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Resolution of a compact group action in the sense described by Albin and Melrose is applied to the conjugation action by the unitary group on self-adjoint matrices. It is shown that the eigenvalues are smooth on the resolved space and that the trivial bundle smoothly decomposes into the direct sum of global one-dimensional eigenspaces.

For a general compact Lie group G acting on a smooth compact manifold with corners M, Albin and Melrose [2011] showed that there is a canonical full resolution such that the group action lifts to the blown-up space $Y(M)$ to have a unique isotropy type. Under this condition, a result of Borel and Ji [2006] applies to show that the orbit space $G \setminus Y(M)$ is smooth.

In this paper, we give an explicit construction of the resolution of the action of the unitary group on the space of self-adjoint matrices

$$S = S(n) = \{X \in M_n(\mathbb{C}) \mid X^* = X\},$$

with the unitary group $\mathrm{U}(n)$ acting by conjugation:

$$u \in \mathrm{U}(n), \quad X \in S, \quad u \cdot X := uXu^{-1}.$$

The orbit of an element $X \in S$, denoted by $\mathrm{U}(n) \cdot X$, consists of the matrices with the same eigenvalues including multiplicities. For a matrix $X \in S$ with m distinct eigenvalues $\{\lambda_j\}_{j=1}^m$ with multiplicities i_k, $k = 1, 2, \ldots, m$, the isotropy group of X is conjugate to a direct sum of smaller unitary groups:

$$\mathrm{U}(n)^X \big(:= \{u \in \mathrm{U}(n) \mid u \cdot X = X\}\big) \cong \bigoplus_{k=1}^{m} U(i_k).$$

The isotropy types are therefore parametrized by the partition of n into integers. Note here that the partition contains information about ordering, for example, the two partitions of 3, $\{i_1 = 1, i_2 = 2\}$ and $\{i_1 = 2, i_2 = 1\}$, are not the same type.

For $n > 1$, the eigenvalues are not smooth functions on S, but are singular where the multiplicities change. Consider the trivial bundle over S, $M := S \times \mathbb{C}^n$, the fiber of which can be decomposed into n eigenspaces of the self-adjoint matrix at the

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base point. This decomposition is not unique at matrices with multiple eigenvalues, and the eigenspaces are not smooth at these base points. We will show that, by doing iterative blow-ups, the singularities are resolved and the eigenvalues become smooth functions on the resolved space. Moreover, by doing a "full" blow-up, the eigenspaces also become smooth.

Recall a lemma on group action resolutions:

Lemma 1 [Albin and Melrose 2011]. *A compact manifold* (*with corners*), M, *with a smooth*, *boundary intersection free*, *action by a compact Lie group*, G, *has a canonical full resolution*, $Y(M)$, *obtained by iterative blow-up of minimal isotropy types.*

In this paper we will discuss two kinds of blow-ups, namely radial and projective blow-ups, which give different results; a projective blow-up of a hypersurface is trivial but a radial blow-up produces a new boundary. A resolution of S involves the choice of blow-up and which centers to blow-up. In this paper, we will discuss three kinds of resolutions:

Definition 2. We define the following three resolutions of S:

(1) radial resolution $\widehat{S}_r$: a radial blow-up of all singular strata {there exists $i \neq j$, $\lambda_i = \lambda_j$} in an order compatible with inclusion of the conjugation class of the isotropy group;

(2) projective resolution $\widehat{S}_p$: a projective blow-up of all singular strata in the same order as radial resolution;

(3) small resolution $\widehat{S}_s$: a radial blow-up of a smaller set of centers

$$\bigcup_{1 \le i < j \le n} \{\lambda_i = \lambda_{i+1} = \cdots = \lambda_j\}$$

with the order determined by complete inclusion.

As pointed out in [Albin and Melrose 2011], a projective blow-up usually requires an extra step of reflection in the iterative scheme in order to obtain smoothness. We will show that, the radial resolution yields that the trivial bundle M decomposes into the direct sum of n one-dimensional eigenspaces. By contrast, after projective resolution or small resolution, the eigenvalues are smooth on the resolved space, and locally we have a smooth decomposition into simple eigenspaces, but the trivial bundle doesn't split into global line bundles.

Remark 3. In theory there is a fourth resolution by doing a projective blow-up of the smaller set of centers introduced in $\widehat{S}_s$. This resolves eigenvalues but does not globally resolve eigenbundles, for the same reason as $\widehat{S}_s$. Therefore for simplicity we do not include this resolution in our discussion below.

To describe the different outcomes of the three resolutions above, we recall the resolution in the sense of Albin and Melrose.

Definition 4 (eigenresolution). By an eigenresolution of S, we mean a manifold with corners $\widehat{S}$, with a surjective smooth map $\beta : \widehat{S} \to S$ such that the self-adjoint matrices have a smooth (local) diagonalization when lifted to $\widehat{S}$. Eigenvalues then lift to n smooth functions f_i on $\widehat{S}$, i.e., for any $X \in \widehat{S}$, $\beta(X)$ has eigenvalues $\{f_i(X)\}_{i=1}^n$.

Note that in the definition we only require the diagonalization to exist locally. To encompass the information of global decomposition of eigenvectors, we introduce the full resolution below.

