpha_m(E_{u_{j_1}}E_{u_{j_2}}\cdots E_{u_{j_k}}) = \sum \mathcal{A}^m_{v_{l_1},u_{j_1}}(\lambda)\cdots \mathcal{A}^m_{v_{l_k},u_{j_k}}(\lambda)E_{v_{l_1}}\cdots E_{v_{l_k}},$$

where the sum runs over all $v_{l_i} \in M(f)$ for each i, $1 \le i \le k$. Since, by above, $|\alpha_m(\tilde{\pi}^{w(u)}x^u)| \le |\tilde{\pi}|^{w(u)(p^{m-1}-1)/p^{m-1}}$ therefore $\mathrm{Sym}^k(\alpha_m)$ is a completely continuous map. The map Υ may be extended to $\mathrm{Sym}^k_{\mathcal{O}_0}(\mathcal{C}_0(\mathcal{O}_0)) \hookrightarrow \mathcal{S}_0(\mathcal{O}_0)$ as follows. For $\boldsymbol{u} = (u_{j_1}, \ldots, u_{j_k})$ an ordered multiset of cardinality k with elements in $M(f)$, set

$$\Upsilon(E_{\boldsymbol{u}}) = \begin{cases} e_{\boldsymbol{u}} & \text{if } j_1 > 0,\\ e_{u_{j_{r+1}}}e_{u_{j_{r+2}}}\cdots e_{u_{j_k}} & \text{if } j_1 = j_2 = \cdots = j_r = 0.\end{cases}$$

Thus $\Upsilon(\mathrm{Sym}^k_{\mathcal{O}_0}\mathcal{C}_0(\mathcal{O}_0))$ consists of all power-series with coefficients in $\mathcal{O}_0$ and support in monomials $e_{\boldsymbol{u}}$ of degree $\le k$, with coefficients going to 0 as $w(\boldsymbol{u}) = w(u_1) + \cdots + w(u_r) \to \infty$.

We have as well a dual variant

$$\mathrm{Sym}^k_{\mathcal{O}_{0,p^m}}\mathcal{C}^*_0(\mathcal{O}_{0,p^m}) := \left\{ \sum_{|\boldsymbol{u}|=k} A_{\boldsymbol{u}}(\lambda)E^*_{\boldsymbol{u}} \,\middle|\, A_{\boldsymbol{u}}(\lambda) \in \mathcal{O}_{0,p^m}\right\},$$

where we denote $E^*_u := \tilde{\pi}^{-w(u)}x^{-u}$ for each $u \in M(f)$, and using the linear order above write for each multiset $\boldsymbol{u} = (u_{j_1}, \ldots, u_{j_k})$ of cardinality k of indices, with $j_1 \le \cdots \le j_k$ we set $E^*_{\boldsymbol{u}} := E^*_{u_1}\cdots E^*_{u_k}$. Then

$$\mathrm{Sym}^k_{\mathcal{O}_{0,p^m}}\mathcal{C}^*_0(\mathcal{O}_{0,p^m}) = \left\{ \sum_{|\boldsymbol{u}|=k} \xi(\boldsymbol{u})E^*_{\boldsymbol{u}} \,\middle|\, \xi(\boldsymbol{u}) \in \mathcal{O}_{0,p^m}\right\},$$

there being no requirement here that the coefficients tend to 0 as $w(\boldsymbol{u}) \to \infty$. Since $\alpha^*_m : \mathcal{C}^*_0(\mathcal{O}_{0,p^m}) \to \mathcal{C}^*_0(\mathcal{O}_{0,p^m})$, we may define for $\boldsymbol{u} = (u_{j_1}, \ldots, u_{j_k})$,

$$\mathrm{Sym}^k(\alpha^*_m)(E^*_{\boldsymbol{u}}) = \sum \mathcal{A}^*_{v_{l_1},u_{j_1}}(\lambda)\mathcal{A}^*_{v_{l_2},u_{j_2}}(\lambda)\cdots \mathcal{A}^*_{v_{l_k},u_{j_k}}(\lambda)E^*_{\boldsymbol{v}},$$

where $\boldsymbol{v} = (v_{l_1}, \dots v_{l_k})$, the sum runs over $v_{l_i} \in \{\tilde{\pi}^{-w(u)} x^{-u} \mid u \in M(f)\}$, and where $\alpha_m^*(\tilde{\pi}^{-w(u)} x^{-u}) = \sum_{v \in M(f)} \mathcal{A}_{u,v}^*(\lambda) \tilde{\pi}^{-w(v)} x^{-v}$. The map $\mathrm{Sym}^k(\alpha_m^*)$ then is defined on $\mathrm{Sym}^k_{\mathcal{O}_{0,p^m}}$ since, as we noted earlier in (11), $|\alpha_m^*(\tilde{\pi}^{-w(u)} x^{-u})| \le |\tilde{\pi}|^{w(u)(p-1)}$.

We extend the pairing above to these symmetric power spaces by "linearly" extending the following: for decomposable elements $\xi = \xi_1 \cdots \xi_k \in \mathrm{Sym}^k_{\mathcal{O}_{0,q_1}} \mathcal{C}_0(\mathcal{O}_{0,q_1})$ and $\xi^* = \xi_1^* \cdots \xi_k^* \in \mathrm{Sym}^k_{\mathcal{O}_{0,q_2}} \mathcal{C}_0^*(\mathcal{O}_{0,q_2})$,

$$(\xi, \xi^*) := (\xi_1 \cdots \xi_k, \xi_1^* \cdots \xi_k^*)_k := \tfrac{1}{k!} \sum_{\sigma \in S_k} \prod_{i=1}^{k} (\xi_i, \xi_{\sigma(i)}^*), \tag{13}$$

where S_k denotes the symmetric group on k letters. This pairing $(\cdot,\cdot)_k$ is well-defined since $\mathcal{A}_0$ is a ring. It follows from (12) that, for $\xi \in \mathrm{Sym}^k \mathcal{C}_0(\mathcal{O}_0)$ and $\xi^* \in \mathrm{Sym}^k \mathcal{C}_0^*(\mathcal{O}_{0,q_{\bar{t}}})$,

$$(\mathrm{Sym}^k \alpha_{ad(\bar{t})} \xi, \xi^*)_k = (\xi, \mathrm{Sym}^k \alpha_{ad(\bar{t})}^* \xi^*)_k. \tag{14}$$

Infinite symmetric powers. Denote by $\mathcal{S}_0^*(\mathcal{O}_0) := \mathcal{O}_0[\![e_u^* : u \in M \setminus \{0\}]\!]$ the formal power series ring over $\mathcal{O}_0$ in the variables $\{e_u^*\}_{u \in M \setminus \{0\}}$, a set of formal symbols indexed by $M \setminus \{0\}$. We endow $\mathcal{S}_0^*(\mathcal{O}_0)$ with the sup-norm on coefficients. Monomials in $\mathcal{S}_0^*(\mathcal{O}_0)$ have the form $e_{\boldsymbol{u}}^* := e_{u_1}^* e_{u_2}^* \cdots e_{u_r}^*$, where $u_1, \dots, u_r \in M(f) \setminus \{0\}$ for $r > 0$, and $e_0^* := 1$ when $r = 0$. Thus, elements in the ring may be described by

$$\mathcal{S}_0^*(\mathcal{O}_0) := \left\{ \xi^* = \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi^*(\boldsymbol{u}) e_{\boldsymbol{u}}^* \,\middle|\, \xi^*(\boldsymbol{u}) \in \mathcal{O}_0 \right\}.$$

Using the same notation as before, define the embedding $\Upsilon : \mathcal{C}_0^*(\mathcal{O}_0) \hookrightarrow \mathcal{S}_0^*(\mathcal{O}_0)$ by $\Upsilon(\tilde{\pi}^{-w(u)} x^{-u}) := e_u^*$ for $u \in M \setminus \{0\}$, and $\Upsilon(1) := e_0^* = 1$. For each $m \ge 1$, recall from Lemma 4.1 that $\alpha_m^*(1) = 1 + \eta_m^*(\lambda, x)$ for some element $\eta_m^* \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ satisfying $|\eta_m^*| < 1$. It follows that $(\Upsilon \circ \alpha_m^*(1))^\tau \in \mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ for any $\tau \in \mathbb{Z}_p$. For $m \ge 1$, we define the map $[\alpha_m^*]_\kappa : \mathcal{S}_0^*(\mathcal{O}_{0,p^m}) \to \mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ by

$$\begin{aligned} &[\alpha_m^*]_\kappa (e_{u_1}^* \cdots e_{u_r}^*) \\ &\quad := (\Upsilon(\alpha_m^*(1)))^{\kappa - r} (\Upsilon(\alpha_m^*(\tilde{\pi}^{-w(u_1)} x^{-u_1}))) \cdots (\Upsilon(\alpha_m^*(\tilde{\pi}^{-w(u_r)} x^{-u_r}))). \end{aligned} \tag{15}$$

The product on the right side makes sense and lives in $\mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ since $\mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ is a ring and each factor is clearly in $\mathcal{S}_0^*(\mathcal{O}_{0,p^m})$. Furthermore,

$$|[\alpha_m^*]_\kappa (e_{\boldsymbol{u}}^*)| \le |\tilde{\pi}^{(p-1) w(\boldsymbol{u})}|. \tag{16}$$

Define the R-module

$$\mathcal{O}_{0,q}^* := \left\{ \zeta^* = \sum_{\gamma \in M(\Gamma)} \zeta^*(\gamma) \tilde{\pi}^{-w_q \Gamma(\gamma)} \lambda^{-\gamma} \,\middle|\, \zeta^*(\gamma) \in R \right\}.$$

Here we do not insist that the coefficients go to 0 and we do not claim $\mathcal{O}_{0,q}^*$ is a ring. As usual we define an absolute value on $\mathcal{O}_{0,q}^*$ by $|\zeta^*| := \sup_{\gamma\in M(\Gamma)} |\zeta^*(\gamma)|$. For series in λ, we define a projection (or truncation) map

$$\mathrm{pr}_{M(\Gamma)} : \quad \sum_{\gamma\in\mathbb{Z}^s} A(\gamma)\lambda^{-\gamma} \longmapsto \sum_{\gamma\in M(\Gamma)} A(\gamma)\lambda^{-\gamma}.$$

Note that for any q a power of the prime p, if γ, γ', and δ all belong to $M(\Gamma)$ with $\gamma-\gamma' = -\delta$ then $w_{q\Gamma}(\gamma) - w_{q\Gamma}(\gamma') \geq -w_{q\Gamma}(\delta)$. It follows that, for $\xi \in \mathcal{O}_{0,q}$ and $\xi^* \in \mathcal{O}_{0,q}^*$,

$$\mathrm{pr}_{M(\Gamma)}(\xi\cdot\xi^*) \in \mathcal{O}_{0,q}^*. \tag{17}$$

Define the R module

$$\mathcal{S}_0^*(\mathcal{O}_0^*) := \left\{ \omega^* = \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} \omega^*(\gamma,\boldsymbol{u})\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma} e_{\boldsymbol{u}}^* \,\middle|\, \omega^*(\gamma,\boldsymbol{u}) \in R \right\}.$$

Define the map Φ_λ by $\lambda \mapsto \lambda^p$. We define an R-linear map

$$\beta_{\kappa,\bar{t}}^* := \mathrm{pr}_{M(\Gamma)} \circ [\alpha_{ad(\bar{t})}^*]_\kappa \circ \Phi_\lambda^{ad(\bar{t})}$$

by "linearly" extending over R the action

$$\beta_{\kappa,\bar{t}}^*(\lambda^{-\gamma} e_{\boldsymbol{u}}^*) = \mathrm{pr}_{M(\Gamma)}(\lambda^{-q_{\bar{t}}\gamma}\cdot[\alpha_{ad(\bar{t})}^*]_\kappa(e_{\boldsymbol{u}}^*)).$$

Lemma 4.2. *$\beta_{\kappa,\bar{t}}^*$ is an R-linear endomorphism of $\mathcal{S}_0^*(\mathcal{O}_0^*)$.*

Proof. We have remarked already that $[\alpha_{ad(\bar{t})}^*]_\kappa$ is a well-defined endomorphism of $\mathcal{S}_0^*(\mathcal{O}_{0,q_{\bar{t}}})$. As such, we may write for each $\boldsymbol{u} \in \mathcal{S}(M)$,

$$[\alpha_{ad(\bar{t})}^*]_\kappa(e_{\boldsymbol{u}}^*) = \sum_{\substack{\sigma\in M(\Gamma)\\ \boldsymbol{v}\in\mathcal{S}(M)}} B_{\boldsymbol{u}}(\sigma,\boldsymbol{v})\tilde{\pi}^{w_{q_{\bar{t}}\Gamma}(\sigma)}\lambda^\sigma e_{\boldsymbol{v}}^* \in \mathcal{S}_0^*(\mathcal{O}_{0,q_{\bar{t}}}),$$

with $B_{\boldsymbol{u}}(\sigma,\boldsymbol{v}) \in R$, and $B_{\boldsymbol{u}}(\sigma,\boldsymbol{v}) \to 0$ as $w_{q_{\bar{t}}\Gamma}(\sigma)+w(\boldsymbol{v}) \to \infty$ using (16). For $\omega^* = \sum_{\gamma\in M(\Gamma),\boldsymbol{u}\in\mathcal{S}(M)} \omega^*(\gamma,\boldsymbol{u})\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma}e_{\boldsymbol{u}}^* \in S_0^*(\mathcal{O}_0^*)$, we have

$$\begin{aligned}
\beta_{\kappa,\bar{t}}^*(\omega^*) &= \mathrm{pr}_{M(\Gamma)}\Bigg(\sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} \omega^*(\gamma,u)\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-q_{\bar{t}}\gamma}\cdot[\alpha_{ad(\bar{t})}^*]_\kappa(e_{\boldsymbol{u}}^*)\Bigg)\\
&= \mathrm{pr}_{M(\Gamma)}\Bigg(\sum_{\gamma\in M(\Gamma)}\lambda^{-q_{\bar{t}}\gamma}\sum_{\boldsymbol{u}\in\mathcal{S}(M)}\omega^*(\gamma,\boldsymbol{u})\sum_{\substack{\sigma\in M(\Gamma)\\ \boldsymbol{v}\in\mathcal{S}(M)}} B_{\boldsymbol{u}}(\sigma,\boldsymbol{v})\tilde{\pi}^{-w_{q_{\bar{t}}\Gamma}(\sigma)}\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^\sigma e_{\boldsymbol{v}}^*\Bigg)\\
&= \sum_{\substack{\tau\in M(\Gamma)\\ \boldsymbol{v}\in\mathcal{S}(M)}} C(\tau,\boldsymbol{v})\tilde{\pi}^{-w_\Gamma(\tau)}\lambda^{-\tau}e_{\boldsymbol{v}}^*,
\end{aligned}$$

where

$$C(\tau, \boldsymbol{v}) := \sum_{\boldsymbol{u}\in\mathcal{S}(M)} \sum_{\substack{\gamma,\sigma\in M(\Gamma)\\ q_{\bar{\imath}}\gamma-\sigma=\tau}} \omega^*(\gamma, \boldsymbol{u}) B_{\boldsymbol{u}}(\sigma, \boldsymbol{v}) \tilde{\pi}^{-w_\Gamma(\gamma)+w_{q_{\bar{\imath}}\Gamma}(\sigma)+w_\Gamma(\tau)}.$$

Observe that the exponent of $\tilde{\pi}$ satisfies

$$-w_\Gamma(\gamma) + w_{q_{\bar{\imath}}\Gamma}(\sigma) + w_\Gamma(\tau) \geq \left(1 - \tfrac{1}{q_{\bar{\imath}}}\right) w_\Gamma(\tau),$$

so that the term $\tilde{\pi}^{-w_\Gamma(\gamma)+w_{q_{\bar{\imath}}\Gamma}(\sigma)+w_\Gamma(\tau)}$ is bounded in norm by 1 since $w(\tau) \geq 0$, and $\omega^*(\gamma, \boldsymbol{u})$ and $B_{\boldsymbol{u}}(\sigma, v) \in R$. On the other hand, $B_{\boldsymbol{u}}(\sigma, \boldsymbol{v}) \to 0$ as $w_\Gamma(\sigma)+w(\boldsymbol{v}) \to \infty$ so that the coefficient $C(\tau, \boldsymbol{v})$ is defined, in R, and $\beta^*_\kappa(\omega^*) \in \mathcal{S}^*_0(\mathcal{O}^*_0)$. Clearly it is R-linear. □

Estimation using finite symmetric powers. It is useful to estimate $\beta_{\kappa,\bar{t}}$ and $\beta^*_{\kappa,\bar{t}}$ using finite symmetric powers. For monomials $e_{\boldsymbol{u}}$ or $e^*_{\boldsymbol{u}}$, with $\boldsymbol{u} \in \mathcal{S}(M)$, $\boldsymbol{u} = (u_1, \dots, u_r) \in (M(f)\setminus 0)^r$, we say as usual that the degree or length of $e_{\boldsymbol{u}}$ or $e^*_{\boldsymbol{u}}$ is r. For $\xi \in \mathcal{S}_0(\mathcal{O}_0)$, define length$(\xi)$ as the supremum of the lengths of the monomials $e_{\boldsymbol{u}}$ in the support of ξ (i.e., those terms appearing with nonzero coefficients). In the case length$(\xi) = r$, we may write $\xi = \sum_{|\boldsymbol{u}|\leq r} \xi(\boldsymbol{u}) e_{\boldsymbol{u}}$, and ξ may be a series (not a polynomial), since $M(f)$ and the set of monomials of degree $\leq r$ are infinite in general. Similarly for $\xi^*_{\boldsymbol{u}}$.

Let k be a positive integer. Define $\mathcal{S}_0^{(k)}(\mathcal{O}_0) := \{\xi \in \mathcal{S}_0(\mathcal{O}_0) \mid \text{length}(\xi) \leq k\}$. Then the map

$$E_0^{k-r} E_{u_1} \cdots E_{u_r} \longmapsto e_{u_1} e_{u_2} \cdots e_{u_r}$$

identifies $\mathrm{Sym}^k\, \mathcal{C}_0(\mathcal{O}_0)$ with $\mathcal{S}_0^{(k)}(\mathcal{O}_0)$ as $\mathcal{O}_0$-submodules in $\mathcal{S}_0(\mathcal{O}_0)$. Similarly, we identify $\mathrm{Sym}^k\, \mathcal{C}^*_0(\mathcal{O}_0)$ in $\mathcal{S}^*_0(\mathcal{O}_0)$ as the $\mathcal{O}_0$-submodule $\mathcal{S}_0^{*(k)}(\mathcal{O}_0)$ of power series in $\{e^*_{\boldsymbol{u}} \mid |\boldsymbol{u}| \leq k\}$ with coefficients in $\mathcal{O}_0$. By transfer of structure, we have a pairing $(\cdot\,,\cdot)_k : \mathcal{S}_0^{(k)}(\mathcal{O}_0) \times \mathcal{S}_0^{*(k)}(\mathcal{O}_0) \to \mathcal{O}_0$.

We now work over R and define a new pairing $\langle\cdot\,,\cdot\rangle_k : \mathcal{S}_0^{(k)}(\mathcal{O}_0) \times \mathcal{S}_0^{*(k)}(\mathcal{O}^*_0) \to \Omega$ as follows. (Here again $\mathcal{S}_0^{*(k)}(\mathcal{O}^*_0)$ is the R-submodule of $\mathcal{S}^*_0(\mathcal{O}^*_0)$ of series with support in monomials of degree $\leq k$, namely $\{e^*_{\boldsymbol{u}} \mid |\boldsymbol{u}| \leq k\}$, with coefficients in $\mathcal{O}^*_0$.) Let

$$\xi := \sum_{\gamma\in M(\Gamma), \boldsymbol{u}\in\mathcal{S}(M)} \xi(\gamma, \boldsymbol{u}) \tilde{\pi}^{w_\Gamma(\gamma)} \lambda^\gamma e_{\boldsymbol{u}} \in \mathcal{S}_0^{(k)}(\mathcal{O}_0),$$

$$\xi^* := \sum_{\sigma\in M(\Gamma), \boldsymbol{v}\in\mathcal{S}(M)} \xi^*(\sigma, \boldsymbol{v}) \tilde{\pi}^{-w_\Gamma(\sigma)} \lambda^{-\sigma} e^*_{\boldsymbol{v}} \in \mathcal{S}_0^{*(k)}(\mathcal{O}^*_0),$$

and set

$$\langle \xi, \xi^* \rangle_k := \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} \xi(\gamma, \boldsymbol{u}) \xi^*(\gamma, \boldsymbol{u}) (e_{\boldsymbol{u}}, e^*_{\boldsymbol{u}})_k,$$

where $(\cdot\,,\cdot)_k$ was defined above. (Observe that as defined, a denominator $k!$ is introduced, so $(e_{\boldsymbol{u}}, e^*_{\boldsymbol{u}})_k$ is a rational number with p-adic valuation bounded below by $-k/(p-1)$. This is independent of $\boldsymbol{u}$, so $\langle \xi, \xi^* \rangle_k$ is well-defined and takes values in the R-submodule of Ω consisting of elements with $\operatorname{ord}_p c \geq -k/(p-1)$.) It is useful to think of $\langle \xi, \xi^* \rangle_k$ as the constant term with respect to λ and the $e_{\boldsymbol{u}}$ and $e^*_{\boldsymbol{u}}$ of the product $\xi \cdot \xi^*$, where the product $e_{\boldsymbol{u}} \cdot e^*_{\boldsymbol{v}}$ is defined to be zero if $\boldsymbol{u} \neq \boldsymbol{v}$ and $(e_{\boldsymbol{u}}, e^*_{\boldsymbol{u}})_k$ if $\boldsymbol{u} = \boldsymbol{v}$.

Let k_m be a sequence of positive integers which tend to infinity (in the usual archimedean sense) and such that $\lim_{m\to\infty} k_m = \kappa$ p-adically. For each m we have a Frobenius map $\operatorname{Sym}^{k_m}(\alpha_{ad(\bar{t})})$ on $\operatorname{Sym}^{k_m} \mathcal{C}_o(\mathcal{O}_0)$, as well as a Frobenius map $\operatorname{Sym}^{k_m}(\alpha^*_{ad(\bar{t})})$ on $\operatorname{Sym}^{k_m} \mathcal{C}^*_0(\mathcal{O}_{0,q_{\bar{t}}})$. By transport of structure, we have then a Frobenius map $[\alpha_{ad(\bar{t})}]_{(\kappa;m)}$ on $\mathcal{S}_0^{(k_m)}(\mathcal{O}_0)$ and a dual Frobenius $[\alpha^*_{ad(\bar{t})}]_{(\kappa;m)}$ on $\mathcal{S}_0^{*(k_m)}(\mathcal{O}_{0,q_{\bar{t}}})$. We extend by zero these maps to all of $\mathcal{S}_0(\mathcal{O}_0)$ and $\mathcal{S}^*_0(\mathcal{O}_{0,q_{\bar{t}}})$, respectively. That is, we define

$$[\alpha_{ad(\bar{t})}]_{(\kappa;m)}(e_{\boldsymbol{u}}) := \begin{cases} [\alpha_{ad(\bar{t})}]_{k_m}(e_{\boldsymbol{u}}) & \text{if } |\boldsymbol{u}| \leq k_m, \\ 0 & \text{otherwise.} \end{cases}$$

To avoid any possible confusion, we note

$$\begin{aligned}
&[\alpha_{ad(\bar{t})}]_{(\kappa;m)}(e_{u_1} \cdots e_{u_r}) \\
&\qquad = (\Upsilon \circ \alpha_{ad(\bar{t})}(1))^{k_m - r} (\Upsilon \circ \alpha_{ad(\bar{t})} \tilde{\pi}^{w(u_1)} x^{u_1}) \cdots (\Upsilon \circ \alpha_{ad(\bar{t})} \tilde{\pi}^{w(u_r)} x^{u_r}) \\
&\qquad \cong (\operatorname{Sym}^{k_m} \alpha_{ad(\bar{t})})(E_0^{k_m - r} E_{u_1} \cdots E_{u_r}),
\end{aligned}$$

when $r \leq k_m$. Similarly

$$[\alpha^*_{ad(\bar{t})}]_{(\kappa;m)}(e^*_{\boldsymbol{u}}) := \begin{cases} [\alpha^*_{ad(\bar{t})}]_{k_m}(e^*_{\boldsymbol{u}}) & \text{if } |\boldsymbol{u}| \leq k_m, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3. $\lim_{m\to\infty} [\alpha_{ad(\bar{t})}]_{(\kappa;m)} = [\alpha_{ad(\bar{t})}]_\kappa$ *as maps from* $\mathcal{S}_0(\mathcal{O}_0) \to \mathcal{S}_0(\mathcal{O}_{0,q_{\bar{t}}})$.

Proof. Write

$$\begin{aligned}
(18)\quad &([\alpha_{ad(\bar{t})}]_{(\kappa;m)} - [\alpha_{ad(\bar{t})}]_\kappa)(e_{u_1} e_{u_2} \cdots e_{u_r}) \\
&\qquad = \big(\Upsilon(\alpha_{ad(\bar{t})}(1))^{k_m - r} - \Upsilon(\alpha_{ad(\bar{t})}(1))^{\kappa - r}\big) \\
&\qquad\qquad \times (\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_1)} x^{u_1}))) \cdots (\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_r)} x^{u_r}))).
\end{aligned}$$

If $r \leq k_m$, then the first factor on the right may itself be factored into

$$-\Upsilon(\alpha_{ad(\bar{t})}(1))^{\kappa - r}(1 - (\Upsilon(\alpha_{ad(\bar{t})}(1))^{k_m - \kappa})).$$

If $\kappa = k_m + p^{\tau(m)} \sigma_m$ (with $\tau(m) \to \infty$ and $\sigma_m \in \mathbb{Z}_p$) then

$$\big|1 - (\Upsilon(\alpha_{ad(\bar{t})}(1))^{k_m - \kappa}\big| \leq \big|\tilde{\pi}^{\tau(m)+1}\big|,$$

as in the proof of [Haessig 2014, Lemma 2.2], and using the estimate (9). If $r > k_m$ then (18) becomes

$$\begin{aligned}\big([\alpha_{ad(\bar{t})}]_{(\kappa;m)} - [\alpha_{ad(\bar{t})}]_\kappa\big)(e_{\boldsymbol{u}}) \\ &= -[\alpha_{ad(\bar{t})}]_\kappa e_{\boldsymbol{u}} \\ &= -\Upsilon(\alpha_{ad(\bar{t})}(1))^{\kappa-r}(\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_1)}x^{u_1})))\cdots(\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_r)}x^{u_r}))).\end{aligned}$$

Applying (9) to the r rightmost factors we see that

$$\big|\big([\alpha_{ad(\bar{t})}]_{(\kappa;m)} - [\alpha_{ad(\bar{t})}]_\kappa\big)e_{\boldsymbol{u}}\big| \le |\tilde{\pi}|^{w(\boldsymbol{u})(p^{ad(\bar{t})-1}-1)/p^{ad(\bar{t})-1}}.$$

But $w(\boldsymbol{u}) \ge r w_0 > k_m w_0$ (where $w_0 := \min\{w(\boldsymbol{u}) \mid \boldsymbol{u} \in M(f)\setminus\{0\}\}$). In terms of the operator norm,

$$\|[\alpha_{ad(\bar{t})}]_\kappa - [\alpha_{ad(\bar{t})}]_{(\kappa;m)}\| \le |\tilde{\pi}|^{\min\{\tau(m)+1,\, k_m w_0(p^{ad(\bar{t})-1}-1)/p^{ad(\bar{t})-1}\}}.$$

As k_m and $\tau(m)$ both tend to infinity as m grows, we see that

$$\lim_{m\to\infty}[\alpha_{ad(\bar{t})}]_{(\kappa;m)} = [\alpha_{ad(\bar{t})}]_\kappa. \qquad \square$$

In an altogether similar manner, for $u \neq 0$ we have, by Lemma 4.1, that $\alpha_m^*(\tilde{\pi}^{-w(u)}x^{-u})$ belongs to $\mathcal{C}_0^*(\mathcal{O}_{0,p^m})$, and (recalling (11))

$$|\alpha_m^*(\tilde{\pi}^{-w(u)}x^{-u})| \le |\tilde{\pi}|^{(p-1)w(u)}.$$

Also $\alpha_m^*(1) = 1 + \eta^*(\lambda)$ with $\eta^*(\lambda) \in \mathcal{O}_{0,p^m}$ and $|\eta^*(\lambda)| \le |\tilde{\pi}|$. With these observations, an entirely similar argument shows $\lim_{m\to\infty}[\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} = [\alpha^*_{ad(\bar{t})}]_\kappa$ as maps from $\mathcal{S}_0^*(\mathcal{O}_{0,q_{\bar{t}}}) \to \mathcal{S}_0^*(\mathcal{O}_{0,q_{\bar{t}}})$. Define

$$\begin{aligned}\beta_{(\kappa;m),\bar{t}} &:= \psi_\lambda^{ad(\bar{t})} \circ [\alpha_{ad(\bar{t})}]_{(\kappa;m)},\\ \beta^*_{(\kappa;m),\bar{t}} &:= \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} \circ \Phi_\lambda^{ad(\bar{t})}.\end{aligned}$$

As ψ_λ and Φ_λ are bounded maps, it follows that as operators on $\mathcal{S}_0(\mathcal{O}_0)$ and $\mathcal{S}_0^*(\mathcal{O}_0^*)$, respectively,

$$\lim_{m\to\infty}\beta_{(\kappa;m),\bar{t}} = \beta_{\kappa,\bar{t}} \quad\text{and}\quad \lim_{m\to\infty}\beta^*_{(\kappa;m),\bar{t}} = \beta^*_{\kappa,\bar{t}}. \tag{19}$$

Lemma 4.4. *For* $\xi \in \mathcal{S}_0^{(k_m)}(\mathcal{O}_0)$ *and* $\xi^* \in \mathcal{S}_0^{*(k_m)}(\mathcal{O}_0^*)$,

$$\langle \beta_{(\kappa;m),\bar{t}}\xi, \xi^*\rangle_{k_m} = \langle \xi, \beta^*_{(\kappa;m),\bar{t}}\xi^*\rangle_{k_m}. \tag{20}$$

Proof. With $\xi \in \mathcal{S}_0^{(k_m)}(\mathcal{O}_0)$ and $\xi^* \in \mathcal{S}_0^{*(k_m)}(\mathcal{O}_{0,q_{\bar{t}}})$, we may rewrite (14) as

$$([\alpha_{ad(\bar{t})}]_{(\kappa,m)}\xi, \xi^*)_{k_m} = (\xi, [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)}\xi^*)_{k_m}. \tag{21}$$

By linearity we only need consider $\xi = \lambda^\gamma e_{\boldsymbol{u}}$ and $\xi^* = \lambda^{-\sigma} e^*_{\boldsymbol{v}}$ where $\gamma, \sigma \in M(\Gamma)$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{S}(M)$. We may write

$$(e_{\boldsymbol{u}}, [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} e^*_{\boldsymbol{v}})_{k_m} = \sum_{\tau \in M(\Gamma)} C(\tau)\lambda^\tau.$$

Next, observe that

$$\langle \psi_\lambda \xi, \xi^* \rangle_{k_m} = \langle \xi, \Phi_\lambda \xi^* \rangle_{k_m}.$$

Hence, in the case $\xi = \lambda^\gamma e_{\boldsymbol{u}}$ and $\xi^* = \lambda^{-\sigma} e^*_{\boldsymbol{v}}$,

$$\begin{aligned}
\langle \beta_{(\kappa;m)} \xi, \xi^* \rangle_{k_m} &= \langle [\alpha_{ad(\bar{t})}]_{(\kappa;m)} \xi, \Phi_\lambda^{ad(\bar{t})} \xi^* \rangle_{k_m} \\
&= \text{the constant term of } [\lambda^{\gamma - q_{\bar{t}}\sigma} ([\alpha_{ad(\bar{t})}]_{(\kappa;m)} e_{\boldsymbol{u}}, e^*_{\boldsymbol{v}})_{k_m}] \\
&= \text{the constant term of } [\lambda^{\gamma - q_{\bar{t}}\sigma} (e_{\boldsymbol{u}}, [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} e^*_{\boldsymbol{v}})_{k_m}] \quad \text{(by (21))} \\
&= \text{the constant term of } \Big[\lambda^{\gamma - q_{\bar{t}}\sigma} \sum_{\tau \in M(\Gamma)} C(\tau)\lambda^\tau\Big] \\
&= \begin{cases} C(q_{\bar{t}}\sigma - \gamma) & \text{if } q_{\bar{t}}\sigma - \gamma \in M(\Gamma), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In the other direction, again setting $\xi = \lambda^\gamma e_{\boldsymbol{u}}$ and $\xi^* = \lambda^{-\sigma} e^*_{\boldsymbol{v}}$,

$$\begin{aligned}
\langle \xi, \beta^*_{(\kappa;m)} \xi^* \rangle_{k_m} &= \text{the constant term of } [\lambda^\gamma \cdot \mathrm{pr}_{M(\Gamma)} (\lambda^{-q_{\bar{t}}\sigma} (e_{\boldsymbol{u}}, [\alpha^*_{ad(\bar{t}}]_{(\kappa;m)} e^*_{\boldsymbol{v}})_{k_m}))] \\
&= \text{the constant term of } \Big[\lambda^\gamma \cdot \mathrm{pr}_{M(\Gamma)} \Big(\sum_{\tau \in M(\Gamma)} C(\tau) \lambda^{-(q_{\bar{t}}\sigma - \tau)} \Big)\Big] \\
&= \text{the constant term of } \Big[\lambda^\gamma \cdot \sum_{\substack{\tau \in M(\Gamma) \text{ such that} \\ q_{\bar{t}}\sigma - \tau \in M(\Gamma)}} C(\tau) \lambda^{-(q_{\bar{t}}\sigma - \tau)} \Big] \\
&= \begin{cases} C(q_{\bar{t}}\sigma - \gamma) & \text{if } q_{\bar{t}}\sigma - \gamma \in M(\Gamma), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Observe that $\beta_{(\kappa;m),\bar{t}}$ and $\beta_{\kappa,\bar{t}}$ are completely continuous operators on the p-adic Banach R-algebra $\mathcal{S}_0(\mathcal{O}_0)$ (viewed as R-algebra) with orthonormal basis $\{\tilde{\pi}^{w_\Gamma(\gamma)} \lambda^\gamma e_{\boldsymbol{u}} \mid \gamma \in M(\Gamma), \boldsymbol{u} \in \mathcal{S}(M)\}$. Let $\mathcal{T}_0(R)$ be $\mathcal{S}_0(\mathcal{O}_0)$ viewed in this way as an R-algebra. Similarly, write $\mathcal{T}_0^*(R)$ for the $b(I)$-space (over R) in Serre's terminology with "basis" $I := \{\tilde{\pi}^{-w_\Gamma(\gamma)} \lambda^{-\gamma} e^*_{\boldsymbol{u}} \mid \gamma \in M(\Gamma), \boldsymbol{u} \in \mathcal{S}(M)\}$ with coefficients in R. Again, $\mathcal{T}_0^*(R)$ is just $\mathcal{S}_0^*(\mathcal{O}_0^*)$ viewed over R. Then

$$\lim_{m \to \infty} \det(1 - \beta_{(\kappa;m),\bar{t}} T) = \det(1 - \beta_{\kappa,\bar{t}} T).$$

Similarly, $\beta^*_{(\kappa;m),\bar{t}}$ is a continuous R-linear endomorphism of $\mathcal{T}_0^*(R)$ to itself. We may consider a matrix $\mathfrak{B}^{*(\kappa;m),\bar{t}}$ with entries in R defined by

$$\beta^*_{(\kappa;m),\bar{t}}(\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma}e^*_{\boldsymbol{u}}) = \sum \mathfrak{B}^{*(\kappa;m),\bar{t}}_{(\delta,\boldsymbol{v}),(\gamma,\boldsymbol{u})}\tilde{\pi}^{-w_\Gamma(\delta)}\lambda^{-\delta}e^*_{\boldsymbol{v}}.$$

Using the matrix $\mathfrak{B}^{*(\kappa;m),\bar{t}}$, we define in the usual way the Fredholm determinant $\det(1-\beta^*_{(\kappa;m),\bar{t}}T) = \sum_{j\geq 0}(-1)^{j+1}C_j(\beta^*_{(\kappa;m),\bar{t}})T^j$ where $C_0 = 1$ and C_j is the series of all principal $j \times j$ subdeterminants of the matrix $\mathfrak{B}^{*(\kappa;m),\bar{t}}$. The $\langle\cdot\,,\cdot\rangle_{k_m}$-adjointness of $\beta_{(\kappa;m),\bar{t}}$ and $\beta^*_{(\kappa;m),\bar{t}}$ implies $C_j(\beta_{(\kappa;m),\bar{t}}) = C_j(\beta^*_{(\kappa;m),\bar{t}})$, so that

$$\det(1-\beta^*_{(\kappa;m),\bar{t}}T) = \det(1-\beta_{(\kappa;m),\bar{t}}T).$$

The uniform convergence $\lim_{m\to\infty}\mathfrak{B}^{*(\kappa;m),\bar{t}} =: \mathfrak{B}^*_{\kappa,\bar{t}}$ over the entries implies that the series $\sum_{j\geq 0}(-1)^{j+1}C_j(\mathfrak{B}^*_{\kappa,\bar{t}})T^j$ is well-defined, and is the coefficient-wise limit of $\det(1-\mathfrak{B}^*_{(\kappa;m),\bar{t}}T)$ as $m\to\infty$. If we define

$$\det(1-\beta^*_{\kappa,\bar{t}}T) := \sum_{j\geq 0}(-1)^{j+1}C_j(\mathfrak{B}^*_{\kappa,\bar{t}})T^j,$$

then we have shown:

Theorem 4.5. $\det(1-\beta_{\kappa,\bar{t}}T) = \det(1-\beta^*_{\kappa,\bar{t}}T)$, *and thus from* (10),

$$L^{(0)}(\kappa,\bar{t},T)^{(-1)^{s+1}} = \det(1-\beta^*_{\kappa,\bar{t}}T)^{\delta^s_{q_{\bar{t}}}}. \tag{22}$$

5. Eigenvector

Recall that

$$G(t,\lambda,x) = f(t,x) + P(\lambda,x) = \sum t_u x^u + \sum A(\gamma,v)\lambda^\gamma x^v$$

in $\mathbb{F}_q[x_1^\pm,\ldots,x_n^\pm,\lambda_1^\pm,\ldots,\lambda_s^\pm,\{t_u\}_{u\in\mathrm{supp}(f)}]$. Let $\hat{A}(\gamma,v)$ be the Teichmüller lift in $\mathbb{Q}_q$ for each $(\gamma,v)\in\mathrm{supp}(P)$, and denote the lifting of G by

$$\hat{G}(t,\lambda,x) := \hat{f}(t,x) + \hat{P}(\lambda,x) = \sum t_u x^u + \sum \hat{A}(\gamma,v)\lambda^\gamma x^v$$

in $\mathbb{Q}_q[x_1^\pm,\ldots,x_n^\pm,\lambda_1^\pm,\ldots,\lambda_s^\pm,\{t_u\}_{u\in\mathrm{supp}(f)}]$. We now replace every coefficient of G (with respect to the variables x and λ) with a new variable Λ:

$$\boldsymbol{f}(\Lambda,x) = \sum_{u\in\mathrm{supp}(f)}\Lambda_u x^u,$$
$$\mathcal{P}(\Lambda,\lambda,x) = \sum_{(\gamma,v)\in\mathrm{supp}(P)}\Lambda_{\gamma,v}\lambda^\gamma x^v,$$
$$H(\Lambda,\lambda,x) := \boldsymbol{f}(\Lambda,x) + \mathcal{P}(\Lambda,\lambda,x).$$

As before, let $\Delta_\infty(H)$ denote the Newton polytope of H at infinity in $\mathbb{R}^{s+n}$ (in λ and x variables). Let $\mathrm{Cone}(H)$ be the cone in $\mathbb{R}^{s+n}$ over $\Delta_\infty(H)$ and $M(H) = \mathrm{Cone}(H) \cap \mathbb{Z}^{s+n}$ be the relevant monoid. Clearly $M(H) \subset M(\Gamma) \times M(f)$. By our hypothesis that the x-support of P is contained in $\Delta_\infty(f)$ we have that the polyhedral weight function on this polytope w_H dominates the total weight $w_\Gamma + w$ relative to the polyhedron $\Gamma \times \Delta_\infty$; more precisely

$$w_\Gamma(\gamma) + w(u) \le w_H(\gamma, u),$$

for all $(\gamma, u) \in M(H)$.

The following definitions extend those in Section 4, by replacing R with the (multivariable) formal power series ring $\mathcal{K} := R[[\Lambda]]$. We equip $\mathcal{K}$ with the sup-norm. Denote by $\mathcal{K}_0$ the subring of $\mathcal{K}$ of power series which converge on the closed unit polydisk $|\Lambda| \le 1$. For q any power of the prime p, define

$$\mathcal{O}_{0,q}(\mathcal{K}) := \left\{ \sum_{\gamma \in M(\Gamma)} C(\gamma)\lambda^\gamma \tilde{\pi}^{w_{q\Gamma}(\gamma)} \,\middle|\, C(\gamma) \in \mathcal{K}, C(\gamma) \to 0 \text{ as } \gamma \to \infty \right\},$$

$$\mathcal{C}_0^*(\mathcal{O}_{0,q}(\mathcal{K})) := \left\{ \sum_{u \in M(f)} \xi(u)\tilde{\pi}^{-w(u)} x^{-u} \,\middle|\, \xi(u) \in \mathcal{O}_{0,q}(\mathcal{K}) \right\},$$

$$\mathcal{S}_0^*(\mathcal{O}_{0,q}(\mathcal{K})) := \left\{ \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi^*(\boldsymbol{u}) e_{\boldsymbol{u}}^* \,\middle|\, \xi^*(\boldsymbol{u}) \in \mathcal{O}_{0,q}(\mathcal{K}) \right\},$$

$$\mathcal{O}_{0,q}^*(\mathcal{K}) := \left\{ \sum_{\gamma \in M(\Gamma)} \zeta^*(\gamma)\tilde{\pi}^{-w_{q\Gamma}(\gamma)}\lambda^{-\gamma} \,\middle|\, \zeta^*(\gamma) \in \mathcal{K} \right\},$$

$$\mathcal{S}_0^*(\mathcal{O}_{0,q}^*(\mathcal{K})) := \left\{ \sum_{\substack{\gamma \in M(\Gamma) \\ \boldsymbol{u} \in \mathcal{S}(M)}} \omega^*(\gamma, \boldsymbol{u})\tilde{\pi}^{-w_{q\Gamma}(\gamma)}\lambda^{-\gamma} e_{\boldsymbol{u}}^* \,\middle|\, \omega^*(\gamma, \boldsymbol{u}) \in \mathcal{K} \right\}.$$

In all cases, the spaces above have versions (with obvious modification of notation), where the ring of coefficients $\mathcal{K}$ is replaced by the subring $\mathcal{K}_0$. Define the maps

$$\mathrm{pr}_{M(f)} : \sum_{u \in \mathbb{Z}^n} C(u)x^{-u} \longmapsto \sum_{u \in M(f)} C(u)x^{-u},$$

and $\Upsilon : \mathcal{C}_0^*(\mathcal{O}_0(\mathcal{K})) \hookrightarrow \mathcal{S}_0^*(\mathcal{O}_0(\mathcal{K}))$ by $\tilde{\pi}^{-w(u)}x^{-u} \mapsto e_u^*$ for $u \in M \backslash \{0\}$ and $\Upsilon(1) := 1$. Next, define a relative Frobenius map as follows. First, set

$$F(\Lambda, \lambda, x) := \prod_{u \in \mathrm{supp}(f)} \theta(\Lambda_u x^u) \cdot \prod_{(\gamma, v) \in \mathrm{supp}(P)} \theta(\Lambda_{\gamma,v}\lambda^\gamma x^v),$$

$$F_m(\Lambda, \lambda, x) := \prod_{i=0}^{m-1} F(\Lambda^{p^i}, \lambda^{p^i}, x^{p^i}),$$

and note that, similar to before,

$$F_m(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M(H)} B_{\gamma,u}(\Lambda)\tilde{\pi}^{w_H(\gamma,u)/p^{m-1}}\lambda^\gamma x^u,$$

with $|B_{\gamma,u}(\Lambda)| \le 1$. It follows that, if we set

$$\alpha^*_{m,\Lambda} := \mathrm{pr}_{M(f)} \circ F_m(\Lambda, \lambda, x) \circ \Phi_x^m,$$

where Φ_x sends $x^u \mapsto x^{pu}$, then an argument similar to Lemma 4.1 shows

$$\alpha^*_{m,\Lambda} : \mathcal{C}_0^*(\mathcal{O}_{0,p^m}(\mathcal{K})) \to \mathcal{C}_0^*(\mathcal{O}_{0,p^m}(\mathcal{K})).$$

For any $\kappa \in \mathbb{Z}_p$, we define $[\alpha^*_{m,\Lambda}]_\kappa : \mathcal{S}_0^*(\mathcal{O}_{0,p^m}(\mathcal{K})) \to \mathcal{S}_0^*(\mathcal{O}_{0,p^m}(\mathcal{K}))$ using (15). By an argument similar to Lemma 4.2, the map

$$\beta^*_{\kappa,\bar{t},\Lambda} : \mathcal{S}_0^*(\mathcal{O}_0^*(\mathcal{K})) \to \mathcal{S}_0^*(\mathcal{O}_0^*(\mathcal{K}))$$

defined by

$$\beta^*_{\kappa,\bar{t},\Lambda} := \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \circ \Phi_\lambda^{ad(\bar{t})},$$

is an endomorphism over $\mathcal{K}$.

