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AUGMENTATIONS AND RULINGS OF LEGENDRIAN LINKS IN $\#^k(S^1 \times S^2)$

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Given a Legendrian link in $\#^k(S^1 \times S^2)$, we extend the definition of a normal ruling from $J^1(S^1)$ given by Lavrov and Rutherford and show that the existence of an augmentation to any field of the Chekanov–Eliashberg differential graded algebra over $\mathbb{Z}[t, t^{-1}]$ is equivalent to the existence of a normal ruling of the front diagram. For Legendrian knots, we also show that any even graded augmentation must send t to -1. We use the correspondence to give nonvanishing results for the symplectic homology of certain Weinstein 4-manifolds. We show a similar correspondence for the related case of Legendrian links in $J^1(S^1)$, the solid torus.

1. Introduction

Augmentations and normal rulings are important tools in the study of Legendrian knot theory, especially in the study of Legendrian knots in $\mathbb{R}^3$. Here, augmentations are augmentations of the Chekanov–Eliashberg differential graded algebra introduced by Chekanov [2002] and Eliashberg [1998]. Chekanov describes the noncommutative differential graded algebra (DGA) over $\mathbb{Z}/2$ associated to a Lagrangian diagram of a Legendrian link in $(\mathbb{R}^3, \xi_{\text{std}})$ combinatorially: The DGA is generated by crossings of the link; the differential is determined by a count of immersed polygons whose corners lie at crossings of the link and whose edges lie on the link. This is called the Chekanov–Eliashberg DGA and Chekanov showed that the homology of this DGA is invariant under Legendrian isotopy. Etnyre, Ng, and Sabloff [Etnyre et al. 2002] defined a lift of the Chekanov–Eliashberg DGA to a DGA over $\mathbb{Z}[t, t^{-1}]$ in. Following ideas introduced by Eliashberg [1987] and motivated by generating families (functions whose critical values generate front diagrams of Legendrian knots), Fuchs [2003] and Chekanov and Pushkar [2005] gave invariants of Legendrian knots in $\mathbb{R}^3$. Fuchs looked at decompositions of these generating families, generally called "normal rulings."

These two invariants are very closely related; Fuchs [2003], Fuchs and Ishkhanov [2004], and Sabloff [2005] showed that the existence of a normal ruling is equivalent

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to the existence of an augmentation to $\mathbb{Z}/2$ of the Chekanov–Eliashberg DGA $\mathcal{A}$ for Legendrian knots in $\mathbb{R}^3$. Here, given a unital ring S, an augmentation of $\mathcal{A}$ is a ring map $\epsilon : \mathcal{A} \to S$ such that $\epsilon \circ \partial = 0$ and $\epsilon(1) = 1$. One of the main results of [Leverson 2016] is that the equivalence remains true when one looks at augmentations to a field of the lift of the Chekanov–Eliashberg DGA from [Etnyre et al. 2002] to the DGA over $\mathbb{Z}[t, t^{-1}]$ for Legendrian knots in $\mathbb{R}^3$. We extend the result to Legendrian *links* in $\mathbb{R}^3$ to prove the main result of this paper.

Theorem 1.1. *Let Λ be an s-component Legendrian link in $\mathbb{R}^3$. Given a field F, the Chekanov–Eliashberg DGA $(\mathcal{A}, \partial)$ over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ has a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ if and only if a front diagram of Λ has a ρ-graded normal ruling. Furthermore, if ρ is even, then $\epsilon(t_1 \cdots t_s) = (-1)^s$.*

The final statement tells us that for all even graded augmentations $\epsilon : \mathcal{A} \to F$, $\epsilon(t_1 \cdots t_s) = (-1)^s$. In particular, if Λ is a knot, then any even graded augmentation sends t to -1.

For $k \geq 0$, an analogous correspondence can be shown for Legendrian links in $\#^k(S^1 \times S^2)$. A Legendrian link in $\#^k(S^1 \times S^2)$ with the standard contact structure is an embedding $\Lambda : \coprod_s S^1 \to \#^k(S^1 \times S^2)$ which is everywhere tangent to the contact planes. We will think of them as Gompf [1998] does. For an example, see Figure 2. In this paper, we extend the definition of normal ruling of a Legendrian link in $\mathbb{R}^3$ to a Legendrian link in $\#^k(S^1 \times S^2)$. We then define the ruling polynomial for a Legendrian link in $\#^k(S^1 \times S^2)$ and show that the ruling polynomial is invariant under Legendrian isotopy. Note that Lavrov and Rutherford [2013] did this previously in the case where $k = 1$.

Theorem 1.2. *The ρ-graded ruling polynomial $R^{\rho}_{(\Lambda,m)}$ with respect to the Maslov potential m (which changes under Legendrian isotopy) is a Legendrian isotopy invariant.*

Ekholm and Ng [2015] extend the definition of the Chekanov–Eliashberg DGA over $\mathbb{Z}[t, t^{-1}]$ to Legendrian links in $\#^k(S^1 \times S^2)$. The main result of this paper uses Theorem 1.1 to extend the correspondence between normal rulings and augmentations to a correspondence for Legendrian links in $\#^k(S^1 \times S^2)$.

Theorem 1.3. *Let Λ be an s-component Legendrian link in $\#^k(S^1 \times S^2)$ for some $k \geq 0$. Given a field F, the Chekanov–Eliashberg DGA $(\mathcal{A}(\Lambda), \partial)$ over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ has a ρ-graded augmentation $\epsilon : \mathcal{A}(\Lambda) \to F$ if and only if a front diagram of Λ has a ρ-graded normal ruling. Furthermore, if ρ is even, then $\epsilon(t_1 \cdots t_s) = (-1)^s$.*

Notice that one can consider Legendrian links in $\mathbb{R}^3$ as being Legendrian links in $\#^0(S^1 \times S^2)$. In this way, this result is a generalization of the correspondence in [Leverson 2016] and Theorem 1.1. An immediate corollary is the following:

Corollary 1.4. *If Λ is a Legendrian link in $\#^k(S^1 \times S^2)$ and there exists ℓ such that N_ℓ is odd, then there does not exist a ρ-graded augmentation of the DGA $\mathcal{A}(\Lambda)$ for any ρ.*

In other words, if Λ has a 1-handle with an odd number of strands going through it, then there does not exist a ρ-graded augmentation of the DGA $\mathcal{A}(\Lambda)$ for any ρ. This follows from the fact that every involution of a set with an odd number of elements has a fixed-point.

Along with the work of Bourgeois, Ekholm, and Eliashberg [Bourgeois et al. 2012], Theorem 1.3 gives nonvanishing results for Weinstein (Stein) 4-manifolds. (Note that proofs of the results in [loc. cit.] have not appeared yet.) In particular:

Corollary 1.5. *If X is the Weinstein 4-manifold obtained from attaching 2-handles along a Legendrian link Λ to $\#^k(S^1 \times S^2)$ and Λ has a graded normal ruling, then the full symplectic homology $S\mathbb{H}(X)$ is nonzero.*

This follows from Theorem 1.3 as the existence of a normal ruling implies the existence of an augmentation to $\mathbb{Q}$, which, by [Bourgeois et al. 2012], is a sufficient condition for the full symplectic homology to be nonzero.

We show a correspondence for Legendrian links in the 1-jet space of the circle $J^1(S^1)$. Ng and Traynor [2004] extend the definition of the Chekanov–Eliashberg DGA to Legendrian links in $J^1(S^1)$. Lavrov and Rutherford [2012] extend the definition of normal ruling to a "generalized normal ruling" of Legendrian links in $J^1(S^1)$ and show that the existence of a generalized normal ruling is equivalent to the existence of an augmentation to $\mathbb{Z}/2$ of the Chekanov–Eliashberg DGA over $\mathbb{Z}/2$ of a Legendrian link in $J^1(S^1)$. In Section 6, we show that this correspondence holds for augmentations to any field of the Chekanov–Eliashberg DGA over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$.

Theorem 1.6. *Suppose that Λ is a Legendrian link in $J^1(S^1)$. Given a field F, the Chekanov–Eliashberg DGA $(\mathcal{A}, \partial)$ over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ has a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ if and only if a front diagram of Λ has a ρ-graded generalized normal ruling.*

1A. ***Outline of the article.*** In Section 2, we recall background on Legendrian links in $\#^k(S^1 \times S^2)$ and $\mathbb{R}^3$. We give definitions of the Chekanov–Eliashberg DGA over $\mathbb{Z}[t, t^{-1}]$, with sign conventions, and augmentations of the DGA in both $\#^k(S^1 \times S^2)$ and $\mathbb{R}^3$. We also define normal rulings for links in $\#^k(S^1 \times S^2)$ and show that the ruling polynomial is invariant under Legendrian isotopy, proving Theorem 1.2. In Section 3, we prove Theorem 1.1. In Section 4, given an augmentation, we construct a normal ruling proving one direction of Theorem 1.3. In Section 5, given a normal ruling, we construct an augmentation, finishing the proof of Theorem 1.3. In Section 6, we prove Theorem 1.6. In the Appendix, we give the nonvanishing symplectic homology result.

2. Background material

2A. ***Legendrian links in*** $\#^k(S^1 \times S^2)$**.** In this section we will briefly discuss necessary concepts of Legendrian links in $\#^k(S^1 \times S^2)$. We will follow the notation in [Ekholm and Ng 2015].

Definition 2.1. Let $A, M > 0$ be fixed. A tangle in $[0, A] \times [-M, M] \times [-M, M]$ is *Legendrian* if it is everywhere tangent to the standard contact structure $dz - y\,dx$. Informally, a Legendrian tangle T in $[0, A] \times [-M, M] \times [-M, M]$ is in *normal form* if

- T meets $x = 0$ and $x = A$ in k groups of strands, where the groups are of size $N_1, \dots, N_k$, from top to bottom in both the xy- and xz-projections,
- and within the ℓ-th group, we label the strands by $1, \dots, N_\ell$ from top to bottom at $x = 0$ in both the xy- and xz-projections and $x = A$ in the xz-projection, and from bottom to top at $x = A$ in the xy-projection.

Every Legendrian tangle in normal form gives a Legendrian link in $\#^k(S^1 \times S^2)$ by attaching k 1-handles which join parts of the xz projection of the tangle at $x = 0$ to the parts at $x = A$. In particular, the ℓ-th 1-handle joins the ℓ-th group at $x = 0$ to the ℓ-th group at $x = A$ and connects the strands in this group with the same label at $x = 0$ and $x = A$ through the 1-handle. See Figure 2.

Every Legendrian link in $\#^k(S^1 \times S^2)$ has an xz-diagram of the form given by Gompf [1998], which we will call *Gompf standard form*. The left diagram of Figure 2 is an example of a link in Gompf standard form. Any link in Gompf standard form can be isotoped to a link whose xy-projection is obtained from the xz-diagram by *resolution*. The resolution of an xz-diagram of a link is obtained by the replacements given in Figure 1. For an example, see Figure 2. By [Ekholm and Ng 2015], an xy-diagram obtained by the resolution of an xz-diagram of a link in Gompf standard form is in normal form. Thus, we can assume that the xy-diagram of any Legendrian link is in normal form.

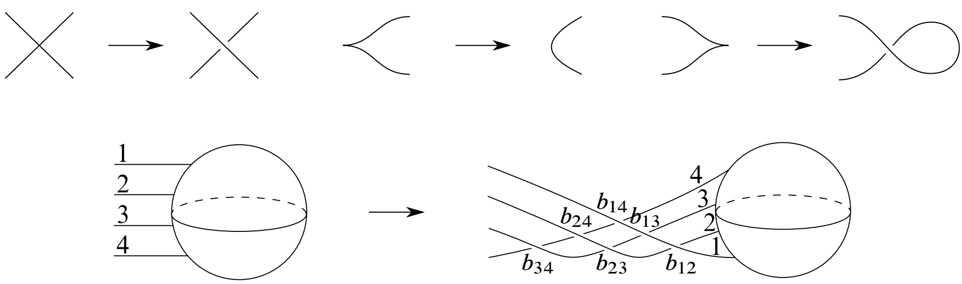


Figure 1. Resolutions of an xz-diagram in Gompf standard form.

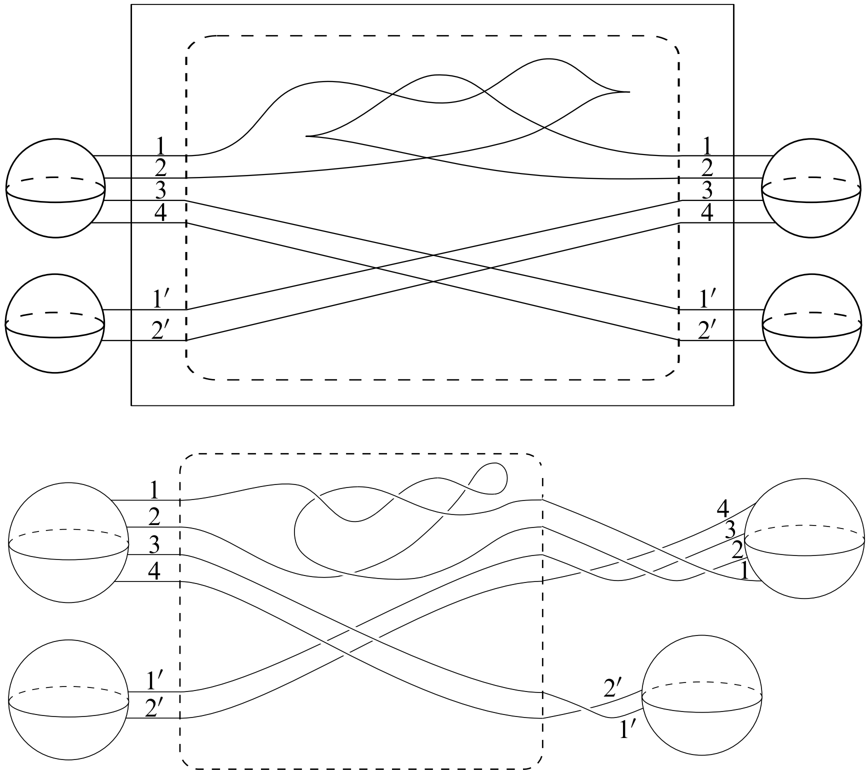


Figure 2. A Legendrian xz-diagram of a link in $\#^2(S^1 \times S^2)$ in Gompf standard form (top), and the resolution of the Legendrian link to an xy-diagram of a Legendrian isotopic link (bottom).

2B. ***Definition of the DGA and augmentations in $\#^k(S^1 \times S^2)$.*** This section contains an overview of the differential graded algebra over the ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ presented by Ekholm and Ng [2015]. Let $\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_n$ be a Legendrian link in $\#^k(S^1 \times S^2)$ in normal form, where the Λ_i denote the components of Λ and $n \leq s$. On each link component Λ_i, label a point by $*_i$ (corresponding to t_i) within the tangle (away from crossings). We will discuss the case where there is more than one basepoint on a given component in Section 2K. Let $N_i \geq 1$ be the number of strands of Λ which go through the i-th 1-handle with $N = \sum N_i$ the total number of strands at $x = 0$.

2C. ***Internal DGA.*** We will define the internal DGA for a Legendrian link in $S^1 \times S^2$, but one can easily extend the definition to the internal DGA for a Legendrian link in $\#^k(S^1 \times S^2)$ by defining the internal DGA as follows for each 1-handle separately.

Let $(r_1, \dots, r_n) \in \mathbb{Z}^n$ be the n-tuple where r_i is the rotation number of the i-th component Λ_i, let $r = \gcd(r_1, \dots, r_n)$, and let $(m(1), \dots, m(N)) \in (\mathbb{Z}/2r)^N$ be the N-tuple of a choice of Maslov potential for each strand passing through the 1-handle (see Section 2E).

