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VANDERSON LIMA

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PARABOLIC MINIMAL SURFACES IN $\mathbb{M}^2 \times \mathbb{R}$

VANDERSON LIMA

Let $\mathbb{M}^2$ be a complete noncompact orientable surface of nonnegative curvature. We prove some theorems involving parabolicity of minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$. First, using a characterization of δ-parabolicity we prove that under additional conditions on $\mathbb{M}$, an embedded minimal surface with bounded Gaussian curvature is proper. The second theorem states that under some conditions on $\mathbb{M}$, if Σ is a properly immersed minimal surface with finite topology and one end in $\mathbb{M} \times \mathbb{R}$, which is transverse to a slice $\mathbb{M} \times \{t\}$ except at a finite number of points, and such that $\Sigma \cap (\mathbb{M} \times \{t\})$ contains a finite number of components, then Σ is parabolic. In the last result, we assume some conditions on $\mathbb{M}$ and prove that if a minimal surface in $\mathbb{M} \times \mathbb{R}$ has height controlled by a logarithmic function, then it is parabolic and has a finite number of ends.

1. Introduction

Let $\mathbb{M}^2$ be a complete noncompact orientable surface with nonnegative curvature. Under these conditions $\mathbb{M} \times \mathbb{R}$ is complete and has nonnegative sectional curvature, in particular nonnegative Ricci curvature. Recently, using some of the results of [Schoen and Yau 1982], G. Liu classified complete noncompact 3-manifolds with nonnegative Ricci curvature.

Theorem [Liu 2013]. *Let N be a complete noncompact 3-manifold with nonnegative Ricci curvature. Then either N is diffeomorphic to $\mathbb{R}^3$ or its universal cover $\widetilde{N}$ is isometric to a Riemannian product $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is a complete surface with nonnegative sectional curvature.*

In particular it follows from the proof of this result that if N is not flat or does not have positive Ricci curvature then its universal cover splits as a product $\mathbb{M} \times \mathbb{R}$. So the spaces $\mathbb{M} \times \mathbb{R}$ are in fact general examples of a very important class of 3-manifolds.

We are interested in minimal surfaces in $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is as above. In particular we want information about the topology and the conformal structure. It is

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important to study under which hypotheses we can guarantee that a minimal surface is proper. Concerning the topology, we know that there is no compact minimal surface in these spaces. So, one can study the genus and the number of ends of such minimal surfaces. Concerning the conformal structure, one important property is *parabolicity*. Our results are inspired by analogous results in $\mathbb{R}^3$.

First we study the problem of properness. Bessa, Jorge and Oliveira-Filho studied this problem for manifolds with nonnegative Ricci curvature and obtained some partial results in $\mathbb{R}^3$.

Theorem [Bessa et al. 2001]. *Let N^3 be a complete Riemannian 3-manifold of bounded geometry and positive Ricci curvature. Let $f : \Sigma^2 \to N^3$ be a complete injective minimal immersion, where Σ is a complete oriented surface with bounded curvature.*

(1) *If N is compact, then Σ is compact.*

(2) *If N is not compact, then f is proper.*

A major breakthrough was the work of Colding and Minicozzi [2008], where it was proved that a complete minimal surface of finite topology embedded in $\mathbb{R}^3$ is proper. After this, Meeks and Rosenberg [2006] proved that if Σ is a complete embedded minimal surface in $\mathbb{R}^3$ which has positive injectivity radius, then Σ is proper. Finally, Meeks and Rosenberg [2008] proved that if $f : \Sigma \to \mathbb{R}^3$ is an injective minimal immersion, with Σ complete and of bounded curvature, then f is proper. We extend the last result to the case of a product $\mathbb{M} \times \mathbb{R}$:

Theorem A. *Let $\mathbb{M}$ be a complete simply connected orientable noncompact surface such that $0 \le K_{\mathbb{M}} \le \kappa$. Let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be an injective minimal immersion of a complete, connected Riemannian surface of bounded curvature. Then the map f is proper.*

Next we focus on surfaces with finite topology and one end. The results in [Colding and Minicozzi 2008; Meeks and Rosenberg 2005] imply that every complete, embedded minimal surface in $\mathbb{R}^3$ of finite genus and one end is properly embedded and intersects some plane transversely in a single component, and so, is parabolic. Meeks and Rosenberg [2008] gave an independent proof that the surface is parabolic without the additional assumption that it is embedded. Namely, they proved:

Theorem [Meeks and Rosenberg 2008]. *Let Σ be a surface of finite topology and one end, and let $f : \Sigma \to \mathbb{R}^3$ be a proper minimal immersion. Suppose that f is transverse to a plane P except at a finite number of points, and $f^{-1}(P)$ contains a finite number of components. Then Σ is parabolic.*

The half-space theorem of Hoffman and Meeks [1990] states that a properly immersed minimal surface in $\mathbb{R}^3$ which is above a plane is a parallel plane. Thus the condition that a minimal surface be transverse to a plane is natural. Rosenberg proved the following half-space theorem for product spaces:

Theorem [Rosenberg 2002]. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $K_{\mathbb{M}} \geq 0$.

(2) *There is a point $p \in M$ such that the geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded.*

If Σ is a properly immersed minimal surface in a half-space $\mathbb{M} \times [t_0, +\infty)$, then Σ is a slice $\mathbb{M} \times \{s\}$ for some $s > t_0$.

Based on these results we prove the following:

Theorem B. *Suppose $\mathbb{M}$ satisfies the conditions of the previous theorem. Let Σ be a surface of finite topology and one end and let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be a proper minimal immersion. Suppose that f is transverse to a slice $\mathbb{M} \times \{t_0\}$ except at a finite number of points and that $f^{-1}(\mathbb{M} \times \{t_0\})$ contains a finite number of components. Then Σ is parabolic.*

Next we focus on surfaces with more than one end. A major breakthrough was the proof of the generalized Nitsche conjecture in $\mathbb{R}^3$:

Theorem [Collin 1997]. *Let Σ be a properly embedded minimal surface in $\mathbb{R}^3$ with at least two ends. Then an annular end of Σ is asymptotic to a plane or to the end of a catenoid.*

Let Σ be as in the last theorem. The set $\mathcal{E}_\Sigma$ of all the ends of Σ has a natural topology that makes it a compact Hausdorff space. The limit points in $\mathcal{E}_\Sigma$ are called the *limit ends* of Σ, and an end which is not a limit end is called a *simple end*. To Σ is associated a unique plane P passing through the origin in $\mathbb{R}^3$ called the limit tangent plane at infinity of Σ [Callahan et al. 1990]. The ends of Σ are linearly ordered by their relative heights over P and this linear ordering, up to reversing it, depends only on the proper ambient isotopy class of Σ in $\mathbb{R}^3$ [Frohman and Meeks 1997]. Since $\mathcal{E}_\Sigma$ is compact and the ordering is linear, there exists a unique *top end* which is the highest end and a unique *bottom end* which is lowest in the associated ordering. The ends of Σ that are neither top nor bottom ends are called *middle ends*. In the proof of the ordering theorem, one shows that every middle end of Σ is contained between two catenoids in the following sense: if E is an end of Σ there are $c_1 > 0$ and $r_1 > 0$ such that $E \subset \{(x_1, x_2, x_3) : |x_3| \leq c_1 \log r,\ r^2 = x_1^2 + x_2^2,\ r \geq r_1\}$.