Eigenvector. In the following, we will define an eigenvector $\Upsilon(\eta)^\kappa$ of $\beta^*_{\kappa,\bar{t},\Lambda}$ whose eigenvalue is $\mathcal{F}_{ad(\bar{t})}(\Lambda)^\kappa$. We will then specialize Λ, proving Theorem 1.1. We start by defining the groups

$$M_0(\Gamma) = M(\Gamma) \cap (-M(\Gamma)) \quad \text{and} \quad M_0(f) = M(f) \cap (-M(f)).$$

Define the projection map

$$\mathrm{pr}_0 : \sum_{\substack{\gamma\in\mathbb{Z}^s \\ u\in\mathbb{Z}^n}} C(\gamma, u)\lambda^\gamma x^u \longmapsto \sum_{\substack{\gamma\in M_0(\Gamma) \\ u\in M_0(f)}} C(\gamma, u)\lambda^\gamma x^u,$$

and write

$$\mathrm{pr}_0 \circ \exp \pi H(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M_0(\Gamma)\times M_0(f)} J_{\gamma,u}(\Lambda)\lambda^\gamma x^u, \tag{23}$$

with $J_{\gamma,u} \in R[\![\Lambda]\!]$. Observe that $J_{0,0} \in 1 + \Lambda R[\![\Lambda]\!]$, and so we may define

$$\eta(\Lambda, \lambda, x) := \tfrac{1}{J_{0,0}(\Lambda)} \mathrm{pr}_0(\exp \pi H(\Lambda, \lambda, x)).$$

We will eventually need to specialize Λ to Teichmüller units. The following lemma demonstrates that this is possible.

Lemma 5.1. *$J_{\gamma,u}(\Lambda)/J_{0,0}(\Lambda) \in \mathcal{K}_0$ for each $(\gamma, u) \in M_0(\Gamma) \times M_0(f)$. Also, $J_{0,0}(\Lambda)/J_{0,0}(\Lambda^p) \in \mathcal{K}_0$.*

Proof. This result is essentially a version of the main result, Proposition 2.15 and its corollaries, in [Adolphson and Sperber 2012]. The proof of the version here necessitates only some minor modifications from that in the above reference. The key difference is that the setup here uses total weight, $w_{\mathrm{tot}} = w_\Gamma + w$ based on $\Gamma \times \Delta_\infty(f)$ rather than the straightforward polyhedral weight w_H based on $\Delta_\infty(H)$. The argument of [Adolphson and Sperber 2012] works here as well. □

Next, we will show that the $\Upsilon(\eta)^\kappa$ is a well-defined element of our dual space.

Lemma 5.2. $\Upsilon(\eta(\Lambda, \lambda, x))^\kappa \in S_0^*(\mathcal{O}_0^*(\mathcal{K}_0))$.

Proof. First, write

$$\eta(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M_0(\Gamma)\times M_0(f)} C_{\gamma,u}(\Lambda)\lambda^\gamma x^u,$$

with $|C_{\gamma,u}| \le 1$ and $C_{\gamma,u} \in \mathcal{K}_0$. Since $u \in M_0(f)$, we may write

$$\eta(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M_0(\Gamma)\times M_0(f)} (C_{\gamma,-u}(\Lambda)\tilde{\pi}^{w(u)})\lambda^\gamma \tilde{\pi}^{-w(u)} x^{-u},$$

and so $\Upsilon(\eta(\Lambda, \lambda, x)) = \sum_{\gamma,u} \tilde{C}_{\gamma,u}(\Lambda)\lambda^\gamma e_u^*$, with $\tilde{C}_{\gamma,u}(\Lambda) := C_{\gamma,-u}(\Lambda)\tilde{\pi}^{w(u)}$. Next, since $\tilde{C}_{0,0}$ is a unit, we may write

$$\begin{aligned}
\Upsilon(\eta)^\kappa &= \Bigl(\tilde{C}_{0,0}(\Lambda) + \sum_{(\gamma,u)\in M\setminus\{0\}} \tilde{C}_{\gamma,u}(\Lambda)\lambda^\gamma e_u^*\Bigr)^\kappa \\
&= \sum_{l=0}^{\infty} \binom{\kappa}{l} \tilde{C}_{0,0}(\Lambda)^{\kappa-l} \Bigl(\sum_{(\gamma,u)\in M\setminus\{0\}} \tilde{C}_{\gamma,u}(\Lambda)\lambda^\gamma e_u^*\Bigr)^l \\
&= \sum_{\substack{\gamma\in M_0(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M_0(f))}} D_{\gamma,\boldsymbol{u}}(\Lambda)\lambda^\gamma e_{\boldsymbol{u}}^*,
\end{aligned}$$

with $D_{\gamma,\boldsymbol{u}} \in \mathcal{K}_0$. Lastly, since $\gamma \in M_0(\Gamma)$, we may rewrite this as

$$\Upsilon(\eta)^\kappa = \sum_{\substack{\gamma\in M_0(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M_0(f))}} (D_{-\gamma,\boldsymbol{u}}(\Lambda)\tilde{\pi}^{w_\Gamma(\gamma)})\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma} e_{\boldsymbol{u}}^*,$$

and thus $\Upsilon(\eta)^\kappa \in S_0^*(\mathcal{O}_0^*(\mathcal{K}_0))$. □

We now consider the action of $\alpha^*_{1,\Lambda}$ on η. Set $\mathcal{F}(\Lambda) := J_{0,0}(\Lambda)/J_{0,0}(\Lambda^p)$. Observe that

$$\begin{aligned}
&\alpha^*_{1,\Lambda}(\eta(\Lambda^p, \lambda^p, x))\\
&= \mathrm{pr}_{M(f)}\big(F(\Lambda, \lambda, x)\,\mathrm{pr}_0\big(\exp \pi H(\Lambda^p, \lambda^p, x^p)/J_{0,0}(\Lambda^p)\big)\big)\\
&= \mathrm{pr}_{M(f)}\big(F(\Lambda, \lambda, x)\big(\exp \pi H(\Lambda^p, \lambda^p, x^p)/J_{0,0}(\Lambda^p) + \hat{\omega}(\Lambda, \lambda, x) + \epsilon(\Lambda.\lambda, x)\big)\big)\\
&= \mathcal{F}(\Lambda)\big(\mathrm{pr}_{M(f)}\big(\exp \pi H(\Lambda, \lambda, x)/J_{0,0}(\Lambda) + \omega^*(\Lambda, \lambda, x)\big)\big)\\
&= \mathcal{F}(\Lambda)\big(\eta(\Lambda, \lambda, x) + \tilde{\omega}(\Lambda, \lambda, x)\big),
\end{aligned}$$

where each $\lambda^\gamma x^u$ appearing in $\hat{\omega}$ (and ω^* and $\tilde{\omega}$) has γ in $M(\Gamma) \setminus M_0(\Gamma)$, and every $\lambda^\gamma x^u$ appearing in ϵ has u in $M(f) \setminus M_0(f)$. Iterating this, if we set

$$\mathcal{F}_m(\Lambda) := \prod_{i=0}^{m-1} \mathcal{F}(\Lambda^{p^i}),$$

then we have

$$\alpha^*_{ad(\bar{t}),\Lambda}\eta(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x) = \mathcal{F}_{ad(\bar{t})}(\Lambda)(\eta(\Lambda, \lambda, x) + \omega(\Lambda, \lambda, x)), \tag{24}$$

where each λ^γ appearing in ω lies in $M(\Gamma) \setminus M_0(\Gamma)$.

For the calculation of the eigenvalue, we will need the following. First, as every λ^γ appearing in $\Upsilon(\omega)$ (from Equation (24)) satisfies $\gamma \in M(\Gamma) \setminus M_0(\Gamma)$, it follows that the same is true for $\Upsilon(\eta)^{\kappa-r}\Upsilon(\omega)^r$ for every $r \in \mathbb{Z}_{\geq 1}$. Hence,

$$\mathrm{pr}_{M(\Gamma)}(\Upsilon(\eta) + \Upsilon(\omega))^\kappa = \mathrm{pr}_{M(\Gamma)} \sum_{r=0}^{\infty} \tbinom{\kappa}{r} \Upsilon(\eta)^{\kappa-r}\Upsilon(\omega)^r = \Upsilon(\eta)^\kappa. \tag{25}$$

We may now finish the proof of Theorem 1.1. For convenience, write $\eta(\Lambda, \lambda, x) = 1 + h(\Lambda, \lambda, x)$ so that $\Upsilon(\eta)^\kappa = (1 + \Upsilon(h))^\kappa = \sum_{l=0}^{\infty} \binom{\kappa}{l} \Upsilon(h)^l$. Observe that

$$\begin{aligned}
&\beta^*_{\kappa,\bar{t},\Lambda}\Upsilon(\eta(\Lambda^{q_{\bar{t}}}, \lambda, x))^\kappa\\
&\qquad = \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \circ \Phi_\lambda^{ad(\bar{t})} \Upsilon(\eta(\Lambda^{q_{\bar{t}}}, \lambda, x))^\kappa\\
&\qquad = \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \Upsilon(\eta(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^\kappa\\
&\qquad = \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \sum_{l=0}^{\infty} \binom{\kappa}{l} \Upsilon(h(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^l\\
&\qquad = \mathrm{pr}_{M(\Gamma)} \sum_{l=0}^{\infty} \binom{\kappa}{l} (\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} \cdot 1)^{\kappa-l} (\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} h(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^l \qquad \text{(by (15))}\\
&\qquad = \mathrm{pr}_{M(\Gamma)}\big(\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} \cdot 1 + \Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} h(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x)\big)^\kappa\\
&\qquad = \mathrm{pr}_{M(\Gamma)}(\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} \eta(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^\kappa
\end{aligned}$$

$$= \mathrm{pr}_{M(\Gamma)}\, \mathcal{F}_{ad(\bar{t})}(\Lambda)^{\kappa} (\Upsilon(\eta(\Lambda, \lambda, x)) + \Upsilon(\omega(\Lambda, \lambda, x)))^{\kappa} \quad \text{(by (24))}$$
$$= \mathrm{pr}_{M(\Gamma)}\, \mathcal{F}_{ad(\bar{t})}(\Lambda)^{\kappa} \Upsilon(\eta(\Lambda, \lambda, x))^{\kappa} \left(1 + \frac{\Upsilon(\omega(\Lambda, \lambda, x))}{\Upsilon(\eta(\Lambda, \lambda, x))}\right)^{\kappa}$$
$$= \mathcal{F}_{ad(\bar{t})}(\Lambda)^{\kappa} \Upsilon(\eta(\Lambda, \lambda, x))^{\kappa} \quad \text{(by (25))}.$$

Finally, we may specialize this equality by taking Λ at the Teichmüller unit coefficients of $\hat{G}(\hat{t}, \lambda, x)$,

$$\Lambda_u = \hat{t}_u \quad \text{and} \quad \Lambda_{\gamma,v} = \hat{A}(\gamma, v)$$

for all u and γ, v in the support of H. Setting

$$\eta_{\mathrm{sp}}(\lambda, x) := \big(\eta(\Lambda, \lambda, x) \text{ specialized at } \Lambda_u = \hat{t}_u \text{ and } \Lambda_{\gamma,v} = \hat{A}(\gamma, v)\big),$$

we see that

$$\beta^*_{\kappa,\bar{t}}\, \Upsilon(\eta_{\mathrm{sp}}(\lambda, x))^{\kappa} = \mathcal{F}_{ad(\bar{t})}(\hat{t})^{\kappa} \Upsilon(\eta_{\mathrm{sp}}(\lambda, x))^{\kappa}. \tag{26}$$

This demonstrates that $\mathcal{F}_{ad(\bar{t})}(\hat{t})^{\kappa}$ is the unique unit root of $L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ by (22), which, together with Theorem 3.1, completes the proof of Theorem 1.1.

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References

[Adolphson and Sperber 2012] A. Adolphson and S. Sperber, "On unit root formulas for toric exponential sums", *Algebra Number Theory* **6**:3 (2012), 573–585. MR Zbl

[Dwork 1973] B. Dwork, "Normalized period matrices, II", *Ann. of Math.* (2) **98** (1973), 1–57. MR Zbl

[Gouvêa and Mazur 1992] F. Gouvêa and B. Mazur, "Families of modular eigenforms", *Math. Comp.* **58**:198 (1992), 793–805. MR Zbl

[Haessig 2014] C. D. Haessig, "Meromorphy of the rank one unit root L-function revisited", *Finite Fields Appl.* **30** (2014), 191–202. MR Zbl

[Haessig 2016] C. D. Haessig, "L-functions of symmetric powers of Kloosterman sums (unit root L-functions and p-adic estimates)", *Math. Ann.* (online publication August 2016), 1–31.

[Haessig and Sperber 2014] C. D. Haessig and S. Sperber, "L-functions associated with families of toric exponential sums", *J. Number Theory* **144** (2014), 422–473. MR Zbl

[Wan 1999] D. Wan, "Dwork's conjecture on unit root zeta functions", *Ann. of Math.* (2) **150**:3 (1999), 867–927. MR Zbl

[Wan 2000a] D. Wan, "Higher rank case of Dwork's conjecture", *J. Amer. Math. Soc.* **13**:4 (2000), 807–852. MR Zbl

[Wan 2000b] D. Wan, "Rank one case of Dwork's conjecture", *J. Amer. Math. Soc.* **13**:4 (2000), 853–908. MR Zbl

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BAVARD'S DUALITY THEOREM ON CONJUGATION-INVARIANT NORMS

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Bavard proved a duality theorem between commutator length and quasimorphisms. Burago, Ivanov and Polterovich introduced the notion of a conjugation-invariant norm which is a generalization of commutator length. Entov and Polterovich proved Oh–Schwarz spectral invariants are subset-controlled quasimorphisms, which are generalizations of quasimorphisms. We prove a Bavard-type duality theorem between subset-controlled quasimorphisms on stable groups and conjugation-invariant (pseudo)norms. We also pose a generalization of our main theorem and prove "stably nondisplaceable subsets of symplectic manifolds are heavy" in a rough sense if that generalization holds.

1. Definitions and results

Definitions. Burago, Ivanov and Polterovich defined the notion of conjugation-invariant (pseudo)norms on groups and they gave a number of its applications.

Definition 1.1 [Burago et al. 2008]. Let G be a group. A function $\nu : G \to \mathbb{R}_{\geq 0}$ is *a conjugation-invariant norm* on G if ν satisfies the following axioms:

(1) $\nu(1) = 0$;

(2) $\nu(f) = \nu(f^{-1})$ for every $f \in G$;

(3) $\nu(fg) \leq \nu(f) + \nu(g)$ for every $f, g \in G$;

(4) $\nu(f) = \nu(gfg^{-1})$ for every $f, g \in G$;

(5) $\nu(f) > 0$ for every $f \neq 1 \in G$.

A function $\nu : G \to \mathbb{R}$ is *a conjugation-invariant pseudonorm* on G if ν satisfies axioms (1), (2), (3) and (4) above.

For a conjugation-invariant pseudonorm ν, let $s\nu$ denote the stabilization of ν, i.e., $s\nu(g) = \lim_{n\to\infty} \nu(g^n)/n$ (this limit exists by Fekete's Lemma).

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For a perfect group G, the commutator length cl on G is a conjugation-invariant norm. Bavard [1991] proved the following famous theorem (see also [Calegari 2009]):

Theorem 1.2 (Corollary of Bavard's [1991] duality theorem). *Let g be an element of a perfect group G. Then* $\mathrm{scl}(g) > 0$ *if and only if there exists a homogeneous quasimorphism ϕ such that $\phi(g) > 0$.*

For interesting applications of Bavard's duality theorem, see [Calegari et al. 2014], [Endo and Kotschick 2001] and [Mimura 2010] for example. After Bavard's work, Calegari and Zhuang [2011] proved a Bavard-type duality theorem on W-length which is also conjugation-invariant. In the present paper, we give a Bavard-type duality theorem on general conjugation-invariant (pseudo)norms for some groups which are stable in some sense.

To state our main theorem, we introduce the notion of subset-controlled quasimorphism (partial quasimorphism, prequasimorphism) which is a generalization of quasimorphism:

Definition 1.3. Let G be a group and H a subgroup of G. We define the fragmentation norm ν_H with respect to H for an element f of G, by

$$\nu_H(f) = \min\{k : \text{ there exist } g_1, \dots, g_k \in G, \text{ and } h_1, \dots, h_k \in H \text{ such that } f = g_1 h_1 g_1^{-1} \cdots g_k h_k g_k^{-1}\}.$$

If there is no such decomposition of f as above, we put $\nu_H(f) = \infty$.

Definition 1.4. Let H be a subgroup of a group G. A function $\phi : G \to \mathbb{R}$ is called an *H-quasimorphism* if there exists a positive number C such that for any $f, g \in G$,

$$|\phi(fg) - \phi(f) - \phi(g)| < C \min\{\nu_H(f), \nu_H(g)\}.$$

The infimum of such C is called *the defect of ϕ* and we denote it by $D(\phi)$. If $\phi(f^n) = n\phi(f)$ for any element f of G and any integer n, ϕ is called *homogeneous*.

Such generalizations of quasimorphisms appeared first in [Entov and Polterovich 2006]. They proved that Oh–Schwarz spectral invariants (for example, see [Schwarz 2000] and [Oh 2006]) are controlled quasimorphisms.

Remark 1.5. In [Kawasaki 2016], H-quasimorphism is called quasimorphism relative to ν_H. Tomohiko Ishida and Tetsuya Ito pointed out that quasimorphism relative to H usually means quasimorphism which vanishes on H. Thus we use a different notation from that work.

Let K be a subset of a group G. For elements f, g of G, let fKg denote the subset $\{fkg; k \in K\}$ of G.

Definition 1.6. Let H be a subgroup of a group G. If for any element g of G, $\nu_H(g) < \infty$, G is said to be *c-generated* by H.

The author essentially proved the following proposition:

Proposition 1.7 [Kawasaki 2016]. *Let G be a group c-generated by a perfect subgroup H (in particular, G is also perfect). If there exists an H-quasimorphism ϕ with $\lim_{k\to\infty} \phi(g^k)/k > 0$ for some g, then there is a conjugation-invariant norm ν with $\mathrm{s}\nu(g) > 0$ (such a norm is called* ***stably unbounded*** *[Burago et al. 2008]).*

Our main theorem (Theorem 1.12) is a converse of the Proposition 1.7.

Remark 1.8. The author [Kawasaki 2016] proved that there exists such a $\mathrm{Ham}(\mathbb{B}^{2n})$-quasimorphism μ_K on $\mathrm{Ham}(\mathbb{R}^{2n})$. Here, $\mathrm{Ham}(\mathbb{B}^{2n})$ and $\mathrm{Ham}(\mathbb{R}^{2n})$ are the group of Hamiltonian diffeomorphisms with compact support of the ball and the Euclidean space with the standard symplectic form, respectively. He also proved that $\mu_K(g) > 0$ for some commutator g. Thus, by Proposition 1.7, $[\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{R}^{2n})]$ admits a stably unbounded norm.

Kimura [2016] proved a similar result on the infinite braid group $B_\infty = \bigcup_{k=1}^{\infty} B_k$ (the existence of a stably unbounded norm on $[B_\infty, B_\infty]$ is also proved by Brandenbursky and Kedra [2015]).

Definition 1.9. Let G be a group, H a subgroup of G and K a subset of G. We define the set $\mathrm{D}^f_H(K)$ of maps *displacing K far away* by

$$\mathrm{D}^f_H(K) = \{h_0 \in G : \text{ for all } g_1, \dots, g_k \in G, \text{ there exists } h \in G \text{ such that } hh_0h^{-1}K(hh_0h^{-1})^{-1} \text{ commutes with } g_1Hg_1^{-1} \cup \dots \cup g_kHg_k^{-1}\}.$$

Let ν be a conjugation-invariant pseudonorm on a group G. For a subset K of G, we define *the far away displacement energy* $\mathrm{E}_{H,\nu}(K)$ of K by

$$\mathrm{E}_{H,\nu}(K) = \inf_{g\in \mathrm{D}^f_H(K)} \nu(g).$$

Definition 1.10. Let G be a group and H a subgroup of G. The pair (G, H) satisfies the property FM if G and H satisfy the following conditions.

(1) G is c-generated by H,

(2) For any elements $h_1, \dots, h_k$ of G, $\mathrm{D}^f_H(h_1Hh_1^{-1} \cup \dots \cup h_kHh_k^{-1}) \neq \varnothing$.

A group G satisfies the property FM if (G, H) satisfies the property FM for some subgroup H.

For a group G, we define the set $\mathsf{FM}(G)$ by

$$\mathsf{FM}(G) = \{H \le G;\ (G, H) \text{ satisfies the property } \mathsf{FM}\}.$$

We give some examples satisfying the property FM.

Proposition 1.11. (1) *For any integer i, the pair (B_∞, B_i) satisfies the property* FM, *and so does the pair* $([B_\infty, B_\infty], [B_i, B_i])$.

(2) *We consider the Riemannian surface $\Sigma_\infty = \bigcup_{k=1}^{\infty} \Sigma_k^1$ where Σ_k^1 is the Riemannian surface which has genus k and* 1 *puncture. The pair of mapping class groups* $(\mathrm{MCG}(\Sigma_\infty), \mathrm{MCG}(\Sigma_i^1))$ *satisfies the property* FM *for any integer i.*

(3) *The pair* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *satisfies the property* FM, *and so does the pair* $([\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{R}^{2n})], [\mathrm{Ham}(\mathbb{B}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n})])$.

Our main theorem is the following one.

Theorem 1.12. *Let G be a group satisfying the property* FM *and ν a conjugation-invariant pseudonorm on G. Then,*

(1) *For any element g of G such that* $s\nu(g) > 0$, *there exists a function $\phi : G \to \mathbb{R}$ which is a homogeneous H-quasimorphism for any element H of* $\mathsf{FM}(G)$ *such that $\phi(g) > 0$.*

(2) *For any element g of the commutator subgroup $[G, G]$ and any $H \in \mathsf{FM}(G)$,*

$$s\nu(g) \leq 8 \sup_{\phi} \frac{\phi(g) \cdot E_{H,\nu}(H)}{D(\phi)},$$

where sup *is taken over the set of homogeneous H-quasimorphisms $\phi : G \to \mathbb{R}$.*

In Section 2, we construct the normed vector space A_ν and prove Theorem 1.12 by applying the Hahn–Banach theorem to A_ν. In Section 3, we prove that A_ν is a normed vector space. In Section 4, we prove Proposition 1.11. In Section 5, we pose a generalization of Theorem 1.12 (Problem 5.6) and give its application to symplectic geometry. There, we prove that "stably nondisplaceable subsets of symplectic manifolds are heavy" in a very rough sense if the positive answer of Problem 5.6 holds.

2. Proof of main theorem

To construct controlled quasimorphisms by using the Hahn–Banach theorem, we consider the normed vector space A_ν which we define here. The idea of our construction comes from [Calegari and Zhuang 2011].

For a group G, we define the set $A_G = \coprod_{k=0}^{\infty} (G \times \mathbb{R})^k$. We denote elements of A_G by $g_1^{s_1} \cdots g_k^{s_k}$, where $g_1, \dots, g_k \in G$ and $s_1, \dots, s_k$ are real numbers.

Let ν be a conjugation-invariant pseudonorm on G. We define the $\mathbb{R}_{\geq 0}$-valued function $\|\cdot\|_\nu : A_G \to \mathbb{R}_{\geq 0}$ by

$$\|g_1^{s_1} \cdots g_k^{s_k}\|_\nu = \lim_{n \to \infty} \frac{1}{n} \cdot \nu(g_1^{[s_1 n]} \dots g_k^{[s_k n]}),$$

where $[\,\cdot\,]$ denotes the integer part. For the trivial element $1 \in (G \times \mathbb{R})^0$ of A_G, we define $\|1\|_\nu = 0$.

Proposition 2.1. *Let ν be a conjugation-invariant pseudonorm on a group G satisfying the property* FM. *Then for any element $g_1^{s_1} \cdots g_k^{s_k}$ of A_G, the above limit $\|g_1^{s_1} \cdots g_k^{s_k}\|_\nu$ exists. Thus $\|\cdot\|_\nu$ is well defined.*

We prove Proposition 2.1 in Section 3. First, we define some operations on A_G. For elements $\mathtt{g} = g_1^{s_1} \cdots g_k^{s_k}$, $\mathtt{h} = h_1^{t_1} \cdots h_l^{t_l}$ of A_G and a real number λ, we define $\mathtt{g} \cdot \mathtt{h}$, $\bar{\mathtt{g}}$ and $\mathtt{g}^{(\lambda)}$ by

$$\mathtt{g} \cdot \mathtt{h} = g_1^{s_1} \cdots g_k^{s_k} h_1^{t_1} \cdots h_l^{t_l}, \quad \bar{\mathtt{g}} = g_k^{-s_k} \cdots g_1^{-s_1} \quad \text{and} \quad \mathtt{g}^{(\lambda)} = g_1^{\lambda s_1} \cdots g_k^{\lambda s_k}.$$

By the definition of conjugation-invariant pseudonorms, we can confirm that the function $\|\cdot\|_\nu : A_G \to \mathbb{R}$ satisfies the following properties easily. For any $\mathtt{g}, \mathtt{h} \in A_G$,

$$\|\mathtt{g} \cdot \mathtt{h}\|_\nu \le \|\mathtt{g}\|_\nu + \|\mathtt{h}\|_\nu, \quad \|\mathtt{h} \cdot \mathtt{g} \cdot \bar{\mathtt{h}}\|_\nu = \|\mathtt{g}\|_\nu \quad \text{and} \quad \|\bar{\mathtt{g}}\|_\nu = \|\mathtt{g}\|_\nu.$$

We define the equivalence relation $\sim$ by $\mathtt{g} \sim \mathtt{h}$ if and only if $\|\mathtt{g} \cdot \bar{\mathtt{h}}\|_\nu = 0$. We denote the set $A_G/\sim$ by A_ν and the function $\|\cdot\|_\nu : A_G \to \mathbb{R}$ on A_G induces the function $\|\cdot\|_\nu : A_\nu \to \mathbb{R}$ on A_ν.

In the present paper, we want to consider A_ν as an $\mathbb{R}$-vector space with the norm $\|\cdot\|_\nu$. We define a sum operation, an inverse operation and an $\mathbb{R}$-action on A_ν. For elements $\mathsf{g} = [\mathtt{g}]$, $\mathsf{h} = [\mathtt{h}]$ of A_ν and a real number λ, we define $\mathsf{g} + \mathsf{h}$ and $\lambda \mathsf{g}$ by

$$\mathsf{g} + \mathsf{h} = [\mathtt{g} \cdot \mathtt{h}] \quad \text{and} \quad \lambda \mathsf{g} = [\mathtt{g}^{(\lambda)}].$$

Proposition 2.2. *Assume that G satisfies the property* FM. *Then the above operations are well defined.*

To use the Hahn–Banach theorem, we prove that A_ν is a normed vector space.

Proposition 2.3. *Assume that G satisfies the property* FM. *Then $(A_\nu, \|\cdot\|_\nu)$ is a normed vector space with respect to the above operations.*

We prove Proposition 2.2 and 2.3 in Section 3.

Let G be a group and ν a conjugation-invariant pseudonorm on G. Let $L(G, \nu)$ denote the set of Lipschitz continuous (linear) homomorphisms from A_ν to $\mathbb{R}$. By the Hahn–Banach theorem, Proposition 2.3 implies the following proposition.

Proposition 2.4. *Assume that G satisfies the property* FM. *Then for any $\mathsf{g} \in A_\nu$,*

$$\|\mathsf{g}\|_\nu = \sup_{\hat{\phi} \in L(G,\nu)} \frac{\hat{\phi}(\mathsf{g})}{l(\hat{\phi})},$$

where $l(\hat{\phi})$ is the optimal Lipschitz constant of $\hat{\phi}$.

For an element $\hat{\phi}$ of $L(G, \nu)$, we define the map $\phi : G \to \mathbb{R}$ by $\phi(g) = \hat{\phi}([g^1])$.

Proposition 2.5. *Let H be an element of* $\mathsf{FM}(G)$. *For any element $\hat{\phi}$ of $L(G, \nu)$, ϕ is a homogeneous H-quasimorphism. Moreover, $D(\phi) \le 8 l(\hat{\phi}) \cdot E_{H,\nu}(H)$.*

To prove Proposition 2.5, we use the following lemmas:

Lemma 2.6. *Let G be a group and H, K subgroups of G. Assume (G, H) satisfies the property* FM. *Then for any $g \in G$ and any element $f \in K$, $\nu([g, f]) \leq 4E_{H,\nu}(K)$.*

Proof. Let f, g and h_0 be elements of K, G and $\mathrm{D}_H^f(K)$, respectively. Since G is c-generated by H and the set $\{f, g\}$ is a finite set, there exist elements $h_1, \dots, h_k$ of G such that $f, g \in \langle h_1 H h_1^{-1}, \dots, h_k H h_k^{-1}\rangle$.

Then, by the definition of $\mathrm{D}_H^f(K)$, there exists an element h of G such that $(hh_0h^{-1})K(hh_0h^{-1})^{-1}$ commutes with $\langle h_1 H h_1^{-1}, \dots, h_k H h_k^{-1}\rangle$. Since $f \in K$ and $f, g \in \langle h_1 H h_1^{-1}, \dots, h_k H h_k^{-1}\rangle$, $(hh_0h^{-1})f(hh_0h^{-1})^{-1}$ commutes with both of f and g and thus $[g, f] = [g, [f, hh_0h^{-1}]]$ holds.

Since ν is a conjugation-invariant pseudonorm,

$$\begin{aligned}\nu([g, f]) &\leq \nu(g[f, hh_0h^{-1}]g^{-1}) + \nu([f, hh_0h^{-1}]^{-1}) = 2\nu([f, hh_0h^{-1}])\\ &\leq 2(\nu(f(hh_0h^{-1})f^{-1}) + \nu((hh_0h^{-1})^{-1}))\\ &= 4\nu(hh_0h^{-1}) = 4\nu(h_0).\end{aligned}$$

By taking the infimum, $\nu([g, f]) \leq 4E_{H,\nu}(K)$. □

Lemma 2.7 [Entov and Polterovich 2006],[Kimura 2016]. *Let G be a group, H a subgroup of G and C a positive real number. Assume that a map $\phi : G \to \mathbb{R}$ satisfies $|\phi(f) + \phi(g) - \phi(fg)| \leq C$ for any elements f, g of G with $\nu_H(f) = 1$. Then ϕ is an H-quasimorphism. Moreover, $D(\phi) \leq 2C$.*

Proof of Proposition 2.5. Let $\hat{\phi}$ be an element of $L(G, \nu)$ and f, g elements of G with $\nu_H(f) = 1$. Since H is a subgroup, $\nu_H(f^i) = 1$ for any nonzero integer i. Since ν is a conjugation-invariant pseudonorm, by Lemma 2.6,

$$\begin{aligned}&|\phi(g) + \phi(f) - \phi(fg)|\\ &\quad= |\hat{\phi}([g^1]) + \hat{\phi}([f^1]) - \hat{\phi}([(fg)^1])|\\ &\quad= |\hat{\phi}([g^1] + [f^1] + (-1)[(fg)^1])|\\ &\quad\leq l(\hat{\phi}) \cdot \lim_m m^{-1} \cdot \nu(g^m f^m (g^{-1} f^{-1})^m)\\ &\quad= l(\hat{\phi}) \cdot \lim_m m^{-1} \cdot \nu((g^{m-1}[g, f^m]g^{-m+1})(g^{m-2}[g, f^{m-1}]g^{-m+2}) \cdots (g^0[g, f]g^0))\\ &\quad\leq l(\hat{\phi}) \cdot \liminf_m m^{-1} \cdot \sum_{i=1}^{m-1} \nu([g, f^i])\\ &\quad\leq l(\hat{\phi}) \cdot \liminf_m m^{-1} \cdot (m-1) \cdot 4E_{H,\nu}(H)\\ &\quad= 4l(\hat{\phi}) \cdot E_{H,\nu}(H).\end{aligned}$$

Thus, by Lemma 2.7, ϕ is an H-quasimorphism and $D(\phi) \le 8l(\hat{\phi}) \cdot E_{H,\nu}(H)$. Since $\hat{\phi}$ is a homomorphism, $\phi : G \to \mathbb{R}$ is a homogeneous H-quasimorphism. □

Proof of Theorem 1.12. Note that $\|[g^1]\|_\nu = \mathrm{s}\nu(g)$ for any element g of G. Then (1) follows from Proposition 2.4 and 2.5. To prove (2), it is sufficient to prove it for an element g of $[G, G]$ with $\mathrm{s}\nu(g) > 0$. Then, by Proposition 2.4 and $\|[g^1]\|_\nu = \mathrm{s}\nu(g)$, there exists an element $\hat{\phi}$ of $L(G, \nu)$ satisfying $\phi(g) = \hat{\phi}([g^1]) \neq 0$. Since $g \in [G, G]$, $D(\phi) > 0$. Thus Proposition 2.5 implies $8l(\hat{\phi})^{-1} \le D(\phi)^{-1} \cdot E_{H,\nu}(H)$. Therefore Proposition 2.4 implies

$$\mathrm{s}\nu(g) \le 8 \sup_{\phi} \frac{\phi(g) \cdot E_{H,\nu}(H)}{D(\phi)}. \qquad \square$$

3. Proof of being a normed vector space

Definition 3.1. Let H be a subgroup of a group G and ν a conjugation-invariant pseudonorm on G. For elements $g_1, \dots, g_k$ of G, we define the far away displacement energy $E_{H,\nu}[g_1, \dots, g_k]$ of $(g_1, \dots, g_k)$ by

$$E_{H,\nu}[g_1, \dots, g_k] = \inf E_{H,\nu}(\langle h_1 H h_1^{-1}, \dots, h_l H h_l^{-1} \rangle),$$

where inf is taken over $h_1, \dots, h_l$ such that $g_1, \dots, g_k \in \langle h_1 H h_1^{-1}, \dots, h_l H h_l^{-1} \rangle$. If (G, H) satisfies the property FM, $E_{H,\nu}[g_1, \dots, g_k] < \infty$ for any $g_1, \dots, g_k \in G$.

To prove Proposition 2.1, 2.2 and 2.3, we use the following lemma:

Lemma 3.2 [Calegari and Zhuang 2011]. *Let ν a conjugation-invariant pseudonorm on a group G. For any elements $g_1, \dots, g_k$ of G and integers $s_1, \dots, s_k, t_1, \dots, t_k$,*

$$\nu((g_1^{s_1} \cdots g_k^{s_k})^{-1}(g_1^{t_1} \cdots g_k^{t_k})) \le \sum_{i=1}^{k} |t_i - s_i| \cdot \nu(g_i).$$

Proof. By using a graphical calculus argument (for example, see 2.2.4 of [Calegari 2009]), there exist elements $h_1, \dots, h_k$ of $\langle g_1, \cdots, g_k \rangle$ such that

$$(g_1^{s_1} \cdots g_k^{s_k})^{-1}(g_1^{t_1} \cdots g_k^{t_k}) = h_k^{-1} g_k^{t_k - s_k} h_k \cdots h_1^{-1} g_1^{t_1 - s_1} h_1.$$

Since ν is a conjugation-invariant pseudonorm,

$$\nu((g_1^{s_1} \cdots g_k^{s_k})^{-1}(g_1^{t_1} \cdots g_k^{t_k})) \le \sum_{i=1}^{k} \nu(h_i^{-1} g_i^{t_i - s_i} h_i) \le \sum_{i=1}^{k} |t_i - s_i| \cdot \nu(g_i). \qquad \square$$

Proof of Proposition 2.1. Fix an element $\mathrm{g} = [g_1^{s_1} \cdots g_k^{s_k}]$ of A_ν. Define a function $F : \mathbb{Z}_{>0} \to \mathbb{R}$ by $F(m) = \nu(g_1^{[s_1 m]} \cdots g_k^{[s_k m]})$. By Fekete's Lemma, it is sufficient to prove that there exists a positive real number C such that $F(m+n) \leq F(m) + F(n) + C$ for any positive integers m, n. By Lemma 3.2,

$$\begin{aligned}F(m+n) &= \nu(g_1^{[s_1(m+n)]} \cdots g_k^{[s_k(m+n)]}) \\ &\leq \nu(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) \\ &\qquad + \nu((g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]})^{-1}(g_1^{[s_1(m+n)]} \cdots g_k^{[s_k(m+n)]})) \\ &\leq \nu(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) + \sum_{i=1}^{k} \nu(g_i).\end{aligned}$$

By using a graphical calculus argument, there exists an integer $l(k)$ which depends only on k and elements $f_1, \ldots, f_{l(k)}, f'_1, \ldots, f'_{l(k)}$ of $\langle g_1, \ldots, g_k \rangle$ such that

$$(g_1^{[s_1m]} \cdots g_k^{[s_km]})^{-1}(g_1^{[s_1n]} \cdots g_k^{[s_kn]})^{-1}(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) = [f_1, f'_1] \cdots [f_{l(k)}, f'_{l(k)}].$$

Fix an element H of $\mathsf{FM}(G)$. Then $E_{H,\nu}[g_1, \ldots, g_k] < \infty$. Thus, by Lemma 2.6,

$$\begin{aligned}&F(m+n) - F(m) - F(n) \\ &\leq \nu(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) + \sum_{i=1}^{k} \nu(g_i) - \nu(g_1^{[s_1m]} \cdots g_k^{[s_km]}) - \nu(g_1^{[s_1n]} \cdots g_k^{[s_kn]}) \\ &\leq \nu((g_1^{[s_1m]} \cdots g_k^{[s_km]})^{-1}(g_1^{[s_1n]} \cdots g_k^{[s_kn]})^{-1}(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]})) + \sum_{i=1}^{k} \nu(g_i) \\ &\leq \nu([f_1, f'_1] \cdots [f_{l(k)}, f'_{l(k)}]) + \sum_{i=1}^{k} \nu(g_i) \\ &\leq \sum_{j=1}^{l(k)} \nu([f_j, f'_j]) + \sum_{i=1}^{k} \nu(g_i) \\ &\leq 4l(k) E_{H,\nu}[g_1, \ldots, g_k] + \sum_{i=1}^{k} \nu(g_i).\end{aligned}$$

Thus we can apply Fekete's Lemma. □

To prove Proposition 2.2 and 2.3, we use the following lemmas.

Lemma 3.3. *Let G be a group satisfying the property* FM *and ν any conjugation-invariant pseudonorm on G. Then for any $\mathrm{g} \in A_G$ and any real numbers λ_1, λ_2,*

$$\|\bar{\mathrm{g}}^{(\lambda_1+\lambda_2)} \cdot \mathrm{g}^{(\lambda_1)} \cdot \mathrm{g}^{(\lambda_2)}\|_\nu = 0.$$

Proof. Assume that g is represented by $g_1^{s_1} g_2^{s_2} \cdots g_k^{s_k} \in A_G$. For any integer n, by using a graphical calculus argument, there exist elements $f_{n,1}, \ldots, f_{n,l(k)}$ and $f'_{n,1}, \ldots, f'_{n,l(k)}$ of $\langle g_1, \ldots, g_k \rangle$ such that

$$\begin{aligned}&(g_1^{[n\lambda_1s_1]+[n\lambda_2s_1]} g_2^{[n\lambda_1s_2]+[n\lambda_2s_2]} \cdots g_k^{[n\lambda_1s_k]+[n\lambda_2s_k]})^{-1} \\ &(g_1^{[n\lambda_1s_1]} g_2^{[n\lambda_1s_2]} \cdots g_k^{[n\lambda_1s_k]})(g_1^{[n\lambda_2s_1]} g_2^{[n\lambda_2s_2]} \cdots g_k^{[n\lambda_2s_k]}) = [f_{n,1}, f'_{n,1}] \cdots [f_{n,l(k)}, f'_{n,l(k)}].\end{aligned}$$

Fix $H \in \mathsf{FM}(G)$. Then $E_{H,\nu}[g_1, \dots, g_k] < \infty$. Thus, by Lemma 3.2 and Lemma 2.6,

$$\begin{aligned}
&\|\bar{\mathrm{g}}^{(\lambda_1+\lambda_2)} \cdot \mathrm{g}^{(\lambda_1)} \cdot \mathrm{g}^{(\lambda_2)}\|_\nu \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \nu\big((g_1^{[n\lambda_1 s_1 + n\lambda_2 s_1]} \cdots g_k^{[n\lambda_1 s_k + n\lambda_2 s_k]})^{-1} \\
&\qquad\qquad (g_1^{[n\lambda_1 s_1]} \cdots g_k^{[n\lambda_1 s_k]})(g_1^{[n\lambda_2 s_1]} \cdots g_k^{[n\lambda_2 s_k]})\big) \\
&\le \lim_{n\to\infty} \frac{1}{n} \cdot \Big(\nu((g_1^{[n\lambda_1 s_1]+[n\lambda_2 s_1]} \cdots g_k^{[n\lambda_1 s_k]+[n\lambda_2 s_k]})^{-1} \\
&\qquad\qquad (g_1^{[n\lambda_1 s_1]} \cdots g_k^{[n\lambda_1 s_k]})(g_1^{[n\lambda_2 s_1]} \cdots g_k^{[n\lambda_2 s_k]})) + \sum_{i=1}^{k} \nu(g_i)\Big) \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \Big(\nu([f_{n,1}, f'_{n,1}] \cdots [f_{n,l(k)}, f'_{n,l(k)}]) + \sum_{i=1}^{k} \nu(g_i)\Big) \\
&\le \lim_{n\to\infty} \frac{1}{n} \cdot \Big(\sum_{j=1}^{l(k)} \nu([f_{n,j}, f'_{n,j}]) + \sum_{i=1}^{k} \nu(g_i)\Big) \\
&\le \lim_{n\to\infty} \frac{1}{n} \cdot \Big(4l(k) E_{H,\nu}[g_1, \dots, g_k] + \sum_{i=1}^{k} \nu(g_i)\Big) \\
&= 0.
\end{aligned}$$

□

Lemma 3.4. *Let G be a group satisfying the property* FM *and ν a conjugation-invariant pseudonorm on G. For $g_1, \dots, g_k \in G$ and real numbers $\lambda, s_1, \dots, s_k$,*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[\lambda s_1 n]} \cdots g_k^{[\lambda s_k n]}) = |\lambda| \lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[s_1 n]} \cdots g_k^{[s_k n]}).$$

Proof. We first prove for the case when λ is a positive rational number, i.e., $\lambda = q/p$ where p, q are positive integers. By the existence of the limits (Proposition 2.1), since the limit of any subsequence equals that of the original sequence,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[\lambda s_1 n]} \cdots g_k^{[\lambda s_k n]}) &= \lim_{n\to\infty} \frac{1}{pn} \cdot \nu(g_1^{[q s_1 n]} \cdots g_k^{[q s_k n]}) \\
&= \lim_{n\to\infty} \frac{q}{pn} \cdot \nu(g_1^{[s_1 n]} \cdots g_k^{[s_k n]}) \\
&= \lambda \lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[s_1 n]} \cdots g_k^{[s_k n]}).
\end{aligned}$$

We prove for the case $\lambda = -1$.

