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ON BISECTIONAL NONPOSITIVELY CURVED COMPACT KÄHLER–EINSTEIN SURFACES

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We prove a conjecture on the pinching of the bisectional curvature of nonpositively curved Kähler–Einstein surfaces. We also prove that any compact Kähler–Einstein surface M is a quotient of the complex two-dimensional unit ball or the complex two-dimensional plane if M has nonpositive Einstein constant and, at each point, the average holomorphic sectional curvature is closer to the minimum than to the maximum.

1. Introduction

In [Siu and Yang 1981] the authors conjectured that any compact Kähler surface with negative bisectional curvature is a quotient of the complex two-dimensional unit ball. They proved that there is a number $a \in \left(\frac{1}{3}, \frac{2}{3}\right)$ such that if, at every point P, $K_{\text{av}} - K_{\min} \leq a[K_{\max} - K_{\min}]$ then M is a quotient of the complex ball. Here $K_{\min}$, $K_{\max}$ and K_{av} is the minimal, maximal and average value of the holomorphic sectional curvature, respectively. The number a they obtained was $a < 2/\left(3\left[1+\sqrt{\frac{6}{11}}\,\right]\right) < 0.38$ (see [Polombo 1992, p. 398]). In [Hong et al. 1988], Yi Hong pointed out that this is also true if $a \leq 2/\left(3\left[1+\sqrt{\frac{1}{6}}\right]\right) < 0.476$. We observed in [Hong et al. 1988, Theorem 2] that if $a \leq \frac{1}{2}$, then there is a ball-like point P. That is $K_{\max} = K_{\min}$ at P. We notice here that $\sqrt{\frac{1}{6}} > \frac{1}{3}$. Therefore, we conjectured in [Hong et al. 1988] that M is a quotient of the complex ball if $a = \frac{1}{2}$. In general, we believe that we might not get a quotient of the complex ball if $a > \frac{1}{2}$. Around 1992 Hong Cang Yang almost proved this conjecture except for some technical difficulties, see the argument of Theorem 1.2 in [Chen et al. 2011]. Polombo [1988; 1992] used a different method and proved that a can be $(3 + (4\sqrt{3})/3)/11 < 0.48$ (according to [Chen et al. 2011, p. 2628 right before Theorem 1.2]), see [Polombo 1988, p. 669] or [Polombo 1992, p. 398]. In [Chen et al. 2011], the authors improved the constant to $a < \frac{1}{2}$ which gave a proof of a weaker version of the conjecture.

We first notice that in the proof of Theorem 2 in [Hong et al. 1988] (for which this author was responsible) we proved that if $K_{\text{av}} - K_{\min} = \frac{1}{2}[K_{\max} - K_{\min}]$ at P,

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then P must be a ball-like point (for this part, any negativity of the curvature is not needed except to use the result from [Siu and Yang 1981] when $A = B$). See the remark after the Theorem 1 in [Hong et al. 1988]. According to [Siu and Yang 1981, p. 485, Proposition 4] the subset of ball-like points is either the whole manifold or a real codimension two analytic subvariety. Since the function considered in Theorem 1.2 of [Chen et al. 2011] is bounded, it can be extended to all of M, is a constant and must be zero. Notice that we only need that the bisectional curvature is nonpositive. With this in mind, we also have the possibility of the flat case. That is, the manifold could also be a quotient of $\mathbb{C}^2$ if the Einstein constant is zero. This case should also be included in the main theorem of [Siu and Yang 1981, p. 472] and Theorems A and 1 of [Hong et al. 1988].

Since [Hong et al. 1988] was only written in Chinese, we provide a mostly self contained account here. Also, Polombo [1988; 1992] had something more general than stated above. Therefore, we generalized our result to the case of nonpositive Einstein constant.

Theorem. *Let M be a connected compact Kähler–Einstein surface with nonpositive scalar curvature, if we have*

$$K_{\mathrm{av}} - K_{\min} \leq \tfrac{1}{2}[K_{\max} - K_{\min}]$$

at every point, then M is a compact quotient of either the complex two-dimensional unit ball or the two-complex-dimensional plane.