Definition 5 (full eigenresolution). A full eigenresolution is an eigenresolution with global eigenbundles. The eigenvalues lift to n smooth functions f_i on $\widehat{S}$, and the trivial n-dimensional complex vector bundle on $\widehat{S}$ is decomposed into n smooth line bundles:

$$\widehat{S} \times \mathbb{C}^n = \bigoplus_{i=1}^{n} E_i$$

such that

$$\beta(X)v_i = f_i(X)v_i \quad \text{for all } v_i \in E_i(X) \quad \text{for all } X \in \widehat{S}.$$

We use the blow-up constructions introduced by Melrose [1996, Chapter 5] and show that we can obtain resolutions in this way and, in particular, a full resolution if we use radial blow-ups.

Theorem 6. *The three types of resolutions given in Definition 2, namely, $\widehat{S}_r$, $\widehat{S}_p$, and $\widehat{S}_s$, each yield an eigenresolution. Only the radial resolution $\widehat{S}_r$ gives a full eigenresolution.*

Remark 7. In particular, the blow-down map $\beta : \widehat{S} \to S$ is a diffeomorphism between the interior of $\widehat{S}$ and the open dense subset of S consisting of the matrices with n-distinct eigenvalues.

Related to the problem of resolving eigenvalues is the problem of desingularization of polynomial roots. In [Kurdyka and Paunescu 2008], generalizing Rellich's result [1937] on one-dimensional analytical families, the perturbation theory of hyperbolic polynomials is discussed using Hironaka's resolution theory. It is applied to perturbation theory of normal operators and resonances; see for example [Rainer 2013] and [Rauch 1980].

The idea of resolution has been used in many geometric problems. The abstract notion of a resolution structure on a manifold with corners is discussed in [Baum et al. 1985]. In [Davis 1978], it is shown that for a general action the induced action on the set of boundary hypersurfaces can be appropriately resolved. The canonical resolution is presented in [Duistermaat and Kolk 2000], and the induced resolution of the orbit space is considered in [Hassell et al. 1995]. In [Albin and Melrose 2011], an iterative procedure is shown to capture the simultaneous resolution of all isotropy types in a "resolution structure" consisting of equivariant iterated fibrations of the boundary faces, which is the procedure we will use in this paper.

1. Proof of Theorem 6

The proof of Theorem 6 proceeds through induction on the dimension. We begin by discussing the first example which is the 2×2 matrices.

Lemma 8 (2×2 case). *For the 2×2 self-adjoint matrices $S(2)$, the eigenvalues and eigenvectors are smooth except at multiples of the identity. After radial resolution, the singularities are resolved and the trivial 2-dimensional bundle splits into the direct sum of two line bundles. The projective resolution also gives smooth eigenvalues, but does not give two global line bundles.*

Remark 9. Note that in the 2×2 case, the radial resolution $\widehat{S}_r$ and the small resolution $\widehat{S}_s$ are the same.

Proof. In this case

$$S = S(2) = \left\{ \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} \,\middle|\, a_{ii} \in \mathbb{R}, z_{12} \in \mathbb{C} \right\} \cong \mathbb{R}^4.$$

The space S is isomorphic to the product of $\mathbb{R}$ and the trace-free subspace

$$S_0 = \left\{ \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} \,\middle|\, a_{11} + a_{22} = 0 \right\}, \tag{1}$$

i.e., there is a bijective linear map:

$$\begin{aligned} \phi :\ & S && \to && S_0 \times \mathbb{R} \\ & A = \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} && \mapsto && \big(A_0 := A - \tfrac{1}{2}(a_{11} + a_{22})I,\ \tfrac{1}{2}(a_{11} + a_{22})\big). \end{aligned} \tag{2}$$

The eigenvalues λ_i and eigenvectors v_i of A are related to those of A_0 by $\lambda_i(A) = \lambda_i(A_0) + \frac{1}{2}\operatorname{tr}(A)$, $v_i(A) = v_i(A_0)$, $i = 1, 2$. Therefore, we can restrict the discussion of resolution to the subspace S_0, since the smoothness of eigenvalues and eigenvectors on the resolution of S follows.

Let $z_{12} = c + di$. The space S_0 can be identified with $\mathbb{R}^3 = \{(a_{11}, c, d)\}$. The eigenvalues of this matrix are:

$$\lambda_{\pm} = \pm\sqrt{a_{11}^2 + c^2 + d^2}. \tag{3}$$

Hence the only singularity of the eigenvalues on S_0 is at the point $a_{11} = c = d = 0$ which represents the zero matrix.

Based on the resolution formula in [Melrose 1996], the radial blow-up can be realized as

$$\widehat{S}_{0,r} = [S_0, \{0\}] = S^+N\{0\} \sqcup (S_0 \setminus \{0\}) \simeq \mathbb{S}^2 \times [0, \infty)_+, \tag{4}$$

where the front face $S^+N\{0\} \simeq \mathbb{S}^2$. Here the radial variable is

$$r = \sqrt{a_{11}^2 + c^2 + d^2}.$$

The blow-down map is

$$\beta : [S_0, \{0\}] \to S_0, \quad (r, \theta) \mapsto r\theta, \quad r \in \mathbb{R}_+, \theta \in \mathbb{S}^2. \tag{5}$$

The radial variable r lifts to be smooth on the blown up space; therefore the two eigenvalues $\lambda_\pm = \pm r$ become smooth functions.