Let g denote the element $g_1^{s_1} g_2^{s_2} \cdots g_k^{s_k}$ of A_G. By Lemma 3.3, $[\mathrm{g}^{(-1)} \cdot \mathrm{g}] = [\mathrm{g}^{(0)}] = [1]$. Recall that $1 \in (G \times \mathbb{R})^0$ is the trivial element of A_G. Thus $[\mathrm{g}^{(-1)}] = [\mathrm{g}^{(-1)} \cdot \mathrm{g} \cdot \bar{\mathrm{g}}] = [1 \cdot \bar{\mathrm{g}}] = [\bar{\mathrm{g}}]$. Therefore $\|(-1)\mathrm{g}\|_\nu = \|\bar{\mathrm{g}}\|_\nu = \|\mathrm{g}\|_\nu$ and we have completed the proof for the case when λ is a rational number.

Since Lemma 3.2 implies that the function $\mathbb{R} \to \mathbb{R}$, $\lambda \mapsto \lim_{n\to\infty}(1/n) \cdot \nu(g_1^{[\lambda s_1 n]} \cdots g_k^{[\lambda s_k n]})$ is continuous, we have completed the proof. □

Proof of Proposition 2.2. Assume that elements $\mathtt{f}_1, \mathtt{f}_2, \mathrm{g}_1, \mathrm{g}_2$ of A_G satisfy $[\mathtt{f}_1] = [\mathtt{f}_2]$ and $[\mathrm{g}_1] = [\mathrm{g}_2]$. Then

$$\begin{aligned}\|(\mathtt{f}_1 \cdot \mathrm{g}_1) \cdot \overline{(\mathtt{f}_2 \cdot \mathrm{g}_2)}\|_\nu &= \|\mathtt{f}_1 \cdot \mathrm{g}_1 \cdot \bar{\mathrm{g}}_2 \cdot \bar{\mathtt{f}}_2\|_\nu \\ &\leq \|\mathtt{f}_1 \cdot \mathrm{g}_1 \cdot \bar{\mathrm{g}}_2 \cdot \bar{\mathtt{f}}_1\|_\nu + \|\mathtt{f}_1 \cdot \bar{\mathtt{f}}_2\|_\nu \\ &= \|\mathrm{g}_1 \cdot \bar{\mathrm{g}}_2\|_\nu + \|\mathtt{f}_1 \cdot \bar{\mathtt{f}}_2\|_\nu = 0.\end{aligned}$$

Thus $[\mathtt{f}_1 \cdot \mathrm{g}_1] = [\mathtt{f}_2 \cdot \mathrm{g}_2]$.

Assume $\mathrm{g}_1, \mathrm{g}_2 \in A_G$ satisfy $[\mathrm{g}_1] = [\mathrm{g}_2]$. For any $\lambda \in \mathbb{R}$, Lemma 3.4 implies $\|\bar{\mathrm{g}}_1^{(\lambda)} \cdot \mathrm{g}_2{}^{(\lambda)}\|_\nu = \|(\bar{\mathrm{g}}_1 \cdot \mathrm{g}_2)^{(\lambda)}\|_\nu = |\lambda| \cdot \|(\bar{\mathrm{g}}_1 \cdot \mathrm{g}_2)\|_\nu = 0$. Thus $[\mathrm{g}_1^{(\lambda)}] = [\mathrm{g}_2^{(\lambda)}]$. □

Lemma 3.5. *Let G be a group satisfying the property* FM *and ν a conjugation-invariant pseudonorm on G. Then for any elements* f, g *of A_ν,*

$$\mathsf{f} + \mathsf{g} = \mathsf{g} + \mathsf{f}.$$

Proof. Assume f, g are represented by $[\mathtt{f}] = [f_1^{s_1} f_2^{s_2} \cdots f_k^{s_k}]$, $[\mathrm{g}] = [g_1^{t_1} g_2^{t_2} \cdots g_l^{t_l}]$, respectively. Fix an element H of $\mathsf{FM}(G)$. Then $E_{H,\nu}[g_1, \ldots, g_l] < \infty$. Since $g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]} \in \langle g_1, \ldots, g_l \rangle$ for any n, Lemma 2.6 implies

$$\begin{aligned}\|\mathtt{f} \cdot \mathrm{g} \cdot \overline{(\mathrm{g} \cdot \mathtt{f})}\|_\nu &= \|\mathtt{f} \cdot \mathrm{g} \cdot \bar{\mathtt{f}} \cdot \bar{\mathrm{g}}\|_\nu \\ &= \lim_{n \to \infty} \frac{1}{n} \cdot \nu\big((f_1^{[s_1 n]} f_2^{[s_2 n]} \cdots f_k^{[s_k n]})(g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]}) \\ &\qquad (f_1^{[s_1 n]} f_2^{[s_2 n]} \cdots f_k^{[s_k n]})^{-1} (g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]})^{-1}\big) \\ &= \lim_{n \to \infty} \frac{1}{n} \cdot \nu\big([f_1^{[s_1 n]} f_2^{[s_2 n]} \cdots f_k^{[s_k n]}, g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]}]\big) \\ &= \lim_{n \to \infty} \frac{1}{n} \cdot 4E_{H,\nu}[g_1, \ldots, g_l] = 0.\end{aligned}$$

Thus $\mathsf{f} + \mathsf{g} = [\mathtt{f} \cdot \mathrm{g}] = [\mathrm{g} \cdot \mathtt{f}] = \mathsf{g} + \mathsf{f}$. □

Proof of Proposition 2.3. By Lemma 3.3, 3.4 and 3.5, for any elements f, g of A_ν and real numbers λ_1, λ_2,

$$(\lambda_1 + \lambda_2)\mathsf{g} = \lambda_1 \mathsf{g} + \lambda_2 \mathsf{g}, \quad \|\lambda_1 \mathsf{g}\|_\nu = |\lambda_1| \cdot \|\mathsf{g}\|_\nu, \quad \text{and} \quad \mathsf{f} + \mathsf{g} = \mathsf{g} + \mathsf{f}.$$

We can confirm the other axioms of a normed vector space easily. Thus we complete the proof of Proposition 2.3. □

4. Proof that examples satisfy the property FM

In the present section, we prove that $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ satisfies the property FM. We can prove other parts of Proposition 1.11 similarly.

We use the following notations. For a diffeomorphism g on a manifold M, let $\mathrm{Supp}(g)$ denote the support of g. For a point p of $\mathbb{R}^{2n}$ and a positive real number R, let $\mathbb{B}^{2n}(p, R)$ denote a subset $\{x \in \mathbb{R}^{2n}; \|x - p\| < R\}$ of $\mathbb{R}^{2n}$.

Proof. For simplicity, let $\mathcal{B}$ denote the subgroup $\mathrm{Ham}(\mathbb{B}^{2n})$ and p_0 denote the point $(3, 0, \dots, 0)$ of $\mathbb{R}^{2n}$.

Let f_0 be a Hamiltonian diffeomorphism on $\mathbb{R}^{2n}$ such that $f_0(\mathbb{B}^{2n}) = \mathbb{B}^{2n}(p_0, 1)$. Fix Hamiltonian diffeomorphisms $g_1, \dots, g_k$ on $\mathbb{R}^{2n}$. Then there exists a positive real number R such that $\mathrm{Supp}(g_1) \cup \cdots \cup \mathrm{Supp}(g_k) \subset \mathbb{B}^{2n}(0, R)$. Since $f_0(\mathbb{B}^{2n}) = \mathbb{B}^{2n}(p_0, 1)$ and $\mathbb{B}^{2n}(p_0, 1) \cap \mathbb{B}^{2n} = \varnothing$, we can take a Hamiltonian diffeomorphism f such that $f(\mathbb{B}^{2n}) = \mathbb{B}^{2n}$ and $ff_0(\mathbb{B}^{2n}) \cap \mathbb{B}^{2n}(0, R) = \varnothing$. Since $(ff_0f^{-1})\mathcal{B}(ff_0f^{-1})^{-1} = \mathrm{Ham}(ff_0f^{-1}(\mathbb{B}^{2n})) = \mathrm{Ham}(ff_0(\mathbb{B}^{2n}))$ and

$$g_1\mathcal{B}g_1^{-1} \cup \cdots \cup g_k\mathcal{B}g_k^{-1} = \mathrm{Ham}(g_1(\mathbb{B}^{2n}) \cup \cdots \cup g_k(\mathbb{B}^{2n})) \subset \mathrm{Ham}(\mathbb{B}^{2n}(0, R)),$$

$ff_0(\mathbb{B}^{2n}) \cap \mathbb{B}^{2n}(0, R) = \varnothing$ implies that $(ff_0f^{-1})\mathcal{B}(ff_0f^{-1})^{-1}$ commutes with $g_1\mathcal{B}g_1^{-1} \cup \cdots \cup g_k\mathcal{B}g_k^{-1}$. Thus $f_0 \in \mathrm{D}^f_{\mathcal{B}}(\mathcal{B})$.

Note that Banyaga's [1978] fragmentation lemma states that for any Hamiltonian diffeomorphism g, there exist Hamiltonian diffeomorphisms $f_1, \dots, f_k$ such that $g \in \langle f_1\mathcal{B}f_1^{-1}, \dots, f_k\mathcal{B}f_k^{-1}\rangle$. Thus $\mathrm{Ham}(\mathbb{R}^{2n})$ is c-generated by $\mathcal{B}$ and the proof is complete. □

5. Are stably nondisplaceable subsets heavy? Bavard's duality in Hofer's geometry

We have considered subgroups which are displaceable far away. We now pose a problem on displaceable subgroups and give its application to symplectic geometry.

On notions related to symplectic geometry, we follow [Entov 2014].

Definition 5.1. Let G be a group, H a subgroup of G and $\mu : G \to \mathbb{R}$ an H-quasimorphism on G. If $\mu(g^n) = n\mu(g)$ for any element g of G and any nonnegative integer n, μ is called *semihomogeneous*.

Let (M, ω) be a $2m$-dimensional closed symplectic manifold. A subset X of (M, ω) is called *displaceable* if $\overline{X} \cap \phi_F^1(X) = \varnothing$ for some Hamiltonian function $F : S^1 \times M \to \mathbb{R}$ where ϕ_F is the Hamiltonian diffeomorphism generated by F and $\overline{X}$ is the topological closure of X. Otherwise, X is *nondisplaceable*. Let $\mathsf{DO}(M)$ denote the set of displaceable open subsets of (M, ω). A subset X of a symplectic manifold M is *stably displaceable* if $X \times S^1$ is displaceable in $M \times T^*S^1$. Otherwise, X is *stably nondisplaceable*.

Entov and Polterovich [2006] defined for an idempotent a of the quantum homology $QH_*(M, \omega)$, the asymptotic spectral invariant $\mu_a : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ on the universal covering $\widetilde{\mathrm{Ham}}(M)$ of the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms in terms of Oh–Schwarz spectral invariants and proved that μ_a is a semihomogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$. Here $\widetilde{\mathrm{Ham}}_U(M)$ is the set of elements of $\widetilde{\mathrm{Ham}}(M)$ which are generated by Hamiltonian functions with support in $S^1 \times U$.

A Hamiltonian function $F : S^1 \times M \to \mathbb{R}$ is *normalized* if $\int_M F_t \omega^m = 0$ for any $t \in S^1$.

Definition 5.2 [Entov and Polterovich 2009]. Let (M, ω) be a closed symplectic manifold and a an idempotent of $QH_*(M, \omega)$. A compact subset X of (M, ω) is *a-heavy* if for any normalized Hamiltonian function $F : S^1 \times M \to \mathbb{R}$,

$$-\mu_a(\phi_F) \geq \mathrm{vol}(M) \cdot \inf_{S^1 \times X} F,$$

where $\mathrm{vol}(M) = \int_M \omega^m$.

In particular, if X is a-heavy, $\mu_a(\phi_F) < 0$ for any normalized Hamiltonian function F with $F|_{S^1 \times X} > 0$.

Remark 5.3. The above definition of heaviness is different from the one of [Entov and Polterovich 2009] and [Entov 2014] (in their definition, they consider only autonomous Hamiltonian functions). However, as remarked in [Seyfaddini 2014], the above definition is known to be equivalent.

Entov and Polterovich [2009] also proved that heavy subsets are stably nondisplaceable. In the present section, we consider the converse problem, "are stably nondisplaceable subsets heavy?"

Definition 5.4. Let G be a group, H a subgroup of G and K a subset of G. We define the set $\mathrm{D}_H(K)$ of maps *displacing* K by

$$\mathrm{D}_H(K) = \{h_0 \in G;\ h_0 K (h_0)^{-1} \text{ commutes with } H\}.$$

Definition 5.5. Let G be a group and H a subgroup of G. The pair (G, H) satisfies the property FD if G and H satisfy the following conditions:

(1) G is c-generated by H,

(2) $\mathrm{D}_H(H) \neq \varnothing$.

A group G satisfies the property FD if (G, H) satisfies the property FD for some subgroup H.

For a group G which satisfies the property FD, we define the set $\mathsf{FD}(G)$ by

$$\mathsf{FD}(G) = \{H \leq G;\ (G, H) \text{ satisfies the property } \mathsf{FD}\}.$$

We pose the following problem.

Problem 5.6. Let G be a group satisfying the property FD, H an element of $\mathsf{FD}(G)$ and ν a conjugation-invariant pseudonorm on G. Prove that for any element g of G such that $s\nu(g) > 0$, there exists a function $\mu : G \to \mathbb{R}$ which is a semihomogeneous H-quasimorphism for any element H of $\mathsf{FD}(G)$ such that $\mu(g) > 0$.

Here, we give an application of Problem 5.6 to symplectic geometry.

Proposition 5.7. *Assume that the positive answer of Problem 5.6 holds.*

Let X be a stably nondisplaceable compact subset of a closed symplectic manifold (M, ω). For any normalized Hamiltonian function $F : S^1 \times M \to \mathbb{R}$ with $F|_{S^1\times X} > 0$, there exists a function $\mu_F : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ which is a semihomogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$ such that $\mu_F(\phi_F) < 0$.

Proposition 5.7 states that “stably nondisplaceable subsets are heavy” in a very rough sense if the positive answer of Problem 5.6 holds.

To prove Proposition 5.7, we use the following theorem, due to Polterovich:

Theorem 5.8 [Polterovich 1998, 2001]. *Let X be a stably nondisplaceable subset of a closed symplectic manifold (M, ω). For any normalized Hamiltonian function $F : S^1 \times M \to \mathbb{R}$ with $F|_{S^1\times X} \geq p$ for some positive number p, $\|\phi_F\|_H \geq p$. Here $\|\cdot\|_H : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ is the Hofer norm which is known to be a conjugation-invariant pseudonorm.*

Proof of Proposition 5.7. Since X is compact, there exists some positive number p with $F|_{S^1\times X} \geq p$. For any positive integer n, we define a Hamiltonian function $F^{(n)} : S^1 \times M \to \mathbb{R}$ by $F^{(n)}(t, x) = n \cdot F(nt, x)$. Note that $\phi_{F^{(n)}} = (\phi_F)^n$. Then, by $F^{(n)}|_{S^1\times X} \geq np$ and Theorem 5.8, $\|(\phi_F)^n\|_H \geq np$ for any positive integer n. Since $\widetilde{\mathrm{Ham}}_U(M) \in \mathsf{FD}(\widetilde{\mathrm{Ham}}(M))$ for any element U of $\mathsf{DO}(M)$, by the positive answer of Problem 5.6, there exists a function $\mu'_F : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ which is a semihomogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$ such that $\mu'_F(\phi_F) > 0$. Then setting $\mu_F = -\mu'_F$ completes the proof. □

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References

[Banyaga 1978] A. Banyaga, “Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique”, *Comment. Math. Helv.* **53**:2 (1978), 174–227. MR Zbl

[Bavard 1991] C. Bavard, “Longueur stable des commutateurs”, *Enseign. Math.* (2) **37**:1-2 (1991), 109–150. MR Zbl

[Brandenbursky and Kędra 2015] M. Brandenbursky and J. Kędra, “Concordance group and stable commutator length in braid groups”, *Algebr. Geom. Topol.* **15**:5 (2015), 2861–2886. MR Zbl

[Burago et al. 2008] D. Burago, S. Ivanov, and L. Polterovich, “Conjugation-invariant norms on groups of geometric origin”, pp. 221–250 in *Groups of diffeomorphisms*, edited by R. Penner et al., Adv. Stud. Pure Math. **52**, Math. Soc. Japan, Tokyo, 2008. MR Zbl

[Calegari 2009] D. Calegari, *scl*, MSJ Memoirs **20**, Math. Soc. Japan, Tokyo, 2009. MR Zbl

[Calegari and Zhuang 2011] D. Calegari and D. Zhuang, “Stable W-length”, pp. 145–169 in *Topology and geometry in dimension three*, edited by W. Li et al., Contemp. Math. **560**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Calegari et al. 2014] D. Calegari, N. Monden, and M. Sato, “On stable commutator length in hyperelliptic mapping class groups”, *Pacific J. Math.* **272**:2 (2014), 323–351. MR Zbl

[Endo and Kotschick 2001] H. Endo and D. Kotschick, “Bounded cohomology and non-uniform perfection of mapping class groups”, *Invent. Math.* **144**:1 (2001), 169–175. MR Zbl

[Entov 2014] M. Entov, “Quasi-morphisms and quasi-states in symplectic topology”, pp. 1147–1171 in *Proceedings of the International Congress of Mathematicians, II: Invited lectures* (Seoul, 2014), edited by S. Y. Jang et al., Kyung Moon Sa, Seoul, 2014.

[Entov and Polterovich 2006] M. Entov and L. Polterovich, “Quasi-states and symplectic intersections”, *Comment. Math. Helv.* **81**:1 (2006), 75–99. MR Zbl

[Entov and Polterovich 2009] M. Entov and L. Polterovich, “Rigid subsets of symplectic manifolds”, *Compos. Math.* **145**:3 (2009), 773–826. MR Zbl

[Kawasaki 2016] M. Kawasaki, “Relative quasimorphisms and stably unbounded norms on the group of symplectomorphisms of the Euclidean spaces”, *J. Symplectic Geom.* **14**:1 (2016), 297–304. MR Zbl

[Kimura 2016] M. Kimura, “Conjugation-invariant norms on the commutator subgroup of the infinite braid group”, preprint, 2016. To appear in *J. Topol. Anal.* arXiv

[Mimura 2010] M. Mimura, “On quasi-homomorphisms and commutators in the special linear group over a Euclidean ring”, *Int. Math. Res. Not.* **2010**:18 (2010), 3519–3529. MR Zbl

[Oh 2006] Y.-G. Oh, “Lectures on Floer theory and spectral invariants of Hamiltonian flows”, pp. 321–416 in *Morse theoretic methods in nonlinear analysis and in symplectic topology*, edited by P. Biran et al., NATO Sci. Ser. II Math. Phys. Chem. **217**, Springer, 2006. MR Zbl

[Polterovich 1998] L. Polterovich, “Hofer’s diameter and Lagrangian intersections”, *Internat. Math. Res. Notices* **1998**:4 (1998), 217–223. MR Zbl

[Polterovich 2001] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Birkhäuser, Basel, 2001. MR Zbl

[Schwarz 2000] M. Schwarz, “On the action spectrum for closed symplectically aspherical manifolds”, *Pacific J. Math.* **193**:2 (2000), 419–461. MR Zbl

[Seyfaddini 2014] S. Seyfaddini, “Unboundedness of the Lagrangian Hofer distance in the Euclidean ball”, *Electron. Res. Announc. Math. Sci.* **21** (2014), 1–7. MR Zbl

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PARABOLIC MINIMAL SURFACES IN $\mathbb{M}^2 \times \mathbb{R}$

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Let $\mathbb{M}^2$ be a complete noncompact orientable surface of nonnegative curvature. We prove some theorems involving parabolicity of minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$. First, using a characterization of δ-parabolicity we prove that under additional conditions on $\mathbb{M}$, an embedded minimal surface with bounded Gaussian curvature is proper. The second theorem states that under some conditions on $\mathbb{M}$, if Σ is a properly immersed minimal surface with finite topology and one end in $\mathbb{M} \times \mathbb{R}$, which is transverse to a slice $\mathbb{M} \times \{t\}$ except at a finite number of points, and such that $\Sigma \cap (\mathbb{M} \times \{t\})$ contains a finite number of components, then Σ is parabolic. In the last result, we assume some conditions on $\mathbb{M}$ and prove that if a minimal surface in $\mathbb{M} \times \mathbb{R}$ has height controlled by a logarithmic function, then it is parabolic and has a finite number of ends.

1. Introduction

Let $\mathbb{M}^2$ be a complete noncompact orientable surface with nonnegative curvature. Under these conditions $\mathbb{M} \times \mathbb{R}$ is complete and has nonnegative sectional curvature, in particular nonnegative Ricci curvature. Recently, using some of the results of [Schoen and Yau 1982], G. Liu classified complete noncompact 3-manifolds with nonnegative Ricci curvature.

Theorem [Liu 2013]. *Let N be a complete noncompact 3-manifold with nonnegative Ricci curvature. Then either N is diffeomorphic to $\mathbb{R}^3$ or its universal cover $\widetilde{N}$ is isometric to a Riemannian product $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is a complete surface with nonnegative sectional curvature.*

In particular it follows from the proof of this result that if N is not flat or does not have positive Ricci curvature then its universal cover splits as a product $\mathbb{M} \times \mathbb{R}$. So the spaces $\mathbb{M} \times \mathbb{R}$ are in fact general examples of a very important class of 3-manifolds.

We are interested in minimal surfaces in $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is as above. In particular we want information about the topology and the conformal structure. It is

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important to study under which hypotheses we can guarantee that a minimal surface is proper. Concerning the topology, we know that there is no compact minimal surface in these spaces. So, one can study the genus and the number of ends of such minimal surfaces. Concerning the conformal structure, one important property is *parabolicity*. Our results are inspired by analogous results in $\mathbb{R}^3$.

First we study the problem of properness. Bessa, Jorge and Oliveira-Filho studied this problem for manifolds with nonnegative Ricci curvature and obtained some partial results in $\mathbb{R}^3$.

Theorem [Bessa et al. 2001]. *Let N^3 be a complete Riemannian 3-manifold of bounded geometry and positive Ricci curvature. Let $f : \Sigma^2 \to N^3$ be a complete injective minimal immersion, where Σ is a complete oriented surface with bounded curvature.*

(1) *If N is compact, then Σ is compact.*

(2) *If N is not compact, then f is proper.*

A major breakthrough was the work of Colding and Minicozzi [2008], where it was proved that a complete minimal surface of finite topology embedded in $\mathbb{R}^3$ is proper. After this, Meeks and Rosenberg [2006] proved that if Σ is a complete embedded minimal surface in $\mathbb{R}^3$ which has positive injectivity radius, then Σ is proper. Finally, Meeks and Rosenberg [2008] proved that if $f : \Sigma \to \mathbb{R}^3$ is an injective minimal immersion, with Σ complete and of bounded curvature, then f is proper. We extend the last result to the case of a product $\mathbb{M} \times \mathbb{R}$:

Theorem A. *Let $\mathbb{M}$ be a complete simply connected orientable noncompact surface such that $0 \leq K_{\mathbb{M}} \leq \kappa$. Let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be an injective minimal immersion of a complete, connected Riemannian surface of bounded curvature. Then the map f is proper.*

Next we focus on surfaces with finite topology and one end. The results in [Colding and Minicozzi 2008; Meeks and Rosenberg 2005] imply that every complete, embedded minimal surface in $\mathbb{R}^3$ of finite genus and one end is properly embedded and intersects some plane transversely in a single component, and so, is parabolic. Meeks and Rosenberg [2008] gave an independent proof that the surface is parabolic without the additional assumption that it is embedded. Namely, they proved:

Theorem [Meeks and Rosenberg 2008]. *Let Σ be a surface of finite topology and one end, and let $f : \Sigma \to \mathbb{R}^3$ be a proper minimal immersion. Suppose that f is transverse to a plane P except at a finite number of points, and $f^{-1}(P)$ contains a finite number of components. Then Σ is parabolic.*

The half-space theorem of Hoffman and Meeks [1990] states that a properly immersed minimal surface in $\mathbb{R}^3$ which is above a plane is a parallel plane. Thus the condition that a minimal surface be transverse to a plane is natural. Rosenberg proved the following half-space theorem for product spaces:

Theorem [Rosenberg 2002]. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $K_{\mathbb{M}} \geq 0$.

(2) *There is a point $p \in M$ such that the geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded.*

If Σ is a properly immersed minimal surface in a half-space $\mathbb{M} \times [t_0, +\infty)$, then Σ is a slice $\mathbb{M} \times \{s\}$ for some $s > t_0$.

Based on these results we prove the following:

Theorem B. *Suppose $\mathbb{M}$ satisfies the conditions of the previous theorem. Let Σ be a surface of finite topology and one end and let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be a proper minimal immersion. Suppose that f is transverse to a slice $\mathbb{M} \times \{t_0\}$ except at a finite number of points and that $f^{-1}(\mathbb{M} \times \{t_0\})$ contains a finite number of components. Then Σ is parabolic.*

Next we focus on surfaces with more than one end. A major breakthrough was the proof of the generalized Nitsche conjecture in $\mathbb{R}^3$:

Theorem [Collin 1997]. *Let Σ be a properly embedded minimal surface in $\mathbb{R}^3$ with at least two ends. Then an annular end of Σ is asymptotic to a plane or to the end of a catenoid.*

Let Σ be as in the last theorem. The set $\mathcal{E}_\Sigma$ of all the ends of Σ has a natural topology that makes it a compact Hausdorff space. The limit points in $\mathcal{E}_\Sigma$ are called the *limit ends* of Σ, and an end which is not a limit end is called a *simple end*. To Σ is associated a unique plane P passing through the origin in $\mathbb{R}^3$ called the limit tangent plane at infinity of Σ [Callahan et al. 1990]. The ends of Σ are linearly ordered by their relative heights over P and this linear ordering, up to reversing it, depends only on the proper ambient isotopy class of Σ in $\mathbb{R}^3$ [Frohman and Meeks 1997]. Since $\mathcal{E}_\Sigma$ is compact and the ordering is linear, there exists a unique *top end* which is the highest end and a unique *bottom end* which is lowest in the associated ordering. The ends of Σ that are neither top nor bottom ends are called *middle ends*. In the proof of the ordering theorem, one shows that every middle end of Σ is contained between two catenoids in the following sense: if E is an end of Σ there are $c_1 > 0$ and $r_1 > 0$ such that $E \subset \{(x_1, x_2, x_3) : |x_3| \leq c_1 \log r,\ r^2 = x_1^2 + x_2^2,\ r \geq r_1\}$.

Collin, Kusner, Meeks and Rosenberg [Collin et al. 2004] proved that if Σ is a properly immersed minimal surface with compact boundary in $\mathbb{R}^3$ which is contained between two catenoids, then Σ has quadratic area growth. Furthermore, Σ has a finite number of ends. As a consequence the middle ends of a properly embedded minimal surface in $\mathbb{R}^3$ are never *limit ends*. We explain what it means for a properly immersed minimal surface of $\mathbb{M}\times\mathbb{R}$ to be contained between two catenoids and generalize the result above:

Theorem C. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $0 \le K_{\mathbb{M}} \le \kappa$.

(2) *$\mathbb{M}$ has a pole p.*

(3) *The geodesic curvatures of all geodesic circles with center p and radius $r \ge 1$ are uniformly bounded.*

Let Σ be a properly immersed minimal surface inside the region of $\mathbb{M}\times\mathbb{R}$ defined by $|h| \le c_2 \log r$ for some constant $c_2 > 0$ and $r \ge 1$. Then Σ is parabolic. Moreover, if Σ has compact boundary, then Σ has quadratic area growth and a finite number of ends.

The paper is organized as follows. In Section 2 we present some results about the geometry of the space $\mathbb{M}\times\mathbb{R}$ and its minimal surfaces. In Sections 3 and 4 we give some well-known definitions and enunciate some results involving parabolicity and laminations. In Section 5 we prove Theorem A. In Section 6 we prove Theorems B and C.

2. The geometry of $\mathbb{M}^2 \times \mathbb{R}$

Some of the results of this section are well known, but we prove them here for completeness.

Lemma 1. *Let $\mathbb{M}$ be a complete noncompact orientable surface with nonnegative sectional curvature. Then $\mathbb{M}$ is homeomorphic to $\mathbb{R}^2$ or isometric to a flat cylinder $\mathbb{S}^1\times\mathbb{R}$.*

Proof. Since $K_{\mathbb{M}}^- \equiv 0$, by Huber's theorem $\mathbb{M}$ has finite topology and

$$0 \le \int_{\mathbb{M}} K_{\mathbb{M}}\, d\mu \le 2\pi(2-2g-n),$$

where g is the genus of M and n its number of ends; see [White 1987]. Since $\mathbb{M}$ is noncompact and $n \ge 1$, we have

$$1 \le n+2g \le 2.$$

But $n+2g$ is an integer; thus the only possibility is $g=0$, $n=1,2$.

If $n=1$, $\mathbb{M}$ is homeomorphic to $\mathbb{R}^2$. If $n=2$, $\mathbb{M}$ has the topology of $\mathbb{S}^1\times\mathbb{R}$ and

$$\int_{\mathbb{M}} K_{\mathbb{M}}\, d\mu = 0,$$

thus $K_{\mathbb{M}} \equiv 0$ and $\mathbb{M}$ is isometric to $\mathbb{S}^1\times\mathbb{R}$ endowed with a flat metric. □

Lemma 2. *Let* $\mathbb{M}$ *be a complete noncompact surface with sectional curvature satisfying* $0 \le K_{\mathbb{M}} \le \kappa$. *Then* $\mathbb{M}$ *has positive injectivity radius*; *in particular the same holds for* $\mathbb{M} \times \mathbb{R}$.

Proof. By the previous lemma either $\mathbb{M}$ is a flat cylinder, which has positive injectivity radius, or $\mathbb{M}$ is homeomorphic to $\mathbb{R}^2$. Suppose in the last case that $\mathrm{inj}_{\mathbb{M}} = 0$. Since $K_{\mathbb{M}} \le \kappa$, the exponential map $\exp_q : B_{\pi/\sqrt{\kappa}}(0) \to \mathbb{M}$ has no critical points for each $q \in \mathbb{M}$. Then for each positive integer j sufficiently large there is a point p_j such that $\exp_{p_j}$ is not injective in the geodesic ball $B_{1/j}(p_j)$, which implies there are two geodesics $\gamma_j, \sigma_j : [0, l] \to \mathbb{M}$ beginning in p_j which meet at the same endpoint q_j in the boundary of $B_{1/j}(p_j)$ with angle equal to π (q_j realizes the distance from p_j to $\mathrm{Cut}(p_j)$; see [do Carmo 1988, Chapter 13, Proposition 2.12]). This gives us a geodesic loop α_j with one angular vertex at p_j which has exterior angle $\theta_j \le \pi$. Since $\mathbb{M}$ is simply connected, α_j bounds a disc D_j in $\mathbb{M}$. By the Gauss–Bonnet theorem

$$2\pi = \int_{D_j} K_{\mathbb{M}}\, d\mu + \theta_j \le \kappa |D_j| + \pi.$$

However, for j sufficiently large, $|D_j|$ is small and $\kappa |D_j| + \pi < 2\pi$, which is a contradiction. Therefore $\mathrm{inj}_{\mathbb{M}} > 0$. □

Lemma 3 [Espinar and Rosenberg 2009]. *Let* $\mathbb{M}$ *be a complete connected nonflat surface. Let* Σ *be a complete totally geodesic surface in* $\mathbb{M} \times \mathbb{R}$. *Then* Σ *is of the form* $\alpha \times \mathbb{R}$, *where* α *is a geodesic of* M, *or* $\Sigma = \mathbb{M} \times \{t\}$ *for some* $t \in \mathbb{R}$.

Proof. Let Π be the projection of $\mathbb{M} \times \mathbb{R}$ to $\mathbb{M}$. Let η be a unit normal to Σ and define $\nu = \langle \eta, \partial_t \rangle$. Since Σ is totally geodesic we have

$$K_\Sigma(p) = K_{\mathbb{M}}(\Pi(p))\nu(p) \quad \forall p \in \Sigma, \tag{1}$$

$$X\langle \eta, \partial_t \rangle = \langle \nabla_X \eta, \partial_t \rangle \equiv 0 \quad \forall X \in T\Sigma, \tag{2}$$

where (1) is just the Gauss equation. So ν is constant, and we can suppose $\nu \ge 0$. If $\nu = 0$, then Σ is of the form $\alpha \times \mathbb{R}$. If $\nu = 1$, then Σ is a slice.

Suppose $0 < \nu < 1$. We know that

$$\Delta_\Sigma \nu + (\mathrm{Ric}(\eta, \eta) + |A|^2)\nu = 0,$$

and by equation (2), $\Delta_\Sigma \nu = 0$. Thus $0 = \mathrm{Ric}(\eta, \eta) = K_{\mathbb{M}}(\Pi(p))(1 - \nu^2)$, which implies $K_{\mathbb{M}}(\Pi(p)) = 0$. It follows from equation (1) that Σ is flat. Then there is a $\delta > 0$ such that for any $p \in \Sigma$ a neighborhood of p in Σ is a graph (in exponential coordinates) over the disc $D_\delta \subset T_p\Sigma$ of radius δ, centered at the origin of $T_p\Sigma$. This graph, denoted by G_p, has bounded geometry. The number δ is independent of p, and the bound on the geometry of G_p is uniform as well.

We claim that $\Pi(\Sigma) = \mathbb{M}$. Suppose the contrary. Then there exists a bounded open set $\Omega \subset \Pi(\Sigma)$ and $q_0 \in \partial\Omega$ such that, for some point $p \in \Pi^{-1}(\Omega)$, a neighborhood of p in Σ is a vertical graph of a function f defined over Ω and this graph does not extend to a minimal graph over any neighborhood of q_0.

We can identify Ω with $\Omega \times \{0\}$. Let $\{q_n\} \subset \Omega$ be a sequence converging to q_0 and $p_n = (q_n, f(q_n))$. Let Σ_n denote the image of G_{p_n} under the vertical translation taking p_n to q_n. There is a subsequence of $\{q_n\}$ (which we also denote by $\{q_n\}$) such that the tangent planes $T_{q_n}(\Sigma_n)$ converge to some vertical plane $P \subset T_{q_0}(\mathbb{M} \times \mathbb{R})$. In fact, if this were not true, for q_n close enough to q_0, the graph of bounded geometry G_{p_n} would extend to a vertical graph beyond q_0. Hence f would extend beyond q_0, a contradiction. So $T_{p_n}\Sigma$ must become almost vertical at p_n for n sufficiently large, which means that $\eta(p_n)$ must become horizontal. But ν is a constant different from 0, a contradiction.

Then $\Pi(\Sigma) = \mathbb{M}$. Since $K_{\mathbb{M}} \circ \Pi \equiv 0$, it follows that $\mathbb{M}$ is a complete flat surface, which contradicts our assumption. □

Lemma 4 [Rosenberg 2002]. *Let Σ be a minimal surface of $\mathbb{M} \times \mathbb{R}$. Then the height function $h : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$, $h(q, t) = t$, is harmonic on Σ.*

Proof. Let E_1, E_2, η be an orthonormal frame in a neighborhood of a point of Σ, where η is normal to Σ. Since ∂_t is a Killing vector field on $\mathbb{M} \times \mathbb{R}$, we have

$$\operatorname{div} \partial_t = 0 = \langle \nabla_\eta \partial_t, \eta \rangle.$$

Write $\partial_t = \nabla h = X + \nabla_\Sigma h$, where X is normal to Σ. Then

$$\begin{aligned} 0 = \Delta h &= \sum_i [\langle \nabla_{E_i} \nabla_\Sigma h, E_i \rangle + \langle \nabla_{E_i} X, E_i \rangle] \\ &= \Delta_\Sigma h - \sum_i \langle X, \nabla_{E_i} E_i \rangle = \Delta_\Sigma h - \langle X, \vec{H} \rangle = \Delta_\Sigma h. \end{aligned}$$ □

Lemma 5 [Rosenberg 2002]. *Suppose that $\mathbb{M}$ has nonnegative sectional curvature and that there exists a point $p \in \mathbb{M}$ such that the geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded. Define $f : \mathbb{M} \setminus (\{p\} \cup \mathrm{Cut}(p)) \times \mathbb{R} \to \mathbb{R}$, $f(q, t) = \log(r(q))$, where r is the distance in $\mathbb{M}$ to the point p. Let Σ be a minimal surface of $\mathbb{M} \times \mathbb{R}$. Then*

$$\Delta_\Sigma f \leq \frac{c_1}{r} |\nabla_\Sigma h|^2$$

for some constant $c_1 > 0$ and $r \geq 1$.

Proof. Denote by ∇f, Δf and Hess f respectively the gradient, the Laplacian and the Hessian of f in $\mathbb{M} \times \mathbb{R}$. Since $\mathbb{M}$ has nonnegative curvature, by the Laplacian comparison theorem we have

$$\Delta_{\mathbb{M}} r \leq \frac{1}{r}.$$

But f does not depend on the height, so

$$\Delta f = \Delta_{\mathbb{M}} f = \frac{\Delta_{\mathbb{M}} r}{r} - \frac{|\nabla_{\mathbb{M}} r|^2}{r^2} \leq 0.$$

Let E_1, E_2, η be an orthonormal frame in a neighborhood of a point of Σ, where η is normal to Σ. Write $\nabla f = \nabla_\Sigma f + \langle \nabla f, \eta\rangle \eta$. Since Σ is minimal we have

$$\begin{aligned}
\Delta f &= \sum_{i=1}^{2} \langle \nabla_{E_i} \nabla f, E_i\rangle + \langle \nabla_\eta \nabla f, \eta\rangle \\
&= \sum_{i=1}^{2} \langle \nabla_{E_i} \nabla_\Sigma f, E_i\rangle + \sum_{i=1}^{2} \langle \nabla f, \eta\rangle \langle \nabla_{E_i} \eta, E_i\rangle + \langle \nabla_\eta \nabla f, \eta\rangle \\
&= \Delta_\Sigma f + \langle \nabla f, \eta\rangle H + \operatorname{Hess} f(\eta, \eta) \\
&= \Delta_\Sigma f + \operatorname{Hess} f(\eta, \eta).
\end{aligned}$$

Now, let V be tangent to $\mathbb{M}$, $\xi = \partial/\partial t$ and Π be the projection of $\mathbb{M} \times \mathbb{R}$ to $\mathbb{M}$. Again, since f does not depend on the height, we have

$$\begin{aligned}
\operatorname{Hess} f(\xi, \xi) &= 0, \\
\operatorname{Hess} f(V, V) &= \operatorname{Hess}_{\mathbb{M}} f(V, V).
\end{aligned}$$

Then

$$\operatorname{Hess} f(\eta, \eta) = \operatorname{Hess} f(\Pi(\eta), \Pi(\eta)) = \operatorname{Hess}_{\mathbb{M}} f(\Pi(\eta), \Pi(\eta)).$$

But $\Delta f \leq 0$, so

$$\Delta_\Sigma f \leq -\operatorname{Hess} f_{\mathbb{M}}(\Pi(\eta), \Pi(\eta)) \leq |\operatorname{Hess}_{\mathbb{M}} f| |\Pi(\eta)|^2. \tag{3}$$

A simple calculation shows that

$$|\Pi(\eta)| = |\nabla_\Sigma h|. \tag{4}$$

Let $q \in \mathbb{M}$, $r(q) = d(q, p)$ and v be a unit tangent vector to $\mathbb{M}$ at q. Thus

$$\operatorname{Hess}_{\mathbb{M}} f(v, v) = \left\langle \nabla_v \left(\frac{\nabla_{\mathbb{M}} r}{r} \right), v \right\rangle = \frac{1}{r} \langle \nabla_v \nabla_{\mathbb{M}} r, v\rangle + v\left(\frac{1}{r}\right) \langle \nabla_{\mathbb{M}} r, v\rangle.$$

When $v = \nabla_{\mathbb{M}} r$,

$$\operatorname{Hess}_{\mathbb{M}} f(v, v) = -\frac{1}{r^2} |\nabla_{\mathbb{M}} r|^2.$$

When $v = T$, the unit tangent vector to the geodesic circle of radius r through the point q,

$$\operatorname{Hess}_{\mathbb{M}} f(v, v) = \frac{1}{r} \langle \nabla_T \nabla_{\mathbb{M}} r, T\rangle = \frac{1}{r} k_g(q),$$

where $k_g(q)$ is the geodesic curvature of the geodesic circle of radius r centered at the point q. By the hypothesis about the geodesic circles of $\mathbb{M}$,

$$|\mathrm{Hess}_{\mathbb{M}} f|^2 = \frac{1}{r^4} + \frac{1}{r^2}k_g^2 \le \frac{C}{r^2}.$$

Using equations (3) and (4), the lemma follows. □

3. Laminations

Definition 6. Let Σ be a complete, embedded surface in a 3-manifold N. A point $p \in N$ is a limit point of Σ if there exists a sequence $\{p_n\} \subset \Sigma$ which diverges to infinity in Σ with respect to the intrinsic Riemannian topology on Σ, but converges in N to p as $n \to \infty$. Let $\mathcal{L}(\Sigma)$ denote the set of all limit points of Σ in N; we call this set the limit set of Σ. In particular, $\mathcal{L}(\Sigma)$ is a closed subset of N and $\overline{\Sigma} \setminus \Sigma \subset \mathcal{L}(\Sigma)$, where $\overline{\Sigma}$ denotes the closure of Σ.

Definition 7. A codimension-1 lamination of a Riemannian n-manifold N is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying the following conditions:

(1) $\mathcal{L}$ is a closed subset of N.

(2) $\mathcal{A} = \{\varphi_\beta : \mathbb{D} \times (0, 1) \to U_\beta\}_\beta$ is an atlas of coordinate charts of N, where $\mathbb{D}$ is the open unit ball in $\mathbb{R}^{n-1}$ and U_β is an open subset of N.

(3) For each β, there is a closed subset C_β of $(0, 1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$.

If all the leaves are minimal hypersurfaces, $(\mathcal{L}, \mathcal{A})$ is called a minimal lamination.

4. Parabolic manifolds

Definition 8. Given a point p on a Riemannian manifold N with boundary, one can define the hitting, or harmonic, measure μ_p of an interval $I \subset \partial N$ as the probability that a Brownian path beginning at p reaches the boundary for the first time at a point in I.

Proposition 9. *Let N be a Riemannian manifold with nonempty boundary. The following are equivalent*:

(1) *Any bounded harmonic function on N is determined by its boundary values.*

(2) *For some $p \in \mathrm{Int}\, N$, the measure μ_p is full on ∂N, i.e, $\int_{\partial N} \mu_p = 1$.*

(3) *If $h : N \to \mathbb{R}$ is a bounded harmonic function, then $h(p) = \int_{\partial N} h(x)\mu_p$.*

If N satisfies any of the conditions above, then it is called parabolic.

An important property is that a proper subdomain of a parabolic manifold is parabolic; hence removing the interior of a compact domain does not alter parabolicity. Moreover, if there exists a proper nonnegative superharmonic function on N, then N is parabolic. For equivalent definitions and properties of parabolic manifolds see [Grigor'yan 1999].

Definition 10. Let N be a Riemannian manifold with nonempty boundary. For $R > 0$, let $N(R) = \{p \in N : d(p, \partial N) < R\}$. We say that N is δ-parabolic if for every $\delta > 0$, $\widetilde{N} = N \setminus N(\delta)$ is parabolic.

The following theorem gives a sufficient condition for a surface to be δ-parabolic.