Let $(\mathcal{A}_N, \partial_N)$ denote the DGA defined as follows. Let $\mathcal{A}$ be the tensor algebra over $R = \mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ generated by c_{ij}^0 for $1 \le i < j \le N$ and c_{ij}^p for $1 \le i, j \le N$ and $p \ge 1$. Set $|t_i| = -2r_i$, $|t_i^{-1}| = 2r_i$, and

$$|c_{ij}^p| = 2p - 1 + m(i) - m(j)$$

for all i, j, p. Define the differential ∂_N on the generators by

$$\begin{aligned}
\partial_N(c_{ij}^0) &= \sum_{\ell=i+1}^{j-1} (-1)^{|c_{i\ell}^0|+1} c_{i\ell}^0 c_{\ell j}^0, \\
\partial_N(c_{ij}^1) &= \delta_{ij} + \sum_{\ell=i+1}^{N} (-1)^{|c_{i\ell}^0|+1} c_{i\ell}^0 c_{\ell j}^1 + \sum_{\ell=1}^{j-1} (-1)^{|c_{i\ell}^1|+1} c_{i\ell}^1 c_{\ell j}^0, \\
\partial_N(c_{ij}^p) &= \sum_{\ell=0}^{p} \sum_{m=1}^{N} (-1)^{|c_{im}^\ell|+1} c_{im}^\ell c_{mj}^{p-\ell},
\end{aligned}$$

where $p \ge 2$, δ_{ij} is the Kronecker delta function, and we set $c_{ij}^0 = 0$ for $i \ge j$. Extend ∂_N to $\mathcal{A}_N$ by the Leibniz rule

$$\partial_N(xy) = (\partial_N x)y + (-1)^{|x|} x(\partial_N y).$$

From [Ekholm and Ng 2015], we know ∂_N has degree -1, $\partial_N^2 = 0$, and $(\mathcal{A}_N, \partial_N)$ is infinitely generated as an algebra, but is a filtered DGA, where c_{ij}^p is a generator of the ℓ-th component of the filtration if $p \le \ell$.

Given a Legendrian link $\Lambda \subset \#^k(S^1 \times S^2)$, we can associate a DGA $(\mathcal{A}_{N_i}, \partial_{N_i})$ to each of the 1-handles. We then call the DGA generated by the collection of generators of $\mathcal{A}_i$ for $1 \le i \le k$ with differential induced by ∂_{N_i}, the *internal DGA* of Λ.

2D. ***Algebra.*** Suppose we have a Legendrian link $\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_n \subset \#^k(S^1 \times S^2)$ in normal form with exactly one point labeled $*_i$ within the tangle (away from crossings) on each link component Λ_i of Λ (corresponding to t_i). We will discuss the case where there is more than one basepoint on a given component in Section 2K.

Notation 2.2. Let $\tilde{a}_1, \dots, \tilde{a}_m$ denote the crossings of the xy tangle diagram in normal form. Label the k 1-handles in the diagram by $1, \dots, k$ from top to bottom. Recall that N_i denotes the number of strands of the tangle going through the i-th 1-handle. For each i, label the strands going through the i-th 1-handle on the left side of the diagram $1, \dots, N_i$ from top to bottom and from bottom to top on the right side, as in Figure 2.

Let $\mathcal{A}(\Lambda)$ be the tensor algebra over $R = \mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ generated by

- $\tilde{a}_1, \dots, \tilde{a}_m$;
- $c^0_{ij;\ell}$ for $1 \le \ell \le k$ and $1 \le i < j \le N_\ell$;
- $c^p_{ij;\ell}$ for $1 \le \ell \le k$, $p \ge 1$, and $1 \le i, j \le N_\ell$.

(In general, we will drop the index ℓ when the 1-handle is clear.)

2E. ***Grading.*** The following are a few preliminary definitions which will allow us to define the grading on the generators of $\mathcal{A}(\Lambda)$.

Definition 2.3. A *path* in $\pi_{xy}(\Lambda)$ is a path that traverses part (or all) of $\pi_{xy}(\Lambda)$ which is connected except for where it enters a 1-handle, meaning, where it approaches $x = 0$ (respectively $x = A$) along a labeled strand and exits the 1-handle along the strand with the same label from $x = A$ (respectively $x = 0$). Note that the tangent vector in $\mathbb{R}^2$ to the path varies continuously as we traverse a path as the strands entering and exiting 1-handles are horizontal.

The *rotation number* $r(\gamma)$ of a path γ is the number of counterclockwise revolutions around S^1 made by the tangent vector $\gamma'(t)/|\gamma'(t)|$ to γ as we traverse γ. Generally this will be a real number, but will be an integer if and only if γ is smooth and closed.

Thus, the rotation number $r_i = r(\Lambda_i)$ is the rotation number of the path in $\pi_{xy}(\Lambda)$ which begins at the basepoint $*_i$ on the link component Λ_i and traverses the link component, following the orientation of the component. In the case where Λ is a link with components $\Lambda_1, \dots, \Lambda_n$, we define

$$r(\Lambda) = \gcd(r_1, \dots, r_n).$$

Define

$$|t_i| = -2r(\Lambda_i).$$

If $\pi_{xy}(\Lambda)$ is the resolution of an xz-diagram of an n-component link in Gompf standard form, then the method assigning gradings follows: Choose a *Maslov potential* m that associates an integer modulo $2r(\Lambda)$ to each strand in the tangle T associated to Λ, minus cusps and basepoints, such that the following conditions hold:

(1) For all $1 \le \ell \le k$ and all $1 \le i \le N_\ell$, the strand labeled i going through the ℓ-th 1-handle at $x = 0$ and the $x = A$ must have the same Maslov potential.

(2) If a strand is oriented to the right, meaning it enters the 1-handle at $x = A$ and exits at $x = 0$, then the Maslov potential of the strand must be even. Otherwise the Maslov potential of the strand must be odd.

(3) At a cusp, the upper strand (strand with higher z-coordinate) has Maslov potential one more than the lower strand.

The Maslov potential is well-defined up to an overall shift by an even integer for knots. (Ekholm and Ng [2015] give another method for defining the gradings using the rotation numbers of specified paths.)

Set $|t_i| = -2r(\Lambda_i)$ and $|c_{ij;\ell}^p| = 2p - 1 + m(i) - m(j)$, where $m(i)$ means the Maslov potential of the strand with label i going through the ℓ-th 1-handle. It remains to define the grading on crossings in the tangle, crossings resulting from resolving right cusps, and crossings from the half-twists in the resolution. If a is a crossing in the tangle T, then define

$$|a| = m(S_o) - m(S_u),$$

where S_o is the strand which crosses over the strand S_u at a in the xy-projection of Λ. If a is a right cusp, define $|a| = 1$ (assuming there is not a basepoint in the loop). If a is a crossing in one of the half-twists in the resolution where strand i crosses over strand j $(i < j)$, then

$$|a| = m(i) - m(j).$$

2F. *Differential.* It suffices to define the differential ∂ on generators and extend by the Leibniz rule. Define $\partial(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]) = 0$. Set $\partial = \partial_{N_\ell}$ on $\mathcal{A}_{N_\ell}$ as in Section 2C.

In [Ekholm and Ng 2015], the DGA on crossings a_i is defined by looking for immersed disks in the xy-diagrams of Legendrian links, (see the left diagram in Figure 3). However, Ekholm and Ng note that it is equivalent to look for immersed disks in dip versions of the diagram, (see the right diagram in Figure 3). See Figure 4 for the labeling of the crossings in Figure 3.

Definition 2.4. Let $a, b_1, \ldots, b_\ell$ be generators. Define $\Delta(a; b_1, \ldots, b_\ell)$ to be the set of orientation-preserving maps

$$f : D^2 \to \mathbb{R}^2$$

(up to smooth reparametrization) that map ∂D^2 to the dip version of Λ such that

(1) f is a smooth immersion except at $a, b_1, \ldots, b_\ell$,

(2) $a, b_1, \ldots, b_\ell$ are encountered as one traverses $f(\partial D^2)$ counterclockwise,

(3) near $a, b_1, \ldots, b_\ell$, $f(D^2)$ covers exactly one quadrant, specifically, a quadrant with positive Reeb sign near a and a quadrant with negative Reeb sign near

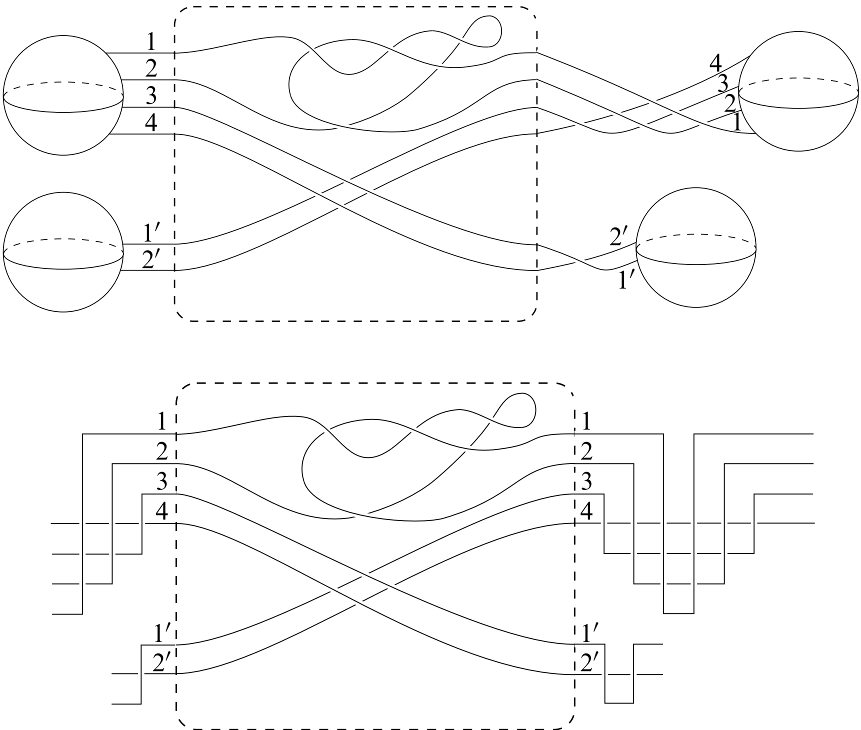


Figure 3. A Legendrian xy-diagram of a link in $\#^2(S^1 \times S^2)$ which has resulted from the resolution of a link in Gompf standard form (top) and the dipped version of the link where the half of a dip on the left side of the dipped version is identified with the right half of the dip on the right side. See Figure 4 for the labeling of the crossings in the dips (bottom).

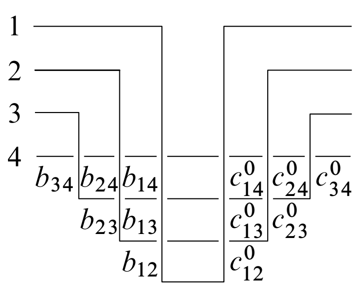


Figure 4. This is the dip at the right of the bottom figure in Figure 3 with strands and crossings labeled. The labels of the partial dip at the left of the bottom figure in Figure 3 are the same as the right half of the dip depicted.

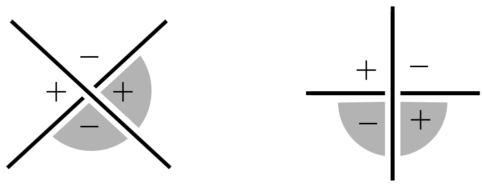


Figure 5. The signs in the figure give the Reeb signs of the quadrants around the crossings. The orientation signs are $+1$ for all quadrants of crossings of odd degree. For crossings of even degree, we use the convention indicated in the left figure if the crossing comes from the xz-projection and the convention in the right figure if the crossing is in a dip, which will be discussed in Section 2J, where the shaded quadrants have orientation sign -1 and the other quadrants have orientation sign $+1$.

$b_1, \ldots, b_\ell$, where the Reeb sign of a quadrant near a crossing is defined as in Figure 5.

To each immersed disk, we can assign a word in $\mathcal{A}(\Lambda)$ by starting with the first corner where the quadrant covered has negative Reeb sign, b_1, and listing the crossing labels of all negative corners as encountered while following the boundary of the immersed polygon counterclockwise, $b_1 \cdots b_\ell$. We associate an *orientation sign* $\delta_{Q,a}$ to each quadrant Q in the neighborhood of a crossing a, defined in Figure 5, and use these to define the sign of a disk $f(D^2)$ to be the product of the orientation signs over all the corners of the disk. We denote this sign by $\delta(f)$. In many cases there is a unique disk with positive corner at a (with respect to Reeb sign) and negative corners at $b_1, \ldots, b_\ell$ and in these we define $\delta(a; b_1, \ldots, b_\ell)$ to be the sign of the unique disk. (In exceptional cases there may be more than one disk with positive corner at a and negative corners at $b_1, \ldots, b_\ell$.)

Define $n_{*_i}(f)$ or $n_{*_i}(a; b_1, \ldots, b_\ell)$ to be the signed count of the number of times one encounters the basepoint $*_i$ while following $f(\partial D^2)$ counterclockwise, where the sign is positive if we encounter the basepoint while following the orientation of the link component and negative if we encounter the basepoint while going against the orientation.

We define

$$\partial(a_i) = \sum_{\ell \geq 0} \sum_{(b_1, \ldots, b_\ell)} \sum_{f \in \Delta(a_i; b_1, \ldots, b_\ell)} \delta(f) t_1^{n_{*_1}(f)} \cdots t_s^{n_{*_s}(f)} b_1 \cdots b_\ell$$

and extend to $\mathcal{A}(\Lambda)$ by the Leibniz rule.

Ekholm and Ng [2015] prove that the map ∂ has degree -1 and is a differential, i.e., $\partial^2 = 0$.

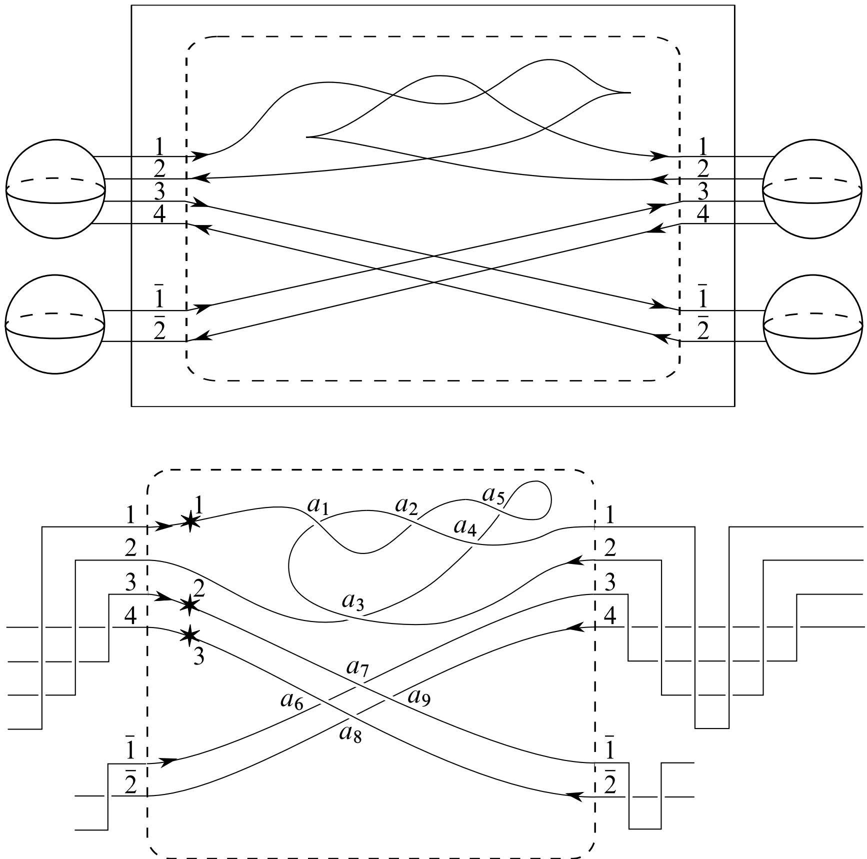


Figure 6. A Legendrian xz-diagram in $\#^2(S^1 \times S^2)$ in Gompf standard form (top) and the dip form of the normal form (bottom). Recall the labels on the crossings in the dips from Figure 4 for the top 1-handle and label the left crossing $\bar{b}_{12}$ and the right $\bar{c}_{12}$ in the dip of the bottom 1-handle.