Now we consider the eigenvectors to the corresponding eigenvalues $\lambda_\pm$:

$$v_\pm = \left(c + di, \pm\sqrt{a_{11}^2 + c^2 + d^2} - a_{11}\right) \in \mathbb{C}^2. \tag{6}$$

Similar to the discussion of the eigenvalues, the only singularity is at $r = 0$, which becomes a smooth function on $[S_0, \{0\}]$. It follows that v_+ and v_- span two smooth line bundles on $[S_0, \{0\}]$.

If we do the projective blow-up instead, which identifies the antipodal points in the front face of $\mathbb{S}^2$ to get $\mathbb{RP}^2$, namely,

$$\widehat{S}_{0,p} = \{(x, l) \mid x \in l\} \subset \mathbb{R}^3 \times \mathbb{RP}^2, \tag{7}$$

which we can cover with three coordinate patches:

$$(x_1, y_1, z_1) = \left(c, \frac{d}{c}, \frac{a_{11}}{c}\right) \in \mathbb{R}^3,$$

and the other two (x_2, y_2, z_2), (x_3, y_3, z_3)=$(d, c/d, a_{11}/d)$, $(a_{11}, c/a_{11}, d/a_{11})$ are similar. The two eigenvalues we get from here are

$$v_\pm = \pm\sqrt{a_{11}^2 + c^2 + d^2} = \pm|x_1|\sqrt{(1 + y_1^2 + z_1^2)},$$

which is smooth across $\{x_1 = 0\}$. Similar discussions hold for the other two coordinate patches.

However, the trivial bundle does not decompose into two line bundles as in the radial case. The nontriviality of eigenbundles can be seen by taking a homotopically nontrivial loop in $\mathbb{RP}^2$

$$l = \beta^{-1}(\{r = 1\}) \subset \widehat{S}_{0,p}.$$

This curve intersects the line $c = d = 0$ twice, which hits at two different places; thus both $a_{11}^\pm = \pm 1$ are on the curve, and equation (6) shows that starting from

$v_- = (0, -2) = (0, -2a_{11}^+)$, this turns into $v_+ = (0, -2) = (0, 2a_{11}^-)$, which means the two eigenvectors are not separated by projective blow-up.

Now that we have done the radial resolution for the trace-free slice S_0, the resolution of S follows. Consider S as a 3-dimensional vector bundle on $\mathbb{R}$ with trace being the projection map. Then at each base point λ, the fiber is $S_0 + \lambda I$. The resolution is $[S_0 + \lambda I; \lambda I] \cong [S_0; \{0\}]$. Since the trace direction is transversal to the blow-up,

$$[S; \mathbb{R}I] = [S_0; \{0\}] \times \mathbb{R}. \tag{8}$$

And because the trace doesn't change the eigenvectors, the smoothness follows. □

To proceed to higher dimensions, we first discuss the partition of eigenvalues into clusters. The basic case is when the eigenvalues are divided into two clusters; then the $\mathrm{U}(n)$ action of the matrices can be decomposed to two commuting actions.

Definition 10 (spectral gap). A connected neighborhood $U \subset S$ has a spectral gap at $c \in \mathbb{R}$, if c is not an eigenvalue of X for any $X \in U$.

Note here that since U is connected, the number of eigenvalues less than c stays the same for all $X \in U$, denoted by k.

Lemma 11 (local eigenspace decomposition). *If a bounded neighborhood $U \subset S(n)$ has a spectral gap at c, then the matrices in U can be decomposed into two smooth self-adjoint commuting matrices*:

$$X = L_X + R_X, \, L_X R_X = R_X L_X.$$

with $\operatorname{rank}(L_X) = k, \ \operatorname{rank}(R_X) = n - k$.

Proof. Let γ be a simple closed curve on $\mathbb{C}$ such that it intersects with $\mathbb{R}$ only at $-R$ and c, where R is a sufficiently large number such that $-R$ is less than any eigenvalues of the matrices contained in U. In this way, for any matrix $X \in U$, the k smallest eigenvalues are contained inside γ. We consider the operator $P_X : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$P_X := -\frac{1}{2\pi i} \oint_\gamma (X - sI)^{-1} \, ds. \tag{9}$$

Since the resolvent is nonsingular on γ, P_X is a well-defined operator and varies smoothly with X, the integral is independent of choice of γ up to homotopy.