Collin, Kusner, Meeks and Rosenberg [Collin et al. 2004] proved that if Σ is a properly immersed minimal surface with compact boundary in $\mathbb{R}^3$ which is contained between two catenoids, then Σ has quadratic area growth. Furthermore, Σ has a finite number of ends. As a consequence the middle ends of a properly embedded minimal surface in $\mathbb{R}^3$ are never *limit ends*. We explain what it means for a properly immersed minimal surface of $\mathbb{M} \times \mathbb{R}$ to be contained between two catenoids and generalize the result above:

Theorem C. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $0 \leq K_{\mathbb{M}} \leq \kappa$.

(2) $\mathbb{M}$ *has a pole* p.

(3) *The geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded.*

Let Σ be a properly immersed minimal surface inside the region of $\mathbb{M} \times \mathbb{R}$ defined by $|h| \leq c_2 \log r$ for some constant $c_2 > 0$ and $r \geq 1$. Then Σ is parabolic. Moreover, if Σ has compact boundary, then Σ has quadratic area growth and a finite number of ends.

The paper is organized as follows. In Section 2 we present some results about the geometry of the space $\mathbb{M} \times \mathbb{R}$ and its minimal surfaces. In Sections 3 and 4 we give some well-known definitions and enunciate some results involving parabolicity and laminations. In Section 5 we prove Theorem A. In Section 6 we prove Theorems B and C.

2. The geometry of $\mathbb{M}^2 \times \mathbb{R}$

Some of the results of this section are well known, but we prove them here for completeness.

Lemma 1. *Let $\mathbb{M}$ be a complete noncompact orientable surface with nonnegative sectional curvature. Then $\mathbb{M}$ is homeomorphic to $\mathbb{R}^2$ or isometric to a flat cylinder $\mathbb{S}^1 \times \mathbb{R}$.*

Proof. Since $K_{\mathbb{M}}^- \equiv 0$, by Huber's theorem $\mathbb{M}$ has finite topology and

$$0 \leq \int_{\mathbb{M}} K_{\mathbb{M}} \, d\mu \leq 2\pi(2 - 2g - n),$$

where g is the genus of M and n its number of ends; see [White 1987]. Since $\mathbb{M}$ is noncompact and $n \geq 1$, we have

$$1 \leq n + 2g \leq 2.$$

But $n + 2g$ is an integer; thus the only possibility is $g = 0$, $n = 1, 2$.

If $n = 1$, $\mathbb{M}$ is homeomorphic to $\mathbb{R}^2$. If $n = 2$, $\mathbb{M}$ has the topology of $\mathbb{S}^1 \times \mathbb{R}$ and

$$\int_{\mathbb{M}} K_{\mathbb{M}} \, d\mu = 0,$$

thus $K_{\mathbb{M}} \equiv 0$ and $\mathbb{M}$ is isometric to $\mathbb{S}^1 \times \mathbb{R}$ endowed with a flat metric. □

Lemma 2. *Let $\mathbb{M}$ be a complete noncompact surface with sectional curvature satisfying $0 \le K_{\mathbb{M}} \le \kappa$. Then $\mathbb{M}$ has positive injectivity radius; in particular the same holds for $\mathbb{M} \times \mathbb{R}$.*

Proof. By the previous lemma either $\mathbb{M}$ is a flat cylinder, which has positive injectivity radius, or $\mathbb{M}$ is homeomorphic to $\mathbb{R}^2$. Suppose in the last case that $\mathrm{inj}_{\mathbb{M}} = 0$. Since $K_{\mathbb{M}} \le \kappa$, the exponential map $\exp_q : B_{\pi/\sqrt{\kappa}}(0) \to \mathbb{M}$ has no critical points for each $q \in \mathbb{M}$. Then for each positive integer j sufficiently large there is a point p_j such that $\exp_{p_j}$ is not injective in the geodesic ball $B_{1/j}(p_j)$, which implies there are two geodesics $\gamma_j, \sigma_j : [0, l] \to \mathbb{M}$ beginning in p_j which meet at the same endpoint q_j in the boundary of $B_{1/j}(p_j)$ with angle equal to π (q_j realizes the distance from p_j to $\mathrm{Cut}(p_j)$; see [do Carmo 1988, Chapter 13, Proposition 2.12]). This gives us a geodesic loop α_j with one angular vertex at p_j which has exterior angle $\theta_j \le \pi$. Since $\mathbb{M}$ is simply connected, α_j bounds a disc D_j in $\mathbb{M}$. By the Gauss–Bonnet theorem

$$2\pi = \int_{D_j} K_{\mathbb{M}}\, d\mu + \theta_j \le \kappa |D_j| + \pi.$$

However, for j sufficiently large, $|D_j|$ is small and $\kappa|D_j| + \pi < 2\pi$, which is a contradiction. Therefore $\mathrm{inj}_{\mathbb{M}} > 0$. □

Lemma 3 [Espinar and Rosenberg 2009]. *Let $\mathbb{M}$ be a complete connected nonflat surface. Let Σ be a complete totally geodesic surface in $\mathbb{M} \times \mathbb{R}$. Then Σ is of the form $\alpha \times \mathbb{R}$, where α is a geodesic of M, or $\Sigma = \mathbb{M} \times \{t\}$ for some $t \in \mathbb{R}$.*

Proof. Let Π be the projection of $\mathbb{M} \times \mathbb{R}$ to $\mathbb{M}$. Let η be a unit normal to Σ and define $\nu = \langle \eta, \partial_t \rangle$. Since Σ is totally geodesic we have

$$K_\Sigma(p) = K_{\mathbb{M}}(\Pi(p))\nu(p) \quad \forall p \in \Sigma, \tag{1}$$

$$X\langle \eta, \partial_t \rangle = \langle \nabla_X \eta, \partial_t \rangle \equiv 0 \quad \forall X \in T\Sigma, \tag{2}$$

where (1) is just the Gauss equation. So ν is constant, and we can suppose $\nu \ge 0$. If $\nu = 0$, then Σ is of the form $\alpha \times \mathbb{R}$. If $\nu = 1$, then Σ is a slice.