This note is written in such a way that experts who are familiar with [Hong et al. 1988; Chen et al. 2011] will be able to understand the proof of the conjecture stated in those works from the present introduction. For those only familiar with the second of those references, the present Section 2 should be enough to understand the proof of the conjecture. Notice that we do not need the nonpositivity of the bisectional curvature except to apply the result of [Siu and Yang 1981] or [Chen et al. 2011] to the case $A = B$. We shall give a complete proof of the conjecture in Section 3, with a simpler explanation than that of [Chen et al. 2011] for the last step, that also explains away the mystery of the negativity. In Section 4, we apply these methods to prove our theorem.

To the author, the conjecture in [Siu and Yang 1981] is very important to complex geometry. This work is heavily dependent on earlier works in this subject. Although we are able to prove the conjecture from [Hong et al. 1988; Chen et al. 2011] and our main theorem, there is more work which needs to be done in the direction of compact complex surfaces with negative holomorphic bisectional, or even real, sectional curvatures. Therefore, the author thinks that it is proper to write this paper with an emphasis on the nonpositive holomorphic bisectional curvature case instead of the case of our main theorem.

2. Existence of ball-like points

Here, we repeat the argument in the proof of Theorem 2 in [Hong et al. 1988].

Proposition 1 [Hong et al. 1988, p. 597–599]. *Suppose that*

$$K_{\text{av}} - K_{\min} \le \tfrac{1}{2}[K_{\max} - K_{\min}]$$

for every point on the compact Kähler–Einstein surface with nonpositive holomorphic bisectional curvatures. There is at least one ball-like point.

Proof. Throughout this paper, as in [Siu and Yang 1981; Chen et al. 2011], we assume that $\{e_1, e_2\}$ is a unitary basis at a given point P with

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= R_{2\bar{2}2\bar{2}} = K_{\min}, \\ R_{1\bar{1}1\bar{2}} &= R_{2\bar{2}2\bar{1}} = 0, \\ A &= 2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} \ge 0, \\ B &= |R_{1\bar{2}1\bar{2}}|. \end{aligned}$$

As in [Siu and Yang 1981], we always have $A \ge |B|$ and we assume that $B = R_{1\bar{2}1\bar{2}}$ (i.e., the latter is nonnegative).

If P is not a ball-like point, according to [Siu and Yang 1981], we can do as above for a neighborhood $U(P)$ of P whenever $A > B$ (Case 1 in [Siu and Yang 1981], page 475). We should handle the case in which $A = B$ at the end of this proof. We write

$$\begin{aligned} \alpha &= e_1 = \sum a_i \partial_i, \\ \beta &= e_2 = \sum b_i \partial_i, \\ S_{1\bar{1}1\bar{1}} &= R(e_1, \bar{e}_1, e_1, \bar{e}_1) = \sum R_{i\bar{j}k\bar{l}} a_i \bar{a}_j a_k \bar{a}_l, \end{aligned}$$

and so on. In particular, we have

$$S_{1\bar{1}1\bar{1}} = S_{2\bar{2}2\bar{2}} = K_{\min}, \qquad S_{1\bar{1}1\bar{2}} = S_{2\bar{2}2\bar{1}} = 0.$$

According to [Siu and Yang 1981], we have

$$\begin{aligned} K_{\max} &= K_{\min} + \tfrac{1}{2}(A+B), \\ K_{\text{av}} &= K_{\min} + \tfrac{1}{3}A, \\ \tfrac{1}{3}[K_{\max} - K_{\min}] &\le K_{\text{av}} - K_{\min} \le \tfrac{2}{3}[K_{\max} - K_{\min}]. \end{aligned}$$

Our condition in Proposition 1 is therefore equivalent to $A \le 3B$. As in [Hong et al. 1988], we let $\Phi_1 = |B|^2/A^2 = \tau^2$.

If there is no ball-like point, since $\frac{1}{3} \le \tau \le 1$, there is a minimal point.

We shall calculate the Laplacian of Φ_1 at a minimal point, which is not ball-like. For example, when $A = 3B$, the minimum $\Phi_1 = \frac{1}{9}$, is achieved. The Laplacian at that point should be nonnegative.