First we show that P_X is a projection operator, i.e.,

$$P_X^2 = P_X. \tag{10}$$

Let γ_s and γ_t be two curves satisfying the above condition with γ_s completely inside γ_t. Then

$$\begin{aligned}
P_X^2 &= -\frac{1}{4\pi^2}\oint_{\gamma_t}(X-tI)^{-1}\,dt\left(\oint_{\gamma_s}(X-sI)^{-1}\,ds\right)\\
&= -\frac{1}{4\pi^2}\oint_{\gamma_t}dt\left[\oint_{\gamma_s}\frac{1}{s-t}(X-sI)^{-1}\,ds-\oint_{\gamma_s}\frac{1}{s-t}(X-tI)^{-1}\,ds\right]\\
&= \mathrm{I}-\mathrm{II},
\end{aligned}$$

where using the fact that s is completely inside γ_t,

$$\mathrm{I} = -\frac{1}{4\pi^2}\oint_{\gamma_s}\frac{1}{X-sI}\,ds\oint_{\gamma_t}\frac{1}{s-t}\,dt = -\frac{1}{4\pi^2}(-2\pi i)\oint_{\gamma_s}\frac{1}{X-sI}\,ds = P_X,$$

and any t on γ_t is outside of the loop γ_s, so

$$\oint_{\gamma_s}\frac{1}{s-t}\,ds = 0,$$

and we have

$$\mathrm{II} = -\frac{1}{4\pi^2}\oint_{\gamma_t}(X-tI)^{-1}\,dt\oint_{\gamma_s}\frac{1}{s-t}\,ds = 0.$$

This proves (10).

Then we show that P_X is self-adjoint. This is because

$$P_X^* = \frac{1}{2\pi i}\int_\gamma((X-sI)^{-1})^*\,d\bar{s} = \frac{1}{2\pi i}\int_{-\bar{\gamma}}(X-sI)\,ds = P_X.$$

P_X maps $\mathbb{R}^n$ to the invariant subspace spanned by the eigenvectors corresponding to eigenvalues that are less than c. We denote this invariant subspace by L and its orthogonal complement by R. Write X as the diagonalization $X = V\Lambda V^{-1}$, where Λ is the eigenvalue matrix and V is the matrix whose columns are the eigenvectors of X. Then L is spanned by the first k columns of V. Take one of the eigenvectors $v_j \in L$, $j = 1, 2, \ldots, k$,

$$\begin{aligned}
P_X v_j &= -\frac{1}{2\pi i}\oint_\gamma(X-sI)^{-1}v_j\,ds = -\frac{1}{2\pi i}\oint V(\Lambda-sI)^{-1}V^{-1}v_j\,ds\\
&= -\frac{1}{2\pi i}v_j\oint\frac{1}{\lambda_j-s}\,ds = v_j.
\end{aligned}$$

Similarly for $v_j \in R$ that corresponds to an eigenvalue greater than c (therefore λ_j is outside the loop),

$$P_X v_j = -\frac{1}{2\pi i}v_j\oint\frac{1}{\lambda_j-s}\,ds = 0;$$

therefore

$$(I - P_X)v_j = v_j \quad \text{for all } \; v_j \in R.$$

Then using the projection P_X we define two operators L_X and R_X as

$$L_X := P_X X P_X \tag{11}$$

and

$$R_X := (I - P_X)X(I - P_X) \tag{12}$$

Since P_X is smooth, the two operators are also smooth. Moreover, using the fact that P_X is a projection onto the invariant subspace L, we have

$$(I - P_X)X P_X = P_X X(I - P_X) = 0;$$

therefore

$$X = L_X + R_X.$$

For an eigenvector $v \in L$,

$$L_X v = X v \quad \text{and} \quad R_X v = 0, \tag{13}$$

i.e., L_X equals X when restricted to L, similarly $R_X|_R = X$. Since $P_X^* = P_X$, L_X and R_X are also self-adjoint. In this way we get two commuting lower-rank matrices L_X and R_X. □

It is natural to have a finer decomposition when there is more than one spectral gap in the neighborhood, and we have the following corollary.

Corollary 12. *If the eigenvalues of matrices in a neighborhood U can be grouped into k clusters, then the matrices can be decomposed into k lower-rank self-adjoint commuting matrices smoothly.*

Proof. Do the decomposition inductively. If $k = 2$, then it is the case in Lemma 11. Suppose the decomposition for $k = l - 1$ is defined. Then for $k = l$, since the eigenvalues can also be divided into two clusters (by combining the smallest $l - 1$ groups of eigenvalues together), then $X = L_X + R_X$, with L_X and R_X corresponding to the two intervals. Then L_X satisfies the separation condition for $l - 1$ clusters, so by induction, $L_X = L_1 + \cdots + L_{l-1}$. Therefore, $X = L_1 + L_2 + \cdots + L_{l-1} + R_X$ is the desired division. □

Using Lemma 11 of decomposition of matrices in a neighborhood, we can now show that locally the trivial bundle $S \times \mathbb{C}^n$ decomposes into two subspaces if there is a spectral gap. Moreover, locally there is a product structure of two lower-dimensional matrices. In order to see this, we need to introduce the Grassmannian.

Let $\mathrm{Gr}_{\mathbb{C}}(n,k)$ denote the Grassmannian, i.e., the set of k-dimensional subspaces in $\mathbb{C}^n$. Consider the tautological vector bundle over the Grassmannian:

$$\pi_k : T_k \to \mathrm{Gr}_{\mathbb{C}}(n,k), \quad \pi^{-1}(p) = V(p),$$

where each fiber is a k-dimensional subspace in $\mathbb{C}^n$, with self-adjoint operators acting on it. Similarly, we define T_{n-k} to be the orthogonal complement of T_k:

$$\pi_{n-k} : T_{n-k} \to \mathrm{Gr}_{\mathbb{C}}(n,k), \quad \pi^{-1}(p) = V(p)^{\perp}.$$

Definition 13 (Operator bundle). Let P_k (resp. P_{n-k}) be the bundles over $\mathrm{Gr}_{\mathbb{C}}(n,k)$ of the fiberwise self-adjoint operators on the tautological bundle T_k (resp. T_{n-k}).