Suppose $0 < \nu < 1$. We know that

$$\Delta_\Sigma \nu + (\mathrm{Ric}(\eta, \eta) + |A|^2)\nu = 0,$$

and by equation (2), $\Delta_\Sigma \nu = 0$. Thus $0 = \mathrm{Ric}(\eta, \eta) = K_{\mathbb{M}}(\Pi(p))(1 - \nu^2)$, which implies $K_{\mathbb{M}}(\Pi(p)) = 0$. It follows from equation (1) that Σ is flat. Then there is a $\delta > 0$ such that for any $p \in \Sigma$ a neighborhood of p in Σ is a graph (in exponential coordinates) over the disc $D_\delta \subset T_p\Sigma$ of radius δ, centered at the origin of $T_p\Sigma$. This graph, denoted by G_p, has bounded geometry. The number δ is independent of p, and the bound on the geometry of G_p is uniform as well.

We claim that $\Pi(\Sigma) = \mathbb{M}$. Suppose the contrary. Then there exists a bounded open set $\Omega \subset \Pi(\Sigma)$ and $q_0 \in \partial\Omega$ such that, for some point $p \in \Pi^{-1}(\Omega)$, a neighborhood of p in Σ is a vertical graph of a function f defined over Ω and this graph does not extend to a minimal graph over any neighborhood of q_0.

We can identify Ω with $\Omega \times \{0\}$. Let $\{q_n\} \subset \Omega$ be a sequence converging to q_0 and $p_n = (q_n, f(q_n))$. Let Σ_n denote the image of G_{p_n} under the vertical translation taking p_n to q_n. There is a subsequence of $\{q_n\}$ (which we also denote by $\{q_n\}$) such that the tangent planes $T_{q_n}(\Sigma_n)$ converge to some vertical plane $P \subset T_{q_0}(\mathbb{M} \times \mathbb{R})$. In fact, if this were not true, for q_n close enough to q_0, the graph of bounded geometry G_{p_n} would extend to a vertical graph beyond q_0. Hence f would extend beyond q_0, a contradiction. So $T_{p_n}\Sigma$ must become almost vertical at p_n for n sufficiently large, which means that $\eta(p_n)$ must become horizontal. But ν is a constant different from 0, a contradiction.

Then $\Pi(\Sigma) = \mathbb{M}$. Since $K_{\mathbb{M}} \circ \Pi \equiv 0$, it follows that $\mathbb{M}$ is a complete flat surface, which contradicts our assumption. □

Lemma 4 [Rosenberg 2002]. *Let Σ be a minimal surface of $\mathbb{M} \times \mathbb{R}$. Then the height function $h : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$, $h(q, t) = t$, is harmonic on Σ.*

Proof. Let E_1, E_2, η be an orthonormal frame in a neighborhood of a point of Σ, where η is normal to Σ. Since ∂_t is a Killing vector field on $\mathbb{M} \times \mathbb{R}$, we have

$$\operatorname{div} \partial_t = 0 = \langle \nabla_\eta \partial_t, \eta \rangle.$$

Write $\partial_t = \nabla h = X + \nabla_\Sigma h$, where X is normal to Σ. Then

$$\begin{aligned} 0 = \Delta h &= \sum_i [\langle \nabla_{E_i} \nabla_\Sigma h, E_i \rangle + \langle \nabla_{E_i} X, E_i \rangle] \\ &= \Delta_\Sigma h - \sum_i \langle X, \nabla_{E_i} E_i \rangle = \Delta_\Sigma h - \langle X, \vec{H} \rangle = \Delta_\Sigma h. \end{aligned}$$ □

Lemma 5 [Rosenberg 2002]. *Suppose that $\mathbb{M}$ has nonnegative sectional curvature and that there exists a point $p \in \mathbb{M}$ such that the geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded. Define $f : \mathbb{M} \setminus \big(\{p\} \cup \mathrm{Cut}(p)\big) \times \mathbb{R} \to \mathbb{R}$, $f(q, t) = \log(r(q))$, where r is the distance in $\mathbb{M}$ to the point p. Let Σ be a minimal surface of $\mathbb{M} \times \mathbb{R}$. Then*

$$\Delta_\Sigma f \leq \frac{c_1}{r} |\nabla_\Sigma h|^2$$

for some constant $c_1 > 0$ and $r \geq 1$.

Proof. Denote by ∇f, Δf and Hess f respectively the gradient, the Laplacian and the Hessian of f in $\mathbb{M} \times \mathbb{R}$. Since $\mathbb{M}$ has nonnegative curvature, by the Laplacian comparison theorem we have

$$\Delta_{\mathbb{M}} r \leq \frac{1}{r}.$$

But f does not depend on the height, so

$$\Delta f = \Delta_{\mathbb{M}} f = \frac{\Delta_{\mathbb{M}} r}{r} - \frac{|\nabla_{\mathbb{M}} r|^2}{r^2} \leq 0.$$

Let E_1, E_2, η be an orthonormal frame in a neighborhood of a point of Σ, where η is normal to Σ. Write $\nabla f = \nabla_\Sigma f + \langle \nabla f, \eta \rangle \eta$. Since Σ is minimal we have

$$\begin{aligned}
\Delta f &= \sum_{i=1}^{2} \langle \nabla_{E_i} \nabla f, E_i \rangle + \langle \nabla_\eta \nabla f, \eta \rangle \\
&= \sum_{i=1}^{2} \langle \nabla_{E_i} \nabla_\Sigma f, E_i \rangle + \sum_{i=1}^{2} \langle \nabla f, \eta \rangle \langle \nabla_{E_i} \eta, E_i \rangle + \langle \nabla_\eta \nabla f, \eta \rangle \\
&= \Delta_\Sigma f + \langle \nabla f, \eta \rangle H + \operatorname{Hess} f(\eta, \eta) \\
&= \Delta_\Sigma f + \operatorname{Hess} f(\eta, \eta).
\end{aligned}$$