Example 2.5. The following is the definition of the DGA $(\mathcal{A}(\Lambda), \partial)$ for the Legendrian link Λ in Figure 6. Here $\mathcal{A}(\Lambda)$ is generated by $a_1, \ldots, a_9, b_{ij}, c_{ij}^p$ over $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$. We set $|t_i| = 2r(\Lambda_i) = 0$ for $i = 1, 2, 3$. Define a Maslov potential m on the strands near $x = 0$ by

i	1	2	3	4	$\bar{1}$	$\bar{2}$
$m(i)$	2	1	0	-1	0	-1

Then we have the following gradings:

$$|a_1| = |a_2| = |a_3| = |a_7| = |a_8| = 0, \quad |a_4| = |a_5| = |a_9| = 1, \quad |a_6| = -1,$$

ij	12	13	14	23	24	34	$\overline{12}$
$\lvert b_{ij}\rvert$	1	2	3	2	2	1	1
$\lvert c^0_{ij}\rvert$	0	1	2	0	1	0	0

	$\lvert c^1_{ij}\rvert$	$j=1$	2	3	4
i	1	1	2	3	4
	2	0	1	2	3
	3	-1	0	1	2
	4	-2	-1	0	1

	$\lvert c^2_{ij}\rvert$	$j=1$	2	3	4
i	1	3	4	5	6
	2	2	3	4	5
	3	1	2	3	4
	4	0	1	2	3

where $\overline{12}$ is the crossing of the strands in the bottom 1-handle. Since $|c^p_{ij}| = 2p-1+m(i)-m(j)$, we know $|c^p_{ij}|>0$ for $p>2$.

For ease of notation, we will use $\bar{c}^p_{12}$ to denote $c^p_{\overline{12}}$. We then have the following differentials:

$$
\begin{aligned}
\partial a_1 &= \partial a_2 = \partial a_3 = \partial a_6 = 0\\
\partial a_4 &= (1+a_2a_1)a_3 - t_1^{-1}a_2c^0_{12}\\
\partial a_5 &= 1-a_1a_3+t_1^{-1}c^0_{12}\\
\partial a_7 &= t_2^{-1}t_3^{-1}c^0_{34}a_6\\
\partial a_8 &= a_6\bar{c}^0_{12}\\
\partial a_9 &= t_2^{-1}t_3^{-1}c^0_{34}a_8 - a_7\bar{c}^0_{12}
\end{aligned}
$$

$$
\begin{aligned}
\partial b_{12} &= 1+a_2a_1-c^0_{12}\\
\partial b_{13} &= (1+a_2a_1)b_{23}+a_4(t_2c^0_{23}a_7+t_3^{-1}c^0_{24}a_6)\\
&\qquad -t_1^{-1}a_2(t_2c^0_{13}a_7+t_3^{-1}c^0_{14}a_6)-c^0_{13}+b_{12}c^0_{23}\\
\partial b_{14} &= (1+a_2a_1)b_{24}\\
&\qquad -\big[a_4(t_2c^0_{23}a_7+t_3^{-1}c^0_{24}a_6)-t_1^{-1}a_2(t_2c^0_{13}a_7+t_3^{-1}c^0_{14}a_6)\big]b_{34}\\
&\qquad +(a_4c^0_{23}-t_1^{-1}a_2c^0_{13})t_2a_9+(a_4c^0_{24}-t_1^{-1}a_2c^0_{14})t_3^{-1}a_8\\
&\qquad -c^0_{14}+b_{12}c^0_{24}-b_{13}c^0_{34}\\
\partial b_{23} &= -a_3(t_2c^0_{23}a_7+t_3^{-1}c^0_{24}a_6)-c^0_{23}\\
\partial b_{24} &= -a_3(t_2c^0_{23}a_7+t_3^{-1}c^0_{24}a_6)b_{34}-t_3^{-1}a_3c^0_{24}a_8\\
&\qquad -c^0_{24}+b_{23}c^0_{34}-t_2a_3c^0_{23}a_9\\
\partial b_{34} &= \bar{c}^0_{12}-c^0_{34}
\end{aligned}
$$

$$\partial \bar{b}_{12} = t_2^{-1} t_3^{-1} c_{34}^0 - \bar{c}_{12}^0$$

$$\partial c_{ij}^p = \delta_{ij}\delta_{1p} + \sum_{\ell=0}^{p}\sum_{m=1}^{4} (-1)^{|c_{im}^\ell|+1} c_{im}^\ell c_{mj}^{p-\ell}$$

$$\partial \bar{c}_{ij}^p = \delta_{ij}\delta_{1p} + \sum_{\ell=0}^{p}\sum_{m=1}^{2} (-1)^{|\bar{c}_{im}^\ell|+1} \bar{c}_{im}^\ell \bar{c}_{mj}^{p-\ell}$$

Definition 2.6. Let $(\mathcal{A},\partial)$ be a semifree DGA over R generated by $\{a_i | i \in I\}$. Let J be a countable (possibly finite) index set. A *stabilization* of $(\mathcal{A},\partial)$ is the semifree DGA $(S(\mathcal{A}),\partial)$, where $S(\mathcal{A})$ is the tensor algebra over R generated by $\{a_i \mid i \in I\} \cup \{\alpha_j \mid j \in J\} \cup \{\beta_j | j \in J\}$ and the grading on a_i is inherited from $\mathcal{A}$ and $|\alpha_j| = |\beta_j| + 1$ for all $j \in J$. Let the differential on $S(\mathcal{A})$ agree with the differential on $\mathcal{A} \subset S(\mathcal{A})$, define

$$\partial(\alpha_j) = \beta_j \text{ and } \partial(\beta_j) = 0$$

for all $j \in J$, and extend by the Leibniz rule.

Definition 2.7 [Ekholm and Ng 2015]. Two semifree DGAs $(\mathcal{A},\partial)$ and $(\mathcal{A}',\partial')$ are *stable tame isomorphic* if some stabilization of $(\mathcal{A},\partial)$ is tamely isomorphic to some stabilization of $(\mathcal{A}',\partial')$.

Theorem 2.8 [op. cit., Theorem 2.18]. *Let Λ and Λ' be Legendrian isotopic Legendrian links in $\#^k(S^1\times S^2)$ in normal form. Let $(\mathcal{A}(\Lambda),\partial)$ and $(\mathcal{A}(\Lambda'),\partial')$ be the semifree DGAs over $R = \mathbb{Z}[t_1^{\pm1},\dots,t_s^{\pm1}]$ associated to the diagrams $\pi_{xy}(\Lambda)$ and $\pi_{xy}(\Lambda')$, which are in normal form. Then $(\mathcal{A}(\Lambda),\partial)$ and $(\mathcal{A}(\Lambda'),\partial')$ are stable tame isomorphic.*

Definition 2.9. Let F be a field. An *augmentation* of $(\mathcal{A}(\Lambda),\partial)$ to F is a ring map $\epsilon : \mathcal{A}(\Lambda) \to F$ such that $\epsilon \circ \partial = 0$ and $\epsilon(1) = 1$. If $\rho \mid 2r(\Lambda)$ and ϵ is supported on generators of degree divisible by ρ, then ϵ is ρ-graded. In particular, if $\rho = 0$, we say it is *graded* and if $\rho = 1$, we say if is *ungraded*. We call a generator a *augmented* if $\epsilon(a) \neq 0$.

Example 2.10. Recalling the DGA associated with the Legendrian link in Figure 6 of Example 2.5, given a field F, one can check that any graded augmentation ϵ to F satisfies: $\epsilon(t_1) = -1$, $\epsilon(t_3) = \epsilon(t_2)^{-1}$ where $\epsilon(t_2) \neq 0$, $\epsilon(b_{ij}) = \epsilon(\bar{b}_{12}) = 0$, and for $a, b, c, d, e, f \in F$ such that $1 + ab, d, e \neq 0$,

i	1	2	3	4	5	6	7	8	9
$\epsilon(a_i)$	a	b	$-b$	0	0	0	c	c	0

ij	12	13	14	23	24	34	$\overline{12}$
$\epsilon(c_{ij}^0)$	$1+ab$	0	0	0	0	d	d

$$\begin{array}{cc|cccc} & & & j & & \\ & |c^1_{ij}| & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 0 & 0 & 0 \\ & 2 & e & 0 & 0 & 0 \\ i & 3 & 0 & f & 0 & 0 \\ & 4 & 0 & 0 & (1+ab)d^{-1}e & 0 \end{array} \qquad \begin{array}{cc|cccc} & & & j & & \\ & |c^2_{ij}| & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 & 0 \\ i & 3 & 0 & 0 & 0 & 0 \\ & 4 & -(1+ab)d^{-1}f & 0 & 0 & 0 \end{array}$$

Note that any augmentation of a stabilization $S(\mathcal{A})$ restricts to an augmentation of the smaller algebra $\mathcal{A}$ and any augmentation of the algebra $\mathcal{A}$ extends to an augmentation of the stabilization $S(\mathcal{A})$ where the augmentation sends β_j to 0 and α_j to an arbitrary element of F if ρ divides $|\alpha_j|$ and 0 otherwise for all $j \in J$.

2G. ***Normal rulings in $\#^k(S^1 \times S^2)$.*** In this section, we extend the definition of a normal ruling from Legendrian links in $\mathbb{R}^3$ to Legendrian links in $\#^k(S^1 \times S^2)$. We formulate the definition similarly to how Lavrov and Rutherford [2012] define normal rulings in the case of Legendrian links in the solid torus.

Consider the tangle portion of the $\pi_{xz}(\Lambda)$ diagram in normal form of a Legendrian link $\Lambda \subset \#^k(S^1 \times S^2)$. A normal ruling can be viewed locally as a decomposition of $\pi_{xz}(\Lambda)$ into pairs of paths.

Let $C \subset S^1$ be the set of x-coordinates of crossings and cusps of $\pi_{xz}(\Lambda)$ where $S^1 = [0, A]/\{0 = A\}$. We can write

$$S^1 \backslash C = \coprod_{\ell=1}^{M} I_\ell$$

where I_ℓ is an open interval (or all of S^1) for each ℓ. We use the convention that $I_0 = I_M$ and the I_ℓ are ordered $I_0, \dots, I_M$ from $x = 0$ to $x = A$ (from left to right in the xz-diagram) so that $I_{\ell-1}$ appears to the left of (has lower x-coordinates than) I_ℓ. Note that $(I_\ell \times [-M, M]) \cap \pi_{xz}(\Lambda)$ consists of some number of nonintersecting components which project homeomorphically onto I_ℓ. We call these components *strands* of $\pi_{xz}(\Lambda)$ and number them from top to bottom by $1, \dots, N(\ell)$. For each ℓ, choose a point $x_\ell \in I_\ell$.

Definition 2.11. A *normal ruling* of $\pi_{xz}(\Lambda)$ is a sequence of involutions $\sigma = (\sigma_1, \dots, \sigma_M)$,

$$\sigma_m : \{1, \dots, N(m)\} \to \{1, \dots, N(m)\}, \qquad (\sigma_m)^2 = \mathrm{id},$$

satisfying:

(1) Each σ_m is fixed-point-free.

(2) If the strands above I_m labeled ℓ and $\ell+1$ meet at a left cusp in the interval (x_{m-1}, x_m), then

$$\sigma_m(i) = \begin{cases} \ell+1 & \text{if } i = \ell, \\ J(\sigma_{m-1}(i)) & \text{if } i < \ell, \\ J(\sigma_{m-1}(i-2)) & \text{if } i > \ell+1, \end{cases}$$

where

$$J(i) = \begin{cases} i, & i < \ell, \\ i+2, & i \geq \ell, \end{cases}$$

and a similar condition at right cusps.

(3) If strands above I_m labeled ℓ and $\ell+1$ meet at a crossing on the interval (x_{m-1}, x_m), then $\sigma_{m-1}(\ell) \neq \ell+1$ and either
- $\sigma_m = (\ell\ \ell+1) \circ \sigma_{m-1} \circ (\ell\ \ell+1)$, where $(\ell\ \ell+1)$ denotes transposition or
- $\sigma_m = \sigma_{m-1}$.

When the second case occurs, we call the crossing *switched*.

(4) (Normality condition) If there is a switched crossing on the interval (x_{m-1}, x_m), then one of the following holds:
- $\sigma_m(\ell+1) < \sigma_m(\ell) < \ell < \ell+1$,
- $\sigma_m(\ell) < \ell < \ell+1 < \sigma_m(\ell)$,
- $\ell < \ell+1 < \sigma_m(\ell+1) < \sigma_m(\ell)$.

(5) Near $x = 0$ and $x = A$, both the strand with label ℓ and the strand with label $\sigma_0(\ell)$ must go through the same 1-handle; in other words, there exists p such that $\sum_{i=1}^{p-1} N_i < \ell, \sigma_0(\ell) \leq \sum_{i=1}^{p} N_i$.

The final condition is the only condition which is different from how normal rulings are defined in [Lavrov and Rutherford 2012] for the case of solid torus knots. This condition ensures the ruling "behaves well" with the 1-handles.

Remark 2.12. As in [loc. cit.], one can equivalently see normal rulings as pairings of strands in the xz-diagram with certain conditions. Here we think of strands i and j being paired for $x_{m-1} \leq x \leq x_m$ if $\sigma_m(i) = j$. In this way, we can cover the xz-diagram with pairs of paths which have monotonically increasing x-coordinate. Note that if a path goes all the way from $x = 0$ to $x = A$, it may end up on a different strand than it started, but strand i is paired with strand j at $x = 0$ if and only if they are paired at $x = A$. Condition (5) also specifies that the paired strands must go through the same 1-handle. The conditions mentioned above are as follows: Paired paths can only meet at a cusp. This also means that at a crossing, the crossings strands must be paired with other strands. These *companion strands* can either lie above or below the crossing. Conditions (3) and (4) specify that near a crossing the pairings must be one of those depicted in Figure 7.

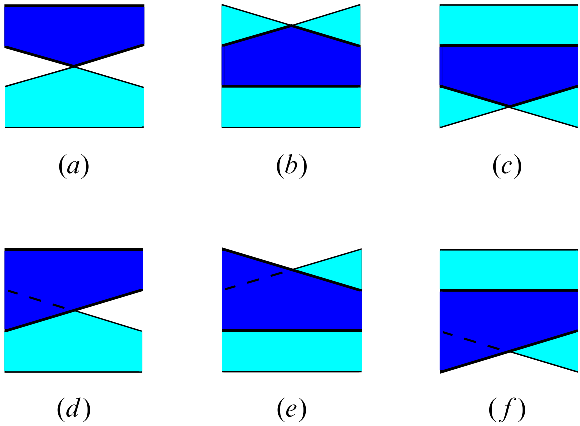


Figure 7. These configurations, along with vertical reflections of (d), (e), and (f), are all possible configurations of a normal ruling near a crossing. The top row contains all possible configurations for switched crossings in a normal ruling. (This figure is taken from [Leverson 2016].)

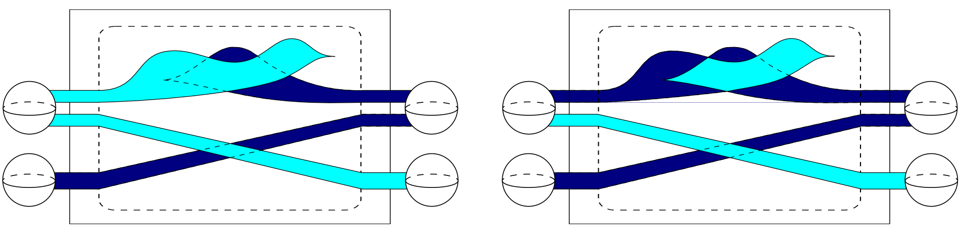


Figure 8. These are the two normal rulings of the Legendrian link of Example 2.5 seen in Figure 6.

Example 2.13. Figure 8 gives the normal rulings of the Legendrian link from Example 2.5.

