Pacific Journal of Mathematics

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Volume 288 No. 1 May 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 288, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.288.1

C^1-UMBILICS WITH ARBITRARILY HIGH INDICES

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We show that C^1-umbilics with arbitrarily high indices exist. This implies that more than C^1-regularity is required to prove Loewner's conjecture.

1. Introduction

The *index* of an isolated umbilic on a given regular surface is the index of the curvature line flow of the surface at that point, which takes values in the set of half-integers. *Loewner's conjecture* asserts that any isolated umbilic on an immersed surface must have index at most 1. *Carathéodory's conjecture* asserts the existence of at least two umbilics on an immersed sphere in $\mathbb{R}^3$, which follows immediately from Loewner's conjecture. Although this problem was investigated mainly on real-analytic surfaces after Hamburger's work [1940; 1941a; 1941b], several geometers recently became interested in nonanalytic cases; see [Ando 2003; Bates 2001; Ghomi and Howard 2012; Gutierrez et al. 1996; Smyth and Xavier 1992]. In particular, Smyth and Xavier [1992] observed that Enneper's minimal surface is inverted to a branched sphere such that the index of the curvature line flow at the branch point is equal to two. Bates [2001] found that the graph of the function

$$B(x,y) := 2 + \frac{xy}{\sqrt{1+x^2}\sqrt{1+y^2}} \tag{1-1}$$

has no umbilics on $\mathbb{R}^2$ and inversion of it gives a genus zero surface without self-intersections, which is differentiable at the image of infinity under that inversion. Ghomi and Howard [2012] gave similar examples of genus zero surfaces using inversion. Moreover, they showed that Carathéodory's conjecture for closed convex surfaces can be reduced to the problem of existence of umbilics of certain entire graphs over $\mathbb{R}^2$. A brief history of Carathéodory's conjecture and recent developments are written also in [Ghomi and Howard 2012]. Recently, Guilfoyle and

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Ando was partly supported by the Grant-in-Aid for Young Scientists (B) 24740048, Japan Society for the Promotion of Science. Umehara was partly supported by the Grant-in-Aid for Scientific Research (A) 262457005, Japan Society for the Promotion of Science.
MSC2010: primary 53A05, 53C45; secondary 57R42, 37C10, 53A30.
Keywords: umbilic, curvature line, Carathéodory's conjecture, Loewner's conjecture.

Klingenberg [2008; 2012] gave an approach to proving the Carathéodory and the Loewner conjectures in the smooth case.

Let $P : U \to \mathbb{R}^3$ be a C^1-immersion defined on an open subset U of $\mathbb{R}^2$ such that P is C^∞-differentiable on $U \setminus \{q\}$ and not C^2-differentiable at q. Then the point $q \in U$ is called a C^1*-umbilic* if the umbilics of P on $U \setminus \{q\}$ do not accumulate to q. At that point q, we can compute the index of the curvature line flow of P. In this paper, we prove the following assertion:

Theorem 1.1. *Let $U_1 \subset \mathbb{R}^2$ be the unit disk centered at the origin. For each positive integer m, there exists a C^1-function $f : U_1 \to \mathbb{R}$ satisfying the following properties*:

(1) *f is real-analytic on $U_1^* := U_1 \setminus \{(0, 0)\}$,*

(2) $(0, 0, f(0, 0))$ *is a C^1-umbilic of the graph of f with index* $1 + (m/2)$.

It should be remarked that the inversion of the graph of Bates' function $B(x, y)$ has a differentiable umbilic of index 2 although not of class C^1 (see Example 2.3). It was classically known that curvature line flows are closely related to the eigenflows of the Hessian matrices of functions (see Appendix A). As an application of the above result, we can show the following:

Corollary 1.2. *For each $m \geq 1$, there exists a C^1-function $\lambda : U_1 \to \mathbb{R}$ satisfying*

(1) *λ is real-analytic on U_1^*, and*

(2) *the eigenflow of the Hessian matrix of λ has an isolated singular point* $(0, 0)$ *with index* $1 + (m/2)$.

When we consider the eigenflow of the Hessian matrix of f, it is well known that the index of the flow at an isolated singular point is equal to half of the index of the vector field

$$d_f := 2f_{xy}\frac{\partial}{\partial x} + (f_{yy} - f_{xx})\frac{\partial}{\partial y}. \tag{1-2}$$

In addition, if $o := (0, 0)$ is an isolated singular point of the eigenflow of the Hessian matrix of f, then its index is equal to $1 + \mathrm{ind}_o(\delta_f)/2$ (see Appendix B), where $\mathrm{ind}_o(\delta_f)$ is the index of the vector field

$$\delta_f := 2(rf_{r\theta} - f_\theta)\frac{\partial}{\partial x} + (-r^2 f_{rr} + rf_r + f_{\theta\theta})\frac{\partial}{\partial y} \tag{1-3}$$

at o, and $x = r\cos\theta,\ y = r\sin\theta$. In order to prove the above theorem, we introduce vector fields D_f and Δ_f analogous to d_f and δ_f, respectively (see Propositions 3.3 and 4.2), and prove the theorem by computing the index of Δ_f at infinity for each of the functions (see Section 5)

$$f = f_m(r, \theta) := 1 + \tanh(r^a \cos m\theta), \quad 0 < a < 1/4,\ m = 1, 2, \ldots. \tag{1-4}$$

We also give an alternative proof of Theorem 1.1 without use of inversion, by an explicit example of λ, see (6-1), satisfying (1) and (2) of Corollary 1.2 (see Section 6).

2. The regularity of the inversion

Let R be a positive number. Consider a function $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$, where

$$\Omega_R := \left\{(x, y) \in \mathbb{R}^2 ;\ \sqrt{x^2+y^2} \leq R\right\}. \tag{2-1}$$

Then $F = (x, y, f(x, y))$ gives a parametrization of the graph of f. The inversion of F is given by $F/(F \cdot F)$, where the dot denotes the inner product on $\mathbb{R}^3$. We consider the following coordinate change:

$$x = \frac{u}{u^2+v^2}, \qquad y = \frac{v}{u^2+v^2}. \tag{2-2}$$

Then

$$\Psi_f := \frac{1}{\rho^2 \hat{f}^2+1}(u, v, \rho^2 \hat{f}\,), \qquad \hat{f}(u, v) := f\left(\frac{u}{\rho^2}, \frac{v}{\rho^2}\right) \tag{2-3}$$

gives a parametrization of the inversion, where $\rho := \sqrt{u^2+v^2}$. The map Ψ_f is defined on the domain

$$U^*_{1/R} := U_{1/R} \setminus \{o\}, \quad \left(U_{1/R} := \left\{(u, v) \in \mathbb{R}^2 ;\ \sqrt{u^2+v^2} < \frac{1}{R}\right\}\right), \tag{2-4}$$

where $o := (0, 0)$. If we set

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{2-5}$$

where $r > 0$, then (2-2) yields

$$\rho = \frac{1}{r}, \qquad u = \rho\cos\theta, \qquad v = \rho\sin\theta. \tag{2-6}$$

In particular, the angular parameter is common in the xy-plane and the uv-plane.

Proposition 2.1. *Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a C^∞-function such that f/r is bounded. Then the inversion $\Psi_f : U^*_{1/R} \to \mathbb{R}^3$ can be continuously extended to* $(0, 0)$, *and moreover*, *if*

$$\left|\frac{f^2 - 2rff_r}{r^2}\right| < 1, \qquad r > R, \tag{2-7}$$

then the image of $\Psi_f = (X, Y, Z)$ can be locally expressed as the graph of a function $Z = Z_f(X, Y)$ on a neighborhood of $(0, 0)$ *in the XY-plane. Under the assumption* (2-7), *the function $Z_f(X, Y)$ is differentiable if and only if*

$$\lim_{r\to\infty} \frac{f}{r} = 0.$$

Proof. We can write

$$\Psi_f(u, v) = \frac{1}{1+\varphi(u, v)^2}\left(u, v, \varphi(u, v)\sqrt{u^2+v^2}\right), \tag{2-8}$$

where

$$\varphi(u,v)=\sqrt{u^2+v^2}\,\hat{f}(u,v)=\frac{f(x,y)}{r}. \tag{2-9}$$

Since f/r is bounded, the function φ is bounded on $U^*_{1/R}$. Thus, using (2-8), we can prove $\lim_{\rho\to 0}\Psi_f=(0,0,0)$, i.e., $\Psi_f(u,v)$ can be continuously extended to $(0,0)$. We denote by $\Pi:\mathbb{R}^3\ni(x,y,z)\mapsto(x,y)\in\mathbb{R}^2$ the orthogonal projection. Setting

$$\psi(\rho,\theta):=\frac{\rho}{1+\varphi(\rho\cos\theta,\rho\sin\theta)^2},$$

it holds that

$$\Pi\circ\Psi_f(u,v)=\bigl(\psi(\rho,\theta)\cos\theta,\ \psi(\rho,\theta)\sin\theta\bigr). \tag{2-10}$$

Since $\hat{f}(\rho\cos\theta,\rho\sin\theta)=f(\cos\theta/\rho,\sin\theta/\rho)$, we have

$$\varphi_\rho=f-rf_r.$$

In particular, it holds that

$$\psi_\rho=\frac{1-(f^2-2rff_r)/r^2}{(1+f^2/r^2)^2}.$$

By (2-7), there exists $\varepsilon>0$ such that $\rho\mapsto\psi(\rho,\theta)$, $|\rho|\le\varepsilon$, is a monotone increasing function for each θ. Thus, by (2-10), we can conclude that $\Pi\circ\Psi_f:\overline{U}_\varepsilon\to\mathbb{R}^2$ is an injection. Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, the inverse map $G:\Omega\to U_\varepsilon$ of $\Pi\circ\Psi_f|_{U_\varepsilon}$ is continuous, where Ω is a neighborhood of the origin of the XY-plane in $\mathbb{R}^3$. Then the graph of

$$Z_f\left(=\frac{\rho\varphi}{1+\varphi^2}\right)=\frac{\varphi(G(X,Y))\rho(G(X,Y))}{1+\varphi(G(X,Y))^2} \tag{2-11}$$

coincides with the image of $\Psi_f=(X,Y,Z)$ around $(0,0,0)$. Then

$$X=\frac{u}{1+\varphi^2},\qquad Y=\frac{v}{1+\varphi^2},\qquad Z=\frac{\rho\varphi}{1+\varphi^2}.$$

