A NEW BICOMMUTANT THEOREM

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We prove an analogue of Voiculescu's theorem: the relative bicommutant of a separable unital subalgebra A of an ultraproduct of simple unital C*-algebras is equal to A.

Ultrapowers[1] $A^{\mathcal{U}}$ of separable C*-algebras are, being subject to well-developed model-theoretic methods, reasonably well-understood; see, e.g., [Farah et al. 2016b, Theorem 1.2] and Section 2. Since the early 1970s and the influential work of McDuff and Connes, central sequence algebras $A' \cap A^{\mathcal{U}}$ play an even more important role than ultrapowers in the classification of II_1 factors and (more recently) C*-algebras. While they do not have a well-studied abstract analogue, in [Farah et al. 2016b, Theorem 1] it was shown that the central sequence algebra of a strongly self-absorbing algebra [Toms and Winter 2007] is isomorphic to its ultrapower if the continuum hypothesis holds. Relative commutants $B' \cap D^{\mathcal{U}}$ of separable subalgebras of ultrapowers of strongly self-absorbing C*-algebras play an increasingly important role in the classification program for separable C*-algebras [Bosa et al. 2016; Matui and Sato 2014, §3]; see also [Tikuisis et al. 2016; Winter 2016]. In the present note we make a step towards better understanding of these algebras.

A C*-algebra is *primitive* if it has a representation that is both faithful and irreducible. We prove an analogue of the well-known consequence of Voiculescu's theorem [1976, Corollary 1.9] and von Neumann's bicommutant theorem [Blackadar 2006, §I.9.1.2].

Theorem 1. *Assume $\prod_{\mathcal{U}} B_j$ is an ultraproduct of primitive C*-algebras and A is a separable C*-subalgebra. In addition, assume A is a unital subalgebra if $\prod_{\mathcal{U}} B_j$ is unital. With $\overline{A}^{\mathrm{WOT}}$ computed in the ultraproduct of faithful irreducible representations of B_js, we have*

$$A = \left(A' \cap \prod_{\mathcal{U}} B_j \right)' = \overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j.$$

[1]Throughout, $\mathcal{U}$ denotes a nonprincipal ultrafilter on $\mathbb{N}$.

A slightly weaker version of the following corollary to Theorem 1 (stated here with Aaron Tikuisis' kind permission) was originally proved using very different methods.

Corollary 2 (Farah and Tikuisis, 2015). *Assume* $\prod_{\mathcal{U}} B_j$ *is an ultraproduct of simple unital C*-algebras and A is a separable unital subalgebra. With* $Z(A)$ *denoting the center of A, we have*

$$Z\big(A' \cap \textstyle\prod_{\mathcal{U}} B_j\big) = Z(A). \hspace{2cm} \square$$

At least two open problems are concerned with bicommutants of separable subalgebras of massive operator algebras. As is well-known, central sequence algebras $M' \cap M^{\mathcal{U}}$ of II$_1$ factors in tracial ultrapowers behave differently from the central sequence algebras of C*-algebras. For a II$_1$ factor M with separable predual, the central sequence algebra $M' \cap M^{\mathcal{U}}$ can be abelian or even trivial. Popa [2014, Conjecture 2.3.1] asked whether if P is a separable subalgebra of an ultraproduct of II$_1$ factors then $\big(P' \cap \prod_{\mathcal{U}} N_i\big)' = P$ implies P is amenable. In the domain of C*-algebras, G. K. Pedersen [1990, Remark 10.11] asked whether the following variant of Theorem 1 is true: if the corona $M(B)/B$ of a σ-unital C*-algebra B is simple and A is a separable unital subalgebra, is $(A' \cap M(B)/B)' = A$? (For the connection between ultraproducts and coronas, see the last paragraph of Section 3.)

The proof of Theorem 1 uses logic of metric structures [Ben Yaacov et al. 2008; Farah et al. 2014] and an analysis of the interplay between C*-algebra B and its second dual B^{**}.

1. Model theory of representations

We expand the language of C*-algebras introduced in [Farah et al. 2014, §2.3.1] to representations of C*-algebras. Readers' familiarity with, or at least easy access to, §2 of that paper is assumed. A structure in the expanded language $\mathcal{L}_{\text{rep}}$ is a C*-algebra together with its representation on a Hilbert space. As in [Farah et al. 2014], the domains of quantification on a C*-algebra are D_n for $n \in \mathbb{N}$ and are interpreted as the n-balls. The domains of quantification on the Hilbert space are D_n^H for $n \in \mathbb{N}$ and are also interpreted as the n-balls. On all domains the metric is $d(x, y) = \|x - y\|$ (we denote both the operator norm on C*-algebras and the ℓ_2-norm on Hilbert spaces by $\|\cdot\|$). As in [Farah et al. 2014, §2.3.1], for every $\lambda \in \mathbb{C}$ we have a unary function symbol λ to be interpreted as multiplication by λ. We also have a binary function $+$ whose interpretation sends $D_m^H \times D_n^H$ to D_{m+n}^H. As the scalar product $(\cdot \mid \cdot)$ is definable from the norm via the polarization identity, we freely use it in our formulas, with the understanding that $(\xi \mid \eta)$ is an abbreviation for $\frac{1}{4} \sum_{j=0}^{3} i^j \|\xi + i^j \eta\|$. The language $\mathcal{L}_{\text{rep}}$ also contains a binary function symbol π whose interpretation sends $D_n \times D_m^H$ to D_{mn}^H for all m and n. It is interpreted as an action of A on H.

Every variable is associated with a sort. In particular, variables x, y, z range over the C*-algebra and variables ξ, η, ζ range over the Hilbert space, all of them decorated with subscripts when needed.

We shall write $\bar{x}$ for a tuple $\bar{x} = (x_1, \ldots, x_n)$ (with n either clear from the context or irrelevant). *Terms* come in two varieties. On the C*-algebra side, a term is a noncommutative *-polynomial in C*-variables. On the Hilbert space side, terms are linear combinations of Hilbert space variables and expressions of the form $\pi(\alpha(\bar{x}))\xi$, where $\alpha(\bar{x})$ is a term in the language of C*-algebras. *Formulas* are defined recursively. Atomic formulas are expressions of the form $\|t\|$ where t is a term.