Definition 2.14. Given ρ such that $\rho \mid 2r(\Lambda)$ and an $\mathbb{Z}/\rho$-valued Maslov potential on Λ, a normal ruling is *ρ-graded* with respect to the $\mathbb{Z}/\rho$-valued Maslov potential if whenever two strands are paired by one of the σ_m, the upper strand (strand with lower label) has Maslov potential one higher than the lower strand (strand with higher label).

Remark 2.15. Note that the condition for being a ρ-graded normal ruling of a Legendrian link in $\#^k(S^1 \times S^2)$ implies that $\rho \,\big|\, |c|$ if the normal ruling is switched at a crossing c. Further, any Legendrian link in $\mathbb{R}^3$ is also a Legendrian link in

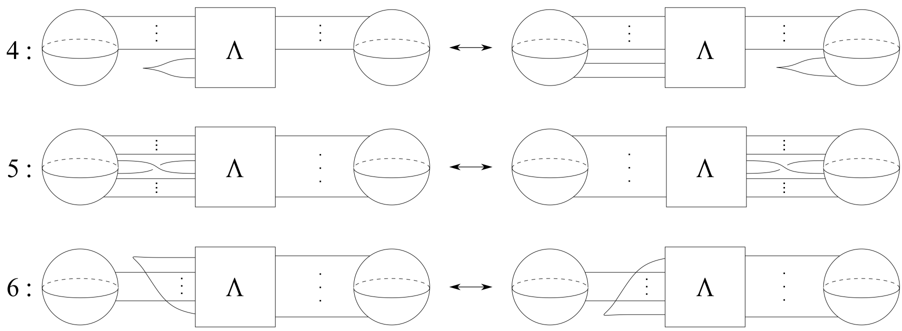


Figure 9. Gompf moves 4, 5, and 6.

$\#^k(S^1 \times S^2)$ for any k (no strands of this link go through any of the 1-handles). We then see that the definition of a ρ-graded normal ruling for the Legendrian link in $\#^k(S^1 \times S^2)$ is equivalent to the definition of a ρ-graded normal ruling for the Legendrian link in $\mathbb{R}^3$.

Similarly to $\mathbb{R}^3$, we can define a ρ-graded ruling polynomial.

Definition 2.16. If m is a $\mathbb{Z}/\rho$-valued Maslov potential for a Legendrian link Λ, then the *ρ-graded ruling polynomial* of Λ with respect to m is

$$R^{\rho}_{(\Lambda,m)} = \sum_{\sigma} z^{j(\sigma)},$$

where the sum is over all ρ-graded normal rulings of Λ and

$$j(\sigma) = \#\,\text{switches} - \#\,\text{right cusps}.$$

Note that in the case where Λ is a knot, the ruling polynomial does not depend on the Maslov potential. Restated from the introduction:

Theorem 1.2. *The ρ-graded ruling polynomial $R^{\rho}_{(\Lambda,m)}$ with respect to the Maslov potential m (which changes under Legendrian isotopy) is a Legendrian isotopy invariant.*

Proof. By Gompf [1998], any Legendrian link in $\#^k(S^1 \times S^2)$ can be represented by an xz-diagram in Gompf standard form and two such xz-diagrams represent links that are Legendrian isotopic if and only if they are related by a sequence of Legendrian Reidemeister moves of the xz-diagram of the tangle inside the rectangle $[0, A] \times [-M, M]$ and three additional moves, which we will, following the nomenclature of [Ekholm and Ng 2015], call Gompf moves 4, 5, and 6 (see Figure 9). By [Pushkar and Chekanov 2005], we know the ruling polynomial is invariant under Legendrian isotopy of the tangle, so we need only show it is invariant under Gompf moves 4, 5, and 6.

Gompf moves 4 and 5 clearly do not change the ruling polynomial. For Gompf move 6, note that any normal ruling cannot pair a strand going through the 1-handle with one of the strands incident to the cusp. Instead, the ruling must pair the two strands incident to the left cusp and not have any switches in the portion of the diagram depicted in Figure 9, thus the ruling polynomial does not change. □

Example 2.17. The normal rulings for the Legendrian link from Example 2.5 are given in Figure 8. Thus the ruling polynomial is

$$R_\Lambda = z^{-1} + z.$$

2H. *Legendrian links in* $\mathbb{R}^3$. The classical invariants for Legendrian isotopy classes of knots in $\mathbb{R}^3$ are: topological knot type, Thurston–Bennequin number, and rotation number; see [Etnyre 2005]. The *Thurston–Bennequin number* of a knot measures the self-linking of a Legendrian knot Λ. Given a push off Λ' of Λ in a direction tangent to the contact structure, then $tb(\Lambda)$ is the linking number of Λ and Λ'. Given the xz-projection of Λ,

$$tb(\Lambda) = \text{writhe}(\Lambda) - \tfrac{1}{2}(\#\,\text{cusps}).$$

The *rotation number* $r(\Lambda)$ of an oriented Legendrian knot Λ is the rotation of its tangent vector field with respect to any global trivialization. (This definition agrees with the definition of the rotation number of a path given earlier.) Given the xz-projection of Λ,

$$r(\Lambda) = \tfrac{1}{2}(\#\,\text{down cusps} - \#\,\text{up cusps}).$$

Given a Legendrian link $\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_n$, we define $tb_i = tb(\Lambda_i)$ and $r_i = r(\Lambda_i)$ for $1 \le i \le n$, and define

$$r(\Lambda) = \gcd(r_1, \dots, r_n).$$

2I. *Satellites, the DGA, and augmentations in* $\mathbb{R}^3$. This section gives the results and notation for Legendrian links in $\mathbb{R}^3$ necessary to prove Theorem 1.3.

We will first extend the idea of satelliting a knot in $J^1(S^1)$ to an unknot (see [Ng and Rutherford 2013]) to satelliting each 1-handle of a knot in $\#^k(S^1 \times S^2)$ around a twice stabilized unknot.

Definition 2.18. Given the xy- or xz-diagram for a Legendrian link Λ in $\#^k(S^1 \times S^2)$, *satellited* Λ is denoted $S(\Lambda)$, the xy-diagram of which is depicted in Figure 10 and the xz-diagram of a Legendrian isotopic link of which is depicted in Figure 12 for the Legendrian link from Figure 6. Label the crossings as indicated, where $i \le j$ and label the basepoints in $S(\Lambda)$ as they are labeled in Λ. Note that the xy- or xz-diagram of Λ defines $S(\Lambda)$ up to Legendrian isotopy.

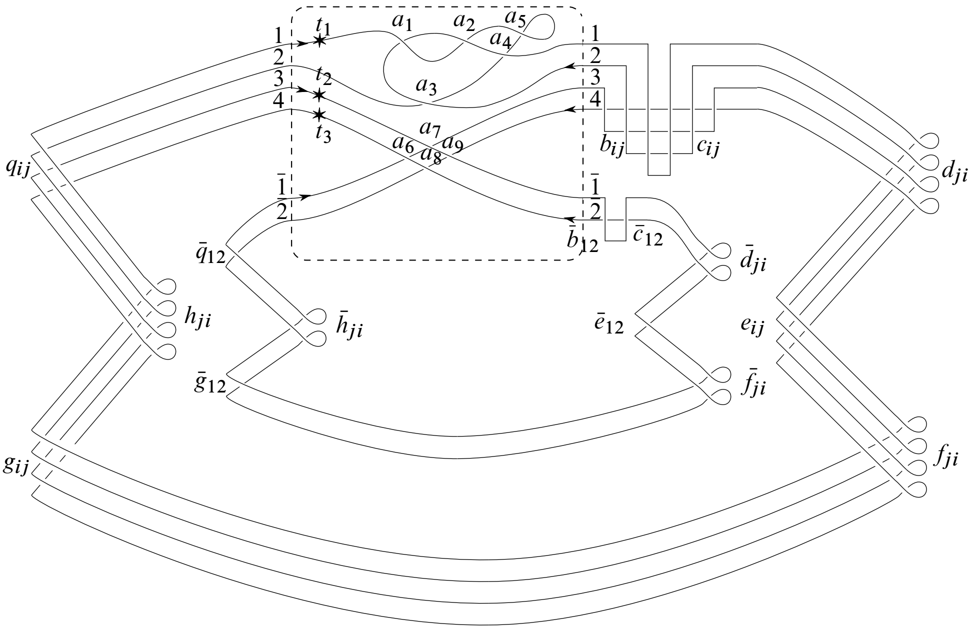


Figure 10. The xy-projection of the satellited link $S(\Lambda)$. The crossings in the c_{ij}-, b_{ij}-, $\bar{c}_{ij}$, and $\bar{b}_{ij}$-lattices are labeled as in Figure 4. The crossings in the d, e, f, g, h, q-lattices are labeled according to Figure 11.

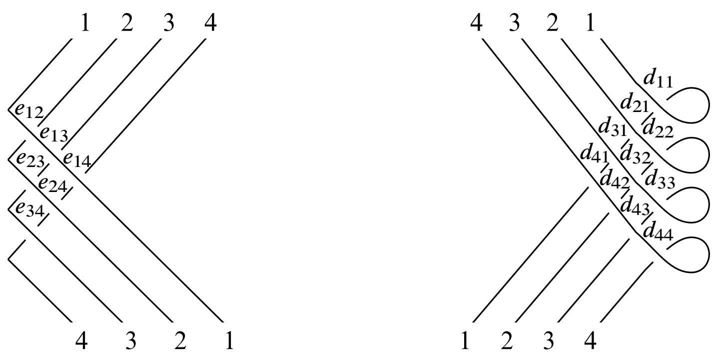


Figure 11. The labels for the crossings in the e- and d-lattices of the satellited link $S(\Lambda)$ as seen in Figure 10. The f- and h-lattices are analogous to the d-lattice. The g- and q-lattices are analogous to the e-lattice.

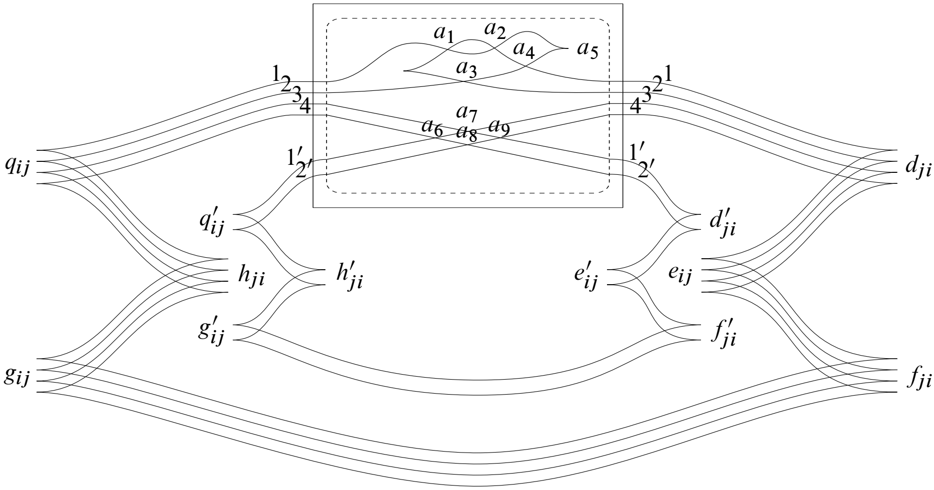


Figure 12. The xz-projection of a link which is Legendrian isotopic to the satellited link $S(\Lambda)$.

Remark 2.19. The Chekanov–Eliashberg DGA was originally defined on Legendrian links in $(\mathbb{R}^3, dz - y\,dx)$; see [Chekanov 2002; Sabloff 2005]. Note that the same DGA results from defining the DGA as we did in $\#^k(S^1 \times S^2)$ where $k = 0$.

2J. ***Dips.*** Dips will be defined analogously to those defined in [Leverson 2016].

Given a diagram $\pi_{xy}(\Lambda)$ in normal form which is the result of resolution, we construct a *dip* in the vertical slice of the diagram between two crossings, a crossing and a cusp, or two cusps, by a sequence of Reidemeister II moves, as seen in Figure 13 in the xz-projection and xy-projection. From the xz-projection, it is clear that the diagram with the dip is Legendrian isotopic to the original diagram. To construct a dip, number the N strands from top to bottom. Using a type II Reidemeister move, push strand $N-1$ over strand N, then strand $N-2$ over strand $N-1$, then strand $N-2$ over strand N, and so on. In this way, strand i is pushed over strand j in antilexicographic order.

Given an xy-diagram for a link $\Lambda \subset \mathbb{R}^3$ in normal form, where all crossings and resolutions of left cusps having distinct x-coordinates, the *dipped diagram* $D(\Lambda)$ is the result of adding a dip between each pair of crossings or resolution of a cusp and crossing. For each Reidemeister II move, we have two new generators. Call the left crossing b_{ij} and the right crossing c_{ij} if strands $i < j$ cross. One can check that $|b_{ij}| = m(j) - m(i)$ and since ∂ lowers degree by 1, we know $|c_{ij}| = |b_{ij}| - 1$.

While dipped diagrams have many more crossings than the original link diagram, the differential ∂ on $\mathcal{A}(D(\Lambda))$ is generally much simpler. In fact, a *totally augmented disk* (a disk from the definition of the differential of the DGA where all crossings at corners are augmented), cannot “go through” or “span” more than one dip.

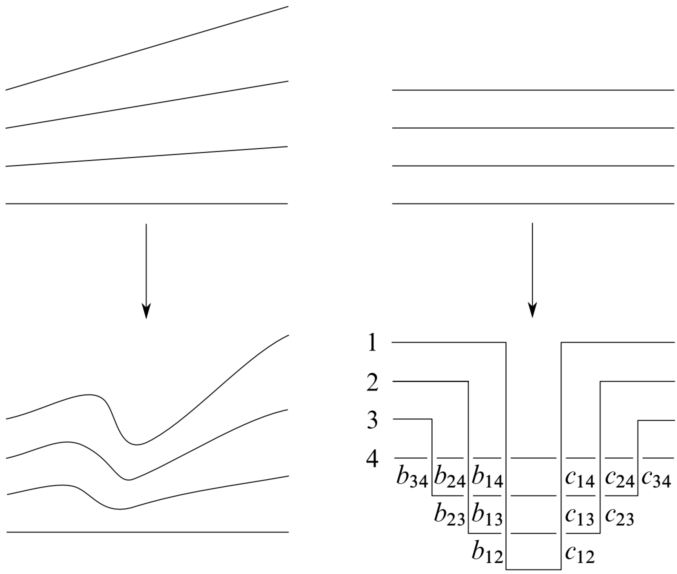


Figure 13. The modification of the xz-diagram when creating a dip (left) and the modification of the xy-diagram (right). (This figure is taken from [Leverson 2016].)

2K. ***Augmentations before and after basepoints and type II moves.*** In certain cases, we will find that adding basepoints will simplify the signs. For Legendrian links in $\mathbb{R}^3$, Ng and Rutherford [2013] give the DGA homomorphisms induced by adding a basepoint to a diagram and by moving a basepoint around a link. One can easily extend their results to $\#^k(S^1 \times S^2)$.

The following theorem is the analog of [op. cit, Theorem 2.21]:

Theorem 2.20. *Let $*_1, \dots, *_k$ and $*'_1, \dots, *'_k$ denote two collections of basepoints on the Lagrangian resolution of the front diagram of a Legendrian knot Λ, each of which is cyclically ordered along Λ, and let $(\mathcal{A}(\Lambda, *_1, \dots, *_k), \partial)$ and $\mathcal{A}(\Lambda, *'_1, \dots, *'_k), \partial')$ denote the corresponding multipointed DGAs. Then there is a DGA isomorphism $\Psi : (\mathcal{A}(\Lambda, *_1, \dots, *_k), \partial) \to (\mathcal{A}(\Lambda, *'_1, \dots, *'_k), \partial')$ such that $\Psi(t_i) = t_i$ for all i.*

In the proof of the theorem, Ψ is defined so that $\Psi(c) = c$ if no basepoint is moved over or under the crossing c. However, if the basepoint $*_i$ is moved over the crossing c, then $\Psi(c) = t_i^{\pm 1} c$, where the sign depends on whether the basepoint is moved along the knot following the orientation of the knot or against the orientation of the knot. If, instead, the basepoint is moved under the crossing c, then $\Psi(c) = c t_i^{\pm 1}$, where the sign, again, depends on the orientation of the knot.