Now, let V be tangent to $\mathbb{M}$, $\xi = \partial/\partial t$ and Π be the projection of $\mathbb{M} \times \mathbb{R}$ to $\mathbb{M}$. Again, since f does not depend on the height, we have

$$\operatorname{Hess} f(\xi, \xi) = 0,$$
$$\operatorname{Hess} f(V, V) = \operatorname{Hess}_{\mathbb{M}} f(V, V).$$

Then

$$\operatorname{Hess} f(\eta, \eta) = \operatorname{Hess} f(\Pi(\eta), \Pi(\eta)) = \operatorname{Hess}_{\mathbb{M}} f(\Pi(\eta), \Pi(\eta)).$$

But $\Delta f \leq 0$, so

$$\Delta_\Sigma f \leq -\operatorname{Hess} f_{\mathbb{M}}(\Pi(\eta), \Pi(\eta)) \leq |\operatorname{Hess}_{\mathbb{M}} f| |\Pi(\eta)|^2. \tag{3}$$

A simple calculation shows that

$$|\Pi(\eta)| = |\nabla_\Sigma h|. \tag{4}$$

Let $q \in \mathbb{M}$, $r(q) = d(q, p)$ and v be a unit tangent vector to $\mathbb{M}$ at q. Thus

$$\operatorname{Hess}_{\mathbb{M}} f(v, v) = \left\langle \nabla_v \left(\frac{\nabla_{\mathbb{M}} r}{r} \right), v \right\rangle = \frac{1}{r} \langle \nabla_v \nabla_{\mathbb{M}} r, v \rangle + v\left(\frac{1}{r}\right) \langle \nabla_{\mathbb{M}} r, v \rangle.$$

When $v = \nabla_{\mathbb{M}} r$,

$$\operatorname{Hess}_{\mathbb{M}} f(v, v) = -\frac{1}{r^2} |\nabla_{\mathbb{M}} r|^2.$$

When $v = T$, the unit tangent vector to the geodesic circle of radius r through the point q,

$$\operatorname{Hess}_{\mathbb{M}} f(v, v) = \frac{1}{r} \langle \nabla_T \nabla_{\mathbb{M}} r, T \rangle = \frac{1}{r} k_g(q),$$

where $k_g(q)$ is the geodesic curvature of the geodesic circle of radius r centered at the point q. By the hypothesis about the geodesic circles of $\mathbb{M}$,

$$|\mathrm{Hess}_{\mathbb{M}} f|^2 = \frac{1}{r^4} + \frac{1}{r^2}k_g^2 \leq \frac{C}{r^2}.$$

Using equations (3) and (4), the lemma follows. □

3. Laminations

Definition 6. Let Σ be a complete, embedded surface in a 3-manifold N. A point $p \in N$ is a limit point of Σ if there exists a sequence $\{p_n\} \subset \Sigma$ which diverges to infinity in Σ with respect to the intrinsic Riemannian topology on Σ, but converges in N to p as $n \to \infty$. Let $\mathcal{L}(\Sigma)$ denote the set of all limit points of Σ in N; we call this set the limit set of Σ. In particular, $\mathcal{L}(\Sigma)$ is a closed subset of N and $\overline{\Sigma} \setminus \Sigma \subset \mathcal{L}(\Sigma)$, where $\overline{\Sigma}$ denotes the closure of Σ.

Definition 7. A codimension-1 lamination of a Riemannian n-manifold N is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying the following conditions:

(1) $\mathcal{L}$ is a closed subset of N.

(2) $\mathcal{A} = \{\varphi_\beta : \mathbb{D} \times (0, 1) \to U_\beta\}_\beta$ is an atlas of coordinate charts of N, where $\mathbb{D}$ is the open unit ball in $\mathbb{R}^{n-1}$ and U_β is an open subset of N.

(3) For each β, there is a closed subset C_β of $(0, 1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$.

If all the leaves are minimal hypersurfaces, $(\mathcal{L}, \mathcal{A})$ is called a minimal lamination.

4. Parabolic manifolds

Definition 8. Given a point p on a Riemannian manifold N with boundary, one can define the hitting, or harmonic, measure μ_p of an interval $I \subset \partial N$ as the probability that a Brownian path beginning at p reaches the boundary for the first time at a point in I.

Proposition 9. *Let N be a Riemannian manifold with nonempty boundary. The following are equivalent*:

(1) *Any bounded harmonic function on N is determined by its boundary values.*

(2) *For some $p \in \mathrm{Int}\, N$, the measure μ_p is full on ∂N, i.e, $\int_{\partial N} \mu_p = 1$.*

(3) *If $h : N \to \mathbb{R}$ is a bounded harmonic function, then $h(p) = \int_{\partial N} h(x)\mu_p$.*

If N satisfies any of the conditions above, then it is called parabolic.

An important property is that a proper subdomain of a parabolic manifold is parabolic; hence removing the interior of a compact domain does not alter parabolicity. Moreover, if there exists a proper nonnegative superharmonic function on N, then N is parabolic. For equivalent definitions and properties of parabolic manifolds see [Grigor'yan 1999].

Definition 10. Let N be a Riemannian manifold with nonempty boundary. For $R > 0$, let $N(R) = \{p \in N : d(p, \partial N) < R\}$. We say that N is δ-parabolic if for every $\delta > 0$, $\widetilde{N} = N \setminus N(\delta)$ is parabolic.

The following theorem gives a sufficient condition for a surface to be δ-parabolic.

Theorem 11 [Meeks and Rosenberg 2008]. *Let N be a complete surface with nonempty boundary and curvature function $K : N \to [0, \infty]$. Suppose that for each $R > 0$, the restricted function $K|_{N(R)}$ is bounded. Then N is δ-parabolic.*

5. Proper minimal immersions

Proposition 12. *Let N be a 3-manifold with nonnegative Ricci curvature and sectional curvature bounded above by $\kappa > 0$. Suppose Σ is a complete, orientable minimal surface with boundary in N, with a Jacobi function u. If $u \geq \epsilon$ for some $\epsilon > 0$, then Σ is δ-parabolic.*

