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SHANGQUAN BU AND GANG CAI

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WELL-POSEDNESS OF SECOND-ORDER DEGENERATE DIFFERENTIAL EQUATIONS WITH FINITE DELAY IN VECTOR-VALUED FUNCTION SPACES

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We give necessary and sufficient conditions of the L^p-well-posedness (respectively, $B^s_{p,q}$-well-posedness) for the second-order degenerate differential equation with finite delay: $(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t)$, ($t \in [0, 2\pi]$) with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, and F and G are bounded linear operators from $L^p([-2\pi, 0]; X)$ (respectively, $B^s_{p,q}([-2\pi, 0]; X)$) into X.

1. Introduction

The purpose of this paper is to study the well-posedness of the following second-order degenerate differential equations with finite delays:

$$(P_2) \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t) & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi), \end{cases}$$

where $\mathbb{T} := [0, 2\pi]$, A and M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, F and G are bounded linear operators from $L^p([-2\pi, 0]; X)$ (resp. $B^s_{p,q}([-2\pi, 0]; X)$) into X, u_t and u'_t are defined on $[-2\pi, 0]$ by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ when $t \in \mathbb{T}$.

Let $1 \le p < \infty$. We say that (P_2) is L^p-well-posed, if for all $f \in L^p(\mathbb{T}; X)$, there exists a unique $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X) \cap L^p(\mathbb{T}; D(A))$, such that $u' \in L^p(\mathbb{T}; D(M))$, $Mu' \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$, and (P_2) is satisfied a.e. on $\mathbb{T}$. Here $D(A)$ and $D(M)$ are equipped with their graph norms so that they become Banach spaces, and $W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$ is the X-valued periodic Sobolev space of order 1. Our main result in this paper gives a necessary and sufficient condition for (P_2) to be L^p-well-posed. Precisely,

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we show that when the underlying Banach space X is a UMD Banach space and $1 < p < \infty$, if the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is Rademacher bounded, then (P_2) is L^p-well-posed if and only if $\rho_p(P_2) = \mathbb{Z}$, and the sets $\{k^2 M N_k : k \in \mathbb{Z}\}$, $\{k N_k : k \in \mathbb{Z}\}$ are Rademacher bounded, where

$$N_k = (k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}, \quad (k \in \mathbb{Z}), \tag{1-1}$$

$F_k, G_k \in \mathcal{L}(X)$ are defined by $F_k x = F(e_k x)$, $G_k x = G(e_k x)$ with $e_k(t) = e^{ikt}$ (see Theorem 2.4). We also study the well-posedness of (P_2) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$, and a necessary and sufficient condition for (P_2) to be $B^s_{p,q}$-well-posed is also given (see Theorem 3.3).

The main tools we will use are operator-valued Fourier multipliers on $L^p(\mathbb{T}; X)$ and $B^s_{p,q}(\mathbb{T}; X)$. Indeed, we will transform the well-posedness of (P_2) to an operator-valued Fourier multiplier problem in the corresponding vector-valued function spaces. Thus the operator-valued Fourier multipliers theorems obtained by Arendt and Bu [2002; 2004] on $L^p(\mathbb{T}; X)$ and $B^s_{p,q}(\mathbb{T}; X)$ are fundamental for us.

The results obtained in this paper recover the known results presented in Bu and Fang [2010] in the nondegenerate case when $M = I_X$ and $\alpha = 0$. Thus our results may be also regarded as generalizations of the previous known results when $M = I_X$ and $F = G = 0$ in the L^p-well-posedness and the $B^s_{p,q}$-well-posedness obtained in [Arendt and Bu 2002; 2004]. Our results also generalize the previous known results obtained by Bu [2013] in the simpler case when $F = G = 0$ and $\alpha = 0$.

A large number of partial differential equations arising in physics and applied sciences, such as in the flow of fluid through fissured rocks, thermodynamics and shear in second-order fluids or in the theory of control of dynamical systems, can be expressed by the model in the form of (P_2). See [Lizama 2006; Bu and Fang 2009; 2010; Lizama and Ponce 2011; 2013; Poblete and Pozo 2013; 2014] for the study of vector-valued degenerate equations with delays. See the monographs by Favini and Yagi [1999] and by Sviridyuk and Fedorov [2003] for detailed studies of abstract degenerate type differential equations.

At the end of this paper, we give concrete examples to which our abstract results may be applied. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $1 < p < \infty$ and m be a nonnegative bounded measurable function defined on Ω; let $X = H^{-1}(\Omega)$, $F, G : L^p([-2\pi, 0]; X) \to X$ be bounded linear operators. If M is the multiplication operator by m on $H^{-1}(\Omega)$ with domain of definition $D(M)$ and $A = \Delta$ is the Laplacian on X with Dirichlet boundary condition and we assume that $D(A) \subset D(M)$, then under suitable assumptions on F and G we obtain the L^p-well-posedness for the corresponding second-order degenerate differential equations with finite delays (see Example 4.1). Our abstract results can also be applied in the following situation: let H be a complex Hilbert space, $1 < p < \infty$ and $F, G \in \mathcal{L}(L^p([-2\pi, 0]; H), H)$ be delay operators, P be a densely defined positive selfadjoint operator on H with $P \geq \delta > 0$. If $M = P - \epsilon$ with $\epsilon < \delta$, and $A = \sum_{i=0}^{k} a_i P^i$ with $a_i \geq 0,\ a_k > 0$. If we assume that $0 \in \rho(M)$, then we obtain the L^p-well-posedness of the corresponding second-order degenerate differential equations with finite delays under suitable assumptions on F and G (see Example 4.2).

This work is organized as follows. In Section 2, we study the well-posedness of (P_2) in $L^p(\mathbb{T}; X)$. In Section 3, we consider the well-posedness of (P_2) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. In Section 4, we give examples of degenerate differential equations with finite delays to which our abstract results may be applied.

2. Well-posedness in Lebesgue–Bochner spaces

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y. If $X = Y$, we will denote it simply by $\mathcal{L}(X)$. Let $1 \leq p < \infty$. We denote by $L^p(\mathbb{T}; X)$ the space of all X-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\|f\|_{L^p} := \left(\int_0^{2\pi} \|f(t)\|^p \,\frac{dt}{2\pi}\right)^{1/p} < \infty.$$

If $f \in L^1(\mathbb{T}; X)$, we define

$$\hat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t) f(t)\,dt,$$

the k-th Fourier coefficient of f, where $k \in \mathbb{Z}$ and $e_k(t) := e^{ikt}$ for $t \in \mathbb{T}$.

Definition. Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X, Y)$ is said to be Rademacher bounded (R-bounded, in short), if there exists $C > 0$ such that

$$\sum_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \epsilon_j T_j x_j \right\| \leq C \sum_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|$$

for all $T_1, \dots, T_n \in \mathbf{T}$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$.

It is clear from the definition that if $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R-bounded, then $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still R-bounded. It is also clear that each R-bounded set is norm bounded. It is known that each norm bounded subset of $\mathcal{L}(X)$ is R-bounded if and only if X is isomorphic to a Hilbert space [Arendt and Bu 2002, Proposition 1.13]. The main tool in the study of L^p-well-posedness of (P_2) is the operator-valued L^p-Fourier multipliers.

