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GEOMETRIC PROPERTIES OF LEVEL CURVES OF HARMONIC FUNCTIONS AND MINIMAL GRAPHS IN 2-DIMENSIONAL SPACE FORMS

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We study the geometric properties of level curves of harmonic functions and minimal graphs in 2-dimensional space forms using the maximum principle. More precisely, we find two auxiliary functions which consist of tangential derivatives of the curvature of level curves and the norms of the gradient of the solution functions. Then we prove that they satisfy certain elliptic partial differential equations.

1. Introduction

The geometric properties of the level surfaces of solutions of elliptic partial differential equations have been studied for a long time. For instance, a book by Ahlfors [1973] contains the well-known result that level curves of the Green function of a 2-dimensional convex domain are convex curves. Gergen [1931] proved the level surfaces of the Green function of a 3-dimensional star-shaped domain are also star-shaped. Shiffman [1956] studied the convexity of the level curves of immersed minimal surfaces in $\mathbb{R}^3$. He proved that if two convex curves in parallel planes in $\mathbb{R}^3$ bound a minimal surface S then the intersections of all other parallel planes with S are also convex curves. In particular, he obtained that if the boundaries are two circles then intermediate level curves are also circles. Gabriel [1957] proved that the level surfaces of the Green function of a 3-dimensional convex domain are strictly convex. Later, Lewis [1977] extended Gabriel's results to p-harmonic functions in high dimensions. For more related extensions and a survey on this subject, see [Bianchini et al. 2009; Caffarelli and Spruck 1982; Kawohl 1985].

There is also a lot of literature on the quantitative curvature estimates of level surfaces of solutions of elliptic partial differential equations. For 2-dimensional harmonic functions, Talenti [1983] got the following result. Let $\Omega \subset \mathbb{R}^2$ be a domain

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and u be a harmonic function with no critical points in Ω. Then the function $\kappa/|\nabla u|$ is harmonic in Ω. Here

$$\kappa = \frac{2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}}{|\nabla u|^3}$$

is the curvature of the level curves of u. Throughout the paper we use subscripts to represent the derivatives with respect to any orthonormal frames. Similar results can also be seen in [Ortel and Schneider 1983; Longinetti 1983]. Recently, Ma, Ou and Zhang [Ma et al. 2010] generalized the above results to n-dimensional harmonic functions ($2 \le n < \infty$) and obtained the sharp Gaussian curvature estimates of the level surfaces. See also [Chang et al. 2010; Ma and Zhang 2013; 2014; Wang and Zhang 2012; Zhang and Zhang 2013].

More recently, Kong and Xu [2015] found that if u is a harmonic function of two variables with no critical points, then the function $(\kappa_1u_2 - \kappa_2u_1)/|\nabla u|^3$ is also harmonic. Using this fact, they proved that all the level curves of solutions of the Laplace equation with homogeneous Dirichlet boundary conditions on an annulus are circles. This result can be viewed as a generalization of Shiffman's result on minimal surfaces. In this paper, we extend Kong and Xu's and Shiffman's results to harmonic functions and minimal graphs in 2-dimensional space forms. More precisely, we obtain the following results.

Theorem 1.1. *Suppose that $M^2(c)$ is a 2-dimensional Riemannian manifold with constant sectional curvature c. Let $\Omega \subset M^2(c)$ be a domain and u be a harmonic function with no critical points in Ω. Let κ be the curvature of the level curves of u. Then the function $\varphi = (\kappa_1u_2 - \kappa_2u_1)/|\nabla u|^3$ is also harmonic in Ω.*

For minimal graphs, we have the following similar result.

Theorem 1.2. *Suppose that $M^2(c)$ is a 2-dimensional Riemannian manifold with constant sectional curvature c. Let $\Omega \subset M^2(c)$ be a domain and u satisfy the minimal surface equation*

$$\sum_{ij} a_{ij}u_{ij} = 0 \quad \text{in } \Omega,$$

where $a_{ij} = (1 + |\nabla u|^2)\delta_{ij} - u_iu_j$. Furthermore, assume that there are no critical points of u in Ω. Let κ be the curvature of the level curves of u. Set

$$\psi = \frac{(1 + |\nabla u|^2)^{3/2}}{|\nabla u|^3} \cdot (\kappa_1u_2 - \kappa_2u_1).$$

Then the function ψ satisfies the differential equation

$$\sum_{ij} a_{ij}\psi_{ij} + \sum_i b_i\psi_i = 0 \quad \text{in } \Omega.$$

Here the b_i are bounded functions.

Based on the above theorems, we have the following characterization of geodesic circles.

Remark 1.3. Since $(\kappa_1 u_2 - \kappa_2 u_1)/|\nabla u|$ is the tangential derivative of the curvature of the level curves, the auxiliary functions φ and ψ are independent of the choice of orthonormal frames. Similar to the case of Euclidean space, by the maximum principle, we know that all the level curves of solutions of the Laplace equation or the minimal surface equation with homogeneous Dirichlet boundary conditions on an annulus are geodesic circles.

Now we give the derivative commutation formulas in Riemannian geometry. Let u be a smooth function and R_{ijkl} be coefficients of the Riemannian curvature tensor under orthonormal frames. Here for 2-dimensional space forms $M^2(c)$, we adopt $R_{1212} = c$. Then we have

$$u_{ij} - u_{ji} = 0, \tag{1-1}$$

$$u_{ijk} - u_{ikj} = \sum_m u_m R_{mijk}, \tag{1-2}$$

$$u_{ijkl} - u_{ijlk} = \sum_m u_{mj} R_{mikl} + \sum_m u_{im} R_{mjkl}. \tag{1-3}$$

For more details, one can consult any book on Riemannian geometry, such as [Chern et al. 1999].

In this paper, all the summation indices i, j, k, l and m run from 1 to 2. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2.

2. Level curves of harmonic functions

In this section, we focus on the calculation of harmonic functions in 2-dimensional space forms.

Let $\Omega \subset M^2(c)$ be a domain and u be a harmonic function defined in Ω with no critical points. Set

$$\varphi = f(|\nabla u|)(\kappa_1 u_2 - \kappa_2 u_1),$$

where κ is the curvature of the level curves and f is a smooth function of one variable defined on the interval $(0, +\infty)$ which will be determined later. For a suitable choice of f, we will prove that φ is also a harmonic function in Ω, i.e., the function φ satisfies

$$\Delta \varphi = 0 \quad \text{in } \Omega. \tag{2-1}$$

In order to prove (2-1) at an arbitrary point $x_0 \in \Omega$, we may choose the orthonormal frames such that

$$u_1(x_0) = 0, \quad u_2(x_0) = |\nabla u|(x_0) > 0.$$

From now on, all the calculations will be done at the fixed point x_0 unless otherwise specified.

By taking the first derivative of φ, we have

$$\varphi_i = f'(|\nabla u|)_i \cdot (\kappa_1 u_2 - \kappa_2 u_1) + f \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_{2i} u_1 - \kappa_2 u_{1i}). \tag{2-2}$$

Differentiating (2-2) once more, we have

$$\begin{aligned}\varphi_{ii} = f''(|\nabla u|)_i^2 \cdot \kappa_1 u_2 + f'(|\nabla u|)_{ii} \cdot \kappa_1 u_2 + 2f'(|\nabla u|)_i \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_2 u_{1i})\\ + f \cdot (\kappa_{1ii} u_2 + 2\kappa_{1i} u_{2i} + \kappa_1 u_{2ii} - 2\kappa_{2i} u_{1i} - \kappa_2 u_{1ii});\end{aligned}$$

hence

$$\begin{aligned}\Delta\varphi = u_2 f \sum_i k_{1ii} &+ \bigl[2u_2 f'(|\nabla u|)_1 + 2f u_{12}\bigr] \cdot \kappa_{11}\\ &+ \bigl[2u_2 f'(|\nabla u|)_2 + 2f u_{22} - 2f u_{11}\bigr] \cdot \kappa_{12} + [-2f u_{12}] \cdot \kappa_{22}\\ &+ \Bigl[u_2 f'' \sum_i (|\nabla u|)_i^2 + u_2 f' \sum_i (|\nabla u|)_{ii}\\ &\qquad + 2f' \sum_i (|\nabla u|)_i u_{2i} + f \sum_i u_{2ii}\Bigr] \cdot \kappa_1\\ &+ \Bigl[-2f' \sum_i (|\nabla u|)_i u_{1i} - f \sum_i u_{1ii}\Bigr] \cdot \kappa_2.\end{aligned} \tag{2-3}$$

