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ON CUSP SOLUTIONS TO A PRESCRIBED MEAN CURVATURE EQUATION

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The nonexistence of “cusp solutions” of prescribed mean curvature boundary value problems in $\Omega \times \mathbb{R}$ when Ω is a domain in $\mathbb{R}^2$ is proven in certain cases and an application to radial limits at a corner is mentioned.

1. Introduction

Let Ω be a domain in $\mathbb{R}^2$ with locally Lipschitz boundary and $\mathcal{O} = (0, 0) \in \partial\Omega$ and $H \in C^{1,\beta}(\overline{\Omega} \times \mathbb{R})$, for some $\beta \in (0, 1)$. Let polar coordinates relative to $\mathcal{O}$ be denoted by r and θ and let $B_\delta(\mathcal{O})$ be the open ball in $\mathbb{R}^2$ of radius δ about $\mathcal{O}$. We shall assume there exist a $\delta^* > 0$ and $\alpha \in (0, \pi)$ such that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of two smooth arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached and for each $\theta \in (-\alpha, \alpha)$, there exists an $r(\theta) > 0$ such that $\{(r\cos\theta, r\sin\theta) : 0 < r < r(\theta)\} \subset \Omega$. Set $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$.

Consider a solution $f \in C^2(\Omega)$ of the prescribed mean curvature equation

$$\operatorname{div}(Tf)(x, y) = 2H(x, y, f(x, y)) \quad \text{for} \quad (x, y) \in \Omega^*, \tag{1}$$

which satisfies the conditions

$$\sup_{(x,y)\in\Omega^*} |f(x, y)| < \infty \quad \text{and} \quad \sup_{(x,y)\in\Omega^*} |H(x, y, f(x, y))| < \infty, \tag{2}$$

where $Tf = \nabla f/\sqrt{1 + |\nabla f|^2}$; examples of such functions might arise as solutions of a Dirichlet or contact angle boundary value problem for (1). We are interested in the radial limits of f:

$$Rf(\theta) \stackrel{\text{def}}{=} \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta), \quad -\alpha < \theta < \alpha. \tag{3}$$

When $\lim_{\partial^+\Omega^*\ni(x,y)\to\mathcal{O}} f(x, y)$ exists, we define $Rf(\alpha)$ to be this limit and when $\lim_{\partial^-\Omega^*\ni(x,y)\to\mathcal{O}} f(x, y)$ exists, we define $Rf(-\alpha)$ to be this limit.

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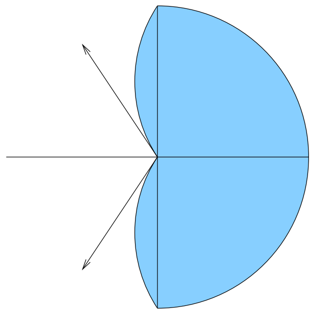

Figure 1. The domain Ω^*.

There are examples in which the radial limits do not exist for any $\theta \in (-\alpha, \alpha)$ [Lancaster 1989; Lancaster and Siegel 1996b]. For solutions of boundary value problems which satisfy appropriate conditions, $Rf(\theta)$ can be proven to exist for $\theta \in [-\alpha, \alpha] \setminus J$, where J is a countable subset of $(-\alpha, \alpha)$; see, e.g., [Entekhabi and Lancaster 2016; 2017; Lancaster 1988; 1991; 2012; Lancaster and Siegel 1996a; 1996b]. We know of no examples in which $J \neq \varnothing$ and we ask if $J = \varnothing$ always holds; this is related to the existence of *cusp solutions*.