Definition. Let X, Y be Banach space and $1 \leq p < \infty$. We say $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique $u \in L^p(\mathbb{T}; Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

It follows easily from the closed graph theorem that when $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier, then there exists a unique $T \in \mathcal{L}(L^p(\mathbb{T}; X), L^p(\mathbb{T}; Y))$, such that $\widehat{Tf}(k) = M_k \hat{f}(k)$ when $f \in L^p(\mathbb{T}; X)$ and $k \in \mathbb{Z}$. The following results were established in [Arendt and Bu 2002]:

Proposition 2.1. *Let X, Y be Banach spaces and assume that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Theorem 2.2. *Let X, Y be* UMD *spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

In this section, we study the following second-order degenerate differential equation with finite delays:

$$(P_2) \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi), \end{cases}$$

where A, M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, and $F, G : L^p([-2\pi, 0]; X) \to X$ are fixed bounded linear operators. Moreover, for fixed $t \in \mathbb{T}$, u_t and u'_t are elements of $L^p([-2\pi, 0]; X)$ defined by $u_t(s) = u(t + s)$, $u'_t(s) = u'(t + s)$ for $-2\pi \le s \le 0$. Here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

To give the definition of the solution space for (P_2), we need to introduce vector-valued periodic Sobolev space of order 1. For $1 \le p < \infty$, we define the periodic "Sobolev" space of order 1 [Arendt and Bu 2002] by:

$$W^{1,p}_{\text{per}}(\mathbb{T}; X) := \{u \in L^p(\mathbb{T}; X) : \text{ there exists } v \in L^p(\mathbb{T}; X) \text{ such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Let $u \in L^p(\mathbb{T}; X)$. Then $u \in W^{1,p}_{\text{per}}(\mathbb{T}; X)$ if and only if u is differentiable a.e. on $\mathbb{T}$ and $u' \in L^p(\mathbb{T}; X)$; in this case, u is actually continuous and $u(0) = u(2\pi)$ [Arendt and Bu 2002, Lemma 2.1].

Let $1 \le p < \infty$. We define the solution space of the L^p-well-posedness for (P_2) by

$$S_p(A, M) := \{u \in L^p(\mathbb{T}; D(A)) \cap W^{1,p}_{\text{per}}(\mathbb{T}; X) : u' \in L^p(\mathbb{T}; D(M)),\ Mu' \in W^{1,p}_{\text{per}}(\mathbb{T}; X)\},$$

here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. When $u \in S_p(A, M)$, then $Fu_\bullet, Gu'_\bullet \in L^p(\mathbb{T}; X)$ as $\|Fu_t\| \le \|F\|\|u\|_p$ and $\|Fu'_t\| \le \|F\|\|u'\|_p$ when $t \in \mathbb{T}$. Thus all terms appearing in (P_2) belong to $L^p(\mathbb{T}; X)$. Moreover $S_p(A, M)$ is a Banach space with the norm

$$\|u\|_{S_p(A,M)} := \|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p}.$$

By [Arendt and Bu 2002, Lemma 2.1], if $u \in S_p(A, M)$, then u and Mu' are X-valued continuous on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$.

Definition. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}; X)$; $u \in S_p(A, M)$ is called a strong L^p-solution of (P_2) if (P_2) is satisfied a.e. on $\mathbb{T}$. We say that (P_2) is L^p-well-posed, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of (P_2).

If (P_2) is L^p-well-posed, there exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T}; X)$, if $u \in S_p(A, M)$ is the unique strong L^p-solution of (P_2), then

$$\|u\|_{S_p(A,M)} \le C\|f\|_{L^p}. \tag{2-1}$$

This is an easy consequence of the closed graph theorem by the closedness of A and M.

Let $F, G \in \mathcal{L}(L^p(-2\pi, 0); X), X)$ and $k \in \mathbb{Z}$. We define the linear operators F_k, G_k on X by

$$F_k x := F(e_k x) \quad \text{and} \quad G_k x := G(e_k x), \quad (x \in X). \tag{2-2}$$

It is clear that $F_k, G_k \in \mathcal{L}(X)$, $\|F_k\| \le \|F\|$ and $\|G_k\| \le \|G\|$ as $\|e_k\|_p = 1$. Moreover when $u \in L^p(\mathbb{T}; X)$,

$$\widehat{Fu_\bullet}(k) = F_k \hat{u}(k) \quad \text{and} \quad \widehat{Gu_\bullet}(k) = G_k \hat{u}(k), \quad (k \in \mathbb{Z}). \tag{2-3}$$

This implies that $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as

$$\|Fu_t\| \le \|F\|\|u_\bullet\|_p = \|F\|\|u\|_p, \quad (t \in \mathbb{T})$$

and thus $Fu_\bullet, Gu_\bullet \in L^p(\mathbb{T}; X)$. We define the resolvent set of (P_2) in the L^p-well-posedness setting by

$$\begin{aligned}\rho_p(P_2) := \{k \in \mathbb{Z} : k^2 M - i\alpha k + ikG_k + F_k + A \quad &\text{is invertible from } D(A) \text{ onto } X \\ \text{and} \quad &(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} \in \mathcal{L}(X)\}.\end{aligned}$$

If $k \in \rho_p(P_2)$, then $M(k^2M - i\alpha k + ikG_k + F_k + A)^{-1}$ and $A(k^2M - i\alpha k + ikG_k + F_k + A)^{-1}$ make sense as $D(A) \subset D(M)$ by assumption, and they belong to $\mathcal{L}(X)$ by the closed graph theorem. We need the following preparation.

Proposition 2.3. *Let A and M be closed linear operators defined on a* UMD *space X satisfying $D(A) \subset D(M)$, $1 < p < \infty$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$. Assume that $\rho_p(P_2) = \mathbb{Z}$ and that the sets $\{k^2 MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ and $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (k^2M - i\alpha k + ikG_k + F_k + A)^{-1}$, F_k and G_k are defined by (2-2) when $k \in \mathbb{Z}$. Then $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$ and $(kMN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.*

Proof. Let $M_k = k^2MN_k$, $S_k = kN_k$ and $T_k = kMN_k$ when $k \in \mathbb{Z}$. The sets $\{G_k : k \in \mathbb{Z}\}$ and $\{F_k : k \in \mathbb{Z}\}$ are R-bounded by [Lizama 2006, Proposition 3.2]. It follows from the R-boundedness of the set $\{I_X/k : k \in \mathbb{Z} \setminus \{0\}\}$ that $\{N_k : k \in \mathbb{Z}\}$ is R-bounded, as the product of R-bounded sets is still R-bounded. Moreover, by the definition of N_k,

$$\begin{aligned} N_{k+1} - N_k &= N_{k+1}(N_k^{-1} - N_{k+1}^{-1})N_k \\ &= N_{k+1}[-(2k+1)M + i\alpha + ikG_k - i(k+1)G_{k+1} + F_k - F_{k+1}]N_k \\ &= -(2k+1)N_{k+1}MN_k + i\alpha N_{k+1}N_k - ikN_{k+1}(G_{k+1} - G_k)N_k \\ &\qquad - iN_{k+1}G_{k+1}N_k - N_{k+1}(F_{k+1} - F_k)N_k. \end{aligned} \tag{2-4}$$