Proof. First note that a Riemannian surface W is δ-parabolic if and only if for all $\delta' > 0$, the surface $W \setminus W(\delta')$ is also δ-parabolic. Thus, without loss of generality, we may assume that Σ has the form $W \setminus W(\delta')$ for some $\delta' > 0$, where W is a stable minimal surface with a positive Jacobi function $u \geq \epsilon$, which exists by [Fischer-Colbrie and Schoen 1980]. By curvature estimates for stable, orientable minimal surfaces [Schoen 1983; Rosenberg et al. 2010], we may assume that Σ has bounded Gaussian curvature. Consider the new Riemannian manifold $\widetilde{\Sigma}$, which is Σ with the metric $\tilde{g} = u\langle \cdot, \cdot \rangle$ on Σ, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on Σ. Since $u \geq \epsilon$ the metric $\tilde{g}$ is complete. Moreover, $\Delta_{\tilde{g}} f = u^{-1} \Delta f$ for any function on Σ which has second derivative. Thus Σ is δ-parabolic if and only if $\widetilde{\Sigma}$ is δ-parabolic. Let E_1, E_2, η be an orthonormal frame in a neighborhood of a point of Σ, where η is normal to Σ. By the Gauss equation,

$$\operatorname{Ric}(\eta, \eta) + |A_\Sigma|^2 = \operatorname{Ric}(E_1, E_1) + \operatorname{Ric}(E_2, E_2) - 2K_\Sigma.$$

Then, as u is a Jacobi function,

$$\Delta_\Sigma u + \big(\operatorname{Ric}(E_1, E_1) + \operatorname{Ric}(E_2, E_2) - 2K_\Sigma\big)u = 0.$$

So,

$$K_{\widetilde{\Sigma}} = \frac{K_\Sigma - \frac{1}{2}\Delta_\Sigma \log u}{u} = \frac{1}{2}\,\frac{\operatorname{Ric}(E_1, E_1) + \operatorname{Ric}(E_2, E_2)}{u} + \frac{1}{2}\,\frac{|\nabla_\Sigma u|^2}{u^3},$$

which implies

$$0 \leq K_{\widetilde{\Sigma}} \leq 2\frac{\kappa}{\epsilon} + \frac{1}{2\epsilon}\frac{|\nabla_\Sigma u|^2}{u^2}.$$

Choose $\delta > 0$ and let $\widetilde{\Omega} = \widetilde{\Sigma} \setminus \widetilde{\Sigma}(\delta)$. Let Ω be the corresponding submanifold on Σ. By the Harnack inequality (see [Moser 1961]), $|\nabla_\Sigma u|/u$ is bounded, and so one has that $K_{\widetilde{\Sigma}}$ is nonnegative and bounded on Ω. It follows from Theorem 11 in Section 4 that $\widetilde{\Omega}$ is parabolic, and hence Ω is parabolic. Since δ was chosen arbitrarily, we conclude that Σ is δ-parabolic. □

Theorem A. *Let $\mathbb{M}$ be a complete simply connected orientable noncompact surface such that $0 \leq K_{\mathbb{M}} \leq \kappa$. Let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be an injective minimal immersion of a complete, connected Riemannian surface of bounded curvature. Then the map f is proper.*

Proof. Since the curvature of $f(\Sigma)$ is bounded, there exists an $\epsilon > 0$ such that for any point $p \in \mathbb{M} \times \mathbb{R}$, every component of $f^{-1}(B_\epsilon(p))$, when pushed forward by f, is a compact disc and a graph over a domain in the tangent plane of any point on it, with a uniform bound on the area. It follows that if p is a limit point of $f(\Sigma)$ coming from distinct components of $f^{-1}(B_\epsilon(p))$, then there is a minimal disc $D(p)$ passing through p that is a graph over its tangent plane at p, and $D(p)$ is a limit of components in $f^{-1}(B_\epsilon(p))$. Let $D'(p)$ be any other such limit disc. Since f is an embedding the unique possibility is that the discs are tangent at p; then the maximum principle implies that the two discs agree near p. This implies that the closure $\mathcal{L}(f(\Sigma))$ of $f(\Sigma)$ has the structure of a minimal lamination.

The immersion f is proper if and only if $\mathcal{L}(f(\Sigma))$ has no limit leaves. Suppose $\mathcal{L}(f(\Sigma))$ has a limit leaf L. Denote by $\widetilde{L}$ the universal cover of L. It was proved in [Meeks et al. 2008] that $\widetilde{L}$ is stable. So, by [Fischer-Colbrie and Schoen 1980] $\widetilde{L}$ is totally geodesic; hence L is totally geodesic. Suppose $\mathbb{M}$ is not flat (the case where $\mathbb{M}$ is flat was proved in [Meeks and Rosenberg 2008]). By Lemma 3 a totally geodesic surface in $\mathbb{M} \times \mathbb{R}$ is a slice $\mathbb{M} \times \{t\}$ or is of the form $\alpha \times \mathbb{R}$, where α is a geodesic of M.

Assume L is a slice. Since Σ is not proper, it is not equal to a slice. We can suppose $L = \mathbb{M} \times \{0\}$ and H^+ is a smallest half-space containing $f(\Sigma)$. Since Σ has bounded curvature, there is an $\epsilon > 0$ such that for every component C of $f(\Sigma)$ in the slab between L and $L_\epsilon = \{t = \epsilon\}$, the Jacobi function $u = \langle \nu, \partial_t \rangle$ satisfies $u \geq \lambda > 0$, where ν is the unit normal to C. Choose $0 < \delta < \epsilon$ such that $C(\delta) = \{p \in C : h \leq \delta\}$ is not empty, where h is the height function. By Proposition 12, $C(\delta)$ is parabolic. But $h|_{C(\delta)}$ is a bounded harmonic function with the same boundary values as the constant function δ. Hence $h|_{C(\delta)}$ is constant, which is a contradiction because $C(\delta)$ is not contained in a slice.

Now, suppose $L = \alpha \times \mathbb{R}$. Consider a one-sided closed ϵ-normal interval bundle $N_\epsilon(L)$ that submerses to $\mathbb{M} \times \mathbb{R}$, with the induced metric. Observe that $N_\epsilon(L)$ is diffeomorphic to $(\alpha \times \mathbb{R}) \times [0, \delta]$, with $L = (\alpha \times \mathbb{R}) \times \{0\}$ as a flat minimal submanifold, and $L(\delta) = (\alpha \times \mathbb{R}) \times \{\delta\}$ having mean curvature vector pointing out of $N_\epsilon(L)$. For ϵ sufficiently small, we may assume that each component of $f(\Sigma) \cap N_\epsilon(L)$ is a normal graph of bounded gradient over the zero section L. Let C be such a component which is a graph over a connected domain Ω of L and let $L_C(\delta)$ be the part of L_δ which is also a normal graph over Ω. Consider the surface $W_\delta = L(\delta) \setminus L_C(\delta)$. Under normal projection to L, $W_\delta \cup C$ is quasi-isometric to the flat plane L. It follows that C is a parabolic Riemann surface with boundary. But the function $d := \operatorname{dist}(\cdot, L)$ is superharmonic, and has constant value δ on the boundary of C. Then C is contained in $L(\delta)$, which contradicts the fact that L is a limit leaf of $\mathcal{L}(f(\Sigma))$. □

6. Parabolicity of minimal surfaces

Theorem B. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

1. $K_{\mathbb{M}} \geq 0$.
2. *There is a point $p \in M$ such that the geodesic curvatures of all geodesic circles with center p and radius $r \geq 1$ are uniformly bounded.*

Let Σ be a surface of finite topology and one end and let $f : \Sigma \to \mathbb{M} \times \mathbb{R}$ be a proper minimal immersion. Suppose that f is transverse to a slice $\mathbb{M} \times \{t_0\}$ except at a finite number of points and that $f^{-1}(\mathbb{M} \times \{t_0\})$ contains a finite number of components. Then Σ is parabolic.