A *cusp solution* for (1) is a domain $\Lambda \subset \mathbb{R}^2$ and a solution f of (1) in Λ such that $\partial\Lambda \setminus \{\mathcal{O}, A, B\} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $A, B, \mathcal{O}$ are distinct points on $\partial\Lambda$, and Γ_1, Γ_2 and Γ_3 are disjoint, smooth (open) arcs with endpoints $\{A, \mathcal{O}\}$, $\{B, \mathcal{O}\}$ and $\{A, B\}$, respectively; where Γ_1 and Γ_2 are tangent at $\mathcal{O}$ (so $\overline{\Lambda}$ has an "outward" cusp at $\mathcal{O}$, such as in Figure 2, which has a cusp at $(0, 0)$); and where $f(x, y) = c_j$ when $(x, y) \in \Gamma_j$ $(j = 1, 2)$, $c_1 < c_2$, and, for each $c \in (c_1, c_2)$, the level curves $\{(x, y) \in \Lambda : f(x, y) = c\}$ are tangent at $\mathcal{O}$; see, e.g., [Lancaster and Siegel 1996b, Section 5]. (Capillary surfaces in cusp regions were studied in [Aoki and Siegel 2012; Scholz 2004].) In cases where cusp solutions do not exist, we know $J = \varnothing$.

In [Lancaster and Siegel 1996a; 1996b], the nonexistence of cusp solutions is proven when (a) $H \in C^{1,\delta}(\overline{\Omega} \times \mathbb{R})$, $\delta \in (0, 1)$, and $H(x, y, z)$ is strictly increasing in z for each $(x, y) \in \overline{\Omega}$ or (b) H is real-analytic. The proof in [Lancaster and Siegel 1996b] for case (a) involves a "local" argument while that for (b) involves a "global" argument which shows (2) is violated. Using a "local" argument, we shall prove:

Theorem 1. *Suppose Ω is a domain in $\mathbb{R}^2$ with locally Lipschitz boundary, $\mathcal{O} = (0, 0) \in \partial\Omega$ and $H \in C^{1,\beta}(\overline{\Omega}^* \times \mathbb{R})$ for some $\beta \in (0, 1)$. Let $f \in C^2(\Omega^*)$ satisfy (1) and (2). Suppose $H(x, y, z)$ is weakly increasing in z for (x, y) in a neighborhood of $(0, 0)$. Then f cannot have a cusp solution (i.e., there is no "cusp region" $\Lambda \subset \Omega$ such that (Λ, f) is a cusp solution).*

We can exclude cusp solutions when H vanishes in the "cusp direction," which we may assume is the direction of the positive x-axis (see Figure 2).

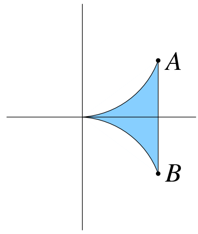

Figure 2. The cusp domain Λ.

Theorem 2. *Suppose Λ is a cusp domain in $\mathbb{R}^2$, $\partial\Lambda$ is tangent to $\vec{i}$ at $\mathcal{O}$, $H \in C^{1,\beta}(\overline{\Lambda} \times \mathbb{R})$ for some $\beta \in (0,1)$, $f \in C^2(\Lambda)$ satisfies (1) and (2) and there exists a $\delta > 0$ such that*

$$H(x,0,z) = 0 \quad \textit{for} \quad (x,z) \in [0,\delta] \times [\liminf_{\Lambda \ni (x,y) \to \mathcal{O}} f(x,y), \limsup_{\Lambda \ni (x,y) \to \mathcal{O}} f(x,y)].$$

Then (Λ, f) cannot be a cusp solution.

What can we say when $H(x,y,z)$ is strictly decreasing in z? Unfortunately, as the following example illustrates, we cannot exclude cusp solutions in this case, even when H is real-analytic; a "global" argument (like in [Lancaster and Siegel 1996b, page 176]) is required to exclude cusp solutions when H is real-analytic. Thus, for example, the reasoning in [Aoki and Siegel 2012, 3B] cannot be used when $\kappa < 0$.

