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MARIO EUDAVE-MUÑOZ AND GRISSEL SANTIAGO-GONZÁLEZ

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KNOTS OF TUNNEL NUMBER ONE AND MERIDIONAL TORI

MARIO EUDAVE-MUÑOZ AND GRISSEL SANTIAGO-GONZÁLEZ

We give a characterization of knots of tunnel number 1 that admit an essential meridional torus with two boundary components. Let K be a knot in S^3, S an essential meridional torus in the exterior of K with two boundary components, and τ an unknotting tunnel for K. We consider the intersections between S and τ. If the intersection is empty, we conclude that the knot K is an iterate of a satellite knot of tunnel number 1 and one of its unknotting tunnels, and then S is knotted as a nontrivial torus knot. If the intersection is nonempty, we simplify it as much as possible, and conclude that the knot K is a $(1,1)$-knot; it follows from known results that in some cases the torus S is knotted as a nontrivial torus knot, while in others cases the torus S is unknotted.

1. Introduction

An important topic in knot theory is that of studying incompressible surfaces in the exterior of knots. We first make a summary of known results for incompressible surfaces for knots of tunnel number 1. There is a classification of satellite knots of tunnel number 1 in S^3, that is, knots that admit in their exterior an incompressible non-∂-parallel torus; this was given by K. Morimoto and M. Sakuma [1991]. Another proof of this classification was given by M. Eudave-Muñoz [1994]. All these knots are $(1,1)$-knots, that is, knots of 1 bridge with respect to a standard torus in S^3; this is a special class of knots of tunnel number 1. Gordon and Reid [1995] proved that knots of tunnel number 1 do not admit any essential planar meridional surface. Regarding surfaces of higher genus, Eudave-Muñoz [1999; 2006] showed that for any $g \geq 2$, there are infinitely many knots of tunnel number 1 whose exterior contains a closed meridionally incompressible surface of genus g, and gave a characterization of $(1,1)$-knots that admit surfaces of this kind. In [Eudave-Muñoz 2000], he showed that for each pair of integers $g \geq 1$ and $n \geq 1$, there are knots k of tunnel number 1 such that there is an essential meridional surface S in the exterior of k, of genus g, and with $2n$ boundary components. Eudave-Muñoz and

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E. Ramírez-Losada [2009] have given a general construction and characterization of $(1,1)$-knots that admit essential meridional surfaces.

In this paper we give a characterization of knots of tunnel number 1 that admit an essential meridional torus with two boundary components. Such knots are either $(1,1)$-knots, and then come from the construction of Eudave-Muñoz and Ramírez-Losada, or are iterates of a satellite knot of tunnel number 1 and one of its unknotting tunnels, i.e., they come from the construction of [Eudave-Muñoz 2000].

In Section 2 we give definitions and statements of the main results. In Section 3 we prove some general lemmas about unknotting tunnels, and in Section 4 we give a proof of the main results.

2. Preliminaries

Let k be a knot in S^3, and denote by $E(k)$ the exterior of k, that is, $E(k) = S^3 - \operatorname{int} N(k)$, where $N(k)$ is a tubular neighborhood of k.

Definition. Let k be a knot in S^3. A surface S properly embedded in $E(k)$ is said to be meridional if ∂S consists of a nonempty collection of meridian curves in $\partial N(k)$.

Definition. Let k be a knot in S^3 and S a surface properly embedded in $E(k)$, which is meridional or disjoint from $\partial N(k)$. We say that S is meridionally compressible in (S^3, k) if there is a disc $D \subset S^3$ such that $D \cap S = \partial D$, D intersects k transversely in one point, and ∂D is essential in S, that is, ∂D does not bound a disc in S and it is not parallel in S to a component of ∂S. The disc D is called a meridional compression disc for S. We say that S is meridionally incompressible in (S^3, k) if S is incompressible and not meridionally compressible in (S^3, k). We say that a meridional surface S is essential if it is meridionally incompressible and not ∂-parallel in $E(k)$.

A meridional surface can be seen as a closed surface $\overline{S}$ in S^3 which a knot intersects transversely in finitely many points. When we say that $\overline{S}$ is a meridional essential surface that intersects a knot k in n points, this means that the surface $S = \overline{S} \cap E(k)$ is a meridional essential surface in $E(k)$ as detailed in the two preceding definitions.

Definition. A knot k in S^3 has tunnel number 1 if there exists an arc τ embedded in S^3 with $\tau \cap k = \partial\tau$, such that $E(k \cup \tau) = S^3 - \operatorname{int} N(k \cup \tau)$ is a genus 2 handlebody. We call τ an unknotting tunnel for k.

Sometimes it is convenient to express a tunnel τ for a knot k as $\tau = \tau_1 \cup \tau_2$, where τ_1 is a simple closed curve in $E(k)$ and τ_2 is an arc in $E(k)$ connecting τ_1 and $\partial N(k)$; by sliding the tunnel we can pass from one expression to the other.

Definition. A knot k in S^3 is a $(1,1)$-knot if there is a standard torus T in S^3 such that k is a 1-bridge knot with respect to T, that is, k intersects T transversely in two points which divide k into two arcs which are parallel to arcs lying on T.

It is not difficult to see that a $(1,1)$-knot k is a knot of tunnel number 1. An unknotting tunnel for k can be seen as $\tau = \tau_1 \cup \tau_2$, where τ_1 is the core of one of the solid tori bounded by T, and τ_2 is a straight arc in that solid torus connecting k and τ_1. Conversely, we have the following result. Though it is well known, we include it for completeness.

Lemma 2.1. *If k is a knot with an unknotting tunnel $\tau = \tau_1 \cup \tau_2$, where τ_1 is a trivial knot in S^3, then k is a $(1,1)$-knot.*

Proof. Note that $E(\tau_1)$ is a solid torus. Slide k over τ_2, until it is an arc k' properly embedded in $E(\tau_1)$. The manifold $E(\tau_1) - \operatorname{int} N(k') \cong E(k \cup \tau)$ has compressible boundary, for it is a handlebody. If every compression disc for $E(\tau_1) - \operatorname{int} N(k')$ intersects a meridian of k', then the manifold obtained by adding a 2-handle along a meridian of k' would have incompressible boundary, by Jaco's addition lemma [1984]. But this is not possible, for the manifold obtained is $E(\tau_1)$, which is a solid torus. Then there is a compression disc disjoint from k'. By compressing along this disc, we get that k' is inside a 3-ball, and then it must be parallel to an arc contained in $\partial E(\tau_1)$. It follows that k can be expressed as $k = k' \cup k''$, where k' is an arc properly embedded in $E(\tau_1)$, and parallel to an arc lying on $\partial E(\tau_1)$, and k'' is an arc contained in $\partial E(\tau_1)$. It follows that k is a 1-bridge knot with respect to the torus $\partial E(\tau_1)$. □

Morimoto and Sakuma's construction [1991] of satellite tunnel number 1 knots is as follows: Let $T(p,q)$ be a torus knot of type (p,q) in S^3, with $|p| \geq 2$, $q \geq 2$, and let $S(\alpha,\beta)$ be a 2-bridge link in S^3 of type (α,β), with $\alpha \geq 4$; that is, $S(\alpha,\beta)$ is neither a trivial link nor a Hopf link. Identify $\partial E(T(p,q))$ and a component of $\partial E(S(\alpha,\beta))$, in such a way that a meridian of $E(S(\alpha,\beta))$ is glued to a fiber of the Seifert fibration of $E(T(p,q))$. The result is the exterior of a satellite knot $K(\alpha,\beta\,;p,q)$ with companion a torus knot, which has tunnel number 1.

The knots $K(\alpha,\beta\,;p,q)$ can also be described in the following way; see [Eudave-Muñoz 1994]. Let T be a standard torus in S^3, and let $A_{p,q} \subset T$ be an annulus so that a component of ∂A is a curve of slope (p,q) on T, $|p| \geq 2$, $q \geq 2$. We say that a knot k belongs to the class of knots $\mathcal{T}$ if k has a 1-bridge presentation with respect to some annulus $A_{p,q}$, that is, k is a 1-bridge knot with respect to T, such that the intersection points of k with T lie in $A_{p,q}$, and the arcs of k are parallel to arcs on $A_{p,q}$. If k is in $\mathcal{T}$ then k can be isotoped to lie in $N(A_{p,q})$, for some $A_{p,q}$. Let $S_{p,q} = \partial N(A_{p,q})$. For any such k that is neither trivial nor the (p,q)-torus knot, the torus S will be essential in the exterior of k. It can be seen that k belongs to $\mathcal{T}$ if and only if it is one of the knots $K(\alpha,\beta\,;p,q)$.

Now we describe the unknotting tunnels for a knot in $\mathcal{T}$. The torus T divides S^3 into two solid tori R_1 and R_2. Let k be a knot in $\mathcal{T}$, such that k has a 1-bridge presentation with respect to an annulus $A_{p,q}$; so $k \subset N(A_{p,q})$. Then k is divided into two arcs k_1 and k_2, which are trivial arcs in R_1 and R_2, respectively. We can consider R_1 as foliated by concentric tori around the core of R_1, and then k_1 as an arc that intersects each of the tori in two or zero points, except for one torus which is tangent to k_1, defining a maximum point in k_1. Similarly we define a minimum of k_2 in R_2. By a straight arc in R_1 or R_2, we mean an arc that intersects each torus in the foliation in at most one point. Take a straight arc ρ_1 which goes from the maximum of k_1 to a point x on $S_{p,q}$. Similarly, take a straight arc ρ_2 which goes from the minimum of k_2 to a point y on $S_{p,q}$. Let ρ_3 be an arc in $S_{p,q}$ joining x and y, which crosses T in one point, and which is disjoint from a meridian of $N(A_{p,q})$. Let τ_x be the union of the core of the solid torus R_1 and a straight arc joining the point x and the core of R_1. Similarly, let τ_y be the union of the core of the solid torus R_2 and a straight arc joining the point y and the core of R_2. Note that τ_x and τ_y are unknotting tunnels for the exterior of $N(A_{p,q})$, that is, for the torus knot $T(p,q)$.