It follows that

$$\begin{aligned} M_{k+1} - M_k &= (k+1)^2 MN_{k+1} - k^2 MN_k \\ &= k^2 M(N_{k+1} - N_k) + (2k+1)MN_{k+1} \\ &= -k^2(2k+1)MN_{k+1}MN_k + i\alpha k^2 MN_{k+1}N_k \\ &\qquad - ik^3 MN_{k+1}(G_{k+1} - G_k)N_k - ik^2 MN_{k+1}G_{k+1}N_k \\ &\qquad - k^2 MN_{k+1}(F_{k+1} - F_k)N_k + (2k+1)MN_{k+1}, \end{aligned} \tag{2-5}$$

$$\begin{aligned} S_{k+1} - S_k &= k(N_{k+1} - N_k)) + N_{k+1} \\ &= -k(2k+1)N_{k+1}MN_k + i\alpha kN_{k+1}N_k - ik^2 N_{k+1}(G_{k+1} - G_k)N_k \\ &\qquad - ikN_{k+1}G_{k+1}N_k - kN_{k+1}(F_{k+1} - F_k)N_k + N_{k+1}, \end{aligned} \tag{2-6}$$

and

$$\begin{aligned} T_{k+1} - T_k &= M(S_{k+1} - S_k) \\ &= -k(2k+1)MN_{k+1}MN_k + i\alpha kMN_{k+1}N_k - ik^2 MN_{k+1}(G_{k+1} - G_k)N_k \\ &\qquad - ikMN_{k+1}G_{k+1}N_k - kMN_{k+1}(F_{k+1} - F_k)N_k + MN_{k+1}. \end{aligned} \tag{2-7}$$

This implies that the sets $\{k(N_{k+1} - N_k) : k \in \mathbb{Z}\}$, $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$, $\{k(S_{k+1} - S_k) : k \in \mathbb{Z}\}$ and $\{k(T_{k+1} - T_k) : k \in \mathbb{Z}\}$ are R-bounded by the R-boundedness of the sets $\{k^2 MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$, $\{F_k : k \in \mathbb{Z}\}$ and $\{G_k : k \in \mathbb{Z}\}$. It follows that $(N_k)_{k\in\mathbb{Z}}$, $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by Theorem 2.2. This completes the proof. $\square$

Our next result gives a necessary and sufficient condition for the L^p-well-posedness of (P_2) when X is a UMD space and $1 < p < \infty$.

Theorem 2.4. *Let X be a* UMD *space*, $1 < p < \infty$ *and let A, M be closed linear operators on X satisfying $D(A) \subset D(M)$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$ be such that the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is R-bounded. Then the following assertions are equivalent.*

(i) (P_2) *is* L^p*-well-posed.*

(ii) $\rho_p(P_2) = \mathbb{Z}$ *and the sets* $\{k^2 M N_k : k \in \mathbb{Z}\}$ *and* $\{k N_k : k \in \mathbb{Z}\}$ *are R-bounded, where* $N_k = (k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$.

Proof. $(i) \Rightarrow (ii)$: Assume that (P_2) is L^p-well-posed. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt} y$ $(t \in \mathbb{T})$. Then $f \in L^p(\mathbb{T}; X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ for $n \neq k$. Since (P_2) is L^p-well-posed, there exists $u \in S_p(A, M)$ such that

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t) \quad \text{a.e. on } \mathbb{T}. \tag{2-8}$$

We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [Arendt and Bu 2002, Lemma 3.1] as $u \in L^p(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides of (2-8), we obtain

$$-(k^2 M - i\alpha k + ikG_k + F_k + A)\hat{u}(k) = y, \tag{2-9}$$

and $-(n^2 M - i\alpha n + inG_n + F_n + A)\hat{u}(n) = 0$ when $n \neq k$. This implies in particular that $k^2 M - i\alpha k + ikG_k + F_k + A$ is surjective. We are going to show that it is also injective. Let $x \in D(A)$ be such that

$$(k^2 M - i\alpha k + ikG_k + F_k + A)x = 0,$$

and let $u(t) = e^{ikt} x$ when $t \in \mathbb{T}$. Then $u \in S_p(A, M)$ and (P_2) holds a.e. on $\mathbb{T}$ when taking $f = 0$. Consequently u is a strong L^p-solution of (P_2) when $f = 0$. We obtain $u = 0$ by the uniqueness assumption and thus $x = 0$. We have shown that $k^2 M - i\alpha k + ikG_k + F_k + A$ is also injective. Therefore $k^2 M - i\alpha k + ikG_k + F_k + A$ is a bijection from $D(A)$ onto X.

Now we show the boundedness of $(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$. For $f(t) = e^{ikt} y$, we let $u \in S_p(A, M)$ be the strong L^p-solution of (P_2). Then

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ -(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} y, & n = k, \end{cases}$$

by (2-9). This means that $u(t) = -e^{ikt}(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} y$. By (2-1), there exists a constant $C > 0$ independent from y and k satisfying

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p} \le C\|f\|_{L^p}.$$

In particular $\|u\|_{L^p} \le C\|f\|_{L^p}$. This implies that $\|(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1} y\| \le C\|y\|$ for all $y \in X$. Thus

$$\|(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}\| \le C.$$

We have shown that $k \in \rho_p(P_2)$. Hence $\rho_p(P_2) = \mathbb{Z}$.

Let $M_k = k^2 M (k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$ and $S_k = ik(k^2 M - i\alpha k + ikG_k + F_k + A)^{-1}$ when $k \in \mathbb{Z}$. We are going to show that $(M_k)_{k \in \mathbb{Z}}$ and $(S_k)_{k \in \mathbb{Z}}$ are L^p-Fourier multipliers. Let $f \in L^p(\mathbb{T}; X)$ be fixed. Then there exists $u \in S_p(A, M)$

strong L^p-solution of (P_2) by assumption. Taking Fourier transforms on both sides of (P_2), we get that $\hat{u}(k) \in D(A)$ by [Arendt and Bu 2002, Lemma 3.1] and

$$-(k^2M - i\alpha k + ikG_k + F_k + A)\hat{u}(k) = \hat{f}(k)$$

when $k \in \mathbb{Z}$. Since $k^2M - i\alpha k + ikG_k + F_k + A$ is invertible, we have

$$\hat{u}(k) = -(k^2M - i\alpha k + ikG_k + F_k + A)^{-1}\hat{f}(k)$$

when $k \in \mathbb{Z}$. We have $\widehat{u'}(k) = ik\hat{u}(k)$ and $\widehat{(Mu')'}(k) = -k^2M\hat{u}(k)$ by [Arendt and Bu 2002, Lemma 3.1]. Consequently

$$\widehat{u'}(k) = -S_k\hat{f}(k), \quad \text{and} \quad \widehat{(Mu')'}(k) = -M_k\hat{f}(k)$$

when $k \in \mathbb{Z}$. We conclude that $(M_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as u', $(Mu')' \in L^p(\mathbb{T}; X)$ by assumption. It follows from Proposition 2.1 that the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{S_k : k \in \mathbb{Z}\}$ are R-bounded.