Take the Whitney sum of the two bundles

$$\pi : P_k \oplus P_{n-k} \to \mathrm{Gr}_{\mathbb{C}}(n,k). \tag{14}$$

Each of its fibers can be identified with $S(k) \oplus S(n-k)$ when we pick a basis. There is a $\mathrm{U}(n)$-action on this bundle:

$$\begin{aligned} g \cdot (p, (p_k, p_{n-k})) &= (g \cdot p, (g \circ p_k \circ g^{-1}, g \circ p_{n-k} \circ g^{-1})), \\ &\qquad p \in \mathrm{Gr}_{\mathbb{C}}(n,k),\ p_k \in P_k(p),\ p_{n-k} \in P_{n-k}(p). \end{aligned} \tag{15}$$

Suppose an open neighborhood $U \in S$ satisfies the spectral gap condition. Let $\mathrm{U}(n) \cdot U$ be the group invariant neighborhood generated by U, that is,

$$\mathrm{U}(n) \cdot U := \bigcup_{g \in \mathrm{U}(n)} g \cdot U. \tag{16}$$

Then $\mathrm{U}(n) \cdot U$ is open and connected, and also satisfies the spectral gap condition as U does, since the $\mathrm{U}(n)$-action preserves the eigenvalues. From the proof of Lemma 11, it is shown that in the neighborhood, the trivial $\mathbb{C}^n$ bundle over U naturally splits into two subbundles $E^k \oplus E^{n-k}$, and this gives a local product structure. We will prove that, for a $\mathrm{U}(n)$-invariant neighborhood, there is actually a group equivariant homeomorphism with the operator bundles defined above.

Lemma 14 (bundle map). *If a point $X_0 \in S$ satisfies the spectral gap condition, then there is a neighborhood $V \subset S$ such that V is homeomorphic to a neighborhood in the product of lower-rank matrices and the Grassmannian, i.e.,*

$$\phi : V \cong V(k) \times V(n-k) \times V_{\mathrm{Gr}} \subset S(k) \times S(n-k) \times \mathrm{Gr}_{\mathbb{C}}(n,k),$$

which is contained in $P_k \oplus P_{n-k}$ as defined in Definition 13. Moreover, $\mathrm{U}(n) \cdot V$ *is homeomorphic to a neighborhood $W \subset P_k \oplus P_{n-k}$ such that $\pi(W) = \mathrm{Gr}_{\mathbb{C}}(n,k)$ and the map ϕ is* $\mathrm{U}(n)$*-equivariant.*

Proof. From the proof of Lemma 11, there is a neighborhood $X_0 \in U \subset S$, such that each element $X \in U$ is decomposed into $L_X + R_X$. Moreover, this induces a decomposition of the trivial bundle $U \times \mathbb{C}^n$ into two subbundles:

$$U \times \mathbb{C}^n = E^k \oplus E^{n-k}, \tag{17}$$

where $E^k(X)$ and $E^{n-k}(X)$ are determined by the projection operator P_X defined in equation (9):

$$E^k(X) = \mathrm{Im}(P_X) \quad \text{and} \quad E^{n-k}(X) = \mathrm{Im}(P_X)^{\perp}. \tag{18}$$

Let $(\xi_1, \ldots, \xi_k)$ be the basis for $E^k(X_0)$. E^k over U is an open neighborhood in $\mathrm{Gr}_{\mathbb{C}}(n, k)$. We can find a neighborhood V of X_0 (possibly smaller than U) such that, for every point in V, the k-dimensional space E^k projects onto $E^k(X_0)$. And an orthonormal basis of $E^k(X)$ is uniquely determined by requiring the projection of the first j vectors to $E^k(X_0)$ spans $(\xi_1, \ldots, \xi_j)$ for every j smaller than k. In this way we find a basis for each fiber of E^k and E^k is trivialized to be a k-dimensional vector bundle on V. Since the action of X on $\mathbb{C}^n$ has been decomposed to L_X and R_X, then with the choice of basis, the action of L_X on $E^k(X)$ gives a $k \times k$ self-adjoint matrix, and by continuity, these matrices form a neighborhood V_k in $S(k)$. And the same argument works for R_X.

Therefore, we have the following map ϕ:

$$\begin{aligned} \phi : V &\to P_k \oplus P_{n-k} \\ X &\mapsto (E^k(X), (L_X|_{E^k(X)}, R_X|_{E^{n-k}(X)})). \end{aligned} \tag{19}$$

We show this map is a homeomorphism between V and $\phi(V)$. It is injective since the actions of the two invariant subspaces uniquely determine the action on $\mathbb{C}^n$, therefore give the unique operator X. Surjectivity is easy to see. The continuity of ϕ and ϕ^{-1} comes from the continuity of the projection operator defined in Theorem 6.