Example 3. Consider the cone $\mathcal{C} = \left\{X(\theta,t) : 0 \le \theta \le \frac{\pi}{2}, 0 < t < \infty\right\}$, where

$$X(\theta,t) = t(\cos\theta, \sin\theta - 1, 1).$$

Set $\Lambda = \left\{t(\cos\theta, \sin\theta - 1) : 0 < \theta < \frac{\pi}{2}, 1 < t < 2\right\}$ and $\mathcal{S} = \mathcal{C} \cap \left(\mathbb{R}^2 \times [1,2]\right)$. A straightforward computation shows that the mean curvature (with respect to the upward normal) is

$$H(\theta,t) = \frac{3-2\sin\theta}{2t(1+(1-\sin\theta)^2)^{3/2}};$$

that is, $H(x,y,z) = (z^2 - 2yz)/\left(2(y^2+z^2)^{3/2}\right)$. Now $y/z = \sin\theta - 1 \in [-1,0]$ and $x = 0$ if and only if $\theta = \pi/2$; another calculation yields

$$2\frac{\partial H}{\partial z}(x,y,z) = -\frac{z^3}{(y^2+z^2)^{5/2}}\left(1 - 4\left(\frac{y}{z}\right) - 2\left(\frac{y}{z}\right)^2 + 2\left(\frac{y}{z}\right)^3\right) < 0.$$

Finally observe that $\mathcal{S}$ is the graph of a cusp solution and satisfies (2) in Λ.

The hypotheses of [Entekhabi and Lancaster 2016] include the assumption that H satisfies one of the conditions which guarantees that cusp solutions do not exist; the following corollary is a consequence of Theorem 1 and that paper. (A second corollary, similar to Corollary 4, follows by applying Theorem 1 to [Entekhabi and Lancaster 2017, Theorems 1 and 2].)

Corollary 4 [Entekhabi and Lancaster 2016]. *Suppose Ω, f and H satisfy the hypotheses of Theorem 1 and either*

(i) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ *or*

(ii) $\alpha \in \left(0, \frac{\pi}{2}\right]$ *and one of $Rf(\alpha)$ or $Rf(-\alpha)$ exists.*

Then $Rf(\theta)$ exists for each $\theta \in (-\alpha, \alpha)$ and $Rf \in C^0\big((-\alpha, \alpha)\big)$. If $Rf(\alpha)$ exists, then $Rf \in C^0\big((-\alpha, \alpha]\big)$. If $Rf(-\alpha)$ exists, then $Rf \in C^0\big([-\alpha, \alpha)\big)$.

2. Proof of Theorem 1

Suppose (Λ, f) is a cusp solution and $\Lambda \subset \{(x, y) \in \mathbb{R}^2 : 0 < x < a,\ |y| < x\}$, $c_1 < c_2$ and the c-level curves of f in Λ are tangent to the positive x-axis at $\mathcal{O}$ for $c_1 \le c \le c_2$, for some $a > 0$ (see Figure 2). Since $H \in C^{1,\beta}(\overline{\Omega} \times \mathbb{R})$, the solution f is an element of $C^3(\Omega)$ and, as in [Lancaster and Siegel 1996a; 1996b], there exist an (open) rectangle $R_0 = (0, a) \times (c_1, c_2)$ and $g \in C^3(R)$, where $R = \overline{R}_0$, such that the graph of f over Λ, $\mathcal{G}$, is the set $\big\{(x, g(x, z), z) : (x, z) \in R_0\big\}$ $\big($i.e., $z = f(x, y)$ if and only if $y = g(x, z)$ for $(x, z) \in R_0$ and $(x, y) \in \Lambda\big)$ and $g(0, z) = \partial g(0, z)/\partial x = 0$ for $c_1 \le z \le c_2$. We may assume that $|\nabla g(x, z)| \le 1$ for $(x, z) \in R$.

The (upward) unit normal to the graph of f, $\mathcal{G}$, is

$$\vec{N}(x, y, z) = \frac{(-f_x(x, y), -f_y(x, y), 1)}{\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}}$$

and $\operatorname{div}(Tf)(x, y) = 2\vec{H}(x, y, z) \cdot \vec{N}(x, y, z)$ for $(x, y, z) \in \mathcal{G}$, where $2\vec{H}$ is the mean curvature vector of $\mathcal{G}$. Then

$$\operatorname{sgn}(g_z(x, z))\vec{N}(x, y, z) = \frac{(g_x(x, z), -1, g_z(x, z))}{\sqrt{1 + g_x^2(x, z) + g_z^2(x, z)}}.$$