We let

$$x_i = \nabla_i \Phi_1 = 2\frac{\tau}{A}[\mathrm{Re}\,\nabla_i S_{1\bar{2}1\bar{2}} + 3\tau \nabla_i S_{1\bar{1}1\bar{1}}].$$

As we pointed out earlier, we first assume that A does not equal B, then we can apply the argument in case 1 of [Siu and Yang 1981, p. 475] at the minimal point since $A > B$.

As in [Siu and Yang 1981; Hong et al. 1988; Chen et al. 2011], we have

$$\Delta R_{1\bar{1}1\bar{1}} = -AR_{1\bar{1}2\bar{2}} + B^2, \quad \Delta R_{1\bar{2}1\bar{2}} = 3(R_{1\bar{1}2\bar{2}} - A)B.$$

At P we have $a_1 = b_2 = 1$, $a_2 = b_1 = 0$, $\nabla a_1 = \nabla b_2 = 0$ and $\nabla a_2 + \nabla \bar{b}_1 = 0$. Therefore, we write $y_{i1} = \nabla_i a_2$ and $y_{i2} = \nabla_i \bar{a}_2$. We also have

$$\Delta(a_1 + \bar{a}_1) = -|\nabla a_2|^2, \quad \Delta(a_2 + \bar{b}_2) = 0, \quad \nabla_i R_{1\bar{1}1\bar{2}} = -Ay_{i1} - By_{i2},$$

since

$$0 = \nabla S_{1\bar{1}1\bar{2}} = \nabla R_{1\bar{1}1\bar{2}} + 2R_{2\bar{1}1\bar{2}}\nabla a_2 + B\nabla \bar{a}_2 + R_{1\bar{1}1\bar{1}}\nabla \bar{b}_1,$$

i.e.,

$$\nabla R_{1\bar{1}1\bar{2}} = -A\nabla a_2 - B\nabla \bar{a}_2.$$

This also gives a similar formula for $\nabla_{\bar{i}} R_{1\bar{1}1\bar{2}}$. Similarly,

$$\begin{aligned}
\nabla S_{1\bar{1}1\bar{1}} &= \nabla R_{1\bar{1}1\bar{1}}, \\
\nabla S_{1\bar{2}1\bar{2}} &= \nabla R_{1\bar{2}1\bar{2}}, \\
\Delta S_{1\bar{1}1\bar{1}} &= -2A\sum |y|^2 - 4B\,\mathrm{Re}\sum y_{i1}\bar{y}_{i2} - AR_{1\bar{1}2\bar{2}} + B^2, \\
\mathrm{Re}\,\Delta S_{1\bar{2}1\bar{2}} &= 4A\sum \mathrm{Re}\, y_{i1}\bar{y}_{i2} + 2B\sum |y|^2 + 3(R_{1\bar{1}2\bar{2}} - A)B., \\
\nabla_{\bar{1}} S_{1\bar{2}1\bar{2}} &= -A\bar{y}_{22} - B\bar{y}_{21}, \\
\nabla_2 S_{1\bar{2}1\bar{2}} &= Ay_{11} + By_{12}, \\
\nabla_1 S_{1\bar{2}1\bar{2}} &= -A(6\tau^2 - 1)y_{22} - 5A\tau y_{21} + x_1, \\
\nabla_{\bar{2}} S_{1\bar{2}1\bar{2}} &= 5A\tau \bar{y}_{12} + A(6\tau^2 - 1)\bar{y}_{11} + \bar{x}_2.
\end{aligned}$$