Direct calculation yields

$$\begin{aligned}(|\nabla u|)_i &= \frac{1}{|\nabla u|} \sum_j u_j u_{ji},\\ (|\nabla u|)_{ii} &= \frac{1}{|\nabla u|} \sum_j u_{ji}^2 + \frac{1}{|\nabla u|} \sum_j u_j u_{jii} - \frac{1}{|\nabla u|^3} \sum_{jk} u_j u_{ji} u_k u_{ki}.\end{aligned} \tag{2-4}$$

Then at the point x_0,

$$(|\nabla u|)_i = u_{2i}, \quad (|\nabla u|)_{ii} = \frac{u_{1i}^2}{u_2} + u_{2ii}. \tag{2-5}$$

By the commutation formulas (1-1)–(1-2), we have

$$\sum_i u_{1ii} = \sum_i u_{i1i} = \sum_i \Bigl[u_{ii1} + \sum_m u_m R_{mi1i}\Bigr] = 0, \tag{2-6}$$

$$\sum_i u_{2ii} = \sum_i u_{i2i} = \sum_i \Bigl[u_{ii2} + \sum_m u_m R_{mi2i}\Bigr] = u_2 \cdot c. \tag{2-7}$$

Putting (2-5)–(2-7) into (2-3), we obtain

$$\Delta\varphi = u_2 f \sum_i \kappa_{1ii} + (2u_2 f' + 2f)u_{12}\cdot\kappa_{11} + (-2u_2 f' - 4f)u_{11}\cdot\kappa_{12} + (-2f)u_{12}\cdot\kappa_{22} + (u_2 f'' + 3f')(u_{11}^2 + u_{12}^2)\cdot\kappa_1 + (u_2^2 f' + u_2 f)\cdot\kappa_1\cdot c. \tag{2-8}$$

To compute the first term in (2-8), we should get the formula for $\Delta\kappa$ at a general point in advance. Recalling the curvature formula for the level curves, we have

$$|\nabla u|^3 \cdot \kappa = 2u_1u_2u_{12} - u_1^2 u_{22} - u_2^2 u_{11}. \tag{2-9}$$

By applying the Laplace operator on both sides of (2-9) and then using (2-4), we obtain

$$\begin{aligned}\Delta\kappa &= \frac{1}{|\nabla u|^3}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} - \frac{6}{|\nabla u|}\sum_i (|\nabla u|)_i\cdot\kappa_i \\ &\quad - \frac{6}{|\nabla u|^2}\sum_i (|\nabla u|)_i^2\cdot\kappa - \frac{3}{|\nabla u|}\sum_i (|\nabla u|)_{ii}\cdot\kappa \\ &= \frac{1}{|\nabla u|^3}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} - \frac{6}{|\nabla u|^2}(u_1u_{11} + u_2u_{12})\cdot\kappa_1 \\ &\quad - \frac{6}{|\nabla u|^2}(u_1u_{12} - u_2u_{11})\cdot\kappa_2 - \frac{1}{|\nabla u|^2}\Big[9(u_{11}^2 + u_{12}^2) + 3\sum_{ij} u_j u_{jii}\Big]\cdot\kappa.\end{aligned} \tag{2-10}$$

Now the commutation formulas (1-1)–(1-2) yield

$$\sum_{ij} u_j u_{jii} = \sum_{ij} u_j\Big[u_{iij} + \sum_m u_m R_{miji}\Big] = \sum_{ijm} u_j u_m R_{miji} = |\nabla u|^2\cdot c. \tag{2-11}$$

By inserting (2-11) into (2-10), we have

$$\begin{aligned}\Delta\kappa &= \frac{1}{|\nabla u|^3}\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} - \frac{6}{|\nabla u|^2}(u_1u_{11} + u_2u_{12})\cdot\kappa_1 \\ &\quad - \frac{6}{|\nabla u|^2}(u_1u_{12} - u_2u_{11})\cdot\kappa_2 - \frac{9}{|\nabla u|^2}(u_{11}^2 + u_{12}^2)\cdot\kappa - 3\kappa\cdot c.\end{aligned} \tag{2-12}$$

Straightforward computation gives

$$\begin{aligned}&\sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} \\ &= \sum_i \big[2u_{1ii}u_2u_{12} + 2u_1u_{2ii}u_{12} + 2u_1u_2u_{12ii} - 2u_1u_{1ii}u_{22} \\ &\qquad - u_1^2u_{22ii} - 2u_2u_{2ii}u_{11} - u_2^2u_{11ii} + 4u_{1i}u_{2i}u_{12} + 4u_{1i}u_2u_{12i} \\ &\qquad + 4u_1u_{2i}u_{12i} - 2u_{1i}^2u_{22} - 4u_1u_{1i}u_{22i} - 2u_{2i}^2u_{11} - 4u_2u_{2i}u_{11i}\big]\end{aligned} \tag{2-13}$$

$$
\begin{aligned}
&= 2(u_2u_{12}+u_1u_{11})\sum_i u_{1ii} + 2(u_1u_{12}-u_2u_{11})\sum_i u_{2ii} \\
&\quad + 2u_1u_2\sum_i u_{12ii} - u_1^2\sum_i u_{22ii} - u_2^2\sum_i u_{11ii} \\
&\quad\quad + 4\sum_i (u_2u_{1i}+u_1u_{2i})u_{12i} + 4\sum_i (u_1u_{1i}-u_2u_{2i})u_{11i} \\
&\triangleq \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{I}_1 &= 2(u_2u_{12}+u_1u_{11})\sum_i u_{1ii} + 2(u_1u_{12}-u_2u_{11})\sum_i u_{2ii}, \\
\mathrm{I}_2 &= 2u_1u_2\sum_i u_{12ii} - u_1^2\sum_i u_{22ii} - u_2^2\sum_i u_{11ii}, \\
\mathrm{I}_3 &= 4\sum_i (u_2u_{1i}+u_1u_{2i})u_{12i} + 4\sum_i (u_1u_{1i}-u_2u_{2i})u_{11i}.
\end{aligned}
$$

We deal with the terms I_1, I_2 and I_3 consecutively. By (2-6)–(2-7), we have

$$
\begin{aligned}
\mathrm{I}_1 &= 2(u_2u_{12}+u_1u_{11})\sum_{im} u_m R_{mi1i} + 2(u_1u_{12}-u_2u_{11})\sum_{im} u_m R_{mi2i} \\
&= 2(u_2u_{12}+u_1u_{11})\cdot u_1\cdot c + 2(u_1u_{12}-u_2u_{11})\cdot u_2\cdot c \\
&= 2|\nabla u|^3\cdot\kappa\cdot c.
\end{aligned}
\tag{2-14}
$$

By the commutation formulas (1-1)–(1-3), we have

$$
u_{jkii} = u_{iijk} + \sum_m u_{mk}R_{miji} + \sum_m u_{mj}R_{miki} + 2\sum_m u_{mi}R_{mjki}.
$$

It follows that

$$
\begin{aligned}
\mathrm{I}_2 &= 2u_1u_2\cdot\left[\sum_{im} u_{m2}R_{mi1i} + \sum_{im} u_{m1}R_{mi2i} + 2\sum_{im} u_{mi}R_{m12i}\right] \\
&\quad - u_1^2\cdot\left[2\sum_{im} u_{m2}R_{mi2i} + 2\sum_{im} u_{mi}R_{m22i}\right] \\
&\quad - u_2^2\cdot\left[2\sum_{im} u_{m1}R_{mi1i} + 2\sum_{im} u_{mi}R_{m11i}\right] \\
&= 2u_1u_2\cdot 4u_{12}\cdot c - u_1^2\cdot(-4u_{11})\cdot c - u_2^2\cdot 4u_{11}\cdot c \\
&= (8u_1u_2u_{12} + 4u_1^2u_{11} - 4u_2^2u_{11})\cdot c.
\end{aligned}
\tag{2-15}
$$