Since $\rho\to 0$ as $(X,Y)\to(0,0)$, we obtain

$$\lim_{(X,Y)\to(0,0)}\frac{Z_f(X,Y)}{\sqrt{X^2+Y^2}}=\lim_{(X,Y)\to(0,0)}\frac{\varphi\rho}{\sqrt{u^2+v^2}}=\lim_{\rho\to 0}\varphi=\lim_{r\to\infty}\frac{f}{r}. \qquad \square \tag{2-12}$$

Corollary 2.2. *Suppose that $f:\mathbb{R}^2\setminus\Omega_R\to\mathbb{R}$ is a bounded C^∞-function satisfying*

$$\lim_{r\to\infty}\frac{f_r}{r}=0. \tag{2-13}$$

Then the inversion $\Psi_f:U^*_{1/R}\to\mathbb{R}^3$ can be continuously extended to $(0,0)$, and moreover, the image of Ψ_f is locally a graph which is differentiable at $(0,0)$.

Example 2.3. Bates' example mentioned in the introduction is differentiable. In fact, $B(x, y)$ in (1-1) is bounded and B_r/r converges to zero as $r \to \infty$. However, the inversion of $(x, y, B(x, y))$ is not C^1. In fact, the unit normal vector field of the graph of B is not continuously extended to the point at infinity. Since the inversion preserves the angle, the unit normal vector field of its inversion cannot be continuously extended to $(0, 0, 0)$.

Example 2.4. Ghomi and Howard [2012] gave an example:

$$f_{\mathrm{GH}} = 1 + \lambda \frac{1 + x + y^2}{\sqrt{1 + (x + y^2)^2}}, \qquad (\lambda > 0). \tag{2-14}$$

The graph of f_{GH} is umbilic-free (see Example 3.5 in Section 3). The function f_{GH} is bounded. In addition, since $(f_{\mathrm{GH}})_r$ is bounded, (2-13) is obvious. Therefore, as pointed out in [Ghomi and Howard 2012], the inversion of $(x, y, f_{\mathrm{GH}}(x, y))$ is differentiable. However, it is not a C^1-map. In fact, the limit of the unit normal vector field along $y = 0$ of the graph of f_{GH} is not equal to that along $x + y^2 = 0$ at the point at infinity.

Next, we give a condition for Ψ_f to be extendable as a C^1-map to $(0, 0)$.

Proposition 2.5. *Suppose that* $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ *is a bounded* C^∞*-function satisfying*

(a) $\lim_{r\to\infty} f_r = 0$,

(b) $\lim_{r\to\infty} f_\theta / r = 0$.

Then $\Psi_f = (X, Y, Z)$ *can be extended to* $(0, 0)$ *as a* C^1*-map. Moreover, the map* $(u, v) \mapsto (X(u, v), Y(u, v))$ *is a* C^1*-diffeomorphism from a neighborhood of the origin in the* uv*-plane onto a neighborhood of the origin in the* XY*-plane.*

To prove this, we prepare the following lemma.

Lemma 2.6. *The conditions* (a) *and* (b) *in Proposition 2.5 are equivalent to the following two conditions, respectively*:

(1) $\lim_{\rho\to 0} \rho^2 \hat{f}_\rho = 0$,

(2) $\lim_{\rho\to 0} \rho \hat{f}_\theta = 0$.

Proof. The equivalency of (2) and (b) is obvious. The equivalency of (1) and (a) follows from the identity $\hat{f}_\rho = -f_r/\rho^2$. □

Proof of Proposition 2.5. We see by Corollary 2.2 that Ψ_f can be extended to $(0, 0)$ as a differentiable map and that the map $(u, v) \mapsto (X(u, v), Y(u, v))$ is a homeomorphism from a neighborhood of $(0, 0)$ onto a neighborhood of $(0, 0)$. We set

$$h := \rho^2 \hat{f} (= \rho\varphi), \quad k := (\rho \hat{f})^2 (= \varphi^2). \tag{2-15}$$

By (2-3), we can write

$$\Psi_f = (X, Y, Z) = \frac{1}{k+1}(u, v, h). \tag{2-16}$$

To show that Ψ_f is a C^1-map at $(0,0)$, it is sufficient to show that h and k are C^1-functions. Since h and k are C^∞-functions on $U^*_{1/R}$, they satisfy

$$\begin{aligned} h_u &= \rho\big((2\hat{f} + \rho\hat{f}_\rho)\cos\theta - \hat{f}_\theta \sin\theta\big), \\ h_v &= \rho\big((2\hat{f} + \rho\hat{f}_\rho)\sin\theta + \hat{f}_\theta \cos\theta\big), \end{aligned} \tag{2-17}$$

$$\begin{aligned} k_u &= 2\hat{f}\rho\big(\cos\theta(\hat{f} + \rho\hat{f}_\rho) - \hat{f}_\theta \sin\theta\big), \\ k_v &= 2\hat{f}\rho\big(\sin\theta(\hat{f} + \rho\hat{f}_\rho) + \hat{f}_\theta \cos\theta\big), \end{aligned} \tag{2-18}$$

on $U^*_{1/R}$. Using (1), (2) in Lemma 2.6, (2-17) and (2-18), one can easily see that

$$\lim_{\rho\to 0} h_u = \lim_{\rho\to 0} h_v = \lim_{\rho\to 0} k_u = \lim_{\rho\to 0} k_v = 0, \tag{2-19}$$

which shows that Ψ_f extends to $(0,0)$ as a C^1-map. By (2-16) and (2-19), we have

$$X_u(0,0) = 1, \quad X_v(0,0) = 0, \quad Y_u(0,0) = 0, \quad Y_v(0,0) = 1.$$

Thus the second assertion follows from the inverse mapping theorem, because the Jacobi matrix of the map $(u, v) \mapsto (X(u, v), Y(u, v))$ is regular at $(0,0)$. □

In Section 5, we need the following:

Proposition 2.7. *Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a bounded C^∞-function satisfying conditions (a) and (b) of Proposition 2.5. If there exists a constant $0 \le c < \frac{1}{2}$ such that*

$$r^{1-c/2} f_r, \quad r^{-c/2} f_\theta, \quad r^{2-c} f_{rr}, \quad r^{1-c} f_{r\theta}, \quad r^{-c} f_{\theta\theta}$$

are bounded on $\mathbb{R}^2 \setminus \Omega_R$, then the map $(u, v) \mapsto (X(u, v), Y(u, v))$ is a C^2-map at $(0,0)$, where $\Psi_f = (X, Y, Z)$.

We prepare the following lemmas:

Lemma 2.8. *The boundedness of the five functions in Proposition 2.7 is equivalent to the boundedness of the functions*

$$\rho^{1+c/2} \hat{f}_\rho, \quad \rho^{c/2} \hat{f}_\theta, \quad \rho^{2+c} \hat{f}_{\rho\rho}, \quad \rho^{1+c} \hat{f}_{\rho\theta}, \quad \rho^{c} \hat{f}_{\theta\theta} \tag{2-20}$$

on $U \setminus \{(0,0)\}$, where U is a sufficiently small neighborhood of $(0,0)$.

Proof. Differentiating $\hat{f} = \hat{f}(\rho\cos\theta, \rho\sin\theta)$ by ρ, we get $\rho\hat{f}_\rho = -r f_r$ and $\rho^2 \hat{f}_{\rho\rho} = 2r f_r + r^2 f_{rr}$, which can be used to check the assertion. □

Lemma 2.9. *Suppose that the five functions in (2-20) are bounded on* $U \setminus \{(0,0)\}$. *Then* $\rho^{2c}k_{uu}$, $\rho^{2c}k_{uv}$ *and* $\rho^{2c}k_{vv}$ *are also bounded on* $U \setminus \{(0,0)\}$, *where k is the function given in (2-15).*

Proof. In fact, each of k_{uu}, k_{uv}, k_{vv} is written as a linear combination of

$$1, \quad \rho\hat{f}_\rho, \quad \hat{f}_\theta, \quad (\rho\hat{f}_\rho)^2, \quad \rho\hat{f}_\rho\hat{f}_\theta, \quad \hat{f}_\theta^2, \quad \rho^2\hat{f}_{\rho\rho}, \quad \rho\hat{f}_{\rho\theta}, \quad \hat{f}_{\theta\theta},$$

with coefficients that are bounded functions. For example,

$$k_{uv} = \sin 2\theta\big(\rho^2\hat{f}_\rho^2+\hat{f}(\rho^2\hat{f}_{\rho\rho}+3\rho\hat{f}_\rho-\hat{f}_{\theta\theta})-\hat{f}_\theta^2\big)+2\cos 2\theta\big(\hat{f}_\theta(\rho\hat{f}_\rho+\hat{f})+\rho\hat{f}\hat{f}_{\rho\theta}\big).$$

Thus, we get the assertion. □

Proof of Proposition 2.7. By Lemmas 2.8 and 2.9, the fact that $2c < 1$ yields that

$$\lim_{\rho\to 0}\rho k_{uu} = \lim_{\rho\to 0}\rho k_{uv} = \lim_{\rho\to 0}\rho k_{vv} = 0. \tag{2-21}$$

Since

$$X_{uu} = \frac{2uk_u^2 - 2(k+1)k_u - u(k+1)k_{uu}}{(k+1)^3},$$

$$X_{uv} = -\frac{k_v(-2uk_u + k + 1) + u(k+1)k_{uv}}{(k+1)^3},$$

$$X_{vv} = -\frac{u\big((k+1)k_{vv} - 2k_v^2\big)}{(k+1)^3},$$

we have that X_{uu}, X_{uv}, X_{vv} tend to 0 as $\rho \to 0$. This implies that X_u, X_v are C^1-functions. Similarly, Y_u, Y_v are also C^1-functions. □

3. The pair of identifiers for umbilics

Let U be a domain on $\mathbb{R}^2$. Consider a flow (i.e., a 1-dimensional foliation) $\mathcal{F}$ defined on $U \setminus \{p_1, \dots, p_n\}$, where $p_1, \dots, p_n$ are distinct points in U. We are interested in the case where $\mathcal{F}$ is

- the curvature line flow of an immersion $P : U \to \mathbb{R}^3$,
- the eigenflow of a matrix-valued function on U, or
- the flow induced by a vector field on U.