Now define $\tau(1,x) = \tau_x \cup \rho_1$, $\tau(2,x) = \tau_x \cup \rho_3 \cup \rho_2$, $\tau(2,y) = \tau_y \cup \rho_2$, $\tau(1,y) = \tau_y \cup \rho_3 \cup \rho_1$. It is not difficult to see that each of these 1-complexes is an unknotting tunnel for k. Furthermore, it follows from [Morimoto and Sakuma 1991], that if τ is an unknotting tunnel for k, then k is one of the tunnels $\tau(1,x)$, $\tau(2,x)$, $\tau(2,y)$, $\tau(1,y)$, up to homeomorphism of $E(k)$. In the same paper, all unknotting tunnels for k up to ambient isotopy of $E(k)$ are also classified. Here we only need the classification up to homeomorphism of $E(k)$, because if two tunnels are homeomorphic, though not isotopic, they will produce the same family of knots when taking iterates of the knot and the tunnels.

Definition. Let k be a knot of tunnel number 1, and τ an unknotting tunnel for k which is an embedded arc with endpoints lying on $\partial N(k)$. Let k^* be a knot formed by the union of two arcs, $k^* = \tau \cup \gamma$, such that γ is contained in $\partial N(k)$. We say that k^* is an iterate of k and τ.

The knot k^* is also a knot with tunnel number 1, where the tunnel is given by the union of k and a straight arc in $N(k)$ connecting k^* and k.

Eudave-Muñoz [2000] showed that there are knots k of tunnel number 1 for which there is an essential meridional torus S in the exterior of k, with two boundary components. These are constructed by taking iterates of satellite knots of tunnel number 1. Here, we recall this construction.

Let k be a satellite knot of tunnel number 1 in S^3. Let $\overline{S}$ be the essential torus lying in the exterior of k as defined above, so $\overline{S}$ is knotted as a torus knot. Let τ be any of the unknotting tunnels $\tau(1,x)$, $\tau(2,x)$, $\tau(2,y)$, $\tau(1,y)$ for k defined above. Note that τ can be expressed as $\tau = \tau_1 \cup \tau_2$, where τ_1 is a simple closed curve, and τ_2 is an arc with endpoints in $\partial N(k)$ and τ_1, such that τ_1 is disjoint from $\overline{S}$ and τ_2 intersects $\overline{S}$ transversely in one point. The torus $\overline{S}$ divides S^3 into two parts, denoted by M_1 and M_2, where, say, k lies in M_2.

Note that $M_2 \cap N(\tau_2)$ is a cylinder $R \cong D^2 \times I$, such that $R \cap \overline{S}$ is a disc $D_1 = D^2 \times \{1\}$, and $R \cap N(k)$ is a disc $D_0 = D^2 \times \{0\}$. Slide τ_1 over τ_2, to get an arc τ with both endpoints in $D_0 \subset \partial N(k)$, such that $\tau \cap M_2$ consists of two straight arcs contained in R, i.e., arcs which intersect each disc $D_2 \times \{x\}$ transversely in one point. The surface $\overline{S}$ and the arc τ then intersect in two points.

Let k^* be an iterate of k and τ as in the previous definition. So $k^* = \tau \cup \lambda$, where λ is contained in $\partial N(k)$. The torus $\overline{S}$ and the knot k^* then intersect in two points. Push the interior of λ into the interior of $N(k)$, such that now λ is a properly embedded arc in $N(k)$ whose endpoints lie in D_0. Recall that the wrapping number of a knot in a solid torus is defined as the minimal number of times that the knot intersects any meridional disc of such a solid torus. We define the wrapping number of the arc λ in $N(k)$ as the wrapping number of the knot obtained by joining the endpoints of λ with an arc in D_0, and then pushing it into the interior of $N(k)$. This is well defined.

Let $\mathcal{D}$ be the family of knots constructed as above and such that any one of the following conditions is satisfied:

(1) k is not a cable knot, and the wrapping number of λ in $N(k)$ is ≥ 2.

(2) Suppose k is a cable knot. Let A be the annulus spanned by k and $\overline{S}$; that is, $A \subset M_2$, then one boundary component of A is in $\partial N(k)$ and the other is a curve on $\overline{S}$. We can assume that the part of τ lying in M_2 is contained in A. Let $B = \partial N(k) \cap N(A)$; this is an annulus in $\partial N(k)$. Assume that $D_0 \subset B$. In this case we assume that the wrapping number of λ in $N(k)$ is ≥ 2, and that the arc λ cannot be isotoped, relative to D_0, to an arc lying in B.

(3) The wrapping number of λ in $N(k)$ is 1. Embed the solid torus $N(k)$ in S^3 in a standard manner. Let $\widehat{\lambda}$ be the knot obtained by joining the endpoints of λ with an arc lying in D_0. The image of this knot in S^3 is a $(1,1)$-knot, in fact a 2-bridge knot. In the present case assume that $\widehat{\lambda}$ is a nontrivial 2-bridge knot.

Note that if none of the above conditions is satisfied then the torus $\overline{S}$ will be compressible in $E(k^*)$.

Theorem 2.2. *Let k^* be a knot in the family $\mathcal{D}$. Then k^* is a knot of tunnel number 1 and $\overline{S}$ is an essential meridional torus which intersects k^* in two points.*

Proof. The knot k^* has tunnel number 1 because it is an iterate of k and τ. By construction k^* intersects $\overline{S}$ in two points. If conditions (1) or (2) are satisfied, then $\overline{S}$ is essential by [Eudave-Muñoz 2000, Theorem 2.1]. If condition (3) is satisfied, then note that k^* is also a $(1,1)$-knot, and then $\overline{S}$ is essential by [Eudave-Muñoz and Ramírez-Losada 2009]. $\square$

Eudave-Muñoz and Ramírez-Losada [2009] have given a general construction of $(1,1)$-knots that admit essential meridional surfaces. In particular, there are three families of knots, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, that consist of $(1,1)$-knots that admit an essential meridional torus intersecting the knot in two points. For two of these families, $\mathcal{A}$ and $\mathcal{B}$, the essential torus is knotted as a torus knot, while for the family $\mathcal{C}$, the essential torus is unknotted. Furthermore, if a $(1,1)$-knot k admits an essential torus intersecting it in two points, then k belongs to one of these families.

In this paper we prove the following:

Theorem 2.3. *Let k be a knot of tunnel number* 1, $\bar{S}$ *a meridional essential torus which intersects the knot in two points, and $\tau = \tau_1 \cup \tau_2$ an unknotting tunnel for k. Then one of the following happens*:

(1) *k is a* $(1,1)$*-knot*; *or*

(2) $\bar{S} \cap \tau = \varnothing$, *and*

   (a) *$\bar{S}$ is knotted as a nontrivial torus knot*,

   (b) *the knot τ_1 is a satellite knot of tunnel number* 1, *and*

   (c) *k is an iterate of τ_1 and of an unknotting tunnel for τ_1.*

From Theorem 2.3 and the results of [Eudave-Muñoz and Ramírez-Losada 2009] we get:

Corollary 2.4. *Let $k \subset S^3$ be a knot of tunnel number* 1, $\bar{S}$ *a meridional essential torus which intersects the knot in two points. Then k belongs to one of the families $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$ defined above.*

3. Some unknotting lemmas

Let M be a compact, orientable, irreducible 3-manifold whose boundary is a torus T. Suppose τ is an unknotting tunnel for M, that is, τ is an arc properly embedded in M such that $H = M - \operatorname{int} N(\tau)$ is a genus 2 handlebody.

Proposition 3.1. *Suppose that τ has been slid* (*over T and over itself*), *in such a way that $\tau = \tau_1 \cup \tau_2$, where τ_1 is a simple closed curve in the interior of M and τ_2 is an arc joining T and τ_1. Suppose that there is no compression disc for T disjoint from τ. Then τ_1 cannot be contained in a* 3*-ball $B \subset M$.*

Proof. Suppose that τ_1 is contained in a 3-ball $B \subset M$. Let β be a curve on $\partial N(\tau)$ which is a cocore of the arc τ_2, i.e., β bounds a disc in $N(\tau)$ which intersects τ_2 in one point. There are two cases, either there is a compression disc for ∂H disjoint from β, or any compression disc intersects β.

Suppose first that D is a compression disc for ∂H disjoint from β. By isotoping D we can assume that ∂D lies in T or in $\partial N(\tau_1)$. If ∂D lies in T, then either T is compressible and there is a compression disc disjoint from τ, or there exists a disc $D' \subset T$, with $\partial D' = \partial D$, such that $D \cup D'$ bounds a 3-ball in which τ lies. If this happens then by cutting H along D we should get two solid tori, as H is a handlebody. But then M is a solid torus and there is a compression disc for T disjoint from τ. In both cases this establishes the proposition.