Thus, If ϵ' is an augmentation of the DGA of the diagram after moving the basepoint $*_i$ over the crossing c, then $\epsilon = \epsilon' \circ \Psi$ is an augmentation of the DGA of the diagram before moving the basepoint.

The following theorem is the analog of [Ng and Rutherford 2013, Theorem 2.22]:

Theorem 2.21. *Let $*_1, \dots, *_k$ be a cyclically ordered collection of basepoints along Λ, and let $*$ be a single basepoint on Λ. Then there is a DGA homomorphism $\phi : (\mathcal{A}(\Lambda, *), \partial) \to (\mathcal{A}(\Lambda, *_1, \dots, *_k), \partial)$ such that $\phi \circ \partial = \partial \circ \phi$ and $\phi(t) = t_1 \cdots t_k$.*

Remark 2.22. In summary, if we have an augmentation $\epsilon : \mathcal{A} \to F$ with $\epsilon(t_i) = -1$, then moving the basepoint $*_i$ through a crossing c only changes the augmentation by changing the sign of the augmentation on the crossing c. Suppose we have a diagram with a basepoint $*$ corresponding to t and the same diagram with basepoints $*_1, \dots, *_s$ associated to $t_1, \dots, t_s$ on the same component of the link and we move all of the basepoints $*_1, \dots, *_s$ to the location of $*$. By the above results, if ϵ is an augmentation to F of the multiple basepoint diagram, there exists an augmentation ϵ' to F of the single basepoint diagram such that for all crossings c there exists $x_c \in F$ such that $\epsilon'(c) = x_c \epsilon(c)$ and

$$\epsilon'(t) = \epsilon(t_1 \cdots t_s) = \prod_{i=1}^{s} \epsilon(t_i).$$

Etnyre, Ng, and Sabloff [Etnyre et al. 2002] give a DGA isomorphism relating the DGA of a diagram of a Legendrian knot in $\mathbb{R}^3$ before and after a Reidemeister II move. One can easily extend this to a similar result for $\#^k(S^1 \times S^2)$, which gives a way to extend an augmentation of the diagram before a Reidemeister II move to an augmentation of the diagram after the move; see [Leverson 2016] for the analogous result in $\mathbb{R}^3$.

3. Correspondence between augmentations and normal rulings for links in $\mathbb{R}^3$

We have the following result for *knots* in $\mathbb{R}^3$:

Theorem 3.1 [Leverson 2016, Theorem 1.1]. *Let Λ be a Legendrian knot in $\mathbb{R}^3$. Given a field F, $(\mathcal{A}, \partial)$ has a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ if and only if any front diagram of Λ has a ρ-graded normal ruling. Furthermore, if ρ is even, then $\epsilon(t) = -1$.*

This result is proven by construction. Using the same method we can prove an analogous result for *links* in $\mathbb{R}^3$. Restating from the introduction:

Theorem 1.1. *Let Λ be an n-component Legendrian link in $\mathbb{R}^3$ with s basepoints (at least one basepoint on each component). Given a field F, the Chekanov–Eliashberg DGA $(\mathcal{A}, \partial)$ over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ has a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ if and* *only if a front diagram of* Λ *has a* ρ*-graded normal ruling. Furthermore, if* ρ *is even, then* $\epsilon(t_1 \cdots t_s) = (-1)^s$.

The following result will be necessary for the proof of Theorem 1.1. Analogous to the knot case in $\mathbb{R}^3$, we have the following extension of [Leverson 2016, Lemma 3.2]:

Lemma 3.2. *If* c *gives the number of right cusps,* sw *is the number of switches in the ruling,* a_- *is the number of* $-(a)$ *crossings, and* n *is the number of components, then*

$$c + sw + a_- \equiv n \mod 2.$$

Proof. As in the knot case, one can easily show each of the following statements:

$$\sum_{i=1}^{n} tb_i + \sum_{i=1}^{n} r_i \equiv n \mod 2 \tag{1}$$

$$\sum_{i=1}^{n} tb_i \equiv c + cr \mod 2 \tag{2}$$

$$cr \equiv sw \mod 2 \tag{3}$$

$$\sum_{i=1}^{n} r_i \equiv a_- \mod 2 \tag{4}$$

where r_i is the rotation number of Λ_i and cr is the number of crossings. Note that if we add these four equations together, we get that

$$c + sw + a_- \equiv n \mod 2$$

as desired. □

Proof of Theorem 1.1. After a series of Legendrian isotopies, we can assume the front diagram of Λ has the following form where from left to right (lowest x-coordinate to highest x-coordinate) we have: all left cusps have the same x-coordinate, no two crossings of Λ have the same x-coordinate, and all right cusps have the same x-coordinate (in [Leverson 2016], this is called plat position). Label the crossings in the right cusps by $q_1, \dots, q_m$ from top to bottom and label the other crossings by $c_1, \dots, c_\ell$ from left to right.

Augmentation to ruling: Beginning with a ρ-graded augmentation of the Chekanov–Eliashberg DGA of the resolution of $\pi_{xz}(\Lambda)$ to a Lagrangian diagram, define a ρ-graded normal ruling of $\pi_{xz}(\Lambda)$ by simultaneously defining a ρ-graded augmentation of the dipped diagram $D(\Lambda)$ as in the knot case, using Figure 14.

Ruling to augmentation: Given a ρ-graded normal ruling of $\pi_{xz}(\Lambda)$, define a ρ-graded augmentation of the dipped diagram $D(\Lambda)$ with basepoints where specified in Figure 14 and at each right cusps as in the knot case, using Figure 14.

Using Lemma 3.2 and the methods in the proof of [Leverson 2016, Theorem 3.1], one can show the final statement of Theorem 1.1. Given a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$, consider the associated ρ-graded normal ruling. If ρ is even, then the ruling is only switched at crossings c_k with $\rho \mid |c_k|$ and so $2 \mid |c_k|$. Thus, any strands paired by the ruling must have opposite orientation. As in the case of knots, this implies that near a crossing where the ruling is switched the crossing must be a positive crossing. Thus each ruling path is an oriented unknot.

If we consider the dipped diagram of the link, by induction we can show that

$$\prod \epsilon(b_{ij}^k)^{\pm 1} = 1,$$

where the product is taken over all paired strands i and j in the ruling between c_k and c_{k+1} and the sign is determined by the orientation of the paired strands as in [op. cit.]. By considering ∂q_k, we see that

$$\begin{aligned}\epsilon(t_1 \cdots t_s) = (-1)^{s-m} \prod_{k=1}^{m} \bigl(-\epsilon(b_{2k,2k-1}^{\ell})^{\pm 1}\bigr)& \\ = (-1)^s \prod_{i<j \text{ paired}} \epsilon(b_{ij}^{\ell})^{\pm 1} = (-1)^s = (-1)^n&\end{aligned}$$

by Lemma 3.2 and the fact that the number of basepoints $s \equiv c + sw + a_- \mod 2$. □

4. Augmentation to ruling

In this section, we will show that a quotient of the DGA of the satellited version of any Legendrian link Λ in $\#^k(S^1 \times S^2)$ is a subalgebra of the DGA of Λ in $\#^k(S^1 \times S^2)$ and use the construction from Theorem 1.1 to construct a ruling of the satellited link in $\mathbb{R}^3$ to then give a normal ruling of Λ in $\#^k(S^1 \times S^2)$. This shows the forward direction of Theorem 1.3.

Given an xy-diagram for the Legendrian link Λ in $\#^k(S^1 \times S^2)$ which results from the resolution of an xz-diagram in normal form with basepoints indicated. We can construct an xy-diagram for $S(\Lambda)$, satellited Λ, (see Figure 10) with basepoints in the same location as they were for Λ.

We will use the notation for Legendrian links in $\#^k(S^1 \times S^2)$ with tildes added for the Legendrian link Λ in $\#^k(S^1 \times S^2)$:

$$\mathcal{A}(\Lambda) = \mathbb{Z}[\tilde{t}_1^{\pm 1}, \dots, \tilde{t}_s^{\pm 1}]\langle \tilde{a}_i, \tilde{b}_{ij;\ell}, \tilde{c}_{ij;\ell}^{p} \rangle$$

with differential $\tilde{\partial}$, where $1 \le \ell \le k$, $i < j$ for all $\tilde{b}_{ij;\ell}$ and $i < j$ for $\tilde{c}_{ij;\ell}^{p}$ if $p = 1$. We will use the notation for Legendrian links from Figure 10 for $S(\Lambda)$:

$$\mathcal{A}(S(\Lambda)) = \mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]\langle a_i, b_{ij;\ell}, c_{ij;\ell}, d_{ji;\ell}, e_{ij;\ell}, f_{ji;\ell}, g_{ij;\ell}, h_{ji;\ell}, q_{ij;\ell} \rangle$$

with differential ∂, where $1 \le \ell \le k$, $1 \le i \le m$ for a_i, $i < j$ for $b_{ij;\ell}$, $c_{ij;\ell}$, $e_{ij;\ell}$, $g_{ij;\ell}$, and $q_{ij;\ell}$, and $i \le j$ for $d_{ji;\ell}$, $f_{ji;\ell}$, and $h_{ji;\ell}$.

Suppose we have a Legendrian link Λ in $\#^k(S^1 \times S^2)$ with associated DGA $(\mathcal{A}(\Lambda), \partial)$. If $(\mathcal{A}(S(\Lambda)), \partial)$ is the DGA associated to satellited Λ, then let $\pi : \mathcal{A}(S(\Lambda)) \to \mathcal{A}(S(\Lambda))/B$ be the quotient algebra homomorphism where B is the ideal in $\mathcal{A}(S(\Lambda))$ generated by

$$\{c_{ij;\ell} - g_{ij;\ell},\ c_{ij;\ell} - q_{ij;\ell},\ c_{ij;\ell} - (-1)^{|e_{ij;\ell}|+1} e_{ij;\ell},\\ h_{ji;\ell} - (-1)^{|f_{ji;\ell}|+1} f_{ji;\ell},\ h_{ji;\ell} - (-1)^{|d_{ji;\ell}|+1} d_{ji;\ell}\}.$$

Define $\gamma : \mathcal{A}(S(\Lambda))/B \to \mathcal{A}(\Lambda)$ by

$$\begin{aligned} \gamma \ : \ \mathcal{A}(S(\Lambda))/B &\longrightarrow \mathcal{A}(\Lambda) \\ [a_i] &\longmapsto \tilde{a}_i \\ [b_{ij;\ell}] &\longmapsto \tilde{b}_{ij;\ell} \\ [c_{ij;\ell}] &\longmapsto \tilde{c}^0_{ij;\ell} \\ [h_{ji;\ell}] &\longmapsto \tilde{c}^1_{ji;\ell} \\ [t_i] &\longmapsto \tilde{t}_i \end{aligned}$$

Proposition 4.1. *If $\phi = \gamma \circ \pi$, then ϕ is a graded algebra homomorphism such that $\tilde{\partial}\phi(c) = \pm\phi\partial(c)$ for all $c \in \{a_i, b_{ij;\ell}, c_{ij;\ell}, d_{ji;\ell}, e_{ij;\ell}, f_{ji;\ell}, g_{ij;\ell}, h_{ji;\ell}, q_{ij;\ell}\}$.*

Proof. Grading: We will first show that π and γ (and thus ϕ) are graded algebra homomorphisms. First, let m be the Maslov potential used to assign the gradings of the crossings of Λ in $\#^k(S^1 \times S^2)$. We will use m to define a Maslov potential μ on $S(\Lambda)$ in $\mathbb{R}^3$ as follows: Define μ on $T \subset S(\Lambda)$ the same as m is defined on $T \subset \Lambda$ and extend μ to the rest of $S(\Lambda)$. Notice that there is only one way to do this which keeps μ of the upper strand (higher z-coordinate) entering a cusp one higher than μ of the lower strand (lower z-coordinate) entering a cusp. The fact that ∂ has degree -1 and properties of the Maslov potential immediately give us that in the p-th 1-handle:

$$\begin{aligned} |\tilde{c}^1_{ji}| = |d_{ji}| = |f_{ji}| = |h_{ji}|, &\quad i \le j \\ |\tilde{c}^0_{ij}| = |c_{ij}| = |e_{ij}| = |g_{ij}| = |q_{ij}|, &\quad i < j \\ -|d_{ji}| = |e_{ij}|, &\quad i < j \\ |b_{ij}| = |c_{ij}| + 1, &\quad i < j \end{aligned} \tag{5}$$

Thus, π and γ are graded algebra homomorphisms and so ϕ is as well.

$\tilde{\partial}\phi(c) = \pm\phi\partial(c)$: From the definition of their gradings, in the p-th 1-handle:

$$|\tilde{c}^0_{ij}| \equiv |\tilde{c}^0_{i\ell}| + |\tilde{c}^0_{\ell j}| \mod 2 \quad \text{and} \quad |\tilde{c}^1_{ji}| \equiv |\tilde{c}^1_{j\ell}| + |\tilde{c}^1_{\ell i}| \mod 2 \tag{6}$$

With (5), we have analogous statements for b_{ij}, c_{ij}, d_{ji}, e_{ij}, f_{ji}, g_{ij}, h_{ji}, and q_{ij}.

By considering the disks which contribute terms to ∂a_i and $\tilde{\partial}\tilde{a}_i$ (and analogously ∂b_{ij} and $\tilde{\partial}\tilde{b}_{ij}$ in the p-th 1-handle for $i < j$), it is clear that

$$\tilde{\partial}\phi(a_i)\tilde{\partial}(\tilde{a}_i) = \phi\partial(a_i) \quad \text{and} \quad \tilde{\partial}\phi(b_{ij})\tilde{\partial}(\tilde{b}_{ij}) = \phi\partial(b_{ij}).$$

Given $1 \le p \le k$ and $1 \le i < j \le N_p$. In the p-th 1-handle:

$$\begin{aligned}
\tilde{\partial}\phi c_{ij} &= \tilde{\partial}\tilde{c}^0_{ij} \\
&= \sum_{i<\ell<j} (-1)^{|\tilde{c}^0_{i\ell}|+1} \tilde{c}^0_{i\ell}\tilde{c}^0_{\ell j} \\
&= \phi\Bigg(\sum_{i<\ell<j} (-1)^{|c_{ij}|+1} c_{i\ell}c_{\ell j}\Bigg) && \text{by (5)} \\
&= \phi\partial c_{ij},
\end{aligned}$$

$$\begin{aligned}
\tilde{\partial}\phi d_{ii} &= \partial(-1)^{|d_{ii}|+1}\tilde{c}^1_{ii} \\
&= (-1)^{|d_{ii}|+1}\Bigg(1 + \sum_{i<\ell\le N_p} (-1)^{|\tilde{c}^0_{i\ell}|+1}\tilde{c}^0_{i\ell}\tilde{c}^1_{\ell i} + \sum_{1\le\ell<i} (-1)^{|\tilde{c}^1_{i\ell}|+1}\tilde{c}^1_{i\ell}\tilde{c}^0_{\ell i}\Bigg) \\
&= 1 + \sum_{i<\ell\le N_p} (-1)^{|\tilde{c}^0_{i\ell}|+|d_{\ell i}|}\phi(c_{i\ell}d_{\ell i}) \\
&\qquad + \sum_{1\le\ell<i} (-1)^{|\tilde{c}^1_{i\ell}|+|d_{i\ell}|+|e_{\ell i}|+1}\phi(d_{i\ell}e_{\ell i}) && \text{since } |d_{ii}| = 1 \\
&= 1 + \sum_{i<\ell\le N_p} \phi(c_{i\ell}d_{\ell i}) + \sum_{1\le\ell<i} (-1)^{|d_{i\ell}|+1}\phi(d_{i\ell}e_{\ell i}) && \text{by (5)} \\
&= \phi\partial d_{ii},
\end{aligned}$$

$$\begin{aligned}
\tilde{\partial}\phi d_{ji} &= \tilde{\partial}(-1)^{|d_{ji}|+1}\tilde{c}^1_{ji} \\
&= (-1)^{|d_{ji}|+1}\Bigg(0 + \sum_{j<\ell\le N_p} (-1)^{|\tilde{c}^0_{j\ell}|+1}\tilde{c}^0_{j\ell}\tilde{c}^1_{\ell i} + \sum_{1\le\ell<i} (-1)^{|\tilde{c}^1_{j\ell}|+1}\tilde{c}^1_{j\ell}c^0_{\ell i}\Bigg) \\
&= \phi\Bigg((-1)^{|d_{ji}|+1}\Bigg(0 + \sum_{j<\ell\le N_p} (-1)^{|\tilde{c}^0_{j\ell}|+|d_{\ell i}|} c_{j\ell}d_{\ell i} \\
&\qquad + \sum_{1\le\ell<i} (-1)^{|\tilde{c}^1_{j\ell}|+|d_{j\ell}|+|e_{\ell i}|+1} d_{j\ell}e_{\ell i}\Bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \phi\Bigg((-1)^{|d_{ji}|+1}\Bigg(0+(-1)^{|d_{ji}|}\sum_{j<\ell\le N_p} c_{j\ell}d_{\ell i}\\
&\qquad\qquad + (-1)^{|d_{ji}|}\sum_{1\le \ell<i}(-1)^{|d_{j\ell}|+1}d_{j\ell}e_{\ell i}\Bigg)\Bigg) && \text{by (5) and (6)}\\
&= -\phi\Bigg(0+\sum_{j<\ell\le N_p} c_{j\ell}d_{\ell i}+\sum_{1\le \ell<i}(-1)^{|d_{j\ell}|+1}d_{j\ell}e_{\ell i}\Bigg)\\
&= -\phi\partial d_{ji}
\end{aligned}
$$