Theorem 11 [Meeks and Rosenberg 2008]. *Let N be a complete surface with nonempty boundary and curvature function $K : N \to [0, \infty]$. Suppose that for each $R > 0$, the restricted function $K|_{N(R)}$ is bounded. Then N is δ-parabolic.*

5. Proper minimal immersions

Proposition 12. *Let N be a 3-manifold with nonnegative Ricci curvature and sectional curvature bounded above by $\kappa > 0$. Suppose Σ is a complete, orientable minimal surface with boundary in N, with a Jacobi function u. If $u \geq \epsilon$ for some $\epsilon > 0$, then Σ is δ-parabolic.*

Proof. First note that a Riemannian surface W is δ-parabolic if and only if for all $\delta' > 0$, the surface $W \setminus W(\delta')$ is also δ-parabolic. Thus, without loss of generality, we may assume that Σ has the form $W \setminus W(\delta')$ for some $\delta' > 0$, where W is a stable minimal surface with a positive Jacobi function $u \geq \epsilon$, which exists by [Fischer-Colbrie and Schoen 1980]. By curvature estimates for stable, orientable minimal surfaces [Schoen 1983; Rosenberg et al. 2010], we may assume that Σ has bounded Gaussian curvature. Consider the new Riemannian manifold $\widetilde{\Sigma}$, which is Σ with the metric $\tilde{g} = u\langle \cdot , \cdot \rangle$ on Σ, where $\langle \cdot , \cdot \rangle$ is the Riemannian metric on Σ. Since $u \geq \epsilon$ the metric $\tilde{g}$ is complete. Moreover, $\Delta_{\tilde{g}} f = u^{-1} \Delta f$ for any function on Σ which has second derivative. Thus Σ is δ-parabolic if and only if $\widetilde{\Sigma}$ is δ-parabolic. Let E_1, E_2, η be an orthonormal frame in a neighborhood of a point of Σ, where η is normal to Σ. By the Gauss equation,

$$\mathrm{Ric}(\eta, \eta) + |A_\Sigma|^2 = \mathrm{Ric}(E_1, E_1) + \mathrm{Ric}(E_2, E_2) - 2K_\Sigma.$$

Then, as u is a Jacobi function,

$$\Delta_\Sigma u + \big(\mathrm{Ric}(E_1, E_1) + \mathrm{Ric}(E_2, E_2) - 2K_\Sigma\big)u = 0.$$

So,

$$K_{\widetilde{\Sigma}} = \frac{K_\Sigma - \frac{1}{2}\Delta_\Sigma \log u}{u} = \frac{1}{2}\,\frac{\mathrm{Ric}(E_1, E_1) + \mathrm{Ric}(E_2, E_2)}{u} + \frac{1}{2}\,\frac{|\nabla_\Sigma u|^2}{u^3},$$

which implies

$$0 \leq K_{\widetilde{\Sigma}} \leq 2\frac{\kappa}{\epsilon} + \frac{1}{2\epsilon}\frac{|\nabla_\Sigma u|^2}{u^2}.$$

Choose $\delta > 0$ and let $\widetilde{\Omega} = \widetilde{\Sigma} \setminus \widetilde{\Sigma}(\delta)$. Let Ω be the corresponding submanifold on Σ. By the Harnack inequality (see [Moser 1961]), $|\nabla_\Sigma u|/u$ is bounded, and so one has that $K_{\widetilde{\Sigma}}$ is nonnegative and bounded on Ω. It follows from Theorem 11 in Section 4 that $\widetilde{\Omega}$ is parabolic, and hence Ω is parabolic. Since δ was chosen arbitrarily, we conclude that Σ is δ-parabolic. □

Theorem A. *Let $\mathbb{M}$ be a complete simply connected orientable noncompact surface such that $0 \leq K_{\mathbb{M}} \leq \kappa$. Let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be an injective minimal immersion of a complete, connected Riemannian surface of bounded curvature. Then the map f is proper.*

Proof. Since the curvature of $f(\Sigma)$ is bounded, there exists an $\epsilon > 0$ such that for any point $p \in \mathbb{M} \times \mathbb{R}$, every component of $f^{-1}(B_\epsilon(p))$, when pushed forward by f, is a compact disc and a graph over a domain in the tangent plane of any point on it, with a uniform bound on the area. It follows that if p is a limit point of $f(\Sigma)$ coming from distinct components of $f^{-1}(B_\epsilon(p))$, then there is a minimal disc $D(p)$ passing through p that is a graph over its tangent plane at p, and $D(p)$ is a limit of components in $f^{-1}(B_\epsilon(p))$. Let $D'(p)$ be any other such limit disc. Since f is an embedding the unique possibility is that the discs are tangent at p; then the maximum principle implies that the two discs agree near p. This implies that the closure $\mathcal{L}(f(\Sigma))$ of $f(\Sigma)$ has the structure of a minimal lamination.

The immersion f is proper if and only if $\mathcal{L}(f(\Sigma))$ has no limit leaves. Suppose $\mathcal{L}(f(\Sigma))$ has a limit leaf L. Denote by $\widetilde{L}$ the universal cover of L. It was proved in [Meeks et al. 2008] that $\widetilde{L}$ is stable. So, by [Fischer-Colbrie and Schoen 1980] $\widetilde{L}$ is totally geodesic; hence L is totally geodesic. Suppose $\mathbb{M}$ is not flat (the case where $\mathbb{M}$ is flat was proved in [Meeks and Rosenberg 2008]). By Lemma 3 a totally geodesic surface in $\mathbb{M} \times \mathbb{R}$ is a slice $\mathbb{M} \times \{t\}$ or is of the form $\alpha \times \mathbb{R}$, where α is a geodesic of M.

Assume L is a slice. Since Σ is not proper, it is not equal to a slice. We can suppose $L = \mathbb{M} \times \{0\}$ and H^+ is a smallest half-space containing $f(\Sigma)$. Since Σ has bounded curvature, there is an $\epsilon > 0$ such that for every component C of $f(\Sigma)$ in the slab between L and $L_\epsilon = \{t = \epsilon\}$, the Jacobi function $u = \langle \nu, \partial_t \rangle$ satisfies $u \geq \lambda > 0$, where ν is the unit normal to C. Choose $0 < \delta < \epsilon$ such that $C(\delta) = \{p \in C : h \leq \delta\}$ is not empty, where h is the height function. By Proposition 12, $C(\delta)$ is parabolic. But $h|_{C(\delta)}$ is a bounded harmonic function with the same boundary values as the constant function δ. Hence $h|_{C(\delta)}$ is constant, which is a contradiction because $C(\delta)$ is not contained in a slice.

Now, suppose $L = \alpha \times \mathbb{R}$. Consider a one-sided closed ϵ-normal interval bundle $N_\epsilon(L)$ that submerses to $\mathbb{M} \times \mathbb{R}$, with the induced metric. Observe that $N_\epsilon(L)$ is diffeomorphic to $(\alpha \times \mathbb{R}) \times [0, \delta]$, with $L = (\alpha \times \mathbb{R}) \times \{0\}$ as a flat minimal submanifold, and $L(\delta) = (\alpha \times \mathbb{R}) \times \{\delta\}$ having mean curvature vector pointing out of $N_\epsilon(L)$. For ϵ sufficiently small, we may assume that each component of $f(\Sigma) \cap N_\epsilon(L)$ is a normal graph of bounded gradient over the zero section L. Let C be such a component which is a graph over a connected domain Ω of L and let $L_C(\delta)$ be the part of L_δ which is also a normal graph over Ω. Consider the surface $W_\delta = L(\delta) \setminus L_C(\delta)$. Under normal projection to L, $W_\delta \cup C$ is quasi-isometric to the flat plane L. It follows that C is a parabolic Riemann surface with boundary. But the function $d := \mathrm{dist}(\,\cdot\,, L)$ is superharmonic, and has constant value δ on the boundary of C. Then C is contained in $L(\delta)$, which contradicts the fact that L is a limit leaf of $\mathcal{L}(f(\Sigma))$. □

6. Parabolicity of minimal surfaces

Theorem B. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $K_{\mathbb{M}} \geq 0$.

(2) *There is a point $p \in M$ such that the geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded.*

Let Σ be a surface of finite topology and one end and let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be a proper minimal immersion. Suppose that f is transverse to a slice $\mathbb{M} \times \{t_0\}$ except at a finite number of points and that $f^{-1}(\mathbb{M} \times \{t_0\})$ contains a finite number of components. Then Σ is parabolic.

Proof. We know from [Rosenberg 2002] that the conditions on $\mathbb{M}$ imply that the surfaces

$$\Sigma(+) := \{(p, t) \in \Sigma : t \geq t_0\},$$
$$\Sigma(-) := \{(p, t) \in \Sigma : t \leq t_0\}$$

are parabolic. Suppose that $\mathcal{E}$ is an annular end representative which does not have conformal representative which is a punctured disc. Then this end has a representative which is conformally diffeomorphic to $\{z \in \mathbb{C} : \epsilon \leq |z| < 1\}$ for some positive $\epsilon < 1$. In this conformal parametrization, the unit circle corresponds to points at infinity on $\mathcal{E}$. After choosing a larger ϵ, we may assume that $f|_{\mathcal{E}}$ intersects $\mathbb{M} \times \{t_0\}$ transversely in a finite positive number of arcs and that each noncompact arc of the intersection has one endpoint on the compact boundary circle $\{z \in \mathbb{C} : |z| = \epsilon\}$.

We claim that it suffices to prove that each of the finite number of noncompact arcs $\alpha_1, \dots, \alpha_n$ in $\mathbb{M} \times \{t_0\}$ has a well-defined limit on the unit circle $\mathbb{S}^1$ of points at infinity. In fact, assume the claim is true; then there is an open arc $\gamma \subset \mathbb{S}^1$

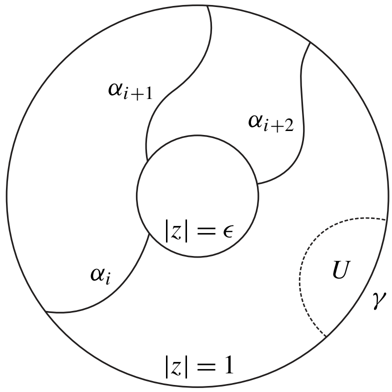

Figure 1. The disc U.

which does not contain limit points of $\alpha_1, \ldots, \alpha_n$. Hence, there would be an open half-disc $U \subset \mathcal{E}$ centered at a point in γ, such that $U \cap (f^{-1}(\mathbb{M} \times \{t_0\})) = \varnothing$; see Figure 1. But U is a proper domain which is contained in one of the parabolic surfaces $\Sigma(+)$ or $\Sigma(-)$, so is parabolic. However, U does not have full harmonic measure, which is a contradiction.

Suppose α_k has two limit points q_1, q_2 in $\mathbb{S}^1$. We first prove that at least one of the two interval components I_1, I_2 of $\mathbb{S}^1 \setminus \{q_1, q_2\}$ consists of limit points of α_k. Suppose not and let $x_1 \in I_1$, $x_2 \in I_2$ be points which are not limit points. Since they are not limit points, there exists a $\delta > 0$ such that the radial arcs β_1 and β_2 in $\mathcal{E}$ of length δ and orthogonal to $\mathbb{S}^1$ at x_1, x_2 respectively, are disjoint from α_k. Since α_k is proper and disjoint from $\beta_1 \cup \beta_2$, the parametrized arc $\alpha_k(s)$ must eventually be in one of the two components of $\{z \in \mathcal{E} \setminus (\beta_1 \cup \beta_2) : |z| \geq 1 - \delta\}$; see Figure 2. Thus, α_k cannot have both q_1 and q_2 as limit points, a contradiction. Now, suppose

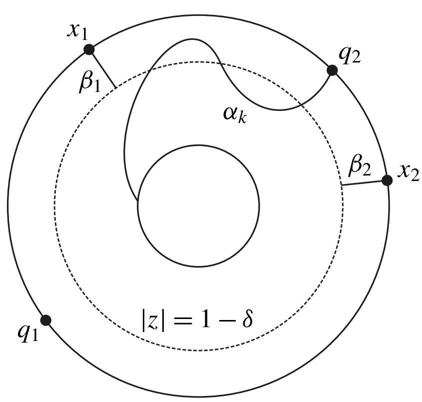

Figure 2. The arc trapped in one component.

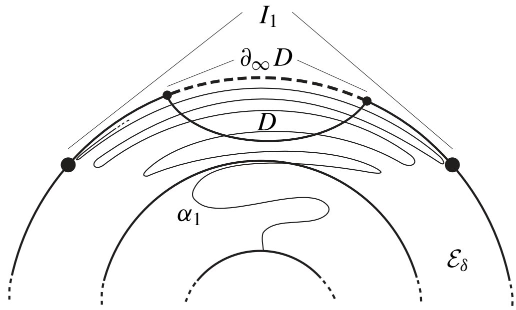

Figure 3. The arc α_1 accumulates in I_1.

one of the intervals, say I_2, contains one point z which is not a limit point of α_k; then by the previous argument the interval I_1 cannot contain any point which is not a limit point. So one of the intervals consists of limit points of α_k.

Since the height function is harmonic on $\mathcal{E}$ and the generator of the homology of $\mathcal{E}$ is a boundary in Σ, by Cauchy's theorem there is a conjugate harmonic function to h, which we denote by h^*. Consider the holomorphic function $g = h + ih^* : \mathcal{E} \to \mathbb{C}$. As the slice $\mathbb{M} \times \{t_0\}$ is transverse to $\mathcal{E}$, we have $\langle \nabla h, \eta \rangle^2 \neq 1$ for all points in an arc α_k and $h = 0$ in this arc, where η is a unit normal to Σ. Moreover, as g is holomorphic we have

$$|\nabla_\Sigma h^*(p)|^2 = |\nabla_\Sigma h(p)|^2 = 1 - \langle \nabla h, \eta \rangle^2(p) > 0 \quad \forall p \in \alpha_k,$$

so $h^*|_{\alpha_k}$ is strictly monotone. Thus g restricted to any of the finite number of components in $(f^{-1}(\mathbb{M} \times \{t_0\})) \cap \mathcal{E}$ monotonically parametrizes an interval on the imaginary axis $\mathbb{R}(i) \subset \mathbb{C}$. Choose a closed half-disc $\bar{D} \subset \bar{\mathcal{E}} = \mathcal{E} \cup \mathbb{S}^1$, centered at a point $p \in I_1$, where I_1, as discussed above, consists entirely of limit points of α_1, and suppose that $\bar{D}$ is chosen sufficiently small so that $\partial_\infty D := \partial D \cap \mathbb{S}^1 \subset I_1$. Since $g|_{\alpha_k}$ is injective we can take a compact interval $J \subset g\big(\bigcup_{k=1}^n \alpha_k\big) \subset \mathbb{R}(i)$ which is disjoint from the endpoints of $g|_{\alpha_k}$ for all k, and choose D sufficiently small such that $\bar{D} \cap (g^{-1}(J)) = \varnothing$.

Observe that g maps D into $\mathbb{C} \setminus J$, so by the Riemann mapping theorem, the function $g|_D$ is essentially bounded in the sense that it maps D into a domain that is conformally equivalent to an open subset of the unit disc. It follows from Fatou's theorem that the holomorphic function $g|_D$ has radial limits almost everywhere, i.e., D is conformally the unit disc, so radial limits are with respect to the radii of the unit disc.

Consider the radial arc β orthogonal to $\mathbb{S}^1$ at the point p (the center of I_1). The arc β divides I_1 into two intervals I_1^- and I_1^+ and separates D into two regions D^- and D^+. Choose $\delta > 0$ small. We can suppose D is inside the region $\mathcal{E}_\delta := \{z \in \mathcal{E} : |z| \geq 1 - \delta\}$. Since α_1 is proper, this arc will eventually be inside of $\mathcal{E}_\delta$. As I_1 is composed of accumulation points of α_1 and $\partial_\infty D$ is not equal to I_1, the arc

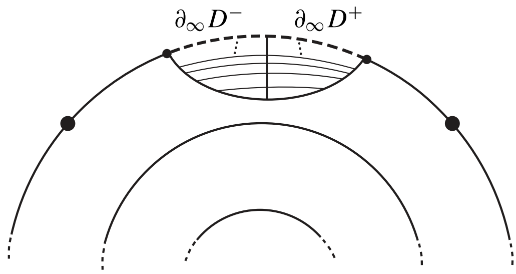

Figure 4. Infinitely many arcs in D^- and D^+.

α_1 leaves D and returns to it an infinite number of times, and it does this crossing the boundaries of D^- and D^+ infinitely many times, in each step getting closer to $\partial_\infty D^-$ and $\partial_\infty D^+$ respectively; see Figure 3. Then there exists an infinite number of arcs in $\alpha_1 \cap D^-$ (respectively $\alpha_1 \cap D^+$) converging to $\partial_\infty D^-$ (respectively $\partial_\infty D^+$); see Figure 4. Thus the points of $\partial_\infty D$ with radial limits for g have a constant value which corresponds to the limiting endpoint of the curve $g \circ \alpha_1$ in $\mathbb{R}(i) \cup \{\infty\}$. However, by Privalov's theorem, a nonconstant meromorphic function on the unit disc cannot have a constant radial limit on a set of $\partial_\infty D$ with positive measure, a contradiction. □

Theorem C. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $0 \le K_{\mathbb{M}} \le \kappa$.

(2) *$\mathbb{M}$ has a pole p.*

(3) *The geodesic curvatures of all geodesic circles with center p and radius $r \ge 1$ are uniformly bounded.*

Let Σ be a properly immersed minimal surface inside the region of $\mathbb{M} \times \mathbb{R}$ defined by $|h| \le c_2 \log r$ for some constant $c_2 > 0$ and $r \ge 1$. Then Σ is parabolic. Moreover, if Σ has compact boundary, then Σ has quadratic area growth and a finite number of ends.

Proof. Let p be the pole of $\mathbb{M}$. Since the map $\exp_p : T_p\mathbb{M} \to \mathbb{M}$ is a diffeomorphism, we have that $\phi : T_p\mathbb{M} \times \mathbb{R} \to \mathbb{M} \times \mathbb{R}$, defined by $\phi(v, s) = (\exp_p v, s)$, is a diffeomorphism and defines a coordinate system.

Let r be the distance to p on $\mathbb{M}$ extended to $\mathbb{M} \times \mathbb{R}$ in the natural way and h be the height function on $\mathbb{M} \times \mathbb{R}$. Let $C_R = \{(q, s) \in \mathbb{M} \times \mathbb{R} : r(q) = R\}$ be the vertical cylinder of radius R and let Σ_R be the part of Σ inside C_R. Let $B_R((p, 0))$ be the ball of $\mathbb{M} \times \mathbb{R}$ of center $(p, 0)$ and radius R. Since $\mathbb{M} \times \mathbb{R}$ has the product metric and p is a pole in M, the point $(p, 0)$ is a pole in $\mathbb{M} \times \mathbb{R}$. Thus $\Sigma \cap B_R((p, 0))$ is inside the interior of C_R. Then it suffices to prove that Σ_R has quadratic area growth as a function of r.

Using these coordinates we can define a horizontal vector field X that is orthogonal to ∇r and ∇h and has norm 1, so $(\nabla r, \nabla h, X)$ is an orthonormal basis at each point of $\mathbb{M} \times \mathbb{R}$. Let η be a unit normal to Σ, so

$$\langle \eta, \nabla r\rangle^2 + \langle \eta, \nabla h\rangle^2 + \langle \eta, X\rangle^2 = 1,$$
$$|\nabla_\Sigma r|^2 = 1 - \langle \eta, \nabla r\rangle^2,$$

and

$$|\nabla_\Sigma h|^2 = 1 - \langle \eta, \nabla h\rangle^2.$$

Hence,

$$|\nabla_\Sigma r|^2 + |\nabla_\Sigma h|^2 = 1 + \langle \eta, X\rangle^2 \geq 1.$$

Thus,

$$\int_{\Sigma_R} d\mu \leq \int_{\Sigma_R} (|\nabla_\Sigma r|^2 + |\nabla_\Sigma h|^2)\, d\mu.$$

Consider the function $f : \Sigma \to \mathbb{R},\ f = -h \arctan(h) + \frac{1}{2}\log(h^2+1)$, where h is the height function on $\mathbb{M} \times \mathbb{R}$. Since h is harmonic on Σ,

$$\Delta_\Sigma f = -\arctan(h)\Delta_\Sigma h - \frac{|\nabla_\Sigma h|^2}{h^2+1} = -\frac{|\nabla_\Sigma h|^2}{h^2+1}.$$

Consider now the function $g = \log r + f$. After rescaling the metric of Σ and removing a compact subset of Σ we may assume that $|h| \leq \frac{1}{2}\log r$. By Lemma 5, g satisfies

$$\Delta_\Sigma g \leq c_1 \frac{|\nabla_\Sigma h|^2}{r} - \frac{|\nabla_\Sigma h|^2}{h^2+1} \leq 0.$$

Since $\log r$ is proper in $\{(q,t) \in \mathbb{M}\times\mathbb{R} : |h| \leq \frac{1}{2}\log r,\ r \geq 1\}$ and Σ is proper, $\log r$ is proper in Σ. Moreover $g \geq \frac{3\pi}{4}\log r$, so g is a nonnegative proper superharmonic function on Σ. This proves that Σ is parabolic.

Suppose $\partial\Sigma$ is compact. There exists $a > 0$ such that $g(\partial\Sigma) \subset [0, a]$. Let $t_2 > t_1 \geq a$. Since g is proper, $g^{-1}([t_1, t_2])$ is compact; then we can apply the divergence theorem and use the fact that g is superharmonic to obtain

$$0 \geq \int_{g^{-1}([t_1,t_2])} \Delta_\Sigma g\, d\mu = -\int_{g^{-1}(t_1)} |\nabla_\Sigma g|\, dL + \int_{g^{-1}(t_2)} |\nabla_\Sigma g|\, dL. \tag{5}$$

It follows that the function $t \mapsto \int_{g^{-1}(t)} |\nabla_\Sigma g|\, dL$ is monotonically decreasing and bounded, so

$$\lim_{t\to\infty} \int_{g^{-1}(t)} |\nabla_\Sigma g|\, dL < \infty. \tag{6}$$

Since $\Sigma = g^{-1}([0,\infty))$ it follows from (5) and (6) that $\Delta_\Sigma g \in \mathrm{L}^1(\Sigma)$. Furthermore, $\Delta_\Sigma g \geq \frac{1}{2}|\Delta_\Sigma f|$ for r large; thus $\Delta_\Sigma f \in \mathrm{L}^1(\Sigma)$. Hence,

$$\int_{\Sigma_R} \Delta_\Sigma f \, d\mu = \int_{\Sigma_R} \frac{|\nabla_\Sigma h|^2}{h^2+1}\, d\mu \leq \int_\Sigma \frac{|\nabla_\Sigma h|^2}{h^2+1}\, d\mu = c_3$$

for some positive constant c_3. Then, for $R \geq 1$,

$$\int_{\Sigma_R} |\nabla_\Sigma h|^2 \, d\mu \leq \int_{\Sigma_R} \left(\frac{(\log R)^2+1}{h^2+1}\right)|\nabla_\Sigma h|^2 \, d\mu \leq ((\log R)^2+1)c_3 \leq c_3 R^2.$$

Since $\Delta_\Sigma f \in \mathrm{L}^1(\Sigma)$ and $|\Delta_\Sigma f| \geq c_4 |\Delta_\Sigma \log r|$ ($c_4 > 0$ a constant), we have $\Delta_\Sigma(\log r) \in \mathrm{L}^1(\Sigma)$. Again by the divergence theorem,

$$\begin{aligned}\int_{\Sigma_R} \Delta_\Sigma \log r \, d\mu &= \int_{\partial\Sigma} \frac{1}{r}\langle \nabla_\Sigma r, \nu\rangle \, dL + \int_{C_R\cap\Sigma} \frac{|\nabla_\Sigma r|}{R}\, dL \\ &= c_5 + \frac{1}{R}\int_{C_R\cap\Sigma} |\nabla_\Sigma r| \, dL,\end{aligned}$$

where ν is the outward conormal to the boundary of Σ. Thus

$$\lim_{R\to\infty} \frac{1}{R}\int_{C_R\cap\Sigma} |\nabla_\Sigma r| \, dL < \infty,$$

which implies there is a constant $c_6 > 0$ such that

$$\int_{C_R\cap\Sigma} |\nabla_\Sigma r| \, dL \leq c_6 R.$$

By the coarea formula

$$\int_{\Sigma_R} |\nabla_\Sigma r|^2 \, d\mu \leq \int_1^R \int_{C_\rho\cap\Sigma} |\nabla_\Sigma r| \, dL \, d\rho \leq c_6 \int_1^R \rho \, d\rho \leq \tfrac{1}{2} c_6 R^2.$$

Therefore Σ has quadratic area growth.

Now, suppose Σ has an infinite number of ends. Let E be an end of Σ. Choose $0 < \delta < \min\{\mathrm{inj}_{\mathbb{M}\times\mathbb{R}}, 1/\sqrt{\kappa}\}$ such that for each positive integer j, there is a distance ball $B_\delta(q_j)$ of $\mathbb{M}\times\mathbb{R}$ inside the region $\mathcal{R}_j$ between C_j and C_{j+1}, with $q_j \in E$. By the monotonicity formula for minimal surfaces (see Chapter 7 of [Colding and Minicozzi 2011]),

$$|E \cap B_\delta(q_j)| \geq \frac{c\delta^2}{e^{2\sqrt{\kappa}\delta}} =: c_7,$$

where $c > 0$ is a constant and $\kappa = \sup K_{\mathbb{M}\times\mathbb{R}}$. Write $E_n = E \cap C_n$. Since in each region $\mathcal{R}_j$, $j < n$, we have a portion of E of area at least c_7 it follows that

$$|E_n| > c_7 n.$$

Then in the cylinder C_{n^2} we have

$$c_7 n^2 \le |E_{n^2}| \le c_8 n^2.$$

Since this holds for each end, choosing n ends we obtain that the area of Σ inside C_{n^2} satisfies

$$c_9 n^3 \le |\Sigma_{n^2}| \le c_{10} n^2,$$

but for n sufficiently large this leads to a contradiction. Hence, Σ has a finite number of ends. □

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References

[Bessa et al. 2001] G. P. Bessa, L. P. Jorge, and G. Oliveira-Filho, "Half-space theorems for minimal surfaces with bounded curvature", *J. Differential Geom.* **57**:3 (2001), 493–508. MR Zbl

[Callahan et al. 1990] M. Callahan, D. Hoffman, and W. H. Meeks, III, "The structure of singly-periodic minimal surfaces", *Invent. Math.* **99**:3 (1990), 455–481. MR Zbl

[do Carmo 1988] M. P. do Carmo, *Geometria riemanniana*, 2nd ed., Instituto de Matemática Pura e Aplicada, Rio de Janiero, 1988. Translated in *Riemannian geometry*, Birkhäuser, Boston, 1992.

[Colding and Minicozzi 2008] T. H. Colding and W. P. Minicozzi, II, "The Calabi–Yau conjectures for embedded surfaces", *Ann. of Math.* (2) **167**:1 (2008), 211–243. MR Zbl

[Colding and Minicozzi 2011] T. H. Colding and W. P. Minicozzi, II, *A course in minimal surfaces*, Graduate Studies in Mathematics **121**, American Mathematical Society, Providence, 2011. MR Zbl

[Collin 1997] P. Collin, "Topologie et courbure des surfaces minimales proprement plongées de $\mathbb{R}^3$", *Ann. of Math.* (2) **145**:1 (1997), 1–31. MR Zbl

[Collin et al. 2004] P. Collin, R. Kusner, W. H. Meeks, III, and H. Rosenberg, "The topology, geometry and conformal structure of properly embedded minimal surfaces", *J. Differential Geom.* **67**:2 (2004), 377–393. MR

[Espinar and Rosenberg 2009] J. M. Espinar and H. Rosenberg, "Complete constant mean curvature surfaces and Bernstein type theorems in $M^2 \times \mathbb{R}$", *J. Differential Geom.* **82**:3 (2009), 611–628. MR Zbl

[Fischer-Colbrie and Schoen 1980] D. Fischer-Colbrie and R. Schoen, "The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature", *Comm. Pure Appl. Math.* **33**:2 (1980), 199–211. MR

[Frohman and Meeks 1997] C. Frohman and W. H. Meeks, III, “The ordering theorem for the ends of properly embedded minimal surfaces”, *Topology* **36**:3 (1997), 605–617. MR Zbl

[Grigor’yan 1999] A. Grigor’yan, “Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds”, *Bull. Amer. Math. Soc.* (*N.S.*) **36**:2 (1999), 135–249. MR Zbl

[Hoffman and Meeks 1990] D. Hoffman and W. H. Meeks, III, “The strong halfspace theorem for minimal surfaces”, *Invent. Math.* **101**:2 (1990), 373–377. MR Zbl

[Liu 2013] G. Liu, “3-manifolds with nonnegative Ricci curvature”, *Invent. Math.* **193**:2 (2013), 367–375. MR Zbl

[Meeks and Rosenberg 2005] W. H. Meeks, III and H. Rosenberg, “The uniqueness of the helicoid”, *Ann. of Math.* (2) **161**:2 (2005), 727–758. MR Zbl

[Meeks and Rosenberg 2006] W. H. Meeks, III and H. Rosenberg, “The minimal lamination closure theorem”, *Duke Math. J.* **133**:3 (2006), 467–497. MR Zbl

[Meeks and Rosenberg 2008] W. H. Meeks, III and H. Rosenberg, “Maximum principles at infinity”, *J. Differential Geom.* **79**:1 (2008), 141–165. MR Zbl

[Meeks et al. 2008] W. H. Meeks, III, J. Pérez, and A. Ros, “Stable constant mean curvature surfaces”, pp. 301–380 in *Handbook of geometric analysis*, vol. 1, edited by L. Ji et al., Adv. Lect. Math. **7**, Int. Press, Somerville, MA, 2008. MR Zbl

[Moser 1961] J. Moser, “On Harnack’s theorem for elliptic differential equations”, *Comm. Pure Appl. Math.* **14** (1961), 577–591. MR Zbl

[Rosenberg 2002] H. Rosenberg, “Minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$”, *Illinois J. Math.* **46**:4 (2002), 1177–1195. MR Zbl

[Rosenberg et al. 2010] H. Rosenberg, R. Souam, and E. Toubiana, “General curvature estimates for stable H-surfaces in 3-manifolds and applications”, *J. Differential Geom.* **84**:3 (2010), 623–648. MR Zbl

[Schoen 1983] R. Schoen, “Estimates for stable minimal surfaces in three-dimensional manifolds”, pp. 111–126 in *Seminar on minimal submanifolds*, edited by E. Bombieri, Ann. of Math. Stud. **103**, Princeton Univ. Press, 1983. MR Zbl

[Schoen and Yau 1982] R. Schoen and S. T. Yau, “Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature”, pp. 209–228 in *Seminar on differential geometry*, edited by S. T. Yau, Ann. of Math. Stud. **102**, Princeton Univ. Press, 1982. MR Zbl

[White 1987] B. White, “Complete surfaces of finite total curvature”, *J. Differential Geom.* **26**:2 (1987), 315–326. Correction in **28**:2 (1988), 359–360. MR Zbl

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REGULARITY CONDITIONS FOR SUITABLE WEAK SOLUTIONS OF THE NAVIER–STOKES SYSTEM FROM ITS ROTATION FORM

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We establish new regularity criteria for suitable weak solutions involving Bernoulli (total) pressure $\Pi = \frac{1}{2}|u|^2 + p$. By the rotation form of the Navier–Stokes equations, we also obtain regularity criteria for suitable weak solutions in terms of either $u \times \omega/|\omega|$ or $\omega \times u/|u|$ with sufficiently small local scaled norm, where ω is the vorticity of the velocity. As a consequence, we extend and refine some known interior regularity criteria for suitable weak solutions.

1. Introduction

Consider the initial boundary-value problem for the incompressible time-dependent Navier–Stokes equations:

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times [0, T), \\ u|_{t=0} = u_0(x) & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{1-1}$$

where the domain $\Omega \subseteq \mathbb{R}^3$ is a bounded regular domain. Here u describes the velocity of the flow, the scalar function p stands for the pressure of the fluid. The initial data $u_0(x)$ satisfies divergence free. Denote by $\omega = \operatorname{curl} u$ the vorticity of the velocity field.

There have been extensive studies on the regularity of suitable weak solutions to the Navier–Stokes equations since the late 1970s (see, e.g., [Caffarelli et al. 1982; Chae et al. 2007; Dong and Du 2007; Dong and Strain 2012; Chae 2010; Gustafson et al. 2007; Wang and Wu 2014; 2016a; 2016b; Struwe 1988; Seregin 2002; 2007; 2014; Wang et al. 2014; Wang and Zhang 2013; 2014; Scheffer 1976; 1977; 1980; Vasseur 2007; Wolf 2008; Lin 1998; Ladyzhenskaya and Seregin 1999; Tian and Xin 1999]). Suitable weak solutions originated with Scheffer [1976; 1977; 1980] in

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studying the potential singular points of solutions to the Navier–Stokes equations and were later developed by Caffarelli, Kohn, Nirenberg [Caffarelli et al. 1982] and Lin [1998]. For convenience, we recall the definition of suitable weak solutions.

Definition (Suitable weak solutions). A pair (u, p) is a suitable weak solution to the Navier–Stokes equations (1-1), provided the following conditions are satisfied

(i) $u \in L^\infty(t, t'; L^2(\Omega)) \cap L^2(t, t'; W^{1,2}(\Omega)),\ \ p \in L^{3/2}(t, t'; L^{3/2}(\Omega))$.

(ii) (u, p) solves (1.1) in $\Omega \times (t, t')$ in the sense of distributions.

(iii) (u, p) obeys the local energy inequality

$$\int_\Omega |u(t',x)|^2\phi\,dx + 2\int_t^{t'}\int_\Omega |\nabla u(s,x)|^2\phi\,dx\,ds \le \int_t^{t'}\int_\Omega |u(s,x)|^2(\partial_s\phi+\Delta\phi)\,dx\,ds + 2\int_t^{t'}\int_\Omega \big(\tfrac{1}{2}|u(s,x)|^2+p(s,x)\big)u(s,x)\cdot\nabla\phi\,dx\,ds \tag{1-2}$$

for any nonnegative function $\phi \in C_0^\infty(\Omega \times (t, t'))$.

A point is said to be a regular point of the Navier–Stokes equations (1-1) if one has an L^∞ bound of u in some neighborhood of this point. Otherwise, they are called singular points. In this direction, the milestone work is that the one-dimensional Hausdorff measure of the possible spacetime singular points of suitable weak solutions to the 3D Navier–Stokes equations is zero, which was shown by Caffarelli, Kohn, Nirenberg in [Caffarelli et al. 1982]. This result relies heavily on the following regularity criteria: if there is an absolute constant ε such that

$$\limsup_{\mu\to 0} \frac{1}{\mu} \iint_{Q(\mu)} |\nabla u|^2\,dx\,dt \le \varepsilon, \tag{1-3}$$

then $(0, 0)$ is a regular point, where $Q(\mu) := B(\mu) \times (-\mu^2, 0)$ and $B(\mu)$ denotes the ball of center 0 and radius μ. Since then, different approaches to show the Caffarelli–Kohn–Nirenberg theorem have been presented. More precisely, based on the blowup method, Lin [1998] provided a simple proof (see also Ladyzenskaja and Seregin [1999] with nonzero external force belonging to parabolic Morrey space). Recently, by means of De Giorgi's iteration technique, Vasseur [2007] provided a constructive proof without external force. In [Wang and Wu 2014], De Giorgi's iteration strategy was applied to the 4D Navier–Stokes equations and the high-dimensional steady Navier–Stokes equations with nonzero external force. In what follows, the local scaled norm of quantity is the one which equips the scale invariant norm similar to (1-3). An alternative proof is offered by Wolf [2008] via establishing a decay estimate of the gradient of the velocity with local scaled norm together with Campanato's Lemma on Hölder continuity. Moreover, notice that regularity condition (1-3) plays a central role in the partial regularity theory of Navier–Stokes. There are a lot of extensions and improvements of (1-3). For instance, Gustafson, Kang and Tsai [Gustafson et al. 2007] obtained the following regularity criteria to suitable weak solutions:

$$\limsup_{\mu\to 0} \mu^{1-\frac{2}{p}-\frac{3}{q}} \|u\|_{L^{p,q}(Q(\mu))} \le \varepsilon, \quad 1 \le \tfrac{2}{p}+\tfrac{3}{q} \le 2, \quad 1 \le p, q \le \infty; \tag{1-4}$$

$$\limsup_{\mu\to 0} \mu^{2-\frac{2}{p}-\frac{3}{q}} \|\nabla u\|_{L^{p,q}(Q(\mu))} \le \varepsilon, \quad 2 \le \tfrac{2}{p}+\tfrac{3}{q} \le 3, \quad 1 \le p, q \le \infty; \tag{1-5}$$

$$\limsup_{\mu\to 0} \mu^{2-\frac{2}{p}-\frac{3}{q}} \|\omega\|_{L^{p,q}(Q(\mu))} \le \varepsilon, \quad 2 \le \tfrac{2}{p}+\tfrac{3}{q} \le 3, \quad 1 \le p, q \le \infty, \tag{1-6}$$

where $(p, q) \ne (1, \infty)$ in (1-6), and where ε is an absolute constant, which extends the work of Tian and Xin [1999]. Employing a blowup procedure, Seregin [2007] improved the regular condition (1-3) to, for any $M > 0$, there exists a positive number $\varepsilon(M)$ such that

$$\limsup_{r\to 0} \frac{1}{r}\iint_{Q(r)} |\nabla u|^2 dx\, dt \le M \quad \text{and} \quad \liminf_{r\to 0} \frac{1}{r}\iint_{Q(r)} |\nabla_3 u|^2 dx\, dt \le \varepsilon(M). \tag{1-7}$$

We also refer the reader to the recent works of Wang and Zhang [2014] and Wang and Wu [2016a; 2016b].

We note that almost all the results mentioned above rest on the Navier–Stokes equations in convective form (1-1). Depending on different expressions of the nonlinear term, the Navier–Stokes equations have several equivalent versions such as the convective form, the skew-symmetric form and the rotation form (see, e.g., [Layton et al. 2009; Zang 1991] and references therein). Thanks to the well-known fact that

$$u \cdot \nabla u = \tfrac{1}{2}\nabla |u|^2 + \omega \times u,$$

the 3D Navier–Stokes equations (1-1) can be equivalently reformulated as the rotation form below:

$$\begin{cases} u_t - \Delta u + w \times u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1-8}$$

where $\Pi = \frac{1}{2}|u|^2 + p$ is called as the Bernoulli (total) pressure, which can be found in [Prandtl 2004; Heywood et al. 1996; Layton et al. 2009; Olshanskii 2002; Zang 1991] and references therein. By means of the Bernoulli pressure Π, the local energy inequality (1-2) can be rewritten as

$$\begin{aligned} \int_\Omega |u(t', x)|^2 \phi dx + 2\int_t^{t'}\int_\Omega |\nabla u(s, x)|^2 \phi\, dx\, ds \\ \le \int_t^{t'}\int_\Omega |u|^2(\phi_s + \Delta\phi)\, dx\, ds + 2\int_t^{t'}\int_\Omega \Pi u \cdot \nabla\phi\, dx\, ds. \end{aligned} \tag{1-9}$$

We refer to the above inequality as the local energy inequality with respect to the 3D Navier–Stokes equations in rotation form (1-8).

The goal of this paper is to derive some new regularity criteria for suitable weak solutions from the Navier–Stokes equations in rotation form (1-8). Notice that the Bernoulli pressure Π not only plays important role in the regular theory of the Navier–Stokes equations (see, e.g., [Frehse and Růžička 1994; 1995; Struwe 1995; Seregin and Šverák 2002; Nečas et al. 1996; Chae 2014; Tsai 1998]), but also can be measurable via numerical simulations (see, e.g., [Heywood et al. 1996; Layton et al. 2009; Prandtl 2004; Olshanskii 2002; Zang 1991]). Seregin and Šverák [2002] showed that the weak solutions to the 3D Navier–Stokes equations are regular provided the positive part of the Bernoulli pressure is controlled. Since the pressure p is nonlocal, it seems difficult to obtain regularity criteria via only the pressure p with sufficiently small local scaled norm. One objective of this paper is to establish the regularity criteria in terms of Bernoulli pressure Π with sufficiently small local scaled norm.

Theorem 1.1. *There exists a constant $\varepsilon_1 > 0$ with the property that if (u, p) is a suitable weak solution of the Navier–Stokes equations such that $\Pi - (\Pi)_{B(\mu)} \in L^{p,q}_{\mathrm{loc}}$ with*

$$\limsup_{\mu\to 0} \mu^{2-\frac{2}{p}-\frac{3}{q}} \left(\int_{-\mu^2}^{0} \left(\int_{B(\mu)} |\Pi - (\Pi)_{B(\mu)}|^q \, dx \right)^{\frac{p}{q}} ds \right)^{\frac{1}{p}} < \varepsilon_1,$$

where $(p, q) \in [1, \infty] \times [1, \infty]$ satisfying

$$2 \le \frac{2}{p} + \frac{3}{q} \le \frac{7}{2} \quad \text{with } 1 \le p \le 2. \tag{1-10}$$

Then u is regular at $(0, 0)$.

Remarks. (1) The range $1 \le p \le 2$ corresponds to the limiting case $2/p+3/q = 7/2$. By means of Hölder's inequality, the range (1-10) can be generalized to

$$\frac{2}{p} + \frac{3}{q} = \begin{cases} \frac{7}{2} - \delta & \text{with } 1-\delta \le 2/p \le 2 (0 \le \delta \le 1), \\ \hbar \in [2, 5/2] & \text{with } 1 \le p \le \infty. \end{cases}$$

(2) Theorem 1.1 also implies the criteria in terms of the gradient of the Bernoulli pressure. Moreover, Theorem 1.1 holds true for nonzero external force f provided that $f \in L^q_{t,x}$ with $q > \frac{5}{2}$.

(3) The same result is valid if $\Pi - (\Pi)_{B(r)}$ is replaced by Π in Theorem 1.1. As a straightforward consequence, a Serrin-type sufficient regularity condition in terms of Bernoulli pressure can be obtained. More precisely, let (u, p) be a suitable weak solution. Then u is regular on $Q(r/2)$ provided Π belongs to $L^{p,q}(Q(r))$ with $2/p + 3/q = 2$.

The key point for proving the above theorem is how to bound the first term on the right hand side of the local energy inequality (1-9). Generally speaking, the magnitude between $\frac{1}{2}|u|^2$ and $\frac{1}{2}|u|^2+p$ is not clear. Resorting to the appropriate test function (backward heat kernel) recently adopted in [Dong and Du 2007; Wang et al. 2014; Wang and Zhang 2013], we could circumvent the direct control. This enables us to obtain

$$\begin{aligned}
&\mu^{-1}\|u\|^2_{L^{\infty,2}(Q(\mu))}+\mu^{-1}\|\nabla u\|^2_{L^2(Q(\mu))}\\
&\le C\Big(\frac{\mu}{\rho}\Big)^2\rho^{-1}\|u\|^2_{L^{\infty,2}(Q(\rho))}\\
&\qquad+C\Big(\frac{\rho}{\mu}\Big)^2\rho^{-2}\|\Pi-(\Pi)_{B(\rho)}\|_{L^{p,q}(Q(\rho))}\big[\|u\|^2_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|^2_{L^2(Q(\rho))}\big]^{1/2},
\end{aligned}$$

which gives the desired iteration. A slight modification of the latter iteration yields

$$\begin{aligned}
&\mu^{-1}\|u\|^2_{L^{\infty,2}(Q(\mu))}+\mu^{-1}\|\nabla u\|^2_{L^2(Q(\mu))}\\
&\le C\Big(\frac{\mu}{\rho}\Big)^2\rho^{-1}\|u\|^2_{L^{\infty,2}(Q(\rho))}+C\Big(\frac{\rho}{\mu}\Big)^2\rho^{-2}\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p^\natural,q^\natural}(Q(\rho))}\big[\|u\|^2_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|^2_{L^2(Q(\rho))}\big].
\end{aligned}$$

This relation leads to the following results:

Theorem 1.2. *There exists a constant $\varepsilon_2>0$ with the property that if (u,p) is a suitable weak solution of the Navier–Stokes equations such that $\Pi/|u|\in L^{p^\natural,q^\natural}_{\mathrm{loc}}$ with*

$$\limsup_{\mu\to0}\mu^{1-\frac{2}{p^\natural}-\frac{3}{q^\natural}}\bigg(\int_{-\mu^2}^0\bigg(\int_{B(\mu)}\Big|\frac{\Pi}{|u|}\Big|^{q^\natural}dx\bigg)^{\frac{p^\natural}{q^\natural}}ds\bigg)^{\frac{1}{p^\natural}}<\varepsilon_2,$$

where $(p^\natural,q^\natural)\in[1,\infty]\times[1,\infty]$ satisfy

$$1\le\frac{2}{p^\natural}+\frac{3}{q^\natural}\le2, \tag{1-11}$$

then u is regular at $(0,0)$.