Hence, we may assume that ∂D lies on $\partial N(\tau_1)$. Let F be a copy of $\partial N(\tau)$ slightly pushed into the interior of H; this is a once-punctured torus properly embedded in H, whose boundary bounds a disc $D' \subset T$, which is a neighborhood of $\tau_2 \cap T$. We can assume that ∂D lies on F, and then by cutting F along D, we get a disc D'' with $\partial D'' = \partial F = \partial D'$. Note that $D'' \cup D'$ must bound a 3-ball in which τ lies. As before, this shows that there is a compression disc for T disjoint from τ.

Suppose now that any compression disc for ∂H must intersect β. By [Jaco 1984] or [Casson and Gordon 1987], it follows that by adding a 2-handle along β, we get an irreducible manifold with incompressible boundary. This is a contradiction, for what we get is $M - \operatorname{int} N(\tau_1)$, which is reducible, for we are assuming that τ_1 lies inside a 3-ball. This completes the proof. □

The next proposition is somehow natural, but it is not so easy to prove because of certain phenomena. If t_1 is a properly embedded arc in a product $T \times I$, where T is a torus, and $T \times I - \operatorname{int} N(t_1)$ is a handlebody, then by a result of Frohman [1989], t_1 is isotopic to a straight arc in $T \times I$. But if t_1 and t_2 are a pair of arcs properly embedded in $T \times I$ such that $T \times I - \operatorname{int} N(t_1 \cup t_2)$ is a handlebody, then t_1 and t_2 may not be straight arcs simultaneously in $T \times I$. Now, let t_1 be an arc properly embedded in $A \times I$, with endpoints in $A \times \{0\}$ and $A \times \{1\}$, where A is an annulus, such that $A \times I - \operatorname{int} N(t_1)$ is a handlebody. Then by Jaco's addition lemma [1984], t_1 is parallel to an arc lying in $\partial(A \times I)$, but it may not be a straight arc in $A \times I$.

Proposition 3.2. *Let M, T and τ be as above, and assume that T is incompressible. Let T' be a torus embedded in M which is parallel to T; that is, T and T' cobound a region homeomorphic to $T \times I$. Suppose that τ intersects T' in two points. Then $(T \times I) \cap \tau$ consists of two straight arcs in $T \times I$, that is, τ can be isotoped, without intersecting T' in more points, such that $(T \times I, (T \times I) \cap \tau) = (T \times I, \{x, y\} \times I)$, where $x, y \in T$.*

Proof. Let $M' = M - \operatorname{int} T \times I$. Then $M = M' \cup (T \times I)$, where $\partial M' = M' \cap (T \times I) = T'$. We have that $H = M - \operatorname{int} N(\tau)$ is a genus 2 handlebody. Note that the arc τ cannot be isotoped to be disjoint from T', for otherwise T' would be an incompressible torus in the handlebody H, which is not possible. Suppose that τ is divided into 3 arcs $\tau = k_1 \cup k_m \cup k_2$, such that $k_1, k_2 \subset T \times I$, and $k_m \subset M'$. Let $\widetilde{T}' = T' \cap H = T' - \operatorname{int} N(k_1) \cup N(k_2)$; this is a twice punctured torus properly embedded in H. Note that $\widetilde{T}'$ is incompressible in H, for otherwise T' would be compressible in M, or the arc τ could be isotoped to be disjoint from T'.

Let D be a compression disc for H. Assume that D and $\widetilde{T}'$ intersect transversely and that this intersection is minimal. Label the endpoints of the arcs of intersection between D and $\widetilde{T}'$ with 1 or 2, depending on whether the endpoint lies in $\partial N(k_1) \cap \widetilde{T}'$ or in $\partial N(k_2) \cap \widetilde{T}'$. Let γ be an outermost arc of intersection in D, then it bounds a disc $D' \subset D$, with $\partial D' = \alpha \cup \gamma$, where α is an arc on ∂H, and the interior of D' is disjoint from $\widetilde{T}'$.

There are several possibilities for the endpoints of the arc γ.

(1) The endpoints of γ are labeled 1 and 2, and α lies in $\partial N(k_m)$. This implies that the arc k_m is isotopic to an arc over T', and then τ can be isotoped to be disjoint from T', which is not possible.

(2) The endpoints of γ are labeled 1 and 2 and D' lies in $T \times I$. Then α is an arc that goes over $N(k_1)$, then over T, and then over $N(k_2)$. This shows that k_1 and k_2 is a pair of parallel arcs in $T \times I$. As H is a handlebody, by cutting H along the incompressible surface $\widetilde{T}'$ we get a pair of handlebodies; one of these is just $T \times I - \operatorname{int} N(k_1 \cup k_2)$. Note that the disc D' is properly embedded in $T \times I - \operatorname{int} N(k_1 \cup k_2)$. Then by cutting this handlebody with D' we get another handlebody, which is homeomorphic to $T \times I - \operatorname{int} N(k_1)$, for k_1 and k_2 are parallel in $T \times I$. This shows that $T \times I - \operatorname{int} N(k_1)$ is a handlebody, and then by a result of Frohman [1989], k_1 is isotopic to a straight arc in $T \times I$. As k_1 and k_2 are parallel, it follows that both are simultaneously straight in $T \times I$.

(3) The endpoints of γ are labeled 1 and 1 (or 2 and 2), and D' lies in $T \times I$. Then α is an arc that goes over $N(k_1)$, then over T, and then again over $N(k_1)$. The arc α cuts $\partial N(k_1)$ into two discs, let F be either of them. Then $D' \cup F$ is an annulus, in $T \times I$, with one boundary component in T and the other in T', and we can assume that k_1 is a spanning arc of the annulus. If γ is a trivial arc in $\widetilde{T}'$, then it bounds a disc $D'' \subset \widetilde{T}'$, such that k_2' intersects D''. But this would imply that k_2 is an arc parallel to k_1, and then, as in the previous case, k_1 and k_2 are simultaneously straight in $T \times I$. Therefore we can assume that the annulus $D' \cup F$ has to be isotopic to an annulus of the form $\delta \times I$, where δ is an essential simple closed curve in T. This shows that k_1 is a straight arc in $T \times I$.

If there is another outermost arc in D with endpoints labeled 2 and 2, then k_2 would also be a straight arc in $T \times I$, and because there would be two disjoint annuli containing k_1 and k_2 respectively, it would follow that both arcs are simultaneously straight arcs in $T \times I$. We can assume then that all outermost arcs in D have endpoints labeled 1 and 1, for otherwise we have finished.

Note that there is a pair of parallel arcs in D, one outermost with endpoints labeled 1 and 1, and one next to it with endpoints labeled 2 and 2. This is because

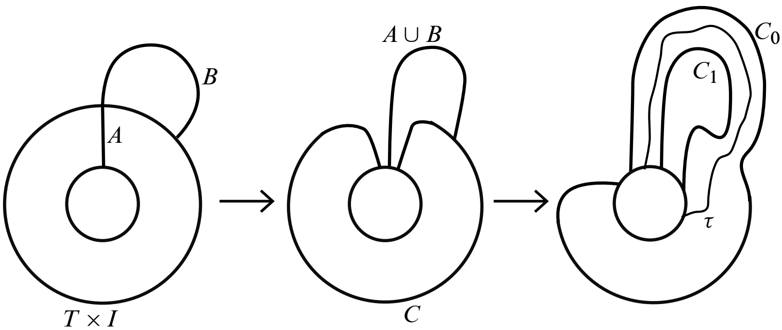

Figure 1. Constructing annuli disjoint from τ.

any outermost arc has endpoints labeled 1 and 1, and next to any label 1 there is a label 2. So assume that there are two arcs γ_1 and γ_2 in D, where γ_1 determines a disc D', with $\partial D' = \alpha \cup \gamma_1$, where α is an arc that goes over $N(k_1)$, then over T, and then again over $N(k_1)$. Then γ_1 is an arc on $\widetilde{T}'$ which goes from $N(k_1)$ to $N(k_1)$, and γ_2 is an arc on $\widetilde{T}'$ which goes from $N(k_2)$ to $N(k_2)$. The arcs γ_1 and γ_2 determine a disc $D'' \subset D$, such that $\partial D'' = \gamma_1 \cup \beta_1 \cup \gamma_2 \cup \beta_2$, where $\beta_1,\ \beta_2$ are arcs on $\partial N(k_m)$. The arc α cuts $\partial N(k_1)$ into two discs, let F be either of them. Then $D' \cup F$ is an annulus A, in $T \times I$, with one boundary component in T and the other in T'. Isotope A in $T \times I$ such that the arc k_1 is a spanning arc of A. The arcs $\beta_1,\ \beta_2$ cut $\partial N(k_m)$ into two discs, let F' be either of them. Then $D'' \cup F'$ is an annulus B, properly embedded in M'. Isotope B in M' such the arc k_m is a spanning arc of B. The annulus B is then incompressible and ∂-incompressible, for otherwise T' would be compressible or the arc k_m would be isotopic to an arc on T'. We can assume that A and B have a boundary component in common; then $A \cup B$ is an annulus, one of its boundaries components lies in T and the other in T'.