Proof. We know from [Rosenberg 2002] that the conditions on $\mathbb{M}$ imply that the surfaces

$$\Sigma(+) := \{(p, t) \in \Sigma : t \geq t_0\},$$
$$\Sigma(-) := \{(p, t) \in \Sigma : t \leq t_0\}$$

are parabolic. Suppose that $\mathcal{E}$ is an annular end representative which does not have conformal representative which is a punctured disc. Then this end has a representative which is conformally diffeomorphic to $\{z \in \mathbb{C} : \epsilon \leq |z| < 1\}$ for some positive $\epsilon < 1$. In this conformal parametrization, the unit circle corresponds to points at infinity on $\mathcal{E}$. After choosing a larger ϵ, we may assume that $f|_{\mathcal{E}}$ intersects $\mathbb{M} \times \{t_0\}$ transversely in a finite positive number of arcs and that each noncompact arc of the intersection has one endpoint on the compact boundary circle $\{z \in \mathbb{C} : |z| = \epsilon\}$.

We claim that it suffices to prove that each of the finite number of noncompact arcs $\alpha_1, \dots, \alpha_n$ in $\mathbb{M} \times \{t_0\}$ has a well-defined limit on the unit circle $\mathbb{S}^1$ of points at infinity. In fact, assume the claim is true; then there is an open arc $\gamma \subset \mathbb{S}^1$

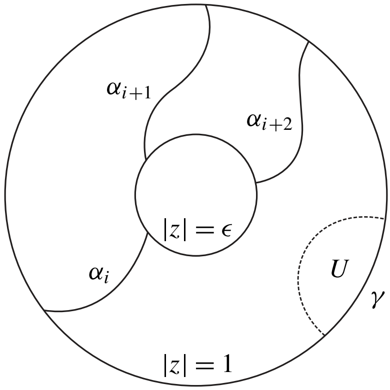

Figure 1. The disc U.

which does not contain limit points of $\alpha_1, \dots, \alpha_n$. Hence, there would be an open half-disc $U \subset \mathcal{E}$ centered at a point in γ, such that $U \cap (f^{-1}(\mathbb{M} \times \{t_0\})) = \varnothing$; see Figure 1. But U is a proper domain which is contained in one of the parabolic surfaces $\Sigma(+)$ or $\Sigma(-)$, so is parabolic. However, U does not have full harmonic measure, which is a contradiction.

Suppose α_k has two limit points q_1, q_2 in $\mathbb{S}^1$. We first prove that at least one of the two interval components I_1, I_2 of $\mathbb{S}^1 \setminus \{q_1, q_2\}$ consists of limit points of α_k. Suppose not and let $x_1 \in I_1$, $x_2 \in I_2$ be points which are not limit points. Since they are not limit points, there exists a $\delta > 0$ such that the radial arcs β_1 and β_2 in $\mathcal{E}$ of length δ and orthogonal to $\mathbb{S}^1$ at x_1, x_2 respectively, are disjoint from α_k. Since α_k is proper and disjoint from $\beta_1 \cup \beta_2$, the parametrized arc $\alpha_k(s)$ must eventually be in one of the two components of $\{z \in \mathcal{E} \setminus (\beta_1 \cup \beta_2) : |z| \geq 1 - \delta\}$; see Figure 2. Thus, α_k cannot have both q_1 and q_2 as limit points, a contradiction. Now, suppose

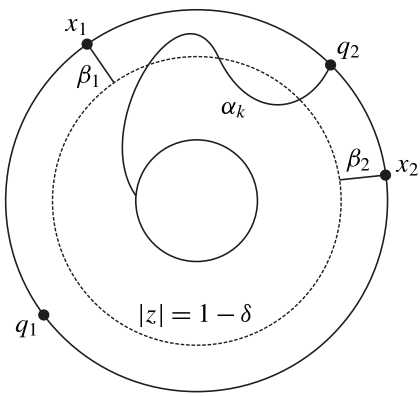

Figure 2. The arc trapped in one component.

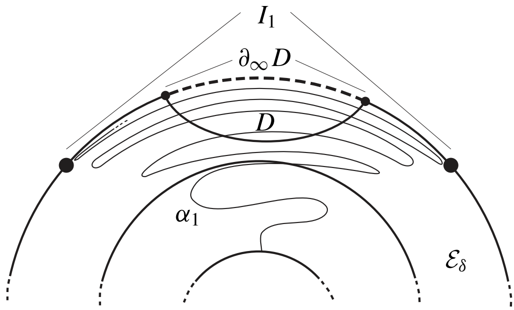

Figure 3. The arc α_1 accumulates in I_1.

one of the intervals, say I_2, contains one point z which is not a limit point of α_k; then by the previous argument the interval I_1 cannot contain any point which is not a limit point. So one of the intervals consists of limit points of α_k.

Since the height function is harmonic on $\mathcal{E}$ and the generator of the homology of $\mathcal{E}$ is a boundary in Σ, by Cauchy's theorem there is a conjugate harmonic function to h, which we denote by h^*. Consider the holomorphic function $g = h + ih^* : \mathcal{E} \to \mathbb{C}$. As the slice $\mathbb{M} \times \{t_0\}$ is transverse to $\mathcal{E}$, we have $\langle \nabla h, \eta \rangle^2 \neq 1$ for all points in an arc α_k and $h = 0$ in this arc, where η is a unit normal to Σ. Moreover, as g is holomorphic we have

$$|\nabla_\Sigma h^*(p)|^2 = |\nabla_\Sigma h(p)|^2 = 1 - \langle \nabla h, \eta \rangle^2(p) > 0 \quad \forall p \in \alpha_k,$$

so $h^*|_{\alpha_k}$ is strictly monotone. Thus g restricted to any of the finite number of components in $(f^{-1}(\mathbb{M} \times \{t_0\})) \cap \mathcal{E}$ monotonically parametrizes an interval on the imaginary axis $\mathbb{R}(i) \subset \mathbb{C}$. Choose a closed half-disc $\overline{D} \subset \bar{\mathcal{E}} = \mathcal{E} \cup \mathbb{S}^1$, centered at a point $p \in I_1$, where I_1, as discussed above, consists entirely of limit points of α_1, and suppose that $\overline{D}$ is chosen sufficiently small so that $\partial_\infty D := \partial D \cap \mathbb{S}^1 \subset I_1$. Since $g|_{\alpha_k}$ is injective we can take a compact interval $J \subset g\left(\bigcup_{k=1}^n \alpha_k\right) \subset \mathbb{R}(i)$ which is disjoint from the endpoints of $g|_{\alpha_k}$ for all k, and choose D sufficiently small such that $\overline{D} \cap (g^{-1}(J)) = \varnothing$.