Now take $\mathrm{U}(n) \cdot V$. Since E^k takes every possible k-subspace of $\mathbb{C}^n$ under the action of $\mathrm{U}(n)$, we know that the first entry of $\phi(\mathrm{U}(n) \cdot V)$ maps onto $\mathrm{Gr}_{\mathbb{C}}(n, k)$. Moreover, since the decomposition respects the action of $\mathrm{U}(n)$, it is easily seen that, for $g \in \mathrm{U}(n)$, $X \in \mathrm{U}(n) \cdot V$,

$$\phi(g \cdot X) = (g \cdot E^k(X), (g \circ L_X \circ g^{-1}, g \circ R_X \circ g^{-1})) = g \cdot (\phi(X)), \tag{20}$$

which means the map is $\mathrm{U}(n)$-equivariant. □

To do the induction, we will need to define an index on the inclusion of isotropy types, so the blow-up procedure could be done in the partial order given by the index. Recall that two matrices have the same isotropy type if they have the same "clustering" of eigenvalues. Now we define the isotropy index of a matrix X as follows.

Definition 15 (Isotropy index). Suppose the eigenvalues of a matrix X are

$$\lambda_1 = \cdots = \lambda_{i_1} < \lambda_{i_1+1} = \cdots = \lambda_{i_2} < \lambda_{i_2+1} = \cdots < \lambda_{i_{k-1}+1} = \cdots = \lambda_n.$$

Then the isotropy index of X is defined as the set

$$I(X) = \{i_0 = 0, i_1, i_2, \ldots, i_{k-1}, i_k = n\}.$$

We denote the set of all matrices with the same isotropy index I as S^I.

There is a partial order of this index on S given by the inclusion. That is, if for two matrices X and Y we have $I(X) \subset I(Y)$, then we say that the order is $X \leq Y$. Note there is an inverse inclusion for isotropy groups. The smallest isotropy index is $I(\lambda I) = \{0, n\}$, while the isotropy group is $\mathrm{U}(n)$, which is the largest. And the largest index is $\{0, 1, 2, \ldots, n-1, n\}$, which corresponds to n distinct eigenvalues, and the isotropy group is the product of n copies of $\mathrm{U}(1)$.

Remark 16. Except the most singular stratum $\{\lambda I\}$, the stratum of other isotropy types are not closed. In fact, the closure of a stratum S^I will include all the stratum $S^{I'}$ with $I' \subset I$. However, the two sets $\{\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_k}\}$ and $\{\lambda_{j_1} = \cdots = \lambda_{j_l}\}$ are transversal once the set $\{\lambda_{\min\{i_1, j_1\}} = \cdots = \lambda_{\max\{i_k, j_l\}}\}$ is blown up. So one can get $\widehat{S}_s$ by blowing up these singular stratum by order of strict inclusion. However, in order to globally decompose the eigenbundle, one needs to blow-up all the intersections first as in $\widehat{S}_r$ (the proof is given later).

For $\widehat{S}_r$ and $\widehat{S}_p$, the total blow-up of $S(n)$ is done by iteratively blowing up the singular strata by the order of isotropy indices. The first step is to blow-up the most singular stratum $S^{\{0,n\}} = \{\mathbb{R}I\}$:

$$[S(n);\, S^{\{0,n\}}].$$

After that we blow-up the second smallest strata $S^{\{0,i,n\}}$, $i = 1, \ldots, n-1$. From the discussion above we know that, for any of such two strata, the intersection of their closure is exactly $S^{\{0,n\}}$ which has been blown up. Therefore one can blow-up these $S^{\{0,i,n\}}$ in any order:

$$\left[S(n);\, S^{\{0,n\}};\, \bigcup_{i=1}^{n-1} S^{\{0,i,n\}}\right].$$

After the second step, the intersection of any two $S^{\{0,i,j,n\}}$ has been blown up. Therefore one can proceed by blowing up those strata in any order. Iteratively, one obtains the following space:

$$\left[S(n);\, S^{\{0,n\}};\, \bigcup_{i=1}^{n-1} S^{\{0,i,n\}};\, \bigcup_{i,j} S^{\{0,i,j,n\}};\, \ldots;\, \bigcup_{0 \leq i_1 < i_2 < \cdots < i_{n-2} \leq n} S^{\{0,i_1,\ldots,i_{n-2},n\}}\right]. \tag{21}$$

In order to do the inductive proof to show this yields the full eigenresolution, the last lemma we need is the compatibility of conjugacy class inclusion and the decomposition to two submatrices, which shows the order of resolution is compatible with the decomposition.

Lemma 17 (Compatibility with conjugacy class). *The partial order of conjugacy class inclusion is compatible with the decomposition in Lemma 11.*

Proof. Suppose a neighborhood $V \subset S(n)$ has a decomposition as Lemma 11. We need to show that, if S^I is the stratum of minimal isotropy type in V, then this stratum corresponds to the minimal isotropy type in $\mathrm{U}(k)$ and $\mathrm{U}(n-k)$.