$(ii) \Rightarrow (i)$: Assume that $\rho_p(P_2) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded. Define $M_k = k^2MN_k$, $S_k = ikN_k$ and $T_k = ikMN_k$ when $k \in \mathbb{Z}$. It follows from Proposition 2.3 that $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. Then for all $f \in L^p(\mathbb{T}; X)$, there exists $u, v, w, g \in L^p(\mathbb{T}; X)$ satisfying

$$\begin{aligned} &\hat{u}(k) = -M_k\hat{f}(k), && \hat{v}(k) = S_k\hat{f}(k), \\ &\hat{w}(k) = N_k\hat{f}(k), && \hat{g}(k) = T_k\hat{f}(k), \quad (k \in \mathbb{Z}). \end{aligned} \tag{2-10}$$

Consequently $\hat{v}(k) = ik\hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$ [Arendt and Bu 2002, Lemma 2.1] and $w' = v$. We note that $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by (2-3). Thus $(ikG_kN_k)_{k\in\mathbb{Z}}$ and $(F_kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as the product of L^p-Fourier multipliers is still an L^p-Fourier multiplier. We have

$$AN_k = I_X - M_k + i\alpha kN_k - ikG_kN_k - F_kN_k, \quad (k \in \mathbb{Z}).$$

It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also an L^p-Fourier multiplier as the sum of L^p-Fourier multipliers is still an L^p-Fourier multiplier. This together with the fact that $(N_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier implies that $N_k \in \mathcal{L}(X, D(A))$. Here we consider $D(A)$ as a Banach space equipped with its graph norm. We have shown that $w \in L^p(\mathbb{T}; D(A))$.

Noticing the facts that $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers, we have that $S_k \in \mathcal{L}(X, D(M))$. Since $\hat{v}(k) = S_k\hat{f}(k)$ when $k \in \mathbb{Z}$ by (2-10), we deduce that $v = w' \in L^p(\mathbb{T}; D(M))$. Again by (2-10),

$$\hat{u}(k) = -k^2MN_k\hat{f}(k) = ik\widehat{Mw'}(k)$$

when $k \in \mathbb{Z}$. Thus we have $Mw' \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$ by [Arendt and Bu 2002, Lemma 2.1]. We have shown that $w \in S_p(A, M)$.

By (2-10), we have

$$\widehat{(Mw')}'(k) + i\alpha k\hat{w}(k) = A\hat{w}(k) + ikG_k\hat{w}(k) + F_k\hat{w}(k) + \hat{f}(k)$$

when $k \in \mathbb{Z}$. This together with the facts $\widehat{Fw_\bullet}(k) = F_k\hat{w}(k)$ and $\widehat{Gw'_\bullet}(k) = ikG_k\hat{w}(k)$ implies that

$$(Mw')'(t) + \alpha u'(t) = Aw(t) + Gw'_t + Fw_t + f(t) \quad \text{a.e. on } \mathbb{T}$$

by the uniqueness theorem [Arendt and Bu 2002, page 314]. Thus w is a strong L^p-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_p(A, M)$ satisfying

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t \quad \text{a.e. on } \mathbb{T}.$$

Taking the Fourier transforms on both sides, we have

$$(k^2M - i\alpha k + ikG_k + F_k + A)\hat{u}(k) = 0, \quad (k \in \mathbb{Z}).$$

Since $\rho_p(P_2) = \mathbb{Z}$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. So (P_2) is L^p-well-posed. This completes the proof. □

Theorem 2.4 recovers the known results presented in Bu and Fang [2010] in the nondegenerate case when $M = I_X$ and $\alpha = 0$. Thus it may be also regarded as generalizations of the previous known results when $M = I_X$, $\alpha = 0$ and $F = G = 0$ in the L^p-well-posedness obtained in [Arendt and Bu 2002]. Our results also generalize the previous known results obtained by Bu [2013] in the simpler case when $F = G = 0$ and $\alpha = 0$.

3. Well-posedness in periodic Besov spaces

In this section we study the $B^s_{p,q}$-well-posedness of (P_2). Firstly, we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [Arendt and Bu 2004]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = \sup_{x\in\mathbb{T}}|f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T}, X) := \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all continuous linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define periodic Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\},\ I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \quad \text{for} \quad x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$,

$$\sup_{x\in\mathbb{R},\, k\in\mathbb{N}_0} 2^{k\alpha} |\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \phi(\mathbb{R})$ be fixed. For $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}; X) := \left\{ f \in \mathcal{D}'(\mathbb{T}, X) : \|f\|_{B^s_{p,q}} := \left(\sum_{j\ge 0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \right\|_p^q \right)^{1/q} < \infty \right\}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T}; X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T}; X)$. Equipping $B^s_{p,q}(\mathbb{T}; X)$ with the norm $\|\cdot\|_{B^s_{p,q}}$ gives a Banach space. See [Arendt and Bu 2004, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T}; X)$. We know that if $s_2 \le s_1$, then $B^{s_1}_{p,q}(\mathbb{T}; X) \subset B^{s_2}_{p,q}(\mathbb{T}; X)$ and the embedding is continuous [Arendt and Bu 2004]. When $s > 0$, it is shown in the same work that $B^s_{p,q}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X)$, $f \in B^{s+1}_{p,q}(\mathbb{T}; X)$ if and only if f is differentiable a.e. on $\mathbb{T}$ and $f' \in B^s_{p,q}(\mathbb{T}; X)$. This implies that if $u \in B^s_{p,q}(\mathbb{T}; X)$ is such that there exists $v \in B^s_{p,q}(\mathbb{T}; X)$ satisfying $\hat{v}(k) = ik\hat{u}(k)$ when $k \in \mathbb{Z}$, then $u \in B^{s+1}_{p,q}(\mathbb{T}; X)$ and $u' = v$ [Arendt and Bu 2004, Lemma 2.1].

The main tool in the study of $B^s_{p,q}$-well-posedness of (P_2) is the operator-valued $B^s_{p,q}$-Fourier multiplier theory established in [Arendt and Bu 2004].

Definition. Let X, Y be Banach spaces, $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We say $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique $u \in B^s_{p,q}(\mathbb{T}; Y)$, such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

The following result, obtained in [Arendt and Bu 2004], gives a sufficient condition for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier.

Theorem 3.1. *Let X, Y be Banach spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We assume*

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty, \tag{3-1}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2} - 2M_{k+1} + M_k)\| < \infty. \tag{3-2}$$

Then for $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. If X is B-convex, then condition (3-1) is already sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-Fourier multiplier.

Recall that a Banach space X is B-convex if it does not contain l_1^n uniformly. This is equivalent to saying that X has Fourier type $1 < p \le 2$, i.e., the Fourier transform is a bounded linear operator from $L^p(\mathbb{R}; X)$ to $l^q(\mathbb{Z}; X)$, where $1/p + 1/q = 1$. It is well known that when $1 < p < \infty$, then $L^p(\mu)$ has Fourier type $\min\{p, p/(p-1)\}$.