We fix a simple closed smooth curve $\gamma : T^1 \to U \setminus \{p_1, \dots, p_n\}$, where $T^1 := \mathbb{R}/2\pi\mathbb{Z}$. We set

$$\partial_x := \frac{\partial}{\partial x}, \qquad \partial_y := \frac{\partial}{\partial y}.$$

Then one can take a smooth vector field

$$V(t) := a(t)\partial_x + b(t)\partial_y$$

along the curve $\gamma(t)$ such that $V(t)$ is a nonzero tangent vector of $\mathbb{R}^2$ at $\gamma(t)$ which points in the direction of the flow $\mathcal{F}$. Then the map

$$\check{V} : T^1 \ni t \mapsto \frac{(a(t), b(t))}{\sqrt{a(t)^2 + b(t)^2}} \in S^1 := \{\boldsymbol{x} \in \mathbb{R}^2\,;\ |\boldsymbol{x}| = 1\} \tag{3-1}$$

is called the *Gauss map* of $\mathcal{F}$ with respect to the curve γ. The mapping degree of the map $\check{V}$ is called the *rotation index* of $\mathcal{F}$ with respect to γ and denoted by $\operatorname{ind}(\mathcal{F}, \gamma)$, which is a half-integer, in general. If γ surrounds only p_j, then $\operatorname{ind}(\mathcal{F}, \gamma)$ is independent of the choice of such a curve γ. So we call it the (*rotation*) *index* of the flow $\mathcal{F}$ at p_j, and it is denoted by $\operatorname{ind}_{p_j}(\mathcal{F})$. If the flow $\mathcal{F}$ is generated by a vector field V defined on $U \setminus \{p_1, \dots, p_n\}$, then $\operatorname{ind}_{p_j}(\mathcal{F})$ is an integer, and we denote it by $\operatorname{ind}_{p_j}(V)$.

We denote by $S_2(\mathbb{R})$ the set of real symmetric 2-matrices. Let U be a domain in $\mathbb{R}^2$, and

$$A = \begin{pmatrix} a_{11}(x, y) & a_{12}(x, y) \\ a_{12}(x, y) & a_{22}(x, y) \end{pmatrix} : U \to S_2(\mathbb{R}),$$

a C^∞-map. A point $p \in U$ is called an *equidiagonal point* of A if $a_{11} = a_{22}$ and $a_{12} = 0$ at p. We now suppose that p is an isolated equidiagonal point. Without loss of generality, we may assume that A has no equidiagonal points on $U \setminus \{p\}$. Since two eigenflows of A are mutually orthogonal, the indices of the two eigenflows of the $S_2(\mathbb{R})$-valued function A are the same half-integer at p. We denote it by $\operatorname{ind}_p(A)$.

It is well known that for an $S_2(\mathbb{R})$-valued function A, the formula

$$\operatorname{ind}_p(A) = \tfrac{1}{2} \operatorname{ind}_p(\boldsymbol{v}_A) \tag{3-2}$$

holds, where $\boldsymbol{v}_A$ is the vector field on U given by

$$\boldsymbol{v}_A := (a_{11} - a_{22})\partial_x + a_{12}\partial_y. \tag{3-3}$$

We shall apply these facts to the computation of the indices of isolated umbilics on regular surfaces in $\mathbb{R}^3$ as follows. Let $f : U \to \mathbb{R}$ be a C^∞-function. The symmetric matrices associated with the first and the second fundamental forms of the graph of f are given by

$$I := \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}, \qquad II := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}. \tag{3-4}$$

We consider a $\mathrm{GL}(2, \mathbb{R})$-valued function

$$P := \begin{pmatrix} 0 & \sqrt{1 + f_x^2} \\ -\sqrt{(1 + f_x^2 + f_y^2)/(1 + f_x^2)} & f_x f_y/\sqrt{1 + f_x^2} \end{pmatrix}, \tag{3-5}$$

which satisfies the identity $PP^T = I$, where P^T is the transpose of P. Then

$$A_f := P^{-1} II (P^T)^{-1} = P^T (I^{-1} II)(P^T)^{-1}$$

is an $S_2(\mathbb{R})$-valued function. The umbilics of the graph of f correspond to the equidiagonal points of A_f. We show the following:

Proposition 3.1. *The symmetric matrix $A_f(p)$ is proportional to the identity matrix at $p \in U$ if and only if p gives an umbilic of the graph of f. Moreover, if p is an isolated umbilic, then* $\operatorname{ind}_p(A_f)$ *coincides with the index of the umbilic p.*

Proof. The first assertion follows from the definition of A_f. Without loss of generality, we may assume that p coincides with the origin $o := (0, 0)$, and the graph of f has no umbilics other than o on U. Take a sufficiently small positive number $\varepsilon > 0$ so that the circle

$$\gamma(t) = \varepsilon(\cos t, \sin t), \qquad 0 \le t \le 2\pi,$$

is null-homotopic in U.

We denote by $(a_1(t), b_1(t))^T$ and $(a_2(t), b_2(t))^T$, eigenvectors of $I^{-1}II$ and A_f at $\gamma(t)$, respectively. We may suppose

$$(a_1(t), b_1(t))P(\gamma(t)) = (a_2(t), b_2(t)), \qquad 0 \le t \le 2\pi.$$

We set

$$\boldsymbol{w}_i(t) := a_i(t)\partial_x + b_i(t)\partial_y, \qquad i = 1, 2.$$

Then $\boldsymbol{w}_1$ points in one of the principal directions of the graph of f. The matrix $P(\gamma(t))$ takes values in the set

$$\mathcal{T} := \left\{ \begin{pmatrix} 0 & x \\ -y & z \end{pmatrix};\ x, y > 0,\ z \in \mathbb{R} \right\}. \tag{3-6}$$

Since the set $\mathcal{T}$ is null-homotopic, the mapping degree of $\check{\boldsymbol{w}}_1(t)$ with respect to the origin is equal to that of $\check{\boldsymbol{w}}_2(t)$. Since the degree of $\check{\boldsymbol{w}}_2(t)$ with respect to o coincides with $\operatorname{ind}_o(A_f)$, we get the second assertion. □

By a straightforward calculation, one can get the following identity:

$$\tilde{A}_f := hk^3 A_f = \begin{pmatrix} f_x f_y (f_x f_y f_{xx} - 2h f_{xy}) + h^2 f_{yy} & lk \\ lk & k^2 f_{xx} \end{pmatrix},$$

where

$$h := 1 + f_x^2, \quad k := \sqrt{1 + f_x^2 + f_y^2}, \quad l := -h f_{xy} + f_x f_y f_{xx}.$$

Then the coefficients of the vector field

$$\boldsymbol{v}_{\tilde{A}_f} = v_1 \partial_x + v_2 \partial_y$$

defined as in (3-3) for $A = \tilde{A}_f$ are given by

$$\begin{aligned} v_1 &= \tilde{a}_{11} - \tilde{a}_{22} = (-1 + f_x^2) f_y^2 f_{xx} - h f_{xx} - 2h f_x f_{xy} f_y + h^2 f_{yy}, \\ v_2 &= \tilde{a}_{12} = -k(h f_{xy} - f_x f_y f_{xx}), \end{aligned}$$

where $\tilde{A}_f = (\tilde{a}_{ij})_{i,j=1,2}$. Hence, we get the following identity:

$$v_1 = \frac{2 f_x f_y}{k} v_2 + h\bigl(-f_{xx}(1 + f_y^2) + (1 + f_x^2) f_{yy}\bigr).$$

Consequently, we get the following fact (see [Ghomi and Howard 2012, (10)]):

Fact 3.2. *The graph of the function $z = f(x, y)$ defined on U has an umbilic at $p \in U$ if and only if the functions*

$$d_1(x, y) := (1 + f_x^2) f_{xy} - f_x f_y f_{xx}, \quad \textit{and} \quad d_2(x, y) := (1 + f_x^2) f_{yy} - f_{xx}(1 + f_y^2)$$

both vanish at p.

We consider the vector field

$$D_f := d_1 \partial_x + d_2 \partial_y$$

defined on the domain U in the xy-plane. Suppose that p is a zero of D_f. The following assertion holds:

Proposition 3.3. *If p gives an isolated umbilic of the graph of f, then half of the index of the vector field D_f at p coincides with the index of the umbilic p.*

Proof. The half of the index of the vector field

$$X := -\boldsymbol{v}_{\hat{A}_f} = (2 f_x f_y d_1 - h d_2)\partial_x + k d_1 \partial_y$$

at p is equal to $\operatorname{ind}_p(\tilde{A}_f)$. We now set

$$X_s := (\partial_x, \partial_y) \begin{pmatrix} \dfrac{2 s f_x f_y}{\sqrt{1 + s(f_x^2 + f_y^2)}} & -1 - s f_x^2 \\ 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \qquad 0 \le s \le 1.$$

Then $X = X_1$ and $X_0 = -d_2 \partial_x + d_1 \partial_y$, and the rotation index of X_s at p does not depend on $s \in [0, 1]$. Since the rotation index of $D_f = (d_1, d_2)$ at p coincides with that of X_0, we can conclude that X has the same rotation index as D_f at p. □

We call d_1, d_2 the *Cartesian umbilic identifiers* of the function f.