As in [Hong et al. 1988, p. 598] at P we have

$$\begin{aligned}
\Delta\Phi_1 &= \frac{2\tau\Delta B}{A} + \frac{6\tau^2}{A}\Delta S_{1\bar{1}1\bar{1}} + \frac{1}{A^2}\sum\bigl(|\nabla S_{1\bar{2}1\bar{2}}|^2 + |\overline{\nabla} S_{1\bar{2}1\bar{2}}|^2\bigr)\\
&\quad + \frac{54\tau^2}{A^2}\sum|\nabla S_{1\bar{1}1\bar{1}}|^2 + \frac{12\tau}{A^2}\sum \mathrm{Re}\bigl(\nabla_i S_{1\bar{1}1\bar{1}}(\nabla_{\bar{i}}(S_{1\bar{2}1\bar{2}} + S_{2\bar{1}2\bar{1}}))\bigr)\\
&= 2\tau\Bigl[3A\tau(\tau^2-1) - 4\tau\sum|y|^2 + 4(1-3\tau^2)\sum\mathrm{Re}(y_{i1}\bar{y}_{i2})\Bigr]\\
&\quad + |y_{22}+\tau y_{21}|^2 + |y_{11}+\tau y_{12}|^2\\
&\quad + \frac{1}{A^2}\Bigl[\bigl|x_1 + A[(1-6\tau^2)y_{22} - 5\tau y_{21}]\bigr|^2\\
&\qquad\qquad + \bigl|x_2 + A\bigl[(6\tau^2-1)y_{11} + 5\tau y_{12}\bigr]\bigr|^2\Bigr]\\
&\quad - 18\tau^2\bigl[|y_{12}+\tau y_{11}|^2 + |y_{21}+\tau y_{22}|^2\bigr]\\
&\quad + \frac{12\tau}{A}\bigl[\mathrm{Re}[(y_{21}+\tau y_{22})\bar{x}_1] - \mathrm{Re}[(y_{21}+\tau y_{11})\bar{x}_2]\bigr]
\end{aligned}$$

Here we notice that $\Delta\Phi_1$ has two general terms. The first term is constant with respect to x and y, and is always nonpositive since $\frac{1}{3} \le \tau \le 1$.

The second term can be regarded as a hermitian form h in x and y. We can separate x and y into two groups: x_1, y_{2j} in one group and x_2, y_{1j} in the other. These two groups of variables are orthogonal to each other with respect to this hermitian form. That is, $h = h_1 + h_2$ where h_1 and h_2 depend only on the first and second group of variables, respectively.

We need to check the nonpositivity for each term.

For x_2, y_{11}, y_{12}, the corresponding matrix of h_2 is

$$\begin{bmatrix} 1/A^2 & -1/A & -\tau/A \\ -1/A & 2(9\tau^2-1)(\tau^2-1) & 0 \\ -\tau/A & 0 & 0 \end{bmatrix},$$

and the matrix for h_1 of x_1, y_{21}, y_{22} is

$$\begin{bmatrix} 1/A^2 & \tau/A & 1/A \\ \tau/A & 0 & 0 \\ 1/A & 0 & 2(9\tau^2-1)(\tau^2-1) \end{bmatrix}.$$

When P is a critical point of Φ_1, then $x_1 = x_2 = 0$. The matrix for y is clearly seminegative. Therefore, if there is no ball-like point, then we have, at the minimal point of Φ_1, that $\tau^2 = 1$ or $A = 0$ since $\tau \ge \frac{1}{3}$.

If $A = 0$, then we have a ball-like point, and we are done.

On the other hand,[1] if $\tau = 1$, we have $A = B$ at P. Since P is a minimal point, this implies that $A = B$ on the whole manifold. According to [Siu and Yang 1981, p. 475, case 2], we have smooth coordinates with $K_{\max} = R_{1\bar{1}1\bar{1}}$. (Fortunately, this works whenever $A = B$. In general, the original argument might not always work since one might not have $A = B$ in a neighborhood. However, as was pointed out in [Siu and Yang 1981, case 1], under our condition the directions for $K_{\max}$ are always isolated. Therefore, it might be better to choose $K_{\max}$ instead of $K_{\min}$ from the very beginning. But this is not in the scope of this paper.) Using this new coordinate, we can define similar functions A_1 and B_1. In general, $B_1 = \frac{1}{2}(A - B)$ and $A_1 = -\frac{1}{2}(A + 3B)$. In our case, $B_1 = 0$ and $A_1 = -2A$. Using this new coordinate, one can do the calculation for any of the functions in [Siu and Yang 1981; Polombo 1988; 1992; Chen et al. 2011] that the set of ball-like points is the whole manifold. If one does not like Polombo's function Φ_α [1992, p. 418] with $\alpha = -\frac{8}{7}$, then one might simply use the function with $\alpha = -1$ (in [Polombo 1988; Polombo 1992], not the vector we mentioned in this paper earlier), i.e., the new function is proportional to $\Phi_2 = (3B - A)A$. In our case, this is just $2A^2$. We can apply $\Phi_2^{1/3}$. This is relatively easy and is left to the reader. We can also use the argument in [Siu and Yang 1981, case 1], in which the minimal vectors are not isolated but they are points in a smooth circle bundle over the manifold so we could choose a smooth section instead.