Let $1 \le p,\ q \le \infty,\ s > 0$ be fixed. We consider the following second-order degenerate differential equation with finite delays:

$$(P_2) \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi), \end{cases}$$

where A, M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, and $F, G : B^s_{p,q}([-2\pi, 0]; X) \to X$ are bounded linear operators. Moreover, for fixed $t \in \mathbb{T}$, u_t and u'_t are elements of $B^s_{p,q}([-2\pi, 0]; X)$ defined by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ for $-2\pi \le s \le 0$. Here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

Let $F, G \in \mathcal{L}(B^s_{p,q}(-2\pi, 0); X), X)$ and $k \in \mathbb{Z}$. We define the linear operators $F_k, G_k \in \mathcal{L}(X)$ by $F_k x := F(e_k \otimes x)$, $G_k x := G(e_k \otimes x)$ for all $x \in X$. It is clear that there exists a constant $C > 0$ such that $\|e_k \otimes x\|_{B^s_{p,q}} \le C\|x\|$ for all $k \in \mathbb{Z}$. Thus

$$\|F_k\| \le C\|F\|, \quad \text{and} \quad \|G_k\| \le C\|G\|, \quad (k \in \mathbb{Z}). \tag{3-3}$$

It is easy to verify that when $u \in B^s_{p,q}(\mathbb{T}; X)$, then

$$\widehat{Fu_\bullet}(k) = F_k \hat{u}(k), \quad \text{and} \quad \widehat{Gu_\bullet}(k) = G_k \hat{u}(k), \quad (k \in \mathbb{Z}).$$

We define the resolvent set of (P_2) in the $B^s_{p,q}$-well-posedness setting by

$$\rho_{p,q,s}(P_2) := \{k \in \mathbb{Z} : k^2 M - ik\alpha + ikG_k + F_k + A \text{ is a bijection from } D(A) \text{ onto } X, \\ \text{and} \quad (k^2 M - ik\alpha + ikG_k + F_k + A)^{-1} \in \mathcal{L}(X)\}.$$

If $k \in \rho_{p,q,s}(P_2)$, then $M(k^2 M + ikG_k + F_k + A)^{-1}$, $A(k^2 M + ikG_k + F_k + A)^{-1}$ make sense as $D(A) \subset D(M)$ by assumption, and they are in $\mathcal{L}(X)$ by the closed graph theorem.

Let $1 \le p,\ q \le \infty,\ s > 0$. We notice that the functions $Fu_\bullet$ and $Gu'_\bullet$ are uniformly bounded on $\mathbb{T}$, but they are not necessarily in $B^s_{p,q}(\mathbb{T}; X)$. We define the solution space of the $B^s_{p,q}$-well-posedness for (P_2) by

$$S_{p,q,s}(A, M) := \{u \in B^s_{p,q}(\mathbb{T}; D(A)) \cap B^{1+s}_{p,q}(\mathbb{T}; X) : u' \in B^s_{p,q}(\mathbb{T}; D(M)), \\ Mu' \in B^{s+1}_{p,q}(\mathbb{T}; X) \quad \text{and} \quad Fu_\bullet, Gu'_\bullet \in B^s_{p,q}(\mathbb{T}; X)\}.$$

Here again we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S_{p,q,s}(A, M)$ is a Banach space with the norm

$$\|u\|_{S_{p,q,s}(A,M)} := \|u\|_{B^{1+s}_{p,q}} + \|Au\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Mu'\|_{B^{1+s}_{p,q}} + \|Fu_\bullet\|_{B^s_{p,q}} + \|Gu'_\bullet\|_{B^s_{p,q}}.$$

From [Arendt and Bu 2002, Lemma 2.1], if $u \in S_{p,q,s}(A, M)$, then u and Mu' are X-valued continuous on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$.

Definition. Let $1 \le p,\ q \le \infty,\ s > 0$ and $f \in B^s_{p,q}(\mathbb{T}; X)$. $u \in S_{p,q,s}(A, M)$ is called a strong $B^s_{p,q}$-solution of (P_2), if (P_2) is satisfied a.e. on $\mathbb{T}$. We say that (P_2) is $B^s_{p,q}$-well-posed, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique strong $B^s_{p,q}$-solution of (P_2).

If (P_2) is $B^s_{p,q}$-well-posed, there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T}; X)$, if $u \in S_{p,q,s}(A, M)$ is the unique strong $B^s_{p,q}$-solution of (P_2), then

$$\|u\|_{S_{p,q,s}(A,M)} \le C\|f\|_{B^s_{p,q}}. \tag{3-4}$$

This can be easily obtained by the closedness of the operators A and M and the closed graph theorem. We need the following preparation:

Proposition 3.2. *Let A and M be closed linear operators defined on a Banach space X satisfying $D(A) \subset D(M)$ and let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. Assume that $\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$, $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$, $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$ when $k \in \mathbb{Z}$. Then $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$, $(kMN_k)_{k\in\mathbb{Z}}$, $(F_kN_k)_{k\in\mathbb{Z}}$ and $(kG_kN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers whenever $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$.*

Proof. Define $M_k = k^2MN_k,\ S_k = kN_k,\ T_k = kMN_k,\ P_k = F_kN_k$ and $Q_k = kG_kN_k$ when $k \in \mathbb{Z}$. We know $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are norm bounded by (3-3). This implies that the sequences $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are norm bounded by assumption. Using the same argument used in the proof of Proposition 2.3, we obtain

$$\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty, \qquad \sup_{k\in\mathbb{Z}} \|k(N_{k+1} - N_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k(S_{k+1} - S_k)\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k(T_{k+1} - T_k)\| < \infty.$$

Moreover, it is easy to see that one has the stronger estimations

$$\sup_{k\in\mathbb{Z}} \|k^2(N_{k+1} - N_k)\| < \infty, \tag{3-5}$$

$$\sup_{k\in\mathbb{Z}} \|k^3M(N_{k+1} - N_k)\| < \infty, \tag{3-6}$$

by using the norm boundedness of $\{k(G_{k+} - G_k) : k \in \mathbb{Z}\}$. For P_k and Q_k, when $k \in \mathbb{Z}$, we have

$$P_{k+1} - P_k = F_{k+1}(N_{k+1} - N_k) + (F_{k+1} - F_k)N_k, \tag{3-7}$$

$$Q_{k+1} - Q_k = G_{k+1}N_{k+1} + k(G_{k+1} - G_k)N_k + kG_k(N_{k+1} - N_k). \tag{3-8}$$

We deduce that

$$\sup_{k\in\mathbb{Z}} \|k(P_{k+1} - P_k)\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k(Q_{k+1} - Q_k)\| < \infty$$

by (3-5) and the boundedness of $(F_k)_{k\in\mathbb{Z}}$, $(G_k)_{k\in\mathbb{Z}}$ and $(k(G_{k+1} - G_k))_{k\in\mathbb{Z}}$.