Remarks. (1) The statement of Theorem 1.2 remains valid if $\Pi/|u|$ is replaced by $\Pi/(\mu^{-1}+|u|)$. This theorem also means the Serrin-type regular condition in terms of $\Pi/|u|$. This theorem corresponds to Beirão da Veiga's [2000] regularity condition that any weak solution u is regular in $\Omega\times(0,T)$ provided

$$\frac{p}{1+|u|}\in L^p(0,T;L^q(\Omega))\quad\text{with }\tfrac{2}{p}+\tfrac{3}{q}=1,\ q>3.$$

(2) The proofs of Theorems 1.1 and 1.2 also yield the regularity conditions involving $\Pi/|u|^\alpha$ with sufficiently small local scaled norm for $0\le\alpha\le1$. Invoking the blowup framework introduced by Seregin [2007], one can improve these results provided $\alpha<1$ in the sense of (1-7).

In the following, we seek out a quantity which can control the Bernoulli pressure from the equations (1-8). Notice that the Bernoulli pressure is determined by

$$\Delta\Pi = -\operatorname{div}(\omega \times u). \tag{1-12}$$

We find that ω and u may be the apposite candidate. Indeed, by virtue of the split of velocity u, Wolf [2008] established the following criteria: assume that u is a suitable weak solution to (1-1). If there exists an absolute constant ε such that

$$\limsup_{\mu\to 0} \frac{1}{\mu} \iint_{Q(\mu)} \Big|\omega \times \frac{u}{|u|}\Big|^2 \,dx\,ds \le \varepsilon, \tag{1-13}$$

then (0, 0) is a regular point. The second goal of this paper is to obtain a regular class in terms of $u\times\omega/|\omega|$ and to extend the integral norms with different exponents in space and time in (1-13).

Theorem 1.3. *Let* (u, p) *be a suitable weak solution to* (1-1) *in* $Q(1)$. *Then* (0, 0) *is regular point provided one of the following conditions holds*:

(1) *There exists a positive constant* ε_3 *such that* $u \times \omega/|\omega| \in L^{i,j}_{\mathrm{loc}}$ *with*

$$\limsup_{\mu\to 0} \mu^{1-\frac{2}{i}-\frac{3}{j}} \bigg(\int_{-\mu^2}^{0} \bigg(\int_{B(\mu)} \Big|u \times \frac{\omega}{|\omega|}\Big|^j \,dx\bigg)^{\frac{i}{j}} ds\bigg)^{\frac{1}{i}} \le \varepsilon_3, \tag{1-14}$$

where $(i, j) \in (2, 4)\times(2, 3)$ *satisfy*

$$1 \le \frac{2}{i} + \frac{3}{j} \le 2 \quad \textit{with } i < 4. \tag{1-15}$$

(2) *There exists a positive constant* ε_3 *such that* $\omega \times u/|u| \in L^{m,n}_{\mathrm{loc}}$ *with*

$$\limsup_{\mu\to 0} \mu^{2-\frac{2}{m}-\frac{3}{n}} \bigg(\int_{-\mu^2}^{0} \bigg(\int_{B(\mu)} \Big|\omega \times \frac{u}{|u|}\Big|^n \,dx\bigg)^{\frac{m}{n}} ds\bigg)^{\frac{1}{m}} \le \varepsilon_3, \tag{1-16}$$

where $(m, n) \in (1, 4)\times(6/5, 3)$ *satisfy*

$$2 \le \frac{2}{m} + \frac{3}{n} \le 3 \quad \textit{with } m < 4. \tag{1-17}$$

Remarks. (1) As noted in the first remark on page 192, in light of Hölder's inequality, one can extend the range of (1-15) and (1-17) to

$$\frac{2}{i} + \frac{3}{j} = \begin{cases} 2-\delta & \text{with } 1-2\delta < \frac{4}{i} < 2 \ \ \big(0 \le \delta \le \frac{1}{2}\big), \\ \ell \in \big[1, \frac{3}{2}\big), & \text{with } 2 < i \le \infty, \end{cases}$$

and

$$\frac{2}{m} + \frac{3}{n} = \begin{cases} 3-\delta & \text{with } 1-2\delta < \frac{4}{m} < 4 \ \ \big(0 \le \delta \le 1/2\big), \\ \ell' \in \big[2, \frac{5}{2}\big), & \text{with } 2 < m \le \infty, \end{cases}$$

respectively.

(2) Theorem 1.3 is an improvement of corresponding results (1-4) and (1-6) proved by Gustafson, Kang and Tsai [Gustafson et al. 2007]. These extensions of (1-4) and (1-6) include their endpoint cases.

(3) As a corollary of Theorem 1.3, one immediately obtains the Serrin-type regularity conditions via $u \times \omega/|\omega|$ or $\omega \times u/|u|$, which was proved in [Chae 2010].

The idea of proving Theorem 1.3 is to establish an effective iteration scheme via local energy inequality (1-9). Therefore, the main target is devoted to deriving the decay-type estimate of $|u|^2$ and the Bernoulli pressure Π in terms of the rotation term $\omega \times u$. In view of (1-12), one can derive the decay-type estimate of the Bernoulli pressure Π in terms of $\omega \times u$. Since there is no direct relationship between $|u|^2$ and $\omega \times u$, the main difficulty of the proof of this theorem lies in the estimate of the first term on the right hand side of the local energy inequality (1-9). One would want to invoke the backward heat kernel as test function utilized in [Dong and Du 2007; Wang et al. 2014; Wang and Zhang 2013] again, which yields the appearance of $(\rho/\mu)^2 > 1$ in the second term on the right hand side of the local energy inequality. However, this breaks down since now neither Π nor u is assumed to be sufficiently small. Our strategy is to utilize the decomposition introduced by Seregin [2002] for studying the partial regularity of the Navier–Stokes equations near the boundary. Precisely, let (v, p_1) be a unique solution to the following initial boundary value problem:

$$\begin{cases} v_t - \Delta v + \nabla p_1 = -w \times u, \operatorname{div} v = 0 & \text{in } Q(\rho) \\ (p_1)_{B(\rho)} = 0 & \text{on } (-\rho^2, 0), \\ v = 0 & \text{on } \{t = -\rho^2\} \times B(\rho) \cup [-\rho^2, 0] \times \partial B_\rho. \end{cases} \tag{1-18}$$

Then $b = u - v$ and $p_2 = \Pi - (\Pi)_{B(\rho/2)} - p_1$ solve the following boundary value problem:

$$\begin{cases} b_t - \Delta b = -\nabla p_2, \operatorname{div} b = 0 & \text{in } Q(\rho) \\ b = u & \text{on } \{t = -\rho^2\} \times B(\rho) \cup [-\rho^2, 0] \times \partial B_\rho. \end{cases} \tag{1-19}$$

This allows us to bound the L^2-norm of u in terms of controlling that of v and b separately. On the one hand, applying the $L^p - L^q$-estimate of solutions to the Stokes system established by Giga and Sohr [1991] to (1-18), we get

$$\|v_t\|_{L^{r,s}(Q(\rho))} + \|A_s v\|_{L^{r,s}(Q(\rho))} + \|\nabla p_1\|_{L^{r,s}(Q(\rho))} \le C\|w \times u\|_{L^{r,s}(Q(\rho))},$$

where $A_s = -\mathbb{P}_s \Delta$ and $\mathbb{P}_s$ is the Leray projection from $L^s(\Omega)^d$ onto $L^s_\sigma(\Omega)$. Then we can apply embedding theorems in mixed norm also shown in the same work to bound $\|v\|_{L^2(Q(\rho))}$ in terms of $\|w \times u\|_{L^{r,s}(Q(\rho))}$. On the other hand, the harmonic function p_2 helps us to get an interior estimate of b below

$$\|b\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^5 \big[\|b\|^2_{L^2(Q(\rho))} + \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big],$$

where $0 < \mu \le \rho/32$. Then we could derive the decay-type estimate

$$\mu^{-3}\|u\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} + C\Big(\frac{\mu}{\rho}\Big)^2\big[\rho^{-3}\|u\|^2_{L^2(Q(\rho))} + \rho^{-3}\|\Pi - (\Pi)_{B(\rho)}\|^2_{L^{r,s'}(Q(\rho))}\big]. \tag{1-20}$$

Remark. The decomposition (1-18)–(1-19) allows us to take full advantage of the structure of the rotation term $\omega \times u$ and the local energy inequality (1-9) to refine regularity criteria (1-4) and (1-6). Roughly speaking, if the rotation term $w \times u$ in (1-18) is replaced by a convective term $u \cdot \nabla u$, then the split (1-18)–(1-19) reduces to Seregin's [2002] original split. However, it seems that, following the pathway of Theorem 1.3, Seregin's original split of the velocity u seems to yield Serrin-type regularity criteria rather than the Caffarelli–Kohn–Nirenberg type regularity conditions via $u \cdot \nabla u/|\nabla u|$ or $u/|u| \cdot \nabla u$.

Finally, we turn our attention to the following stationary Navier–Stokes equations in $\mathbb{R}^d$ for $d = 5, 6$:

$$-\Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0, \quad x \in \Omega. \tag{1-21}$$

First, we also present the definition of suitable weak solutions to the stationary case.

Definition. A pair (u, p) is said to be a suitable weak solution to the stationary Navier–Stokes equations (1-21) if and only if

(1) $u \in W^{1,2}(\Omega)$, $p \in L^{3/2}(\Omega)$.

(2) (u, p) solves (1-21) in the sense of distributions.

(3) (u, p) verifies the local energy inequality

$$2\int_\Omega |\nabla u|^2\psi\, dx \le \int_\Omega |u|^2\Delta\psi\, dx + 2\int_\Omega \big(\tfrac{1}{2}|u|^2 + p\big)u\cdot\nabla\psi\, dx + 2\int_\Omega uf\psi\, dx, \tag{1-22}$$

for $\psi \in C_0^\infty(\Omega)$, in the sense of distributions.

According to the dimensional analysis of the Navier–Stokes equations in [Caffarelli et al. 1982], nonstationary Navier–Stokes equations in $\mathbb{R}^d$ may be viewed as stationary Navier–Stokes equations $\mathbb{R}^{d+2}$. The analogue of the Caffarelli–Kohn–Nirenberg criteria (1-3) for suitable weak solutions to the stationary Navier–Stokes equations in $\mathbb{R}^5$ and $\mathbb{R}^6$ were proved by Struwe [1995] and by Dong and Strain [2012], respectively. By means of an observation that both the local energy inequality for the time-dependent Navier–Stokes equations and the stationary case can be dealt with by the unified approach in [Wang and Wu 2014], one can show the analogue theorem of Theorem 1.1 to system (1-21). To make our paper more self-contained and more readable, we outline the proof of the stationary case with the external force f ($\operatorname{div} f = 0$).

Theorem 1.4. *Suppose that* (u, p) *is a suitable weak solution of the Navier–Stokes equations* (1-21) *and the external force* f *belongs to* $L^q(\Omega)$ *with* $q > \frac{1}{2}d$. *There is a constant* ε_4 *such that if the condition*

$$\limsup_{\mu\to 0} \mu^{-\frac{d-2}{2}} \left(\int_{B(\mu)} |\Pi - (\Pi)_{B(\mu)}|^{\frac{2d}{d+2}} \, dx \right)^{\frac{2-d}{2}} < \varepsilon_4, \quad d = 5, 6,$$

holds, then u *is regular at origin.*

As a byproduct, Hölder's inequality and absolute continuity of Lebesgue's integral immediately yield the following result:

Corollary 1.5. *Let* (u, p) *be a suitable weak solution of the stationary Navier–Stokes equations* (1-21). *If*

$$\tfrac{1}{2}|u|^2 + p \in L^{d/2}_{\mathrm{loc}}(\Omega), \quad \text{with } d = 5, 6, \tag{1-23}$$

then one has $u \in L^\infty_{\mathrm{loc}}(\Omega)$.

Remark. Frehse and Růžička [1994] showed that if the weak solutions satisfy

$$\left(\tfrac{1}{2}|u|^2 + p\right)_+ \in L^q_{\mathrm{loc}}(\Omega) \quad \text{with } q > \tfrac{1}{2}d, \ d \geq 5,$$

and the local energy inequality (1-22), then u is regular. Compared with Frehse and Růžička's regularity condition, the regular class (1-23) is scaling-invariant with respect to system (1-21).

The remainder of the paper is organized as follows. In the next section, we recall some helpful results and give some useful auxiliary lemmas such as the decay estimate involving the Bernoulli pressure and $|u|^2$. The last section will be devoted to proving theorems.

Notation. Throughout this paper, we denote

$$B(x, \mu) = \{y \in \mathbb{R}^d \mid |x - y| \leq \mu\}, \quad B(\mu) := B(0, \mu),$$
$$Q(x, t, \mu) = B(x, \mu) \times (t - \mu^2, t), \quad Q(\mu) := Q(0, 0, \mu).$$

For $p \in [1, \infty]$, the notation $L^p((0, T); X)$ stands for the set of measurable functions f on the interval $(0, T)$ with values in X such that $\|f(t, \cdot)\|_X$ belongs to $L^p(0, T)$. For simplicity, we write

$$\|f\|_{L^{p,q}(Q(\mu))} := \|f\|_{L^p(-\mu^2, 0; L^q(B(\mu)))} \quad \text{and} \quad \|f\|_{L^p(Q(\mu))} := \|f\|_{L^{p,p}(Q(\mu))}.$$

Denote by $L^q_\sigma(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^q(\Omega)^d$, where $C^\infty_{0,\sigma}(\Omega)$ denotes the set $\{u \in C^\infty_0(\Omega)^d : \operatorname{div} u = 0\}$. The classical Sobolev space $W^{1,2}(\Omega)$ is equipped with the norm $\|f\|_{W^{1,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}$. We will also use the summation convention on repeated indices. C is an absolute constant which may be different from line to line unless otherwise stated. According to the natural scaling property of the Navier–Stokes equations [Caffarelli et al. 1982], we introduce the following dimensionless quantities for the nonstationary case

$$E(u,\mu)=\mu^{-1}\|u\|^2_{L^{\infty,2}(Q(\mu))}, \qquad E_*(u,\mu)=\mu^{-1}\|\nabla u\|^2_{L^2(Q(\mu))},$$

$$U_{p,q}(\times,\mu)=\mu^{1-\frac{2}{p}-\frac{3}{q}}\Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{p,q}(Q(\mu))}, \quad E_{p,q}(u,\mu)=\mu^{1-\frac{2}{p}-\frac{3}{q}}\|u\|_{L^{p,q}(Q(\mu))},$$

$$W_{p,q}(\times,\mu)=\mu^{2-\frac{2}{p}-\frac{3}{q}}\Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{p,q}(Q(\mu))}, P_{p,q}\Big(\frac{\Pi}{|u|},\mu\Big)=\mu^{1-\frac{2}{p}-\frac{3}{q}}\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p,q}(Q(\mu))},$$

$$P_{p,q}(\Pi-(\Pi)_{B(\mu)},\mu)=\mu^{2-\frac{2}{p}-\frac{3}{q}}\|\Pi-(\Pi)_{B(\mu)}\|_{L^{p,q}(Q(\mu))},$$

$$E_2(u,r)=\mu^{-3}\|u\|^2_{L^2(Q(\mu))},$$

and for the stationary Navier–Stokes equations,

$$\tilde{E}_p(u,\mu)=\mu^{p-d}\|u\|^p_{L^p(B(\mu))}, \qquad \tilde{E}_*(u,\mu)=\mu^{4-d}\|\nabla u\|^2_{L^2(B(\mu))},$$

$$\tilde{P}_{\frac{2d}{2+d}}(\Pi-(\Pi),\mu)=\mu^{-\frac{d-2}{2}}\|\Pi-(\Pi)\|_{L^{\frac{2d}{2+d}}(B(\mu))}, \tilde{F}_q(f,\mu)=\mu^{3q-d}\|f\|^q_{L^q(B(\mu))}.$$

2. Preliminaries and main lemma

Before proceeding further with the decay-type estimate, we shall recall the L^p-L^q-estimate of solutions to the linear Stokes system and an associated interpolation inequality.

Proposition 2.1 [Giga and Sohr 1991]. *Let Ω be a bounded domain and $r,s\in(1,\infty)$. Then for every $f\in L^r(0,T;L^s(\Omega))$, there exists a unique solution $(v,\nabla p_1)$ to the Stokes system below*:

$$\begin{cases} v_t-\Delta v+\nabla p_1=f, \operatorname{div} v=0 & \textit{in } (0,T)\times\Omega,\\ v|_{\partial\Omega}=0, & \\ (p_1)_\Omega=0, & t\in(0,T),\\ v|_{t=0}=0. & \end{cases}$$

satisfying

$$\|v_t\|_{L^r(0,T;L^s(\Omega))}+\|A_sv\|_{L^r(0,T;L^s(\Omega))}+\|\nabla p_1\|_{L^r(0,T;L^s(\Omega))}\le C\|f\|_{L^r(0,T;L^s(\Omega))},$$

where $C=C(q,s,\Omega)$.

Lemma 2.2 [Giga and Sohr 1991]. *Let $D(A_s)=\{v\in L^s_\sigma(\Omega);\ \partial_l\partial_kv\in L^s_\sigma(\Omega)^d;\ 1\le l,\ k\le d,\ v|_{\partial\Omega}=0\}$. Suppose that $1<s<3/2$, $s<h^*<\infty$, and $1<r\le\rho<\infty$. Assume that*

$$\frac{2}{r}+\frac{3}{s}=2+\frac{3}{h^*}+\frac{2}{\rho}.$$

Then there are constants C such that

$$\|v\|_{L^{\rho}(0,T;L^{h^*}(\Omega))} \le C(\|v_t\|_{L^r(0,T;L^s(\Omega))} + \|A_s v\|_{L^r(0,T;L^s(\Omega))}),$$

for all $v \in L^r(0,T;D(A_s))$ satisfying $v_t, A_s v \in L^r(0,T;L^s(\Omega))$, and $v(0)=0$.

Applying Proposition 2.1 to system (1-18), we immediately get, by Lemma 2.2,

$$\|v\|^2_{L^2(Q(\rho))} \le C(\|v_t\|^2_{L^{r,s}(Q(\rho))} + \|A_s v\|^2_{L^{r,s}(Q(\rho))}) \le C\|\omega \times u\|^2_{L^{r,s}(Q(\rho))}, \tag{2-1}$$

provided that r, s satisfy

$$\frac{2}{r} + \frac{3}{s} = \frac{9}{2}, \quad \text{with } 1 < s < \frac{6}{5}.$$

We recall a well-known interpolation inequality, which will be frequently used later. For every $2 \le \kappa \le \infty$ and $2 \le \tau \le 6$ satisfying $(2/\kappa) + (3/\tau) = \frac{3}{2}$, by Hölder's inequality, Sobolev's inequality and Young's inequality, we see that

$$\begin{aligned} \|u\|_{L^{\kappa,\tau}(Q(\mu))} &\le C\|u\|^{1-2/\kappa}_{L^{\infty,2}(Q(\mu))}\|u\|^{2/\kappa}_{L^{2,6}(Q(\mu))} \\ &\le C\|u\|^{1-2/\kappa}_{L^{\infty,2}(Q(\mu))}(\|u\|_{L^{\infty,2}(Q(\mu))} + \|\nabla u\|_{L^2(Q(\mu))})^{2/\kappa} \\ &\le C(\|u\|_{L^{\infty,2}(Q(\mu))} + \|\nabla u\|_{L^2(Q(\mu))}). \end{aligned} \tag{2-2}$$

The following lemma will play a crucial role in the proof of Theorem 1.1.

Lemma 2.3. *For $\mu \le \frac{1}{2}\rho$, there exists a constant C independent of μ and ρ such that*

$$\begin{aligned} E(u,\mu) + E_*(u,\mu) \le C\Big(\frac{\mu}{\rho}\Big)^2 E(u,\rho) \\ + \Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho)[E(u,\rho) + E_*(u,\rho)]^{1/2}, \end{aligned} \tag{2-3}$$

$$\begin{aligned} E(u,\mu) + E_*(u,\mu) \le C\Big(\frac{\mu}{\rho}\Big)^2 E(u,\rho) \\ + C\Big(\frac{\rho}{\mu}\Big)^2 P_{p^\natural,q^\natural}\Big(\frac{\Pi}{|u|}, \rho\Big)[E(u,\rho) + E_*(u,\rho)], \end{aligned} \tag{2-4}$$

where (p,q) and $(p^\natural, q^\natural)$ satisfy

$$\frac{2}{p} + \frac{3}{q} = \frac{7}{2} \quad \textit{and} \quad \frac{2}{p^\natural} + \frac{3}{q^\natural} = 2 \quad \textit{with } 1 \le p \le 2,\ 1 \le p^\natural \le \infty. \tag{2-5}$$

Proof. Consider the following smooth cutoff function

$$\psi(x,t) = \begin{cases} 1, & (x,t) \in Q(\rho/2), \\ 0, & (x,t) \in Q^c(\rho); \end{cases}$$

which satisfies $0 \le \psi(x,t) \le 1$, $|\psi_t(x,t)|+|\Delta\psi(x,t)| \le C/\rho^2$ and $|\nabla\psi(x)| \le C/\rho$. We denote the backward heat kernel

$$\Gamma(x,t) = \frac{1}{4\pi(\mu^2-t)^{3/2}} e^{-\frac{|x|^2}{4(\mu^2-t)}}.$$

Plugging $\phi = \psi(x,t)\Gamma(x,t)$ into the local energy inequality (1-9) and using that $\Gamma_t + \Delta\Gamma = 0$, we know that

$$\begin{aligned}
&\sup_{-\rho^2\le t\le 0}\int_{B(\rho)} |u(x,t)|^2\Gamma(t,x)\psi(x,t)\,dx + 2\iint_{Q(\rho)} |\nabla u|^2\Gamma(x,s)\psi(x,s)\,dx\,ds \\
&\le \iint_{Q(\rho)} |u|^2[\Gamma(x,s)\psi_s(x,s)+\Gamma(x,s)\Delta\psi(x,s)+2\nabla\psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds \\
&\qquad + \iint_{Q(\rho)} (\Pi-(\Pi)_{B(\rho)})u\cdot[\Gamma\nabla\psi(x,s)+\psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds.
\end{aligned} \tag{2-6}$$

This inequality in turn implies

$$\begin{aligned}
&\sup_{-\mu^2\le t\le 0}\int_{B(\mu)} |u(x,s)|^2\Gamma(x,t)\,dx + 2\iint_{Q(\mu)} |\nabla u|^2\Gamma(x,s)\,dx\,ds \\
&\le \iint_{Q(\rho)\setminus Q(\rho/2)} |u|^2[\Gamma(x,s)\psi_s(x,s)+\Gamma(x,s)\Delta\psi(x,s)+2\nabla\psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds \\
&\qquad + \iint_{Q(\rho)} (\Pi-(\Pi)_{B(\rho)})u\cdot[\Gamma\nabla\psi(x,s)+\psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds,
\end{aligned} \tag{2-7}$$

where we have used the fact that $\operatorname{supp}(\psi_s, \partial_i\psi) \subset Q(2\rho)\setminus Q(\rho)$.

To proceed further, we list some properties of the test function $\phi(x,t)$ whose deduction rests on elementary calculations.

(i) There is a constant $c > 0$ independent of μ such that, for any $(x,t) \in Q(\mu)$,

$$\Gamma(x,t) \ge c\mu^{-3}.$$

(ii) It is clear that, for any $(x,t) \in Q(\rho)$,

$$|\Gamma(x,t)\psi(x,t)| \le C\mu^{-3},\ |\nabla\psi(x,t)\Gamma(x,t)| \le C\mu^{-4},$$

and

$$\partial_i\Gamma(x,t) = -\frac{1}{4\pi(\mu^2-t)^{3/2}} e^{-\frac{|x|^2}{4(\mu^2-t)}} \frac{2x_i}{4(\mu^2-t)},$$

which in turn yields

$$|\psi(x,t)\nabla\Gamma(x,t)| \le C\mu^{-4}.$$

(iii) For any $(x,t) \in Q(\rho)\setminus Q(\rho/2)$, one can deduce that

$$\Gamma(x,t) \le C\rho^{-3}, \quad \partial_i\Gamma(x,t) \le C\rho^{-4},$$

which leads to

$$|\Gamma(x,t)\partial_t\psi(x,t)|+|\Gamma(x,t)\Delta\psi(x,t)|+|\nabla\psi(x,t)\nabla\Gamma(x,t)|\le C\rho^{-5}.$$

Take $1/\kappa=1-1/p$ and $1/\tau=1-1/q$. Then, in light of (2-7), the Hölder inequality, (2-2) and (2-5), we see that

$$\begin{aligned}E(u,\mu)+E_*(u,\mu)&\le C\Big(\frac{\mu}{\rho}\Big)^2E_2(u,\rho)+\Big(\frac{\rho}{\mu}\Big)^2P_{p,q}(\Pi-(\Pi)_{B(\rho)},\rho)E_{\kappa,\tau}(u,\rho)\\&\le C\Big(\frac{\mu}{\rho}\Big)^2E(u,\rho)+\Big(\frac{\rho}{\mu}\Big)^2P_{p,q}(\Pi-(\Pi)_{B(\rho)},\rho)[E(u,\rho)+E_*(u,\rho)]^{1/2},\end{aligned}$$

which means (2-3).

Choose $1/p^\sharp=1-1/p^\natural$ and $1/q^\sharp=1-1/q^\natural$. Then we derive from (2-5) that

$$\frac{2}{2p^\sharp}+\frac{3}{2q^\sharp}=\frac{3}{2}.$$

This together with Hölder's inequality and interpolation inequality (2-2) yields that

$$\begin{aligned}\iint_{Q(\rho)}|\Pi||u|\,dx\,dt&\le\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p^\natural,q^\natural}(Q(\rho))}\|u\|^2_{L^{2p^\sharp,2q^\sharp}(Q(\rho))}\\&\le\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p^\natural,q^\natural}}(\|u\|^2_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|^2_{L^2(Q(\rho))}).\end{aligned}$$

Collecting these estimates leads to (2-4). This completes the proof. □

Next, we derive the decay estimate of the Bernoulli pressure.

Lemma 2.4. *Let $0<4\mu\le\rho$ and i, j, m, n be defined as the limiting case of* (1-15) *and* (1-17). *There exists an absolute constant C independent of μ and ρ such that*

$$P_{r,s'}(\Pi-(\Pi)_{B(\mu)},\mu)\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}U_{i,j}(\times,\rho)E_*^{1/2}(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^{3-\frac{2}{r}}P_{r,s'}(\Pi-(\Pi)_{B(\rho)},\rho),\tag{2-8}$$

$$P_{r,s'}(\Pi-(\Pi)_{B(\mu)},\mu)\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}W_{m,n}(\times,\rho)[E_*(u,\rho)+E(u,\rho)]^{1/2}+C\Big(\frac{\mu}{\rho}\Big)^{3-\frac{2}{r}}P_{r,s'}(\Pi-(\Pi)_{B(\rho)},\rho),\tag{2-9}$$

where the pair (r,s') satisfies

$$\frac{2}{r}+\frac{3}{s'}=\frac{7}{2}\quad\textit{with } 1<r<\frac{4}{3},\ \frac{3}{2}<s'<2.$$

Proof. Utilizing that $(p_1)_{B(\rho)} = 0$ and the Poincaré–Sobolev inequality and applying Proposition 2.1 to system (1-18), we get

$$\|p_1\|_{L^{r,s'}(Q(\rho))} \le C\|\nabla p_1\|_{L^{r,s}(Q(\rho))} \le C\|\omega \times u\|_{L^{r,s}(Q(\rho))}, \tag{2-10}$$

where

$$\frac{3}{s} = 1 + \frac{3}{s'}. \tag{2-11}$$

Since $\Delta p_2 = 0$ on $B(\rho/4)$, then, by the interior estimate of harmonic functions and Hölder's inequality, we see that, for every $x_0 \in B(\rho/4)$,

$$|\nabla p_2(x_0)| \le \frac{C}{\rho^{3+1}}\|p_2\|_{L^1(B_{x_0}(\rho/4))} \le \frac{C}{\rho^{3+1}}\|p_2\|_{L^1(B(\rho/2))} \le \frac{C}{\rho^{3+1}}\rho^{3(1-1/s')}\|p_2\|_{L^{s'}(B(\rho/2))},$$

which in turn implies

$$\|\nabla p_2\|^{s'}_{L^\infty(B(\rho/4))} \le C\rho^{-3-s'}\|p_2\|^{s'}_{L^{s'}(B(\rho/2))}.$$

The latter inequality together with the mean value theorem leads to

$$\begin{aligned}\|p_2 - (p_2)_{B(\mu)}\|^{s'}_{L^{s'}(B(\mu))} &\le C\mu^3\|p_2 - (p_2)_{B(\mu)}\|^{s'}_{L^\infty(B(\mu))}\\ &\le C\mu^3(2\mu)^{s'}\|\nabla p_2\|^{s'}_{L^\infty(B(\rho/4))}\\ &\le C\Big(\frac{\mu}{\rho}\Big)^{3+s'}\|p_2\|^{s'}_{L^{s'}(B(\rho/2))}.\end{aligned}$$

Integrating this inequality in time, we obtain

$$\|p_2 - (p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^{\frac{3}{s'}+1}\|p_2\|_{L^{r,s'}(Q(\rho/2))}.$$

With the help of the triangle inequality and (2-10), we infer that

$$\begin{aligned}\|p_2\|_{L^{r,s'}(Q(\rho/2))} &\le \|\Pi - (\Pi)_{B_{\rho/2}}\|_{L^{r,s'}(Q(\rho/2))} + \|p_1\|_{L^{r,s'}(Q(\rho/2))}\\ &\le \|\Pi - (\Pi)_{B_\rho}\|_{L^{r,s'}(Q(\rho/2))} + \|p_1\|_{L^{r,s'}(Q(\rho))}\\ &\le \|\Pi - (\Pi)_{B_\rho}\|_{L^{r,s'}(Q(\rho))} + \|\omega \times u\|_{L^{r,s}(Q(\rho))},\end{aligned} \tag{2-12}$$

which in turns yields

$$\|p_2 - (p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^{\frac{3}{s'}+1}(\|\Pi - (\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))} + \|\omega \times u\|_{L^{r,s}(Q(\rho))}).$$

It follows from (2-10) and the last estimate that

$$
\begin{aligned}
(2\text{-}13)\quad & \|\Pi-(\Pi)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} \\
&\le \|p_1-(p_1)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}+\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} \\
&\le \|p_1\|_{L^{r,s'}(Q(\mu))}+\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} \\
&\le \|p_1\|_{L^{r,s'}(Q(\rho))}+\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} \\
&\le C\|\omega\times u\|_{L^{r,s}(Q(\rho))}+C\Big(\frac{\mu}{\rho}\Big)^{\frac{3}{s'}+1}\|\Pi-(\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}.
\end{aligned}
$$

Now, we bound $\omega\times u$ in two different ways.

Case I: The Hölder inequality and hypothesis (1-15) in Theorem 1.3 ensure that

$$
\begin{aligned}
(2\text{-}14)\qquad \|\omega\times u\|_{L^{r,s}(Q(\rho))} &\le \Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{i,j}(Q(\rho))}\|\omega\|_{L^2(Q(\rho))} \\
&\le C\Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{i,j}(Q(\rho))}\|\nabla u\|_{L^2(Q(\rho))},
\end{aligned}
$$

where the pair (r,s) satisfies

$$\frac{2}{r}+\frac{3}{s}=\frac{9}{2},$$

and

$$(2\text{-}15)\qquad \frac{1}{2}<\frac{1}{r}=\frac{1}{2}+\frac{1}{i}<1,\quad \frac{2}{3}<\frac{1}{s}=\frac{1}{2}+\frac{1}{j}<1,$$

which guarantees that Proposition 2.1 and Lemma 2.2 work. Substituting (2-14) into (2-13), we conclude that

$$
\begin{aligned}
\mu^{-3/2}\|\Pi-(\Pi)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} &\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}\rho^{-1}\Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{i,j}(Q(\rho))}\rho^{-1/2}\|\nabla u\|_{L^2(Q(\rho))} \\
&\quad +C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}\rho^{-3/2}\|\Pi-(\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}.
\end{aligned}
$$

where we have used the fact $2/r+3/s'=7/2$.

Case II: Using Hölder's inequality, (1-16) and (2-2), we see that

$$
\begin{aligned}
(2\text{-}16)\quad \|\omega\times u\|_{L^{r,s}(Q(\rho))} &\le \Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{m,n}(Q(\rho))}\|u\|_{L^{\kappa,\tau}(Q(\rho))} \\
&\le \Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{m,n}(Q(\rho))}(\|u\|_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|_{L^2(Q(\rho))}),
\end{aligned}
$$

where the pair (r,s) satisfies

$$\frac{2}{r}+\frac{3}{s}=\frac{9}{2}.$$

Just as (2-15), it suffices to verify that

$$\frac{1}{2}<\frac{1}{r}=\frac{1}{m}+\frac{1}{\kappa}<1 \quad\text{and}\quad \frac{2}{3}<\frac{1}{s}=\frac{1}{n}+\frac{1}{\tau}<1. \tag{2-17}$$

Indeed, for $1<m\le 2$, we choose

$$\kappa=\frac{3m}{2m-2}\quad\text{and}\quad \tau=\frac{18m}{m+8}.$$

For $2<m<4$, we pick up $\kappa=2,\quad \tau=6$.

Inserting (2-16) into (2-13), we know that

$$\begin{aligned}\mu^{-3/2}&\|\Pi-(\Pi)_{B(\mu)}\|_{L^{r,s'}(B(\mu))}\\ &\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}\rho^{-1}\Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{m,n}(Q(\rho))}\rho^{-1/2}(\|u\|_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|_{L^2(Q(\rho))})\\ &\qquad+C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}\rho^{-3/2}\|\Pi-(\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}.\end{aligned}$$

This finishes the proof. □

Taking full advantage of the interior estimate of harmonic functions, we can extend Lemma 2.1 in [Wolf 2008] and present its proof arguing as with the heat equation.

Lemma 2.5. *Assume that b is the solution of* (1-19). *Then, for $\mu\le\rho/32$, there is a constant C independent of μ and ρ such that*

$$\mu^{-3}\|b\|^2_{L^2(Q(\mu))}\le C\Big(\frac{\mu}{\rho}\Big)^2\big(\rho^{-3}\|b\|^2_{L^2(Q(\rho/2))}+C\rho^{-3}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big), \tag{2-18}$$

where the pair (r,s') has been defined as in Lemma 2.4.

Proof. Consider the following smooth cutoff functions:

$$\xi(t)=\begin{cases}1, & t\ge-(\rho/8)^2,\\ 0, & t\le-(\rho/4)^2;\end{cases}\quad\text{and}\quad \eta(x)=\begin{cases}1, & x\in B(\rho/8),\\ 0, & x\in B^c(\rho/4),\end{cases}$$

which satisfy

$$0\le\xi(t),\eta(x)\le1,\quad |\xi'(t)|\le\frac{C}{\rho^2}\quad\text{and}\quad |\nabla\eta(x)|\le\frac{C}{\rho}.$$

Taking the inner product of (1-19) with $\xi^2\eta^2 b$ over $(-(\rho/4)^2,t)\times B(\rho/4)$, $(t\le0)$, we arrive at

$$\int_{-(\rho/4)^2}^t\int_{B(\rho/4)}\xi^2\eta^2 bb_s\,dx\,ds-\int_{-(\rho/4)^2}^t\int_{B(\rho/4)}\xi^2\eta^2 b\Delta b\,dx\,ds =-\int_{-(\rho/4)^2}^t\int_{B(\rho/4)}\xi^2\eta^2 b\nabla p_2\,dx\,ds.$$

Integrating by parts and the Cauchy–Schwarz inequality, we infer that

$$\begin{aligned}
&\frac{1}{2}\int_{B(\rho/4)}\xi^2(t)\eta^2(x)b^2(t,x)\,dx+\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi^2(s)\eta^2(x)|\nabla b|^2\,dx\,ds\\
&=\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi'\xi\eta^2b^2\,dx\,ds-2\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi^2\nabla\eta\eta b\nabla b\,dx\,ds\\
&\qquad\qquad-\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi^2\eta^2b\nabla p_2\,dx\,ds\\
&\le C\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}\xi(\xi|\nabla\eta|^2+|\xi'|\eta^2)b^2dx\,ds+\frac{1}{2}\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}\xi^2(s)\eta^2(x)|\nabla b|^2dx\,ds\\
&\qquad+C\left(\int_{-(\rho/4)^2}^{0}\left(\int_{B(\rho/4)}\xi^2\eta^2|\nabla p_2|^2\,dx\right)^{1/2}ds\right)^2+\frac{1}{4}\|\xi\eta b\|^2_{L^{\infty,2}(Q(\rho/4))},
\end{aligned}$$

which in turn implies

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-(\rho/4)^2\le t<0}\frac{1}{2}\int_{B(\rho/4)}\xi^2(t)\eta^2(x)b^2(t,x)\,dx+\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}\xi^2(s)\eta(x)^2|\nabla b|^2dx\,ds\\
&\le\frac{C}{\rho^2}\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}|b|^2dx\,ds+C\left(\int_{-(\rho/4)^2}^{0}\left(\int_{B(\rho/4)}|\nabla p_2|^2dx\right)^{1/2}ds\right)^2+\frac{1}{4}\|\xi\eta b\|^2_{L^{\infty,2}(Q(\rho/4))}.
\end{aligned}$$

Consequently,

$$\|b\|^2_{L^{\infty,2}(Q(\rho/8))}+\|\nabla b\|^2_{L^2(Q(\rho/8))}\le C\rho^{-2}\|b\|^2_{L^2(Q(\rho/4))}+C\|\nabla p_2\|^2_{L^{1,2}(Q(\rho/4))}.\tag{2-19}$$

Notice that the system (1-19) is linear, thus, a slight variant of the proof above provides the estimates

$$\|\nabla b\|^2_{L^{\infty,2}(Q(\rho/16))}+\|\nabla^2 b\|^2_{L^2(Q(\rho/16))}\le C\rho^{-2}\|\nabla b\|^2_{L^2(Q(\rho/8))}+\|\nabla^2 p_2\|^2_{L^{1,2}(Q(\rho/8))},$$

and

$$\|\nabla^2 b\|^2_{L^{\infty,2}(Q(\rho/32))}+\|\nabla^3 b\|^2_{L^2(Q(\rho/32))}\le C\rho^{-2}\|\nabla^2 b\|^2_{L^2(Q(\rho/16))}+\|\nabla^3 p_2\|^2_{L^{1,2}(Q(\rho/16))}.$$

Collecting the above estimates, we find

$$\begin{aligned}
&\|\nabla^2 b\|^2_{L^{\infty,2}(Q(\rho/32))}+\|\nabla^3 b\|^2_{L^2(Q(\rho/32))}\\
&\quad\le C\rho^{-2}\big\{C\rho^{-2}\|\nabla b\|^2_{L^2(Q(\rho/8))}+\|\nabla^2 p_2\|^2_{L^{1,2}(Q(\rho/8))}\big\}+C\|\nabla^3 p_2\|^2_{L^{1,2}(Q(\rho/16))}\\
&\quad\le C\rho^{-2}\big\{C\rho^{-2}\big[C\rho^{-2}\|b\|^2_{L^2(Q(\rho/4))}+\|\nabla p_2\|^2_{L^{1,2}(Q(\rho/4))}\big]+C\|\nabla^2 p_2\|^2_{L^{1,2}(Q(\rho/8))}\big\}\\
&\qquad\qquad+C\|\nabla^3 p_2\|^2_{L^{1,2}(Q(\rho/16))}.
\end{aligned}\tag{2-20}$$

By virtue of the interior estimate of harmonic functions, for every $k\in\mathbb{N}^+$, we have

$$|\nabla^k p_2(x_0)|\le C\rho^{-3-k}\|p_2\|_{L^1(B_{x_0}(\rho/4))}\le\rho^{-3-k}\|p_2\|_{L^1(B(\rho/2))},$$

for any $x_0 \in B(\rho/4)$, from which it follows that

$$\begin{aligned}\|\nabla^{k+1} p_2\|_{L^2(B(\rho/4))} &\le C\rho^{\frac{3}{2}} \|\nabla^{k+1} p_2\|_{L^\infty(B(\rho/4))}\\ &\le C\rho^{\frac{3}{2}} \rho^{-(k+1+3)} \|p_2\|_{L^1(B(\rho/2))}\\ &\le C\rho^{-(k+1)} \rho^{\frac{3}{2}-\frac{3}{s'}} \|p_2\|_{L^{s'}(B(\rho/2))}.\end{aligned}$$

Integrating the last inequality in time yields

$$\|\nabla^{k+1} p_2\|_{L^{r,2}(Q(\rho/4))} \le C\rho^{-(k+1)} \rho^{\frac{3}{2}-\frac{3}{s'}} \|p_2\|_{L^{r,s'}(Q(\rho/2))}.$$

Utilizing Hölder's inequality, we discover

$$\|\nabla^{k+1} p_2\|_{L^{1,2}(Q(\rho/4))} \le C\rho^{-(k+1)} \|p_2\|_{L^{r,s'}(Q(\rho/2))},$$

where we have used the fact $2/r + 3/s' = 7/2$. Plugging this inequality into bounds (2-19) and (2-20) gives

$$\|b\|^2_{L^{\infty,2}(Q(\rho/8))} + \|\nabla b\|^2_{L^2(Q(\rho/8))} \le \frac{C}{\rho^2} \|b\|^2_{L^2(Q(\rho/4))} + \frac{C}{\rho^2} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))},$$

and

$$\|\nabla^2 b\|^2_{L^{\infty,2}(Q(\rho/32))} + \|\nabla^3 b\|^2_{L^2(Q(\rho/32))} \le \frac{C}{\rho^6} \|b\|^2_{L^2(Q(\rho/4))} + \frac{C}{\rho^6} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}.$$

By the Gagliardo–Nirenberg inequality and the latter inequalities, we infer that

$$\begin{aligned}\|b\|^2_{L^2(Q(\mu))} &\le C\mu^5 \|b\|^2_{L^\infty(Q(\rho/32))}\\ &\le C\mu^5 \big(\|b\|^{2\cdot(1/4)}_{L^{\infty,2} Q(\rho/32)} \|\nabla^2 b\|^{2\cdot(3/4)}_{L^{\infty,2}(Q(\rho/32))} + \frac{C}{\rho^3} \|b\|^2_{L^{\infty,2}(Q(\rho/32))}\big)\\ &\le C\mu^5 \big(\rho^{-2} \|b\|^2_{L^2(Q(\rho/4))} + C\rho^{-2} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big)^{1/4}\\ &\quad \times \big(\rho^{-6} \|b\|^2_{L^2(Q(\rho/4))} + C\rho^{-6} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big)^{3/4}\\ &\qquad + C\mu^5 \frac{C}{\rho^3} \big(\rho^{-2} \|b\|^2_{L^2(Q(\rho/4))} + C\rho^{-2} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big)\\ &\le C\Big(\frac{\mu}{\rho}\Big)^5 \big(\|b\|^2_{L^2(Q(\rho/4))} + C\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big),\end{aligned}$$

which means that

$$\mu^{-3} \|b\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^2 \big(\rho^{-3} \|b\|^2_{L^2(Q(\rho/2))} + C\rho^{-3} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big),$$

which is the desired result. □

This lemma entails the desired decay estimate (1-20), that is,

$$E_2(u,\mu) \le C\Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} + C\Big(\frac{\mu}{\rho}\Big)^2[E_2(u,\rho)+P^2_{r,s'}(\Pi-(\Pi)_{B_\rho},\rho)]. \tag{2-21}$$

Indeed, it is enough to bound the right hand of the following inequality:

$$\begin{aligned}\mu^{-3}\|u\|^2_{L^2(Q(\mu))} &\le \mu^{-3}\|v\|^2_{L^2(Q(\mu))} + \mu^{-3}\|b\|^2_{L^2(Q(\mu))}\\ &\le \Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|v\|^2_{L^2(Q(\rho))} + \mu^{-3}\|b\|^2_{L^2(Q(\mu))}\\ &\le C\Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} + \mu^{-3}\|b\|^2_{L^2(Q(\mu))},\end{aligned}$$

where we have used (2-1). To end this, first, by triangle inequality and (2-1) again, we see that

$$\begin{aligned}\rho^{-3}\|b\|^2_{L^2(Q(\rho))} &\le \rho^{-3}\|u\|^2_{L^2(Q(\rho))} + \rho^{-3}\|v\|^2_{L^2(Q(\rho))}\\ &\le \rho^{-3}\|u\|^2_{L^2(Q(\rho))} + C\rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))}.\end{aligned}$$

Then, we insert the latter estimate and (2-12) into (2-18) to obtain

$$\mu^{-3}\|b\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^2\big(\rho^{-3}\|u\|^2_{L^2(Q(\rho))} + \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} + C\rho^{-3}\|\Pi-(\Pi)_{B_\rho}\|^2_{L^{r,s'}(Q(\rho))}\big).$$

This inequality yields the desired estimate (2-21).