Also, the preceding paragraph is not needed in Corollary 2 and Lemma 3 since, in those two propositions, we already have $A = 3B$. With $A = B$, one could readily get that $A = B = 0$.

If $A = 0$, $K_{\max} = K_{\min}$ and P is a ball-like point, then we have a contradiction. Therefore, the set of ball-like points is not empty. □

Observe that if $A = 3B$ at P, then Φ_1 achieves the minimal value at P and $A \neq B$ unless P is a ball-like point. That is the first part of the proof of Proposition 1 goes through. That is, P must be a ball-like point.

Corollary 2. *Assume the above, if $K_{\mathrm{av}} - K_{\min} = \frac{1}{2}[K_{\max} - K_{\min}]$ at P, then P is a ball-like point.*

Therefore, we have:

Lemma 3. *If $K_{\mathrm{av}} - K_{\min} \leq \frac{1}{2}[K_{\max} - K_{\min}]$ on M, then $K_{\mathrm{av}} - K_{\min} < \frac{1}{2}[K_{\max} - K_{\min}]$ on $M - N$, where N is the subset of all the ball-like points.*

Therefore, we can apply the argument of [Chen et al. 2011]. To do that one needs the following proposition:

Proposition 4 (see [Siu and Yang 1981] and [Hong et al. 1988, Theorem 3]). *If $N \neq M$, then N is a real analytic subvariety and* $\operatorname{codim} N \geq 2$.

[1] This paragraph is not needed for the proofs of Corollary 2 and Lemma 3. Also, in this special case, the original frame in [Siu and Yang 1981] works. So, one could apply [Siu and Yang 1981].

As in [Siu and Yang 1981], Proposition 4 gives us a way to the conjecture by finding a superharmonic function on M which was obtained by Hong Cang Yang around 1992. In [Siu and Yang 1981; Hong et al. 1988], the authors used $\Phi = 6B^2 - A^2$. Polombo [1992, p. 417, Lemma] used $(11A - 3B)(B - A) + 16AB$. One might ask why do we need another function, why do we not use Φ_1? The answer is that by a power of Φ_1 we can only correct the Laplacian by $|\nabla\Phi_1|^2$. But that could only change the upper left coefficients of our matrices as it only provides $|x|^2$ terms. In the case of Φ_1, it does not work since $\tau/A \neq 0$ but the coefficients of $|y_{12}|^2$ and $|y_{21}|^2$ are zeros. Therefore, we need another function, which was provided by Hong Cang Yang.

Remark 5. Whenever there is a bounded continuous nonnegative function f on M such that $f(N) = 0$, f is real analytic on $M - N$ and $\Delta f \leq 0$ on $M - N$, then $f = 0$. *Here N could be just a codimension two subset.* This is in general true for extending continuous superharmonic functions over a codimension two subset, see [Siu and Yang 1981; Hong et al. 1988; Chen et al. 2011]. Here, we would like to give our own reasons why this is true in these special cases. If we define $M_s = \{x \in M|_{\mathrm{dist}(x,N)\geq s}\}$ and $h_s = \partial M_s$, then the measure of h_s is smaller than $O(s)$ when s tends to zero. Therefore,

$$0 \geq \ln 2 \int_{M_{2\delta}} \Delta f \omega^n \geq \int_{\delta}^{2\delta} \left[\int_{M_s} \Delta f \omega^n\right] s^{-1}\, ds = \int_{\delta}^{2\delta} \left[\int_{h_s} \frac{\partial f}{\partial n}\, d\tau\right] s^{-1}\, ds.$$

But, by applying an integration by parts to the single variable integral, the last term is about $(\delta)^{-1} \int_{h_{2\delta}} (f - g)\, d\tau \to 0$, since f is bounded and $f - g$ tends to 0 near N, where g is the f value of the corresponding point on h_δ. For example, if $f = r^a$ with $a > 0$, then

$$\frac{\partial f}{\partial n} = ar^{a-1} = as^{a-1} \quad \text{and} \quad \int_{h_s} \frac{\partial f}{\partial n}\, d\tau = O(s^a) \to 0.$$

Therefore, $\Delta f = 0$ on $M - N$. Hence f extends over N as a harmonic function. This implies that $f = 0$ on M.