By (2-3) we have

$$N_{k+1} - N_k = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} + I_k^{(5)},$$

where

$$\begin{aligned} I_k^{(1)} &:= -(2k+1)N_{k+1}MN_k, \\ I_k^{(2)} &:= i\alpha N_{k+1}N_k, \\ I_k^{(3)} &:= -ikN_{k+1}(G_{k+1} - G_k)N_k, \\ I_k^{(4)} &:= -iN_{k+1}G_{k+1}N_k, \\ I_k^{(5)} &:= -N_{k+1}(F_{k+1} - F_k)N_k. \end{aligned}$$

We have

$$\begin{aligned} I_{k+1}^{(1)} - I_k^{(1)} &= -(2k+3)N_{k+2}MN_{k+1} + (2k+1)N_{k+1}MN_k \\ &= -2N_{k+2}MN_{k+1} - (2k+1)(N_{k+2} - N_{k+1})MN_{k+1} \\ &\qquad - (2k+1)N_{k+1}M(N_{k+1} - N_k). \end{aligned} \tag{3-9}$$

This implies that

$$\sup_{k\in\mathbb{Z}} \|k^3(I_{k+1}^{(1)} - I_k^{(1)})\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k^4M(I_{k+1}^{(1)} - I_k^{(1)})\| < \infty$$

using (3-5) and (3-6). A similar argument shows that

$$\sup_{k\in\mathbb{Z}} \|k^3(I_{k+1}^{(i)} - I_k^{(i)})\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k^4M(I_{k+1}^{(i)} - I_k^{(i)})\| < \infty$$

when $i = 2, 3, 4, 5$ using inequalities (3-5), (3-6) and the norm boundedness of $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ and $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$. We have shown that

$$\sup_{k\in\mathbb{Z}} \|k^3(N_{k+2} - 2N_{k+1} + N_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^4M(N_{k+2} - 2N_{k+1} + N_k)\| < \infty. \tag{3-10}$$

In particular,

$$\sup_{k\in\mathbb{Z}} \|k^2(N_{k+2} - 2N_{k+1} + N_k)\| < \infty.$$

By (2-4), (2-5), (3-7), (3-8) and (3-10), and using similar argument used in the proof of (3-10), we show that

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2}-2M_{k+1}+M_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^2(S_{k+2}-2S_{k+1}+S_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k^2(T_{k+2}-2T_{k+1}+T_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^2(P_{k+2}-2P_{k+1}+P_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k^2(Q_{k+2}-2Q_{k+1}+Q_k)\| < \infty.$$

Thus $(N_k)_{k\in\mathbb{Z}}$, $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers by Theorem 3.1. □

Now we give a necessary and sufficient condition for (P_2) to be $B^s_{p,q}$-well-posed.

Theorem 3.3. *Let X be a Banach space, $1 \le p, q \le \infty$, $s > 0$ and let A and M be closed linear operators on X satisfying $D(A) \subset D(M)$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that the sets $\{k(F_{k+2}-2F_{k+1}+F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1}-G_k) : k \in \mathbb{Z}\}$ and $\{k^2(G_{k+2}-2G_{k+1}+G_k) : k \in \mathbb{Z}\}$ are norm bounded. Then the following assertions are equivalent*:

(i) *(P_2) is $B^s_{p,q}$-well-posed.*

(ii) *$\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$.*

Proof. (*i*) ⇒ (*ii*): Assume that (P_2) is $B^s_{p,q}$-well-posed. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed, we define $f(t) = e^{ikt}y$ when $t \in \mathbb{T}$. Then $f \in B^s_{p,q}(\mathbb{T}; X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ for $n \neq k$. Since (P_2) is $B^s_{p,q}$-well-posed, there exists a unique $u \in S_{p,q,s}(A, M)$ satisfying

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), \quad \text{a.e. on } \mathbb{T}.$$

We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [Arendt and Bu 2002, Lemma 3.1] as $u \in B^s_{p,q}(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides, we obtain

$$-(k^2M - ik\alpha + ikG_k + F_k + A)\hat{u}(k) = y \tag{3-11}$$

and $-(k^2M + ikG_k + F_k + A)\hat{u}(n) = 0$ when $n \neq k$. This implies that the operator $k^2M - ik\alpha + ikG_k + F_k + A$ is surjective as the vector $y \in X$ is arbitrary. To show that $k^2M - ik\alpha + ikG_k + F_k + A$ is also injective, we let $x \in D(A)$ satisfying

$$(k^2M - ik\alpha + ikG_k + F_k + A)x = 0.$$

Let $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$. Then $u \in S_{p,q,s}(A, M)$ and (P_2) holds a.e. on $\mathbb{T}$ when $f = 0$. Thus u is a strong $B^s_{p,q}$-solution of (P_2) when $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that $k^2M - ik\alpha + ikG_k + F_k + A$ is injective. Thus it is bijective from $D(A)$ onto X.

Next we show that $(k^2M - ik\alpha + ikG_k + F_k + A)^{-1} \in \mathcal{L}(X)$. For $y \in X$ and $f(t) = e^{ikt}y$, we let $u \in S_{p,q,s}(A, M)$ be the unique strong $B^s_{p,q}$-solution of (P_2). Then taking Fourier coefficients on both sides of (P_2), we obtain by (3-11)

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ -(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}y, & n = k. \end{cases}$$

Consequently, $u(t) = -e^{ikt}(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}y$ when $t \in \mathbb{T}$. By (3-4) there exists a constant $C > 0$ independent from y and k, such that

$$\|u\|_{B^{1+s}_{p,q}} + \|Au\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Mu'\|_{B^{1+s}_{p,q}} + \|Fu_\bullet\|_{B^s_{p,q}} + \|Gu'_\bullet\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}.$$

The estimation

$$\|u'\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}$$

implies that $\|k(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}y\| \le C\|y\|$ for all $y \in X$. Therefore

$$\|k(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}\| \le C.$$

We have shown that $k \in \rho_{p,q,s}(P_2)$ for all $k \in \mathbb{Z}$. Thus $\rho_{p,q,s}(P_2) = \mathbb{Z}$.

Next we show that $(M_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are norm bounded, where $M_k = k^2MN_k$ and $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$ when $k \in \mathbb{Z}$. For this it will suffice to show that $(M_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ define $B^s_{p,q}$-Fourier multipliers by [Arendt and Bu 2004]. Let $f \in B^s_{p,q}(\mathbb{T}; X)$. Then there exists $u \in S_{p,q,s}(A, M)$ which is a strong $B^s_{p,q}$-solution of (P_2) by assumption. Taking Fourier coefficients on both sides of (P_2), we get that $\hat{u}(k) \in D(A)$ and

$$-(k^2M - ik\alpha + ikG_k + F_k + A)\hat{u}(k) = \hat{f}(k),$$

or equivalently,

$$\hat{u}(k) = -(k^2M - ik\alpha + ikG_k + F_k + A)^{-1}\hat{f}(k), \quad (k \in \mathbb{Z}).$$

It follows from $u \in S_{p,q,s}(A, M)$ that $\widehat{(Mu')'}(k) = -k^2M\hat{u}(k)$ and $\widehat{u'}(k) = ik\hat{u}(k)$. We obtain

$$\widehat{(Mu')'}(k) = -k^2M\hat{u}(k) = -M_k\hat{f}(k), \quad \text{and} \quad \widehat{u'}(k) = -ikN_k\hat{f}(k), \quad (k \in \mathbb{Z}).$$

We conclude that $(M_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ define $B^s_{p,q}$-Fourier multipliers as $(Mu')'$, $u' \in B^s_{p,q}(\mathbb{T}; X)$.