Since V satisfies the spectral gap condition, the isotropy groups for any elements in V would be subgroups of $\mathrm{U}(k) \oplus \mathrm{U}(n-k)$. Suppose the minimal stratum corresponds to the index $I = \{0, i_1, \dots, i_m\}$ which must contain k as one element because of the spectral gap condition. Then the isotropy type of two subgroups are $\{0, i_1, \dots, k\}$ and $\{i_j - k = 0, i_{j+1} - k, \dots, n-k\}$. They would still be the minimal in each subgroup, otherwise when the two smallest elements are combined it will give a smaller index than I, which is a contradiction. $\square$

Now we can finally prove Theorem 6 using the above lemmas.

Proof of Theorem 6. We prove the theorem by induction on the matrix size. Except special remarks, the discussion below about $\widehat{S}$ applies to all three kinds of resolutions. The 2×2 case is shown in Lemma 8. Suppose the claim holds for all the cases up to $n-1$ dimensions. Now we claim that, by an iterative blow-up, we can get $\widehat{S}(n)$ with eigenvalues and eigenbundles lifted to satisfy the eigenresolution properties.

As in the 2×2 example, we shall first consider the trace-free slice $S_0(n)$ since other slices have the same behavior in terms of smoothness of eigenvalues and eigenbundles, that is, $\widehat{S}(n) = \widehat{S}_0(n) \times \mathbb{R}$. Take the smallest index $I = \{0, n\}$ with the largest possible isotropy group $\mathrm{U}(n)$, and the stratum in $S_0(n)$ with such an isotropy group is a single point, the zero matrix. After blowing up, we get $[S_0; \{0\}]$ as the first step. And in the total $S(n)$ space, this step corresponds to $[S; S^{\{0,n\}} = \{\mathbb{R}I\}] = [S_0; \{0\}] \times \mathbb{R}$.

For any other point $X \notin \{\mathbb{R}I\}$, one can find a bounded neighborhood W such that the matrices in W have a spectral gap as defined in Definition 10. Assume the first k eigenvalues are uniformly bounded below c, then by Lemma 14 there is a fibration structure

$$
\begin{tikzcd}
V(k) \times V(n-k) \arrow[r] & W \arrow[d, "\pi"] \\
& \mathrm{Gr}_{\mathbb{C}}(n,k).
\end{tikzcd} \tag{22}
$$

And the trivial bundle $W \times \mathbb{C}^n$ naturally splits to the sum $E^k \oplus E^{n-k}$ as in (17). Because of the spectral gap, there are two smallest strata of type $\{\lambda_{i_1} = \cdots = \lambda_{i_j}\}$ and $\{\lambda_{i'_1} = \cdots = \lambda_{i'_j}\}$, with $i_j \leq k$ and $i'_1 \geq k+1$, therefore the two strata are transversal as discussed in the Remark 16, and can be blown up at the same time. This give the iteration step for $\widehat{S}_s$.

Now we consider the radial and projective resolution. For each fiber of π in (22), consider the resolved space $\widehat{V}(k) \times \widehat{V}(n-k) \subset \widehat{S}(k) \times \widehat{S}(n-k)$, where the resolution is done by blowing up all the singular stratum *inside* $V(k)$ and $V(n-k)$. By induction the resolution $\widehat{V}(k)$ resolves the singularity for the first k eigenvalues, and $\widehat{V}(n-k)$ resolves the other $n-k$ eigenvalues. For example, take a point $X \in S(5)$ with eigenvalues $\{\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 = \lambda_5\}$. Near this point there is a product decomposition $V(2) \times V(3) \times \mathrm{Gr}_{\mathbb{C}}(5, 2)$. After the resolution, $\widehat{V}(2) \times \widehat{V}(3)$ resolves the isotropy type $(\{0, 2\} \cup \{0, 1, 2\}) \times (\{0, 3\} \cup \{0, 1, 3\} \cup \{0, 2, 3\} \cup \{0, 1, 2, 3\})$, which, after adjusting numbering of eigenvalues, includes all the isotropy types that could occur with this spectral gap in W. Let $\widehat{W}$ be the this resolved space and denote the blow-down map as

$$\begin{tikzcd}
\beta : \widehat{W} \arrow[r] \arrow[dr, "\hat{\pi}"] & W \arrow[d, "\pi"] \\
 & \mathrm{Gr}_{\mathbb{C}}(n, k).
\end{tikzcd}$$

Consider the two subbundles E^k and E^{n-k} under the pullback map from β:

$$\begin{tikzcd}
\widehat{E}^k \oplus \widehat{E}^{n-k} \arrow[r, "\beta"] \arrow[d, "\hat{\phi}"] & E^k \oplus E^{n-k} \arrow[d, "\phi"] \\
\widehat{W} \arrow[r, "\beta"] & W.
\end{tikzcd} \tag{23}$$

By the induction assumptions, $\widehat{V}(k)$ and $\widehat{V}(n-k)$ are eigenresolutions, hence $\widehat{E}^k$ splits into line bundles $\bigoplus_{i=1}^k E_i$ over $\widehat{V}(k)$ and the same for $\widehat{E}^{n-k} = \bigoplus_{i=k+1}^n E_i$ over $\widehat{V}(n-k)$. With the local product structure of π, the Whitney sum $\widehat{E}^k \oplus \widehat{E}^{n-k}$ splits into n eigenbundles locally.