Example 3.4. For a function $f(x, y) := \operatorname{Re}(z^3) = x^3 - 3xy^2$ $(z = x + iy)$, the Cartesian umbilic identifiers are given by $d_1 = -6y\varphi_1,\ d_2 = -6x\varphi_2$, where

$$\varphi_1 := -9x^4 + 9y^4 + 1, \qquad \varphi_2 := 9x^4 + 18x^2y^2 + 9y^4 + 2.$$

Since $\varphi_i,\ i=1,2$, are positive at the origin $(0,0)$, the vector field D_f can be continuously deformed into the vector field $-y\partial_x - x\partial_y$ preserving the property that the origin is an isolated zero. Thus D_f is of index -1, and the graph of the function f has an isolated umbilic of index $-\frac{1}{2}$ at the origin.

Example 3.5. Bates' function $B(x,y)$ has no umbilics since $d_1 > 0$ on $\mathbb{R}^2$. On the other hand, the identifier d_1 with respect to Ghomi and Howard's function $f_{\mathrm{GH}}(x,y)$ in (2-14) vanishes if and only if $y=0$ or $x=-y^2$. Since d_2 never vanishes on these two sets, the graph of f_{GH} also has no umbilics on $\mathbb{R}^2$.

4. The pair of polar identifiers for umbilics

Let U be a domain in the xy-plane, and $f : U \to \mathbb{R}$ a C^∞-function. Let (r,θ) be the polar coordinate system associated to (x,y) as in (2-5). Then

$$F(r,\theta) := (r\cos\theta, r\sin\theta, f(r\cos\theta, r\sin\theta))$$

gives a parametrization of the graph of f with the unit normal vector

$$\nu := \frac{1}{\sqrt{f_\theta^2 + r^2\left(1+f_r^2\right)}}\left(f_\theta\sin\theta - rf_r\cos\theta,\ -rf_r\sin\theta - f_\theta\cos\theta,\ r\right).$$

Then

$$\widehat{I} := \begin{pmatrix} 1+f_r^2 & f_r f_\theta \\ f_r f_\theta & r^2+f_\theta^2 \end{pmatrix}$$

is the symmetric matrix consisting of the coefficients of the first fundamental form of F. If we set

$$Q = \begin{pmatrix} 0 & \sqrt{1+f_r^2} \\ -\sqrt{f_\theta^2 + r^2\left(1+f_r^2\right)}/\sqrt{1+f_r^2} & f_r f_\theta/\sqrt{1+f_r^2} \end{pmatrix},$$

then $QQ^T = \widehat{I}$. The symmetric matrix consisting of the coefficients of the second fundamental form is given by

$$\widehat{II} := \frac{1}{\sqrt{f_\theta^2 + r^2\left(1+f_r^2\right)}}\begin{pmatrix} rf_{rr} & rf_{r\theta}-f_\theta \\ rf_{r\theta}-f_\theta & r(f_{\theta\theta}+rf_r) \end{pmatrix}.$$

Then the symmetric matrix

$$B_f = Q^{-1}\widehat{II}(Q^{-1})^T = Q^T(\widehat{I}^{-1}\widehat{II})(Q^T)^{-1}$$

satisfies

$$\tilde{B}_f = \hat{h}\hat{k}^3 B_f = \begin{pmatrix} rf_r^2 f_\theta^2 f_{rr} + \hat{h} f_r\left(-2rf_\theta f_{r\theta} + 2f_\theta^2 + r^2\hat{h}\right) + r\hat{h}^2 f_{\theta\theta} & \hat{l}\hat{k} \\ \hat{l}\hat{k} & r\hat{k}^2 f_{rr} \end{pmatrix},$$

where

$$\hat{h} := 1 + f_r^2, \quad \hat{k} := \sqrt{f_\theta^2 + r^2\big(1 + f_r^2\big)}, \quad \hat{l} := f_\theta(\hat{h} + r f_r f_{rr}) - r\hat{h} f_{r\theta}.$$

The following holds:

Proposition 4.1. *The symmetric matrix $\tilde{B}_f(p)$ is proportional to the identity matrix at $p \in U \setminus \{o\}$ if and only if p gives an umbilic of the graph of f. Moreover, if o is an isolated umbilic of the graph of f, then the index of the umbilic at o is equal to* $1 + \mathrm{ind}_o(\tilde{B}_f)$.

Proof. The first assertion follows from the above discussions. So we now prove the second assertion. Suppose o is an isolated umbilic. We take a simple closed smooth curve $\gamma(t)$ in the xy-plane, where $0 \le t \le 2\pi$, which surrounds the origin o anticlockwisely, and does not surround any other umbilics. Let $\boldsymbol{w}_1 : [0, 2\pi] \to \mathbb{R}^2$ be a vector field along γ such that $\boldsymbol{w}_1(t)$ is an eigenvector of the matrix $I^{-1}II$ at $\gamma(t)$ for each $t \in [0, 2\pi]$. Since

$$\begin{aligned} \partial_r &= \cos\theta\,\partial_x + \sin\theta\,\partial_y, \\ \partial_\theta &= -r\sin\theta\,\partial_x + r\cos\theta\,\partial_y, \end{aligned}$$

we have that

$$(\partial_r, \partial_\theta) = (\partial_x, \partial_y)T_0, \qquad T_0 := \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Then, it holds that

$$\widehat{I}^{-1}\widehat{II} = (T_0)^{-1}(I^{-1}II)T_0.$$

In particular,

$$\boldsymbol{w}_2(t) := T_0(\gamma(t))^{-1}\boldsymbol{w}_1(t), \qquad 0 \le t \le 2\pi,$$

gives an eigenvector of the matrix $\widehat{I}^{-1}\widehat{II}$ at $\gamma(t)$. Let $T_s : U \to \mathrm{GL}(2, \mathbb{R}),\ 0 \le s \le 1$, be a map defined by

$$T_s := \begin{pmatrix} \cos\theta & -(r(1-s)+s)\sin\theta \\ \sin\theta & (r(1-s)+s)\cos\theta \end{pmatrix}, \qquad 0 \le s \le 1.$$

Then it gives a continuous deformation of T_0 to the rotation matrix T_1. Since the winding number of the curve $\gamma(t)$ with respect to the origin o is equal to 1, the difference between the rotation indices of $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ is equal to 1. Since the eigenflow of the symmetric matrix $\tilde{B}_f$ is associated with that of the matrix $\widehat{I}^{-1}\widehat{II}$ by Q, the fact that Q takes values in the set $\mathcal{T}$ in Section 3 yields that the index of the umbilic o is equal to $1 + \mathrm{ind}_o(\tilde{B}_f)$. □

We now set

$$\delta_1 := -\tilde{b}_{12}/\hat{k} = -f_\theta(1 + f_r^2 + r f_r f_{rr}) + r\big(1 + f_r^2\big) f_{r\theta},$$

where $\tilde{B}_f = (\tilde{b}_{ij})_{i,j=1,2}$. Then we have

$$\tilde{b}_{11} - \tilde{b}_{22} = -2 f_r f_\theta \delta_1 + r\bigl(1 + f_r^2\bigr)\delta_2,$$

where

$$\delta_2 := \bigl(1 + f_r^2\bigr)(r f_r + f_{\theta\theta}) - f_{rr}\bigl(r^2 + f_\theta^2\bigr).$$

Thus, as in the proof of Proposition 3.3, we get the following assertion:

Proposition 4.2. *Let U be a neighborhood of the origin $o := (0,0)$. Let $f : U \to \mathbb{R}$ be a C^∞-function. Then the graph of f has an umbilic at $p \in U \setminus \{o\}$ if and only if the two functions $\delta_1(r,\theta)$, $\delta_2(r,\theta)$ both vanish at p, where $x = r\cos\theta$ and $y = r\sin\theta$. Further, if o is an isolated umbilic, then half of the index of the vector field*

$$\Delta_f := \delta_1 \partial_x + \delta_2 \partial_y$$

at o equals $-1 + I_f(o)$, where $I_f(o)$ is the index of the umbilic o.

We call δ_1, δ_2 the *polar umbilic identifiers* of the function f.

Example 4.3. Consider the function (where $z = x + iy$)

$$f(x,y) := \mathrm{Re}(z^2\bar{z}) = x^3 + xy^2 = r^3\cos\theta.$$

By straightforward calculations, we have

$$\delta_1 = -2r^3\sin\theta, \qquad \delta_2 = -2r^3(2 - 3r^4 - 6r^4\cos 2\theta)\cos\theta.$$

Since $2 - 3r^4 - 6r^4\cos 2\theta$ is positive for sufficiently small $r > 0$, the vector field Δ_f can be continuously deformed into the vector field $-\sin\theta\,\partial_r - \cos\theta\,\partial_\theta$ preserving the property that the origin is an isolated zero. Thus the rotation index of Δ_f at o is equal to -1, and $I_f(o) = 1 - \frac{1}{2} = \frac{1}{2}$.