One can similarly show that for $i<j$

$$
\begin{aligned}
&\tilde{\partial}\phi e_{ij} = \phi\partial e_{ij},\\
&\tilde{\partial}\phi f_{ii} = \phi\partial f_{ii}, \qquad \tilde{\partial}\phi f_{ji} = -\phi\partial f_{ji},\\
&\tilde{\partial}\phi g_{ij} = \phi\partial g_{ij}, \qquad \tilde{\partial}\phi h_{ii} = \phi\partial h_{ii},\\
&\tilde{\partial}\phi h_{ji} = \phi\partial h_{ji},\\
&\tilde{\partial}\phi q_{ij} = \phi\partial q_{ij}.
\end{aligned}
$$

□

Given a field F and a ρ-graded augmentation $\tilde{\epsilon}:\mathcal{A}(\Lambda)\to F$, we will construct a ρ-graded augmentation $\epsilon:\mathcal{A}(S(\Lambda))\to F$. Define $\epsilon=\tilde{\epsilon}\circ\phi$. Thus, on the generators of $\mathcal{A}(S(\Lambda))$ in the p-th 1-handle,

$$
\epsilon(c)=\begin{cases}
\tilde{\epsilon}(\tilde{a}_i) & \text{if } c=a_i\\
\tilde{\epsilon}(\tilde{b}_{ij}) & \text{if } c=b_{ij}\\
\tilde{\epsilon}(\tilde{c}^0_{ij}) & \text{if } c\in\{c_{ij},g_{ij},q_{ij}\}\\
(-1)^{|\tilde{c}^0_{ij}|+1}\tilde{\epsilon}(\tilde{c}^0_{ij}) & \text{if } c=e_{ij}\\
\tilde{\epsilon}(\tilde{c}^1_{ji}) & \text{if } c=h_{ji}\\
(-1)^{|\tilde{c}^1_{ji}|+1}\tilde{\epsilon}(\tilde{c}^1_{ji}) & \text{if } c\in\{d_{ji},f_{ji}\}\\
\tilde{\epsilon}(\tilde{t}_i) & \text{if } c=t_i.
\end{cases}
$$

We see that ϵ is an augmentation because on any generator c of $\mathcal{A}(S(\Lambda))$,

$$
\begin{aligned}
\epsilon\partial(c) &= \tilde{\epsilon}\phi\partial(c)\\
&= \pm\tilde{\epsilon}\tilde{\partial}\phi(c) && \text{by Proposition 4.1}\\
&= 0,
\end{aligned}
$$

since ϵ' is an augmentation. And, since ϵ' is a ρ-graded augmentation and ϕ is a graded algebra homomorphism, ϵ is a ρ-graded augmentation.

Thus an augmentation $\tilde{\epsilon}:\mathcal{A}(\Lambda)\to F$ of the DGA of Λ in $\#^k(S^1\times S^2)$ gives an augmentation $\epsilon:\mathcal{A}(S(\Lambda))\to F$ of the DGA of $S(\Lambda)$ in $\mathbb{R}^3$. By [Leverson 2016, Theorem 1.1], the augmentation ϵ gives an augmentation of the DGA of $S(\Lambda)$ with dips in $\mathbb{R}^3$, which gives a normal ruling of $S(\Lambda)$ with no dips in $\mathbb{R}^3$. We must check that if two strands are paired in this normal ruling, then they go through the same 1-handle. Clearly this normal ruling must be *thin*, meaning outside of the tangle T associated to Λ the ruling only has switches at crossings where the crossing strands go through the same 1-handle. By restricting the ρ-graded normal ruling of $S(\Lambda)$ in $\mathbb{R}^3$ to a ρ-graded normal ruling of T, we get a ρ-graded normal ruling of Λ in $\#^k(S^1 \times S^2)$.

5. Ruling to augmentation

Let F be a field. We will now prove the existence of a ρ-graded normal ruling implies the existence of a ρ-graded augmentation, the backward direction of Theorem 1.3, by constructing a ρ-graded augmentation $\epsilon : \mathcal{A}(D(\Lambda)) \to F$ given a ρ-graded normal ruling of Λ in $\#^k(S^1 \times S^2)$.

Given an xz-diagram of a Legendrian link Λ in $\#^k(S^1 \times S^2)$ in normal form, we will consider the resolution to an xy-diagram of a Legendrian isotopic link. Using Legendrian isotopy, we can ensure all crossings, left cusps, and right cusps have different x coordinates and all right cusps occur "above" (have higher y or z coordinate than) the remaining strands of the tangle at that x coordinate. Place a basepoint on every strand at $x = 0$ and one in every loop coming from the resolution of a right cusp.

Define the augmentation $\epsilon : \mathcal{A}(D(\Lambda)) \to F$ of the DGA for the dipped diagram $D(\Lambda)$ on generators as follows: If the ruling is switched at a crossing a_ℓ, then set $\epsilon(a_\ell) = 1$. If not, set $\epsilon(a_\ell) = 0$. (Note that we can augment the switched crossings to any nonzero element of F and still get an augmentation. But in the case where Λ is a knot, by augmenting the switched crossing to 1, we will be able to ensure $\epsilon(t) = -1$.) Add basepoints and augment the crossings in the dips following Figure 14. On the remaining generators, set

$$\epsilon(c_{ij}^\ell) = \begin{cases} 1 & \text{if } \ell = 0 \text{ and strands } i < j \text{ are paired in the normal ruling} \\ & \qquad \text{and go through the } p\text{-th 1-handle} \\ (-1)^{|c_{ij}^\ell|} & \text{if } \ell = 1,\ i > j, \text{ and strands } i, j \text{ are paired in the normal} \\ & \qquad \text{ruling and go through the } p\text{-th 1-handle} \\ 0 & \text{otherwise.} \end{cases}$$

Augment all basepoints to -1.

By considering Figure 14, it is involved but straightforward to check that ϵ is an augmentation on the a_ℓ and the crossings in the dips.

Notation 5.1. $c_{\{ij\}}^\ell = c_{\min(i,j),\,\max(i,j)}^\ell$

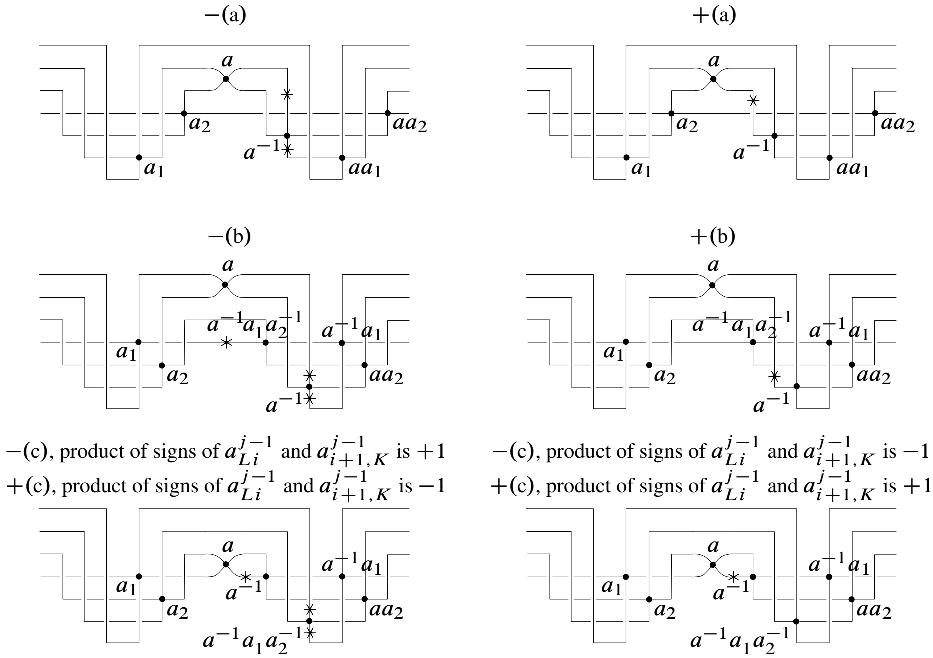


Figure 14. In the diagrams, $*$ denotes a basepoint. A dot denotes the specified crossing is augmented and the augmentation sends the crossing to the labeled value. For example, in the left dip of the $-$(a) configuration, $\epsilon(c_{12}) = a_1$ and $\epsilon(c_{34}) = a_2$. All other crossings are sent to 0 by the augmentation. Here $-/+$(a) denotes a negative/positive crossing where the ruling has configuration (a) and the rest are defined analogously. See Figure 15 for configurations (d), (e), and (f). (This figure is taken from [Leverson 2016].)

We will now check that ϵ is an augmentation on the c^ℓ_{ij} generators from the p-th 1-handle.

$\underline{\epsilon \partial c^0_{ij} = 0}$: For any ruling, at the left end of the diagram, each strand is paired with another strand going through the same 1-handle. So for each strand i going through the p-th 1-handle, there exists a strand $j \neq i$ such that strand i and j are paired and $1 \leq i, j \leq N_p$. So if $i < j$, then $\epsilon(c^0_{ij}) = 1$, $\epsilon(c^0_{\{i\ell\}}) = 0$ for all $\ell \neq j$, and $\epsilon(c^0_{\{j\ell\}}) = 0$ for all $\ell \neq i$. Suppose $i < r < \ell$. Then $\epsilon(c^0_{ir}) = 0$ if $r \neq j$ and $\epsilon(c^0_{r\ell}) = 0$ if $r = j$. Thus $\epsilon(c^0_{ir} c^0_{r\ell}) = 0$ for all $i < r < \ell$ and so for $i < \ell$,

$$\epsilon \partial c^0_{i\ell} = \sum_{i<r<\ell} (-1)^{|c^0_{ir}|+1} \epsilon(c^0_{ir} c^0_{r\ell}) = 0.$$

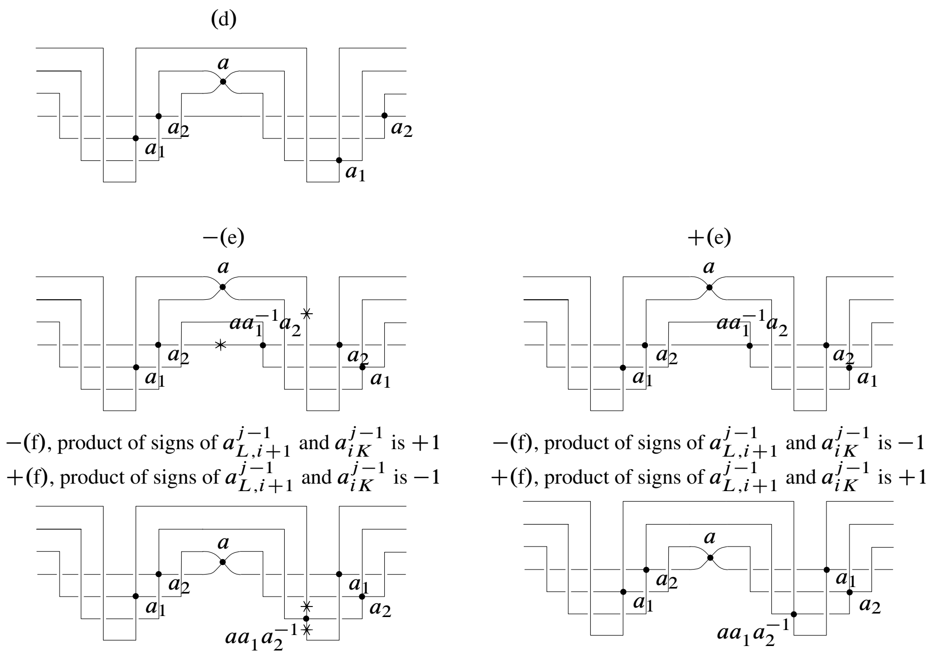


Figure 15. A continuation of Figure 14. (This figure is taken from [Leverson 2016].)

$\underline{\epsilon\partial c^1_{ij} = 0}$: Recall that in the p-th 1-handle

$$\partial c^1_{ij} = \delta_{ij} + \sum_{i<\ell\le N_p} (-1)^{|c^0_{i\ell}|+1} c^0_{i\ell} c^1_{\ell j} + \sum_{1\le \ell<j} (-1)^{|c^1_{i\ell}|+1} c^1_{i\ell} c^0_{\ell j}.$$

If $i \neq j$, then $\epsilon(c^0_{i\ell}c^1_{\ell j}) = 0$ and $\epsilon(c^1_{i\ell}c^0_{\ell j}) = 0$ for all ℓ since it is not possible for strand i to be paired with strand ℓ and for strand ℓ to be paired with strand j when $i \neq j$. Thus

$$\epsilon\partial c^1_{ij} = \sum_{i<\ell\le N_p} (-1)^{|c^0_{i\ell}|+1} \epsilon(c^0_{i\ell} c^1_{\ell j}) + \sum_{1\le \ell<j} (-1)^{|c^1_{i\ell}|+1} \epsilon(c^1_{i\ell} c^0_{\ell j}) = 0.$$

To show $\epsilon\partial c^1_{ii} = 0$, suppose strand i is paired with strand ℓ through the p-th 1-handle. Then by (5),

$$\begin{aligned}\epsilon\partial c^1_{ii} &= \begin{cases} 1 + (-1)^{|c^0_{i\ell}|+1}\epsilon(c^0_{i\ell}c^1_{\ell i}), & i<\ell, \\ 1 + (-1)^{|c^1_{i\ell}|+1}\epsilon(c^1_{i\ell}c^0_{\ell i}), & i>\ell, \end{cases} \\ &= \begin{cases} 1 + (-1)^{|c^0_{i\ell}|+1}(-1)^{|c^1_{\ell i}|}, & i<\ell, \\ 1 + (-1)^{|c^1_{i\ell}|+1}(-1)^{|c^1_{i\ell}|}, & i>\ell, \end{cases} \\ &= 0.\end{aligned}$$

$\underline{\epsilon\partial c_{ij}^{\ell}=0 \text{ for } 1<\ell}$: Recall

$$\partial c_{ij}^{\ell}=\sum_{r=0}^{\ell}\sum_{s=1}^{N_p}(-1)^{|c_{is}^r|+1}c_{is}^r c_{sj}^{\ell-r}$$

for $1<\ell$, $1\le p\le k$, and $1\le i,j\le N_p$. We will show that

$$\epsilon(c_{is}^r c_{sj}^{\ell-r})=0,$$

which implies that $\epsilon\partial c_{ij}^{\ell}=0$. If $\ell>2$, then for all $0\le r\le \ell$, either $r>1$ or $\ell-r>1$, so $\epsilon(c_{is}^r c_{sj}^{\ell-r})=0$ for all i,j,s. If $\ell=2$, then $r>1$, $\ell-r>1$, or $r=1=\ell-r$. The first and second case clearly imply $\epsilon(c_{is}^r c_{sj}^{\ell-r})=0$. In the final case, this is also clearly true, unless $i=j$ and strands i and s are paired in the ruling. In this case, either $i<s$ or $s<i=j$, so either $\epsilon(c_{is}^1)=0$ or $\epsilon(c_{sj}^1)=0$. So

$$\epsilon\partial c_{ii}^{\ell}=\sum_{r=0}^{\ell}\sum_{s=1}^{N_p}(-1)^{|c_{is}^r|+1}\epsilon(c_{is}^r c_{si}^{\ell-r})=0$$

for all $1\le p\le k$, $1\le i\le N_p$, and $\ell>1$. So for $1<\ell$

$$\epsilon\partial c_{ij}^{\ell}=0.$$

<u>Grading</u>: From the definition, a_i is augmented only if the ρ-graded normal ruling is switched at a_i and thus $\rho \,\big|\, |a_i|$. Since $|a_i|=|\tilde a_i|$, we have $\rho \,\big|\, |a_i|$. By definition, if $c_{ij;p}^{\ell}$ is augmented, then either $\ell=0$, $i<j$, and strands i and j are paired by the normal ruling and go through the p-th 1-handle or $\ell=1$, $i>j$, and strands i and j are paired in the normal ruling and go through the p-th 1-handle. In the first case, $\mu(i)\equiv\mu(j)+1 \mod \rho$ and so

$$|c_{ij;p}^0|=2(0)-1+\mu(i)-\mu(j)\equiv 0 \mod \rho.$$

In the second case, $\mu(j)\equiv\mu(i)+1 \mod \rho$ and so

$$|c_{ij;p}^1|=2(1)-1+\mu(i)-\mu(j)\equiv 0 \mod \rho.$$

Following arguments similar to those in [Leverson 2016], one can also check that if a crossing c in a dip is augmented then $\rho \,\big|\, |c|$.