Before we state the auxiliary results to the stationary Navier–Stokes equations, we first recall the Caffarelli–Kohn–Nirenberg regular condition below to the steady Navier–Stokes equations.

Proposition 2.6 [Struwe 1995; Dong and Strain 2012; Wang and Wu 2014]. *Suppose (u, p) is a suitable weak solution to* (1-21) *and the external force $f \in L^q(\Omega)$ with $q > \frac{1}{2}d$. Then the origin* 0 *is a regular point for $u(x)$ if the following condition holds*:

$$\limsup_{\mu\to 0}\frac{1}{\mu^{d-4}}\int_{B(\mu)}|\nabla u|^2\,dx < \varepsilon, \quad d=5,6, \tag{2-22}$$

for a universal constant $\varepsilon > 0$.

To show Theorem 1.4, we need to prove the following lemma:

Lemma 2.7. *Let* $0 < 2r < \rho$. *It holds that*

$$\begin{aligned}\text{(2-23)}\quad &\tilde{E}(u,\mu)+\tilde{E}_*(u,\mu)\\ &\le C\Big(\frac{\mu}{\rho}\Big)^2\tilde{E}(u,\rho)\\ &+C\Big(\frac{\rho}{\mu}\Big)^{d-3}\big(\tilde{P}_{\frac{2d}{d+2}}(\Pi-(\Pi)_{B(\rho)},\rho)+\tilde{F}_{\frac{2d}{d+2}}(f,\rho)\big)[\tilde{E}(u,\rho)+\tilde{E}_*(u,\rho)]^{1/2},\end{aligned}$$

where the constant C *is independent of* μ *and* ρ.

Proof. The conclusion can be derived by a slight change of the proof of Lemma 2.3 as follows. In the spirit of the backward heat kernel for the time-dependent case, we modify slightly the fundamental solution of Laplace equations to set

$$\Gamma(x)=\frac{1}{(\mu^2+|x|^2)^{(d-2)/2}},\quad d=5,6.$$

An easy computation gives

$$\partial_i\Gamma(x)=-\frac{(d-2)x_i}{(\mu^2+|x|^2)^{d/2}}\quad\text{and}\quad\Delta\Gamma(x)=\frac{-d(d-2)\mu^2}{(\mu^2+|x|^2)^{(d+2)/2}}.$$

Consider the smooth cutoff function

$$\eta(x)=\begin{cases}1, & x\in B(\rho/2),\\ 0, & x\in B^c(\rho),\end{cases}$$

which satisfies

$$0\le\eta(x)\le 1,\quad |\nabla\eta(x)|\le\frac{C}{\rho},\quad\text{and}\quad|\Delta\eta(x)|\le\frac{C}{\rho^2}.$$

The desired estimate turns out to be a consequence of the following properties of the test function $\eta(x)\Gamma(x)$:

(i) For every $x\in B(\mu)$, straightforward calculations yield

$$-\Delta\Gamma\ge C\mu^{-d},\quad\Gamma\ge C\mu^{-(d-2)}.$$

(ii) For every $x\in B(\rho)$, it is easy to verify that

$$|\eta(x)\Gamma|\le C\mu^{-(d-2)},\quad|\eta(x)\nabla\Gamma|+|\Gamma\nabla\eta(x)|\le C\mu^{-(d-1)},$$

(iii) For every $\rho/2\le|x|\le\rho$, we know that

$$|\Gamma\Delta\eta(x)|+|\nabla\eta(x)\cdot\nabla\Gamma|\le C\rho^{-d}.$$

Inserting $\phi = \eta(x)\Gamma(x)$ into the local energy inequality (1-22), we see that

$$
\begin{aligned}
-\int_{B(\rho)} |u|^2 \eta \Delta\Gamma\, dx + 2\int_{B(\rho)} |\nabla u|^2 \eta\Gamma\, dx \\
\le \int_{B(\rho)} |u|^2 (\Gamma\Delta\eta + 2\nabla\eta\cdot\nabla\Gamma)\, dx + 2\int_{B(\rho)} (\Pi - (\Pi)_{B(\rho)}) u\cdot(\eta\nabla\Gamma + \Gamma\nabla\eta) \\
+2\int_{B(\rho)} f\cdot u\eta\Gamma\, dx.
\end{aligned}
$$

This inequality implies

$$
\begin{aligned}
&-\int_{B(\mu)} |u|^2 \Delta\Gamma\, dx + 2\int_{B(\mu)} |\nabla u|^2 \Gamma\, dx \\
&\le \int_{B(\rho)\setminus B(\rho/2)} |u|^2 (\Gamma\Delta\eta + 2\nabla\eta\cdot\nabla\Gamma)\, dx + 2\int_{B(\rho)} (\Pi - (\Pi)_{B(\rho)}) u\cdot(\eta\nabla\Gamma + \Gamma\nabla\eta)\, dx \\
&\qquad +2\int_{B(\rho)} f\cdot u\eta\Gamma\, dx.
\end{aligned}
$$

The property of test functions and Hölder's inequality yield that

$$
\begin{aligned}
&\frac{C}{\mu^{d-2}}\int_{B(\mu)} |u|^2\, dx + \frac{C}{\mu^{d-4}}\int_{B(\mu)} |\nabla u|^2\, dx \\
&\le \Big(\frac{\mu}{\rho}\Big)^2 \frac{1}{\rho^{d-2}} \int_{B(\rho)\setminus B(\rho/2)} |u|^2\, dx \\
&\quad + C\Big(\frac{\rho}{\mu}\Big)^{d-3} \Big(\frac{1}{\rho^{\frac{d^2-2d}{d+2}}} \int_{B(\rho)} |(\Pi - (\Pi)_{B(\rho)})|^{\frac{2d}{d+2}}\, dx \Big)^{\frac{d+2}{2d}} \Big(\frac{1}{\rho^{\frac{d^2-4d}{d-2}}} \int_{B(\rho)} |u|^{\frac{2d}{d-2}}\, dx \Big)^{\frac{d-2}{2d}} \\
&\quad + C\Big(\frac{\rho}{\mu}\Big)^{d-4} \Big(\frac{1}{\rho^{\frac{d^2-4d}{d+2}}} \int_{B(\rho)} |f|^{\frac{2d}{d+2}}\, dx \Big)^{\frac{d+2}{2d}} \Big(\frac{1}{\rho^{\frac{d^2-4d}{d-2}}} \int_{B(\rho)} |u|^{\frac{2d}{d-2}}\, dx \Big)^{\frac{d-2}{2d}}.
\end{aligned}
$$

Combining this estimate with the Sobolev embedding

$$
\|u\|_{L^{2d/(d-2)}(B(\rho))} \le C(\|\nabla u\|_{L^2(B(\rho))} + \rho^{-1}\|u\|_{L^2(B(\rho))}), \quad x\in\mathbb{R}^d \tag{2-24}
$$

with $d = 5, 6$, we derive the desired estimate (2-23). □

3. Proofs of theorems

This section is devoted to the proofs of Theorem 1.1–1.4.

Proof of Theorem 1.1. In the light of Hölder's inequality, it suffices to deal with the case $2/p + 3/q = 7/2$. According to the hypothesis of Theorem 1.1, we know that there exists a constant $r_0 > 0$ such that

$$P_{p,q}(\Pi - (\Pi)_{B(\mu)}, \mu) \le \varepsilon_1, \quad \text{for any } \mu \le r_0.$$

Before going further, we set

$$G_1(\mu) = E(u,\mu) + E_*(u,\mu) \quad \text{and} \quad \lambda = \mu/\rho \quad (\lambda \le 1/4).$$

By (2-3) in Lemma 2.3 and Young's inequality, we derive that

$$\begin{aligned}
G_1(\mu) &\le C\Big(\frac{\mu}{\rho}\Big)^2 E(u,\rho) + C\Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho)[E(u,\rho) + E_*(u,\rho)]^{1/2} \\
&\le C\Big(\frac{\mu}{\rho}\Big)^2 G_1(\rho) + C\Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho) G_1(\rho) + \Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho) \\
&\le C_2\lambda^2 G_1(\rho) + C_1\lambda^{-2}\varepsilon_1 G_1(\rho) + \lambda^{-2}\varepsilon_1.
\end{aligned}$$

Choosing λ, ε_1 such that $q = 2C_2\lambda^2 < 1$ and $\varepsilon_1 = \min\{q\lambda^2/(2C_1), (1-q)\lambda^3\varepsilon/2\}$, we obtain

$$G_1(\lambda\rho) \le qG_1(\rho) + \lambda^{-2}\varepsilon_1$$

Iterating the latter inequality, we deduce that

$$G_1(\lambda^k\rho) \le q^k G_1(\rho) + \tfrac{1}{2}\lambda\varepsilon.$$

From the definition of $G_1(\mu)$, there exists a positive number K_0 such that

$$q^{K_0} G_1(r_0) \le 2\frac{C(\|u\|_{L^\infty L^2}, \|\nabla u\|_{L^2})}{r_0} q^{K_0} \le \frac{1}{2}\lambda\varepsilon.$$

Let $r_2 := \lambda^{K_0} r_0$. For every $0 < r \le r_2$, there exists $k \ge K_0$ such that $\lambda^{k+1} r_0 \le r \le \lambda^k r_0$. An easy computation yields that

$$E_*(r) \le \frac{1}{\lambda^{k+1} r_0} \iint_{Q(\lambda^k r_0)} |\nabla u|^2 \, dx\, dt \le \frac{1}{\lambda} G_1(\lambda^k r_0) \le \frac{1}{\lambda}\Big(q^{k-K_0} q^{K_0} G_1(r_0) + \frac{1}{2}\lambda\varepsilon\Big) \le \varepsilon.$$

This together with (1-3) completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Thanks to Hölder's inequality, without loss of generality, we just consider the endpoint case $2/p^\natural + 3/q^\natural = 2$. With the estimate (2-4) in hand, arguing as with the iteration method above, we can finish the proof. □

Proof of Theorem 1.4. It follows from Hölder's inequality that

$$\tilde{F}_p(f,\mu) = \mu^{3p-d} \int_{B(\mu)} |f(x)|^p \, dx \le \mu^{3p - \frac{p}{q}d} \Big(\int_\Omega |f(x)|^q \, dx \Big)^{p/q}, \tag{3-1}$$

which together with the integrability hypothesis on the force f implies that

$$\tilde{F}_p(f,\mu) \text{ tends to 0 as } \mu \to 0,$$

where $p < \frac{1}{2}d < q$. Therefore, we see that there is a constant r_1 such that for any $\mu \le r_1$, $\tilde{F}_{2d/(d+2)}(f, \mu) \le \varepsilon_2$. Owing to the assumption, there exists a constant $r_2 \le r_1$ such that

$$\tilde{P}_{2d/(d+2)}(\Pi - (\Pi)_{B(\mu)}, \mu) \le \varepsilon_2, \quad \text{for any } \mu \le r_2.$$

Based on this inequality and (2-23) in Lemma 2.7, we complete the proof in the same way as in the proof of Theorem 1.1. □

Proof of Theorem 1.3. This will occupy the remainder of the section. We start with some preliminaries. Recall the symbols r, s' defined ias n Lemma 2.4, which correspond to the borderline cases of (1-15) and (1-17). Set

$$\frac{1}{r^\sharp} = 1 - \frac{1}{r} \quad \text{and} \quad \frac{1}{s^\sharp} = 1 - \frac{1}{s'}.$$

Then it is obvious that

$$\frac{2}{r^\sharp} + \frac{3}{s^\sharp} = \frac{3}{2} \quad \text{with } r^\sharp \in [2, \infty), s^\sharp \in (2, 6).$$

It follows from (2-2) that

$$\|u\|_{L^{r^\sharp,s^\sharp}(Q(\rho))} \le C(\|u\|_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|_{L^2(Q(\rho))}).$$

Consider the usual cutoff function $\varphi(x, t) \in C_0^\infty(Q(2\mu))$ satisfying $\varphi \equiv 1$ in $Q(\mu)$, $0 \le \varphi \le 1$, $|\nabla\varphi| \le C\mu^{-1}$ and $|\partial_t\varphi| + |\Delta\varphi| \le C\mu^{-2}$. By the divergence-free condition div $= 0$, Hölder's inequality and the latter inequality, for $32\mu \le \rho$, we infer that

$$\begin{aligned}\iint_{Q(2\mu)} u\cdot\nabla\varphi\Pi \, dx\, ds &= \iint_{Q(2\mu)} u\cdot\nabla\varphi(\Pi - (\Pi)_{B(2\mu)}) \, dx\, ds \\ &\le C\mu^{-1}\|\Pi - (\Pi)_{B(2\mu)}\|_{L^{r,s'}(Q(2\mu))}\|u\|_{L^{r^\sharp,s^\sharp}(Q(2\mu))} \\ &\le C\mu^{-1}\|\Pi - (\Pi)_{B(2\mu)}\|_{L^{r,s'}(Q(2\mu))}(\|u\|_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|_{L^2(Q(\rho))}).\end{aligned}$$

Choosing $\varphi(x, t)$ as the test function in (1-9) and using the latter relation, we see that

$$\begin{aligned}&E(u, \mu) + E_*(u, \mu) \\ &\quad \le E_2(u, 2\mu) + P_{r,s'}(\Pi - (\Pi), 2\mu)\Big(\frac{\rho}{\mu}\Big)^{1/2}(E(u, \rho) + E_*(u, \rho))^{1/2}.\end{aligned} \tag{3-2}$$

This concludes the preliminaries. The proof proper is divided into two steps.

(1) Substituting (2-14) into (2-21), we have

$$E_2(u,\mu) \le C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)E_*(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^2[E_2(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)]. \tag{3-3}$$

Plugging (3-3) and (2-8) into (3-2), we infer that

$$E(u,\mu)+E_*(u,\mu)\le C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)E_*(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^2[E(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)]$$
$$+\Big[C\Big(\frac{\rho}{\mu}\Big)^{3/2}U_{i,j}(\times,\rho)E_*^{1/2}(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}P_{r,s'}(\Pi-(\Pi)_{B_\rho},\rho)\Big]$$
$$\times\Big(\frac{\rho}{\mu}\Big)^{1/2}[E(u,\rho)+E_*(u,\rho)]^{1/2}.$$

We define $G_2(\mu)=E(u,\mu)+E_*(u,\mu)+P_{r,s'}^2(\Pi-(\Pi)_{B_\mu},\mu)$. Then the last inequality and (2-8) in Lemma 2.4 lead to

$$\text{(3-4)}\quad G_2(\mu)\le C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)G_2(\rho)+C\Big(\frac{\mu}{\rho}\Big)^2 G_2(\rho)$$
$$+C\Big(\frac{\rho}{\mu}\Big)^2 U_{i,j}(\times,\rho)G_2(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r}G_2(\rho)$$
$$+C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)E_*(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^{6-4/r}P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)$$
$$\le C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}(\times,\rho)G_2(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r}G_2(\rho).$$

Now, by an argument completely analogous to that in the proof of Theorem 1.1, we can complete the first part of the proof of Theorem 1.3.

(2) Substituting (2-16) into (2-21), we get

$$\text{(3-5)}\quad E_2(u,\mu)\le\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}^2(\times,\rho)[E(u,\rho)+E_*(u,\rho)]$$
$$+C\Big(\frac{\mu}{\rho}\Big)^2[E(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)].$$

Plugging (3-5) and (2-9) into (3-2), we infer that

$$E(u,\mu)+E_*(u,\mu)$$
$$\le\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}^2(\times,\rho)[E(u,\rho)+E_*(u,\rho)]+C\Big(\frac{\mu}{\rho}\Big)^2[E(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)]$$
$$+\Big\{C\Big(\frac{\rho}{\mu}\Big)^{3/2}W_{m,n}(\times,\rho)[E(u,\rho)+E_*(u,\rho)]^{1/2}+C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}P_{r,s'}(\Pi-(\Pi)_{B_\rho},\rho)\Big\}$$
$$\times\Big(\frac{\rho}{\mu}\Big)^{1/2}[E(u,\rho)+E_*(u,\rho)]^{1/2}.$$

Let

$$G_3(\mu)=E(u,\mu)+E_*(u,\mu)+P_{r,s'}^2(\Pi-(\Pi)_{B_\mu},\mu).$$

Then the latter relation and (2-9) allow us to obtain

$$
\begin{aligned}
(3\text{-}6)\qquad G_3(\mu) \le & C\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}^2(\times,\rho)G_3(\rho)+C\Big(\frac{\mu}{\rho}\Big)^2 G_3(\rho)\\
&+C\Big(\frac{\rho}{\mu}\Big)^2 W_{m,n}(\times,\rho)G_3(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r} G_3(\rho)\\
&+C\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}^2(\times,\rho)G_3(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{6-\frac{4}{r}} G_3(\rho)\\
\le & C\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}(\times,\rho)G_3(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r} G_3(\rho).
\end{aligned}
$$

Combining equations (3-4) and (3-6) and iterating as in the proof of Theorem 1.1 completes the second part of the proof of Theorem 1.3. □

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References

[Beirão da Veiga 2000] H. Beirão da Veiga, “A sufficient condition on the pressure for the regularity of weak solutions to the Navier–Stokes equations”, *J. Math. Fluid Mech.* **2**:2 (2000), 99–106. MR Zbl

[Caffarelli et al. 1982] L. Caffarelli, R. Kohn, and L. Nirenberg, “Partial regularity of suitable weak solutions of the Navier–Stokes equations”, *Comm. Pure Appl. Math.* **35**:6 (1982), 771–831. MR Zbl

[Chae 2010] D. Chae, “On the regularity conditions of suitable weak solutions of the 3D Navier–Stokes equations”, *J. Math. Fluid Mech.* **12**:2 (2010), 171–180. MR Zbl

[Chae 2014] D. Chae, “Localized energy equalities for the Navier–Stokes and the Euler equations”, *Adv. Math.* **254** (2014), 69–78. MR Zbl

[Chae et al. 2007] D. Chae, K. Kang, and J. Lee, “On the interior regularity of suitable weak solutions to the Navier–Stokes equations”, *Comm. Partial Differential Equations* **32**:7-9 (2007), 1189–1207. MR Zbl

[Dong and Du 2007] H. Dong and D. Du, “Partial regularity of solutions to the four-dimensional Navier–Stokes equations at the first blow-up time”, *Comm. Math. Phys.* **273**:3 (2007), 785–801. MR Zbl

[Dong and Strain 2012] H. Dong and R. M. Strain, “On partial regularity of steady-state solutions to the 6D Navier–Stokes equations”, *Indiana Univ. Math. J.* **61**:6 (2012), 2211–2229. MR Zbl

[Frehse and Růžička 1994] J. Frehse and M. Růžička, “Regularity for the stationary Navier–Stokes equations in bounded domains”, *Arch. Rational Mech. Anal.* **128**:4 (1994), 361–380. MR Zbl

[Frehse and Růžička 1995] J. Frehse and M. Růžička, “Existence of regular solutions to the stationary Navier–Stokes equations”, *Math. Ann.* **302**:4 (1995), 699–717. MR Zbl

[Giga and Sohr 1991] Y. Giga and H. Sohr, "Abstract L^p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains", *J. Funct. Anal.* **102**:1 (1991), 72–94. MR Zbl

[Gustafson et al. 2007] S. Gustafson, K. Kang, and T.-P. Tsai, "Interior regularity criteria for suitable weak solutions of the Navier–Stokes equations", *Comm. Math. Phys.* **273**:1 (2007), 161–176. MR Zbl

[Heywood et al. 1996] J. G. Heywood, R. Rannacher, and S. Turek, "Artificial boundaries and flux and pressure conditions for the incompressible Navier–Stokes equations", *Internat. J. Numer. Methods Fluids* **22**:5 (1996), 325–352. MR Zbl

[Ladyzhenskaya and Seregin 1999] O. A. Ladyzhenskaya and G. A. Seregin, "On partial regularity of suitable weak solutions to the three-dimensional Navier–Stokes equations", *J. Math. Fluid Mech.* **1**:4 (1999), 356–387. MR Zbl

[Layton et al. 2009] W. Layton, C. C. Manica, M. Neda, M. Olshanskii, and L. G. Rebholz, "On the accuracy of the rotation form in simulations of the Navier–Stokes equations", *J. Comput. Phys.* **228**:9 (2009), 3433–3447. MR

[Lin 1998] F. Lin, "A new proof of the Caffarelli–Kohn–Nirenberg theorem", *Comm. Pure Appl. Math.* **51**:3 (1998), 241–257. MR Zbl

[Nečas et al. 1996] J. Nečas, M. Růžička, and V. Šverák, "On Leray's self-similar solutions of the Navier–Stokes equations", *Acta Math.* **176**:2 (1996), 283–294. MR Zbl

[Olshanskii 2002] M. A. Olshanskii, "A low order Galerkin finite element method for the Navier–Stokes equations of steady incompressible flow: a stabilization issue and iterative methods", *Comput. Methods Appl. Mech. Engrg.* **191**:47-48 (2002), 5515–5536. MR Zbl

[Prandtl 2004] L. Prandtl, *Essentials of fluid mechanics*, 2nd ed., edited by H. Oertel, Applied Mathematical Sciences **158**, Springer, 2004. MR Zbl

[Scheffer 1976] V. Scheffer, "Partial regularity of solutions to the Navier–Stokes equations", *Pacific J. Math.* **66**:2 (1976), 535–552. MR Zbl

[Scheffer 1977] V. Scheffer, "Hausdorff measure and the Navier–Stokes equations", *Comm. Math. Phys.* **55**:2 (1977), 97–112. MR Zbl

[Scheffer 1980] V. Scheffer, "The Navier–Stokes equations on a bounded domain", *Comm. Math. Phys.* **73**:1 (1980), 1–42. MR Zbl

[Seregin 2002] G. A. Seregin, "Local regularity of suitable weak solutions to the Navier–Stokes equations near the boundary", *J. Math. Fluid Mech.* **4**:1 (2002), 1–29. MR Zbl

[Seregin 2007] G. A. Seregin, "Local regularity of suitable weak solutions of the Navier–Stokes equations", *Uspekhi Mat. Nauk* **62**:3 (2007), 149–168. In Russian; translated in *Russian Math. Surveys* **62**:3 (2007), 595–614. MR Zbl

[Seregin 2014] G. Seregin, *Lecture notes on regularity theory for the Navier–Stokes equations*, World Scientific, Hackensack, NJ, 2014. MR Zbl

[Seregin and Šverák 2002] G. Seregin and V. Šverák, "Navier–Stokes equations with lower bounds on the pressure", *Arch. Ration. Mech. Anal.* **163**:1 (2002), 65–86. MR Zbl

[Struwe 1988] M. Struwe, "On partial regularity results for the Navier–Stokes equations", *Comm. Pure Appl. Math.* **41**:4 (1988), 437–458. MR Zbl

[Struwe 1995] M. Struwe, "Regular solutions of the stationary Navier–Stokes equations on $\mathbb{R}^5$", *Math. Ann.* **302**:4 (1995), 719–741. MR Zbl

[Tian and Xin 1999] G. Tian and Z. Xin, "Gradient estimation on Navier–Stokes equations", *Comm. Anal. Geom.* **7**:2 (1999), 221–257. MR Zbl

[Tsai 1998] T.-P. Tsai, "On Leray's self-similar solutions of the Navier–Stokes equations satisfying local energy estimates", *Arch. Rational Mech. Anal.* **143**:1 (1998), 29–51. MR Zbl

[Vasseur 2007] A. F. Vasseur, "A new proof of partial regularity of solutions to Navier–Stokes equations", *NoDEA Nonlinear Differential Equations Appl.* **14**:5-6 (2007), 753–785. MR Zbl

[Wang and Wu 2014] Y. Wang and G. Wu, "A unified proof on the partial regularity for suitable weak solutions of non-stationary and stationary Navier–Stokes equations", *J. Differential Equations* **256**:3 (2014), 1224–1249. MR Zbl

[Wang and Wu 2016a] Y. Wang and G. Wu, "Anisotropic regularity conditions for the suitable weak solutions to the 3D Navier–Stokes equations", *J. Math. Fluid Mech.* **18**:4 (2016), 699–716.

[Wang and Wu 2016b] Y. Wang and G. Wu, "Local regularity criteria of the 3D Navier–Stokes and related equations", *Nonlinear Anal.* **140** (2016), 130–144. MR Zbl

[Wang and Zhang 2013] W. Wang and Z. Zhang, "On the interior regularity criteria for suitable weak solutions of the magnetohydrodynamics equations", *SIAM J. Math. Anal.* **45**:5 (2013), 2666–2677. MR Zbl

[Wang and Zhang 2014] W. Wang and Z. Zhang, "On the interior regularity criteria and the number of singular points to the Navier–Stokes equations", *J. Anal. Math.* **123** (2014), 139–170. MR Zbl

[Wang et al. 2014] W. Wang, L. Zhang, and Z. Zhang, "On the interior regularity criteria of the 3-D Navier–Stokes equations involving two velocity components", preprint, 2014. arXiv

[Wolf 2008] J. Wolf, "A direct proof of the Caffarelli–Kohn–Nirenberg theorem", pp. 533–552 in *Parabolic and Navier–Stokes equations, II*, edited by J. Rencławowicz and W. M. Zajączkowski, Banach Center Publ. **81**, Polish Acad. Sci. Inst. Math., Warsaw, 2008. MR Zbl

[Zang 1991] T. A. Zang, "On the rotation and skew-symmetric forms for incompressible flow simulations", *Appl. Numer. Math.* **7**:1 (1991), 27–40. Zbl

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GEOMETRIC PROPERTIES OF LEVEL CURVES OF HARMONIC FUNCTIONS AND MINIMAL GRAPHS IN 2-DIMENSIONAL SPACE FORMS

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We study the geometric properties of level curves of harmonic functions and minimal graphs in 2-dimensional space forms using the maximum principle. More precisely, we find two auxiliary functions which consist of tangential derivatives of the curvature of level curves and the norms of the gradient of the solution functions. Then we prove that they satisfy certain elliptic partial differential equations.

1. Introduction

The geometric properties of the level surfaces of solutions of elliptic partial differential equations have been studied for a long time. For instance, a book by Ahlfors [1973] contains the well-known result that level curves of the Green function of a 2-dimensional convex domain are convex curves. Gergen [1931] proved the level surfaces of the Green function of a 3-dimensional star-shaped domain are also star-shaped. Shiffman [1956] studied the convexity of the level curves of immersed minimal surfaces in $\mathbb{R}^3$. He proved that if two convex curves in parallel planes in $\mathbb{R}^3$ bound a minimal surface S then the intersections of all other parallel planes with S are also convex curves. In particular, he obtained that if the boundaries are two circles then intermediate level curves are also circles. Gabriel [1957] proved that the level surfaces of the Green function of a 3-dimensional convex domain are strictly convex. Later, Lewis [1977] extended Gabriel's results to p-harmonic functions in high dimensions. For more related extensions and a survey on this subject, see [Bianchini et al. 2009; Caffarelli and Spruck 1982; Kawohl 1985].

There is also a lot of literature on the quantitative curvature estimates of level surfaces of solutions of elliptic partial differential equations. For 2-dimensional harmonic functions, Talenti [1983] got the following result. Let $\Omega \subset \mathbb{R}^2$ be a domain

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and u be a harmonic function with no critical points in Ω. Then the function $\kappa/|\nabla u|$ is harmonic in Ω. Here

$$\kappa = \frac{2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}}{|\nabla u|^3}$$

is the curvature of the level curves of u. Throughout the paper we use subscripts to represent the derivatives with respect to any orthonormal frames. Similar results can also be seen in [Ortel and Schneider 1983; Longinetti 1983]. Recently, Ma, Ou and Zhang [Ma et al. 2010] generalized the above results to n-dimensional harmonic functions ($2 \le n < \infty$) and obtained the sharp Gaussian curvature estimates of the level surfaces. See also [Chang et al. 2010; Ma and Zhang 2013; 2014; Wang and Zhang 2012; Zhang and Zhang 2013].

More recently, Kong and Xu [2015] found that if u is a harmonic function of two variables with no critical points, then the function $(\kappa_1u_2 - \kappa_2u_1)/|\nabla u|^3$ is also harmonic. Using this fact, they proved that all the level curves of solutions of the Laplace equation with homogeneous Dirichlet boundary conditions on an annulus are circles. This result can be viewed as a generalization of Shiffman's result on minimal surfaces. In this paper, we extend Kong and Xu's and Shiffman's results to harmonic functions and minimal graphs in 2-dimensional space forms. More precisely, we obtain the following results.

Theorem 1.1. *Suppose that $M^2(c)$ is a 2-dimensional Riemannian manifold with constant sectional curvature c. Let $\Omega \subset M^2(c)$ be a domain and u be a harmonic function with no critical points in Ω. Let κ be the curvature of the level curves of u. Then the function $\varphi = (\kappa_1u_2 - \kappa_2u_1)/|\nabla u|^3$ is also harmonic in Ω.*

For minimal graphs, we have the following similar result.

Theorem 1.2. *Suppose that $M^2(c)$ is a 2-dimensional Riemannian manifold with constant sectional curvature c. Let $\Omega \subset M^2(c)$ be a domain and u satisfy the minimal surface equation*

$$\sum_{ij} a_{ij}u_{ij} = 0 \quad \text{in } \Omega,$$

where $a_{ij} = (1+|\nabla u|^2)\delta_{ij} - u_iu_j$. Furthermore, assume that there are no critical points of u in Ω. Let κ be the curvature of the level curves of u. Set

$$\psi = \frac{(1+|\nabla u|^2)^{3/2}}{|\nabla u|^3} \cdot (\kappa_1u_2 - \kappa_2u_1).$$

Then the function ψ satisfies the differential equation

$$\sum_{ij} a_{ij}\psi_{ij} + \sum_i b_i\psi_i = 0 \quad \text{in } \Omega.$$

Here the b_i are bounded functions.

Based on the above theorems, we have the following characterization of geodesic circles.

Remark 1.3. Since $(\kappa_1 u_2 - \kappa_2 u_1)/|\nabla u|$ is the tangential derivative of the curvature of the level curves, the auxiliary functions φ and ψ are independent of the choice of orthonormal frames. Similar to the case of Euclidean space, by the maximum principle, we know that all the level curves of solutions of the Laplace equation or the minimal surface equation with homogeneous Dirichlet boundary conditions on an annulus are geodesic circles.

Now we give the derivative commutation formulas in Riemannian geometry. Let u be a smooth function and R_{ijkl} be coefficients of the Riemannian curvature tensor under orthonormal frames. Here for 2-dimensional space forms $M^2(c)$, we adopt $R_{1212} = c$. Then we have

$$u_{ij} - u_{ji} = 0, \tag{1-1}$$

$$u_{ijk} - u_{ikj} = \sum_m u_m R_{mijk}, \tag{1-2}$$

$$u_{ijkl} - u_{ijlk} = \sum_m u_{mj} R_{mikl} + \sum_m u_{im} R_{mjkl}. \tag{1-3}$$

For more details, one can consult any book on Riemannian geometry, such as [Chern et al. 1999].

In this paper, all the summation indices i, j, k, l and m run from 1 to 2. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2.

2. Level curves of harmonic functions

In this section, we focus on the calculation of harmonic functions in 2-dimensional space forms.

Let $\Omega \subset M^2(c)$ be a domain and u be a harmonic function defined in Ω with no critical points. Set

$$\varphi = f(|\nabla u|)(\kappa_1 u_2 - \kappa_2 u_1),$$

where κ is the curvature of the level curves and f is a smooth function of one variable defined on the interval $(0, +\infty)$ which will be determined later. For a suitable choice of f, we will prove that φ is also a harmonic function in Ω, i.e., the function φ satisfies

$$\Delta\varphi = 0 \quad \text{in } \Omega. \tag{2-1}$$

In order to prove (2-1) at an arbitrary point $x_0 \in \Omega$, we may choose the orthonormal frames such that

$$u_1(x_0) = 0, \quad u_2(x_0) = |\nabla u|(x_0) > 0.$$

From now on, all the calculations will be done at the fixed point x_0 unless otherwise specified.

By taking the first derivative of φ, we have

$$\varphi_i = f'(|\nabla u|)_i \cdot (\kappa_1 u_2 - \kappa_2 u_1) + f \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_{2i} u_1 - \kappa_2 u_{1i}). \tag{2-2}$$

Differentiating (2-2) once more, we have

$$\begin{aligned}\varphi_{ii} = f''(|\nabla u|)_i^2 \cdot \kappa_1 u_2 + f'(|\nabla u|)_{ii} \cdot \kappa_1 u_2 + 2 f'(|\nabla u|)_i \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_2 u_{1i}) \\ + f \cdot (\kappa_{1ii} u_2 + 2\kappa_{1i} u_{2i} + \kappa_1 u_{2ii} - 2\kappa_{2i} u_{1i} - \kappa_2 u_{1ii});\end{aligned}$$

hence

$$\begin{aligned}\Delta\varphi = u_2 f \sum_i k_{1ii} + \big[2u_2 f'(|\nabla u|)_1 + 2 f u_{12}\big] \cdot \kappa_{11} \\ + \big[2u_2 f'(|\nabla u|)_2 + 2 f u_{22} - 2 f u_{11}\big] \cdot \kappa_{12} + [-2 f u_{12}] \cdot \kappa_{22} \\ + \bigg[u_2 f'' \sum_i (|\nabla u|)_i^2 + u_2 f' \sum_i (|\nabla u|)_{ii} \\ + 2 f' \sum_i (|\nabla u|)_i u_{2i} + f \sum_i u_{2ii}\bigg] \cdot \kappa_1 \\ + \bigg[-2 f' \sum_i (|\nabla u|)_i u_{1i} - f \sum_i u_{1ii}\bigg] \cdot \kappa_2.\end{aligned} \tag{2-3}$$

Direct calculation yields

$$\begin{aligned}(|\nabla u|)_i &= \frac{1}{|\nabla u|} \sum_j u_j u_{ji}, \\ (|\nabla u|)_{ii} &= \frac{1}{|\nabla u|} \sum_j u_{ji}^2 + \frac{1}{|\nabla u|} \sum_j u_j u_{jii} - \frac{1}{|\nabla u|^3} \sum_{jk} u_j u_{ji} u_k u_{ki}.\end{aligned} \tag{2-4}$$

Then at the point x_0,

$$(|\nabla u|)_i = u_{2i}, \quad (|\nabla u|)_{ii} = \frac{u_{1i}^2}{u_2} + u_{2ii}. \tag{2-5}$$

By the commutation formulas (1-1)–(1-2), we have

$$\sum_i u_{1ii} = \sum_i u_{i1i} = \sum_i \Big[u_{ii1} + \sum_m u_m R_{mi1i}\Big] = 0, \tag{2-6}$$

$$\sum_i u_{2ii} = \sum_i u_{i2i} = \sum_i \Big[u_{ii2} + \sum_m u_m R_{mi2i}\Big] = u_2 \cdot c. \tag{2-7}$$

Putting (2-5)–(2-7) into (2-3), we obtain

$$\Delta\varphi = u_2 f \sum_i \kappa_{1ii} + (2u_2 f' + 2f)u_{12}\cdot\kappa_{11} + (-2u_2 f' - 4f)u_{11}\cdot\kappa_{12} + (-2f)u_{12}\cdot\kappa_{22} + (u_2 f'' + 3f')(u_{11}^2 + u_{12}^2)\cdot\kappa_1 + (u_2^2 f' + u_2 f)\cdot\kappa_1\cdot c. \tag{2-8}$$

To compute the first term in (2-8), we should get the formula for $\Delta\kappa$ at a general point in advance. Recalling the curvature formula for the level curves, we have

$$|\nabla u|^3\cdot\kappa = 2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}. \tag{2-9}$$

By applying the Laplace operator on both sides of (2-9) and then using (2-4), we obtain

$$\begin{aligned}\Delta\kappa &= \frac{1}{|\nabla u|^3}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} - \frac{6}{|\nabla u|}\sum_i (|\nabla u|)_i\cdot\kappa_i \\ &\quad - \frac{6}{|\nabla u|^2}\sum_i (|\nabla u|)_i^2\cdot\kappa - \frac{3}{|\nabla u|}\sum_i (|\nabla u|)_{ii}\cdot\kappa \\ &= \frac{1}{|\nabla u|^3}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} - \frac{6}{|\nabla u|^2}(u_1u_{11} + u_2u_{12})\cdot\kappa_1 \\ &\quad - \frac{6}{|\nabla u|^2}(u_1u_{12} - u_2u_{11})\cdot\kappa_2 - \frac{1}{|\nabla u|^2}\Big[9(u_{11}^2 + u_{12}^2) + 3\sum_{ij} u_j u_{jii}\Big]\cdot\kappa.\end{aligned} \tag{2-10}$$

Now the commutation formulas (1-1)–(1-2) yield

$$\sum_{ij} u_j u_{jii} = \sum_{ij} u_j\Big[u_{iij} + \sum_m u_m R_{miji}\Big] = \sum_{ijm} u_j u_m R_{miji} = |\nabla u|^2\cdot c. \tag{2-11}$$

By inserting (2-11) into (2-10), we have

$$\begin{aligned}\Delta\kappa &= \frac{1}{|\nabla u|^3}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} - \frac{6}{|\nabla u|^2}(u_1u_{11} + u_2u_{12})\cdot\kappa_1 \\ &\quad - \frac{6}{|\nabla u|^2}(u_1u_{12} - u_2u_{11})\cdot\kappa_2 - \frac{9}{|\nabla u|^2}(u_{11}^2 + u_{12}^2)\cdot\kappa - 3\kappa\cdot c.\end{aligned} \tag{2-12}$$

Straightforward computation gives

$$\begin{aligned}&\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} \\ &\quad = \sum_i \big[2u_{1ii}u_2u_{12} + 2u_1u_{2ii}u_{12} + 2u_1u_2u_{12ii} - 2u_1u_{1ii}u_{22} \\ &\qquad - u_1^2u_{22ii} - 2u_2u_{2ii}u_{11} - u_2^2u_{11ii} + 4u_{1i}u_{2i}u_{12} + 4u_{1i}u_2u_{12i} \\ &\qquad + 4u_1u_{2i}u_{12i} - 2u_{1i}^2u_{22} - 4u_1u_{1i}u_{22i} - 2u_{2i}^2u_{11} - 4u_2u_{2i}u_{11i}\big]\end{aligned} \tag{2-13}$$

$$
\begin{aligned}
&= 2(u_2u_{12}+u_1u_{11})\sum_i u_{1ii} + 2(u_1u_{12}-u_2u_{11})\sum_i u_{2ii} \\
&\quad + 2u_1u_2\sum_i u_{12ii} - u_1^2\sum_i u_{22ii} - u_2^2\sum_i u_{11ii} \\
&\quad + 4\sum_i (u_2u_{1i}+u_1u_{2i})u_{12i} + 4\sum_i (u_1u_{1i}-u_2u_{2i})u_{11i} \\
&\triangleq \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{I}_1 &= 2(u_2u_{12}+u_1u_{11})\sum_i u_{1ii} + 2(u_1u_{12}-u_2u_{11})\sum_i u_{2ii}, \\
\mathrm{I}_2 &= 2u_1u_2\sum_i u_{12ii} - u_1^2\sum_i u_{22ii} - u_2^2\sum_i u_{11ii}, \\
\mathrm{I}_3 &= 4\sum_i (u_2u_{1i}+u_1u_{2i})u_{12i} + 4\sum_i (u_1u_{1i}-u_2u_{2i})u_{11i}.
\end{aligned}
$$

We deal with the terms I_1, I_2 and I_3 consecutively. By (2-6)–(2-7), we have

$$
\begin{aligned}
\mathrm{I}_1 &= 2(u_2u_{12}+u_1u_{11})\sum_{im} u_m R_{mi1i} + 2(u_1u_{12}-u_2u_{11})\sum_{im} u_m R_{mi2i} \\
&= 2(u_2u_{12}+u_1u_{11})\cdot u_1\cdot c + 2(u_1u_{12}-u_2u_{11})\cdot u_2\cdot c \\
&= 2|\nabla u|^3\cdot\kappa\cdot c.
\end{aligned}
\tag{2-14}
$$

By the commutation formulas (1-1)–(1-3), we have

$$
u_{jkii} = u_{iijk} + \sum_m u_{mk}R_{miji} + \sum_m u_{mj}R_{miki} + 2\sum_m u_{mi}R_{mjki}.
$$

It follows that

$$
\begin{aligned}
\mathrm{I}_2 &= 2u_1u_2\cdot\Big[\sum_{im} u_{m2}R_{mi1i} + \sum_{im} u_{m1}R_{mi2i} + 2\sum_{im} u_{mi}R_{m12i}\Big] \\
&\quad - u_1^2\cdot\Big[2\sum_{im} u_{m2}R_{mi2i} + 2\sum_{im} u_{mi}R_{m22i}\Big] \\
&\quad - u_2^2\cdot\Big[2\sum_{im} u_{m1}R_{mi1i} + 2\sum_{im} u_{mi}R_{m11i}\Big] \\
&= 2u_1u_2\cdot 4u_{12}\cdot c - u_1^2\cdot(-4u_{11})\cdot c - u_2^2\cdot 4u_{11}\cdot c \\
&= (8u_1u_2u_{12} + 4u_1^2u_{11} - 4u_2^2u_{11})\cdot c.
\end{aligned}
\tag{2-15}
$$

By the commutation formulas (1-1)–(1-2),

$$u_{121} = u_{112} + \sum_m u_m R_{m121} = u_{112} + u_2 \cdot c, \tag{2-16}$$

$$u_{122} = u_{221} + \sum_m u_m R_{m212} = -u_{111} + u_1 \cdot c. \tag{2-17}$$

Then we have

$$\begin{aligned} \mathrm{I}_3 = 8(u_1u_{11} - u_2u_{12})u_{111} + 8(u_1u_{12} + u_2u_{11})u_{112} \\ + (8u_1u_2u_{12} + 4u_2^2u_{11} - 4u_1^2u_{11}) \cdot c. \end{aligned} \tag{2-18}$$

Combining (2-13)–(2-15) and (2-18), we get

$$\begin{aligned} \sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} \\ = 8(u_1u_{11} - u_2u_{12})u_{111} + 8(u_1u_{12} + u_2u_{11})u_{112} \\ + 2|\nabla u|^3 \cdot \kappa \cdot c + 16u_1u_2u_{12} \cdot c. \end{aligned} \tag{2-19}$$

Now let us explore the relations between u_{111}, u_{112} and κ_1, κ_2. Taking the first derivative on both sides of (2-9) and using (2-4), (2-16) and (2-17), we obtain

$$\begin{aligned} (u_1^2 - u_2^2) \cdot u_{111} + 2u_1u_2 \cdot u_{112} \\ = |\nabla u|^3 \cdot \kappa_1 + 3|\nabla u|(u_1u_{11} + u_2u_{12}) \cdot \kappa - 2u_1(u_{11}^2 + u_{12}^2) - 2u_1u_2^2 \cdot c, \\ - 2u_1u_2 \cdot u_{111} + (u_1^2 - u_2^2) \cdot u_{112} \\ = |\nabla u|^3 \cdot \kappa_2 + 3|\nabla u|(u_1u_{12} - u_2u_{11}) \cdot \kappa - 2u_2(u_{11}^2 + u_{12}^2) - 2u_1^2u_2 \cdot c. \end{aligned}$$