We give a generalization of Proposition 4.2 for the computation of the index of the curvature line flow of a surface along an arbitrarily given simple closed curve surrounding several umbilics as follows. Let $z = f(x,y)$ be a C^∞-function defined on $\mathbb{R}^2$ admitting only isolated umbilics. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a C^∞-map satisfying $\gamma(t + 2\pi) = \gamma(t)$ which gives a simple closed curve in the xy-plane such that it surrounds a bounded domain containing the origin o anticlockwisely. Moreover, we assume that $\gamma(t)$ does not pass through any points corresponding to umbilics of the graph of f. We denote by $I_f(\gamma)$ (resp. $\mathrm{ind}_\gamma(\Delta_f)$) the rotation index of the curvature line flow (resp. of the vector field Δ_f) along the simple closed curve γ. Then the formula

$$I_f(\gamma) = 1 + \frac{\mathrm{ind}_\gamma(\Delta_f)}{2} \tag{4-1}$$

can be proved by modifying the proof of Proposition 4.2. Suppose that there exist at most finitely many points $t = t_1, \dots, t_k \in [0, 2\pi]$ such that $\delta_1(\gamma(t))$ vanishes

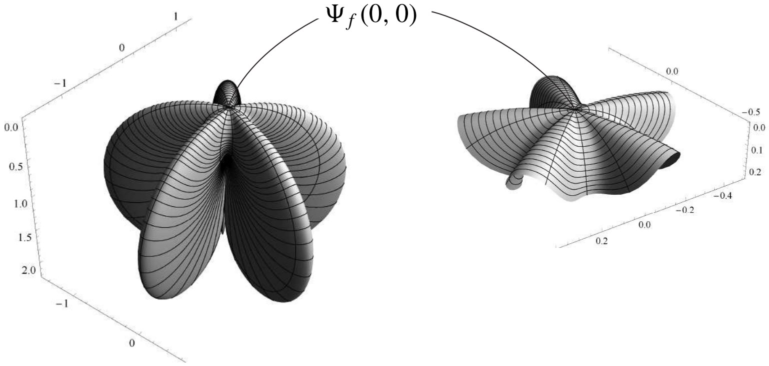

Figure 1. The inversion of the graph f_5 for $a = \frac{1}{5}$ (left) and its enlarged view (right). In these two figures, the z-axis points toward the downward direction.

at $t = t_j$. We now assume that $\delta_1'(\gamma(t)) := d\delta_1(\gamma(t))/dt$ does not vanish at $t = t_j$, for $j = 1, \ldots, k$. We set

$$\varepsilon(t_j) = \begin{cases} 0 & \text{for } \delta_2(\gamma(t_j)) < 0, \\ 1 & \text{for } \delta_1'(\gamma(t_j)) > 0 \text{ and } \delta_2(\gamma(t_j)) > 0, \\ -1 & \text{for } \delta_1'(\gamma(t_j)) < 0 \text{ and } \delta_2(\gamma(t_j)) > 0. \end{cases}$$

Then, it holds that

$$\text{(4-2)} \qquad \operatorname{ind}_\gamma(\Delta_f) = -\sum_{j=1}^{k} \varepsilon(t_j).$$

5. Proof of the main theorem

In this section, using the function $f = f_m$ ($m = 1, 2, 3, \ldots$) given in (1-4), we prove Theorem 1.1 and Corollary 1.2 in the introduction. More generally, we consider the function

$$\text{(5-1)} \quad g = g_m(r, \theta) := 1 + F(r^a \cos m\theta), \qquad 0 < a < 1/4,\ m = 1, 2, 3, \ldots,$$

which is defined on $\{(r, \theta)\,;\ r > R\}$, where R is an arbitrarily fixed positive number, and $F : \mathbb{R} \to \mathbb{R}$ is a bounded C^∞-function satisfying the following conditions:

(i) $F(x)$ is an odd function, that is, it satisfies $F(-x) = -F(x)$,

(ii) the derivative $F'(x)$ of F is a positive-valued bounded function on $\mathbb{R}$,

(iii) the second derivative $F''(x)$ is a bounded function on $\mathbb{R}$ such that $F''(x) < 0$ for $x > 0$,

(iv) there exist three constants α, β and γ ($\beta \neq 0$, $\gamma > 0$) such that

$$\lim_{x\to\infty} e^{\gamma x} F'(x) = \alpha, \qquad \lim_{x\to\infty} e^{\gamma x} F''(x) = \beta.$$

One can easily construct a bounded C^∞-function $F(x)$ satisfying properties (i–iv). For example, one can construct an odd C^∞-function satisfying (ii) and (iii) so that

$$F(x) = 1 - e^{-x}, \qquad x \in [M, \infty),$$

for a positive number M. Then it satisfies (iv) also. However, to prove Theorem 1.1, we must choose the function $F(x)$ to be real-analytic, and

$$F(x) := \tanh x$$

satisfies all of the properties required. In this case, $g_m = f_m$ holds. From now on, we shall prove Theorem 1.1 and Corollary 1.2 using only the above four properties of $F(x)$.

The function g can be considered as a C^∞-function on $\mathbb{R}^2 \setminus \Omega_R$ in the xy-plane for any $R > 0$. The graph of g lies between two parallel planes orthogonal to the z-axis, and is symmetric under rotation by the angle $2\pi/m$ with respect to the z-axis (the entire figure of the inversion of the graph of f_5 is given in the left-hand side of Figure 1). The partial derivatives of the function g are given by

$$\begin{aligned} g_r &= ar^{a-1} c_m F'(r^a c_m), \\ g_\theta &= -mr^a s_m F'(r^a c_m), \\ g_{rr} &= ar^{a-2} c_m \bigl(ar^a c_m F''(r^a c_m) + (a-1) F'(r^a c_m)\bigr), \\ g_{r\theta} &= -amr^{a-1} s_m \bigl(r^a c_m F''(r^a c_m) + F'(r^a c_m)\bigr), \\ g_{\theta\theta} &= m^2 r^a \bigl(r^a s_m^2 F''(r^a c_m) - c_m F'(r^a c_m)\bigr), \end{aligned} \tag{5-2}$$

where

$$c_m := \cos m\theta, \qquad s_m := \sin m\theta. \tag{5-3}$$

Since $F(x)$ is a bounded function, g is bounded and satisfies (2-13), since $a < 2$. Therefore, the inversion Ψ_g can be expressed as a graph near $(0, 0, 0)$. Since $0 < a < 1$, the function g satisfies (a) and (b) of Proposition 2.5. Then $Z = Z_f(X, Y)$ as in (2-11), where $f := g$ is a C^1-function at $(0, 0)$. The graph of Z_g for $g = f_5$ near $(0, 0, 0)$ is indicated in the right-hand side of Figure 1. To prove Theorem 1.1, it is sufficient to show that $(0, 0, 0)$ is a C^1-umbilic of the graph of $Z_g(X, Y)$ with index $1 + (m/2)$. In the following discussions, we would like to show that there exists a positive number R such that the graph of g has no umbilics if $r > R$. We then compute the index $I_g(\Gamma)$ with respect to the circle

$$\Gamma(\theta) := (r\cos\theta, r\sin\theta), \qquad 0 \le \theta \le 2\pi, \ r > R, \tag{5-4}$$

using (4-1) and (4-2), which does not depend on the choice of $r > R$, as follows. We set

$$\check{\delta}_j(\theta) := \delta_j(\Gamma(\theta)), \qquad j = 1, 2. \tag{5-5}$$

The first polar identifier is given by

$$\delta_1 = -mr^a s_m\bigl(ar^a c_m F''(r^a c_m) + (a-1)F'(r^a c_m)\bigr). \tag{5-6}$$

Since $0 < a < 1$, condition (ii) yields that

$$(a-1)F'(r^a c_m) < 0. \tag{5-7}$$

On the other hand, by (i) and (iii), it holds that

$$xF''(x) \le 0, \qquad x := r^a c_m. \tag{5-8}$$

By (5-7) and (5-8), we can conclude that $\check{\delta}_1(\theta)$ changes sign only at the zeros of the function $\sin m\theta$. Since the function g is symmetric with respect to rotation by angle $2\pi/m$, to compute the rotation index of Δ_g along Γ, it is sufficient to check the sign changes of $\check{\delta}_i(\theta)$, $i = 1, 2$, for $\theta = 0$ and $\theta = \pi/m$. By (5-6), (5-7) and (5-8), we get the following:

$$\left.\frac{d\check{\delta}_1}{d\theta}\right|_{\theta=0} > 0, \qquad \left.\frac{d\check{\delta}_1}{d\theta}\right|_{\theta=\pi/m} < 0. \tag{5-9}$$

The second polar identifier δ_2 is given by

$$\begin{aligned} r^{2-3a}\delta_2 = &- r^{2-a}(a^2c_m^2 - m^2s_m^2)F''(c_m r^a) + ac_m(a^2c_m^2 - am^2 + m^2s_m^2)F'(c_m r^a)^3 \\ &- c_m r^{2-2a}(a^2 - 2a + m^2)F'(c_m r^a). \end{aligned}$$

We need the sign of $\check{\delta}_2(\theta)$ at $\theta \in (\pi/m)\mathbb{Z}$. In this case, $s_m = 0$ and $c_m = \pm 1$. Substituting these relations and using the fact that F' (resp. F'') is an even function (resp. an odd function), we have

$$r^{2-3a}\delta_2 = \mp r^{2-a}a^2F''(r^a) \pm a^2(a - m^2)F'(r^a)^3 \mp r^{2-2a}(a^2 - 2a + m^2)F'(r^a).$$

Since F' is bounded, the middle term is bounded. Hence, by (iv) and by the fact that $0 < a < 1$, there exists a positive number R such that the sign of δ_2 is determined by the sign of the first term $\mp r^{2-a}a^2F''(r^a)$ whenever $r > R$. Then, we have

$$-\check{\delta}_2(\pi/m) = \check{\delta}_2(0) > 0. \tag{5-10}$$

In particular, the image of the graph of g has no umbilics when $r > R$. By the $2\pi/m$-symmetry of g, (4-2), (5-9), and (5-10), the index $\operatorname{ind}_\Gamma(\Delta_g)$ is equal to $-m$. Then the index of the curvature line flow along Γ is equal to $I_g(\Gamma) = 1 - m/2$

by (4-1). Then after inversion, the Poincaré–Hopf index formula yields that the index I_0 of the umbilic of Ψ_g at the origin is

$$I_0 = 2 - I_g(\Gamma) = 1 + m/2.$$

If we choose $F(x) := \tanh x$, then the function $Z_g(X, Y)$ satisfies the properties of Theorem 1.1.