By the commutation formulas (1-1)–(1-2),

$$u_{121} = u_{112} + \sum_m u_m R_{m121} = u_{112} + u_2 \cdot c, \tag{2-16}$$

$$u_{122} = u_{221} + \sum_m u_m R_{m212} = -u_{111} + u_1 \cdot c. \tag{2-17}$$

Then we have

$$\begin{aligned} \mathrm{I}_3 = 8(u_1 u_{11} - u_2 u_{12}) u_{111} + 8(u_1 u_{12} + u_2 u_{11}) u_{112} \\ + (8u_1 u_2 u_{12} + 4u_2^2 u_{11} - 4u_1^2 u_{11}) \cdot c. \end{aligned} \tag{2-18}$$

Combining (2-13)–(2-15) and (2-18), we get

$$\begin{aligned} \sum_i (2u_1 u_2 u_{12} - u_1^2 u_{22} - u_2^2 u_{11})_{ii} \\ = 8(u_1 u_{11} - u_2 u_{12}) u_{111} + 8(u_1 u_{12} + u_2 u_{11}) u_{112} \\ + 2|\nabla u|^3 \cdot \kappa \cdot c + 16 u_1 u_2 u_{12} \cdot c. \end{aligned} \tag{2-19}$$

Now let us explore the relations between u_{111}, u_{112} and κ_1, κ_2. Taking the first derivative on both sides of (2-9) and using (2-4), (2-16) and (2-17), we obtain

$$\begin{aligned} &(u_1^2 - u_2^2) \cdot u_{111} + 2u_1 u_2 \cdot u_{112} \\ &\qquad = |\nabla u|^3 \cdot \kappa_1 + 3|\nabla u|(u_1 u_{11} + u_2 u_{12}) \cdot \kappa - 2u_1(u_{11}^2 + u_{12}^2) - 2u_1 u_2^2 \cdot c, \\ &-2u_1 u_2 \cdot u_{111} + (u_1^2 - u_2^2) \cdot u_{112} \\ &\qquad = |\nabla u|^3 \cdot \kappa_2 + 3|\nabla u|(u_1 u_{12} - u_2 u_{11}) \cdot \kappa - 2u_2(u_{11}^2 + u_{12}^2) - 2u_1^2 u_2 \cdot c. \end{aligned}$$

Thus we have

$$\begin{aligned} u_{111} = \frac{u_1^2 - u_2^2}{|\nabla u|} \cdot \kappa_1 - \frac{2u_1 u_2}{|\nabla u|} \cdot \kappa_2 + \frac{3}{|\nabla u|}(u_1 u_{11} - u_2 u_{12}) \cdot \kappa \\ - \frac{2u_1(u_1^2 - 3u_2^2)}{|\nabla u|^4}(u_{11}^2 + u_{12}^2) + \frac{2u_1 u_2^2}{|\nabla u|^2} \cdot c, \end{aligned} \tag{2-20}$$

and

$$\begin{aligned} u_{112} = \frac{2u_1 u_2}{|\nabla u|} \cdot \kappa_1 + \frac{u_1^2 - u_2^2}{|\nabla u|} \cdot \kappa_2 + \frac{3}{|\nabla u|}(u_2 u_{11} + u_1 u_{12}) \cdot \kappa \\ - \frac{2u_2(3u_1^2 - u_2^2)}{|\nabla u|^4}(u_{11}^2 + u_{12}^2) - \frac{2u_1^2 u_2}{|\nabla u|^2} \cdot c. \end{aligned} \tag{2-21}$$

Hence the formula (2-19) reduces to

$$\begin{aligned}\text{(2-22)}\quad \sum_i (2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ii} &= 8|\nabla u|(u_1u_{11} + u_2u_{12}) \cdot \kappa_1 + 8|\nabla u|(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ &\quad + 8|\nabla u|(u_{11}^2 + u_{12}^2) \cdot \kappa + 2|\nabla u|^3 \cdot \kappa \cdot c.\end{aligned}$$

By (2-12) and (2-22), we have

$$\begin{aligned}\text{(2-23)}\quad \Delta\kappa = \frac{2}{|\nabla u|^2}(u_1u_{11} + u_2u_{12}) \cdot \kappa_1 + \frac{2}{|\nabla u|^2}(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ - \frac{1}{|\nabla u|^2}(u_{11}^2 + u_{12}^2) \cdot \kappa - \kappa \cdot c.\end{aligned}$$

Then at the point x_0, we take the first derivative of (2-23). With (2-5) and (2-16) in hand, we obtain

$$\begin{aligned}\text{(2-24)}\quad (\Delta\kappa)_1 = \frac{2}{u_2}u_{12} \cdot \kappa_{11} - \frac{2}{u_2}u_{11} \cdot \kappa_{12} + \left[\frac{2}{u_2}u_{112} + \frac{1}{u_2^2}u_{11}^2 - \frac{3}{u_2^2}u_{12}^2\right] \cdot \kappa_1 \\ + \left[-\frac{2}{u_2}u_{111} + \frac{4}{u_2^2}u_{11}u_{12}\right] \cdot \kappa_2 \\ + \left[-\frac{2}{u_2^2}(u_{11}u_{111} + u_{12}u_{112}) + \frac{2}{u_2^3}(u_{11}^2 + u_{12}^2)u_{12}\right] \cdot \kappa \\ + \left[\kappa_1 - \frac{2}{u_2}u_{12} \cdot \kappa\right] \cdot c.\end{aligned}$$

Now, the equations (2-20) and (2-21) are simplified as

$$\text{(2-25)}\qquad u_{111} = -u_2\kappa_1 + \frac{3}{u_2}u_{11}u_{12},$$

$$\text{(2-26)}\qquad u_{112} = -u_2\kappa_2 - \frac{1}{u_2}u_{11}^2 + \frac{2}{u_2}u_{12}^2.$$

Putting (2-25)–(2-26) into (2-24), one achieves

$$\begin{aligned}(\Delta\kappa)_1 = \frac{2}{u_2}u_{12} \cdot \kappa_{11} - \frac{2}{u_2}u_{11} \cdot \kappa_{12} + \left[-\frac{3}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\right] \cdot \kappa_1 + \left[-\frac{4}{u_2^2}u_{11}u_{12}\right] \cdot \kappa_2 \\ + \left[-\frac{2}{u_2^3}(u_{11}^2 + u_{12}^2)u_{12}\right] \cdot \kappa + \left[\kappa_1 - \frac{2}{u_2}u_{12} \cdot \kappa\right] \cdot c.\end{aligned}$$

Therefore, by the commutation formulas(1-1)–(1-2), we get

$$
\begin{aligned}
\text{(2-27)}\quad & \sum_i \kappa_{1ii} \\
&= \sum_i \Big[\kappa_{ii1} + \sum_m \kappa_m R_{mi1i}\Big] \\
&= (\Delta\kappa)_1 + \kappa_1 \cdot c \\
&= \frac{2}{u_2} u_{12}\cdot\kappa_{11} - \frac{2}{u_2} u_{11}\cdot\kappa_{12} + \Big[-\frac{3}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\Big]\cdot\kappa_1 \\
&\quad + \Big[-\frac{4}{u_2^2}u_{11}u_{12}\Big]\cdot\kappa_2 + \Big[-\frac{2}{u_2^3}(u_{11}^2+u_{12}^2)u_{12}\Big]\cdot\kappa + \Big[2\kappa_1 - \frac{2}{u_2}u_{12}\cdot\kappa\Big]\cdot c.
\end{aligned}
$$

Thanks to (2-27), the formula (2-8) reduces to

$$
\begin{aligned}
\Delta\varphi = & \big[(2u_2f' + 4f)u_{12}\big]\cdot\kappa_{11} + \big[(-2u_2f' - 6f)u_{11}\big]\cdot\kappa_{12} + [-2fu_{12}]\cdot\kappa_{22} \\
& + \Big[\Big(u_2f'' + 3f' - \frac{3f}{u_2}\Big)u_{11}^2 + \Big(u_2f'' + 3f' + \frac{f}{u_2}\Big)u_{12}^2\Big]\cdot\kappa_1 \\
& + \Big[-\frac{4f}{u_2}u_{11}u_{12}\Big]\cdot\kappa_2 + \Big[-\frac{2f}{u_2^2}(u_{11}^2+u_{12}^2)u_{12}\Big]\cdot\kappa \\
& + \big[(u_2^2f' + 3u_2f)\cdot\kappa_1 - 2fu_{12}\cdot\kappa\big]\cdot c.
\end{aligned}
$$