The set of all formulas is the smallest set $\mathbb{F}$ containing all atomic formulas with the properties that

(i) for every n, all continuous $f : [0, \infty)^n \to [0, \infty)$ and all $\varphi_1, \ldots, \varphi_n$ in $\mathbb{F}$, the expression $f(\varphi_1, \ldots, \varphi_n)$ belongs to $\mathbb{F}$, and

(ii) if $\varphi \in \mathbb{F}$, and x and ξ are variable symbols, then each of $\sup_{\|\xi\| \leq m} \varphi$, $\inf_{\|\xi\| \leq m} \varphi$, $\sup_{\|x\| \leq m} \varphi$, and $\inf_{\|x\| \leq m} \varphi$ belongs to $\mathbb{F}$; see [Farah et al. 2014, §2.4] or [Farah et al. 2016a, Definition 2.1.1].

Suppose $\pi : A \to B(H)$ is a representation of a C*-algebra A on Hilbert space H. To (A, H, π) we associate the natural metric structure $\mathcal{M}(A, H, \pi)$ in the above language.

Suppose $\varphi(\bar{x}, \bar{\xi})$ is a formula whose free variables are included among $\bar{x}$ and $\bar{\xi}$. If $\pi : A \to B(H)$ is a representation of a C*-algebra on Hilbert space, $\bar{a}$ are elements of A and $\bar{\xi}$ are elements of H,[2] then the *interpretation* $\varphi(\bar{a}, \bar{\xi})^{\mathcal{M}(A,H,\pi)}$ is defined by recursion on the complexity of φ in the obvious way; see [Ben Yaacov et al. 2008, §3].

Proposition 1.1. *Triples (A, H, π) such that π is a representation of A on H form an axiomatizable class.*

Proof. As in [Farah et al. 2014, Definition 3.1], we need to define an $\mathcal{L}_{\text{rep}}$-theory T_{rep} such that the category of triples (A, H, π), where $\pi : A \to B(H)$ is a representation of a C*-algebra A, is equivalent to the category of metric structures that are models of T_{rep}, via the map

$$(A, H, \pi) \mapsto \mathcal{M}(A, H, \pi).$$

We use the axiomatization of C*-algebras from [Farah et al. 2014, §3.1]. In addition to the standard Hilbert space axioms, we need the following two axioms assuring

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[2]Symbols $\xi, \eta, \zeta, \ldots$ denote both Hilbert space variables and vectors in Hilbert space due to the font shortage; this shall not lead to a confusion.

that the interpretation of D_n^H equals the n-ball of the underlying Hilbert space for all n:

$$\sup_{\xi \in D_1} \|\xi\| \leq n,$$

(∗)
$$\sup_{\xi \in D_n} \max\big((1 \div \|\xi\|),\ \inf_{\eta \in D_1} \|\xi - \eta\|\big),$$

where $s \div t := \max(s - t, 0)$. The standard axioms,

$$\pi(xy)\xi = \pi(x)\pi(y)\xi,$$

$$\pi(x + y)\xi = \pi(x)\xi + \pi(y)\xi,$$

$$(\pi(x)\xi \mid \eta) = (\xi \mid \pi(x^*)\eta)$$

are expressible as first-order sentences.[3] The axioms described here comprise theory T_{rep}.

One needs to check that the category of models of T_{rep} is equivalent to the category of triples (A, H, π). Every triple (A, H, π) uniquely defines a model $\mathcal{M}(A, H, \pi)$. Conversely, assume $\mathcal{M}$ is a model of T_{rep}. The algebra $A_{\mathcal{M}}$ obtained from the first component of $\mathcal{M}$ is a C*-algebra by [Farah et al. 2014, Proposition 3.2]. Also, the linear space $H_{\mathcal{M}}$ obtained from the second component of $\mathcal{M}$ is a Hilbert space and the third component gives a representation $\pi_{\mathcal{M}}$ of A on H.

To see that this provides an equivalence of categories, we need to check that $\mathcal{M}(A_{\mathcal{M}}, H_{\mathcal{M}}, \pi_{\mathcal{M}}) \cong \mathcal{M}$ for every model $\mathcal{M}$ of T_{rep}. We need to show that the domains on $\mathcal{M}$ are determined by $A_{\mathcal{M}}$ and $H_{\mathcal{M}}$. The former was proved in the second paragraph of [Farah et al. 2014, Proposition 3.2], and the latter follows by (∗).   □

Proposition 1.1 gives us full access to the model-theoretic toolbox, such as Łoś's theorem (see Section 2) and the Löwenheim–Skolem theorem [Farah et al. 2014, Theorem 4.6]. From now on, we shall identify triple (A, H, π) with the associated metric structure $\mathcal{M}(A, H, \pi)$ and stop using the latter notation. We shall also write $\sup_{\|\xi\| \leq n}$ and $\inf_{\|\xi\| \leq n}$ instead of $\sup_{\xi \in D_n}$ and $\inf_{\xi \in D_n}$, respectively.

Lemma 1.2. *The following properties of a representation π of A are axiomatizable*:

(1) π *is faithful.*

(2) π *is irreducible.*

Proof. We explicitly write the axioms for each of the properties of π. Fix a representation π. It is faithful if and only if it is isometric, which can be expressed as

$$\sup_{\|x\| \leq 1} \inf_{\|\xi\| \leq 1} \big| \|x\| - \|\pi(x)\xi\| \big| = 0.$$

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[3]Our conventions are as described in [Farah et al. 2014, p. 485]. In particular $\alpha(x, \xi) = \beta(x, \xi)$ is an abbreviation for $\sup_{\xi \in D_n} \sup_{\xi} \|\alpha(x, \xi) - \beta(x, \xi)\| = 0$, for all n.