We next prove the corollary. We set

$$\lambda := \frac{Z\sqrt{1 + Z_X^2 + Z_Y^2}}{1 + \sqrt{1 + Z_X^2 + Z_Y^2}}, \tag{5-11}$$

where $Z := Z_g$ is the function given in (2-11). Suppose that λ and $\lambda\nu$ are a C^1-function and a C^1-vector field defined on a sufficiently small neighborhood of $(0, 0)$ in the XY-plane, respectively, where ν is a unit normal vector field of the graph of Z_g. Then the map

$$\Phi : (X, Y) \mapsto (\xi(X, Y), \eta(X, Y))$$

given by (A-4) for $f = Z_{f_m}$ is a local C^1-diffeomorphism, and is real-analytic on $U \setminus \{(0, 0)\}$. Then the proof of Fact A.1 in Appendix A is valid in our situation, and we can conclude that the eigenflow of the Hessian matrix of $\lambda(\xi, \eta)$ is equal to the curvature line flow of the map $P(\xi, \eta)$ given by (A-8). Since the image of $P(\xi, \eta)$ coincides with that of $\Psi_{f_m}(u, v)$, we get the proof of the corollary in the introduction.

Thus, it is sufficient to show that λ and $\lambda\nu$ are C^1 at $(X, Y) = (0, 0)$. By (5-11), we have the following expression

$$\lambda\nu = \frac{(ZZ_X, ZZ_Y, -Z)}{1 + \sqrt{1 + Z_X^2 + Z_Y^2}}. \tag{5-12}$$

By (5-11) and (5-12), we can say that $\lambda(X, Y)$ and $\lambda(X, Y)\nu(X, Y)$ are C^1 at $(0, 0)$ if

$$\lim_{(X,Y)\to(0,0)} ZZ_{XX} = \lim_{(X,Y)\to(0,0)} ZZ_{XY} = \lim_{(X,Y)\to(0,0)} ZZ_{YY} = 0 \tag{5-13}$$

hold. So to prove the corollary, it is sufficient to show (5-13). It can be easily seen that all of $r^{1-a}g_r, r^{-a}g_\theta, r^{2-2a}g_{rr},\ r^{1-2a}g_{r\theta}$ and $r^{-2a}g_{\theta\theta}$ are bounded functions on $\mathbb{R}^2 \setminus \Omega_R$. Since $0 < a < \frac{1}{4}$, Proposition 2.7 yields that the map $(u, v) \mapsto (X, Y) = \Pi \circ \Psi_g(u, v)$ is a C^2-map. Then (5-13) is equivalent to

$$\lim_{(u,v)\to(0,0)} ZZ_{uu} = \lim_{(u,v)\to(0,0)} ZZ_{uv} = \lim_{(u,v)\to(0,0)} ZZ_{vv} = 0. \tag{5-14}$$

Since $Z = h/(k + 1)$, (5-14) follows from (2-19), (2-21) and the fact that

$$\lim_{\rho\to 0} \rho h_{uu} = \lim_{\rho\to 0} \rho h_{uv} = \lim_{\rho\to 0} \rho h_{vv} = 0.$$

6. An alternative proof of the main theorem

In the previous section, we have proved Corollary 1.2. However, it is natural to expect that one can give an explicit description of the function with the desired properties. The function λ given in (5-11) does not have a simple expression. On the other hand, we will see that functions

$$\Lambda = \Lambda_m := r^2 \tanh(r^{-a}\cos m\theta), \qquad m = 1, 2, 3, \dots, \tag{6-1}$$

satisfy (1) and (2) of Corollary 1.2 if $0 < a < 1$. We set

$$(\lambda :=)\, \lambda_m := r^2 F(r^{-a}\cos m\theta),$$

where $\xi = r\cos\theta$, $\eta = r\sin\theta$, and $F : \mathbb{R} \to \mathbb{R}$ is a function satisfying the properties (i–iv) given in the beginning of Section 5. Then Λ_m is a special case of λ_m for $F(x) := \tanh x$. It holds that

$$\begin{aligned}
\lambda_r &= r\big(2F(r^{-a}c_m) - ac_m r^{-a}F'(r^{-a}c_m)\big),\\
\lambda_\theta &= -mr^{2-a}s_m F'(r^{-a}c_m),\\
\lambda_{rr} &= 2F(r^{-a}c_m) + ar^{-2a}c_m\big((a-3)r^a F'(r^{-a}c_m) + ac_m F''(r^{-a}c_m)\big),\\
\lambda_{r\theta} &= ms_m r^{1-2a}\big((a-2)r^a F'(r^{-a}c_m) + ac_m F''(r^{-a}c_m)\big),\\
\lambda_{\theta\theta} &= -m^2 r^{2-2a}\big(r^a c_m F'(r^{-a}c_m) - s_m^2 F''(r^{-a}c_m)\big),
\end{aligned}$$

where c_m and s_m are defined in (5-3). We set

$$\zeta_1 := 2(r\lambda_{r\theta} - \lambda_\theta), \qquad \zeta_2 := -r^2\lambda_{rr} + r\lambda_r + \lambda_{\theta\theta}.$$

Then each component of the vector field $\delta_\lambda := \zeta_1\partial_x + \zeta_2\partial_y$ is an identifier for the eigenflow of the Hessian matrix of λ at the origin given in the introduction; see (1-3). By a direct calculation, we have

$$\begin{aligned}
\zeta_1 &= 2mr^{2-2a}s_m\big(ac_m F''(r^{-a}c_m) + (a-1)r^a F'(r^{-a}c_m)\big),\\
\zeta_2 &= -r^{2-2a}(a^2c_m^2 - m^2s_m^2)F''(r^{-a}c_m) - (a^2 - 2a + m^2)r^{2-a}c_m F'(r^{-a}c_m).
\end{aligned}$$

By the property (ii) of F, $(a-1)r^a F'(r^{-a}c_m)$ is negative, and by (ii) and (iii), $c_m F''(r^{-a}c_m)$ is also negative. So ζ_1 is positively proportional to $-s_m (= -\sin m\theta)$. In particular, ζ_1 vanishes only when $s_m = 0$. Moreover, for fixed r, it holds that $d\zeta_1/d\theta < 0$ (resp. $d\zeta_1/d\theta > 0$) if $c_m = 1$ (resp. $c_m = -1$).

On the other hand, if $s_m = 0$ and r tends to zero, then $c_m = \pm 1$ and $F'(\pm r^{-a})$ and $F''(\pm r^{-a})$ tend to zero with exponential order (see condition (iv) for $F(x)$). Therefore, the leading term of ζ_2 for small r is $-r^{2-2a}(a^2c_m^2 - m^2s_m^2)F''(r^{-a}c_m)$. Hence, for a fixed sufficiently small r, the function ζ_2 is positive (resp. negative) if $c_m = 1$ (resp. $c_m = -1$). Summarizing these facts, one can easily show that the

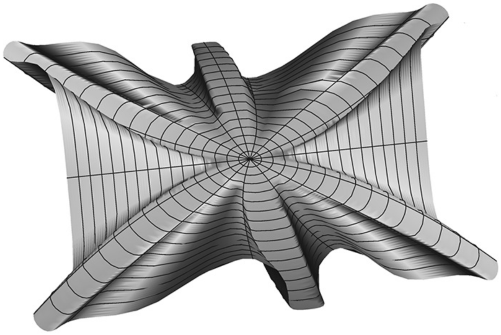

Figure 2. The image of P $\left(r \le \frac{1}{2}\right)$ for $m = 2$ and $a = \frac{1}{2}$.

index of the vector field δ_λ at $o := (0, 0)$ is equal to m. So the index of the eigenflow of the Hessian matrix of λ at o is equal to $1 + m/2$ (see Appendix B). One can easily check that λ is a C^1-function at o and the function λ satisfies (1) and (2) of Corollary 1.2. Since Λ is a special case of λ, we proved that Λ satisfies the desired properties.