Proposition 5.2. *If $\Lambda\subset\#^k(S^1\times S^2)$ is an n-component link, $\rho\mid 2r(\Lambda)$ is even, and Λ has a ρ-graded normal ruling, then the ρ-graded augmentation $\epsilon:\mathcal{A}(\Lambda)\to F$ constructed above sends $t_1\cdots t_s$ to $(-1)^n$.*

Proof. Given a ρ-graded ruling of Λ in $\#^k(S^1\times S^2)$, there is a unique way to extend it to a normal ruling of $S(\Lambda)$ by switching at $d_{ji}, e_{ij}, f_{ji}, g_{ij}, h_{ji}, q_{ij}$ if and only if strands $i<j$ are paired in the ruling of Λ. Let $\tilde\epsilon:\mathcal{A}(\Lambda)\to F$ be the ρ-graded augmentation resulting from the ρ-graded normal ruling and let $\epsilon:\mathcal{A}(S(\Lambda))\to F$

be the ρ-graded augmentation resulting from the ρ-graded normal ruling of $S(\Lambda)$ as constructed in [Leverson 2016] in $\mathbb{R}^3$. Note that

$$\frac{\epsilon(t_1\cdots t_s)}{\tilde{\epsilon}(\tilde{t}_1\cdots\tilde{t}_r)} = \left(\prod_{1\leq p\leq k}(-1)^{3N_p}\right)\prod_{i,j \text{ paired}}(-1)^6.$$

If strands $i<j$ are paired near $x=0$ in the ruling of Λ, then the ruling of $S(\Lambda)$ must be switched at d_{ji}, e_{ij}, f_{ji}, g_{ij}, h_{ji}, and q_{ij} with configuration $+$(a) since the ruling is ρ-graded and ρ is even. So there is one additional basepoint augmented to -1 per crossing. Thus, there are six additional basepoints augmented to -1 for each pair of strands. Each right cusp contributes one extra basepoint augmented to -1 and there are three additional right cusps for each strand. However, N_p is even for all $1\leq p\leq k$ by Corollary 1.4 and $\epsilon(t_1\cdots t_s)=(-1)^n$ by Theorem 1.1, so

$$\frac{(-1)^n}{\tilde{\epsilon}(\tilde{t}_1\cdots\tilde{t}_r)} = 1$$

and so $\tilde{\epsilon}(\tilde{t}_1\cdots\tilde{t}_r)=(-1)^n$. ☐

All that remains to be proven is the final statement of Theorem 1.3, which says:

Proposition 5.3. *Given a field F, if Λ is an n component link in $\#^k(S^1\times S^2)$, $\epsilon(t)=(-1)^n$ for all even-graded augmentations $\epsilon:\mathcal{A}(\Lambda)\to F$.*

Proof. Suppose that $\tilde{\epsilon}:\mathcal{A}(\Lambda)\to F$ is an even-graded augmentation (ρ-graded augmentation where $2\mid\rho$). As in Section 4, we construct a ρ-graded augmentation $\epsilon:\mathcal{A}(S(\lambda))\to F$. By definition, $\epsilon(t_i)=\tilde{\epsilon}(\tilde{t}_i)$ for all $1\leq i\leq s$ and so

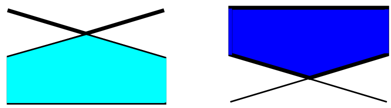
$$\tilde{\epsilon}(\tilde{t}_1\cdots\tilde{t}_s)=\epsilon(t_1\cdots t_s)=(-1)^n,$$

where the final equality follows from Theorem 1.1. ☐

6. Correspondence for links in $J^1(S^1)$

Recall that the 1-jet space of the circle, $J^1(S^1)$, is diffeomorphic to the solid torus $S^1_x\times\mathbb{R}^2_{y,z}$ with contact structure given by $\xi=\ker(dz-y\,dx)$. As in [Ng and Traynor 2004], by viewing S^1 as a quotient of the unit interval, $S^1=[0,1]/(0\sim 1)$, we can see Legendrian links in $J^1(S^1)$ as quotients of arcs in $I\times\mathbb{R}^2$ with boundary conditions which are everywhere tangent to the contact planes. Given a Legendrian link $\Lambda\subset J^1(S^1)$ we will use the methods of Lavrov and Rutherford [2012] to show the following theorem, restated from the introduction:

Theorem 1.6. *Suppose Λ is a Legendrian link in $J^1(S^1)$. Given a field F, the Chekanov–Eliashberg DGA $(\mathcal{A},\partial)$ over $\mathbb{Z}[t_1^{\pm 1},\dots,t_s^{\pm 1}]$ has a ρ-graded augmentation $\epsilon:\mathcal{A}\to F$ if and only if a front diagram of Λ has a ρ-graded generalized normal ruling.*



(g) (h)

Figure 16. These figures give the configuration of a generalized normal ruling near a switched crossing involving exactly one self-paired strand. With the top row of configurations in Figure 7, these are all possible configurations of a generalized normal ruling near a switched crossing.

We recall the definition of generalized normal ruling.

Definition 6.1 [Lavrov and Rutherford 2012]. A *generalized normal ruling* is a sequence of involutions $\sigma = (\sigma_1, \dots, \sigma_M)$ as in Definition 2.11 with the following differences:

(1) Remove the requirement that σ_m is fixed-point-free and the condition about 1-handles.

(2) If strands ℓ and $\ell+1$ cross in the interval (x_{m-1}, x_m) above I_{m-1}, where exactly one of the crossing strands is a fixed point of σ_m, then the crossing is a switch if σ_m satisfies the conditions in (3) of Definition 2.11. If crossing is a switch, then we require an additional normality condition:

$$\sigma_m(\ell) = \ell < \ell+1 < \sigma_m(\ell+1) \quad \text{or} \quad \sigma_m(\ell) < \ell < \ell+1 = \sigma_m(\ell+1).$$

A *strictly generalized normal ruling* is a generalized normal ruling which is not a normal ruling, in other words, a generalized normal ruling with at least one fixed point.

Thus, near a crossing, a generalized normal ruling looks like the crossings in Figure 7 or Figure 16.

Remark 6.2. (1) If a crossing involving strands ℓ and $\ell+1$ occurs in the interval (x_{m-1}, x_m) and both crossing strands are fixed by the ruling, self-paired, in other words, $\sigma_{m-1}(\ell) = \ell$ and $\sigma_{m-1}(\ell+1) = \ell+1$, then $\sigma_m = (\ell \quad \ell+1) \circ \sigma_{m-1} \circ (\ell \quad \ell+1)$ and so we will not consider such crossings to be switched.

(2) Note that the number of generalized normal rulings of a Legendrian link is not invariant under Legendrian isotopy.

The definition of the Chekanov–Eliashberg DGA of a Legendrian link in $\mathbb{R}^3$ can be extended to Legendrian links in $J^1(S^1)$. (One can find the full definition of the Chekanov–Eliashberg DGA of a Legendrian link in $J^1(S^1)$ in [Ng and Traynor 2004].) Note that given an augmentation of the Chekanov–Eliashberg DGA over

$\mathbb{Z}[t, t^{-1}]$ of a Legendrian link in $S^1 \times S^2$, one can define an augmentation of the DGA of the analogous link (where if a strand goes through the 1-handle with $y = y_0$ at $x = 0$, then it is paired with the strand going through the 1-handle with $y = y_0$ at $x = A$) in $J^1(S^1)$ and similarly for normal rulings. (The resulting normal ruling of the link in $J^1(S^1)$ will not have any self-paired strands.) However, there is no reason to think the converse is true.

6A. ***Matrix definition of the DGA in $J^1(S^1)$.*** Ng and Traynor [2004] define a version of the Chekanov–Eliashberg DGA $\mathcal{A}$ over $R = \mathbb{Z}[t, t^{-1}]$ in. For ease of definition, note that we can assume all left and right cusps involve the two strands with lowest z-coordinate (and thus highest labels) and that there is one basepoint at $x = 0$ on each strand with the basepoint on strand i corresponding to t_i, and one basepoint in each loop resulting from the resolution of a right cusp. We give the definition of the DGA for the dipped version Λ, $D(\Lambda)$ as in [Lavrov and Rutherford 2012] with an extra dip immediately to the right of the basepoints at $x = 0$. Label the dips as in Figure 13 with b_{ij}^m and c_{ij}^m in the dip at x_m. Place these generators in upper triangular matrices

$$B_m = (b_{ij}^m) \text{ and } C_m = (c_{ij}^m).$$

Note that since the x-coordinate is S^1-valued, we need to add the convention that $B_0 = B_M$ and $C_0 = C_M$. We then see that

$$\begin{aligned}
\partial C_m &= (\Sigma_m C_m)^2, \\
\partial B_1 &= T C_0 T^{-1}(I + B_1) - \Sigma_1 (I + B_1) \Sigma_1 C_1, \\
\partial B_m &= \widetilde{C}_{m-1}(I + B_m) - \Sigma_m (I + B_m) \Sigma_m C_m,
\end{aligned}$$

where Σ_m is the diagonal matrix with $(-1)^{\mu_m(i)}$ the i-th entry on the diagonal for Maslov potential μ_m at $x = x_m$, T is the diagonal matrix with $t_i^{o_1(i)}$ the i-th entry on the diagonal where

$$o_m(i) = \begin{cases} -1 & \text{if strand } i \text{ is oriented to the right at } x = x_m, \\ 1 & \text{otherwise,} \end{cases}$$

and I is the appropriately sized identity matrix. The form of $\widetilde{C}_m$ will depend on the tangle appearing in the interval (x_{m-1}, x_m).

If (x_{m-1}, x_m) contains a crossing a_m of strands k and $k+1$, then

$$\begin{aligned}
\partial a_m &= c_{k,k+1}^{m-1}, \\
\widetilde{C}_{m-1} &= U_{k,k+1} \widehat{C}_{m-1} V_{k,k+1},
\end{aligned}$$

where $U_{k,k+1}$ and $V_{k,k+1}$ are the identity matrix with the 2×2 block in rows k and $k+1$ and columns k and $k+1$ replaced with

$$\begin{pmatrix} 0 & 1 \\ 1 & (-1)^{|a_m|+1}a_m \end{pmatrix}$$

for $U_{k,k+1}$ and

$$\begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix}$$

for $V_{k,k+1}$, and $\widehat{C}_{m-1}$ is C_{m-1} with 0 replacing the entry $c^{m-1}_{k,k+1}$.

If (x_{m-1}, x_m) contains a left cusp, by assumption strands $N(m)-1$ and $N(m)$ are incident to the cusp. In this case,

$$\widetilde{C}_{m-1} = JC_{m-1}J^T + W,$$

where J is the $N(m-1)\times N(m-1)$ identity matrix with two rows of zeroes added to the bottom and W is $N(m)\times N(m)$ matrix where the $(N(m)-1, N(m))$-entry is 1 and all other entries are zero.

Finally, if (x_{m-1}, x_m) contains a right cusp a_m with basepoint $*_\alpha$ corresponding to t_α in the loop, by assumption strands $N(m)-1$ and $N(m)$ are incident to the cusp. In this case

$$\begin{aligned} \partial a_m &= t_\alpha^{o_{m-1}(N(m-1)-1)} + c^{m-1}_{N(m-1)-1,N(m-1)}, \\ \widetilde{C}_{m-1} &= KC_{m-1}K^T, \end{aligned}$$

where K is the $N(m-1)\times N(m-1)$ identity matrix with two columns of zeroes added to the right.

6B. ***Proof of correspondence.*** We will use the methods of [Lavrov and Rutherford 2012] to prove Theorem 1.3. Given an involution σ of $\{1,\dots,N\}$, $\sigma^2 = \mathrm{id}$, we define $A_\sigma = (a_{ij})$ the $N\times N$ matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if } i < \sigma(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Ruling to augmentation: Given a generalized normal ruling $\sigma = (\sigma_1, \dots, \sigma_M)$, we will define a ρ-graded augmentation $\epsilon : \mathcal{A}(D(\Lambda)) \to F$ satisfying Property (R) (as in [Sabloff 2005]) by defining ϵ on the crossings in the dip involving crossings b^0_{ij} and c^0_{ij} and extending to the right.

Property (R): In any dip, the generator c^m_{rs} is augmented (to 1) if and only if $\sigma_m(r) = s$.

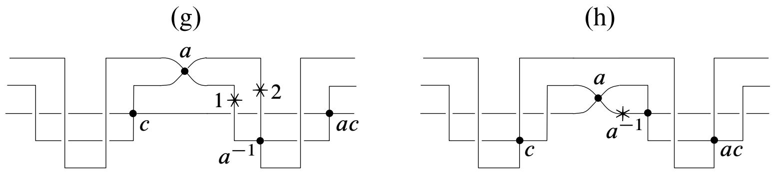


Figure 17. In the diagrams, $*_i$ denotes the basepoint associated to t_i. A dot denotes the specified crossing is augmented and the augmentation sends the crossing to the label. In configuration (g), $\epsilon(t_1) = (-1)^{|a|+1}$ and $\epsilon(t_2) = (-1)^{|c_{i,i+1}|+1}$. In configuration (h), $\epsilon(t) = -1$.

Add a basepoint to the loop in each resolution of a right cusp. Augment all basepoints to -1. Given a crossing a, set

$$\epsilon(a) = \begin{cases} 1 & \text{if the ruling is switched at } a, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\epsilon(B_0) = 0$ and $\epsilon(C_0) = A_{\sigma_0}$. We will now extend ϵ to the right. Suppose ϵ is defined on all crossings in the interval $(0, x_{m-1})$. If (x_{m-1}, x_m) contains a crossing, define ϵ on crossings b_{ij}^m and c_{ij}^m and add basepoints as in Figure 14 and Figure 17. If (x_{m-1}, x_m) contains a left cusp, set

$$\epsilon(B_m) = J\epsilon(B_{m-1})J^T + W.$$

If (x_{m-1}, x_m) contains a right cusp, set

$$\epsilon(B_m) = K\epsilon(B_{m-1})K^T.$$

It is easy to check that by our definition the augmentation satisfies Property (R), which tells us $\epsilon(B_0) = \epsilon(B_M)$ and $\epsilon(C_0) = \epsilon(C_M)$, and our augmentation is a ρ-graded augmentation.