Now, let $f = (3B - A)^a$, this is natural after the proof of Proposition 1, we will show in the next section that $\Delta f \leq 0$ for $a \leq \frac{1}{3}$ (see the proof in [Chen et al. 2011]). Therefore, $A = 3B$ always. By Corollary 2, we have $A = B = 0$. This function is also related to the functions in [Polombo 1992, p. 417] with $a_1 = a_3 = 0$. Polombo had to pick up functions with $a_1 = a_2$ to avoid a complication of the singularities. See page 406 and the first paragraph in page 418 in [Polombo 1992] and the last paragraph of page 668 in [Polombo 1988]. We shall completely resolve the difficulty in the next section.

3. Hong Cang Yang's function

Let $\Psi = 3B - A$. Around 1992 Hong Cang Yang considered $f = \Psi^{1/3}$.

Lemma 6 [Chen et al. 2011, p. 2630 (13)]. *We have*

$$\Delta(3B-A)=3\big[\Psi R_{1\bar{1}2\bar{2}}+B(B-3A)\big]+\tfrac{3}{B}|\nabla(\operatorname{Im} R_{1\bar{2}1\bar{2}})|^2+6(B-A)\sum|y_{i1}-y_{i2}|^2.$$

Let $z_i = \nabla_i \Psi$, then

$$\begin{aligned}
z_1 = \nabla_1(3B - A) &= \tfrac{3}{2}\nabla_1(R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}} + 2R_{1\bar{1}1\bar{1}}),\\
\sqrt{-1}\nabla_1(\operatorname{Im} R_{1\bar{2}1\bar{2}}) &= \tfrac{1}{2}\nabla_1(R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}})\\
&= \tfrac{1}{3}z_1 - \nabla_1 R_{2\bar{1}2\bar{1}} - \nabla_1 R_{1\bar{1}1\bar{1}}\\
&= \tfrac{1}{3}z_1 - \nabla_2 \overline{R}_{1\bar{1}1\bar{2}} + \nabla_2 R_{1\bar{1}1\bar{2}}\\
&= \tfrac{1}{3}z_1 + (A - B)y_{22} + (B - A)y_{21},
\end{aligned}$$

$$\begin{aligned}
z_2 = \nabla_2(3B - A) &= \tfrac{3}{2}\nabla_2(R_{2\bar{1}2\bar{1}} + R_{1\bar{2}1\bar{2}} + 2R_{1\bar{1}1\bar{1}}),\\
\sqrt{-1}\nabla_2(\operatorname{Im} R_{1\bar{2}1\bar{2}}) &= \tfrac{1}{2}\nabla_2(R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}})\\
&= -\tfrac{1}{3}z_2 + \nabla_2 R_{1\bar{1}1\bar{1}} + \nabla_2 R_{1\bar{2}1\bar{2}}\\
&= -\tfrac{1}{3}z_2 + \nabla_1 R_{2\bar{1}1\bar{1}} - \nabla_1 R_{1\bar{1}1\bar{2}}\\
&= -\tfrac{1}{3}z_2 + (B - A)y_{12} + (A - B)y_{11}.
\end{aligned}$$

We can write the formula in the Lemma 6 as

$$\begin{aligned}
\Delta\Psi = 3\big[\Psi R_{1\bar{1}2\bar{2}} + B(B - 3A)\big] - 3\frac{A-B}{B}\Psi\sum|y_{i1} - y_{i2}|^2&\\
+ 2\frac{A-B}{B}\operatorname{Re}\big[(y_{12} - y_{11})\bar{z}_2 + (y_{22} - y_{21})\bar{z}_1\big] + \sum\frac{1}{3B}|z|^2&.
\end{aligned}$$