A representation π is irreducible if and only if for all vectors ξ and η in H such that $\|\eta\| \leq 1$ and $\|\xi\| = 1$, the expression $\|\eta - \pi(a)\xi\|$ can be made arbitrarily small when a ranges over the unit ball of A. In symbols,

$$\sup_{\|\xi\| \leq 1} \sup_{\|\eta\| \leq 1} \inf_{\|x\| \leq 1} \left| \|\xi\| \dotminus 1 \right| \left\| \eta - \pi(x)\xi \right\| = 0.$$

The interpretation of this sentence in (A, H, π) is 0 if and only if the representation π is irreducible. $\qquad \square$

A triple (D, θ, K) is an *elementary submodel* of (B, π, H), and (B, π, H) is an *elementary extension* of (D, θ, K), if $D \subseteq B$, $K \subseteq H$, $\theta(d) = \pi(d) \restriction H$ for all $d \in D$, and

$$\varphi(\bar{a})^{(D,\theta,K)} = \varphi(\bar{a})^{(B,\pi,H)}$$

for all formulas φ and all $\bar{a}$ in (D, θ, K) of the appropriate sort. Axiomatizable properties, such as being irreducible or faithful, transfer between elementary submodels and elementary extensions. Therefore the downward Löwenheim–Skolem theorem [Farah et al. 2014, Theorem 4.6] and Lemma 1.2 together imply, e.g., that if φ is a pure state of a nonseparable C*-algebra B then B is an inductive limit of separable subalgebras D such that the restriction of φ to D is pure. This fact was proved in [Akemann and Weaver 2004] and its slightly more precise version will be used in the proof of Lemma 3.2.

Some other properties of representations (such as not being faithful) are axiomatizable, but we shall concentrate on proving Theorem 1.

2. Saturation and representations

It has been known to logicians since the 1960s that the two defining properties of ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ in axiomatizable categories are Łoś's theorem [Farah et al. 2014, Proposition 4.3] and countable saturation [Farah et al. 2014, Proposition 4.11]. By the former, the diagonal embedding of a metric structure M into its ultrapower is elementary. More generally, if $\varphi(\bar{x})$ is a formula and $\bar{a}(j) \in M_j$ are of the appropriate sort then

$$\varphi(\bar{a})^{\prod_{\mathcal{U}} M_j} = \lim_{j \to \mathcal{U}} \varphi(\bar{a}(j))^{M_j}.$$

In order to define countable saturation, we recall the notion of a type from the logic of metric structures [Farah et al. 2014, §4.3]. A *closed condition* (or simply a *condition*; we shall not need any other conditions) is any expression of the form $\varphi \leq r$ for formula φ and $r \geq 0$ and a *type* is a set of conditions [Farah et al. 2014, §4.3]. As every expression of the form $\varphi = r$ is equivalent to the condition $\max(\varphi, r) \leq r$ and every expression of the form $\varphi \geq r$ is equivalent to the condition $\min(0, r - \varphi) \leq 0$, we shall freely refer to such expressions as conditions. For m

and n in $\mathbb{N}$ such that $m + n \geq 1$, an (m, n)-*type* is a type t such that all free variables occurring in conditions of t are among $\{x_1, \ldots, x_m\} \cup \{\xi_1, \ldots, \xi_n\}$.

Given a structure (A, H, π) and a subset X of $A \cup H$, we expand the language $\mathcal{L}_{\text{rep}}$ by adding constants for the elements of X (as in [Farah et al. 2014, §2.4.1]). The new language is denoted $(\mathcal{L}_{\text{rep}})_X$. C*-terms in $(\mathcal{L}_{\text{rep}})_X$ are *-polynomials in C*-variables and constants from $X \cap A$. Hilbert space terms are linear combinations of Hilbert space variables, constants in $X \cap H$, and expressions of the form $\pi(\alpha)\xi$, where α is a C*-term in the expanded language. The interpretation of an $(\mathcal{L}_{\text{rep}})_X$-formula is defined recursively in the natural way; see, e.g., the paragraph after Definition 2.1.1 in [Farah et al. 2016a].

A type *over* X is a type in $(\mathcal{L}_{\text{rep}})_X$. Such a type is *realized* in some elementary extension of (A, H, π) if the latter contains a tuple satisfying all conditions from the type. A type is *consistent* if it is realized in some ultrapower of (A, H, π), where the ultrafilter is taken over an arbitrary, not necessarily countable, set. This is equivalent to the type being realized in some elementary extension of (A, H, π).

By Łoś's theorem, a type t is consistent if and only every finite subset of t is ε-realized in (A, H, π) for every $\varepsilon > 0$ [Farah et al. 2014, Proposition 4.8].

A structure (A, H, π) is said to be *countably saturated* if every consistent type over a countable (or equivalently, norm-separable) set is realized in (A, H, π). Ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ are always countably saturated [Farah et al. 2014, Proposition 4.11]. A standard transfinite back-and-forth argument shows that a structure of density character $\aleph_1$ is countably saturated if and only if it is an ultraproduct. (The *density character* is the smallest cardinality of a dense subset.)

In the case when $A = B(H)$, we have

$$(B(H), H)^{\mathcal{U}} = (B(H)^{\mathcal{U}}, H^{\mathcal{U}});$$

in particular $B(H)^{\mathcal{U}}$ is identified with a subalgebra of $B(H^{\mathcal{U}})$. These two algebras are equal (still assuming $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$) if and only if H is finite-dimensional. As a matter of fact, no projection $p \in B(H^{\mathcal{U}})$ with a separable, infinite-dimensional range belongs to $B(H)^{\mathcal{U}}$ (this is proved by a standard argument, see, e.g., the last two paragraphs of the proof of Proposition 4.6 in [Farah et al. 2013]).

In the following, π will always be faithful and clear from the context and we shall identify A with $\pi(A)$ and suppress writing π. We shall therefore write (A, H) in place of (A, H, id).

The following two lemmas are standard (they were used in the proof of Corollary 2 on p. 344 of [Arveson 1977]) but we sketch the proofs for the reader's convenience.

Lemma 2.1. *Suppose A is a* C*-*algebra and φ is a functional on A. Then there are a representation $\pi : A \to B(K)$ and vectors ξ and η in K such that $\varphi(a) = (\pi(a)\,\xi \mid \eta)$ for all $a \in A$.*

Proof. Let $\bar{\varphi}$ be the unique extension of φ to a normal functional of the von Neumann algebra A^{**}. By Sakai's polar decomposition for normal linear functionals (see, for example, [Pedersen 1979, Proposition 3.6.7]) there exists a normal state ψ of A^{**} and a partial isometry v such that $\varphi(a) = \psi(av)$ for all $a \in A^{**}$. Let $\pi : A^{**} \to B(K)$ be the GNS representation corresponding to ψ. If η is the corresponding cyclic vector and $\xi = v\eta$, then the restriction of π to A is as required. $\square$

Lemma 2.2. *Suppose A is a proper unital subalgebra of $C = C^*(A, b)$. Then there exists a representation $\pi : C \to B(K)$ and a projection q in $\pi(A)' \cap B(K)$ such that $[q, b] \neq 0$.*