Since $\operatorname{div}(Tg) = 2\vec{H} \cdot (-g_x, 1, -g_z)/\sqrt{1 + g_x^2 + g_z^2}$, we see that

$$\operatorname{div}(Tg)(x, z) = 2\vec{H}(x, y, z) \cdot \big(-\operatorname{sgn}(g_z(x, z))\big)\vec{N}(x, y, z) \quad \text{for} \quad (x, y, z) \in \mathcal{G}.$$

(Of course, if $g_z(x, z) = 0$ for some $(x, z) \in R$ with $x > 0$, then $\mathcal{G}$ has a horizontal unit normal at an interior point of Ω, which contradicts our hypothesis $f \in C^2(\Omega)$; hence $g_z(x, z) \ne 0$ when $(x, z) \in R$ with $x > 0$.)

Let us assume $\operatorname{sgn}(g_z(x, z)) = \operatorname{sgn}\big(f_y(x, g(x, z))\big) = +1$ for $(x, z) \in R$ with $x > 0$; the opposite choice will lead to the same (eventual) conclusion that cusp solutions do not exist. Then

$$Mg(x, z) = -2H(x, g(x, z), z),$$

where $Mg = \nabla \cdot Tg = \operatorname{div}(Tg)$. Suppose there exist a $\delta_1 > 0$ such that $H(x, y, z)$ is weakly increasing in z for each $(x, y) \in \Lambda$ and $z \in [c_1, c_2]$ when $x^2 + y^2 \le \delta_1^2$. We may assume $a \le \delta_1$.

Fix $\epsilon \in \big(0, \frac{1}{2}(c_2 - c_1)\big)$ and set $\tilde{c}_1 = c_1 + \epsilon$ and $\tilde{c}_2 = c_2 - \epsilon$; notice that $\tilde{c}_2 > \tilde{c}_1$. Set

$$g_j(x, z) := g(x, z + \tilde{c}_j) \quad \text{for} \quad 0 \le x \le a,\ -\epsilon \le z \le \epsilon,\ j = 1, 2, \tag{4}$$

and define $h = g_1 - g_2$.

If $h(x_0, z_0) = 0$ for some $(x_0, z_0) \in (0, a] \times [-\epsilon, \epsilon]$, then the graph of f fails the vertical line test since $(x_0, y_0, z_0 + \tilde{c}_1)$ and $(x_0, y_0, z_0 + \tilde{c}_2)$ are both points on the graph of f, where $y_0 = g_1(x_0, z_0) = g_2(x_0, z_0)$. Thus $h(x, z) \ne 0$ for all $0 < x \le a,\ -\epsilon \le z \le \epsilon$. Since $\operatorname{sgn}(g_z(x, z)) = +1$ when $(x, z) \in (0, a] \times [-\epsilon, \epsilon]$, we see that $h(x, z) < 0$ for all $(x, z) \in (0, a] \times [-\epsilon, \epsilon]$. (This is essentially the argument at the bottom of page 175 in [Lancaster and Siegel 1996b] since $h(0, z) > 0$ is the only option available there.)

Define

$$K(x, y) = 2H(x, y, \tilde{c}_1 + \epsilon), \quad 0 \le x \le a,\ (x, y) \in \Lambda,$$

and $d(x, z) = 2H(x, g(x, z), \tilde{c}_1 + \epsilon) - 2H(x, g(x, z), z)$. Notice that $d(x, z + \tilde{c}_1) \ge 0$ and $d(x, z + \tilde{c}_2) \le 0$ when $(x, z) \in [0, a] \times [-\epsilon, \epsilon]$. Now, for each $j = 1, 2,\ g_j$ is a solution of the Cauchy problem

$$\begin{aligned} Mg_j(x, z) &= -K(x, g_j(x, z)) + d(x, z + \tilde{c}_j) && \text{for } (x, z) \in [0, a] \times [-\epsilon, \epsilon], \\ g_j(0, z) &= \frac{\partial g_j}{\partial x}(0, z) = 0 && \text{for } z \in [-\epsilon, \epsilon]. \end{aligned}$$