To give an alternative proof of Theorem 1.1, we consider the real analytic map $P : \mathbb{R}^2 \setminus \{o\} \to \mathbb{R}^3$ defined (see (A-8)) by

$$P(\xi, \eta) := (\xi, \eta, \Lambda(\xi, \eta)) - \Lambda(\xi, \eta)\nu(\xi, \eta),$$

where

$$\nu := \frac{1}{\Lambda_\xi^2 + \Lambda_\eta^2 + 1}(2\Lambda_\xi, 2\Lambda_\eta, \Lambda_\xi^2 + \Lambda_\eta^2 - 1). \tag{6-2}$$

One can easily verify that

$$\Lambda_\xi = r^{1-a}\big((ms_1 s_m - ac_1 c_m)\operatorname{sech}^2(r^{-a}c_m) + 2r^a c_1 \tanh(r^{-a}c_m)\big),$$
$$\Lambda_\eta = r^{1-a}\big(2r^a s_1 \tanh(r^{-a}c_m) - (as_1 c_m + mc_1 s_m)\operatorname{sech}^2(r^{-a}c_m)\big),$$

where $c_1 = \cos\theta$ and $s_1 = \sin\theta$. Using them, one can get the expressions

$$\Lambda_{\xi\xi} = \frac{1}{r^{2a}}h_1(r, \theta), \quad \Lambda_{\xi\eta} = \frac{1}{r^{2a}}h_2(r, \theta), \quad \Lambda_{\eta\eta} = \frac{1}{r^{2a}}h_3(r, \theta), \tag{6-3}$$

where $h_i(r, \theta)$, $i = 1, 2, 3$, are continuous functions defined on $\mathbb{R}^2$. Using (6-2), (6-3) and the fact $\lim_{r\to 0} \Lambda/r^{2a} = 0$, we have

$$\lim_{r\to 0} \Lambda\nu_\xi = \lim_{r\to 0} \frac{\Lambda}{r^{2a}}(r^{2a}\nu_\xi) = 0, \tag{6-4}$$

and also

$$\lim_{r\to 0} \Lambda\nu_\eta = 0. \tag{6-5}$$

Using (6-4), (6-5) and the fact

$$d(\Lambda\nu) = (d\Lambda)\nu + \Lambda d\nu,$$

we can conclude that $\Lambda\nu$ can be extended as a C^1-function at o. Thus $P(\xi, \eta)$ can also be extended as a C^1-differentiable map at o. One can also easily check that

$$P_\xi(0,0) = (1,0,0), \qquad P_\eta(0,0) = (0,1,0).$$

Hence P is an immersion at o, and

$$\Phi : (\xi, \eta) \mapsto (X(\xi,\eta), Y(\xi,\eta))$$

is a local C^1-diffeomorphism, where $P = (X, Y, Z)$. In particular,

$$Z_\Lambda := Z(\Phi^{-1}(X, Y))$$

gives a function defined on a neighborhood of $(X, Y) = (0, 0)$. By Fact A.1 in Appendix A, the index of the curvature line flow at $(0, 0)$ of the graph of Z_Λ is equal to the index of the eigenflow of the Hessian matrix of Λ, which implies Theorem 1.1. The image of P for $m = 3$ and $a = \frac{1}{2}$ is given in Figure 2.

7. The duality of indices

At the end of this paper, we consider the index at infinity for eigenflows of Hessian matrices. Let

$$f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}, \qquad g : U_{1/R} \setminus \{o\} \to \mathbb{R}$$

be C^2-functions, where Ω_R and $U_{1/R}$ are disks defined in Section 2. Let $\mathcal{H}_f$ (resp. $\mathcal{H}_g$) be the eigenflow of the Hessian matrix of f (resp. g). If the Hessian matrix of f has no equidiagonal points, then we can consider the index $\mathrm{ind}(\mathcal{H}_f, \Gamma)$ with respect to the circle Γ given in (5-4) and it is independent of the choice of $r > R$. So we denote it by $\mathrm{ind}_\infty(\mathcal{H}_f)$. Similarly, if the Hessian matrix of g has no equidiagonal points, then we can consider the index $\mathrm{ind}(\mathcal{H}_g, \Gamma')$ with respect to the circle $\Gamma'(\theta) := (\rho\cos\theta, \rho\sin\theta)$, $0 \le \theta \le 2\pi$, $\rho < 1/R$. Since it is independent of the choice of $\rho < 1/R$, we denote it by $\mathrm{ind}_o(\mathcal{H}_g)$. Consider the plane-inversion

$$\iota : \mathbb{R}^2 \in (u, v) \mapsto \frac{1}{u^2 + v^2}(u, v) \in \mathbb{R}^2.$$

Then the following assertion holds:

Proposition 7.1 (duality of indices). *Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a C^2-function whose Hessian matrix has no equidiagonal points. Then the function $g : \Omega_R \to \mathbb{R}$ defined by*

$$g(x, y) := (u^2 + v^2) f \circ \iota(u, v)$$

(*called the dual of* f) *satisfies*

$$\mathrm{ind}_o(\mathcal{H}_g) + \mathrm{ind}_\infty(\mathcal{H}_f) = 2.$$

Proof. Using the identification of (u, v) and $z = u + iv$, it holds that $u = (z + \bar{z})/2$ and $v = (z - \bar{z})/(2i)$. In particular, f can be considered as a function of variables z and $\bar{z}$, and can be denoted by $f = f(z, \bar{z})$. Since $\iota(z) = 1/\bar{z}$, we can write

$$g(z, \bar{z}) := z\bar{z} f(1/\bar{z}, 1/z).$$

Then

$$g_{zz}(z, \bar{z}) = \frac{\bar{z} f_{\bar{z}\bar{z}}(1/\bar{z}, 1/z)}{z^3}$$

holds, where

$$\frac{\partial}{\partial z} := \tfrac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big), \quad \frac{\partial}{\partial \bar{z}} := \tfrac{1}{2}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big).$$

Since $\Gamma(\theta) = re^{i\theta}$, we have that

$$g_{zz}(\Gamma(\theta)) = \frac{f_{\bar{z}\bar{z}}(\iota \circ \Gamma(\theta))}{r^2 e^{4i\theta}}.$$

Thus, it holds that

$$\mathrm{ind}_o(g_{zz}, \Gamma) = -4 + \mathrm{ind}_o(f_{\bar{z}\bar{z}}, \iota \circ \Gamma).$$

By (B-1), we have

$$\begin{aligned} \mathrm{ind}_o(g_{zz}, \Gamma) &= -2\,\mathrm{ind}_o(\mathcal{H}_g), \\ \mathrm{ind}_o(f_{\bar{z}\bar{z}}, \iota \circ \Gamma) &= -\,\mathrm{ind}_o(f_{zz}, \iota \circ \Gamma) = 2\,\mathrm{ind}_\infty(\mathcal{H}_f). \end{aligned}$$

Thus we get the assertion. □

Applying Proposition 7.1 for the function $g = \Lambda_m$, see (6-1), we get the following:

Corollary 7.2. *For each* $m \geq 1$, *there exists a* C^1*-function* $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ *satisfying*

1. f *is real-analytic on* $\mathbb{R}^2 \setminus \Omega_R$,
2. *the eigenflow of the Hessian matrix of* f *has no singular points*, *and*
3. *the index at infinity of the eigenflow of* H_f *is equal to* $1 - m/2$.

The function Λ_m used in the second proof of Theorem 1.1 coincides with the dual of the function $f_m - 1$ given in (1-4).

Appendix A: The classical reduction

In this appendix we show the existence of a special coordinate system (ξ, η) of the graph of a function $f(x, y)$ which reduces the curvature line flow to the Hessian of a certain function, called Ribaucour's parametrization (Umehara learned this from Konrad Voss at the conference of Thessaloniki 1997). Although, the existence of such a coordinate system was classically known, and a proof is in the appendix of [Scherbel 1993], the authors will give the proof here for the sake of convenience. We set $P = (x, y, f(x, y))$, and suppose that $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Consider a sphere which is tangent to the graph of f at P and also tangent to the xy-plane at a point Q. Then, it holds that

$$Q + \lambda \boldsymbol{e}_3 = P + \lambda \nu, \tag{A-1}$$

where $\boldsymbol{e}_3 = (0, 0, 1)$ and $\nu = (f_x, f_y, -1)/\sqrt{1 + f_x^2 + f_y^2}$. Taking the third component of (A-1), we get

$$\lambda = \frac{f\sqrt{1 + f_x^2 + f_y^2}}{1 + \sqrt{1 + f_x^2 + f_y^2}}. \tag{A-2}$$

In particular, $\lambda(0, 0) = 0$. Since $f_x(0, 0) = f_y(0, 0) = 0$, we have that

$$d\lambda(0, 0) = df(0, 0) = 0. \tag{A-3}$$

Taking the exterior derivative of (A-1), and using (A-3) and $\lambda(0, 0) = 0$, we have $dP(0, 0) = dQ(0, 0)$. So, if we set $Q = (\xi(x, y), \eta(x, y), 0)$, then it holds that

$$\begin{aligned}(\xi_x(0, 0)dx + \xi_y(0, 0)dy,\ \eta_x(0, 0)dx + \eta_y(0, 0)dy,\ 0) &= dQ \\ &= dP = (dx, dy, f_x(0, 0)dx + f_y(0, 0)dy) = (dx, dy, 0),\end{aligned}$$

which implies that the Jacobi matrix of the map

$$\Phi : (x, y) \mapsto (\xi(x, y), \eta(x, y)) \tag{A-4}$$

is the identity matrix at $(0, 0)$. So we can take (ξ, η) as a new local coordinate system. Differentiating (A-1) by ξ and η, we get the following two identities:

$$Q_\xi + \lambda_\xi \boldsymbol{e}_3 = P_\xi + \lambda_\xi \nu + \lambda \nu_\xi, \qquad Q_\eta + \lambda_\eta \boldsymbol{e}_3 = P_\eta + \lambda_\eta \nu + \lambda \nu_\eta.$$

Taking the inner products of them and ν, these two equations yield

$$Q_\xi \cdot \nu + \lambda_\xi \nu_3 = \lambda_\xi, \qquad Q_\eta \cdot \nu + \lambda_\eta \nu_3 = \lambda_\eta, \tag{A-5}$$

where we set $\nu = (\nu_1, \nu_2, \nu_3)$. Since $Q = (\xi, \eta, 0)$, we have that $Q_\xi = (1, 0, 0)$ and $Q_\eta = (0, 1, 0)$. So $Q_\xi \cdot \nu = \nu_1$ and $Q_\eta \cdot \nu = \nu_2$. Substituting this into (A-5), we have

$$\lambda_\xi = \frac{\nu_1}{1 - \nu_3}, \qquad \lambda_\eta = \frac{\nu_2}{1 - \nu_3}. \tag{A-6}$$

This implies that $(\lambda_\xi, \lambda_\eta)$ is the image of ν via the stereographic projection, and

$$\nu = \frac{1}{1+\lambda_\xi^2+\lambda_\eta^2}(2\lambda_\xi, 2\lambda_\eta, \lambda_\xi^2+\lambda_\eta^2-1). \tag{A-7}$$