Proof. By the Hahn–Banach separation theorem, there exists a functional φ on C of norm 1 such that φ annihilates A and $\varphi(b) = \text{dist}(A, b)$. Let $\pi : C \to B(K)$, and η and ξ be as guaranteed by Lemma 2.1. Let L be the norm-closure of $\pi(A)\xi$. Since A is unital, $L \neq \{0\}$. As $0 = \varphi(a) = (\pi(a)\xi \mid \eta)$ for all $a \in A$, η is orthogonal to L and therefore the projection p to L is nontrivial. Clearly $p \in \pi(A)' \cap B(K)$. Since $(\pi(b)\xi \mid \eta) = \varphi(b) \neq 0$, $\pi(b)$ does not commute with p and we therefore have $q \in \pi(A)' \cap B(K)$ such that $\|[\pi(b), q]\| > 0$. $\square$

The proof of Theorem 1 would be much simpler if Lemma 2.2 provided an irreducible representation. This is impossible in general, as the following example shows. Let A be the unitization of the algebra of compact operators $\mathcal{K}(H)$ on an infinite-dimensional Hilbert space and let b be a projection in $B(H)$ which is Murray–von Neumann equivalent to $1 - b$. Then $C = C^*(A, b)$ has (up to equivalence) three irreducible representations. Two of those representations annihilate A and send b to a scalar, and the third representation is faithful and the image of b is in the weak operator closure of the image of A.

It is well-known that for a Banach space X, the second dual X^{**} can be embedded into an ultrapower of X [Heinrich 1980, Proposition 6.7]. In general, the second dual A^{**} of a C*-algebra A cannot be embedded into an ultrapower of A by a *-homomorphism for at least two reasons. First, A^{**} is a von Neumann algebra [Blackadar 2006, §III.5.2] and it therefore has real rank zero, while A may have no nontrivial projections at all. Since being projectionless is axiomatizable [Farah et al. 2016a, Theorem 2.5.1], if A is projectionless then Łoś's theorem implies that $A^{\mathcal{U}}$ is projectionless as well and A^{**} cannot be embedded into it. The referee pointed out another, much subtler, obstruction. In the context of Banach spaces, the embeddability of X^{**} into $X^{\mathcal{U}}$ is equivalent to a finitary statement, the so-called *local reflexivity* of Banach spaces, the C*-algebraic version of which does not hold for all C*-algebras [Effros and Haagerup 1985, §5]. In particular, for a large class of C*-algebras the diagonal embedding of A into $A^{\mathcal{U}}$ cannot be extended even to a unital completely positive map from A^{**} into $A^{\mathcal{U}}$. The referee also pointed out that a result of J. M. G. Fell is closely related to results of the present section. It

is a standard fact that a representation of a discrete group is weakly contained in another representation of the same group if and only if it can be embedded into an ultrapower of the direct sum of infinitely many copies of the latter representation. In [Fell 1960, Theorem 1.2] it was essentially proved that this equivalence carries over to arbitrary C*-algebras.

All this said, Lemma 2.3 is a poor man's C*-algebraic variant of the fact that Banach space X^{**} embeds into $X^{\mathcal{U}}$. As in [Pedersen 1979, 3.3.6], we say that two representations π_1 and π_2 of A are *equivalent* if the identity map on A extends to an isomorphism between $\pi_1(A)''$ and $\pi_2(A)''$.

Lemma 2.3. *Assume $\left(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\right)$ is an ultraproduct of faithful irreducible representations of unital C*-algebras and C is a unital separable subalgebra of $B^{\mathcal{U}}$.*

(1) *If $C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right) = \{0\}$ then the induced representation of C on $\prod_{\mathcal{U}} H_j$ is equivalent to the universal representation of C.*

(2) *In general, if*

$$p = \bigvee \left\{ q : q \text{ is a projection in } C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right) \right\}$$

then $p \in C' \cap B\left(\prod_{\mathcal{U}} H_j\right)$ and $c \mapsto (1 - p)c$ is equivalent to the universal representation of $C/\left(C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)\right)$ on $(1 - p) \prod_{\mathcal{U}} H_j$.

Proof. For a state ψ on C the $(0, 1)$-type $t_\psi(\xi)$ of a vector ξ implementing ψ consists of all conditions of the form $(a\xi \mid \xi) = \psi(a)$ for $a \in C$ and $\|\xi\| = 1$.

(1) Fix a state ψ on C. By Glimm's lemma [Davidson 1996, Lemma II.5.1], the type t_ψ is consistent with the theory of $\left(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\right)$. By the separability of C and countable saturation, there exists a unit vector $\eta \in \prod_{\mathcal{U}} H_j$ such that $\psi(c) = (c\eta \mid \eta)$ for all $c \in C$. Let L be the norm-closure of $C\eta$ in $\prod_{\mathcal{U}} H_j$. Then L is a reducing subspace for C and the induced representation of C on L is spatially isomorphic to the GNS representation of C corresponding to ψ. Since ψ was arbitrary, by [Pedersen 1979, Theorem 3.8.2] this completes the proof.

(2) For every $a \in C$ we have $pa \in C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)$ and therefore $pa(1 - p) = 0$. Similarly $(1 - p)ap = 0$, and therefore $p \in C' \cap B\left(\prod_{\mathcal{U}} H_j\right)$. Let p_n, for $n \in \mathbb{N}$, be a maximal family of orthogonal projections in $C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)$. It is countable by the separability of C and $p = \bigvee_n p_n$. Let ψ be a state of C that annihilates $C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)$. Let $t_\psi^+(\xi)$ be the type obtained from $t_\psi(\xi)$ by adding to it all conditions of the form $p_n\xi = 0$ for $n \in \mathbb{N}$. By Glimm's lemma (as stated in [Davidson 1996, Lemma II.5.1]) the type $t_\psi^+(\xi)$ is consistent, and by the countable saturation we can find $\xi_1 \in \prod_{\mathcal{U}} H_j$ that realizes this type. Then $p\xi_1 = 0$ and therefore $\xi_1 \in (1 - p) \prod_{\mathcal{U}} H_j$. Therefore every GNS representation of $C/\left(C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)\right)$ is spatially equivalent to a subrepresentation of $c \mapsto (1 - p)c$, and by [Pedersen 1979, Theorem 3.8.2] this concludes the proof. $\qquad\square$

3. Second dual and Day's trick

The natural embedding of a C*-algebra B into its second dual B^{**} is rarely elementary. For example, having real rank zero is axiomatizable [Farah et al. 2016a, Theorem 2.5.1] and B^{**}, being a von Neumann algebra, has real rank zero while B may have no nontrivial projections at all. However, we shall see that there is a restricted degree of elementarity between B and B^{**}, and it will suffice for our purposes.