Thus we have

$$\begin{aligned} u_{111} = \frac{u_1^2 - u_2^2}{|\nabla u|} \cdot \kappa_1 - \frac{2u_1u_2}{|\nabla u|} \cdot \kappa_2 + \frac{3}{|\nabla u|}(u_1u_{11} - u_2u_{12}) \cdot \kappa \\ - \frac{2u_1(u_1^2 - 3u_2^2)}{|\nabla u|^4}(u_{11}^2 + u_{12}^2) + \frac{2u_1u_2^2}{|\nabla u|^2} \cdot c, \end{aligned} \tag{2-20}$$

and

$$\begin{aligned} u_{112} = \frac{2u_1u_2}{|\nabla u|} \cdot \kappa_1 + \frac{u_1^2 - u_2^2}{|\nabla u|} \cdot \kappa_2 + \frac{3}{|\nabla u|}(u_2u_{11} + u_1u_{12}) \cdot \kappa \\ - \frac{2u_2(3u_1^2 - u_2^2)}{|\nabla u|^4}(u_{11}^2 + u_{12}^2) - \frac{2u_1^2u_2}{|\nabla u|^2} \cdot c. \end{aligned} \tag{2-21}$$

Hence the formula (2-19) reduces to

$$\begin{aligned}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} &= 8|\nabla u|(u_1u_{11} + u_2u_{12}) \cdot \kappa_1 + 8|\nabla u|(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ &\quad + 8|\nabla u|(u_{11}^2 + u_{12}^2) \cdot \kappa + 2|\nabla u|^3 \cdot \kappa \cdot c.\end{aligned} \tag{2-22}$$

By (2-12) and (2-22), we have

$$\begin{aligned}\Delta\kappa = \frac{2}{|\nabla u|^2}(u_1u_{11} + u_2u_{12}) \cdot \kappa_1 + \frac{2}{|\nabla u|^2}(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ - \frac{1}{|\nabla u|^2}(u_{11}^2 + u_{12}^2) \cdot \kappa - \kappa \cdot c.\end{aligned} \tag{2-23}$$

Then at the point x_0, we take the first derivative of (2-23). With (2-5) and (2-16) in hand, we obtain

$$\begin{aligned}(\Delta\kappa)_1 = \frac{2}{u_2}u_{12} \cdot \kappa_{11} - \frac{2}{u_2}u_{11} \cdot \kappa_{12} + \left[\frac{2}{u_2}u_{112} + \frac{1}{u_2^2}u_{11}^2 - \frac{3}{u_2^2}u_{12}^2\right] \cdot \kappa_1 \\ + \left[-\frac{2}{u_2}u_{111} + \frac{4}{u_2^2}u_{11}u_{12}\right] \cdot \kappa_2 \\ + \left[-\frac{2}{u_2^2}(u_{11}u_{111} + u_{12}u_{112}) + \frac{2}{u_2^3}(u_{11}^2 + u_{12}^2)u_{12}\right] \cdot \kappa \\ + \left[\kappa_1 - \frac{2}{u_2}u_{12} \cdot \kappa\right] \cdot c.\end{aligned} \tag{2-24}$$

Now, the equations (2-20) and (2-21) are simplified as

$$u_{111} = -u_2\kappa_1 + \frac{3}{u_2}u_{11}u_{12}, \tag{2-25}$$

$$u_{112} = -u_2\kappa_2 - \frac{1}{u_2}u_{11}^2 + \frac{2}{u_2}u_{12}^2. \tag{2-26}$$

Putting (2-25)–(2-26) into (2-24), one achieves

$$\begin{aligned}(\Delta\kappa)_1 = \frac{2}{u_2}u_{12} \cdot \kappa_{11} - \frac{2}{u_2}u_{11} \cdot \kappa_{12} + \left[-\frac{3}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\right] \cdot \kappa_1 + \left[-\frac{4}{u_2^2}u_{11}u_{12}\right] \cdot \kappa_2 \\ + \left[-\frac{2}{u_2^3}(u_{11}^2 + u_{12}^2)u_{12}\right] \cdot \kappa + \left[\kappa_1 - \frac{2}{u_2}u_{12} \cdot \kappa\right] \cdot c.\end{aligned}$$

Therefore, by the commutation formulas(1-1)–(1-2), we get

$$
\begin{aligned}
\text{(2-27)}\quad & \sum_i \kappa_{1ii} \\
&= \sum_i \Big[\kappa_{ii1} + \sum_m \kappa_m R_{mi1i}\Big] \\
&= (\Delta\kappa)_1 + \kappa_1 \cdot c \\
&= \frac{2}{u_2} u_{12}\cdot\kappa_{11} - \frac{2}{u_2} u_{11}\cdot\kappa_{12} + \Big[-\frac{3}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\Big]\cdot\kappa_1 \\
&\quad + \Big[-\frac{4}{u_2^2}u_{11}u_{12}\Big]\cdot\kappa_2 + \Big[-\frac{2}{u_2^3}(u_{11}^2+u_{12}^2)u_{12}\Big]\cdot\kappa + \Big[2\kappa_1 - \frac{2}{u_2}u_{12}\cdot\kappa\Big]\cdot c.
\end{aligned}
$$

Thanks to (2-27), the formula (2-8) reduces to

$$
\begin{aligned}
\Delta\varphi &= \big[(2u_2 f' + 4f)u_{12}\big]\cdot\kappa_{11} + \big[(-2u_2 f' - 6f)u_{11}\big]\cdot\kappa_{12} + [-2fu_{12}]\cdot\kappa_{22} \\
&\quad + \Big[\Big(u_2 f'' + 3f' - \frac{3f}{u_2}\Big)u_{11}^2 + \Big(u_2 f'' + 3f' + \frac{f}{u_2}\Big)u_{12}^2\Big]\cdot\kappa_1 \\
&\quad + \Big[-\frac{4f}{u_2}u_{11}u_{12}\Big]\cdot\kappa_2 + \Big[-\frac{2f}{u_2^2}(u_{11}^2+u_{12}^2)u_{12}\Big]\cdot\kappa \\
&\quad + \big[(u_2^2 f' + 3u_2 f)\cdot\kappa_1 - 2fu_{12}\cdot\kappa\big]\cdot c.
\end{aligned}
$$

At the point x_0, by (2-23), we have

$$
\kappa_{22} = -\kappa_{11} + \frac{2}{u_2}u_{12}\cdot\kappa_1 - \frac{2}{u_2}u_{11}\cdot\kappa_2 - \frac{1}{u_2^2}(u_{11}^2+u_{12}^2)\cdot\kappa - \kappa\cdot c.
$$

Thus

$$
\begin{aligned}
\text{(2-28)}\quad \Delta\varphi &= (2u_2 f' + 6f)u_{12}\cdot\kappa_{11} + (-2u_2 f' - 6f)u_{11}\cdot\kappa_{12} \\
&\quad + \Big(u_2 f'' + 3f' - \frac{3f}{u_2}\Big)(u_{11}^2+u_{12}^2)\cdot\kappa_1 + (u_2^2 f' + 3u_2 f)\cdot\kappa_1\cdot c.
\end{aligned}
$$

By (2-2), we have

$$
\text{(2-29)}\qquad \kappa_{11} = \frac{1}{u_2 f}\big[\varphi_1 - (u_2 f' + f)u_{12}\cdot\kappa_1 + fu_{11}\cdot\kappa_2\big],
$$

$$
\text{(2-30)}\qquad \kappa_{12} = \frac{1}{u_2 f}\big[\varphi_2 + (u_2 f' + f)u_{11}\cdot\kappa_1 + fu_{12}\cdot\kappa_2\big].
$$

Putting (2-29)–(2-30) into (2-28), we finally get

$$\text{(2-31)}\quad \Delta\varphi = \left(\frac{2f'}{f} + \frac{6}{u_2}\right) \cdot (u_{12}\varphi_1 - u_{11}\varphi_2) \\ + \left(u_2 f'' - \frac{2u_2 f'^2}{f} - 5f' - \frac{9f}{u_2}\right)(u_{11}^2 + u_{12}^2)\cdot\kappa_1 + (u_2^2 f' + 3u_2 f)\cdot\kappa_1\cdot c.$$

If we let

$$f(t) = t^{-3},$$

then all of the terms on the right-hand side of (2-31) vanish. This completes the proof of Theorem 1.1. □

3. Level curves of minimal graphs

Along the same lines as in Section 2, in this section we deal with the minimal graphs in 2-dimensional space forms.

Let $\Omega \subset M^2(c)$ be a domain and u be a solution with no critical points of the minimal surface equation

$$\text{(3-1)}\qquad \sum_{ij} a_{ij} u_{ij} = 0 \quad \text{in } \Omega,$$

where

$$a_{ij} = (1 + |\nabla u|^2)\delta_{ij} - u_i u_j.$$

Set

$$\psi = g(|\nabla u|)(\kappa_1 u_2 - \kappa_2 u_1),$$

where κ is the curvature of the level curves and g is a smooth function of one variable defined on the interval $(0, +\infty)$ to be determined later. For a suitable choice of g, we will prove that the function ψ satisfies

$$\text{(3-2)}\qquad \sum_{ij} a_{ij}\psi_{ij} + \sum_i b_i \psi_i = 0 \quad \text{in } \Omega.$$

Here the b_i are bounded functions.

In order to prove (3-2) at an arbitrary point $x_0 \in \Omega$, we may choose the orthonormal frames such that

$$u_1(x_0) = 0,$$

$$u_2(x_0) = |\nabla u|(x_0) > 0.$$

From now on, all the calculations will be done at the fixed point x_0 unless otherwise specified.

By taking the derivative of ψ, we have

$$\psi_i = g'(|\nabla u|)_i \cdot (\kappa_1 u_2 - \kappa_2 u_1) + g \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_{2i} u_1 - \kappa_2 u_{1i}). \tag{3-3}$$

Differentiating (3-3) once more, we have

$$\begin{aligned}\psi_{ij} = {} & g''(|\nabla u|)_j (|\nabla u|)_i \cdot \kappa_1 u_2 + g'(|\nabla u|)_{ij} \cdot \kappa_1 u_2 + g'(|\nabla u|)_i \cdot (\kappa_{1j} u_2 + \kappa_1 u_{2j} - \kappa_2 u_{1j}) \\ & + g'(|\nabla u|)_j \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_2 u_{1i}) \\ & \quad + g \cdot (\kappa_{1ij} u_2 + \kappa_{1i} u_{2j} + \kappa_{1j} u_{2i} + \kappa_1 u_{2ij} - \kappa_{2i} u_{1j} - \kappa_{2j} u_{1i} - \kappa_2 u_{1ij});\end{aligned}$$

hence

$$\begin{aligned}\sum_{ij} a_{ij} \psi_{ij} = {} & u_2 g \sum_{ij} a_{ij} k_{1ij} + \Big[2u_2 g' \sum_j a_{1j} (|\nabla u|)_j + 2g \sum_j a_{1j} u_{2j} \Big] \cdot \kappa_{11} \\ & + \Big[2u_2 g' \sum_j a_{2j} (|\nabla u|)_j + 2g \sum_{2j} a_{2j} u_{2j} - 2g \sum_j a_{1j} u_{1j} \Big] \cdot \kappa_{12} \\ & + \Big[-2g \sum_j a_{2j} u_{1j} \Big] \cdot \kappa_{22} \\ & + \Big[u_2 g'' \sum_{ij} a_{ij} (|\nabla u|)_i (|\nabla u|)_j + u_2 g' \sum_{ij} a_{ij} (|\nabla u|)_{ij} \\ & \qquad + 2g' \sum_{ij} a_{ij} (|\nabla u|)_j u_{2i} + g \sum_{ij} a_{ij} u_{2ij} \Big] \cdot \kappa_1 \\ & + \Big[-2g' \sum_{ij} a_{ij} (|\nabla u|)_j u_{1i} - g \sum_{ij} a_{ij} u_{1ij} \Big] \cdot \kappa_2.\end{aligned} \tag{3-4}$$

Direct calculation yields

$$\begin{aligned}(|\nabla u|)_i &= \frac{1}{|\nabla u|} \sum_k u_k u_{ki}, \\ (|\nabla u|)_{ij} &= \frac{1}{|\nabla u|} \sum_k u_{kj} u_{ki} + \frac{1}{|\nabla u|} \sum_k u_k u_{kij} - \frac{1}{|\nabla u|^3} \sum_{kl} u_l u_{lj} u_k u_{ki}.\end{aligned} \tag{3-5}$$

Then at the point x_0,

$$\begin{aligned}(|\nabla u|)_i &= u_{2i}, \\ (|\nabla u|)_{ij} &= \frac{u_{1j} u_{1i}}{u_2} + u_{2ij}.\end{aligned} \tag{3-6}$$

By the commutation formulas (1-1)–(1-2), we obtain

$$\begin{aligned}\sum_{ij} a_{ij}u_{1ij} &= \sum_{ij} a_{ij}u_{i1j}\\ &= \sum_{ij} a_{ij}\Big[u_{ij1} + \sum_m u_m R_{mi1j}\Big]\\ &= -\sum_{ij} a_{ij,1}u_{ij} + \sum_{ijm} a_{ij}u_m R_{mi1j}\\ &= -2u_1(u_{11}u_{22} - u_{12}^2) + u_1(1+|\nabla u|^2)\cdot c\end{aligned} \tag{3-7}$$

and

$$\begin{aligned}\sum_{ij} a_{ij}u_{2ij} &= \sum_{ij} a_{ij}u_{i2j}\\ &= \sum_{ij} a_{ij}\Big[u_{ij2} + \sum_m u_m R_{mi2j}\Big]\\ &= -\sum_{ij} a_{ij,2}u_{ij} + \sum_{ijm} a_{ij}u_m R_{mi2j}\\ &= -2u_2(u_{11}u_{22} - u_{12}^2) + u_2(1+|\nabla u|^2)\cdot c.\end{aligned} \tag{3-8}$$

Hence at the point x_0, we have

$$\sum_{ij} a_{ij}u_{1ij} = 0, \tag{3-9}$$

$$\sum_{ij} a_{ij}u_{2ij} = -2u_2(u_{11}u_{22} - u_{12}^2) + u_2(1+u_2^2)\cdot c. \tag{3-10}$$

On the other hand,

$$a_{11} = 1+u_2^2, \quad a_{12} = 0, \quad a_{21} = 0, \quad a_{22} = 1, \tag{3-11}$$

$$u_{22} = -(1+u_2^2)u_{11}. \tag{3-12}$$

Inserting (3-6), (3-9)–(3-12) into (3-4), we obtain

$$\begin{aligned}\sum_{ij} a_{ij}\psi_{ij} &\\ = u_2 g\sum_{ij} a_{ij}k_{1ij} &+ \big[2u_2(1+u_2^2)g' + 2(1+u_2^2)g\big]u_{12}\cdot\kappa_{11}\\ &+ \big[-2u_2(1+u_2^2)g' - 4(1+u_2^2)g\big]u_{11}\cdot\kappa_{12} + [-2g]u_{12}\cdot\kappa_{22}\\ &+ \big[u_2(1+u_2^2)g'' + (3+4u_2^2)g' + 2u_2 g\big]\cdot\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big]\cdot\kappa_1\\ &+ \big[u_2^2(1+u_2^2)g' + u_2(1+u_2^2)g\big]\cdot\kappa_1\cdot c.\end{aligned} \tag{3-13}$$

To compute the first term in (3-13), we should get the formula for $\sum_{ij} a_{ij}\kappa_{ij}$ at a general point in advance. Recalling the curvature formula for the level curves, we have

$$|\nabla u|^3 \cdot \kappa = 2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}. \tag{3-14}$$

On both sides of (3-14), we take the second derivative with respect to ij, multiply by a_{ij}, and then sum with respect to ij. We obtain

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{ij} = {} & \frac{1}{|\nabla u|^3}\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\ & - \frac{6}{|\nabla u|}\sum_{ij} a_{ij}(|\nabla u|)_j \cdot \kappa_i - \frac{6}{|\nabla u|^2}\sum_{ij} a_{ij}(|\nabla u|)_i(|\nabla u|)_j \cdot \kappa \\ & - \frac{3}{|\nabla u|}\sum_{ij} a_{ij}(|\nabla u|)_{ij} \cdot \kappa.\end{aligned} \tag{3-15}$$

Recalling the minimal surface equation (3-1), we have

$$a_{11} = 1 + u_2^2, \quad a_{12} = -u_1u_2, \quad a_{21} = -u_1u_2, \quad a_{22} = 1 + u_1^2, \tag{3-16}$$

and

$$u_{22} = \frac{2u_1u_2u_{12} - (1 + u_2^2)u_{11}}{1 + u_1^2}. \tag{3-17}$$

Inserting (3-5) and (3-16)–(3-17) into (3-15), we get

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{ij} \\ = {} & \frac{1}{|\nabla u|^3}\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\ & - \frac{6(1 + |\nabla u|^2)}{|\nabla u|^2(1 + u_1^2)}\big[(1 + u_2^2)u_1u_{11} + (1 - u_1^2)u_2u_{12}\big] \cdot \kappa_1 \\ & - \frac{6(1 + |\nabla u|^2)}{|\nabla u|^2}(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ & - \frac{1}{|\nabla u|^2}\bigg\{\frac{9 + 6|\nabla u|^2}{1 + u_1^2}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big] \\ & \qquad + 3\sum_{ijk} a_{ij}u_ku_{kij}\bigg\} \cdot \kappa.\end{aligned} \tag{3-18}$$

Now the commutation formulas (1-1)–(1-2) and relations (3-16)–(3-17) yield

$$
\begin{aligned}
\text{(3-19)}\quad \sum_{ijk} a_{ij}u_k u_{kij} &= \sum_{ijk} a_{ij}u_k u_{ijk} + \sum_{ijkm} a_{ij}u_k u_m R_{mikj} \\
&= -\sum_{ijk} a_{ij,k}u_k u_{ij} + \sum_{ijkm} a_{ij}u_k u_m R_{mikj} \\
&= \frac{2|\nabla u|^2}{1+u_1^2}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big] \\
&\qquad + |\nabla u|^2(1+|\nabla u|^2)\cdot c.
\end{aligned}
$$

Putting (3-19) into (3-18), we have

$$
\begin{aligned}
\text{(3-20)}\quad &\sum_{ij} a_{ij}\kappa_{ij} \\
&= \frac{1}{|\nabla u|^3}\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\
&\quad - \frac{6(1+|\nabla u|^2)}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_1u_{11} + (1-u_1^2)u_2u_{12}\big]\cdot\kappa_1 \\
&\quad - \frac{6(1+|\nabla u|^2)}{|\nabla u|^2}(u_1u_{12} - u_2u_{11})\cdot\kappa_2 \\
&\quad - \frac{9+12|\nabla u|^2}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big]\cdot\kappa \\
&\quad - 3(1+|\nabla u|^2)\cdot\kappa\cdot c.
\end{aligned}
$$

Straightforward computation gives

$$
\begin{aligned}
\text{(3-21)}\quad &\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\
&= 2u_1u_2\sum_{ij} a_{ij}u_{12ij} - u_1^2\sum_{ij} a_{ij}u_{22ij} - u_2^2\sum_{ij} a_{ij}u_{11ij} \\
&\quad + 2(u_2u_{12} - u_1u_{22})\sum_{ij} a_{ij}u_{1ij} + 2(u_1u_{12} - u_2u_{11})\sum_{ij} a_{ij}u_{2ij} \\
&\quad - 4u_2\sum_{ij} a_{ij}u_{2i}u_{11j} - 4u_1\sum_{ij} a_{ij}u_{1i}u_{22j} + 4u_2\sum_{ij} a_{ij}u_{1i}u_{12j} \\
&\quad + 4u_1\sum_{ij} a_{ij}u_{2i}u_{12j} + 4u_{12}\sum_{ij} a_{ij}u_{1i}u_{2j} \\
&\quad - 2u_{22}\sum_{ij} a_{ij}u_{1i}u_{1j} - 2u_{11}\sum_{ij} a_{ij}u_{2i}u_{2j} \\
&\triangleq \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4,
\end{aligned}
$$

where

$$J_1 = 2u_1u_2 \sum_{ij} a_{ij}u_{12ij} - u_1^2 \sum_{ij} a_{ij}u_{22ij} - u_2^2 \sum_{ij} a_{ij}u_{11ij},$$

$$J_2 = 2(u_2u_{12} - u_1u_{22}) \sum_{ij} a_{ij}u_{1ij} + 2(u_1u_{12} - u_2u_{11}) \sum_{ij} a_{ij}u_{2ij},$$

$$J_3 = -4u_2 \sum_{ij} a_{ij}u_{2i}u_{11j} - 4u_1 \sum_{ij} a_{ij}u_{1i}u_{22j} + 4u_2 \sum_{ij} a_{ij}u_{1i}u_{12j} + 4u_1 \sum_{ij} a_{ij}u_{2i}u_{12j},$$

$$J_4 = 4u_{12} \sum_{ij} a_{ij}u_{1i}u_{2j} - 2u_{22} \sum_{ij} a_{ij}u_{1i}u_{1j} - 2u_{11} \sum_{ij} a_{ij}u_{2i}u_{2j}.$$

We deal with the terms J_1, J_2, J_3 and J_4 consecutively. If we differentiate the minimal surface equation (3-1) twice, then we have

$$\sum_{ij} a_{ij,k}u_{ij} + \sum_{ij} a_{ij}u_{ijk} = 0, \tag{3-22}$$

$$\sum_{ij} a_{ij,kl}u_{ij} + \sum_{ij} a_{ij,k}u_{ijl} + \sum_{ij} a_{ij,l}u_{ijk} + \sum_{ij} a_{ij}u_{ijkl} = 0. \tag{3-23}$$

By the commutation formulas (1-1)–(1-3), we have

$$u_{klij} = u_{ijkl} + \sum_m u_{mi}R_{mklj} + \sum_m u_{mj}R_{mkli} + \sum_m u_{mk}R_{milj} + \sum_m u_{ml}R_{mikj}.$$

It follows that

$$\sum_{ij} a_{ij}u_{klij} = -\Big[\sum_{ij} a_{ij,kl}u_{ij} + \sum_{ij} a_{ij,k}u_{ijl} + \sum_{ij} a_{ij,l}u_{ijk}\Big] + \sum_{ijm} a_{ij}u_{mi}R_{mklj} + \sum_{ijm} a_{ij}u_{mj}R_{mkli} + \sum_{ijm} a_{ij}u_{mk}R_{milj} + \sum_{ijm} a_{ij}u_{ml}R_{mikj}. \tag{3-24}$$

Note that $a_{ij} = (1+|\nabla u|^2)\delta_{ij} - u_iu_j$. It is easy to get

$$a_{ij,k} = 2\sum_m u_mu_{mk}\delta_{ij} - u_{ik}u_j - u_iu_{jk},$$

$$a_{ij,l} = 2\sum_m u_mu_{ml}\delta_{ij} - u_{il}u_j - u_iu_{jl},$$

$$a_{ij,kl} = 2\sum_m u_{ml}u_{mk}\delta_{ij} + 2\sum_m u_mu_{mkl}\delta_{ij} - u_{ikl}u_j - u_{ik}u_{jl} - u_{il}u_{jk} - u_iu_{jkl}.$$

Thus we have

$$\sum_{ij} a_{ij,kl}u_{ij} = 2(u_1u_{22}-u_2u_{12})u_{1kl}+2(u_2u_{11}-u_1u_{12})u_{2kl} + 2u_{1l}u_{1k}u_{22}+2u_{2l}u_{2k}u_{11}-2u_{1k}u_{2l}u_{12}-2u_{2k}u_{1l}u_{12}, \tag{3-25}$$

$$\sum_{ij} a_{ij,k}u_{ijl} = 2u_2u_{2k}u_{11l}+2u_1u_{1k}u_{22l}-2(u_1u_{2k}+u_2u_{1k})u_{12l}, \tag{3-26}$$

$$\sum_{ij} a_{ij,l}u_{ijk} = 2u_2u_{2l}u_{11k}+2u_1u_{1l}u_{22k}-2(u_1u_{2l}+u_2u_{1l})u_{12k}. \tag{3-27}$$

Putting (3-25)–(3-27) into (3-24), we obtain

$$\begin{aligned}\sum_{ij} a_{ij}u_{klij} = -\big[&2(u_1u_{22}-u_2u_{12})u_{1kl}+2(u_2u_{11}-u_1u_{12})u_{2kl}+2u_{1l}u_{1k}u_{22} \\ &+2u_{2l}u_{2k}u_{11}-2u_{1k}u_{2l}u_{12}-2u_{2k}u_{1l}u_{12}+2u_2u_{2k}u_{11l} \\ &+2u_1u_{1k}u_{22l}-2(u_1u_{2k}+u_2u_{1k})u_{12l}+2u_2u_{2l}u_{11k} \\ &+2u_1u_{1l}u_{22k}-2(u_1u_{2l}+u_2u_{1l})u_{12k}\big] \\ &+\sum_{ijm} a_{ij}u_{mi}R_{mklj}+\sum_{ijm} a_{ij}u_{mj}R_{mkli} \\ &+\sum_{ijm} a_{ij}u_{mk}R_{milj}+\sum_{ijm} a_{ij}u_{ml}R_{mikj}.\end{aligned} \tag{3-28}$$

By the commutation formulas (1-1)–(1-2),

$$u_{121} = u_{112}+\sum_m u_mR_{m121} = u_{112}+u_2\cdot c, \tag{3-29}$$

$$u_{122} = u_{221}+\sum_m u_mR_{m212} = u_{221}+u_1\cdot c. \tag{3-30}$$

With (3-16) and (3-29)–(3-30) in hand, formula (3-28) is equivalent to

$$\begin{aligned}\sum_{ij} a_{ij}u_{12ij} = &-2u_2u_{22}u_{111}+2u_2u_{12}u_{112}+2u_1u_{12}u_{221} \\ &-2u_1u_{11}u_{222}-2u_{12}(u_{11}u_{22}-u_{12}^2) \\ &+\big[u_1u_2u_{11}+(4+5u_1^2+5u_2^2)u_{12}+u_1u_2u_{22}\big]\cdot c,\end{aligned}$$

$$\begin{aligned}\sum_{ij} a_{ij}u_{22ij} = &-4u_2u_{22}u_{112}+2(3u_2u_{12}+u_1u_{22})u_{221} \\ &-2(u_2u_{11}+u_1u_{12})u_{222}-2u_{22}(u_{11}u_{22}-u_{12}^2) \\ &+\big[-2(1+u_2^2)u_{11}+10u_1u_2u_{12}+2(1+u_1^2+u_2^2)u_{22}\big]\cdot c,\end{aligned}$$

$$\begin{aligned}\sum_{ij} a_{ij}u_{11ij} = &-2(u_2u_{12}+u_1u_{22})u_{111}+2(u_2u_{11}+3u_1u_{12})u_{112} \\ &-4u_1u_{11}u_{221}-2u_{11}(u_{11}u_{22}-u_{12}^2) \\ &+\big[2(1+u_1^2+u_2^2)u_{11}+10u_1u_2u_{12}-2(1+u_1^2)u_{22}\big]\cdot c.\end{aligned}$$

Therefore,

(3-31)

$$
\begin{aligned}
J_1 = {} & \big[2u_2^3u_{12} - 2u_1u_2^2u_{22}\big]\cdot u_{111} + \big[-2u_2^3u_{11} - 2u_1u_2^2u_{12} + 4u_1^2u_2u_{22}\big]\cdot u_{112}\\
& + \big[4u_1u_2^2u_{11} - 2u_1^2u_2u_{12} - 2u_1^3u_{22}\big]\cdot u_{221} + \big[-2u_1^2u_2u_{11} + 2u_1^3u_{12}\big]\cdot u_{222}\\
& - 2(u_{11}u_{22} - u_{12}^2)\cdot(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})\\
& + \big[2(u_1^2 - u_2^2 + u_1^2u_2^2 - u_2^4)u_{11} + 8u_1u_2u_{12} + 2(-u_1^2 + u_2^2 + u_1^2u_2^2 - u_1^4)u_{22}\big]\cdot c.
\end{aligned}
$$

Let us handle the term J_2. By (1-1)–(1-2), (3-16) and (3-22), we have

$$
\begin{aligned}
\sum_{ij} a_{ij}u_{1ij} &= \sum_{ij} a_{ij}u_{ij1} + \sum_{ijm} a_{ij}u_m R_{mi1j}\\
&= -\sum_{ij} a_{ij,1}u_{ij} + \sum_{ijm} a_{ij}u_m R_{mi1j}\\
&= -2u_1(u_{11}u_{22} - u_{12}^2) + u_1(1 + u_1^2 + u_2^2)\cdot c
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{ij} a_{ij}u_{2ij} &= \sum_{ij} a_{ij}u_{ij2} + \sum_{ijm} a_{ij}u_m R_{mi2j}\\
&= -\sum_{ij} a_{ij,2}u_{ij} + \sum_{ijm} a_{ij}u_m R_{mi2j}\\
&= -2u_2(u_{11}u_{22} - u_{12}^2) + u_2(1 + u_1^2 + u_2^2)\cdot c.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\text{(3-32)}\quad J_2 = {} & -4(u_{11}u_{22} - u_{12}^2)\cdot(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})\\
& + 2(1 + u_1^2 + u_2^2)\cdot(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})\cdot c.
\end{aligned}
$$

For the term J_3, by (3-16) and (3-29)–(3-30), we have

$$
\begin{aligned}
\text{(3-33)}\quad J_3 = {} & \big[(-4u_2 - 4u_2^3)u_{12} + 4u_1u_2^2u_{22}\big]\cdot u_{111}\\
& + \big[(4u_2 + 4u_2^3)u_{11} + (4u_1 + 4u_1u_2^2)u_{12} + (-4u_2 - 8u_1^2u_2)u_{22}\big]\cdot u_{112}\\
& + \big[(-4u_1 - 8u_1u_2^2)u_{11} + (4u_2 + 4u_1^2u_2)u_{12} + (4u_1 + 4u_1^3)u_{22}\big]\cdot u_{221}\\
& + \big[4u_1^2u_2u_{11} + (-4u_1 - 4u_1^3)u_{12}\big]\cdot u_{222}\\
& + \big[(4u_2^2 - 4u_1^2u_2^2 + 4u_2^4)u_{11} + 8u_1u_2u_{12} + (4u_1^2 - 4u_1^2u_2^2 + 4u_1^4)u_{22}\big]\cdot c.
\end{aligned}
$$

Moreover, straightforward computation yields

$$
\text{(3-34)}\qquad J_4 = -2(u_{11}u_{22} - u_{12}^2)\big[(1 + u_2^2)u_{11} - 2u_1u_2u_{12} + (1 + u_1^2)u_{22}\big] = 0.
$$

Combining (3-21) and (3-31)–(3-34), we obtain

$$\begin{aligned}
\text{(3-35)}\quad \sum_{ij} a_{ij}(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})_{ij}
&= \big[(-4u_2-2u_2^3)u_{12}+2u_1u_2^2u_{22}\big]\cdot u_{111}\\
&\quad+\big[(4u_2+2u_2^3)u_{11}+(4u_1+2u_1u_2^2)u_{12}+(-4u_2-4u_1^2u_2)u_{22}\big]\cdot u_{112}\\
&\quad+\big[(-4u_1-4u_1u_2^2)u_{11}+(4u_2+2u_1^2u_2)u_{12}+(4u_1+2u_1^3)u_{22}\big]\cdot u_{221}\\
&\quad+\big[2u_1^2u_2u_{11}+(-4u_1-2u_1^3)u_{12}\big]\cdot u_{222}\\
&\quad-6(u_{11}u_{22}-u_{12}^2)\cdot(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})\\
&\quad+\big[(2u_1^2-4u_1^2u_2^2)u_{11}+(20u_1u_2+4u_1u_2^3+4u_1^3u_2)u_{12}\\
&\qquad+(2u_2^2-4u_1^2u_2^2)u_{22}\big]\cdot c.
\end{aligned}$$

Now let us explore the relations between $u_{111}, u_{112}, u_{221}, u_{222}$ and κ_1, κ_2. If we take the first derivative on both sides of (3-14) and (3-1), respectively, then using (3-5), (3-29)–(3-30) and (3-16), we obtain

$$\begin{aligned}
&-u_2^2\cdot u_{111}+2u_1u_2u_{12}\cdot u_{112}-u_1^2\cdot u_{221}-|\nabla u|^3\cdot\kappa_1\\
&\qquad-3|\nabla u|(u_1u_{11}+u_2u_{12})\cdot\kappa-2u_1(u_{11}u_{22}-u_{12}^2)+2u_1u_2^2\cdot c=0,\\
&-u_2^2\cdot u_{112}+2u_1u_2u_{12}\cdot u_{221}-u_1^2\cdot u_{222}-|\nabla u|^3\cdot\kappa_2\\
&\qquad-3|\nabla u|(u_1u_{12}+u_2u_{22})\cdot\kappa-2u_2(u_{11}u_{22}-u_{12}^2)+2u_1^2u_2\cdot c=0,
\end{aligned}$$

and

$$\begin{aligned}
&(1+u_2^2)\cdot u_{111}-2u_1u_2\cdot u_{112}+(1+u_1^2)\cdot u_{221}+2u_1(u_{11}u_{22}-u_{12}^2)-2u_1u_2^2\cdot c=0,\\
&(1+u_2^2)\cdot u_{112}-2u_1u_2\cdot u_{221}+(1+u_1^2)\cdot u_{222}+2u_2(u_{11}u_{22}-u_{12}^2)-2u_1^2u_2\cdot c=0.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\text{(3-36)}\quad u_{111} &= (-u_2^2+u_1^2+3u_1^2u_2^2+u_1^4)|\nabla u|^{-1}\cdot\kappa_1+(-2u_1u_2-2u_1^3u_2)|\nabla u|^{-1}\cdot\kappa_2\\
&\quad+3|\nabla u|^{-3}\big[(-u_1u_2^2+u_1^3+3u_1^3u_2^2+u_1^5)u_{11}\\
&\qquad+(-u_2^3-u_1^2u_2+3u_1^2u_2^3-u_1^4u_2)u_{12}\\
&\qquad+(-2u_1u_2^2-2u_1^3u_2^2)u_{22}\big]\cdot\kappa\\
&\quad+(-6u_1u_2^2+2u_1^3)|\nabla u|^{-4}\cdot(u_{11}u_{22}-u_{12}^2)+2u_1u_2^2|\nabla u|^{-2}\cdot c,
\end{aligned}$$

$$\begin{aligned}
\text{(3-37)}\quad u_{112} &= (2u_1u_2+2u_1u_2^3)|\nabla u|^{-1}\cdot\kappa_1+(-u_2^2+u_1^2-u_1^2u_2^2+u_1^4)|\nabla u|^{-1}\cdot\kappa_2\\
&\quad+3|\nabla u|^{-3}\big[(2u_1^2u_2+2u_1^2u_2^3)u_{11}\\
&\qquad+(u_1u_2^2+u_1^3+2u_1u_2^4-u_1^3u_2^2+u_1^5)u_{12}\\
&\qquad+(-u_2^3+u_1^2u_2-u_1^2u_2^3+u_1^4u_2)u_{22}\big]\cdot\kappa\\
&\quad+(-2u_2^3+6u_1^2u_2)|\nabla u|^{-4}\cdot(u_{11}u_{22}-u_{12}^2)-2u_1^2u_2|\nabla u|^{-2}\cdot c,
\end{aligned}$$

and

$$u_{221} = (u_2^2 + u_2^4 - u_1^2 - u_1^2 u_2^2)|\nabla u|^{-1} \cdot \kappa_1 + (2u_1u_2 + 2u_1^3 u_2)|\nabla u|^{-1} \cdot \kappa_2 \\ + 3|\nabla u|^{-3}\big[(u_1u_2^2 + u_1u_2^4 - u_1^3 - u_1^3u_2^2)u_{11} \\ + (u_2^3 + u_2^5 + u_1^2u_2 - u_1^2u_2^3 + 2u_1^4u_2)u_{12} \\ + (2u_1u_2^2 + 2u_1^3u_2^2)u_{22}\big] \cdot \kappa \\ + (6u_1u_2^2 - 2u_1^3)|\nabla u|^{-4} \cdot (u_{11}u_{22} - u_{12}^2) - 2u_1u_2^2|\nabla u|^{-2} \cdot c, \tag{3-38}$$

$$u_{222} = (-2u_1u_2 - 2u_1u_2^3)|\nabla u|^{-1} \cdot \kappa_1 + (u_2^2 + u_2^4 - u_1^2 + 3u_1^2u_2^2)|\nabla u|^{-1} \cdot \kappa_2 \\ + 3|\nabla u|^{-3}\big[(-2u_1^2u_2 - 2u_1^2u_2^3)u_{11} \\ + (-u_1u_2^2 - u_1u_2^4 - u_1^3 + 3u_1^3u_2^2)u_{12} \\ + (u_2^3 + u_2^5 - u_1^2u_2 + 3u_1^2u_2^3)u_{22}\big] \cdot \kappa \\ + (2u_2^3 - 6u_1^2u_2)|\nabla u|^{-4} \cdot (u_{11}u_{22} - u_{12}^2) + 2u_1^2u_2|\nabla u|^{-2} \cdot c. \tag{3-39}$$

Inserting (3-36)–(3-39) and (3-17) into (3-35), after some tedious calculation, we get

$$\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\ = \frac{2|\nabla u|(4+3|\nabla u|^2)}{1+u_1^2}\big[(1+u_2^2)u_1u_{11} + (1-u_1^2)u_2u_{12}\big] \cdot \kappa_1 \\ + 2|\nabla u|(4+3|\nabla u|^2)(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ + \frac{4|\nabla u|(2+3|\nabla u|^2)}{1+u_1^2}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big] \cdot \kappa \\ + 2|\nabla u|^3(1+|\nabla u|^2) \cdot \kappa \cdot c. \tag{3-40}$$

By (3-20) and (3-40), we have

$$\sum_{ij} a_{ij}\kappa_{ij} = \frac{2}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_1u_{11} + (1-u_1^2)u_2u_{12}\big] \cdot \kappa_1 \\ + \frac{2}{|\nabla u|^2}(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ - \frac{1}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big] \cdot \kappa \\ - (1+|\nabla u|^2)\kappa \cdot c. \tag{3-41}$$

Then at the point x_0, we take the first derivative of (3-41). Note that

$$\kappa(x_0) = -\frac{u_{11}}{u_2}.$$

With (3-6), (3-16) and (3-29) in hand, we obtain

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{ij1} = {} & \frac{2(1-u_2^2)}{u_2}u_{12}\cdot\kappa_{11} - \frac{2(1-u_2^2)}{u_2}u_{11}\cdot\kappa_{12} \\ & + \left[\frac{2}{u_2}u_{112} + \frac{1+u_2^2}{u_2^2}u_{11}^2 - \frac{3}{u_2^2}u_{12}^2\right]\cdot\kappa_1 \\ & + \left[-\frac{2}{u_2}u_{111} + \frac{4}{u_2^2}u_{11}u_{12}\right]\cdot\kappa_2 + \frac{2(1+u_2^2)}{u_2^3}u_{11}^2u_{111} \\ & + \frac{2}{u_2^3}u_{11}u_{12}u_{112} - \frac{2(1+u_2^2)}{u_2^4}u_{11}^3u_{12} - \frac{2}{u_2^4}u_{11}u_{12}^3 \\ & + \left[(1-u_2^2)\cdot\kappa_1 + \frac{2(1+u_2^2)}{u_2^2}u_{11}u_{12}\right]\cdot c.\end{aligned} \tag{3-42}$$

Now, the equations (3-36) and (3-37) are simplified as

$$u_{111} = -u_2\kappa_1 + \frac{3}{u_2}u_{11}u_{12}, \tag{3-43}$$

$$u_{112} = -u_2\kappa_2 - \frac{1+u_2^2}{u_2}u_{11}^2 + \frac{2}{u_2}u_{12}^2. \tag{3-44}$$

Putting (3-43)–(3-44) into (3-42), one obtains

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{ij1} = {} & \frac{2(1-u_2^2)}{u_2}u_{12}\cdot\kappa_{11} - \frac{2(1-u_2^2)}{u_2}u_{11}\cdot\kappa_{12} + \left[-\frac{3(1+u_2^2)}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\right]\cdot\kappa_1 \\ & - \frac{4}{u_2^2}u_{11}u_{12}\cdot\kappa_2 + \frac{2(1+u_2^2)}{u_2^4}u_{11}^3u_{12} + \frac{2}{u_2^4}u_{11}u_{12}^3 \\ & + \left[(1-u_2^2)\cdot\kappa_1 + \frac{2(1+u_2^2)}{u_2^2}u_{11}u_{12}\right]\cdot c.\end{aligned}$$

Therefore, by commutation formulas (1-1)–(1-2), we get

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{1ij} & = \sum_{ij} a_{ij}\left[\kappa_{ij1} + \sum_m \kappa_m R_{mi1j}\right] = \sum_{ij} a_{ij}\kappa_{ij1} + \kappa_1\cdot c \\ & = \frac{2(1-u_2^2)}{u_2}u_{12}\cdot\kappa_{11} - \frac{2(1-u_2^2)}{u_2}u_{11}\cdot\kappa_{12} \\ & \quad + \left[-\frac{3(1+u_2^2)}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\right]\cdot\kappa_1 - \frac{4}{u_2^2}u_{11}u_{12}\cdot\kappa_2 + \frac{2(1+u_2^2)}{u_2^4}u_{11}^3u_{12} \\ & \quad + \frac{2}{u_2^4}u_{11}u_{12}^3 + \left[(2-u_2^2)\cdot\kappa_1 + \frac{2(1+u_2^2)}{u_2^2}u_{11}u_{12}\right]\cdot c.\end{aligned} \tag{3-45}$$

Thanks to (3-45), the formula (3-13) reduces to

$$
\begin{aligned}
\text{(3-46)}\quad & \sum_{ij} a_{ij}\psi_{ij} \\
&= \big[2u_2(1+u_2^2)g' + 4g\big]u_{12}\cdot\kappa_{11} \\
&\quad - \big[2u_2(1+u_2^2)g' + (6+2u_2^2)g\big]u_{11}\cdot\kappa_{12} - 2gu_{12}\cdot\kappa_{22} \\
&\quad + \Bigg\{\bigg[u_2(1+u_2^2)^2g'' + (3+4u_2^2)(1+u_2^2)g' - \frac{(3-2u_2^2)(1+u_2^2)}{u_2}g\bigg]u_{11}^2 \\
&\qquad + \bigg[u_2(1+u_2^2)g'' + (3+4u_2^2)g' + \frac{1+2u_2^2}{u_2}g\bigg]u_{12}^2\Bigg\}\cdot\kappa_1 \\
&\quad - \frac{4g}{u_2}u_{11}u_{12}\cdot\kappa_2 + \frac{2g}{u_2^3}\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big]u_{11}u_{12} \\
&\quad + \bigg\{\big[u_2^2(1+u_2^2)g' + 3u_2g\big]\cdot\kappa_1 + \frac{2(1+u_2^2)g}{u_2}u_{11}u_{12}\bigg\}\cdot c.
\end{aligned}
$$

By (3-3) and (3-41), we have

$$
\text{(3-47)}\qquad \kappa_{11} = \frac{1}{u_2g}\big[\psi_1 - (u_2g' + g)u_{12}\cdot\kappa_1 + gu_{11}\cdot\kappa_2\big],
$$

$$
\text{(3-48)}\qquad \kappa_{12} = \frac{1}{u_2g}\big[\psi_2 + (u_2g' + g)(1+u_2^2)u_{11}\cdot\kappa_1 + gu_{12}\cdot\kappa_2\big],
$$

and

$$
\begin{aligned}
\text{(3-49)}\quad \kappa_{22} = \frac{1}{u_2g}\bigg\{&-(1+u_2^2)\psi_1 + \big[u_2(1+u_2^2)g' + (3+u_2^2)g\big]u_{12}\cdot\kappa_1 \\
&- (3+u_2^2)gu_{11}\cdot\kappa_2 + \frac{g}{u_2^2}u_{11}\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big] \\
&+ (1+u_2^2)gu_{11}\cdot c\bigg\}.
\end{aligned}
$$

Putting (3-47)–(3-49) into (3-46), we finally get

$$
\begin{aligned}
\text{(3-50)}\quad & \sum_{ij} a_{ij}\psi_{ij} \\
&= \bigg[\frac{2(1+u_2^2)g'}{g} + \frac{6+2u_2^2}{u_2}\bigg]\cdot(u_{12}\psi_1 - u_{11}\psi_2) \\
&\quad + \bigg[u_2(1+u_2^2)g'' - \frac{2u_2(1+u_2^2)g'^2}{g} - 5g' - \frac{9g}{u_2}\bigg]\cdot\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big]\cdot\kappa_1 \\
&\quad + \big[u_2^2(1+u_2^2)g' + 3u_2g\big]\cdot\kappa_1\cdot c.
\end{aligned}
$$

If we let

$$g(t) = \frac{(1+t^2)^{3/2}}{t^3},$$

then the last two terms on the right-hand side of (3-50) vanish. Namely,

$$\sum_{ij} a_{ij}\psi_{ij} = 2u_2 \cdot (u_{12}\psi_1 - u_{11}\psi_2).$$

This completes the proof of Theorem 1.2. □

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References

[Ahlfors 1973] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill, New York, 1973. MR Zbl

[Bianchini et al. 2009] C. Bianchini, M. Longinetti, and P. Salani, "Quasiconcave solutions to elliptic problems in convex rings", *Indiana Univ. Math. J.* **58**:4 (2009), 1565–1589. MR Zbl

[Caffarelli and Spruck 1982] L. A. Caffarelli and J. Spruck, "Convexity properties of solutions to some classical variational problems", *Comm. Partial Differential Equations* **7**:11 (1982), 1337–1379. MR Zbl

[Chang et al. 2010] S.-Y. A. Chang, X.-N. Ma, and P. Yang, "Principal curvature estimates for the convex level sets of semilinear elliptic equations", *Discrete Contin. Dyn. Syst.* **28**:3 (2010), 1151–1164. MR Zbl

[Chern et al. 1999] S. S. Chern, W. H. Chen, and K. S. Lam, *Lectures on differential geometry*, Series on University Mathematics **1**, World Scientific, River Edge, NJ, 1999. MR Zbl

[Gabriel 1957] R. M. Gabriel, "A result concerning convex level surfaces of 3-dimensional harmonic functions", *J. London Math. Soc.* **32** (1957), 286–294. MR Zbl

[Gergen 1931] J. J. Gergen, "Note on the green function of a star-shaped three dimensional region", *Amer. J. Math.* **53**:4 (1931), 746–752. MR Zbl

[Kawohl 1985] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics **1150**, Springer, 1985. MR Zbl

[Kong and Xu 2015] S. Kong and J. Xu, "A remark on the level sets of the graph of harmonic functions bounded by two circles in parallel planes", *J. Partial Differ. Equ.* **28**:3 (2015), 197–207. MR Zbl

[Lewis 1977] J. L. Lewis, "Capacitary functions in convex rings", *Arch. Rational Mech. Anal.* **66**:3 (1977), 201–224. MR Zbl

[Longinetti 1983] M. Longinetti, "Convexity of the level lines of harmonic functions", *Boll. Un. Mat. Ital. A* (6) **2**:1 (1983), 71–75. In Italian. MR Zbl

[Ma and Zhang 2013] X.-N. Ma and W. Zhang, "The concavity of the Gaussian curvature of the convex level sets of p-harmonic functions with respect to the height", *Commun. Math. Stat.* **1**:4 (2013), 465–489. MR Zbl

[Ma and Zhang 2014] X.-N. Ma and Y. Zhang, "The convexity and the Gaussian curvature estimates for the level sets of harmonic functions on convex rings in space forms", *J. Geom. Anal.* **24**:1 (2014), 337–374. MR Zbl

[Ma et al. 2010] X.-N. Ma, Q. Ou, and W. Zhang, "Gaussian curvature estimates for the convex level sets of p-harmonic functions", *Comm. Pure Appl. Math.* **63**:7 (2010), 935–971. MR Zbl

[Ortel and Schneider 1983] M. Ortel and W. Schneider, "Curvature of level curves of harmonic functions", *Canad. Math. Bull.* **26**:4 (1983), 399–405. MR Zbl

[Shiffman 1956] M. Shiffman, "On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes", *Ann. of Math.* (2) **63** (1956), 77–90. MR Zbl

[Talenti 1983] G. Talenti, "On functions, whose lines of steepest descent bend proportionally to level lines", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **10**:4 (1983), 587–605. MR Zbl

[Wang and Zhang 2012] P. Wang and W. Zhang, "Gaussian curvature estimates for the convex level sets of solutions for some nonlinear elliptic partial differential equations", *J. Partial Differ. Equ.* **25**:3 (2012), 239–275. MR Zbl

[Zhang and Zhang 2013] T. Zhang and W. Zhang, "On convexity of level sets of p-harmonic functions", *J. Differential Equations* **255**:7 (2013), 2065–2081. MR Zbl

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EIGENVALUE RESOLUTION OF SELF-ADJOINT MATRICES

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Resolution of a compact group action in the sense described by Albin and Melrose is applied to the conjugation action by the unitary group on self-adjoint matrices. It is shown that the eigenvalues are smooth on the resolved space and that the trivial bundle smoothly decomposes into the direct sum of global one-dimensional eigenspaces.