Observe that g maps D into $\mathbb{C} \setminus J$, so by the Riemann mapping theorem, the function $g|_D$ is essentially bounded in the sense that it maps D into a domain that is conformally equivalent to an open subset of the unit disc. It follows from Fatou's theorem that the holomorphic function $g|_D$ has radial limits almost everywhere, i.e., D is conformally the unit disc, so radial limits are with respect to the radii of the unit disc.

Consider the radial arc β orthogonal to $\mathbb{S}^1$ at the point p (the center of I_1). The arc β divides I_1 into two intervals I_1^- and I_1^+ and separates D into two regions D^- and D^+. Choose $\delta > 0$ small. We can suppose D is inside the region $\mathcal{E}_\delta := \{z \in \mathcal{E} : |z| \geq 1 - \delta\}$. Since α_1 is proper, this arc will eventually be inside of $\mathcal{E}_\delta$. As I_1 is composed of accumulation points of α_1 and $\partial_\infty D$ is not equal to I_1, the arc

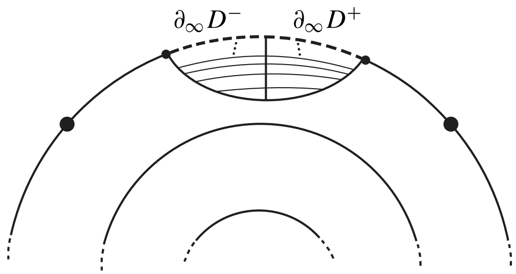

Figure 4. Infinitely many arcs in D^- and D^+.

α_1 leaves D and returns to it an infinite number of times, and it does this crossing the boundaries of D^- and D^+ infinitely many times, in each step getting closer to $\partial_\infty D^-$ and $\partial_\infty D^+$ respectively; see Figure 3. Then there exists an infinite number of arcs in $\alpha_1 \cap D^-$ (respectively $\alpha_1 \cap D^+$) converging to $\partial_\infty D^-$ (respectively $\partial_\infty D^+$); see Figure 4. Thus the points of $\partial_\infty D$ with radial limits for g have a constant value which corresponds to the limiting endpoint of the curve $g \circ \alpha_1$ in $\mathbb{R}(i) \cup \{\infty\}$. However, by Privalov's theorem, a nonconstant meromorphic function on the unit disc cannot have a constant radial limit on a set of $\partial_\infty D$ with positive measure, a contradiction. □

Theorem C. *Let $\mathbb{M}$ be a complete noncompact surface satisfying the following conditions*:

(1) $0 \le K_{\mathbb{M}} \le \kappa$.

(2) *$\mathbb{M}$ has a pole p.*

(3) *The geodesic curvatures of all geodesic circles with center p and radius $r \ge 1$ are uniformly bounded.*

Let Σ be a properly immersed minimal surface inside the region of $\mathbb{M} \times \mathbb{R}$ defined by $|h| \le c_2 \log r$ for some constant $c_2 > 0$ and $r \ge 1$. Then Σ is parabolic. Moreover, if Σ has compact boundary, then Σ has quadratic area growth and a finite number of ends.

Proof. Let p be the pole of $\mathbb{M}$. Since the map $\exp_p : T_p\mathbb{M} \to \mathbb{M}$ is a diffeomorphism, we have that $\phi : T_p\mathbb{M} \times \mathbb{R} \to \mathbb{M} \times \mathbb{R}$, defined by $\phi(v, s) = (\exp_p v, s)$, is a diffeomorphism and defines a coordinate system.

Let r be the distance to p on $\mathbb{M}$ extended to $\mathbb{M} \times \mathbb{R}$ in the natural way and h be the height function on $\mathbb{M} \times \mathbb{R}$. Let $C_R = \{(q, s) \in \mathbb{M} \times \mathbb{R} : r(q) = R\}$ be the vertical cylinder of radius R and let Σ_R be the part of Σ inside C_R. Let $B_R((p, 0))$ be the ball of $\mathbb{M} \times \mathbb{R}$ of center $(p, 0)$ and radius R. Since $\mathbb{M} \times \mathbb{R}$ has the product metric and p is a pole in M, the point $(p, 0)$ is a pole in $\mathbb{M} \times \mathbb{R}$. Thus $\Sigma \cap B_R((p, 0))$ is inside the interior of C_R. Then it suffices to prove that Σ_R has quadratic area growth as a function of r.

Using these coordinates we can define a horizontal vector field X that is orthogonal to ∇r and ∇h and has norm 1, so $(\nabla r, \nabla h, X)$ is an orthonormal basis at each point of $\mathbb{M} \times \mathbb{R}$. Let η be a unit normal to Σ, so

$$\langle \eta, \nabla r\rangle^2 + \langle \eta, \nabla h\rangle^2 + \langle \eta, X\rangle^2 = 1,$$
$$|\nabla_\Sigma r|^2 = 1 - \langle \eta, \nabla r\rangle^2,$$

and

$$|\nabla_\Sigma h|^2 = 1 - \langle \eta, \nabla h\rangle^2.$$

Hence,

$$|\nabla_\Sigma r|^2 + |\nabla_\Sigma h|^2 = 1 + \langle \eta, X\rangle^2 \geq 1.$$

Thus,

$$\int_{\Sigma_R} d\mu \leq \int_{\Sigma_R} (|\nabla_\Sigma r|^2 + |\nabla_\Sigma h|^2)\, d\mu.$$