We shall consider the language $(\mathcal{L}_{\mathrm{rep}})_B$ obtained by adding new constants for parameters in B; see Section 2. Term $\alpha(x)$ in the extended language is *linear* if it is of the form

$$\alpha(x) = xa + bx$$

for some parameters a and b.

A *restricted B-linear formula* is a formula of the form

(1) $\max_{j \leq m} \|\alpha_j(x) - b_j\| + \max_{j \leq n}(r_j \mathbin{\dot{-}} \|\beta_j(x)\|)$,

where

(2) all b_j, for $1 \leq j \leq m$, are parameters in B,

(3) all r_j, for $1 \leq j \leq n$, are positive real numbers,

(4) all α_j, for $1 \leq j \leq m$, are linear terms with parameters in B, and

(5) all β_j, for $1 \leq j \leq n$, are linear terms with parameters in B.

The proof of the following is based on an application of the Hahn–Banach separation theorem first used by Day [1957]; see also [Elliott 1977, Section 2] for some uses of this method in the theory of C*-algebras.

Lemma 3.1. *Suppose B is a unital C*-algebra and*

$$\gamma(x) = \max_{j \leq m} \|\alpha_j(x) - b_j\| + \max_{j \leq n}\big(r_j \mathbin{\dot{-}} \|\beta_j(x)\|\big).$$

is a restricted B-linear formula. Then the following are equivalent:

(6) $\inf_{x \in B} \gamma(x) = 0$,

(7) $\inf_{x \in B^{**}} \gamma(x) = 0$.

Proof. Condition (6) implies (7) because B is isomorphic to a unital subalgebra of B^{**} and therefore $\inf_{x \in B^{**}} \gamma(x) \leq \inf_{x \in B} \gamma(x)$.

Assume (7) holds. Let a_j and c_j, for $j \leq n$, be such that $\alpha_j(x) = a_j x + x c_j$. For each j we identify α_j with its interpretation, a linear map from B to B. The second adjoint $\alpha_j^{**} : B^{**} \to B^{**}$ also satisfies $\alpha_j^{**}(x) = a_j x + x c_j$, hence $\alpha_j^{**}(x)$ is the interpretation of the term $\alpha_j(x)$ in B^{**}. The set

$$Z := \langle \alpha_j(x) : x \in B_{\leq 1} \rangle,$$

being an image of a convex set under a linear map, is a convex subset of B^m and by the Hahn–Banach theorem,

$$Z_1 := B^m \cap \langle \alpha_j(x) : x \in B^{**}_{\leq 1} \rangle$$

is included in the norm-closure of Z. By (7) we have $(b_1, \ldots, b_m) \in Z_1$.

Fix $\varepsilon > 0$ and let

$$X_1 := \left\{ x \in B_{\leq 1} : \max_{j \leq m} \| \alpha_j(x) - b_j \| \leq \varepsilon \right\}.$$

By the above, this is a convex subset of the unit ball of B and (by using the Hahn–Banach separation theorem again) the weak*-closure of X_1 in B^{**} is equal to $\{ x \in B^{**}_{\leq 1} : \max_{j \leq m} \| \alpha_j(x) - b_j \| \leq \varepsilon \}$.

Let $c \in B^{**}_{\leq 1}$ be such that $\gamma(c) < \varepsilon$. Then c belongs to the weak*-closure of X_1. For each $j \leq n$ we have $\| \beta_j(c) \| > r_j - \varepsilon$. Fix a norming functional $\varphi_j \in B^*$ such that $\| \varphi_j \| = 1$ and $\varphi_j(\beta_j(c)) > r_j - \varepsilon$. Then

$$U := \{ x \in B^{**} : \varphi_j(\beta_j(x)) > r_j - \varepsilon \text{ for all } j \}$$

is a weak*-open neighborhood of c and, as c belongs to the weak*-closure of X_1, $U \cap X_1$ is a nonempty subset of $B_{\leq 1}$. Any $b \in U \cap X_1$ satisfies $\gamma(b) < \varepsilon$. As $\varepsilon > 0$ was arbitrary, this shows that (6) holds. $\qquad\square$

In the following, $A \subseteq \prod_{\mathcal{U}} B_j$ is identified with a subalgebra of $B\left(\prod_{\mathcal{U}} H_j \right)$.

Lemma 3.2. *Suppose (B_j, H_j) is an irreducible representation of B_j on H_j for $j \in \mathbb{N}$ and A is a separable subalgebra of $\prod_{\mathcal{U}} B_j$.*

(1) *For every $b \in \prod_{\mathcal{U}} B_j$, we have that $b \in \left(A' \cap B\left(\prod_{\mathcal{U}} H_j \right) \right)'$ if and only if $b \in \left(A' \cap \prod_{\mathcal{U}} B_j \right)'$. Equivalently,*

$$\left(A' \cap B\left(\prod\nolimits_{\mathcal{U}} H_j \right) \right)' \cap \prod\nolimits_{\mathcal{U}} B_j = \left(A' \cap \prod\nolimits_{\mathcal{U}} B_j \right)' \cap \prod\nolimits_{\mathcal{U}} B_j.$$

(2) $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = \left(A' \cap \prod_{\mathcal{U}} B_j \right)'.$

Proof. (1) Since $\prod_{\mathcal{U}} B_j \subseteq B\left(\prod_{\mathcal{U}} H_j \right)$, we clearly have $\left(A' \cap B\left(\prod_{\mathcal{U}} H_j \right) \right)' \subseteq \left(A' \cap \prod_{\mathcal{U}} B_j \right)'$. In order to prove the converse inclusion, fix $b \in \prod_{\mathcal{U}} B_j$ and suppose that there exists $q \in A' \cap B\left(\prod_{\mathcal{U}} H_j \right)$ such that $\| [q, b] \| = r > 0$. We need to find $d \in A' \cap \prod_{\mathcal{U}} B_j$ satisfying $[d, b] \neq 0$.