Take a product neighborhood $A \times I$ of A, where A is identified with $A \times \left\{\frac{1}{2}\right\}$. Consider the annulus $C = (T' - \partial A \times I) \cup (A \times \{0\}) \cup (A \times \{1\})$; note that C is properly embedded in M, it is ∂-parallel in M and it intersects τ in one point. Note that $A \cup B$ and C intersect in a simple closed curve, namely, the boundary component of $A \cup B$ lying in T'. Now take a product neighborhood $(A \cup B) \times I$ of $A \cup B$, where $A \cup B$ is identified with $(A \cup B) \times \left\{\frac{1}{2}\right\}$, which intersects C only in a neighborhood of the curve $(A \cup B) \cap C$. Consider the pair of annuli $C_0 \cup C_1 = (C - \partial(A \cup B) \times I) \cup ((A \cup B) \times \{0\}) \cup ((A \cup B) \times \{1\})$. Note that C_0 and C_1 are in fact a pair of annuli properly embedded in M, which are parallel in M, i.e., they cobound a product region $C_0 \times I$, where $C_0 = C_0 \times \{0\}$ and $C_1 = C_0 \times \{1\}$, such that τ is disjoint from C_0 and C_1, but it lies inside the product region $C_0 \times I$ (see Figure 1). Note that C_0 and C_1 are incompressible and ∂-incompressible in M, for these are just extensions of B via $T \times I$ to M. Then C_0 and C_1 are incompressible annuli in H, but they are ∂-compressible in H, for H is a handlebody. Then there is a disc E in H, such that $\partial E = \rho_0 \cup \rho_1$, where ρ_0 is a spanning arc of C_0, say, and ρ_1 lies on ∂H, and furthermore, $E \cap C_0 = \rho_0$, $E \cap C_1 = \varnothing$. The disc E must lie in $H' = C_0 \times I - \operatorname{int} N(\tau)$, for otherwise C_0 would be ∂-compressible in M. Note that H' is a handlebody, for it is one of the components obtained by cutting H along $C_0 \cup C_1$.

Take two parallel copies of E, and join them by the disc $\overline{C_0 - N(E)}$, and push the interior of the resulting disc into the interior of H'. We get a disc E', properly embedded in H', whose boundary is disjoint from $C_0 \cup C_1$. Note that $\partial E'$ is a nontrivial curve in $\partial H'$. Let ξ_0 and ξ_1 be the cores of the annuli C_0 and C_1 respectively. Note that E' is a compression disc for $\partial H' - \xi_0 \cup \xi_1$. Let J be a cocore of τ, that is, a curve in $\partial N(\tau)$ which bounds a disc in $N(\tau)$ intersecting τ in one point. Note that $C_0 \times I$ is the manifold obtained by attaching a 2-handle to H' along J. Then there is a compression disc E'' in $C_0 \times I$ which intersects $\xi_0 \cup \xi_1$ in two points, i.e., E'' is a 2-compression disc for $\partial(C_0 \times I)$ with respect to $\xi_0 \cup \xi_1$, as defined in [Wu 1992]. Then by Theorem 1 of that paper, there is a compression disc G for H' disjoint from J, which intersects $\xi_0 \cup \xi_1$ in at most two points. As ∂G is disjoint from J, we can assume that ∂G lies in $\partial(C_0 \times I)$. There are two possibilities for G, either ∂G is a meridian of $C_0 \times I$ intersecting each ξ_i once, or ∂G is a trivial curve in $\partial(C_0 \times I)$ intersecting ξ_0 or ξ_1 twice. In the latter case ∂G bounds a disc G' in $\partial(C_0 \times I)$ such that $G \cup G'$ is a sphere bounding a 3-ball which must contain τ. Then there is another 2-compression disc for $C_0 \times I$ which is a meridian of $C_0 \times I$. In any case, it follows that there is a meridian disc G of $C_0 \times I$, disjoint from $N(\tau)$. By cutting H' along this disc we get a solid torus. But by cutting $C_0 \times I$ along G, we get a 3-ball containing τ; it follows that τ is an unknotted arc in the 3-ball. This shows that τ is an arc parallel to an arc on C_0, and then that k_1 and k_2 are parallel straight arcs in $T \times I$. $\square$

4. Main proofs

In this section we give a proof of Theorem 2.3.

Proposition 4.1. *Let k be a knot of tunnel number* 1, *$\overline{S}$ a meridional essential torus which intersects the knot in two points, and $\tau = \tau_1 \cup \tau_2$ an unknotting tunnel for k, where τ_1 is a simple closed curve and τ_2 is an arc connecting τ_1 and $\partial N(k)$. Suppose that $\overline{S}$ and τ cannot be made disjoint. Then one of the following happens*:

(1) *τ_1 is a trivial knot*; *or*

(2) *there is a meridional essential torus $\overline{S}'$ which intersects the knot in two points, such that $\overline{S}' \cap \tau = \varnothing$, and such that $\overline{S}'$ bounds a solid torus with τ_1 as its core.*

Let k be a knot of tunnel number 1 and $\overline{S}$ a meridional essential torus which intersects k in two points. So $S = \overline{S} \cap E(k)$ is a meridional essential surface in

$E(k)$ whose boundary consists of two meridians of k. Let τ be an unknotting tunnel for k, so that τ may have been slid over itself so that it can be expressed as $\tau = \tau_1 \cup \tau_2$, where τ_1 is a simple closed curve and τ_2 is an arc connecting τ_1 and $\partial N(k)$. Let ν be the intersection point between τ_1 and τ_2.

Assume that S has been isotoped such that it intersects τ transversely in a finite number of points, say, S meets τ_1 in n points and τ_2 in m points, $n+m>0$, and that this intersection is minimal.

Denote by $\alpha_1, \alpha_2, \ldots, \alpha_n$ the discs of intersection between S and $N(\tau_1)$, numbered in order along τ_1 as they intersect S, starting at ν with an arbitrary choice of direction. Denote by $\beta_1, \beta_2, \ldots, \beta_m$ the discs of intersection between S and $N(\tau_2)$, numbered in order along τ_2 as they intersect S, starting at ν, going from ν to $\partial N(k)$. Denote by s_1 and s_2 the boundary components of S (or rather, the discs of intersection of $N(k)$ with $\overline{S}$).

Let $M = S^3 - \operatorname{int} N(k \cup \tau)$, so M is a genus 2 handlebody. Let $\widetilde{S} = S \cap M$.

Lemma 4.2. *$\widetilde{S}$ is incompressible in M.*

Proof. Suppose that $\widetilde{S}$ is compressible. Then there exists a compression disc E for $\widetilde{S}$, so that ∂E bounds a disc E' in S, because S is incompressible in $E(k)$, and E' must intersect τ. If we exchange E' by E we obtain a surface S' isotopic to S but with fewer intersections with τ, which cannot happen because the intersection of S and τ is minimal. $\square$

Let D be a compression disc of M. Assume D has been isotoped to intersect $\widetilde{S}$ transversely and that it has minimal intersection with $\widetilde{S}$ among all compression discs for M. If $D \cap \widetilde{S}$ contains a simple closed curve, an innermost disc argument can eliminate it, for $\widetilde{S}$ is incompressible. So we may assume that $D \cap \widetilde{S}$ consists of a collection of arcs. Note that any such arc of intersection is not ∂-parallel in $\widetilde{S}$, for otherwise, if an arc δ in $\widetilde{S}$ is ∂-parallel, then by cutting D with the disc determined by δ in $\widetilde{S}$, we get a compression disc of M with fewer intersections with $\widetilde{S}$, a contradiction.

Label the endpoints of the arcs of intersection in D with the labels of the discs of $S \cap N(\tau)$ or the component of ∂S in which the points lie. Parts of the proof of the following lemma are similar to that of Proposition 2.3 in [Eudave-Muñoz 1994]; we include here a proof for completeness.

Lemma 4.3. *The number n is 0. Further, if δ is any arc of intersection between D and $\widetilde{S}$, which is outermost in D, then both ends of δ have labels β_1, and the arc γ of ∂D determined by such an outermost arc wraps at least once around $N(\tau_1)$.*

Proof. Let δ be an outermost arc on D. Then δ cuts a disc $D' \subset D$ with $D' \cap \widetilde{S} = \delta$ and $\partial D' = \delta \cup \gamma$, where γ is an arc on $\partial N(k \cup \tau)$.

There are several possible cases for δ:

Case 1. One endpoint of δ has label α_i, and the other α_{i+1}, $1 \le i < n$ (or β_j and β_{j+1}, $1 \le j < m$), and γ is disjoint from $N(\nu)$ and from $\partial N(k)$.

In this case the surface S can be pushed along D' to eliminate α_i and α_{i+1}.

Case 2. One endpoint of δ has label α_1, and the other α_n, $n \neq 1$.

Suppose that γ meets either $\partial N(k)$ or $N(\nu)$, for otherwise this would be a special case of Case 1, when $n = 2$. If $m \neq 0$, push S along D'. With this move α_1 and α_n convert into a new β_1, reducing $m + n$.

If $m = 0$ and γ does not meet $\partial N(k)$, then push S as before, creating a new β_1. If γ meets $\partial N(k)$, slide τ_1 over τ_2, then slide τ_1 over $\partial N(k)$ and then again slide over τ_2 following γ, without introducing new intersections with S. So D' is transformed into a disc as in the previous case, where $m = 0$ and $\partial D' \cap \partial N(k) = \varnothing$.

Case 3. Both endpoints of δ are labeled α_1 (or both are labeled α_n).