$(ii) \Rightarrow (i)$: Let $\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ be norm bounded, where $N_k = (k^2M - ik\alpha + ikG_k + F_k + A)^{-1}$. Define $M_k = k^2MN_k$, $S_k = ikN_k$, $T_k = kMN_k$, $P_k = F_kN_k$ and $Q_k = ikG_kN_k$ when $k \in \mathbb{Z}$. It follows from Proposition 3.2 that $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$

are $B^s_{p,q}$-Fourier multipliers. Then for all $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists $u, v, w \in B^s_{p,q}(\mathbb{T}; X)$ satisfying

$$\hat{u}(k) = -k^2 M N_k \hat{f}(k), \quad \hat{v}(k) = ikN_k \hat{f}(k) \quad \text{and} \quad \hat{w}(k) = N_k \hat{f}(k), \tag{3-12}$$

when $k \in \mathbb{Z}$. We deduce from the facts that $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers that $Fw_\bullet, Gw'_\bullet \in B^s_{p,q}(\mathbb{T}; X)$ as

$$\widehat{Fw_\bullet}(k) = F_k \hat{w}(k) = F_k N_k \hat{f}(k) = P_k \hat{f}(k), \quad (k \in \mathbb{Z})$$

and

$$\widehat{Gw'_\bullet}(k) = G_k \widehat{w'}(k) = ikG_k \hat{w}(k) = ikG_k N_k \hat{f}(k) = Q_k \hat{f}(k), \quad (k \in \mathbb{Z}).$$

On the other hand, $\hat{v}(k) = ik\hat{w}(k)$ when $k \in \mathbb{Z}$ by (3-12). Therefore w is differentiable a.e. on $\mathbb{T}$ and $w' = v$. This implies that $w \in B^{1+s}_{p,q}(\mathbb{T}; X)$ as $v \in B^s_{p,q}(\mathbb{T}; X)$ [Arendt and Bu 2002, Lemma 2.1].

We note that

$$AN_k = M_k + \alpha S_k - P_k - Q_k + I_X, \quad (k \in \mathbb{Z}).$$

It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-Fourier multiplier. Therefore there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ satisfying

$$\hat{g}(k) = AN_k \hat{f}(k), \quad (k \in \mathbb{Z}). \tag{3-13}$$

Thus $\hat{g}(k) = A\hat{w}(k)$ when $k \in \mathbb{Z}$. This implies $w \in B^s_{p,q}(\mathbb{T}; D(A))$ by [Arendt and Bu 2002, Lemma 3.1].

Since $(T_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, there exists $h \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\hat{h}(k) = ikMN_k \hat{f}(k) = M\widehat{w'}(k), \quad (k \in \mathbb{Z}).$$

Thus $w' \in B^s_{p,q}(\mathbb{T}; D(M))$ by [Arendt and Bu 2002, Lemma 3.1]. In view of (3-12), we obtain

$$\hat{u}(k) = -k^2 MN_k \hat{f}(k) = -k^2 M\hat{w}(k) = ik\widehat{Mw'}(k), \quad (k \in \mathbb{Z})$$

which implies that $Mw' \in B^{s+1}_{p,q}(\mathbb{T}; X)$ by [Arendt and Bu 2002, Lemma 2.1]. We have shown that $u \in S_{p,q,s}(A, M)$.

By (3-12), we have

$$\widehat{(Mw')'}(k) + \alpha\widehat{w'}(k) = A\hat{w}(k) + ikG_k\hat{w}(k) + F_k\hat{w}(k) + \hat{f}(k), \quad (k \in \mathbb{Z}).$$

It follows that $(Mw')'(t) + \alpha w'(t) = Aw(t) + Gw'_t + Fw_t + f(t)$ a.e. on $\mathbb{T}$ by the uniqueness theorem [Arendt and Bu 2002, page 314]. Thus w is a strong $B^s_{p,q}$-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_{p,q,s}(A, M)$ satisfy

$$(Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t$$

a.e. on $\mathbb{T}$. Taking the Fourier coefficients on both sides, we have

$$-(k^2 M - \alpha S_k + ikG_k + F_k + A)\hat{u}(k) = 0$$

for all $k \in \mathbb{Z}$. Since $\rho_{p,q,s}(P_2) = \mathbb{Z}$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. So (P_2) is $B^s_{p,q}$-well-posed. This finishes the proof. □

By the proof of Theorem 2.4 and using Theorem 3.1, one can obtain the following result.

Theorem 3.4. *Let X be a B-convex Banach space, $1 \le p,\ q \le \infty$, $s > 0$ and let A, M be closed linear operators on X satisfying $D(A) \subset D(M)$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is norm bounded. Then the following assertions are equivalent*:

(i) (P_2) *is $B^s_{p,q}$-well-posed.*

(ii) $\rho_{p,q,s}(P_2) = \mathbb{Z}$ *and the sets* $\{k^2 M N_k : k \in \mathbb{Z}\}$ *and* $\{kN_k : k \in \mathbb{Z}\}$ *are norm bounded, where* $N_k = (k^2 M - ik\alpha + ikG_k + F_k + A)^{-1}$.

4. Applications

In the last section, we give some examples to which our abstract results (Theorem 2.4 and Theorem 3.3) may be applied.

Example 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and m be a nonnegative bounded measurable function defined on Ω. Let f be a given function on $[0, 2\pi] \times \Omega$ and $X = H^{-1}(\Omega)$. We consider the following periodic degenerate differential equations with finite delay:

$$(P)\quad \begin{cases} \frac{\partial^2}{\partial t^2}(m(x)u(t,x)) + \alpha\frac{\partial}{\partial t}u(t,x) + \Delta u = Fu_t + Gu'_t + f(t,x), & (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = 0, & (t,x) \in [0,2\pi] \times \partial\Omega, \\ u(0,x) = u(2\pi,x), & x \in \Omega, \\ \frac{\partial u(t,x)}{\partial t}|_{t=0} = \frac{\partial u(t,x)}{\partial t}|_{t=2\pi}, & x \in \Omega, \end{cases}$$

where $\alpha \in \mathbb{C}$ is fixed, $u_t(s, x) := u(t + s, x)$, $u'_t(s, x) := u'(t + s, x)$ when $s \in [-2\pi, 0]$ and $x \in \Omega$, the delay operators $F, G : L^p([-2\pi, 0]; X) \to X$ are bounded linear operators for some fixed $1 < p < \infty$.