Consider the $(1, 0)$-type $\boldsymbol{t}(x)$ consisting of all conditions of the form

$$\| [x, b] \| \geq r \quad \text{and} \quad [x, a] = 0$$

for $a \in A$. This type is satisfied in $B\left(\prod_{\mathcal{U}} H_j \right)$ by q. Since all formulas in $\boldsymbol{t}(x)$ are quantifier-free, their interpretation is unchanged when passing to a larger algebra.

Fix a finite subset of $\boldsymbol{t}(x)$ and let $F \subseteq A$ be the set of parameters occurring in this subset. Then

$$\gamma_F(x) := \inf_x \max_{a \in F} \| [x, a] \| + (r \div \| [x, b] \|)$$

is a restricted $\prod_{\mathcal{U}} B_j$-linear formula. Since A is separable, we can find projection p in $C^*(A, q)' \cap B\left(\prod_{\mathcal{U}} H_j\right)$ with separable range such that $q_1 := pq$ satisfies $\|[q_1, b]\| = r$. To find this p, take a separable elementary submodel (C, H_0) of

$$\left(B\left(\prod_{\mathcal{U}} H_j\right), \prod_{\mathcal{U}} H_j\right)$$

such that $A \subseteq C$ and let p be the projection to H_0.

By the downward Löwenheim–Skolem theorem [Farah et al. 2014, Theorem 4.6] there exists a separable elementary submodel (D, K) of $\left(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\right)$ such that $C^*(A, b) \subseteq D$ and the range of p is included in K. Part (2) of Lemma 1.2 and Łoś's theorem imply that $\overline{\prod_{\mathcal{U}} B_j}^{\text{WOT}} = B\left(\prod_{\mathcal{U}} H_j\right)$ and $\overline{p_K D p_K}^{\text{WOT}} = B\left(p_K \prod_{\mathcal{U}} H_j\right)$, where p_K denotes the projection to K. We can therefore identify p_K with a minimal central projection in D^{**}. Via this identification we have $q_1 \in D^{**}$. Since $\gamma_F(q_1) = 0$, Lemma 3.1 implies $\inf_{x \in D, \|x\| \leq 1} \gamma_F(x) = 0$ and $\inf_{x \in \prod_{\mathcal{U}} B_j, \|x\| \leq 1} \gamma_F(x) = 0$ (since γ_F is quantifier-free).

Since F was an arbitrary finite subset of A, the type $\boldsymbol{t}(x)$ is consistent with the theory of $\prod_{\mathcal{U}} B_j$. Since A is separable, by the countable saturation there exists $d \in A' \cap \prod_{\mathcal{U}} B_j$ satisfying $\|[d, b]\| \geq r$.

(2) By the von Neumann bicommutant theorem, $\overline{A}^{\text{WOT}} = \left(A' \cap B\left(\prod_{\mathcal{U}} H_j\right)\right)'$ and therefore (1) implies $\overline{A}^{\text{WOT}} \cap \prod_{\mathcal{U}} B_j = \left(A' \cap \prod_{\mathcal{U}} B_j\right)'$. $\qquad\square$

4. Proof of Theorem 1

Suppose (B_j, H_j) is a faithful irreducible representation of B_j on H_j for $j \in \mathbb{N}$ and A is a separable subalgebra of $\prod_{\mathcal{U}} B_j$. By Lemma 1.2, $\left(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\right)$ is an irreducible faithful representation of $\prod_{\mathcal{U}} B_j$.

By (2) of Lemma 3.2, we have $\overline{A}^{\text{WOT}} \cap \prod_{\mathcal{U}} B_j = \left(A' \cap \prod_{\mathcal{U}} B_j\right)'$. Then, since $A \subseteq \left(A' \cap \prod_{\mathcal{U}} B_j\right)'$, it remains to prove $\left(A' \cap \prod_{\mathcal{U}} B_j\right)' \subseteq A$. Fix $b \in \prod_{\mathcal{U}} B_j$ such that $r := \text{dist}(b, A) > 0$. By (1) of Lemma 3.2, it suffices to find $d \in A' \cap B\left(\prod_{\mathcal{U}} H_j\right)$ such that $[d, b] \neq 0$. Let

$$C := C^*(A, b).$$

Lemma 4.1. *With A, b, C, r and $\prod_{\mathcal{U}} B_j$ as above, there exists a representation*

$$\pi : C/\left(C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)\right) \to B(K)$$

and $q \in \pi(A)' \cap B(K)$ such that $[q, \pi(b)] \neq 0$.

Since the proof of Lemma 4.1 is on the long side, let us show how it completes the proof of Theorem 1. Lemma 2.3 implies that if

$$p = \bigvee\left\{q : q \text{ is a projection in } C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)\right\}$$

then $p \in C' \cap B\left(\prod_{\mathcal{U}} H_j\right)$ and $c \mapsto (1-p)c$ is equivalent to the universal representation of $C/\left(C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)\right)$ on $(1-p)\prod_{\mathcal{U}} H_j$. Therefore q as in the conclusion

of Lemma 4.1 can be found in $A' \cap B\left(\prod_{\mathcal{U}} H_j\right)$, implying $b \notin \left(A' \cap B\left(\prod_{\mathcal{U}} H_j\right)\right)'$. By Lemma 3.2 this implies $b \notin \left(A' \cap \prod_{\mathcal{U}} B_j\right)'$, reducing the proof of Theorem 1 to the following.

Proof of Lemma 4.1. An easy special case is worth noting. If $C \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right) = \{0\}$ then Lemma 2.2 implies the existence of a representation $\pi : C \to B(K)$ and $q \in \pi(A)' \cap B(K)$ such that $[q, \pi(b)] \neq 0$.

In the general case, let q_n, for $n \in J$, be an enumeration of a maximal orthogonal set of minimal projections in $A \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)$. The index-set J is countable (and possibly finite or even empty) since A is separable. Let $p_n := \bigvee_{j \leq n} q_j$.

Suppose for a moment that there exists n such that $p_n b p_n \notin A$. Since the range of p_n is finite-dimensional, by von Neumann's bicommutant theorem [Blackadar 2006, §I.9.1.2] and the Kadison transitivity theorem [Blackadar 2006, Theorem II.6.1.13] there exists $d \in A' \cap B(p_n \prod_{\mathcal{U}} H_j)$ such that $[d, b] \neq 0$. Lemma 3.2 now implies $p_n b p_n \notin \left(A' \cap \prod_{\mathcal{U}} B_j\right)'$ and $b \notin \left(A' \cap \prod_{\mathcal{U}} B_j\right)'$.