At the point x_0, by (2-23), we have

$$
\kappa_{22} = -\kappa_{11} + \frac{2}{u_2}u_{12}\cdot\kappa_1 - \frac{2}{u_2}u_{11}\cdot\kappa_2 - \frac{1}{u_2^2}(u_{11}^2+u_{12}^2)\cdot\kappa - \kappa\cdot c.
$$

Thus

$$
\begin{aligned}
\text{(2-28)}\quad \Delta\varphi = & (2u_2f' + 6f)u_{12}\cdot\kappa_{11} + (-2u_2f' - 6f)u_{11}\cdot\kappa_{12} \\
& + \Big(u_2f'' + 3f' - \frac{3f}{u_2}\Big)(u_{11}^2+u_{12}^2)\cdot\kappa_1 + (u_2^2f' + 3u_2f)\cdot\kappa_1\cdot c.
\end{aligned}
$$

By (2-2), we have

$$
\kappa_{11} = \frac{1}{u_2 f}\big[\varphi_1 - (u_2f' + f)u_{12}\cdot\kappa_1 + fu_{11}\cdot\kappa_2\big], \tag{2-29}
$$

$$
\kappa_{12} = \frac{1}{u_2 f}\big[\varphi_2 + (u_2f' + f)u_{11}\cdot\kappa_1 + fu_{12}\cdot\kappa_2\big]. \tag{2-30}
$$

Putting (2-29)–(2-30) into (2-28), we finally get

$$\begin{aligned}\Delta\varphi = &\left(\frac{2f'}{f}+\frac{6}{u_2}\right)\cdot(u_{12}\varphi_1-u_{11}\varphi_2)\\ &+\left(u_2f''-\frac{2u_2f'^2}{f}-5f'-\frac{9f}{u_2}\right)(u_{11}^2+u_{12}^2)\cdot\kappa_1+(u_2^2f'+3u_2f)\cdot\kappa_1\cdot c.\end{aligned}\tag{2-31}$$

If we let

$$f(t)=t^{-3},$$

then all of the terms on the right-hand side of (2-31) vanish. This completes the proof of Theorem 1.1. □

3. Level curves of minimal graphs

Along the same lines as in Section 2, in this section we deal with the minimal graphs in 2-dimensional space forms.

Let $\Omega\subset M^2(c)$ be a domain and u be a solution with no critical points of the minimal surface equation

$$\sum_{ij}a_{ij}u_{ij}=0\quad\text{in }\Omega,\tag{3-1}$$

where

$$a_{ij}=(1+|\nabla u|^2)\delta_{ij}-u_iu_j.$$

Set

$$\psi=g(|\nabla u|)(\kappa_1u_2-\kappa_2u_1),$$

where κ is the curvature of the level curves and g is a smooth function of one variable defined on the interval $(0,+\infty)$ to be determined later. For a suitable choice of g, we will prove that the function ψ satisfies

$$\sum_{ij}a_{ij}\psi_{ij}+\sum_i b_i\psi_i=0\quad\text{in }\Omega.\tag{3-2}$$

Here the b_i are bounded functions.

In order to prove (3-2) at an arbitrary point $x_0\in\Omega$, we may choose the orthonormal frames such that

$$u_1(x_0)=0,$$

$$u_2(x_0)=|\nabla u|(x_0)>0.$$

From now on, all the calculations will be done at the fixed point x_0 unless otherwise specified.

By taking the derivative of ψ, we have

$$\psi_i = g'(|\nabla u|)_i \cdot (\kappa_1 u_2 - \kappa_2 u_1) + g \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_{2i} u_1 - \kappa_2 u_{1i}). \tag{3-3}$$

Differentiating (3-3) once more, we have

$$\begin{aligned}\psi_{ij} = {} & g''(|\nabla u|)_j (|\nabla u|)_i \cdot \kappa_1 u_2 + g'(|\nabla u|)_{ij} \cdot \kappa_1 u_2 + g'(|\nabla u|)_i \cdot (\kappa_{1j} u_2 + \kappa_1 u_{2j} - \kappa_2 u_{1j}) \\ & + g'(|\nabla u|)_j \cdot (\kappa_{1i} u_2 + \kappa_1 u_{2i} - \kappa_2 u_{1i}) \\ & \quad + g \cdot (\kappa_{1ij} u_2 + \kappa_{1i} u_{2j} + \kappa_{1j} u_{2i} + \kappa_1 u_{2ij} - \kappa_{2i} u_{1j} - \kappa_{2j} u_{1i} - \kappa_2 u_{1ij});\end{aligned}$$

hence

$$\begin{aligned}\sum_{ij} a_{ij} \psi_{ij} = u_2 g \sum_{ij} a_{ij} k_{1ij} & + \Big[2u_2 g' \sum_j a_{1j} (|\nabla u|)_j + 2g \sum_j a_{1j} u_{2j} \Big] \cdot \kappa_{11} \\ & + \Big[2u_2 g' \sum_j a_{2j} (|\nabla u|)_j + 2g \sum_{2j} a_{2j} u_{2j} - 2g \sum_j a_{1j} u_{1j} \Big] \cdot \kappa_{12} \\ & + \Big[-2g \sum_j a_{2j} u_{1j} \Big] \cdot \kappa_{22} \\ & + \Big[u_2 g'' \sum_{ij} a_{ij} (|\nabla u|)_i (|\nabla u|)_j + u_2 g' \sum_{ij} a_{ij} (|\nabla u|)_{ij} \\ & \qquad + 2g' \sum_{ij} a_{ij} (|\nabla u|)_j u_{2i} + g \sum_{ij} a_{ij} u_{2ij} \Big] \cdot \kappa_1 \\ & + \Big[-2g' \sum_{ij} a_{ij} (|\nabla u|)_j u_{1i} - g \sum_{ij} a_{ij} u_{1ij} \Big] \cdot \kappa_2.\end{aligned} \tag{3-4}$$

Direct calculation yields

$$\begin{aligned}(|\nabla u|)_i &= \frac{1}{|\nabla u|} \sum_k u_k u_{ki}, \\ (|\nabla u|)_{ij} &= \frac{1}{|\nabla u|} \sum_k u_{kj} u_{ki} + \frac{1}{|\nabla u|} \sum_k u_k u_{kij} - \frac{1}{|\nabla u|^3} \sum_{kl} u_l u_{lj} u_k u_{ki}.\end{aligned} \tag{3-5}$$

Then at the point x_0,

$$\begin{aligned}(|\nabla u|)_i &= u_{2i}, \\ (|\nabla u|)_{ij} &= \frac{u_{1j} u_{1i}}{u_2} + u_{2ij}.\end{aligned} \tag{3-6}$$

By the commutation formulas (1-1)–(1-2), we obtain

$$\begin{aligned}
\sum_{ij} a_{ij}u_{1ij} &= \sum_{ij} a_{ij}u_{i1j} \\
&= \sum_{ij} a_{ij}\Big[u_{ij1} + \sum_m u_m R_{mi1j}\Big] \\
&= -\sum_{ij} a_{ij,1}u_{ij} + \sum_{ijm} a_{ij}u_m R_{mi1j} \\
&= -2u_1(u_{11}u_{22} - u_{12}^2) + u_1(1+|\nabla u|^2)\cdot c
\end{aligned} \tag{3-7}$$

and

$$\begin{aligned}
\sum_{ij} a_{ij}u_{2ij} &= \sum_{ij} a_{ij}u_{i2j} \\
&= \sum_{ij} a_{ij}\Big[u_{ij2} + \sum_m u_m R_{mi2j}\Big] \\
&= -\sum_{ij} a_{ij,2}u_{ij} + \sum_{ijm} a_{ij}u_m R_{mi2j} \\
&= -2u_2(u_{11}u_{22} - u_{12}^2) + u_2(1+|\nabla u|^2)\cdot c.
\end{aligned} \tag{3-8}$$

Hence at the point x_0, we have

$$\sum_{ij} a_{ij}u_{1ij} = 0, \tag{3-9}$$

$$\sum_{ij} a_{ij}u_{2ij} = -2u_2(u_{11}u_{22} - u_{12}^2) + u_2(1+u_2^2)\cdot c. \tag{3-10}$$