By (A-1), we have

$$P = (\xi, \eta, 0) - \lambda\nu + (0, 0, \lambda). \tag{A-8}$$

We prove the following:

Fact A.1. *The curvature line flow of the graph $z = f(x, y)$ coincides with the eigenflow of the Hessian of the function $\lambda(\xi, \eta)$ given by (A-2).*

Proof. Noticing (A-8), we set

$$\Delta_{(\xi,\eta)} := \det\begin{pmatrix} \nu \\ dP \\ d\nu \end{pmatrix} = \det\begin{pmatrix} \nu \\ d\xi, d\eta, d\lambda \\ d\nu \end{pmatrix}.$$

Then this gives a map $\Delta_{(\xi,\eta)} : T_{(\xi,\eta)}\mathbb{R}^2 \to \mathbb{R}$ such that

$$\Delta_{(\xi,\eta)}\Big(a\frac{\partial}{\partial\xi} + b\frac{\partial}{\partial\eta}\Big) = \det\big(\nu, aP_\xi(\xi,\eta) + bP_\eta(\xi,\eta), a\nu_\xi(\xi,\eta) + b\nu_\eta(\xi,\eta)\big) \in \mathbb{R}.$$

It is well known that $\boldsymbol{w} \in T_{(\xi,\eta)}\mathbb{R}^2$ points in a principal direction of P at (ξ, η) if and only if $\Delta_{(\xi,\eta)}(\boldsymbol{w}) = 0$. Since $(\nu_1)^2 + (\nu_2)^2 + (\nu_3)^2 = 1$, (A-6) yields that

$$\lambda_\xi\nu_1 + \lambda_\eta\nu_2 = \frac{(\nu_1)^2 + (\nu_2)^2}{1-\nu_3} = \frac{1-(\nu_3)^2}{1-\nu_3} = 1 + \nu_3,$$

which implies $\nu_3 = \lambda_\xi\nu_1 + \lambda_\eta\nu_2 - 1$. We now set $\mu = 2/(1+\lambda_\xi^2+\lambda_\eta^2)$. Differentiating (A-7), we have

$$d\nu = \frac{d\mu}{\mu}\nu + \mu(d\lambda_\xi, d\lambda_\eta, \lambda_\xi d\lambda_\xi + \lambda_\eta d\lambda_\eta).$$

The first term of the right-hand side of the above equation is proportional to ν and does not affect the computation of $\Delta_{(\xi,\eta)}$. So we have that

$$\begin{aligned}
\Delta_{(\xi,\eta)} &= \mu\begin{vmatrix} \nu_1 & \nu_2 & \lambda_\xi\nu_1 + \lambda_\eta\nu_2 - 1 \\ d\xi & d\eta & \lambda_\xi d\xi + \lambda_\eta d\eta \\ d\lambda_\xi & d\lambda_\eta & \lambda_\xi d\lambda_\xi + \lambda_\eta d\lambda_\eta \end{vmatrix} \\
&= \mu\begin{vmatrix} \nu_1 & \nu_2 & -1 \\ d\xi & d\eta & 0 \\ d\lambda_\xi & d\lambda_\eta & 0 \end{vmatrix} = -\mu\begin{vmatrix} d\xi & d\eta \\ d\lambda_\xi & d\lambda_\eta \end{vmatrix} \\
&= \mu\big((\lambda_{\xi\xi} - \lambda_{\eta\eta})d\xi d\eta - \lambda_{\xi\eta}(d\xi^2 - d\eta^2)\big).
\end{aligned}$$

Fact A.1 follows from this representation of $\Delta_{(\xi,\eta)}$. □

Appendix B: Indices of eigenflows of Hessian matrices

Let $g : \Omega_R \setminus \{o\} \to \mathbb{R}$ be a C^2-function, where Ω_R is the closed disk of radius R centered at the origin $o := (0, 0)$; see (2-1). The Hessian matrix of g is given by

$$H_g := \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}.$$

We denote by $\mathcal{H}_g$ the eigenflow of H_g. A point $p \in \Omega_R \setminus \{o\}$ is called an *equidiagonal point* of $\mathcal{H}_g$ if $H_g(p)$ is proportional to the identity matrix. Consider the circle

$$\Gamma(\theta) := r(\cos\theta, \sin\theta), \qquad 0 \le \theta < 2\pi, r < R.$$

If there are no equidiagonal points on $\Omega_R \setminus \{o\}$, then we can define the index $\operatorname{ind}(\mathcal{H}_g, \Gamma)$ of the eigenflow $\mathcal{H}_g$ with respect to Γ, which does not depend on the choice of r. We call it the index of $\mathcal{H}_g$ at the origin and denote it by $\operatorname{ind}_o(\mathcal{H}_g)$. Consider the vector field

$$d_g := 2g_{xy}\frac{\partial}{\partial x} + (g_{yy} - g_{xx})\frac{\partial}{\partial y}.$$

It is well known that the mapping degree of the Gauss map, see (3-1),

$$\check{d}_g : T^1 := \mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto \frac{d_g(\Gamma(\theta))}{|d_g(\Gamma(\theta))|} \in S^1 := \{(x, y) \in \mathbb{R}^2 ;\ x^2 + y^2 = 1\}$$

is equal to $2\operatorname{ind}_o(\mathcal{H}_g)$. Using the correspondence $(x, y) \mapsto x + iy$, we identify $\mathbb{R}^2$ with $\mathbb{C}$, where $i = \sqrt{-1}$. Then

$$g_z = \tfrac{1}{2}(g_x - ig_y),$$
$$g_{zz} = \tfrac{1}{4}((g_{xx} - g_{yy}) - 2ig_{xy}),$$

where $g_z := \partial g/\partial z,\ g_{zz} := \partial^2 g/\partial z^2$ and

$$\frac{\partial}{\partial z} := \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

Thus, d_g can be identified with the right-angle rotation of $\overline{g_{zz}}$. In particular,

$$\operatorname{ind}_o(\mathcal{H}_g) = -\tfrac{1}{2}\operatorname{ind}_o(g_{zz}). \tag{B-1}$$

Here g_{zz} is considered as a vector field and $\operatorname{ind}_o(g_{zz})$ is its index at the origin. Let (r, θ) be as in (2-5). Then $z = re^{i\theta}$ and

$$g_z = \frac{e^{-i\theta}}{2r}(rg_r - ig_\theta),$$
$$g_{zz} = \frac{e^{-2i\theta}}{4r^2}\big((r^2 g_{rr} - rg_r - g_{\theta\theta}) + 2i(g_\theta - rg_{r\theta})\big).$$

We consider the vector field defined by

$$\delta_g := 2(rg_{r\theta} - g_\theta)\frac{\partial}{\partial x} + (-r^2 g_{rr} + rg_r + g_{\theta\theta})\frac{\partial}{\partial y}. \tag{B-2}$$

Since, from [Klotz 1959, (18)],

$$\operatorname{ind}_o(\overline{g_{zz}}) = 2 + \operatorname{ind}_o(\delta_g),$$

we obtain the following:

Lemma B.1. *The identity* $\operatorname{ind}_o(\mathcal{H}_g) = 1 + \operatorname{ind}_o(\delta_g)/2$ *holds.*

Acknowledgements

The authors thank Wayne Rossman for valuable comments. Umehara also thanks Udo Hertrich-Jeromin for fruitful conversations.

References

[Ando 2003] N. Ando, "An umbilical point on a non-real-analytic surface", *Hiroshima Math. J.* **33**:1 (2003), 1–14. MR Zbl

[Bates 2001] L. Bates, "A weak counterexample to the Carathéodory conjecture", *Differential Geom. Appl.* **15**:1 (2001), 79–80. MR Zbl

[Ghomi and Howard 2012] M. Ghomi and R. Howard, "Normal curvatures of asymptotically constant graphs and Carathéodory's conjecture", *Proc. Amer. Math. Soc.* **140**:12 (2012), 4323–4335. MR Zbl

[Guilfoyle and Klingenberg 2008] B. Guilfoyle and W. Klingenberg, "Proof of the Carathéodory conjecture", preprint, 2008. arXiv

[Guilfoyle and Klingenberg 2012] B. Guilfoyle and W. Klingenberg, "From global to local: an index bound for umbilic points on smooth convex surfaces", preprint, 2012. arXiv

[Gutierrez et al. 1996] C. Gutierrez, F. Mercuri, and F. Sánchez-Bringas, "On a conjecture of Carathéodory: analyticity versus smoothness", *Experiment. Math.* **5**:1 (1996), 33–37. MR Zbl

[Hamburger 1940] H. Hamburger, "Beweis einer Caratheodoryschen Vermutung, I", *Ann. of Math.* (2) **41** (1940), 63–86. MR Zbl

[Hamburger 1941a] H. L. Hamburger, "Beweis einer Caratheodoryschen Vermutung, II", *Acta Math.* **73** (1941), 175–228. MR Zbl

[Hamburger 1941b] H. L. Hamburger, "Beweis einer Caratheodoryschen Vermutung, III", *Acta Math.* **73** (1941), 229–332. MR Zbl

[Klotz 1959] T. Klotz, "On G. Bol's proof of Carathéodory's conjecture", *Comm. Pure Appl. Math.* **12** (1959), 277–311. MR Zbl

[Scherbel 1993] H. Scherbel, *A new proof of Hamburger's index theorem on umbilical points*, Ph.D. thesis, Eidgenöessische Technische Hochschule Zürich, 1993, http://search.proquest.com/docview/304065217. MR

[Smyth and Xavier 1992] B. Smyth and F. Xavier, "A sharp geometric estimate for the index of an umbilic on a smooth surface", *Bull. London Math. Soc.* **24**:2 (1992), 176–180. MR Zbl

Received August 7, 2015. Revised September 25, 2016.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 288 No. 1 May 2017