For a general compact Lie group G acting on a smooth compact manifold with corners M, Albin and Melrose [2011] showed that there is a canonical full resolution such that the group action lifts to the blown-up space $Y(M)$ to have a unique isotropy type. Under this condition, a result of Borel and Ji [2006] applies to show that the orbit space $G \setminus Y(M)$ is smooth.

In this paper, we give an explicit construction of the resolution of the action of the unitary group on the space of self-adjoint matrices

$$S = S(n) = \{X \in M_n(\mathbb{C}) \mid X^* = X\},$$

with the unitary group $\mathrm{U}(n)$ acting by conjugation:

$$u \in \mathrm{U}(n), \quad X \in S, \quad u \cdot X := uXu^{-1}.$$

The orbit of an element $X \in S$, denoted by $\mathrm{U}(n) \cdot X$, consists of the matrices with the same eigenvalues including multiplicities. For a matrix $X \in S$ with m distinct eigenvalues $\{\lambda_j\}_{j=1}^m$ with multiplicities i_k, $k = 1, 2, \ldots, m$, the isotropy group of X is conjugate to a direct sum of smaller unitary groups:

$$\mathrm{U}(n)^X \big(:= \{u \in \mathrm{U}(n) \mid u \cdot X = X\}\big) \cong \bigoplus_{k=1}^{m} U(i_k).$$

The isotropy types are therefore parametrized by the partition of n into integers. Note here that the partition contains information about ordering, for example, the two partitions of 3, $\{i_1 = 1, i_2 = 2\}$ and $\{i_1 = 2, i_2 = 1\}$, are not the same type.

For $n > 1$, the eigenvalues are not smooth functions on S, but are singular where the multiplicities change. Consider the trivial bundle over S, $M := S \times \mathbb{C}^n$, the fiber of which can be decomposed into n eigenspaces of the self-adjoint matrix at the

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base point. This decomposition is not unique at matrices with multiple eigenvalues, and the eigenspaces are not smooth at these base points. We will show that, by doing iterative blow-ups, the singularities are resolved and the eigenvalues become smooth functions on the resolved space. Moreover, by doing a "full" blow-up, the eigenspaces also become smooth.

Recall a lemma on group action resolutions:

Lemma 1 [Albin and Melrose 2011]. *A compact manifold (with corners), M, with a smooth, boundary intersection free, action by a compact Lie group, G, has a canonical full resolution, $Y(M)$, obtained by iterative blow-up of minimal isotropy types.*

In this paper we will discuss two kinds of blow-ups, namely radial and projective blow-ups, which give different results; a projective blow-up of a hypersurface is trivial but a radial blow-up produces a new boundary. A resolution of S involves the choice of blow-up and which centers to blow-up. In this paper, we will discuss three kinds of resolutions:

Definition 2. We define the following three resolutions of S:

(1) radial resolution $\widehat{S}_r$: a radial blow-up of all singular strata {there exists $i \neq j$, $\lambda_i = \lambda_j$} in an order compatible with inclusion of the conjugation class of the isotropy group;

(2) projective resolution $\widehat{S}_p$: a projective blow-up of all singular strata in the same order as radial resolution;

(3) small resolution $\widehat{S}_s$: a radial blow-up of a smaller set of centers

$$\bigcup_{1 \leq i < j \leq n} \{\lambda_i = \lambda_{i+1} = \cdots = \lambda_j\}$$

with the order determined by complete inclusion.

As pointed out in [Albin and Melrose 2011], a projective blow-up usually requires an extra step of reflection in the iterative scheme in order to obtain smoothness. We will show that, the radial resolution yields that the trivial bundle M decomposes into the direct sum of n one-dimensional eigenspaces. By contrast, after projective resolution or small resolution, the eigenvalues are smooth on the resolved space, and locally we have a smooth decomposition into simple eigenspaces, but the trivial bundle doesn't split into global line bundles.

Remark 3. In theory there is a fourth resolution by doing a projective blow-up of the smaller set of centers introduced in $\widehat{S}_s$. This resolves eigenvalues but does not globally resolve eigenbundles, for the same reason as $\widehat{S}_s$. Therefore for simplicity we do not include this resolution in our discussion below.

To describe the different outcomes of the three resolutions above, we recall the resolution in the sense of Albin and Melrose.

Definition 4 (eigenresolution). By an eigenresolution of S, we mean a manifold with corners $\widehat{S}$, with a surjective smooth map $\beta : \widehat{S} \to S$ such that the self-adjoint matrices have a smooth (local) diagonalization when lifted to $\widehat{S}$. Eigenvalues then lift to n smooth functions f_i on $\widehat{S}$, i.e., for any $X \in \widehat{S}$, $\beta(X)$ has eigenvalues $\{f_i(X)\}_{i=1}^n$.

Note that in the definition we only require the diagonalization to exist locally. To encompass the information of global decomposition of eigenvectors, we introduce the full resolution below.

Definition 5 (full eigenresolution). A full eigenresolution is an eigenresolution with global eigenbundles. The eigenvalues lift to n smooth functions f_i on $\widehat{S}$, and the trivial n-dimensional complex vector bundle on $\widehat{S}$ is decomposed into n smooth line bundles:

$$\widehat{S} \times \mathbb{C}^n = \bigoplus_{i=1}^{n} E_i$$

such that

$$\beta(X)v_i = f_i(X)v_i \quad \text{for all } v_i \in E_i(X) \quad \text{for all } X \in \widehat{S}.$$

We use the blow-up constructions introduced by Melrose [1996, Chapter 5] and show that we can obtain resolutions in this way and, in particular, a full resolution if we use radial blow-ups.

Theorem 6. *The three types of resolutions given in Definition 2, namely*, $\widehat{S}_r$, $\widehat{S}_p$, *and* $\widehat{S}_s$, *each yield an eigenresolution. Only the radial resolution* $\widehat{S}_r$ *gives a full eigenresolution.*

Remark 7. In particular, the blow-down map $\beta : \widehat{S} \to S$ is a diffeomorphism between the interior of $\widehat{S}$ and the open dense subset of S consisting of the matrices with n-distinct eigenvalues.

Related to the problem of resolving eigenvalues is the problem of desingularization of polynomial roots. In [Kurdyka and Paunescu 2008], generalizing Rellich's result [1937] on one-dimensional analytical families, the perturbation theory of hyperbolic polynomials is discussed using Hironaka's resolution theory. It is applied to perturbation theory of normal operators and resonances; see for example [Rainer 2013] and [Rauch 1980].

The idea of resolution has been used in many geometric problems. The abstract notion of a resolution structure on a manifold with corners is discussed in [Baum et al. 1985]. In [Davis 1978], it is shown that for a general action the induced action on the set of boundary hypersurfaces can be appropriately resolved. The canonical resolution is presented in [Duistermaat and Kolk 2000], and the induced resolution of the orbit space is considered in [Hassell et al. 1995]. In [Albin and Melrose 2011], an iterative procedure is shown to capture the simultaneous resolution of all isotropy types in a "resolution structure" consisting of equivariant iterated fibrations of the boundary faces, which is the procedure we will use in this paper.

1. Proof of Theorem 6

The proof of Theorem 6 proceeds through induction on the dimension. We begin by discussing the first example which is the 2×2 matrices.

Lemma 8 (2×2 case). *For the 2×2 self-adjoint matrices $S(2)$, the eigenvalues and eigenvectors are smooth except at multiples of the identity. After radial resolution, the singularities are resolved and the trivial 2-dimensional bundle splits into the direct sum of two line bundles. The projective resolution also gives smooth eigenvalues, but does not give two global line bundles.*

Remark 9. Note that in the 2×2 case, the radial resolution $\widehat{S}_r$ and the small resolution $\widehat{S}_s$ are the same.

Proof. In this case

$$S = S(2) = \left\{ \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} \middle| a_{ii} \in \mathbb{R}, z_{12} \in \mathbb{C} \right\} \cong \mathbb{R}^4.$$

The space S is isomorphic to the product of $\mathbb{R}$ and the trace-free subspace

$$S_0 = \left\{ \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} \middle| a_{11} + a_{22} = 0 \right\}, \tag{1}$$

i.e., there is a bijective linear map:

$$\begin{aligned} \phi : \; S & \;\to\; S_0 \times \mathbb{R} \\ A = \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} & \mapsto \big(A_0 := A - \tfrac{1}{2}(a_{11}+a_{22})I,\; \tfrac{1}{2}(a_{11}+a_{22})\big). \end{aligned} \tag{2}$$

The eigenvalues λ_i and eigenvectors v_i of A are related to those of A_0 by $\lambda_i(A) = \lambda_i(A_0) + \frac{1}{2}\operatorname{tr}(A)$, $v_i(A) = v_i(A_0)$, $i = 1, 2$. Therefore, we can restrict the discussion of resolution to the subspace S_0, since the smoothness of eigenvalues and eigenvectors on the resolution of S follows.

Let $z_{12} = c + di$. The space S_0 can be identified with $\mathbb{R}^3 = \{(a_{11}, c, d)\}$. The eigenvalues of this matrix are:

$$\lambda_{\pm} = \pm\sqrt{a_{11}^2 + c^2 + d^2}. \tag{3}$$

Hence the only singularity of the eigenvalues on S_0 is at the point $a_{11} = c = d = 0$ which represents the zero matrix.

Based on the resolution formula in [Melrose 1996], the radial blow-up can be realized as

$$\widehat{S}_{0,r} = [S_0, \{0\}] = S^+N\{0\} \sqcup (S_0 \setminus \{0\}) \simeq \mathbb{S}^2 \times [0, \infty)_+, \tag{4}$$

where the front face $S^+N\{0\} \simeq \mathbb{S}^2$. Here the radial variable is

$$r = \sqrt{a_{11}^2 + c^2 + d^2}.$$

The blow-down map is

$$\beta : [S_0, \{0\}] \to S_0, \quad (r, \theta) \mapsto r\theta, \quad r \in \mathbb{R}_+, \theta \in \mathbb{S}^2. \tag{5}$$

The radial variable r lifts to be smooth on the blown up space; therefore the two eigenvalues $\lambda_\pm = \pm r$ become smooth functions.

Now we consider the eigenvectors to the corresponding eigenvalues $\lambda_\pm$:

$$v_\pm = \big(c + di, \pm\sqrt{a_{11}^2 + c^2 + d^2} - a_{11}\big) \in \mathbb{C}^2. \tag{6}$$

Similar to the discussion of the eigenvalues, the only singularity is at $r = 0$, which becomes a smooth function on $[S_0, \{0\}]$. It follows that v_+ and v_- span two smooth line bundles on $[S_0, \{0\}]$.

If we do the projective blow-up instead, which identifies the antipodal points in the front face of $\mathbb{S}^2$ to get $\mathbb{RP}^2$, namely,

$$\widehat{S}_{0,p} = \{(x, l) \mid x \in l\} \subset \mathbb{R}^3 \times \mathbb{RP}^2, \tag{7}$$

which we can cover with three coordinate patches:

$$(x_1, y_1, z_1) = \Big(c, \frac{d}{c}, \frac{a_{11}}{c}\Big) \in \mathbb{R}^3,$$

and the other two (x_2, y_2, z_2), (x_3, y_3, z_3)=$(d, c/d, a_{11}/d)$, $(a_{11}, c/a_{11}, d/a_{11})$ are similar. The two eigenvalues we get from here are

$$v_\pm = \pm\sqrt{a_{11}^2 + c^2 + d^2} = \pm|x_1|\sqrt{(1 + y_1^2 + z_1^2)},$$

which is smooth across $\{x_1 = 0\}$. Similar discussions hold for the other two coordinate patches.

However, the trivial bundle does not decompose into two line bundles as in the radial case. The nontriviality of eigenbundles can be seen by taking a homotopically nontrivial loop in $\mathbb{RP}^2$

$$l = \beta^{-1}(\{r = 1\}) \subset \widehat{S}_{0,p}.$$

This curve intersects the line $c = d = 0$ twice, which hits at two different places; thus both $a_{11}^\pm = \pm 1$ are on the curve, and equation (6) shows that starting from

$v_- = (0, -2) = (0, -2a_{11}^+)$, this turns into $v_+ = (0, -2) = (0, 2a_{11}^-)$, which means the two eigenvectors are not separated by projective blow-up.

Now that we have done the radial resolution for the trace-free slice S_0, the resolution of S follows. Consider S as a 3-dimensional vector bundle on $\mathbb{R}$ with trace being the projection map. Then at each base point λ, the fiber is $S_0 + \lambda I$. The resolution is $[S_0 + \lambda I; \lambda I] \cong [S_0; \{0\}]$. Since the trace direction is transversal to the blow-up,

$$[S; \mathbb{R}I] = [S_0; \{0\}] \times \mathbb{R}. \tag{8}$$

And because the trace doesn't change the eigenvectors, the smoothness follows. □

To proceed to higher dimensions, we first discuss the partition of eigenvalues into clusters. The basic case is when the eigenvalues are divided into two clusters; then the $\mathrm{U}(n)$ action of the matrices can be decomposed to two commuting actions.

Definition 10 (spectral gap). A connected neighborhood $U \subset S$ has a spectral gap at $c \in \mathbb{R}$, if c is not an eigenvalue of X for any $X \in U$.

Note here that since U is connected, the number of eigenvalues less than c stays the same for all $X \in U$, denoted by k.

Lemma 11 (local eigenspace decomposition). *If a bounded neighborhood $U \subset S(n)$ has a spectral gap at c, then the matrices in U can be decomposed into two smooth self-adjoint commuting matrices*:

$$X = L_X + R_X, L_X R_X = R_X L_X.$$

with $\operatorname{rank}(L_X) = k,\ \operatorname{rank}(R_X) = n - k.$

Proof. Let γ be a simple closed curve on $\mathbb{C}$ such that it intersects with $\mathbb{R}$ only at $-R$ and c, where R is a sufficiently large number such that $-R$ is less than any eigenvalues of the matrices contained in U. In this way, for any matrix $X \in U$, the k smallest eigenvalues are contained inside γ. We consider the operator $P_X : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$P_X := -\frac{1}{2\pi i} \oint_\gamma (X - sI)^{-1}\, ds. \tag{9}$$

Since the resolvent is nonsingular on γ, P_X is a well-defined operator and varies smoothly with X, the integral is independent of choice of γ up to homotopy.

First we show that P_X is a projection operator, i.e.,

$$P_X^2 = P_X. \tag{10}$$

Let γ_s and γ_t be two curves satisfying the above condition with γ_s completely inside γ_t. Then

$$\begin{aligned}P_X^2 &= -\frac{1}{4\pi^2}\oint_{\gamma_t}(X-tI)^{-1}\,dt\left(\oint_{\gamma_s}(X-sI)^{-1}\,ds\right)\\ &= -\frac{1}{4\pi^2}\oint_{\gamma_t}dt\left[\oint_{\gamma_s}\frac{1}{s-t}(X-sI)^{-1}\,ds-\oint_{\gamma_s}\frac{1}{s-t}(X-tI)^{-1}\,ds\right]\\ &= \mathrm{I}-\mathrm{II},\end{aligned}$$

where using the fact that s is completely inside γ_t,

$$\mathrm{I} = -\frac{1}{4\pi^2}\oint_{\gamma_s}\frac{1}{X-sI}\,ds\oint_{\gamma_t}\frac{1}{s-t}\,dt = -\frac{1}{4\pi^2}(-2\pi i)\oint_{\gamma_s}\frac{1}{X-sI}\,ds = P_X,$$

and any t on γ_t is outside of the loop γ_s, so

$$\oint_{\gamma_s}\frac{1}{s-t}\,ds = 0,$$

and we have

$$\mathrm{II} = -\frac{1}{4\pi^2}\oint_{\gamma_t}(X-tI)^{-1}\,dt\oint_{\gamma_s}\frac{1}{s-t}\,ds = 0.$$

This proves (10).

Then we show that P_X is self-adjoint. This is because

$$P_X^* = \frac{1}{2\pi i}\int_{\gamma}((X-sI)^{-1})^*\,d\bar{s} = \frac{1}{2\pi i}\int_{-\bar{\gamma}}(X-sI)\,ds = P_X.$$

P_X maps $\mathbb{R}^n$ to the invariant subspace spanned by the eigenvectors corresponding to eigenvalues that are less than c. We denote this invariant subspace by L and its orthogonal complement by R. Write X as the diagonalization $X = V\Lambda V^{-1}$, where Λ is the eigenvalue matrix and V is the matrix whose columns are the eigenvectors of X. Then L is spanned by the first k columns of V. Take one of the eigenvectors $v_j \in L$, $j = 1, 2, \ldots, k$,

$$\begin{aligned}P_X v_j &= -\frac{1}{2\pi i}\oint_{\gamma}(X-sI)^{-1}v_j\,ds = -\frac{1}{2\pi i}\oint V(\Lambda-sI)^{-1}V^{-1}v_j\,ds\\ &= -\frac{1}{2\pi i}v_j\oint\frac{1}{\lambda_j-s}\,ds = v_j.\end{aligned}$$

Similarly for $v_j \in R$ that corresponds to an eigenvalue greater than c (therefore λ_j is outside the loop),

$$P_X v_j = -\frac{1}{2\pi i}v_j\oint\frac{1}{\lambda_j-s}\,ds = 0;$$

therefore

$$(I - P_X)v_j = v_j \quad \text{for all } v_j \in R.$$

Then using the projection P_X we define two operators L_X and R_X as

$$L_X := P_X X P_X \tag{11}$$

and

$$R_X := (I - P_X)X(I - P_X) \tag{12}$$

Since P_X is smooth, the two operators are also smooth. Moreover, using the fact that P_X is a projection onto the invariant subspace L, we have

$$(I - P_X)X P_X = P_X X(I - P_X) = 0;$$

therefore

$$X = L_X + R_X.$$

For an eigenvector $v \in L$,

$$L_X v = X v \quad \text{and} \quad R_X v = 0, \tag{13}$$

i.e., L_X equals X when restricted to L, similarly $R_X|_R = X$. Since $P_X^* = P_X$, L_X and R_X are also self-adjoint. In this way we get two commuting lower-rank matrices L_X and R_X. □

It is natural to have a finer decomposition when there is more than one spectral gap in the neighborhood, and we have the following corollary.

Corollary 12. *If the eigenvalues of matrices in a neighborhood U can be grouped into k clusters, then the matrices can be decomposed into k lower-rank self-adjoint commuting matrices smoothly.*

Proof. Do the decomposition inductively. If $k = 2$, then it is the case in Lemma 11. Suppose the decomposition for $k = l - 1$ is defined. Then for $k = l$, since the eigenvalues can also be divided into two clusters (by combining the smallest $l - 1$ groups of eigenvalues together), then $X = L_X + R_X$, with L_X and R_X corresponding to the two intervals. Then L_X satisfies the separation condition for $l - 1$ clusters, so by induction, $L_X = L_1 + \cdots + L_{l-1}$. Therefore, $X = L_1 + L_2 + \cdots + L_{l-1} + R_X$ is the desired division. □

Using Lemma 11 of decomposition of matrices in a neighborhood, we can now show that locally the trivial bundle $S \times \mathbb{C}^n$ decomposes into two subspaces if there is a spectral gap. Moreover, locally there is a product structure of two lower-dimensional matrices. In order to see this, we need to introduce the Grassmannian.

Let $\mathrm{Gr}_{\mathbb{C}}(n,k)$ denote the Grassmannian, i.e., the set of k-dimensional subspaces in $\mathbb{C}^n$. Consider the tautological vector bundle over the Grassmannian:

$$\pi_k : T_k \to \mathrm{Gr}_{\mathbb{C}}(n,k), \quad \pi^{-1}(p) = V(p),$$

where each fiber is a k-dimensional subspace in $\mathbb{C}^n$, with self-adjoint operators acting on it. Similarly, we define T_{n-k} to be the orthogonal complement of T_k:

$$\pi_{n-k} : T_{n-k} \to \mathrm{Gr}_{\mathbb{C}}(n,k), \quad \pi^{-1}(p) = V(p)^{\perp}.$$

Definition 13 (Operator bundle). Let P_k (resp. P_{n-k}) be the bundles over $\mathrm{Gr}_{\mathbb{C}}(n,k)$ of the fiberwise self-adjoint operators on the tautological bundle T_k (resp. T_{n-k}).

Take the Whitney sum of the two bundles

$$\pi : P_k \oplus P_{n-k} \to \mathrm{Gr}_{\mathbb{C}}(n,k). \tag{14}$$

Each of its fibers can be identified with $S(k) \oplus S(n-k)$ when we pick a basis. There is a $\mathrm{U}(n)$-action on this bundle:

$$g \cdot (p, (p_k, p_{n-k})) = (g \cdot p, (g \circ p_k \circ g^{-1}, g \circ p_{n-k} \circ g^{-1})), \qquad p \in \mathrm{Gr}_{\mathbb{C}}(n,k),\ p_k \in P_k(p),\ p_{n-k} \in P_{n-k}(p). \tag{15}$$

Suppose an open neighborhood $U \in S$ satisfies the spectral gap condition. Let $\mathrm{U}(n) \cdot U$ be the group invariant neighborhood generated by U, that is,

$$\mathrm{U}(n) \cdot U := \bigcup_{g \in \mathrm{U}(n)} g \cdot U. \tag{16}$$

Then $\mathrm{U}(n) \cdot U$ is open and connected, and also satisfies the spectral gap condition as U does, since the $\mathrm{U}(n)$-action preserves the eigenvalues. From the proof of Lemma 11, it is shown that in the neighborhood, the trivial $\mathbb{C}^n$ bundle over U naturally splits into two subbundles $E^k \oplus E^{n-k}$, and this gives a local product structure. We will prove that, for a $\mathrm{U}(n)$-invariant neighborhood, there is actually a group equivariant homeomorphism with the operator bundles defined above.

Lemma 14 (bundle map). *If a point $X_0 \in S$ satisfies the spectral gap condition, then there is a neighborhood $V \subset S$ such that V is homeomorphic to a neighborhood in the product of lower-rank matrices and the Grassmannian, i.e.,*

$$\phi : V \cong V(k) \times V(n-k) \times V_{\mathrm{Gr}} \subset S(k) \times S(n-k) \times \mathrm{Gr}_{\mathbb{C}}(n,k),$$

which is contained in $P_k \oplus P_{n-k}$ as defined in Definition 13. Moreover, $\mathrm{U}(n) \cdot V$ *is homeomorphic to a neighborhood $W \subset P_k \oplus P_{n-k}$ such that $\pi(W) = \mathrm{Gr}_{\mathbb{C}}(n,k)$ and the map ϕ is* $\mathrm{U}(n)$*-equivariant.*

Proof. From the proof of Lemma 11, there is a neighborhood $X_0 \in U \subset S$, such that each element $X \in U$ is decomposed into $L_X + R_X$. Moreover, this induces a decomposition of the trivial bundle $U \times \mathbb{C}^n$ into two subbundles:

$$U \times \mathbb{C}^n = E^k \oplus E^{n-k}, \tag{17}$$

where $E^k(X)$ and $E^{n-k}(X)$ are determined by the projection operator P_X defined in equation (9):

$$E^k(X) = \mathrm{Im}(P_X) \quad \text{and} \quad E^{n-k}(X) = \mathrm{Im}(P_X)^{\perp}. \tag{18}$$

Let $(\xi_1, \ldots, \xi_k)$ be the basis for $E^k(X_0)$. E^k over U is an open neighborhood in $\mathrm{Gr}_{\mathbb{C}}(n, k)$. We can find a neighborhood V of X_0 (possibly smaller than U) such that, for every point in V, the k-dimensional space E^k projects onto $E^k(X_0)$. And an orthonormal basis of $E^k(X)$ is uniquely determined by requiring the projection of the first j vectors to $E^k(X_0)$ spans $(\xi_1, \ldots, \xi_j)$ for every j smaller than k. In this way we find a basis for each fiber of E^k and E^k is trivialized to be a k-dimensional vector bundle on V. Since the action of X on $\mathbb{C}^n$ has been decomposed to L_X and R_X, then with the choice of basis, the action of L_X on $E^k(X)$ gives a $k \times k$ self-adjoint matrix, and by continuity, these matrices form a neighborhood V_k in $S(k)$. And the same argument works for R_X.

Therefore, we have the following map ϕ:

$$\begin{aligned} \phi : V &\to P_k \oplus P_{n-k} \\ X &\mapsto (E^k(X), (L_X|_{E^k(X)}, R_X|_{E^{n-k}(X)})). \end{aligned} \tag{19}$$

We show this map is a homeomorphism between V and $\phi(V)$. It is injective since the actions of the two invariant subspaces uniquely determine the action on $\mathbb{C}^n$, therefore give the unique operator X. Surjectivity is easy to see. The continuity of ϕ and ϕ^{-1} comes from the continuity of the projection operator defined in Theorem 6.

Now take $\mathrm{U}(n) \cdot V$. Since E^k takes every possible k-subspace of $\mathbb{C}^n$ under the action of $\mathrm{U}(n)$, we know that the first entry of $\phi(\mathrm{U}(n) \cdot V)$ maps onto $\mathrm{Gr}_{\mathbb{C}}(n, k)$. Moreover, since the decomposition respects the action of $\mathrm{U}(n)$, it is easily seen that, for $g \in \mathrm{U}(n)$, $X \in \mathrm{U}(n) \cdot V$,

$$\phi(g \cdot X) = (g \cdot E^k(X), (g \circ L_X \circ g^{-1}, g \circ R_X \circ g^{-1})) = g \cdot (\phi(X)), \tag{20}$$

which means the map is $\mathrm{U}(n)$-equivariant. □

To do the induction, we will need to define an index on the inclusion of isotropy types, so the blow-up procedure could be done in the partial order given by the index. Recall that two matrices have the same isotropy type if they have the same "clustering" of eigenvalues. Now we define the isotropy index of a matrix X as follows.

Definition 15 (Isotropy index). Suppose the eigenvalues of a matrix X are

$$\lambda_1 = \cdots = \lambda_{i_1} < \lambda_{i_1+1} = \cdots = \lambda_{i_2} < \lambda_{i_2+1} = \cdots < \lambda_{i_{k-1}+1} = \cdots = \lambda_n.$$

Then the isotropy index of X is defined as the set

$$I(X) = \{i_0 = 0, i_1, i_2, \ldots, i_{k-1}, i_k = n\}.$$

We denote the set of all matrices with the same isotropy index I as S^I.

There is a partial order of this index on S given by the inclusion. That is, if for two matrices X and Y we have $I(X) \subset I(Y)$, then we say that the order is $X \leq Y$. Note there is an inverse inclusion for isotropy groups. The smallest isotropy index is $I(\lambda I) = \{0, n\}$, while the isotropy group is $\mathrm{U}(n)$, which is the largest. And the largest index is $\{0, 1, 2, \ldots, n-1, n\}$, which corresponds to n distinct eigenvalues, and the isotropy group is the product of n copies of $\mathrm{U}(1)$.

Remark 16. Except the most singular stratum $\{\lambda I\}$, the stratum of other isotropy types are not closed. In fact, the closure of a stratum S^I will include all the stratum $S^{I'}$ with $I' \subset I$. However, the two sets $\{\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_k}\}$ and $\{\lambda_{j_1} = \cdots = \lambda_{j_l}\}$ are transversal once the set $\{\lambda_{\min\{i_1, j_1\}} = \cdots = \lambda_{\max\{i_k, j_l\}}\}$ is blown up. So one can get $\widehat{S}_s$ by blowing up these singular stratum by order of strict inclusion. However, in order to globally decompose the eigenbundle, one needs to blow-up all the intersections first as in $\widehat{S}_r$ (the proof is given later).

For $\widehat{S}_r$ and $\widehat{S}_p$, the total blow-up of $S(n)$ is done by iteratively blowing up the singular strata by the order of isotropy indices. The first step is to blow-up the most singular stratum $S^{\{0,n\}} = \{\mathbb{R}I\}$:

$$[S(n);\, S^{\{0,n\}}].$$

After that we blow-up the second smallest strata $S^{\{0,i,n\}}$, $i = 1, \ldots, n-1$. From the discussion above we know that, for any of such two strata, the intersection of their closure is exactly $S^{\{0,n\}}$ which has been blown up. Therefore one can blow-up these $S^{\{0,i,n\}}$ in any order:

$$\left[S(n);\, S^{\{0,n\}};\, \bigcup_{i=1}^{n-1} S^{\{0,i,n\}}\right].$$

After the second step, the intersection of any two $S^{\{0,i,j,n\}}$ has been blown up. Therefore one can proceed by blowing up those strata in any order. Iteratively, one obtains the following space:

$$\left[S(n);\, S^{\{0,n\}};\, \bigcup_{i=1}^{n-1} S^{\{0,i,n\}};\, \bigcup_{i,j} S^{\{0,i,j,n\}};\, \ldots;\, \bigcup_{0 \leq i_1 < i_2 < \cdots < i_{n-2} \leq n} S^{\{0,i_1,\ldots,i_{n-2},n\}}\right]. \tag{21}$$

In order to do the inductive proof to show this yields the full eigenresolution, the last lemma we need is the compatibility of conjugacy class inclusion and the decomposition to two submatrices, which shows the order of resolution is compatible with the decomposition.

Lemma 17 (Compatibility with conjugacy class). *The partial order of conjugacy class inclusion is compatible with the decomposition in Lemma 11.*

Proof. Suppose a neighborhood $V \subset S(n)$ has a decomposition as Lemma 11. We need to show that, if S^I is the stratum of minimal isotropy type in V, then this stratum corresponds to the minimal isotropy type in $\mathrm{U}(k)$ and $\mathrm{U}(n-k)$.

Since V satisfies the spectral gap condition, the isotropy groups for any elements in V would be subgroups of $\mathrm{U}(k) \oplus \mathrm{U}(n-k)$. Suppose the minimal stratum corresponds to the index $I = \{0, i_1, \ldots, i_m\}$ which must contain k as one element because of the spectral gap condition. Then the isotropy type of two subgroups are $\{0, i_1, \ldots, k\}$ and $\{i_j - k = 0, i_{j+1} - k, \ldots, n-k\}$. They would still be the minimal in each subgroup, otherwise when the two smallest elements are combined it will give a smaller index than I, which is a contradiction. □

Now we can finally prove Theorem 6 using the above lemmas.

Proof of Theorem 6. We prove the theorem by induction on the matrix size. Except special remarks, the discussion below about $\widehat{S}$ applies to all three kinds of resolutions. The 2×2 case is shown in Lemma 8. Suppose the claim holds for all the cases up to $n-1$ dimensions. Now we claim that, by an iterative blow-up, we can get $\widehat{S}(n)$ with eigenvalues and eigenbundles lifted to satisfy the eigenresolution properties.

As in the 2×2 example, we shall first consider the trace-free slice $S_0(n)$ since other slices have the same behavior in terms of smoothness of eigenvalues and eigenbundles, that is, $\widehat{S}(n) = \widehat{S}_0(n) \times \mathbb{R}$. Take the smallest index $I = \{0, n\}$ with the largest possible isotropy group $\mathrm{U}(n)$, and the stratum in $S_0(n)$ with such an isotropy group is a single point, the zero matrix. After blowing up, we get $[S_0; \{0\}]$ as the first step. And in the total $S(n)$ space, this step corresponds to $[S; S^{\{0,n\}} = \{\mathbb{R}I\}] = [S_0; \{0\}] \times \mathbb{R}$.

For any other point $X \notin \{\mathbb{R}I\}$, one can find a bounded neighborhood W such that the matrices in W have a spectral gap as defined in Definition 10. Assume the first k eigenvalues are uniformly bounded below c, then by Lemma 14 there is a fibration structure

$$
(22) \qquad \begin{array}{ccc} V(k) \times V(n-k) & \longrightarrow & W \\ & & \Big\downarrow \pi \\ & & \mathrm{Gr}_{\mathbb{C}}(n,k). \end{array}
$$

And the trivial bundle $W \times \mathbb{C}^n$ naturally splits to the sum $E^k \oplus E^{n-k}$ as in (17). Because of the spectral gap, there are two smallest strata of type $\{\lambda_{i_1} = \cdots = \lambda_{i_j}\}$ and $\{\lambda_{i'_1} = \cdots = \lambda_{i'_j}\}$, with $i_j \leq k$ and $i'_1 \geq k+1$, therefore the two strata are transversal as discussed in the Remark 16, and can be blown up at the same time. This give the iteration step for $\widehat{S}_s$.

Now we consider the radial and projective resolution. For each fiber of π in (22), consider the resolved space $\widehat{V}(k) \times \widehat{V}(n-k) \subset \widehat{S}(k) \times \widehat{S}(n-k)$, where the resolution is done by blowing up all the singular stratum *inside* $V(k)$ and $V(n-k)$. By induction the resolution $\widehat{V}(k)$ resolves the singularity for the first k eigenvalues, and $\widehat{V}(n-k)$ resolves the other $n-k$ eigenvalues. For example, take a point $X \in S(5)$ with eigenvalues $\{\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 = \lambda_5\}$. Near this point there is a product decomposition $V(2) \times V(3) \times \mathrm{Gr}_{\mathbb{C}}(5, 2)$. After the resolution, $\widehat{V}(2) \times \widehat{V}(3)$ resolves the isotropy type $(\{0, 2\} \cup \{0, 1, 2\}) \times (\{0, 3\} \cup \{0, 1, 3\} \cup \{0, 2, 3\} \cup \{0, 1, 2, 3\})$, which, after adjusting numbering of eigenvalues, includes all the isotropy types that could occur with this spectral gap in W. Let $\widehat{W}$ be the this resolved space and denote the blow-down map as

$$\begin{array}{ccc} \beta : \widehat{W} & \longrightarrow & W \\ & \searrow^{\hat{\pi}} & \downarrow \pi \\ & & \mathrm{Gr}_{\mathbb{C}}(n, k). \end{array}$$

Consider the two subbundles E^k and E^{n-k} under the pullback map from β:

$$\begin{array}{ccc} \widehat{E}^k \oplus \widehat{E}^{n-k} & \xrightarrow{\ \beta\ } & E^k \oplus E^{n-k} \\ \downarrow \hat{\phi} & & \downarrow \phi \\ \widehat{W} & \xrightarrow{\ \beta\ } & W. \end{array} \tag{23}$$

By the induction assumptions, $\widehat{V}(k)$ and $\widehat{V}(n-k)$ are eigenresolutions, hence $\widehat{E}^k$ splits into line bundles $\bigoplus_{i=1}^k E_i$ over $\widehat{V}(k)$ and the same for $\widehat{E}^{n-k} = \bigoplus_{i=k+1}^n E_i$ over $\widehat{V}(n-k)$. With the local product structure of π, the Whitney sum $\widehat{E}^k \oplus \widehat{E}^{n-k}$ splits into n eigenbundles locally.

For the radial resolution $\widehat{S}_r$, since the local product structure is $\mathrm{U}(n)$-equivariant, extending to $\bigoplus_{i=1}^n \mathrm{U}(n) \cdot E_i$, we get that the splitting of eigenbundles is global over $\widehat{W}$. We have already shown in Lemma 8 that the projective resolution does not give a global eigendecomposition. Similarly, for the small resolution $\widehat{S}_s$, one can find a closed curve in the base such that one eigenvector switches to another around the curve. We prove this by giving an example: consider the curve of 4×4 matrices of the form $X(t) = U(t)\Lambda(t)U(t)^{-1}$, $0 \leq t \leq 1$, where $U(t)$ is unitary for all t, switching from the identity to its column permutation,

$$U(t) = \begin{cases} (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) & 0 \le t \le \frac{1}{3} \\ U(t) & \frac{1}{3} \le t \le \frac{2}{3} \\ (\vec{e}_3, \vec{e}_4, \vec{e}_1, \vec{e}_2) & \frac{2}{3} \le t \le 1 \end{cases},$$

which smoothly permutes the eigenspace decomposition. On the other hand, $\Lambda(t)$ is always diagonal, going through $\{\lambda_1 = \lambda_2\}$ and $\{\lambda_3 = \lambda_4\}$:

$$\Lambda(t) = \begin{cases} \operatorname{diag}\{-1, -1, 1, 1\} & t = 0 \\ \operatorname{diag}\{-1, -1, 1-t, 1+t\} & 0 \le t \le \frac{1}{3} \\ \operatorname{diag}\{-1-t, -1+t, \frac{1}{3}+t, \frac{5}{3}-t\} & \frac{1}{3} \le t \le \frac{2}{3} \\ \operatorname{diag}\{-2+t, -t, 1, 1\} & \frac{2}{3} \le t \le 1 \\ \operatorname{diag}\{-1, -1, 1, 1\} & t = 1 \end{cases}.$$

With $X(t)$ defined above, one can see that $X(0) = X(1)$ in the stratum that is not blown up in $\widehat{S}_s$. Now consider the lift of the curve to $\widehat{S}_s$, which is still a closed curve. Now one can immediately see that as t goes from 0 to 1, the eigenspace for the first two eigenvalues switches from $\{e_1, e_2\}$ to $\{e_3, e_4\}$. So one cannot obtain a global decomposition.

Even though the eigenbundles do not always split, the three resolutions all resolve eigenvalues. Since the blow-down map β is injective on a dense open set, the eigenvalues extend to the front face to be n smooth functions f_i on $\widehat{W}$ and the splitting of eigendata extends to $\widehat{E}^{n-k} \oplus \widehat{E}^{n-k}$ from nearby such that

$$\beta(X)v_i = f_i(X)v_i \quad \text{for all } v_i \in E_i(X) \quad \text{for all } X \in \widehat{W}.$$

According to Lemma 17 the isotropy index order is preserved when decomposed into two subspaces. By induction, to obtain the global eigenresolution, we have iteratively blown up the strata according to isotropy indices to get $\widehat{S}_r$ as in (21). □

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References

[Albin and Melrose 2011] P. Albin and R. Melrose, "Resolution of smooth group actions", pp. 1–26 in *Spectral theory and geometric analysis*, edited by M. Braverman et al., Contemp. Math. **535**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Baum et al. 1985] P. Baum, J.-L. Brylinski, and R. MacPherson, "Cohomologie équivariante délocalisée", *C. R. Acad. Sci. Paris Sér. I Math.* **300**:17 (1985), 605–608. MR Zbl

[Borel and Ji 2006] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser, Boston, 2006. MR Zbl

[Davis 1978] M. Davis, “Smooth G-manifolds as collections of fiber bundles”, *Pacific J. Math.* **77**:2 (1978), 315–363. MR Zbl

[Duistermaat and Kolk 2000] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Springer, 2000. MR Zbl

[Hassell et al. 1995] A. Hassell, R. Mazzeo, and R. B. Melrose, “Analytic surgery and the accumulation of eigenvalues”, *Comm. Anal. Geom.* **3**:1-2 (1995), 115–222. MR Zbl

[Kurdyka and Paunescu 2008] K. Kurdyka and L. Paunescu, “Hyperbolic polynomials and multiparameter real-analytic perturbation theory”, *Duke Math. J.* **141**:1 (2008), 123–149. MR Zbl

[Melrose 1996] R. B. Melrose, “Differential analysis on manifolds with corners”, book in progress, 1996, available at http://math.mit.edu/~rbm/book.html.

[Rainer 2013] A. Rainer, “Perturbation theory for normal operators”, *Trans. Amer. Math. Soc.* **365**:10 (2013), 5545–5577. MR Zbl

[Rauch 1980] J. Rauch, “Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians”, *J. Funct. Anal.* **35**:3 (1980), 304–315. MR Zbl

[Rellich 1937] F. Rellich, “Störungstheorie der Spektralzerlegung”, *Math. Ann.* **113**:1 (1937), 600–619. MR Zbl

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