Note that both endpoints of γ are in the same side of α_1, since $\widetilde{S}$ is a 2-sided surface. Suppose first that $m \neq 0$. We can isotope γ to be completely contained in $N(\tau_1)$. If γ does not meet $N(\nu)$, then the intersection between ∂D and $\widetilde{S}$ is not minimal.

If γ meets $N(\nu)$, then we find a disc E in $N(\tau_1 \cup \tau_2)$ such that E meets τ_1 once and does not intersect τ_2, and $\partial E = \gamma \cup \alpha_1'$, where α_1' is a subarc of $\partial\alpha_1$. Let $E' = E \cup D'$, then $E' \cap S = \partial E' = \delta \cup \alpha_1'$. As E' is contained in $E(k)$ and S is incompressible, $\partial E'$ bounds a disc E'' in S. There are two cases, depending whether α_1 is contained in E'' or it is not. In any case, there must be at least one intersection of τ with E'', other than α_1, for otherwise the arc δ in $\widetilde{S}$ would be ∂-parallel. By exchanging E' by E'' we obtain a surface S' isotopic to S. Suppose first that the disc α_1 is not contained in E''. As E' intersects τ once, and E'' intersects τ at least once, the new surface has at most as many intersections with τ as S. Note that $S' \cap N(\tau)$ contains the disc $E \cup \alpha_1$, which intersects τ in two points. Then by isotoping S', the disc $E \cup \alpha_1$ becomes a new β_1, intersecting τ just once. Then S' has fewer intersections with τ than S, which is a contradiction. Suppose now that the disc α_1 is contained in E''. In this case, E'' intersects τ in at least two points, and E' intersects τ just once. So, S' has fewer intersections with τ than S. Note that in this case the intersection of S' with $N(\tau)$ contains the disc E. So, we are eliminating α_1 and some other α_i or β_j, and getting a new α_n.

Suppose now that $m = 0$. If we can isotope γ such that it is contained in $\partial N(\tau)$, then the proof is identical to the previous case. In the other case, a subarc of γ is contained in $\partial N(k)$ and does not intersect ∂S. Slide τ_1 over τ_2, such that τ is a properly embedded arc in $E(k)$. This can be done following γ such that no new intersections between $\widetilde{S}$ and D are created. There is a disc E contained in $N(k \cup \tau)$, $\partial E = \gamma \cup \alpha_1$, where α_1' is a subarc of $\partial\alpha_1$, and such that E meets k

once. Let $E' = E \cup D'$, then $E' \cap S = \partial E' = \delta \cup \alpha_1$. Since E' meets k once and S is meridionally incompressible, $\partial E'$ bounds a disc F in $\overline{S}$ which intersects k in one point, say it intersects $N(k)$ in s_1. Then $E' \cup F$ is a sphere which bounds a ball that intersects k in an unknotted spanning arc, for k is a prime knot. Let $\overline{S}' = (\overline{S} - F) \cup E'$; this is a surface intersecting k in two points, so that the corresponding meridional surface $S' = \overline{S}' \cap E(k)$ is isotopic to S. By slightly isotoping the tunnel τ, we see that S' has fewer intersections with τ than S, since at least we eliminated α_1, which is a contradiction.

Case 4. Both endpoints of δ are labeled β_m (and if $m = 1$, suppose that γ is on the side of β_1 closest to $\partial N(k)$).

If γ can be isotoped on ∂M such that it is contained in $\partial N(\tau)$, then the intersection between ∂D and $\widetilde{S}$ is not minimal. Otherwise, a subarc of γ is contained in $\partial N(k)$ and does not meet ∂S. Now the proof is identical to that of Case 3 when $m = 0$, with β_m in place of α_1.

Case 5. One endpoint of δ is labeled α_1, α_n or β_m, and the other s_i, $i = 1, 2$.

Suppose first that one endpoint of δ is labeled α_1 (or α_n); note that in this case $m = 0$. Slide τ_1 over τ_2, following γ, without introducing new intersections between S and τ, until τ is an arc properly embedded in $E(k)$. Now pushing τ along D', the disc of intersection α_1 is eliminated. If one endpoint of δ is labeled β_m, then push τ as before to eliminate β_m.

Case 6. One endpoint of δ is labeled s_1, and the other s_2.

As $m + n \neq 0$, γ can be made disjoint from $\partial N(\tau)$, by sliding τ if necessary. This implies that S is ∂-compressible, a contradiction.

Case 7. Both endpoints of δ are labeled s_i, $i = 1, 2$.

Again, we can assume that γ does not intersect $\partial N(\tau)$. As S is ∂-incompressible, δ cuts a disc E from S, which may contain some α_i or β_j. Note that $\partial E = \delta \cup s_i'$, where s_i' is a subarc of s_i. Then $D' \cup E$, glued along δ, is a disc whose boundary is in $N(k)$, and because $\partial N(k)$ is incompressible in $E(k)$, it bounds a disc E' in $\partial N(k)$. Note that E' must intersect τ, for otherwise D can be isotoped along E', to reduce the number of intersections between ∂D and S, which is not possible. So, $D' \cup E \cup E'$ bounds a 3-ball in $E(k)$. As τ intersects E', it must also intersect E in at least one point. Now exchange E with D', to get an essential surface S' isotopic to S in $E(k)$, with fewer intersections with τ. Note that one boundary component of S' is $\gamma \cup s_i''$, where s_i'' is the other subarc of s_i, and that, in fact, $\gamma \cup s_i''$ is a meridian of $N(k)$.

Case 8. One endpoint of δ is labeled β_1, and the other α_1 (or α_n).

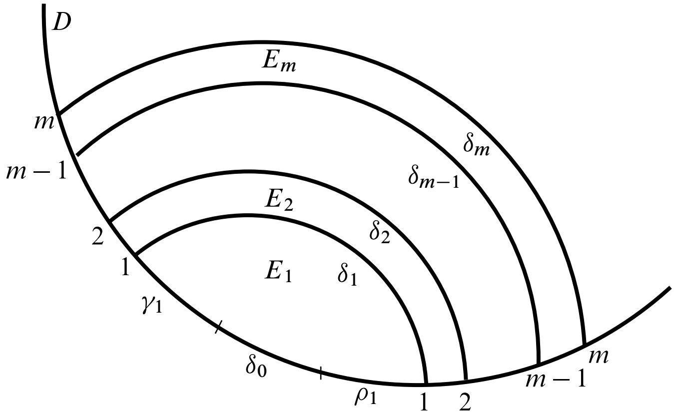

Figure 2. Outermost arcs in D.

Pushing S along D', α_1 and β_1 convert into a curve parallel to α_n, and this reduces $n+m$.

Case 9. Both endpoints of δ are labeled β_1, and the arc δ can be isotoped into $N(\tau_2) \cup N(\nu)$.

If γ is disjoint from $N(\nu)$, then the intersection between ∂D and $\widetilde{S}$ is not minimal. If γ meets $N(\nu)$, then it can be arranged such that γ intersects $N(\tau_2)$ in two arcs. There exists a disc E contained in $N(\tau)$ such that $\partial E = \gamma \cup \beta_1'$, where β_1' is a subarc of β_1. Let $E' = D' \cup E$, then $\partial E' = \delta \cup \beta_1'$ is contained in S, and because of the incompressibility of S, it bounds a disc D'' in S. We can choose the discs E and D'' such that τ_2 meets D'' in a point corresponding to β_1, and τ intersects E' once. The disc D'' necessarily intersects τ in more points, for otherwise the arc δ would be ∂-parallel in $\widetilde{S}$. Exchanging D'' with E' we get a surface S' isotopic to S, with $m' + n' < m + n$.

With this we have already considered all the possible cases for the arc δ, except if the ends of δ are in β_1 and the arc γ cannot be isotoped to $N(\tau_2) \cup N(\nu)$, i.e., γ is wrapped one or more times around $N(\tau_1)$, but this is possible only if $n = 0$, that is, S intersects τ only in the arc τ_2. □

Lemma 4.4. *There is a collection of m arcs, say $\delta_1, \delta_2, \dots, \delta_m$ in $D \cap \widetilde{S}$, which are parallel in D and δ_1 is an outermost arc in D.*

Proof. $D \cap \widetilde{S}$ consists of a collection of arcs in D. We construct a tree in D as follows: assign a vertex for each region of $D - \widetilde{S}$, then connect two vertices if their respective regions are adjacent, that is, they have an arc of $D \cap \widetilde{S}$ common. The resultant graph G is a tree, because D is a disc. The ends of the tree, (that is, the vertices of degree 1), correspond to the outermost regions of D.

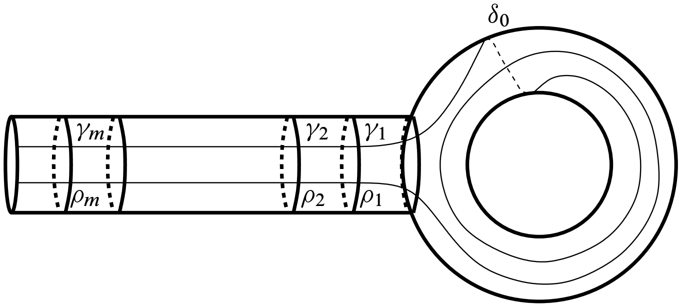

Figure 3. Curves in $\partial N(\tau)$.