On the other hand,

$$a_{11} = 1+u_2^2, \quad a_{12} = 0, \quad a_{21} = 0, \quad a_{22} = 1, \tag{3-11}$$

$$u_{22} = -(1+u_2^2)u_{11}. \tag{3-12}$$

Inserting (3-6), (3-9)–(3-12) into (3-4), we obtain

$$\begin{aligned}
\sum_{ij} a_{ij}\psi_{ij}
&= u_2 g\sum_{ij} a_{ij}k_{1ij} + \big[2u_2(1+u_2^2)g' + 2(1+u_2^2)g\big]u_{12}\cdot\kappa_{11} \\
&\quad + \big[-2u_2(1+u_2^2)g' - 4(1+u_2^2)g\big]u_{11}\cdot\kappa_{12} + [-2g]u_{12}\cdot\kappa_{22} \\
&\quad + \big[u_2(1+u_2^2)g'' + (3+4u_2^2)g' + 2u_2 g\big]\cdot\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big]\cdot\kappa_1 \\
&\quad + \big[u_2^2(1+u_2^2)g' + u_2(1+u_2^2)g\big]\cdot\kappa_1\cdot c.
\end{aligned} \tag{3-13}$$

To compute the first term in (3-13), we should get the formula for $\sum_{ij} a_{ij}\kappa_{ij}$ at a general point in advance. Recalling the curvature formula for the level curves, we have

$$|\nabla u|^3 \cdot \kappa = 2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}. \tag{3-14}$$

On both sides of (3-14), we take the second derivative with respect to ij, multiply by a_{ij}, and then sum with respect to ij. We obtain

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{ij} = {} & \frac{1}{|\nabla u|^3}\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\ & - \frac{6}{|\nabla u|}\sum_{ij} a_{ij}(|\nabla u|)_j \cdot \kappa_i - \frac{6}{|\nabla u|^2}\sum_{ij} a_{ij}(|\nabla u|)_i(|\nabla u|)_j \cdot \kappa \\ & - \frac{3}{|\nabla u|}\sum_{ij} a_{ij}(|\nabla u|)_{ij} \cdot \kappa.\end{aligned} \tag{3-15}$$

Recalling the minimal surface equation (3-1), we have

$$a_{11} = 1 + u_2^2, \quad a_{12} = -u_1u_2, \quad a_{21} = -u_1u_2, \quad a_{22} = 1 + u_1^2, \tag{3-16}$$

and

$$u_{22} = \frac{2u_1u_2u_{12} - (1 + u_2^2)u_{11}}{1 + u_1^2}. \tag{3-17}$$

Inserting (3-5) and (3-16)–(3-17) into (3-15), we get

$$\begin{aligned}\sum_{ij} a_{ij}\kappa_{ij} & \\ = {} & \frac{1}{|\nabla u|^3}\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\ & - \frac{6(1 + |\nabla u|^2)}{|\nabla u|^2(1 + u_1^2)}\big[(1 + u_2^2)u_1u_{11} + (1 - u_1^2)u_2u_{12}\big] \cdot \kappa_1 \\ & - \frac{6(1 + |\nabla u|^2)}{|\nabla u|^2}(u_1u_{12} - u_2u_{11}) \cdot \kappa_2 \\ & - \frac{1}{|\nabla u|^2}\left\{\frac{9 + 6|\nabla u|^2}{1 + u_1^2}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big]\right. \\ & \left. \qquad\qquad + 3\sum_{ijk} a_{ij}u_ku_{kij}\right\} \cdot \kappa.\end{aligned} \tag{3-18}$$

Now the commutation formulas (1-1)–(1-2) and relations (3-16)–(3-17) yield

$$
\begin{aligned}
(3\text{-}19)\quad \sum_{ijk} a_{ij}u_k u_{kij} &= \sum_{ijk} a_{ij}u_k u_{ijk} + \sum_{ijkm} a_{ij}u_k u_m R_{mikj} \\
&= -\sum_{ijk} a_{ij,k}u_k u_{ij} + \sum_{ijkm} a_{ij}u_k u_m R_{mikj} \\
&= \frac{2|\nabla u|^2}{1+u_1^2}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big] \\
&\qquad + |\nabla u|^2(1+|\nabla u|^2)\cdot c.
\end{aligned}
$$

Putting (3-19) into (3-18), we have

$$
\begin{aligned}
(3\text{-}20)\quad & \sum_{ij} a_{ij}\kappa_{ij} \\
&= \frac{1}{|\nabla u|^3}\sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\
&\quad - \frac{6(1+|\nabla u|^2)}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_1u_{11} + (1-u_1^2)u_2u_{12}\big]\cdot\kappa_1 \\
&\quad - \frac{6(1+|\nabla u|^2)}{|\nabla u|^2}(u_1u_{12} - u_2u_{11})\cdot\kappa_2 \\
&\quad - \frac{9+12|\nabla u|^2}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_{11}^2 - 2u_1u_2u_{11}u_{12} + (1+u_1^2)u_{12}^2\big]\cdot\kappa \\
&\quad - 3(1+|\nabla u|^2)\cdot\kappa\cdot c.
\end{aligned}
$$

Straightforward computation gives

$$
\begin{aligned}
(3\text{-}21)\quad & \sum_{ij} a_{ij}(2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11})_{ij} \\
&= 2u_1u_2\sum_{ij} a_{ij}u_{12ij} - u_1^2\sum_{ij} a_{ij}u_{22ij} - u_2^2\sum_{ij} a_{ij}u_{11ij} \\
&\quad + 2(u_2u_{12} - u_1u_{22})\sum_{ij} a_{ij}u_{1ij} + 2(u_1u_{12} - u_2u_{11})\sum_{ij} a_{ij}u_{2ij} \\
&\quad - 4u_2\sum_{ij} a_{ij}u_{2i}u_{11j} - 4u_1\sum_{ij} a_{ij}u_{1i}u_{22j} + 4u_2\sum_{ij} a_{ij}u_{1i}u_{12j} \\
&\quad + 4u_1\sum_{ij} a_{ij}u_{2i}u_{12j} + 4u_{12}\sum_{ij} a_{ij}u_{1i}u_{2j} \\
&\quad - 2u_{22}\sum_{ij} a_{ij}u_{1i}u_{1j} - 2u_{11}\sum_{ij} a_{ij}u_{2i}u_{2j} \\
&\triangleq \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4,
\end{aligned}
$$

where

$$J_1 = 2u_1u_2 \sum_{ij} a_{ij}u_{12ij} - u_1^2 \sum_{ij} a_{ij}u_{22ij} - u_2^2 \sum_{ij} a_{ij}u_{11ij},$$

$$J_2 = 2(u_2u_{12} - u_1u_{22}) \sum_{ij} a_{ij}u_{1ij} + 2(u_1u_{12} - u_2u_{11}) \sum_{ij} a_{ij}u_{2ij},$$

$$J_3 = -4u_2 \sum_{ij} a_{ij}u_{2i}u_{11j} - 4u_1 \sum_{ij} a_{ij}u_{1i}u_{22j} + 4u_2 \sum_{ij} a_{ij}u_{1i}u_{12j} + 4u_1 \sum_{ij} a_{ij}u_{2i}u_{12j},$$

$$J_4 = 4u_{12} \sum_{ij} a_{ij}u_{1i}u_{2j} - 2u_{22} \sum_{ij} a_{ij}u_{1i}u_{1j} - 2u_{11} \sum_{ij} a_{ij}u_{2i}u_{2j}.$$

We deal with the terms J_1, J_2, J_3 and J_4 consecutively. If we differentiate the minimal surface equation (3-1) twice, then we have

$$\sum_{ij} a_{ij,k}u_{ij} + \sum_{ij} a_{ij}u_{ijk} = 0, \tag{3-22}$$

$$\sum_{ij} a_{ij,kl}u_{ij} + \sum_{ij} a_{ij,k}u_{ijl} + \sum_{ij} a_{ij,l}u_{ijk} + \sum_{ij} a_{ij}u_{ijkl} = 0. \tag{3-23}$$

By the commutation formulas (1-1)–(1-3), we have

$$u_{klij} = u_{ijkl} + \sum_m u_{mi}R_{mklj} + \sum_m u_{mj}R_{mkli} + \sum_m u_{mk}R_{milj} + \sum_m u_{ml}R_{mikj}.$$