Then, as in [Gilbarg and Trudinger 1983, pages 263–264], we have

$$\begin{aligned} 0 &= Mg_1(x, z) - Mg_2(x, z) + 2H(x, g_1(x, z), z + \tilde{c}_1) - 2H(x, g_2(x, z), z + \tilde{c}_2) \\ &= Lh(x, z) - d(x, z + \tilde{c}_1) + d(x, z + \tilde{c}_2), \end{aligned}$$

where, setting $D_1 := \partial/\partial x$ and $D_2 := \partial/\partial z$,

$$Lh = \sum_{i,j=1}^{2} a^{i,j} D_{ij} h + \sum_{i=1}^{2} b^i D_i h + ch; \tag{5}$$

here

$$a^{i,j}(x, z) = e^{i,j}(Dg_1(x, z)) \quad \text{for} \quad i, j = 1, 2, \tag{6}$$

with

$$\begin{aligned} e^{1,1}(p, q) &= (1 + q^2)W^{-3} \quad e^{1,2}(p, q) = e^{2,1}(p, q) = -pqW^{-3}, \\ e^{2,2}(p, q) &= (1 + p^2)W^{-3} \quad W = W(p, q) = \sqrt{1 + p^2 + q^2}, \end{aligned}$$

$$b^1(x,z)=\sum_{i,j=1}^{2} D_{ij}g_2(x,z)\frac{\partial e^{i,j}}{\partial p}(\xi_1,(g_1)_z(x,z)), \tag{7}$$

$$b^2(x,z)=\sum_{i,j=1}^{2} D_{ij}g_2(x,z)\frac{\partial e^{i,j}}{\partial q}((g_2)_x(x,z),\xi_2) \tag{8}$$

and $c(x,z)=\partial K(x,\xi)/\partial y=2\partial H(x,\xi,\tilde{c}_1+\epsilon)/\partial y$, for some ξ between $g_1(x,z)$ and $g_2(x,z)$, ξ_1 between $(g_1)_x(x,z)$ and $(g_2)_x(x,z)$ and ξ_2 between $(g_1)_z(x,z)$ and $(g_2)_z(x,z)$.

Notice that $a^{i,j}\in C^1(R)$ for $i,j\in\{1,2\}$, $b^i\in L^\infty(R)$ for $i\in\{1,2\}$ and $c\in L^\infty(R)$. Now $h(0,z)=\partial h(0,z)/\partial x=0$ for $|z|\le\epsilon$ and

$$Lh(x,z)=d(x,z+\tilde{c}_1)-d(x,z+\tilde{c}_2)\ge 0,\qquad (x,z)\in[0,a]\times[-\epsilon,\epsilon]. \tag{9}$$

From (9) and the Hopf boundary point lemma (see, e.g., [Gilbarg and Trudinger 1983, Lemma 3.4]), we have

$$\frac{\partial h}{\partial x}(0,z)<0\quad\text{for each}\quad z\in(-\epsilon,\epsilon)$$

and this contradicts the fact that $h_x(0,z)=0$ if $z\in[-\epsilon,\epsilon]$. Thus we have proven Theorem 1. □

Remark 5. The assumption that H is weakly increasing in z is equivalent to one in the (weak) comparison principle (see, e.g., [Gilbarg and Trudinger 1983, Theorem 10.1] or [Finn 1986, Theorem 5.1]), which plays a critical role here.

3. Proof of Theorem 2

Suppose (Λ,f) is a cusp solution and $\Lambda\subset\{(x,y)\in\mathbb{R}^2:0<x<a,\ |y|<x\}$, $c_1<c_2$ and the c-level curves of f in Λ are tangent to the positive x-axis at $\mathcal{O}$ for $c_1\le c\le c_2$, for some $a>0$ (see Figure 2). As before, there exist an (open) rectangle $R_0=(0,a)\times(c_1,c_2)$ and $g\in C^3(R)$ such that the graph of f over Λ, $\mathcal{G}$, is the set $\{(x,g(x,z),z):(x,z)\in R_0\}$ and $g(0,z)=\partial g(0,z)/\partial x=0$ for $c_1\le z\le c_2$. We shall assume that $|\nabla g(x,z)|\le 1$ for $(x,z)\in R$.