Let M be the multiplication operator by m on $H^{-1}(\Omega)$ with domain $D(M)$. Then it follows from [Favini and Yagi 1999, Section 3.7] that if we consider the Laplacian operator Δ on X with Dirichlet boundary condition, then there exists a constant $C > 0$ such that

$$\|M(zM - \Delta)^{-1}\| \leq \frac{C}{1+|z|},$$

when $\mathrm{Re}(z) \geq -\beta(1 + |\mathrm{Im}(z)|)$ for some positive constant β depending only on m, which implies that

$$\|M(k^2 M - \Delta)^{-1}\| \leq \frac{C}{1+|k|^2}, \quad (k \in \mathbb{Z}). \tag{4-1}$$

If we assume that m^{-1} is regular enough so that the multiplication operator by the function m^{-1} is bounded on $H^{-1}(\Omega)$, then there exists a constant C_1 such that

$$\|(k^2 M - \Delta)^{-1}\| \leq \frac{C_1}{1+|k|^2}, \quad (k \in \mathbb{Z}). \tag{4-2}$$

Assume that $D(\Delta) \subset D(M)$, that the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is norm bounded, and that $\rho_p(P) = \mathbb{Z}$, so that for all $k \in \mathbb{Z}$ the operator $-k^2 M + i\alpha k + \Delta - F_k - ikG_k$ is a bijection from $D(\Delta)$ onto X, and $(-k^2 M + i\alpha k + \Delta - F_k - ikG_k)^{-1} \in \mathcal{L}(X)$. We observe that

$$-k^2 M + i\alpha k + \Delta - F_k - ikG_k = (I - (F_k + ikG_k - i\alpha k)(-k^2 M + \Delta)^{-1})(-k^2 M + \Delta)$$

for $k \in \mathbb{Z}$. From (4-2) we get $\lim_{k\to\infty} \|(F_k + ikG_k - i\alpha k)(-k^2 M + \Delta)^{-1}\| = 0$ using the norm boundedness of $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$. This implies that the operator $I - (-k^2 M + \Delta)^{-1}(F_k + ikG_k - i\alpha k)$ is invertible when $|k|$ is big enough. For such k we have

$$\begin{aligned}(-k^2 M &+ i\alpha k + \Delta - F_k - ikG_k)^{-1} \\ &= (-k^2 M + \Delta)^{-1}(I - (F_k + ikG_k - i\alpha k)(-k^2 M + \Delta)^{-1})^{-1}.\end{aligned}$$

It follows from (4-1) and (4-2) that

$$\sup_{k\in\mathbb{Z}} \|k(-k^2 M + i\alpha k + \Delta - F_k - ikG_k)^{-1}\| < \infty,$$

and

$$\sup_{k\in\mathbb{Z}} \|k^2 M(-k^2 M + i\alpha k + \Delta - F_k - ikG_k)^{-1}\| < \infty.$$

As a consequence, the sets $\{k(-k^2 M + i\alpha k + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\}$ and $\{k^2 M(-k^2 M + i\alpha k + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\}$ are R-bounded. Here we used the fact that when the underlying Banach space X is a Hilbert space, then each norm bounded subset of $\mathcal{L}(X)$ is R-bounded [Arendt and Bu 2002, Proposition 1.13]. We deduce from Theorem 2.4 that (P) is L^p-well-posed when $X = H^{-1}(\Omega)$.

If we consider $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$, we may also apply Theorem 3.3 and Theorem 3.4 to obtain the $B^s_{p,q}$-well-posedness of (P) under suitable assumptions on F and G.

Example 4.2. Let H be a complex Hilbert space, let $1 < p < \infty$ and let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; H), H)$ be delay operators. Let P be a densely defined positive self-adjoint operator on H with $P \geq \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = \sum_{i=0}^{k} a_i P^i$ with $a_i \geq 0,\ a_k > 0$. Then there exists a constant $C > 0$, such that

$$\|M(zM + A)^{-1}\| \leq \frac{C}{1+|z|}$$

whenever $\operatorname{Re} z \geq -\beta(1 + |\operatorname{Im} z|)$ for some positive constant β depending only on A and M by [Favini and Yagi 1999, page 73]. This implies in particular that

$$\sup_{k\in\mathbb{Z}} \|k^2 M(k^2 M + A)^{-1}\| < \infty.$$

If we assume $0 \in \rho(M)$, then

$$\sup_{k\in\mathbb{Z}} \|k^2 (k^2 M + A)^{-1}\| < \infty.$$

Furthermore we assume that the set $\{k(G_{k+1} - G_k : k \in \mathbb{Z})\}$ is norm bounded. Then the argument used in the example on page 43 our first example shows that the degenerate differential equations with finite delay

$$(P') \qquad \begin{cases} (Mu')'(t) + \alpha u'(t) = Au(t) + Gu'_t + Fu_t + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi) \end{cases}$$

is L^p-well-posed when $\rho_p(P') = \mathbb{Z}$. Under suitable assumptions on F, G, we may also apply Theorem 3.3 to (P') to obtain the $B^s_{p,q}$-well-posedness of (P') for all $1 \leq p,\ q \leq \infty,\ s > 0$.

We can also give a concrete example of (P'). We consider the following problem:

$$\begin{cases} \frac{\partial^2}{\partial t^2}\big(1 - \frac{\partial^2}{\partial x^2}\big)u(t,x) + \alpha\frac{\partial}{\partial t}u(t,x) = \frac{\partial^4}{\partial x^4}u(t,x) + Fu_t(\,\cdot\,,x) + G\big(\frac{\partial u}{\partial t}\big)_t(\,\cdot\,,x) + f(t,x), \\ u(t,0) = u(t,1) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,1) = 0, \\ u(0,x) = u(2\pi,x), \quad \big(1 - \frac{\partial^2}{\partial x^2}\big)u(0,x) = \big(1 - \frac{\partial^2}{\partial x^2}\big)u(2\pi,x), \\ \frac{\partial}{\partial t}\big(1 - \frac{\partial^2}{\partial x^2}\big)u(0,x) = \frac{\partial}{\partial t}\big(1 - \frac{\partial^2}{\partial x^2}\big)u(2\pi,x), \end{cases}$$

where $x \in \Omega,\ t \in (0, 2\pi)$ in the first equation, and $t \in [0, 2\pi]$ in the second equation. Here, $\Omega = (0,1)$, $F, G \in \mathcal{L}(L^p([-2\pi, 0]; L^2(\Omega)), L^2(\Omega))$ and $u_t(s,x) := u(t+s, x)$ when $t \in [0, 2\pi]$ and $s \in [-2\pi, 0]$. Let $X = L^2(\Omega)$ and let $P = -\partial^2/\partial x^2$ with domain $D(P) = H^2(\Omega) \cap H^1_0(\Omega)$, i.e., P is the Laplacian on $L^2(\Omega)$ with Dirichlet boundary conditions. Then P is positive self adjoint on X. Let $M = P + I_X$ and $A = P^2$. It is clear that $-P$ generates an contraction semigroup on $L^2(\Omega)$ [Arendt et al. 2001, Example 3.4.7], hence $1 \in \rho(-P)$, or equivalently $M = I_X + P$ has a bounded inverse, i.e., $0 \in \rho(M)$. Then the abstract results obtained above for the problem (P') may be applied.

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SHANGQUAN BU
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
100084 BEIJING
CHINA
sbu@math.tsinghua.edu.cn

GANG CAI
SCHOOL OF MATHEMATICAL SCIENCES
CHONGQING NORMAL UNIVERSITY
401331 CHONGQING
CHINA
caigang-aaaa@163.com

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