A branch of G is a trajectory that begins at an end of G and finishes in a vertex of degree > 2, such that the intermediate vertices of the branch are all of degree 2. If all the vertices of G are of degree 1 or 2, then all the arcs are parallel, and there are at least $2m$ such arcs. Otherwise, let G' be the graph obtained by eliminating the branches, that is, by clearing the vertices of degree 1 and 2 of branches and the corresponding edges. Let V be a vertex of degree 1 of G' (if vertices of degree 1 do not exist, let V be the unique vertex of G'). Then at least two branches arrive at V, say r_1 and r_2 are two adjacent branches that arrive at V. Let η_1 and η_2 be the outermost arcs corresponding to r_1 and r_2, respectively. The endpoints of η_1 and η_2 are labeled β_1 and β_1, by Lemma 4.3. Let ϕ be an arc of ∂D that goes from one endpoint of η_1 to one endpoint of η_2. Then ϕ must cross labels $\beta_1, \beta_2, \dots, \beta_m, \beta_m, \beta_{m-1}, \dots, \beta_2, \beta_1$, and perhaps more labels between β_m and β_m. Any arc of intersection that leaves these labels corresponds to an edge of r_1 or r_2, by the selection of the branches. This implies that $r_1 \cup r_2$ has at least $2m$ edges, and then at least one of the branches has m or more edges corresponding to m parallel arcs. □

Label with i the endpoints of δ_i for $1 \le i \le m$. Call $E_1 \subset D$ the disc determined by δ_1. Let β_0 be a disc in $N(\tau)$ which intersects τ just in the point ν, such that $\partial\beta_0$ is a curve on $\partial N(\tau_2)$ parallel to $\partial\beta_1$. $E_1 \cap \partial N(\tau)$ can be isotoped so that it intersects β_0 in two points which divide $E_1 \cap \partial N(\tau)$ into three arcs, say γ_1, ρ_1 and δ_0, where γ_1, ρ_1 are in $\partial N(\tau_2)$ and δ_0 is in $\partial N(\tau_1)$ (see Figure 2).

Denote by γ_i and ρ_i the arcs in ∂D with endpoints $i-1$ and i. Call $E_i \subset D$ the disc determined by $\delta_i, \delta_{i-1}, \rho_i$ and γ_i, for $2 \le i \le m$. The arcs γ_i and ρ_i are contained in $\partial N(\tau_2)$ and decompose $\partial\beta_i$ into two arcs, call them β_i^1 and β_i^2, for $0 \le i \le m$. Note that $\beta_i^1, \beta_{i-1}^1, \gamma_i$ and ρ_i, for $1 \le i \le m$, determine a disc in $\partial N(\tau_2)$, call it C_i, and $\beta_i^2, \beta_{i-1}^2, \gamma_i$ and ρ_i also determine a disc, call it C_i' (see Figure 3).

Lemma 4.5. *There is an annulus A with interior disjoint from S, such that one of the boundary components is $\delta_1 \cup \beta_1^1 \subset S$, and the other is $\delta_0 \cup \beta_0^1 \subset \partial N(\tau_1)$ with some slope p/q, where $q \ge 2$.*

Proof. Note that $E_1 \cup C_1$ is an annulus A, where one of its boundary components is $\delta_1 \cup \beta_1^1 \subset S$, and the other boundary component is $\delta_0 \cup \beta_0^1$, which is contained in $\partial N(\tau_1)$, with some slope p/q. If $q = 1$, that is, $\delta_0 \cup \beta_0^1$ only turns once around $N(\tau_1)$, then τ_1 is isotopic to $\delta_1 \cup \beta_1^1$ on S, so we can push the tunnel through S, using the annulus A, eliminating one intersection with S corresponding to β_1. Thus $q \geq 2$. □

Since $\overline{S}$ is a torus in S^3, it is boundary of a solid torus R. We have two cases, depending whether τ_1 is contained in R or not.

Case 1. Suppose that τ_1 is not contained in R.

In this case the interior of the annulus A is disjoint from R. One boundary component of A lies in $\partial N(\tau)$, and the other in $\partial R = \overline{S}$.

Lemma 4.6. *The core of R is a cable around τ_1 and ∂A is a longitude of R, or the core of R and τ_1 form a Hopf link.*

Proof. The component of ∂A in $N(\tau_1)$ is a curve with slope p/q and $q \geq 2$ by Lemma 4.5. If the component of ∂A in R is a curve with slope r/s and $s \geq 2$, then the unique possibility is that τ_1 and the core of R form a Hopf link, by [Eudave-Muñoz and Uchida 1996, Theorem 1(iv)]. Otherwise, the slope of ∂A in R is longitudinal, in which case the core of R is a cable around τ_1. □

If the core of R and τ_1 form a Hopf link, then τ_1 is a trivial knot and we are done. So, we suppose now that the core of R is a cable around τ_1 and ∂A is a longitude of R.

Lemma 4.7. *The number of points of intersection, m, is* 1.

Proof. Suppose that $m \geq 2$, and consider the annulus $F = E_2 \cup C_2$, where E_2 and C_2 are glued along γ_2 and ρ_2, with its boundary lying on S. We have that $F \subset R$, and ∂F consists of two longitudes of R, so one of these boundary components is $\delta_1 \cup \beta_1$, which is contained in ∂A. The annulus F divides R into two solid tori, only one of which intersects the knot, and we can push the arc τ_2 along the other solid torus to eliminate at least two intersections with it, which is a contradiction. □

Suppose then that $S \cap \tau_2$ is one point. Let $N(A)$ be a neighborhood of A such that $z_1 = N(A) \cap R$ is a neighborhood of $\delta_1 \cup \beta_1^1$ in S, and $N(A) \cap N(\tau_1)$ is a neighborhood of $\delta_0 \cup \beta_0^1$ in $\partial N(\tau_1)$. We can assume that $N(A)$ and k are disjoint.

Let $W = R \cup N(A) \cup N(\tau_1)$. Then W is a solid torus and τ_1 is a core of W. Let $T_1 = \partial W$. The surface T_1 is a torus which intersects k in two points.

Lemma 4.8. *Either the punctured surface $T_1 - k$ is incompressible in $S^3 - k$, or τ_1 is a trivial knot.*

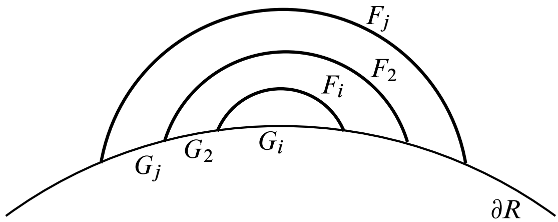

Figure 4. Constructing parallel annuli.

Proof. We prove first that $T_1 - k$ is incompressible in $W - k$. Note that z_1 is an annulus properly embedded in W, with slope p/q, which does not meet k. Suppose Q is a compression disc for $T_1 - k$. Then $Q \cap z_1$ consists of simple closed curves and arcs, and the simple closed curves can be eliminated, because z_1 is essential in W. Now we take an outermost arc η in Q. If η is trivial in z_1, then we can isotope Q to eliminate intersections with this annulus. If η is essential in z_1, then the outermost disc determined by z_1 in Q is contained in R, since $q \geq 2$. This implies that S is compressible in $R - k$, which is not possible.

If $T_1 - k$ is compressible in $S^3 - \operatorname{int} W$, we have two cases, either the boundary of a compression disc Q is essential in the torus T_1, or is trivial in that torus. If the curve ∂Q is essential in T_1, we have that the solid torus W is unknotted and then τ_1 is a trivial knot.

If the curve ∂Q is trivial in T_1, then it bounds a disc $Q' \subset T_1$, which meets k in two points. If W is unknotted, then τ_1 is a trivial knot. Suppose that W is knotted; exchanging Q' for Q, we have a bigger torus T_1', parallel to T_1, which does not touch the knot. The torus T_1' is incompressible in $S^3 - \operatorname{int} N(k \cup \tau)$, since it bounds a knotted solid torus and τ_1 is a core of W, but this cannot happen because $S^3 - \operatorname{int} N(k \cup \tau)$ is a handlebody. □

In this case we concluded that either τ_1 is a trivial knot, or that there is another meridional essential torus which intersects k in two points that is disjoint from τ, and such that τ_1 is a core of the solid torus bounded by T_1.

Case 2. Suppose that τ_1 is contained in R. In this case τ_1 is a core of R.

Lemma 4.9. *Either $m = 1$, or τ_1 is a trivial knot.*

Proof. Suppose that $m \geq 2$. Let F_2 be defined as before, $F_2 = E_2 \cup C_2$, where E_2 and C_2 are glued along γ_2 and ρ_2, with its boundary lying on S. Now F_2 is not contained in R. Note that ∂F_2 consists of two curves in ∂R, with slope p/q and $q \geq 2$. That is, F_2 is an annulus in the exterior of R, and F_2 is parallel to an annulus $G_2 \subset \partial R$, since the slope of its boundary is not integral. If k is not in the region bounded by $F_2 \cup G_2$, we can eliminate two intersections with τ, by pushing τ_2 through the solid torus with boundary $F_2 \cup G_2$. Suppose then that k is in such a

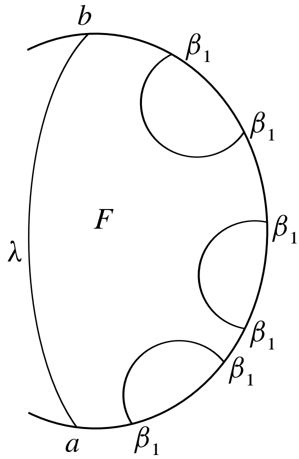

Figure 5. Outermost arcs in D when $m = 1$.

region. Consider any other of the annuli F_i defined as before, $F_i = E_i \cup C_i$, where E_i and C_i are glued along γ_i and ρ_i, with its boundary lying on S. Suppose that F_i is not contained in R. Again, F_i is parallel to an annulus $G_i \subset \partial R$ and k must be contained in the region bounded by $F_i \cup G_i$. This shows that F_2 and F_i must be parallel (see Figure 4).