Consider the function $f : \Sigma \to \mathbb{R}$, $f = -h \arctan(h) + \frac{1}{2}\log(h^2+1)$, where h is the height function on $\mathbb{M} \times \mathbb{R}$. Since h is harmonic on Σ,

$$\Delta_\Sigma f = -\arctan(h)\Delta_\Sigma h - \frac{|\nabla_\Sigma h|^2}{h^2+1} = -\frac{|\nabla_\Sigma h|^2}{h^2+1}.$$

Consider now the function $g = \log r + f$. After rescaling the metric of Σ and removing a compact subset of Σ we may assume that $|h| \leq \frac{1}{2}\log r$. By Lemma 5, g satisfies

$$\Delta_\Sigma g \leq c_1 \frac{|\nabla_\Sigma h|^2}{r} - \frac{|\nabla_\Sigma h|^2}{h^2+1} \leq 0.$$

Since $\log r$ is proper in $\left\{(q,t) \in \mathbb{M}\times\mathbb{R} : |h| \leq \frac{1}{2}\log r,\ r \geq 1\right\}$ and Σ is proper, $\log r$ is proper in Σ. Moreover $g \geq \frac{3\pi}{4}\log r$, so g is a nonnegative proper superharmonic function on Σ. This proves that Σ is parabolic.

Suppose $\partial\Sigma$ is compact. There exists $a > 0$ such that $g(\partial\Sigma) \subset [0, a]$. Let $t_2 > t_1 \geq a$. Since g is proper, $g^{-1}([t_1, t_2])$ is compact; then we can apply the divergence theorem and use the fact that g is superharmonic to obtain

$$0 \geq \int_{g^{-1}([t_1,t_2])} \Delta_\Sigma g\, d\mu = -\int_{g^{-1}(t_1)} |\nabla_\Sigma g|\, dL + \int_{g^{-1}(t_2)} |\nabla_\Sigma g|\, dL. \tag{5}$$

It follows that the function $t \mapsto \int_{g^{-1}(t)} |\nabla_\Sigma g|\, dL$ is monotonically decreasing and bounded, so

$$\lim_{t\to\infty} \int_{g^{-1}(t)} |\nabla_\Sigma g|\, dL < \infty. \tag{6}$$

Since $\Sigma = g^{-1}([0,\infty))$ it follows from (5) and (6) that $\Delta_\Sigma g \in \mathrm{L}^1(\Sigma)$. Furthermore, $\Delta_\Sigma g \geq \frac{1}{2}|\Delta_\Sigma f|$ for r large; thus $\Delta_\Sigma f \in \mathrm{L}^1(\Sigma)$. Hence,

$$\int_{\Sigma_R} \Delta_\Sigma f\, d\mu = \int_{\Sigma_R} \frac{|\nabla_\Sigma h|^2}{h^2+1}\, d\mu \leq \int_\Sigma \frac{|\nabla_\Sigma h|^2}{h^2+1}\, d\mu = c_3$$

for some positive constant c_3. Then, for $R \geq 1$,

$$\int_{\Sigma_R} |\nabla_\Sigma h|^2\, d\mu \leq \int_{\Sigma_R} \left(\frac{(\log R)^2+1}{h^2+1}\right)|\nabla_\Sigma h|^2\, d\mu \leq ((\log R)^2+1)c_3 \leq c_3 R^2.$$

Since $\Delta_\Sigma f \in \mathrm{L}^1(\Sigma)$ and $|\Delta_\Sigma f| \geq c_4 |\Delta_\Sigma \log r|$ ($c_4 > 0$ a constant), we have $\Delta_\Sigma(\log r) \in \mathrm{L}^1(\Sigma)$. Again by the divergence theorem,

$$\begin{aligned}\int_{\Sigma_R} \Delta_\Sigma \log r\, d\mu &= \int_{\partial\Sigma} \frac{1}{r}\langle \nabla_\Sigma r, \nu\rangle\, dL + \int_{C_R\cap\Sigma} \frac{|\nabla_\Sigma r|}{R}\, dL \\ &= c_5 + \frac{1}{R}\int_{C_R\cap\Sigma} |\nabla_\Sigma r|\, dL,\end{aligned}$$

where ν is the outward conormal to the boundary of Σ. Thus

$$\lim_{R\to\infty} \frac{1}{R}\int_{C_R\cap\Sigma} |\nabla_\Sigma r|\, dL < \infty,$$

which implies there is a constant $c_6 > 0$ such that

$$\int_{C_R\cap\Sigma} |\nabla_\Sigma r|\, dL \leq c_6 R.$$

By the coarea formula

$$\int_{\Sigma_R} |\nabla_\Sigma r|^2\, d\mu \leq \int_1^R \int_{C_\rho\cap\Sigma} |\nabla_\Sigma r|\, dL\, d\rho \leq c_6 \int_1^R \rho\, d\rho \leq \tfrac{1}{2} c_6 R^2.$$

Therefore Σ has quadratic area growth.

Now, suppose Σ has an infinite number of ends. Let E be an end of Σ. Choose $0 < \delta < \min\{\mathrm{inj}_{\mathbb{M}\times\mathbb{R}}, 1/\sqrt{\kappa}\}$ such that for each positive integer j, there is a distance ball $B_\delta(q_j)$ of $\mathbb{M}\times\mathbb{R}$ inside the region $\mathcal{R}_j$ between C_j and C_{j+1}, with $q_j \in E$. By the monotonicity formula for minimal surfaces (see Chapter 7 of [Colding and Minicozzi 2011]),

$$|E \cap B_\delta(q_j)| \geq \frac{c\delta^2}{e^{2\sqrt{\kappa}\delta}} =: c_7,$$

where $c > 0$ is a constant and $\kappa = \sup K_{\mathbb{M}\times\mathbb{R}}$. Write $E_n = E \cap C_n$. Since in each region $\mathcal{R}_j$, $j < n$, we have a portion of E of area at least c_7 it follows that

$$|E_n| > c_7 n.$$

Then in the cylinder C_{n^2} we have

$$c_7 n^2 \leq |E_{n^2}| \leq c_8 n^2.$$

Since this holds for each end, choosing n ends we obtain that the area of Σ inside C_{n^2} satisfies

$$c_9 n^3 \leq |\Sigma_{n^2}| \leq c_{10} n^2,$$

but for n sufficiently large this leads to a contradiction. Hence, Σ has a finite number of ends. $\square$

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VANDERSON LIMA
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DO ESTADO DO RIO DE JANEIRO (UERJ)
RUA SÃO FRANCISCO XAVIER, 524
PAVILHÃO REITOR JOÃO LYRA FILHO, 6° ANDAR - BLOCO B
20550-900 RIO DE JANEIRO-RJ
BRAZIL

vanderson@ime.uerj.br

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