It follows that

$$\begin{aligned} \sum_{ij} a_{ij}u_{klij} = -\Big[\sum_{ij} a_{ij,kl}u_{ij} + \sum_{ij} a_{ij,k}u_{ijl} + \sum_{ij} a_{ij,l}u_{ijk}\Big] &+ \sum_{ijm} a_{ij}u_{mi}R_{mklj} + \sum_{ijm} a_{ij}u_{mj}R_{mkli} \\ &+ \sum_{ijm} a_{ij}u_{mk}R_{milj} + \sum_{ijm} a_{ij}u_{ml}R_{mikj}. \end{aligned} \tag{3-24}$$

Note that $a_{ij} = (1 + |\nabla u|^2)\delta_{ij} - u_iu_j$. It is easy to get

$$a_{ij,k} = 2\sum_m u_mu_{mk}\delta_{ij} - u_{ik}u_j - u_iu_{jk},$$

$$a_{ij,l} = 2\sum_m u_mu_{ml}\delta_{ij} - u_{il}u_j - u_iu_{jl},$$

$$a_{ij,kl} = 2\sum_m u_{ml}u_{mk}\delta_{ij} + 2\sum_m u_mu_{mkl}\delta_{ij} - u_{ikl}u_j - u_{ik}u_{jl} - u_{il}u_{jk} - u_iu_{jkl}.$$

Thus we have

$$\sum_{ij} a_{ij,kl}u_{ij} = 2(u_1u_{22}-u_2u_{12})u_{1kl}+2(u_2u_{11}-u_1u_{12})u_{2kl} + 2u_{1l}u_{1k}u_{22}+2u_{2l}u_{2k}u_{11}-2u_{1k}u_{2l}u_{12}-2u_{2k}u_{1l}u_{12}, \tag{3-25}$$

$$\sum_{ij} a_{ij,k}u_{ijl} = 2u_2u_{2k}u_{11l}+2u_1u_{1k}u_{22l}-2(u_1u_{2k}+u_2u_{1k})u_{12l}, \tag{3-26}$$

$$\sum_{ij} a_{ij,l}u_{ijk} = 2u_2u_{2l}u_{11k}+2u_1u_{1l}u_{22k}-2(u_1u_{2l}+u_2u_{1l})u_{12k}. \tag{3-27}$$

Putting (3-25)–(3-27) into (3-24), we obtain

$$\begin{aligned}\sum_{ij} a_{ij}u_{klij} = -\big[&2(u_1u_{22}-u_2u_{12})u_{1kl}+2(u_2u_{11}-u_1u_{12})u_{2kl}+2u_{1l}u_{1k}u_{22}\\ &+2u_{2l}u_{2k}u_{11}-2u_{1k}u_{2l}u_{12}-2u_{2k}u_{1l}u_{12}+2u_2u_{2k}u_{11l}\\ &+2u_1u_{1k}u_{22l}-2(u_1u_{2k}+u_2u_{1k})u_{12l}+2u_2u_{2l}u_{11k}\\ &+2u_1u_{1l}u_{22k}-2(u_1u_{2l}+u_2u_{1l})u_{12k}\big]\\ &+\sum_{ijm} a_{ij}u_{mi}R_{mklj}+\sum_{ijm} a_{ij}u_{mj}R_{mkli}\\ &+\sum_{ijm} a_{ij}u_{mk}R_{milj}+\sum_{ijm} a_{ij}u_{ml}R_{mikj}.\end{aligned} \tag{3-28}$$

By the commutation formulas (1-1)–(1-2),

$$u_{121} = u_{112}+\sum_m u_mR_{m121} = u_{112}+u_2\cdot c, \tag{3-29}$$

$$u_{122} = u_{221}+\sum_m u_mR_{m212} = u_{221}+u_1\cdot c. \tag{3-30}$$

With (3-16) and (3-29)–(3-30) in hand, formula (3-28) is equivalent to

$$\begin{aligned}\sum_{ij} a_{ij}u_{12ij} = &-2u_2u_{22}u_{111}+2u_2u_{12}u_{112}+2u_1u_{12}u_{221}\\ &-2u_1u_{11}u_{222}-2u_{12}(u_{11}u_{22}-u_{12}^2)\\ &+\big[u_1u_2u_{11}+(4+5u_1^2+5u_2^2)u_{12}+u_1u_2u_{22}\big]\cdot c,\end{aligned}$$

$$\begin{aligned}\sum_{ij} a_{ij}u_{22ij} = &-4u_2u_{22}u_{112}+2(3u_2u_{12}+u_1u_{22})u_{221}\\ &-2(u_2u_{11}+u_1u_{12})u_{222}-2u_{22}(u_{11}u_{22}-u_{12}^2)\\ &+\big[-2(1+u_2^2)u_{11}+10u_1u_2u_{12}+2(1+u_1^2+u_2^2)u_{22}\big]\cdot c,\end{aligned}$$

$$\begin{aligned}\sum_{ij} a_{ij}u_{11ij} = &-2(u_2u_{12}+u_1u_{22})u_{111}+2(u_2u_{11}+3u_1u_{12})u_{112}\\ &-4u_1u_{11}u_{221}-2u_{11}(u_{11}u_{22}-u_{12}^2)\\ &+\big[2(1+u_1^2+u_2^2)u_{11}+10u_1u_2u_{12}-2(1+u_1^2)u_{22}\big]\cdot c.\end{aligned}$$

Therefore,

(3-31)
$$
\begin{aligned}
\mathrm{J}_1 = {} & \big[2u_2^3u_{12}-2u_1u_2^2u_{22}\big]\cdot u_{111}+\big[-2u_2^3u_{11}-2u_1u_2^2u_{12}+4u_1^2u_2u_{22}\big]\cdot u_{112}\\
& +\big[4u_1u_2^2u_{11}-2u_1^2u_2u_{12}-2u_1^3u_{22}\big]\cdot u_{221}+\big[-2u_1^2u_2u_{11}+2u_1^3u_{12}\big]\cdot u_{222}\\
& -2(u_{11}u_{22}-u_{12}^2)\cdot(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})\\
& +\big[2(u_1^2-u_2^2+u_1^2u_2^2-u_2^4)u_{11}+8u_1u_2u_{12}+2(-u_1^2+u_2^2+u_1^2u_2^2-u_1^4)u_{22}\big]\cdot c.
\end{aligned}
$$

Let us handle the term J_2. By (1-1)–(1-2), (3-16) and (3-22), we have

$$
\begin{aligned}
\sum_{ij}a_{ij}u_{1ij} &= \sum_{ij}a_{ij}u_{ij1}+\sum_{ijm}a_{ij}u_mR_{mi1j}\\
&= -\sum_{ij}a_{ij,1}u_{ij}+\sum_{ijm}a_{ij}u_mR_{mi1j}\\
&= -2u_1(u_{11}u_{22}-u_{12}^2)+u_1(1+u_1^2+u_2^2)\cdot c
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{ij}a_{ij}u_{2ij} &= \sum_{ij}a_{ij}u_{ij2}+\sum_{ijm}a_{ij}u_mR_{mi2j}\\
&= -\sum_{ij}a_{ij,2}u_{ij}+\sum_{ijm}a_{ij}u_mR_{mi2j}\\
&= -2u_2(u_{11}u_{22}-u_{12}^2)+u_2(1+u_1^2+u_2^2)\cdot c.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\text{(3-32)}\quad \mathrm{J}_2 = {} & -4(u_{11}u_{22}-u_{12}^2)\cdot(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})\\
& +2(1+u_1^2+u_2^2)\cdot(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})\cdot c.
\end{aligned}
$$

For the term J_3, by (3-16) and (3-29)–(3-30), we have

$$
\begin{aligned}
\text{(3-33)}\quad \mathrm{J}_3 = {} & \big[(-4u_2-4u_2^3)u_{12}+4u_1u_2^2u_{22}\big]\cdot u_{111}\\
& +\big[(4u_2+4u_2^3)u_{11}+(4u_1+4u_1u_2^2)u_{12}+(-4u_2-8u_1^2u_2)u_{22}\big]\cdot u_{112}\\
& +\big[(-4u_1-8u_1u_2^2)u_{11}+(4u_2+4u_1^2u_2)u_{12}+(4u_1+4u_1^3)u_{22}\big]\cdot u_{221}\\
& +\big[4u_1^2u_2u_{11}+(-4u_1-4u_1^3)u_{12}\big]\cdot u_{222}\\
& +\big[(4u_2^2-4u_1^2u_2^2+4u_2^4)u_{11}+8u_1u_2u_{12}+(4u_1^2-4u_1^2u_2^2+4u_1^4)u_{22}\big]\cdot c.
\end{aligned}
$$