For the radial resolution $\widehat{S}_r$, since the local product structure is $\mathrm{U}(n)$-equivariant, extending to $\bigoplus_{i=1}^n \mathrm{U}(n) \cdot E_i$, we get that the splitting of eigenbundles is global over $\widehat{W}$. We have already shown in Lemma 8 that the projective resolution does not give a global eigendecomposition. Similarly, for the small resolution $\widehat{S}_s$, one can find a closed curve in the base such that one eigenvector switches to another around the curve. We prove this by giving an example: consider the curve of 4×4 matrices of the form $X(t) = U(t)\Lambda(t)U(t)^{-1}$, $0 \leq t \leq 1$, where $U(t)$ is unitary for all t, switching from the identity to its column permutation,

$$U(t) = \begin{cases} (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) & 0 \le t \le \frac{1}{3} \\ U(t) & \frac{1}{3} \le t \le \frac{2}{3} \\ (\vec{e}_3, \vec{e}_4, \vec{e}_1, \vec{e}_2) & \frac{2}{3} \le t \le 1 \end{cases},$$

which smoothly permutes the eigenspace decomposition. On the other hand, $\Lambda(t)$ is always diagonal, going through $\{\lambda_1 = \lambda_2\}$ and $\{\lambda_3 = \lambda_4\}$:

$$\Lambda(t) = \begin{cases} \operatorname{diag}\{-1, -1, 1, 1\} & t = 0 \\ \operatorname{diag}\{-1, -1, 1-t, 1+t\} & 0 \le t \le \frac{1}{3} \\ \operatorname{diag}\{-1-t, -1+t, \frac{1}{3}+t, \frac{5}{3}-t\} & \frac{1}{3} \le t \le \frac{2}{3} \\ \operatorname{diag}\{-2+t, -t, 1, 1\} & \frac{2}{3} \le t \le 1 \\ \operatorname{diag}\{-1, -1, 1, 1\} & t = 1 \end{cases}.$$

With $X(t)$ defined above, one can see that $X(0) = X(1)$ in the stratum that is not blown up in $\widehat{S}_s$. Now consider the lift of the curve to $\widehat{S}_s$, which is still a closed curve. Now one can immediately see that as t goes from 0 to 1, the eigenspace for the first two eigenvalues switches from $\{e_1, e_2\}$ to $\{e_3, e_4\}$. So one cannot obtain a global decomposition.

Even though the eigenbundles do not always split, the three resolutions all resolve eigenvalues. Since the blow-down map β is injective on a dense open set, the eigenvalues extend to the front face to be n smooth functions f_i on $\widehat{W}$ and the splitting of eigendata extends to $\widehat{E}^{n-k} \oplus \widehat{E}^{n-k}$ from nearby such that

$$\beta(X)v_i = f_i(X)v_i \quad \text{for all } v_i \in E_i(X) \quad \text{for all } X \in \widehat{W}.$$

According to Lemma 17 the isotropy index order is preserved when decomposed into two subspaces. By induction, to obtain the global eigenresolution, we have iteratively blown up the strata according to isotropy indices to get $\widehat{S}_r$ as in (21). □

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References

[Albin and Melrose 2011] P. Albin and R. Melrose, "Resolution of smooth group actions", pp. 1–26 in *Spectral theory and geometric analysis*, edited by M. Braverman et al., Contemp. Math. **535**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Baum et al. 1985] P. Baum, J.-L. Brylinski, and R. MacPherson, "Cohomologie équivariante délocalisée", *C. R. Acad. Sci. Paris Sér. I Math.* **300**:17 (1985), 605–608. MR Zbl

[Borel and Ji 2006] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser, Boston, 2006. MR Zbl

[Davis 1978] M. Davis, "Smooth G-manifolds as collections of fiber bundles", *Pacific J. Math.* **77**:2 (1978), 315–363. MR Zbl

[Duistermaat and Kolk 2000] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Springer, 2000. MR Zbl

[Hassell et al. 1995] A. Hassell, R. Mazzeo, and R. B. Melrose, "Analytic surgery and the accumulation of eigenvalues", *Comm. Anal. Geom.* **3**:1-2 (1995), 115–222. MR Zbl

[Kurdyka and Paunescu 2008] K. Kurdyka and L. Paunescu, "Hyperbolic polynomials and multiparameter real-analytic perturbation theory", *Duke Math. J.* **141**:1 (2008), 123–149. MR Zbl

[Melrose 1996] R. B. Melrose, "Differential analysis on manifolds with corners", book in progress, 1996, available at http://math.mit.edu/~rbm/book.html.

[Rainer 2013] A. Rainer, "Perturbation theory for normal operators", *Trans. Amer. Math. Soc.* **365**:10 (2013), 5545–5577. MR Zbl

[Rauch 1980] J. Rauch, "Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians", *J. Funct. Anal.* **35**:3 (1980), 304–315. MR Zbl

[Rellich 1937] F. Rellich, "Störungstheorie der Spektralzerlegung", *Math. Ann.* **113**:1 (1937), 600–619. MR Zbl

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