Let F_j be the annulus not contained in R, bounding a maximal parallel region between F_j and G_j. Let $T = (\partial R - G_j) \cup F_j$. By slightly pushing T, we have that $T \cap \tau = \varnothing$, and $T \cap k = \varnothing$. The torus T bounds a solid torus R' with τ_1 as its core. If τ_1 is not the trivial knot, then T is incompressible in $S^3 - N(k \cup \tau)$, which is not possible, for $S^3 - N(k \cup \tau)$ is a handlebody. Then τ_1 is a trivial knot. □

Suppose now that $m = 1$. Remember that D denotes a meridian disc of $S^3 - \operatorname{int} N(k \cup \tau)$. By Lemma 4.3 we have that $n = 0$, and we can suppose that the intersections of the disc D with S consist of collections of arcs in D, where the outermost arcs have ends in β_1.

We construct a tree in D as in the proof of Lemma 4.4. Consider the graph obtained by cutting the outermost vertices, and choose one of the outermost vertices in the new graph. Now we consider the region F associated with this vertex. This disc is bordered by intersection arcs where all the arcs are outermost arcs except at most one, which we denote by λ.

The outermost arcs have endpoints in β_1, and the endpoints $\{a, b\}$ of the arc λ are one of the pairs from the set $\{\{s_1, s_2\}, \{s_i, s_i\}, \{s_i, \beta_1\}, \{\beta_1, \beta_1\}\}$, with $i = 1$ or 2 (see Figure 5).

Case 1. The arc λ in the region F has its endpoints in $\{s_1, s_2\}$. The arc λ connects s_1 with s_2. Let γ be an arc in $\partial N(k)$, lying in the part of $N(k)$ which is in the solid torus R, so that $\partial\gamma = \partial\lambda$. Let L the link formed by τ_1 and $\gamma \cup \lambda$. Note that $\lambda \subset \partial R$, and that the interior of γ is inside R. We will show that L has an unknotting tunnel.

Let k' be the arc of k lying in the exterior or R. Let k_i be an arc in $\partial N(k')$ that connects s_i and the point $\tau_2 \cap N(k')$, $i = 1, 2$. Assume that $k_1 \cap k_2$ is just the point $\tau_2 \cap N(k')$. Suppose that $N(k') = N(k_1) \cup N(k_2)$. An unknotting tunnel $\hat{\tau}$ for L is formed by the union of τ_2 and k_1. Let F' be the disc in D cut by λ and which contains F. Note that $\partial F' = \lambda \cup \rho$, where ρ is an arc in $N(k') \cup N(\tau)$, and furthermore $\rho = \rho_1 \cup \rho_2 \cup \rho_3$, where $\rho_1 \subset \partial N(k_1)$ and $\rho_3 \subset \partial N(k_2)$. We slide λ along $\hat{\tau}$, following ρ, by first sliding λ over $N(k_1)$, then sliding λ over $N(\tau_2)$, then sliding λ over $N(\tau_1)$, and so on. We do this according to $\partial F'$, until we get to the point $\rho_2 \cap \rho_3$. Now we push the previous arc (equivalent to $\lambda \cup \rho_1 \cup \rho_2$) through F', deforming it into ρ_3. We see that a neighborhood of the complex

$$L \cup \hat{\tau} = \lambda \cup \gamma \cup k_1 \cup \tau_2 \cup \tau_1$$

is deformed into a neighborhood of the complex

$$k_2 \cup k_1 \cup \gamma \cup \tau_2 \cup \tau_1 = k \cup \tau.$$

This proves that $\hat{\tau}$ is a tunnel for L.

We can isotope the link L into R, since $\lambda \subset \partial R$ and the interior of γ is inside R. This link has a tunnel number 1 and does not meet $\overline{S}$. By the classification [Eudave-Muñoz and Uchida 1996] of links which have tunnel number 1 and contain an incompressible torus in their exteriors, this cannot happen unless τ_1 is the trivial knot, and in this case we have the first assertion of Proposition 4.1.

In what follows, suppose that the arc τ_2 is very short, that is, isotope τ_2 until it is almost contained in the boundary of the solid torus R. Let R' be the solid torus $R' = R \cup N(\tau_2)$, and let $S' = \partial R'$. Note that S' intersects k in four points, and then there are two arcs of k in the complement of R', say k^1 and k^2, where k^i is the arc with one endpoint in s_i, $i = 1, 2$.

Case 2. The arc λ in the region F has its ends in $\{s_i, s_i\}$, $i = 1, 2$.

Suppose without loss of generality that the arc λ in S connects s_1 with s_1. In S we have a collection of arcs with ends in β_1, which correspond to the outermost arcs determined by F. These arcs are parallel in S, since each outermost disc determines an annulus with boundary in $\overline{S}$ and $\partial N(\tau_1)$, like in Lemma 4.5. Furthermore the boundary of each of these annuli in $\overline{S}$ is a curve with slope p/q, with $q \geq 2$. Since the arc λ is disjoint from these curves, there are two possibilities for this arc. Either it bounds a disc or punctured disc D' in S, or with a subarc of s_1 is a curve of slope p/q in S.

If λ bounds such a disc D', then there is an intersection arc between S and D, which is trivial and outermost in S. This is clear if s_2 is not contained in D''. If s_2 is contained in D'', then there is a trivial arc with endpoints in s_2, as ∂D intersects s_1 and s_2 in the same number of points. This is not possible.

Then we have that λ with a subarc of s_1 is a curve of slope p/q in S. We can consider F as a disc whose boundary consists of the arc λ, two arcs μ_1 and μ_2 in $N(k^1)$, plus one arc λ' in S'. Note that μ_1 and μ_2 are parallel in $N(k^1)$; that is, there is a disc G in $\partial N(k^1)$, such that $F \cap G = \mu_1 \cup \mu_2$. Let $H = F \cup G$. This is an annulus whose boundary is contained in S', and each of these curves has slope p/q. Then H is an annulus properly embedded in the exterior of R' and its boundary consists of curves with nonintegral slope. Then H is parallel to an annulus H' contained in S', that is, H and H' bound a solid torus. Let

$$T = H \cup (S' - H'),$$

and push this torus slightly such that the arc k^1 is contained in the interior of the solid torus bounded by H and H'.

We have two possibilities: 1) T is disjoint from k and τ. This case is not possible if R is knotted, for T would be an incompressible torus in the handlebody $S^3 - N(k \cup \tau)$, which is not possible. So, τ_1 must be a trivial knot. 2) The torus T intersects k in two points and does not meet τ. We claim that T is incompressible in $E(k)$ or that τ_1 is a trivial knot. Note that T and S cobound a product region, and each of these tori intersects k in two points. So T must be incompressible in the region containing R. Suppose that there is a compression disc E lying in the region not containing R. Let $\gamma = \partial E$. Then we have two cases: γ is essential in T or γ is trivial in T (without considering the intersections with k). If γ is essential in T, then T is not knotted, so τ_1 is a trivial knot. If γ is trivial in T, then it bounds a disc $E' \subset T$. Since γ is essential in $T - N(k)$, E' must contain the intersection points between k and T, then the arc of k is contained in the ball bounded by $E \cup E'$. Now,

$$T' = (T - E') \cup E$$

is a torus which intersects neither k nor τ. If T' is incompressible in $S^3 - N(k \cup \tau)$, then there would be an incompressible torus in $S^3 - N(k \cup \tau)$, which cannot happen. If T' is compressible, then it is not knotted, so τ_1 is a trivial knot.

We conclude that either τ_1 is a trivial knot, or that there is another torus T intersecting k in two points, incompressible in $E(k)$, disjoint from τ, but such that τ_1 is a core of the solid torus bounded by T.

Case 3. The arc λ in the region F has its ends in $\{s_i, \beta_1\}$, $i = 1, 2$. Suppose without loss of generality that the arc λ connects s_1 with β_1. We can suppose that ∂F consist of the arc λ, an arc μ_1 in $N(k^1)$, plus an arc in S'. Push the arc k^1, using the disc F, until it is in a neighborhood of R'. We can take a bigger torus T, which does not intersect the tunnel and meets k twice.

We claim that T is incompressible in $E(k)$ or that τ_1 is a trivial knot. The proof is similar to the proof in the previous case.

Case 4. The arc λ in the region F has the ends in $\{\beta_1, \beta_1\}$. In this case all the arcs have the ends in β_1. We can assume that ∂F lies in the torus S'. If the boundary of the disc F is nontrivial in S', the torus R' cannot be knotted and then τ_1 is a trivial knot. If the boundary of F is trivial in S', then for homological reasons, the arc λ must be parallel in S' to the other arcs with ends in β_1, and there are in total an even number of arcs with ends in β_1. It follows that F bounds a disc E in S', which contains the two points of intersection of k with $\partial N(\tau_2)$. Then both arcs k^1 and k^2 are inside the 3-ball bounded by $F \cup E$, and by exchanging F for E, we find a bigger torus which does not intersect k nor τ. As before, the torus is unknotted, i.e., τ_1 is a trivial knot.