Moreover, straightforward computation yields

$$
\text{(3-34)}\qquad \mathrm{J}_4 = -2(u_{11}u_{22}-u_{12}^2)\big[(1+u_2^2)u_{11}-2u_1u_2u_{12}+(1+u_1^2)u_{22}\big]=0.
$$

Combining (3-21) and (3-31)–(3-34), we obtain

$$
\begin{aligned}
\text{(3-35)}\quad \sum_{ij} a_{ij}(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})_{ij}
&= \big[(-4u_2-2u_2^3)u_{12}+2u_1u_2^2u_{22}\big]\cdot u_{111}\\
&\quad +\big[(4u_2+2u_2^3)u_{11}+(4u_1+2u_1u_2^2)u_{12}+(-4u_2-4u_1^2u_2)u_{22}\big]\cdot u_{112}\\
&\quad +\big[(-4u_1-4u_1u_2^2)u_{11}+(4u_2+2u_1^2u_2)u_{12}+(4u_1+2u_1^3)u_{22}\big]\cdot u_{221}\\
&\quad +\big[2u_1^2u_2u_{11}+(-4u_1-2u_1^3)u_{12}\big]\cdot u_{222}\\
&\quad -6(u_{11}u_{22}-u_{12}^2)\cdot(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})\\
&\quad +\big[(2u_1^2-4u_1^2u_2^2)u_{11}+(20u_1u_2+4u_1u_2^3+4u_1^3u_2)u_{12}\\
&\qquad\qquad +(2u_2^2-4u_1^2u_2^2)u_{22}\big]\cdot c.
\end{aligned}
$$

Now let us explore the relations between $u_{111}, u_{112}, u_{221}, u_{222}$ and κ_1, κ_2. If we take the first derivative on both sides of (3-14) and (3-1), respectively, then using (3-5), (3-29)–(3-30) and (3-16), we obtain

$$
\begin{aligned}
&-u_2^2\cdot u_{111}+2u_1u_2u_{12}\cdot u_{112}-u_1^2\cdot u_{221}-|\nabla u|^3\cdot\kappa_1\\
&\qquad -3|\nabla u|(u_1u_{11}+u_2u_{12})\cdot\kappa-2u_1(u_{11}u_{22}-u_{12}^2)+2u_1u_2^2\cdot c=0,\\
&-u_2^2\cdot u_{112}+2u_1u_2u_{12}\cdot u_{221}-u_1^2\cdot u_{222}-|\nabla u|^3\cdot\kappa_2\\
&\qquad -3|\nabla u|(u_1u_{12}+u_2u_{22})\cdot\kappa-2u_2(u_{11}u_{22}-u_{12}^2)+2u_1^2u_2\cdot c=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(1+u_2^2)\cdot u_{111}-2u_1u_2\cdot u_{112}+(1+u_1^2)\cdot u_{221}+2u_1(u_{11}u_{22}-u_{12}^2)-2u_1u_2^2\cdot c=0,\\
&(1+u_2^2)\cdot u_{112}-2u_1u_2\cdot u_{221}+(1+u_1^2)\cdot u_{222}+2u_2(u_{11}u_{22}-u_{12}^2)-2u_1^2u_2\cdot c=0.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\text{(3-36)}\quad u_{111}&=(-u_2^2+u_1^2+3u_1^2u_2^2+u_1^4)|\nabla u|^{-1}\cdot\kappa_1+(-2u_1u_2-2u_1^3u_2)|\nabla u|^{-1}\cdot\kappa_2\\
&\quad +3|\nabla u|^{-3}\big[(-u_1u_2^2+u_1^3+3u_1^3u_2^2+u_1^5)u_{11}\\
&\qquad\qquad +(-u_2^3-u_1^2u_2+3u_1^2u_2^3-u_1^4u_2)u_{12}\\
&\qquad\qquad\qquad +(-2u_1u_2^2-2u_1^3u_2^2)u_{22}\big]\cdot\kappa\\
&\quad +(-6u_1u_2^2+2u_1^3)|\nabla u|^{-4}\cdot(u_{11}u_{22}-u_{12}^2)+2u_1u_2^2|\nabla u|^{-2}\cdot c,
\end{aligned}
$$

$$
\begin{aligned}
\text{(3-37)}\quad u_{112}&=(2u_1u_2+2u_1u_2^3)|\nabla u|^{-1}\cdot\kappa_1+(-u_2^2+u_1^2-u_1^2u_2^2+u_1^4)|\nabla u|^{-1}\cdot\kappa_2\\
&\quad +3|\nabla u|^{-3}\big[(2u_1^2u_2+2u_1^2u_2^3)u_{11}\\
&\qquad\qquad +(u_1u_2^2+u_1^3+2u_1u_2^4-u_1^3u_2^2+u_1^5)u_{12}\\
&\qquad\qquad\qquad +(-u_2^3+u_1^2u_2-u_1^2u_2^3+u_1^4u_2)u_{22}\big]\cdot\kappa\\
&\quad +(-2u_2^3+6u_1^2u_2)|\nabla u|^{-4}\cdot(u_{11}u_{22}-u_{12}^2)-2u_1^2u_2|\nabla u|^{-2}\cdot c,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{221} = {} & (u_2^2+u_2^4-u_1^2-u_1^2u_2^2)|\nabla u|^{-1}\cdot\kappa_1+(2u_1u_2+2u_1^3u_2)|\nabla u|^{-1}\cdot\kappa_2 \\
& +3|\nabla u|^{-3}\big[(u_1u_2^2+u_1u_2^4-u_1^3-u_1^3u_2^2)u_{11} \\
& \qquad +(u_2^3+u_2^5+u_1^2u_2-u_1^2u_2^3+2u_1^4u_2)u_{12} \\
& \qquad\qquad +(2u_1u_2^2+2u_1^3u_2^2)u_{22}\big]\cdot\kappa \\
& +(6u_1u_2^2-2u_1^3)|\nabla u|^{-4}\cdot(u_{11}u_{22}-u_{12}^2)-2u_1u_2^2|\nabla u|^{-2}\cdot c,
\end{aligned}
\tag{3-38}
$$

$$
\begin{aligned}
u_{222} = {} & (-2u_1u_2-2u_1u_2^3)|\nabla u|^{-1}\cdot\kappa_1+(u_2^2+u_2^4-u_1^2+3u_1^2u_2^2)|\nabla u|^{-1}\cdot\kappa_2 \\
& +3|\nabla u|^{-3}\big[(-2u_1^2u_2-2u_1^2u_2^3)u_{11} \\
& \qquad +(-u_1u_2^2-u_1u_2^4-u_1^3+3u_1^3u_2^2)u_{12} \\
& \qquad\qquad +(u_2^3+u_2^5-u_1^2u_2+3u_1^2u_2^3)u_{22}\big]\cdot\kappa \\
& +(2u_2^3-6u_1^2u_2)|\nabla u|^{-4}\cdot(u_{11}u_{22}-u_{12}^2)+2u_1^2u_2|\nabla u|^{-2}\cdot c.
\end{aligned}
\tag{3-39}
$$

Inserting (3-36)–(3-39) and (3-17) into (3-35), after some tedious calculation, we get

$$
\begin{aligned}
& \sum_{ij} a_{ij}(2u_1u_2u_{12}-u_1^2u_{22}-u_2^2u_{11})_{ij} \\
& \quad = \frac{2|\nabla u|(4+3|\nabla u|^2)}{1+u_1^2}\big[(1+u_2^2)u_1u_{11}+(1-u_1^2)u_2u_{12}\big]\cdot\kappa_1 \\
& \qquad +2|\nabla u|(4+3|\nabla u|^2)(u_1u_{12}-u_2u_{11})\cdot\kappa_2 \\
& \qquad +\frac{4|\nabla u|(2+3|\nabla u|^2)}{1+u_1^2}\big[(1+u_2^2)u_{11}^2-2u_1u_2u_{11}u_{12}+(1+u_1^2)u_{12}^2\big]\cdot\kappa \\
& \qquad +2|\nabla u|^3(1+|\nabla u|^2)\cdot\kappa\cdot c.
\end{aligned}
\tag{3-40}
$$