We may therefore assume $p_n b p_n \in A$, for all n. Let $p = \bigvee_n p_n$. Lemma 2.3 (2) implies $p \in A' \cap B\left(\prod_{\mathcal{U}} H_j\right)$, and we may therefore assume $[b, p] = 0$. Since $C = \mathrm{C}^*(A, b)$ this implies $p \in C' \cap B\left(\prod_{\mathcal{U}} H_j\right)$. Since $p_n b p_n \in A$ for all n we have $A \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right) = pCp \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)$. If $c \in C$, then for every n, we have $p_n c(1 - p) = 0$ and similarly $(1 - p)cp_n = 0$. Since the sequence p_n, for $n \in \mathbb{N}$, is an approximate unit for $A \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right)$, the latter is an ideal of C. Let $\theta : C \to C/(A \cap \mathcal{K})$ be the quotient map. We claim that $\mathrm{dist}(\theta(b), \theta(A)) = \mathrm{dist}(b, A) > 0$. Fix $a \in A$. We need to show that $\|\theta(a - b)\| \geq r$.

Consider the $(0, 1)$-type $t(\xi)$ consisting of all conditions of the form

$$\|\xi\| = 1, \quad \|(a - b)\xi\| \geq r, \quad p_n \xi = 0,$$

for $n \in J$. To see this type is consistent fix a finite $F \subseteq J$. Let $m \geq \max(F)$ and

$$a' := (1 - p_m)a(1 - p_m) + p_m b p_m.$$

As both summands belong to A, we have $a' \in A$ and therefore $\|a' - b\| \geq r$. Fix $\varepsilon > 0$. If $\xi \in \prod_{\mathcal{U}} H_j$ is a vector of norm ≤ 1 such that $\|(a' - b)\xi\| > r - \varepsilon$ then $\xi' = (1 - p_m)\xi$ has the same property since $(a' - b)p_m = 0$. Since $\varepsilon > 0$ was arbitrary, $t(\xi)$ is consistent. By the countable saturation there exists a unit vector $\xi \in \prod_{\mathcal{U}} H_j$ which realizes $t(\xi)$. Since $p_n \xi = 0$ for all n, we have $p\xi = 0$ and therefore $\|\theta(a - b)\| \geq \left\|(1 - p)(a - b)(1 - p)\right\| \geq r$. Since $a \in A$ was arbitrary, we conclude that $\mathrm{dist}(\theta(b), \theta(A)) = r$.

Suppose for a moment that $(1 - p)C(1 - p) \cap \mathcal{K}\left(\prod_{\mathcal{U}} H_j\right) = \{0\}$. By (2) of Lemma 2.3 the representation

$$C \ni c \mapsto (1 - p)c \in B\left((1 - p) \prod_{\mathcal{U}} H_j\right)$$

is equivalent to the universal representation of C. Hence by Lemma 2.2 we can find $d \in (1 - p)\big(A' \cap B\big(\prod_{\mathcal{U}} H_j\big)\big)$ that does not commute with b, and by the above, this concludes the proof in this case.

We may therefore assume that

$$(1 - p)C(1 - p) \cap \mathcal{K}\big(\textstyle\prod_{\mathcal{U}} H_j\big) \neq \{0\}.$$

By the spectral theorem for self-adjoint compact operators and continuous functional calculus, there exists a nonzero projection $q \in (1 - p)C(1 - p)$ of finite rank. Fix $c \in C$ such that $(1 - p)c(1 - p) = q$.

By Lemma 3.2, it suffices to find $q \in A' \cap (1 - p)B\big(\prod_{\mathcal{U}} H_j\big)(1 - p)$ such that $[q, c] \neq 0$. Suppose otherwise, so that $c \in \big(A' \cap \prod_{\mathcal{U}} B_j\big)'$. Lemma 3.2 (2) implies that $c \in \overline{A}^{\text{WOT}}$. By the Kaplansky density theorem [Blackadar 2006, Theorem I.9.1.3] there is a net of positive contractions in A converging to c in the weak operator topology. By the continuous functional calculus and the Kadison transitivity theorem [Blackadar 2006, Theorem II.6.1.13], we may choose this net among the members of

$$Z := \{a \in A_+ : \|a\| = 1, qaq = q\}.$$

Consider the $(0, 1)$-type $t_1(\xi)$ consisting of all conditions of the form

$$\|\xi\| = 1, \quad a\xi = \xi,$$
$$q\xi = 0, \quad p_n\xi = 0,$$

for $n \in \mathbb{N}$ and $a \in Z$.

We claim that $t_1(\xi)$ is consistent. Fix $\varepsilon > 0$ and $a_1, a_2, \ldots, a_n$ in Z. Let

$$a := a_1 a_2 \cdots a_{n-1} a_n a_{n-1} \cdots a_2 a_1.$$

Then $a \in Z$ and $q \leq a$. By the choice of p the operator $(1 - p)(a - s)_+$ is not compact for any $s < 1$. Therefore there exists a unit vector ξ_0 in $(1 - p - q)\prod_{\mathcal{U}} H_j$ such that $\|\xi_0 - a\xi_0\|$ is arbitrarily small. By the countable saturation there exists a unit vector $\xi_1 \in (1 - (p + q))\prod_{\mathcal{U}} H_j$ such that $a\xi_1 = \xi_1$. As each a_j is a positive contraction, we have $a_j\xi_1 = \xi_1$ for $1 \leq j \leq n$. Since $a_1, \ldots, a_n$ was an arbitrary subset of Z, this shows that $t_1(\xi)$ is consistent.

Since Z is separable, by the countable saturation there exists $\xi \in \prod_{\mathcal{U}} H_j$ realizing $t_1(\xi)$. Then ξ is a unit vector in $(1 - (p + q))\prod_{\mathcal{U}} H_j$ such that $a\xi = \xi$ for all $a \in Z$. As $c\xi = 0$, this contradicts c being in the weak operator topology closure of Z.

Therefore there exists

$$q \in A' \cap (1 - p)B\big(\textstyle\prod_{\mathcal{U}} H_j\big)(1 - p)$$

such that $[q, c] \neq 0$. Since $c \in C = \mathrm{C}^*(A, b)$ we have $[q, b] \neq 0$, and this concludes the proof. $\square$

5. Concluding remarks

In the following infinitary form of the Kadison transitivity theorem, p_K denotes projection to a closed subspace K of $\prod_{\mathcal{U}} H_j$.