This completes the proof of Proposition 4.1. □

Let k be a knot of tunnel number 1 and $\overline{S}$ a meridional essential torus for (S^3, k), which intersects the knot in two points. As before, let $S = \overline{S} \cap E(k)$. Let $\tau = \tau_1 \cup \tau_2$ be an unknotting tunnel for k such that $S \cap \tau = \varnothing$. The surface $\overline{S}$ divides S^3 in two parts $S^3 = V \cup W$, and one of them is a solid torus. Suppose that τ is contained in V. Let $M = S^3 - \operatorname{int} N(k \cup \tau)$. Then M is a handlebody, and S divides M in two handlebodies, say $M = V' \cup W'$, where $V' = V - \operatorname{int} N(k \cup \tau)$ and $W' = W - \operatorname{int} N(k)$.

Lemma 4.10. *V is a solid torus and W is not a solid torus.*

Proof. Suppose that W is a solid torus. As W' is a handlebody, $\partial W'$ is compressible. Let c be the boundary of a meridian disc of k which is in W. Note that $\partial W' - c$ is incompressible in W', for otherwise S would be compressible. Applying Jaco's addition lemma [1984], we have that $W'[c]$ has incompressible boundary (where $W'[c]$ denotes W' with a 2-handle attached along the curve c). On the other hand $W'[c] = W$ which has compressible boundary, and this is not possible. Therefore W cannot be a solid torus, and then V is a solid torus. □

This implies that V is knotted in S^3. As V is a solid torus, we have 3 cases:

(a) τ_1 is inside a 3-ball in V;

(b) τ_1 is a core of V; or

(c) τ_1 is essential in V (that is, cases (a) and (b) do not happen).

Lemma 4.11. *Case (b) cannot happen, and if case (a) happens, τ_1 is a trivial knot.*

Proof. Suppose that case (a) happens; that is, τ_1 is inside a 3-ball B contained in V. Then $k \cap V$ consists of an arc k' properly embedded in V. Let $k' = k_1 \cup k_2$, where k_1 and k_2 are arcs such that $k_1 \cap k_2 = k \cap \tau_2$. Let D be a compression disc for M. The intersection between S and D consists of simple closed curves and arcs, and the simple closed curves can be deleted as usual, because $S \cap M$ is incompressible in M. Let γ be an outermost arc in D, so γ cuts a disc F. If

F were contained in W', it would be a ∂-compression disc for S, which is not possible. Then $F \subset V'$. Note that $\partial F = \gamma \cup \beta$, $\gamma \subset S$ and $\beta \subset N(k \cup \tau)$. Then $\beta = \beta_1 \cup \beta_2 \cup \beta_3$, where β_1 is contained in $\partial N(k_i)$, β_2 is contained in $\partial N(\tau_2 \cup \tau_1)$ and β_3 is contained in $\partial N(k_j)$. Suppose first that $i \neq j$. Shrink τ_2 into τ_1, such that k_1 and k_2 can be seen as arcs with one endpoint in $\partial N(\tau_1)$. Then the arc β can be seen as $\beta = \beta_1 \cup \beta_2 \cup \beta_3$, where β_1 and β_3 are as before and β_2 is an arc on $\partial N(\tau_1)$. By sliding k_1 along $\partial N(\tau_1)$ following β_2, we see that k_1 and k_2 are parallel arcs; that is, there is a disc F' in V' such that $\partial F' = \gamma \cup \beta_1 \cup \beta_2' \cup \beta_3$, where β_2' is an arc in $N(\tau)$, disjoint from a meridian of τ_1. Cut V' along F, to get a handlebody V'', which is homeomorphic to $V - N(\tau_1 \cup k'')$, where k'' is an arc with endpoints on S and τ_1 (it can be considered as k_1). This is not possible by Proposition 3.1.

Suppose now that β_1 and β_3 are both contained in $\partial N(k_1)$. Shrink τ_2 into τ_1 again, such that k_1 and k_2 can be seen as arcs each with one endpoint in $\partial N(\tau_1)$. There is a disc $C \subset \partial N(k_1)$ such that $C \cup F$ is an annulus with one boundary component, say C_1, lying on S, and the other boundary component, C_2, lying on $\partial N(\tau_1)$. The closed curve C_2 is either trivial in $\partial N(\tau_1)$ or it is essential. Suppose first that C_2 is trivial in $\partial N(\tau_1)$. Then it bounds a disc $E \subset \partial N(\tau_1)$ which contains an endpoint of k_2. If C_1 is trivial on S, then k_2 must be an arc parallel to k_1, and we proceed as in the previous case. If C_1 is nontrivial on S, then it must be a meridian of S, for $C \cup F \cup E$ is a disc in V with boundary C_1. By taking a copy of $C \cup F \cup E$ and pushing it to be disjoint from $k_1 \cup \tau_1$, we get a disc whose boundary is a meridian of S and which intersects k_2 in one point, and then it is a meridian disc that intersects k in one point. This is not possible because S is meridionally incompressible.

Suppose now that C_2 is essential in $\partial N(\tau_1)$. Assume that the annulus $C \cup F$ and the sphere ∂B intersect transversely, and note that $\partial(C \cup F)$ is disjoint from ∂B. Let α be an innermost curve of intersection on ∂B. If α is a trivial curve in $C \cup F$, we can find another 3-ball containing τ_1 whose boundary has fewer intersections with $C \cup F$. If α is essential in $C \cup F$, then by cutting $C \cup F$ with the disc in ∂B bounded by α we get an embedded disc whose boundary is C_1. If C_1 is not a longitudinal curve in $\partial N(\tau_1)$, this implies that there is a punctured lens space embedded in V, which is impossible. So, C_1 must be a longitude of $\partial N(\tau_1)$, and then τ_1 must be a trivial knot.

Suppose now that case (b) happens; that is, τ_1 is a core of V. As above, let k' be the arc $k \cap V$, such that $k' = k_1 \cup k_2$, where k_1 and k_2 are arcs with $k_1 \cap k_2 = k \cap \tau_2$. Slide k_2 over τ_2, getting two arcs, k_1' and k_2', each with one endpoint on S and one in τ_1. By Proposition 3.2, it follows that k_1' and k_2' are a pair of simultaneously straight arcs in the space product $V - N(\tau_1)$. By sliding back k_2' over k_1', we see that k' is an arc in V that is isotopic to an arc contained in ∂V. This implies that T is compressible in $S^3 - k$, a contradiction. □

Proof of Theorem 2.3. Let k be a knot of tunnel number 1, $\overline{S}$ a meridional essential torus which intersects the knot in two points and $\tau = \tau_1 \cup \tau_2$ an unknotting tunnel for k. Suppose first that τ cannot be made disjoint from $\bar{S}$. Then by Proposition 4.1, either τ_1 is a trivial knot, or there is another essential meridional torus $\bar{S}'$, which intersects k in two points, is disjoint from τ, and such that τ_1 is a core of the solid torus bounded by $\bar{S}'$. However, the existence of such a torus contradicts Lemma 4.11, so this case is not possible. Therefore, τ_1 is a trivial knot, and by Lemma 2.1, k is a $(1,1)$-knot.

Suppose now that τ and $\bar{S}$ are disjoint. By Lemma 4.10, $\bar{S}$ bounds a solid torus V in which τ lies. Then by Lemma 4.11, either τ_1 is a trivial knot, and then k is a $(1,1)$-knot, or we have case (c), that is, τ_1 is an essential curve in V. So, suppose that case (c) happens. Then $\bar{S}$ is essential in $S^3 - \tau_1$ and $\tau_2 \cup k$ is a tunnel for τ_1. Then τ_1 is a satellite knot with tunnel number 1, and this implies that $\bar{S}$ is knotted as a torus knot, by the result of Morimoto and Sakuma [1991]. Slide k over τ_2 until it becomes an arc k' with endpoints on τ_1. Then k' has to be one of the unknotting tunnels for τ_1 as classified by Morimoto and Sakuma [1991]; that is, by sliding k' over $\partial N(\tau_1)$ we get an arc ρ which is one of the tunnels $\tau(1,x)$, $\tau(2,x)$, $\tau(2,y)$ or $\tau(1,y)$ for τ_1, as defined in Section 2. To get k from ρ, we have to slide ρ over $\partial N(\tau_1)$ and then over itself, but this is equivalent to taking an arc on $\partial N(\tau_1)$ joining the endpoints of ρ, in fact the arc γ determined by the sliding of ρ over $\partial N(\tau_1)$, and then taking the iterate of ρ and τ_1 using the arc γ.

This completes the proof of Theorem 2.3. □

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MARIO EUDAVE-MUÑOZ
INSTITUTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
CIRCUITO EXTERIOR
CIUDAD UNIVERSITARIA
04510 MÉXICO DF
MEXICO

and

CIMAT
CALLEJÓN JALISCO S/N
COL. VALENCIANA 36240
GUANAJUATO
MEXICO

mario@matem.unam.mx

GRISSEL SANTIAGO-GONZÁLEZ
FACULTAD DE CIENCIAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
CIRCUITO EXTERIOR
CIUDAD UNIVERSITARIA
04510 MÉXICO, DF
MEXICO

gissel@matem.unam.mx

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