By (3-20) and (3-40), we have

$$
\begin{aligned}
\sum_{ij} a_{ij}\kappa_{ij} = {} & \frac{2}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_1u_{11}+(1-u_1^2)u_2u_{12}\big]\cdot\kappa_1 \\
& +\frac{2}{|\nabla u|^2}(u_1u_{12}-u_2u_{11})\cdot\kappa_2 \\
& -\frac{1}{|\nabla u|^2(1+u_1^2)}\big[(1+u_2^2)u_{11}^2-2u_1u_2u_{11}u_{12}+(1+u_1^2)u_{12}^2\big]\cdot\kappa \\
& -(1+|\nabla u|^2)\kappa\cdot c.
\end{aligned}
\tag{3-41}
$$

Then at the point x_0, we take the first derivative of (3-41). Note that

$$\kappa(x_0) = -\frac{u_{11}}{u_2}.$$

With (3-6), (3-16) and (3-29) in hand, we obtain

$$
\begin{aligned}
\sum_{ij} a_{ij}\kappa_{ij1} = {}& \frac{2(1-u_2^2)}{u_2}u_{12}\cdot\kappa_{11} - \frac{2(1-u_2^2)}{u_2}u_{11}\cdot\kappa_{12} \\
&+ \left[\frac{2}{u_2}u_{112} + \frac{1+u_2^2}{u_2^2}u_{11}^2 - \frac{3}{u_2^2}u_{12}^2\right]\cdot\kappa_1 \\
&+ \left[-\frac{2}{u_2}u_{111} + \frac{4}{u_2^2}u_{11}u_{12}\right]\cdot\kappa_2 + \frac{2(1+u_2^2)}{u_2^3}u_{11}^2u_{111} \\
&+ \frac{2}{u_2^3}u_{11}u_{12}u_{112} - \frac{2(1+u_2^2)}{u_2^4}u_{11}^3u_{12} - \frac{2}{u_2^4}u_{11}u_{12}^3 \\
&+ \left[(1-u_2^2)\cdot\kappa_1 + \frac{2(1+u_2^2)}{u_2^2}u_{11}u_{12}\right]\cdot c.
\end{aligned}
\tag{3-42}
$$

Now, the equations (3-36) and (3-37) are simplified as

$$
u_{111} = -u_2\kappa_1 + \frac{3}{u_2}u_{11}u_{12},
\tag{3-43}
$$

$$
u_{112} = -u_2\kappa_2 - \frac{1+u_2^2}{u_2}u_{11}^2 + \frac{2}{u_2}u_{12}^2.
\tag{3-44}
$$

Putting (3-43)–(3-44) into (3-42), one obtains

$$
\begin{aligned}
\sum_{ij} a_{ij}\kappa_{ij1} = {}& \frac{2(1-u_2^2)}{u_2}u_{12}\cdot\kappa_{11} - \frac{2(1-u_2^2)}{u_2}u_{11}\cdot\kappa_{12} + \left[-\frac{3(1+u_2^2)}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\right]\cdot\kappa_1 \\
&- \frac{4}{u_2^2}u_{11}u_{12}\cdot\kappa_2 + \frac{2(1+u_2^2)}{u_2^4}u_{11}^3u_{12} + \frac{2}{u_2^4}u_{11}u_{12}^3 \\
&+ \left[(1-u_2^2)\cdot\kappa_1 + \frac{2(1+u_2^2)}{u_2^2}u_{11}u_{12}\right]\cdot c.
\end{aligned}
$$

Therefore, by commutation formulas (1-1)–(1-2), we get

(3-45)

$$
\begin{aligned}
\sum_{ij} a_{ij}\kappa_{1ij} &= \sum_{ij} a_{ij}\left[\kappa_{ij1} + \sum_m \kappa_m R_{mi1j}\right] = \sum_{ij} a_{ij}\kappa_{ij1} + \kappa_1\cdot c \\
&= \frac{2(1-u_2^2)}{u_2}u_{12}\cdot\kappa_{11} - \frac{2(1-u_2^2)}{u_2}u_{11}\cdot\kappa_{12} \\
&\quad + \left[-\frac{3(1+u_2^2)}{u_2^2}u_{11}^2 + \frac{1}{u_2^2}u_{12}^2\right]\cdot\kappa_1 - \frac{4}{u_2^2}u_{11}u_{12}\cdot\kappa_2 + \frac{2(1+u_2^2)}{u_2^4}u_{11}^3u_{12} \\
&\quad + \frac{2}{u_2^4}u_{11}u_{12}^3 + \left[(2-u_2^2)\cdot\kappa_1 + \frac{2(1+u_2^2)}{u_2^2}u_{11}u_{12}\right]\cdot c.
\end{aligned}
$$

Thanks to (3-45), the formula (3-13) reduces to

$$
\begin{aligned}
\text{(3-46)}\quad & \sum_{ij} a_{ij}\psi_{ij} \\
&= \big[2u_2(1+u_2^2)g' + 4g\big]u_{12}\cdot\kappa_{11} \\
&\quad - \big[2u_2(1+u_2^2)g' + (6+2u_2^2)g\big]u_{11}\cdot\kappa_{12} - 2gu_{12}\cdot\kappa_{22} \\
&\quad + \Bigg\{\bigg[u_2(1+u_2^2)^2g'' + (3+4u_2^2)(1+u_2^2)g' - \frac{(3-2u_2^2)(1+u_2^2)}{u_2}g\bigg]u_{11}^2 \\
&\qquad\qquad + \bigg[u_2(1+u_2^2)g'' + (3+4u_2^2)g' + \frac{1+2u_2^2}{u_2}g\bigg]u_{12}^2\Bigg\}\cdot\kappa_1 \\
&\quad - \frac{4g}{u_2}u_{11}u_{12}\cdot\kappa_2 + \frac{2g}{u_2^3}\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big]u_{11}u_{12} \\
&\quad + \bigg\{\big[u_2^2(1+u_2^2)g' + 3u_2g\big]\cdot\kappa_1 + \frac{2(1+u_2^2)g}{u_2}u_{11}u_{12}\bigg\}\cdot c.
\end{aligned}
$$

By (3-3) and (3-41), we have

$$\text{(3-47)}\qquad \kappa_{11} = \frac{1}{u_2g}\big[\psi_1 - (u_2g'+g)u_{12}\cdot\kappa_1 + gu_{11}\cdot\kappa_2\big],$$

$$\text{(3-48)}\qquad \kappa_{12} = \frac{1}{u_2g}\big[\psi_2 + (u_2g'+g)(1+u_2^2)u_{11}\cdot\kappa_1 + gu_{12}\cdot\kappa_2\big],$$

and

$$
\begin{aligned}
\text{(3-49)}\quad \kappa_{22} = \frac{1}{u_2g}\bigg\{&-(1+u_2^2)\psi_1 + \big[u_2(1+u_2^2)g' + (3+u_2^2)g\big]u_{12}\cdot\kappa_1 \\
&- (3+u_2^2)gu_{11}\cdot\kappa_2 + \frac{g}{u_2^2}u_{11}\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big] \\
&+ (1+u_2^2)gu_{11}\cdot c\bigg\}.
\end{aligned}
$$

Putting (3-47)–(3-49) into (3-46), we finally get

$$
\begin{aligned}
\text{(3-50)}\quad & \sum_{ij} a_{ij}\psi_{ij} \\
&= \bigg[\frac{2(1+u_2^2)g'}{g} + \frac{6+2u_2^2}{u_2}\bigg]\cdot(u_{12}\psi_1 - u_{11}\psi_2) \\
&\quad + \bigg[u_2(1+u_2^2)g'' - \frac{2u_2(1+u_2^2)g'^2}{g} - 5g' - \frac{9g}{u_2}\bigg]\cdot\big[(1+u_2^2)u_{11}^2 + u_{12}^2\big]\cdot\kappa_1 \\
&\quad + \big[u_2^2(1+u_2^2)g' + 3u_2g\big]\cdot\kappa_1\cdot c.
\end{aligned}
$$

If we let

$$g(t) = \frac{(1+t^2)^{3/2}}{t^3},$$

then the last two terms on the right-hand side of (3-50) vanish. Namely,

$$\sum_{ij} a_{ij}\psi_{ij} = 2u_2 \cdot (u_{12}\psi_1 - u_{11}\psi_2).$$

This completes the proof of Theorem 1.2. □

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