Proposition 5.1. *Assume* $\left(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\right)$ *is an ultraproduct of faithful and irreducible representations of unital* C*-algebras. *Also assume* K *is a separable closed subspace of* $\prod_{\mathcal{U}} H_j$ *and* $T \in B(K)$.

(1) *There exists* $b \in \prod_{\mathcal{U}} B_j$ *such that* $\|b\| = \|T\|$ *and* $p_K b p_K = T$.

(2) *If* T *is self-adjoint, positive, or unitary in* $B(K)$, *then* b *can be chosen to be self-adjoint, positive, or unitary, respectively, in* $B\left(\prod_{\mathcal{U}} H_j\right)$.

Proof. (1) This is a consequence of the Kadison transitivity theorem and countable saturation of the structure $\left(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j\right)$. Let p_n, for $n \in \mathbb{N}$, be an increasing sequence of finite-dimensional projections converging to p_K in the strong operator topology and let a_n, for $n \in \mathbb{N}$, be a dense subset of A. We need to check that the type $t(x)$ consisting of all conditions of the form

$$\|p_n(x - T)p_n\| = 0, \qquad \|x\| = \|T\|,$$

for $n \in \mathbb{N}$ is consistent. Since the representation of $\prod_{\mathcal{U}} B_j$ on $\prod_{\mathcal{U}} H_j$ is irreducible by Lemma 1.2, every finite subset of $t(x)$ is consistent by the Kadison transitivity theorem. We can therefore find $b \in \prod_{\mathcal{U}} B_j$ that satisfies $t(x)$ and thus $p_K b p_K = T$ and $\|b\| = \|T\|$.

(2) If T is self-adjoint, add the condition $x = x^*$ to $t(x)$. By [Pedersen 1979, Theorem 2.7.5] the corresponding type is consistent, and the assertion again follows by countable saturation. The case when T is unitary uses the same theorem. $\square$

An important consequence of Voiculescu's theorem is that any two unital representations $\pi_j : A \rightarrow B(H)$ of a separable unital C*-algebra A on H such that $\ker(\pi_1) = \ker(\pi_2)$ and $\pi_1(A) \cap \mathcal{K}(H) = \pi_2(A) \cap \mathcal{K}(H) = \{0\}$ are approximately unitarily equivalent [Voiculescu 1976, Corollary 1.4]. The analogous statement is in general false for the ultraproducts. Let $B_n = M_n(\mathbb{C})$ for $n \in \mathbb{N}$ and let $A = \mathbb{C}^2$. The group $K_0\left(\prod_{\mathcal{U}} M_n(\mathbb{C})\right)$ is isomorphic to $\mathbb{Z}^{\mathbb{N}}$ with the natural ordering and the identity function id as the order-unit. Every unital representation of A corresponds to an element of this group that lies between 0 and id, and there are $2^{\aleph_0}$ inequivalent representations. Also, $K_0\left(\prod_{\mathcal{U}} M_n(\mathbb{C})\right)$ is isomorphic to the ultraproduct $\prod_{\mathcal{U}} \mathbb{Z}$ and $2^{\aleph_0}$ of these extensions remain inequivalent even after passing to the ultraproduct.

We return to Pedersen's question [1990, Remark 10.11], whether a bicommutant theorem $(A' \cap M(B)/B)' = A$ is true for a separable unital subalgebra A of a corona $M(B)/B$ of a σ-unital C*-algebra B? A simple and unital C*-algebra C is *purely infinite* if for every nonzero $a \in C$ there are x and y such that $x a y = 1$.

Question 5.2. Suppose C is unital, simple, purely infinite, and separable and A is a unital subalgebra of C. Is $(A' \cap C^{**})' \cap C = A$?

Let us prove that a positive answer to Question 5.2 would imply a positive answer to Pedersen's question. If A is a separable and unital subalgebra of $M(B)/B$ and $b \in (M(B)/B) \setminus A$, then there exists a separable elementary submodel C of $M(B)/B$ containing b. By [Lin 2004], $M(B)/B$ is simple if and only if it is purely infinite, and since being simple and purely infinite is axiomatizable [Farah et al. 2016a, Theorem 2.5.1], C is simple and purely infinite. If $(A' \cap C^{**})' \cap C = A$ then Proposition 5.3 implies that there exists $d \in A' \cap M(B)/B$ such that $[d, b] \neq 0$.

Proposition 5.3. *Suppose B is a* C*-algebra, *A is a separable subalgebra of B, $b \in B$ and $r \geq 0$. If B is an ultraproduct or a corona of a σ-unital, nonunital* C*-algebra then*

$$\sup_{d \in (A' \cap B)_+, \|d\| \leq 1} \|[d, b]\| = \sup_{d \in (A' \cap B^{**})_+, \|d\| \leq 1} \|[d, b]\|.$$

Proof. The only property of B used in this proof is that of being countably degree-1 saturated [Farah and Hart 2013, Theorem 1]. Since $B \subseteq B^{**}$, it suffices to prove "$\geq$" in the above inequality. Suppose $b \in B$ and $d \in (A' \cap B^{**})_+$ are such that $\|d\| = 1$ and $r \doteq \|[b, d]\|$. Consider the type $t(x)$ consisting of conditions $\|x\| = 1$, $x \geq 0$, $\|xb - bx\| \geq r$, and $\|[x, a]\| = 0$ for a in a countable dense subset of A. This is a countable degree-1 type. If $\varphi_j = 0$, for $j < n$, is a finite subset of $t(x)$ then $\gamma(x) := \max_{j<n} \varphi_j(x)$ is a restricted B-linear formula and Lemma 2.3 implies that it is approximately satisfied in B. By the countable degree-1 saturation of B [Farah and Hart 2013, Theorem 1] we can find a realization d' of $t(x)$ in B. Clearly $d' \in (A' \cap B)_+$, $\|d'\| = 1$, and $\|[d', b]\| \geq r$, completing the proof. $\square$

Some information on a special case of Pedersen's conjecture can also be found in [Elliott and Kucerovsky 2007].

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ILIJAS FARAH
DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
4700 KEELE STREET
TORONTO, ON M3J1P3
CANADA

ifarah@mathstat.yorku.ca

PACIFIC JOURNAL OF MATHEMATICS