As in the last section, we have two general terms, the first is negative as is the constant term of z with respect to y. The second is a hermitian form in z and y. We can actually let $w_i = y_{i^*1} - y_{i^*2}$ with $i^* \neq i$. Then the second term is a sum of two hermitian forms. One of them is on w_1, z_1 and the other is on w_2, z_2. We notice that the second term is also nonpositive on y (or nonpositive on w, if we assume that $z = 0$). We can modify the coefficient of $|z|^2$ (only) by taking the power of Ψ. More precisely, if we let $g = \Psi^a$, to make sure that $\Delta g < 0$, after taking out a factor $3(A - B)/B$ we need

$$\begin{vmatrix} -\Psi & \frac{1}{3} \\ \frac{1}{3} & \frac{1-3\Psi^{-1}(1-a)B}{9(A-B)} \end{vmatrix} \geq 0.$$

That is,

$$A - 3B + 3(1 - a)B - A + B = (3(1 - a) - 2)B \geq 0.$$

We have $1-3a \geq 0$. So, $a \leq \frac{1}{3}$.

Therefore, we have:

Lemma 7. *$\Delta g < 0$ for $a \leq \frac{1}{3}$ on $M - N$.*

This is exactly the same as in [Chen et al. 2011]. Actually, the number $\frac{1}{6}$ was already in [Siu and Yang 1981; Hong et al. 1988; Polombo 1988; 1992] for those quadratic functions.

So, finally we have:

Theorem 8. *If $K_{\text{av}} - K_{\min} \leq \frac{1}{2}[K_{\max} - K_{\min}]$, then M has a constant holomorphic sectional curvature.*

Remark 9. The reason we did not get this earlier was that there was a difficulty when $A = B$. In that case, the argument in [Siu and Yang 1981, p. 475, case 2] seems not to work. Polombo resolved the problem by using a function which is symmetric about $\lambda_1 = -A/3$ and $\lambda_2 = A - 3B/6$ (see [Polombo 1992] the first paragraph of page 418 and the end of page 397). However, Hong Cang Yang's function Ψ is only $-6\lambda_2$ and therefore is not symmetric after all. To overcome this difficulty, we let $\Omega = \{x \in M|_{A=B}\}$. Then according to [Siu and Yang 1981], all our calculation are good on $M - \Omega$ since $N \subset \Omega$. In [Chen et al. 2011, p. 2632] there was a suggestion on how to prove that $\operatorname{codim} \Omega \leq 2$, although it was not very well explained. By doing this, everything went through. The relation was that if we use the argument in [Siu and Yang 1981, p. 475, case 2] using the maximum instead of the minimum, and we let $B_1 = |R_{1\bar{2}1\bar{2}}|$ then $2B_1 = A - B$. That is $\Omega = \{x \in M|_{B_1=0}\}$. The argument goes as follows:

Case 1: If Ω is a closed region, we have

$$0 \geq \int_{M-\Omega} \Delta g = a \int_{-\partial\Omega} \Psi^{a-1} \frac{\partial(-A_1 - 3B_1)}{\partial n}$$
$$\geq a \int_{-\partial\Omega} (2A)^{a-1} \frac{\partial(-A_1)}{\partial n} = -\int_{\Omega} \Delta F_1 \geq 0,$$

where F_1 can be chosen from one of the functions in [Polombo 1992] which satisfy the symmetric condition on M, e.g., a power of Φ_2 from the proof of Proposition 1, or one of our functions with a calculation using the new smooth coordinate in [Siu and Yang 1981, p. 475] with $R_{1\bar{1}1\bar{1}} = K_{\max}$. Actually, A_1 itself is proportional to the λ_2 in [Polombo 1992] and is symmetric in the sense of Polombo. On Ω, F_1 is just our g since $B_1 = 0$. We notice that there is a sign difference for the Laplacian operator in [Polombo 1992]. Again, on Ω, since $A = B$ on a neighborhood, the set of minimum directions is an S^1 bundle over Ω, therefore one can choose a smooth section of it locally such that the calculation of [Siu and Yang 1981] still works in our case. That is, one could simply choose F_1 to be g.

Case 2: If Ω is a hypersurface then the same argument goes through except that

$$\int_{\partial(M-\Omega)} (A)^{a-1} \frac{\partial A}{\partial n} = 0,$$

since $A \neq 0$ outside a codimension one subset and on $\Omega_1 = \{x \in \Omega|_{A\neq 0}\}$, the integral is integrated from both sides.

Therefore, Ω is a subset of codimension two and we can apply Remark 5. By the calculation in Remark 5, we see that g is harmonic on $M - \Omega$. Now, by Lemma 6, that implies that $B(B-3A) = 0$ and hence $A = B = 0$ by our assumptions.