Let us assume there exist $\delta\in(0,a]$ and $d_1,d_2\in[c_1,c_2]$ with $d_1<d_2$ such that $H(x,0,z)=0$ for $0\le x\le\delta$, $d_1\le z\le d_2$. Now $g_{xx}(0,z)=0$ for all $z\in[c_1,c_2]$ $\big($since $\triangle g(0,z)=Mg(0,z)=-2H(0,0,z)=0\big)$ and

$$H(x,g(x,z),z)=H(x,0,z)+\frac{\partial H}{\partial y}(x,\xi,z)g(x,z)=\frac{\partial H}{\partial y}(x,\xi,z)g(x,z)$$

for some ξ between 0 and $g(x,z)$. We may extend g as an even function in x by setting $g(x,z)=g(-x,z)$ for $-a\le x<0$, $c_1\le z\le c_2$, so that $g\in C^2(R\cup R^-)$, where $R^- = \{(-x, z) : (x, z) \in R\}$. Then

$$0 = Mg(x, z) + 2H(x, g(x, z), z) = \tilde{L}g(x, z),$$

where

$$a^{1,1}(x, z) = \frac{1 + g_z^2(x, z)}{W^3}, \qquad a^{1,2}(x, z) = -\frac{g_x(x, z)g_z(x, z)}{W^3},$$
$$a^{2,2}(x, z) = \frac{1 + g_x^2(x, z)}{W^3}, \qquad W(x, z) = \sqrt{1 + g_x^2(x, z) + g_z^2(x, z)},$$
$$a^{1,2} = a^{2,1}, \qquad \tilde{c}(x, z) = 2H_y(x, \xi, z)$$

and

$$\tilde{L}u = \sum_{i,j=1}^{2} a^{i,j} D_{ij}u + \tilde{c}u.$$

Since $|\nabla g(x, z)| \le 1$ for $(x, z) \in R$, $\tilde{L}$ is uniformly elliptic in R. Notice that $a^{i,j} \in C^1(R)$ for $i, j = 1, 2$ and $\tilde{c} \in C^0(R)$. Since $g \in C^2(R \cup R^-)$, Theorems 1* and 2* of [Hartman and Wintner 1953] imply that for each $z \in (d_1, d_2)$, there exist a natural number n and real constants e_1 and e_2, not both zero, such that

$$g_x(\rho\cos\theta, z + \rho\sin\theta) = \rho^n(e_1\cos(n\theta) + e_2\sin(n\theta)) + o(\rho^n)$$

and

$$g_z(\rho\cos\theta, z + \rho\sin\theta) = \rho^n(e_2\cos(n\theta) - e_1\sin(n\theta)) + o(\rho^n)$$

as $\rho \to 0$. Since $g_x(0, z) = 0$ and $g_z(0, z) = 0$ for $z \in [c_1, c_2]$, we see that

$$e_1\cos(n\pi/2) + e_2\sin(n\pi/2) = 0, \qquad e_2\cos(n\pi/2) - e_1\sin(n\pi/2) = 0$$

and so $e_1 = e_2 = 0$. This contradicts the fact that at least one of e_1 or e_2 is nonzero. Thus we have proven Theorem 2. □

4. Radial limits

When radial limits for (1) exist, they behave in a different manner than do radial limits of, for example, Laplace's equation; see, e.g., [Bear and Hile 1983]. In particular, if f is a solution of (1) and the radial limits $Rf(\theta)$ exist for $\theta \in (-\alpha, \alpha)$, then they behave in one of the following ways:

(i) $Rf : (-\alpha, \alpha) \to \mathbb{R}$ is a constant function (i.e., f has a nontangential limit at $\mathcal{O}$).

(ii) There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on (α_1, α_2).

(iii) There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $(-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha)$ and is either strictly increasing on $(\alpha_1, \alpha_L]$ and strictly decreasing on $[\alpha_R, \alpha_2)$ or strictly decreasing on $(\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_R, \alpha_2)$.

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