4. The generalization

Actually, in the first section of [Siu and Yang 1981], the authors did not require any negativity. We also see that in Section 2, we do not really need any negativity except when we apply the formula from Lemma 6 in the Section 3.

In the first section of [Siu and Yang 1981], they also consider the coordinate in which $R_{1\bar{1}1\bar{1}}$ achieves the maximum instead of the minimum. We let $C = R_{1\bar{1}2\bar{2}}$ from the earlier sections and C_1 be the bisectional curvature for the maximal case. Then

$$K_{\min} + C = K_{\max} + C_1$$

is the Einstein constant Q,

$$C_1 - C = K_{\min} - K_{\max} = -\tfrac{1}{2}(A+B)$$

and

$$C_1 = C - \tfrac{1}{2}(A+B) = \tfrac{1}{2}(R_{1\bar{1}1\bar{1}} - B) = \tfrac{1}{2}\big(Q - C_1 - \tfrac{1}{2}(A+B) - B\big).$$

Therefore

$$3C_1 = Q - \tfrac{1}{2}(A+B) - B \leq 0,$$

always. Also, $C_1 = 0$ implies that $A = B = Q = 0$.

The constant term in Lemma 6 is

$$\begin{aligned} 3\big[(3B-A)C - B(3A-B)\big] &= 3\big[(3B-A)\big(C_1 + \tfrac{1}{2}(A+B)\big) - B(3A-B)\big] \\ &= \tfrac{3}{2}[2\Psi C_1 - (A-B)(A+5B)] \\ &\leq 0, \end{aligned}$$

always. Therefore, we have the same result only if $Q \leq 0$, unless $C_1 = 0$. As above if $C_1 = 0$ we have $A = B = 0$, then $C = 0$ and therefore $K_{\min} = Q = 0$. The manifold is flat.

Thus we conclude the general case. One might conjecture that our theorem is also true in the higher dimensional cases.

Remark 10. Notice that this generalization basically covers the results in [Polombo 1988; Polombo 1992] for the Kähler–Einstein case (see Corollary on page 398 of [Polombo 1992]). See also [Derdziński 1983, p. 415, Proposition 2] for the W^+ for a Kähler surface. One might ask whether our result could be generalized to the Riemannian manifolds with closed half Weyl curvature tensors. This is out of the scope of this paper although a similar result is true, i.e., $\lambda_2 \leq 1$ at every point. To make the relation between this paper and [Polombo 1988; Polombo 1992] clearer to the reader, we mention that any one of the half Weyl tensors is harmonic if and only if it is closed since the tensor is dual to either itself or the negative of itself. Remark (i) in [Polombo 1992, p. 397] notes that if M is Riemannian–Einstein, then the second Bianchi identity says that the half Weyl tensors are closed (see also [Derdziński 1983] page 408 formula (9) and page 411 Remark 1).

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References

[Chen et al. 2011] D. Chen, Y. Hong, and H. Yang, "Kähler–Einstein surface and symmetric space", *Sci. China Math.* **54**:12 (2011), 2627–2634. MR Zbl

[Derdziński 1983] A. Derdziński, "Self-dual Kähler manifolds and Einstein manifolds of dimension four", *Compositio Math.* **49**:3 (1983), 405–433. MR Zbl

[Hong et al. 1988] Y. Hong, Z. D. Guan, and H. C. Yang, "A remark on Kähler–Einstein manifolds", *Acta Math. Sinica* **31**:5 (1988), 595–602. MR Zbl

[Polombo 1988] A. Polombo, "Condition d'Einstein et courbure négative en dimension 4", *C. R. Acad. Sci. Paris Sér. I Math.* **307**:12 (1988), 667–670. MR Zbl

[Polombo 1992] A. Polombo, "De nouvelles formules de Weitzenböck pour des endomorphismes harmoniques, applications géométriques", *Ann. Sci. École Norm. Sup.* (4) **25**:4 (1992), 393–428. MR Zbl

[Siu and Yang 1981] Y. T. Siu and P. Yang, "Compact Kähler–Einstein surfaces of nonpositive bisectional curvature", *Invent. Math.* **64**:3 (1981), 471–487. MR Zbl

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