Augmentation to ruling: This direction of the proof follows that of the $\mathbb{Z}/2$ case in [Lavrov and Rutherford 2012] and is based on canonical form results from linear algebra due to Barannikov [1994].

Definition 6.3. An *M-complex* $(V, \mathcal{B}, d)$ is a vector space V over a field F with an ordered basis $\mathcal{B} = \{v_1, \dots, v_N\}$ and a differential $d : V \to V$ of the form $dv_i = \sum_{j=i+1}^N a_{ij} v_j$ satisfying $d^2 = 0$.

The following two propositions are essentially in [Lavrov and Rutherford 2012, Propositions 5.4 and 5.6] and [Barannikov 1994, Lemmas 2 and 4].

Proposition 6.4. *Suppose that $(V, \mathcal{B}, d)$ is an M-complex, then there exists a triangular change of basis $\{\tilde{v}_1, \dots, \tilde{v}_N\}$ with $\tilde{v}_i = \sum_{j=i}^N a_{ij} v_j$ and an involution*

$\tau : \{1, \dots, N\} \to \{1, \dots, N\}$ *such that*

$$d\tilde{v}_i = \begin{cases} \tilde{v}_j, & \text{if } i < \tau(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the involution τ is unique.

Remark 6.5. We have the following properties of the involution:

(1) If the basis elements v_i have been assigned degrees $|v_i| \in \mathbb{Z}/\rho$ such that V is $\mathbb{Z}/\rho$-graded and d has degree -1, then it can be assumed that the change of basis preserves degree. Thus, if $i < \tau(i) = j$, then $|v_i| = |v_j| + 1$.

(2) The set $\{[\tilde{v}_i] : \tau(i) = i\}$ forms a basis for the homology $H(V, d)$.

(3) In matrix formulation, according to Proposition 6.4, there is a unique function $D \mapsto \tau(D)$ which assigns an involution $\tau = \tau(D)$ to each strictly upper triangular matrix D with $D^2 = 0$ and there is an invertible upper triangular matrix P so that $PDP^{-1} = A_\tau$. The uniqueness statement tells us that $\tau(QDQ^{-1}) = \tau(D)$ if Q is a nonsingular upper triangular matrix.

Proposition 6.6. *Suppose $(V, \mathcal{B}, d)$ is an M-complex and $k \in \{1, \dots, N\}$ such that $dv_k = \sum_{j=k+2}^{N} a_{kj} v_j$ so the triple $(V, \mathcal{B}', d)$ with $\mathcal{B}' = \{v_1, \dots, v_{k+1}, v_k, \dots, v_N\}$ is also an M-complex. Then the associated involutions τ and τ' from Proposition 6.4 are related as follows*:

(1) *If*

$$\begin{aligned} &\tau(k+1) < \tau(k) < k < k+1, \\ &\tau(k) < k < k+1 < \tau(k+1), \\ &k < k+1 < \tau(k+1) < \tau(k), \\ &\tau(k) < k < k+1 = \tau(k+1), \\ &\tau(k) = k < k+1 < \tau(k+1), \end{aligned}$$

then either $\tau' = \tau$ or $\tau' = (k\ \ k+1) \circ \tau \circ (k\ \ k+1)$.

(2) *Otherwise $\tau' = (k\ \ k+1) \circ \tau \circ (k\ \ k+1)$.*

Augmentation to ruling: This part of the proof is the same as the analogous statement in [Lavrov and Rutherford 2012] with $\Sigma_{m-1}\epsilon(C_{m-1})$ replacing $\epsilon(Y_{m-1})$.

Suppose $\epsilon : \mathcal{A}(D(\Lambda)) \to F$ is a ρ-graded augmentation. Then for all m, $\epsilon(C_m)$ is an $N(m) \times N(m)$ strictly upper triangular matrix such that

$$0 = \epsilon\partial C_m = (\Sigma_m \epsilon(C_m))^2.$$

As in Remark 6.5, we can set $\tau_m = \tau(\Sigma_m C_m)$ and obtain the sequence $\tau = \{\tau_0, \dots, \tau_M\}$ of involutions where τ_m is an involution of $\{1, \dots, N(m)\}$. We will show that τ satisfies the requirements of a generalized normal ruling (see Definition 6.1).

We also have $N(m) \times N(m)$ strictly upper triangular matrices $\epsilon(B_m)$ which satisfy

$$0 = \epsilon\partial B_1 = T\epsilon(C_0)T^{-1}(I+\epsilon(B_1)) - \Sigma_1(I+\epsilon(B_1))\Sigma_1\epsilon(C_1),$$
$$0 = \epsilon\partial B_m = \epsilon(\widetilde{C}_{m-1})(I+\epsilon(B_m)) - \Sigma_m(I+\epsilon(B_m))\Sigma_m\epsilon(C_m).$$

In the case where $m = 1$, this tells us

$$\begin{aligned}\Sigma_1\epsilon(C_1) &= (I+\epsilon(B_1))^{-1}\Sigma_1 T\epsilon(C_0)T^{-1}(I+\epsilon(B_1))\\ &= (I+\epsilon(B_1))^{-1}T\Sigma_1\epsilon(C_0)T^{-1}(I+\epsilon(B_1))\end{aligned}$$

since T and Σ_1 are diagonal matrices. So Remark 6.5 tells us

$$\begin{aligned}\tau_1 = \tau(\Sigma_1\epsilon(C_1)) &= \tau((I+\epsilon(B_1))^{-1}T\Sigma_1\epsilon(C_0)T^{-1}(I+\epsilon(B_1)))\\ &= \tau(\Sigma_1\epsilon(C_0)) = \tau(\Sigma_0\epsilon(C_0)) = \tau_0\end{aligned}$$

since $\Sigma_1 = \Sigma_0$ and $T^{-1}(I+\epsilon(B_1))$ is a nonsingular upper triangular matrix. Thus τ_1 satisfies the definition of generalized normal ruling since τ_0 does.

More generally, for $m > 1$, we have

$$\Sigma_m\epsilon(C_m) = (I+\epsilon(B_m))^{-1}\Sigma_m\epsilon(\widetilde{C}_{m-1})(I+\epsilon(B_m)).$$

So Remark 6.5 tells us

$$\tau_m = \tau(\Sigma_m\epsilon(C_m)) = \tau(\Sigma_m\epsilon(\widetilde{C}_{m-1})).$$

Recall that $\widetilde{C}_{m-1}$ depends on whether the interval (x_{m-1}, x_m) contains a left cusp, right cusp, or crossing.

Crossing: In the case where the interval (x_{m-1}, x_m) contains a crossing a_m of strands k and $k+1$, recall that $0 = \epsilon\partial(a_m) = \epsilon(c^{m-1}_{k,k+1})$. In this case,

$$\widetilde{C}_{m-1} = U_{k,k+1}\widehat{C}_{m-1}V_{k,k+1},$$

where $\widehat{C}_{m-1}$ is C_{m-1} with 0 replacing the entry $c^{m-1}_{k,k+1}$. Thus $\epsilon(\widehat{C}_{m-1}) = \epsilon(C_{m-1})$. So $\epsilon(\widetilde{C}_{m-1}) = \epsilon(U_{k,k+1}C_{m-1}V_{k,k+1})$. Note that $\mu_{m-1}(k) = \mu_m(k+1)$ and $\mu_{m-1}(k+1) = \mu_m(k)$, so $\Sigma_{m-1} = P_{k,k+1}\Sigma_m P_{k,k+1}$. We also see that

$$\begin{aligned}\Sigma_m U_{k,k+1} &= \Sigma_m P_{k,k+1}(I+(-1)^{|a_m|+1}\epsilon(a_m)E_{k,k+1})\\ &= P_{k,k+1}(I-\epsilon(a_m)E_{k,k+1})P_{k,k+1}\Sigma_m P_{k,k+1}\\ &= P_{k,k+1}(I-\epsilon(a_m)E_{k,k+1})\Sigma_{m-1},\\ V_{k,k+1} &= (I+\epsilon(a_m)E_{k,k+1})P_{k,k+1},\end{aligned}$$

where $E_{k,k+1}$ is a matrix with a single nonzero entry of 1 in the $(k, k+1)$ position. Thus

$$\begin{aligned}\Sigma_m\epsilon(\widetilde{C}_{m-1})\\ &= P_{k,k+1}(I - \epsilon(a_m)E_{k,k+1})\Sigma_{m-1}\epsilon(C_{m-1})(I + \epsilon(a_m)E_{k,k+1})P_{k,k+1}.\end{aligned}$$

Since the $(k, k+1)$-entry of $(I-\epsilon(a_m)E_{k,k+1})\Sigma_{m-1}\epsilon(C_{m-1})(I+\epsilon(a_m)E_{k,k+1})$ is 0 no matter the value of $\epsilon(a_m)$, the matrix $\Sigma_m\epsilon(\widetilde{C}_{m-1})$ is strictly upper triangular. Therefore

$$\tau_m = \tau(\Sigma_m\epsilon(C_m)) = \tau(\Sigma_m\epsilon(\widetilde{C}_{m-1}))$$

and

$$\begin{aligned}\tau((I - \epsilon(a_m)E_{k,k+1})\Sigma_{m-1}\epsilon(C_{m-1})(I + \epsilon(a_m)E_{k,k+1}))\\ = \tau(\Sigma_{m-1}\epsilon(C_{m-1})) = \tau_{m-1}\end{aligned}$$

are related as in Proposition 6.6. So, as τ_{m-1} satisfies the conditions of a generalized normal ruling, so does τ_m. The left and right cusp cases follow similarly.

As in Remark 6.5, $\Sigma_m\epsilon(C_m)$ denotes the matrix of an M-complex with basis $v_1, \dots, v_{N(m)}$ corresponding to the strands of Λ at x_m. If ϵ is ρ-graded with respect to μ, then we can assign the gradings $|v_i| = \mu_m(i)$ and the differential will have degree -1. So (1) of Remark 6.5 tells us that the resulting involution $\tau_m = \tau(\Sigma_m\epsilon(C_m))$ is ρ-graded and thus τ is ρ-graded.

6C. *Corollary.* The following proposition uses certain techniques from the proof of Theorem 1.6 to show that

$$\mathrm{Aug}_\rho(\Lambda) = F\backslash 0$$

for any field F and any ρ if Λ has a strictly generalized normal ruling.

Proposition 6.7. *Given a field F and a Legendrian link $\Lambda \subset J^1(S^1)$ with n components and a strictly generalized normal ruling, for all $0 \neq x \in F$ there exists an augmentation $\epsilon : \mathcal{A} \to F$ such that*

$$\epsilon(t_1 \cdots t_s) = x.$$

Proof. Fix $0 \neq x \in F$. Given a generalized normal ruling $\sigma = (\sigma_1, \dots, \sigma_M)$ for Λ with a self-paired strand, we will construct an augmentation $\epsilon : \mathcal{A}(D(\Lambda)) \to F$ such that $\epsilon(t_1 \cdots t_s) = x$.

Suppose k is the label at $x = 0$ of a self-paired strand of the generalized normal ruling σ, in other words, $\sigma_0(k) = k$. We can assume that $D(\Lambda)$ has one basepoint corresponding to t_i on strand i at $x = 0$ and one basepoint in the loop in the resolution of each right cusp, and no other basepoints. Define

$$\epsilon(t_i) = \begin{cases} (-1)^{N+c-1}x & \text{if } i = k, \\ -1 & \text{otherwise,} \end{cases}$$

where c is the number of right cusps and N is the number of strands at $x = 0$.

Define ϵ on all crossings as in the proof of ruling to augmentation in Theorem 1.6. Note that t_k does not appear on the boundary of any totally augmented disks and so ϵ is still an augmentation, but now

$$\epsilon(t_1 \cdots t_s) = x$$

as desired. □

Remark 6.8. For any link $\Lambda \subset J^1(S^1)$, one can consider the analogous link $\Lambda' \subset S^1 \times S^2$. Note that $\mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda')$ where the map is inclusion. Thus, any augmentation $\epsilon' : \Lambda' \to F$ gives an augmentation $\epsilon : \Lambda \to F$. As one would expect from Theorems 1.3 and 1.6, it is also clear that any normal ruling of $\Lambda' \subset S^1 \times S^2$ gives a generalized normal ruling of $\Lambda \subset J^1(S^1)$.

Appendix

The appendix will address Corollary 1.5 which follows from

(1) Theorem 1.3 over $\mathbb{Q}$, and

(2) the result that if a graded augmentation to the rationals exists then the full symplectic homology is nonzero.

The second result is known to experts; assumes the results of [Bourgeois et al. 2012]. We will outline the proof here for completeness. Statement (2) is a straight forward consequence of work of Bourgeois, Ekholm, and Eliashberg [Bourgeois et al. 2012] and has previously been observed in [Lidman and Sivek 2016].

Every connected Weinstein (Stein) 4-manifold X can be decomposed into 1- and 2-handle attachments to D^4 along $\partial D^4 = S^3$. Thus, for each such 4-manifold there exists a Legendrian link Λ in $\#^k(S^1 \times S^2)$ so that attaching 2-handles along Λ to $\#^k(S^1 \times S^2)$ results in X.

Results of Bourgeois, Ekholm, and Eliashberg (using their notation) tell us the following:

Proposition A.1 [Bourgeois et al. 2012, Corollary 5.7].

$$S\mathbb{H}(X) = L\mathbb{H}^{\text{Ho}}(\Lambda),$$

where $L\mathbb{H}^{\text{Ho}}(\Lambda)$ is the homology of the Hochschild complex associated to the Chekanov–Eliashberg differential graded algebra over $\mathbb{Q}$.

Therefore, if the DGA for Λ has a graded augmentation to $\mathbb{Q}$, then $S\mathbb{H}(X)$ is nonzero. By Theorem 1.3, we know that the DGA for Λ has a graded augmentation to $\mathbb{Q}$ if and only if Λ has a graded normal ruling. Thus, restated from the introduction:

Corollary 1.5. *If X is the Weinstein 4-manifold that results from attaching 2-handles along a Legendrian link Λ to $\#^k(S^1 \times S^2)$ and Λ has a graded normal ruling, then the full symplectic homology $S\mathbb{H}(X)$ is nonzero.*

For completeness, we give an outline of the proof of statement (2). Recall that full symplectic homology is a symplectic invariant of Weinstein 4-manifolds which coincides with the Floer–Hofer symplectic homology.

We will show that given a graded augmentation ϵ' of the Chekanov–Eliashberg DGA of a Legendrian link Λ over $\mathbb{Z}[t, t^{-1}]$ to $\mathbb{Q}$, one can define a graded augmentation $\epsilon : LH^{\mathrm{Ho}}(\Lambda) \to \mathbb{Q}$, where the homology of $LH^{\mathrm{Ho}}(\Lambda)$ is $L\mathbb{H}^{\mathrm{Ho}}(\Lambda)$. Recall that

$$LH^{\mathrm{Ho}}(\Lambda) = \widecheck{\mathrm{LHO}^+}(\Lambda) \oplus \mathbb{Q}\langle \tau_1, \dots, \tau_n \rangle \oplus \widehat{\mathrm{LHO}^+}(\Lambda)$$

is generated by elements of the form $\check{w}$, τ_i, and $\hat{v}$, where $w, v \in \mathrm{LHO}(\Lambda) \subset \mathrm{LHA}(\Lambda)$ and n is the number of components of the link. Define

$$\epsilon : LH^{\mathrm{Ho}}(\Lambda) \to \mathbb{Q}$$

by $\check{w} \mapsto \epsilon'(w)$, $\tau_i \mapsto 1$, $\hat{v} \mapsto 0$. Let us check that this gives an augmentation. It suffices to check the generators. Clearly $\epsilon \circ d_{\mathrm{Ho}}(\tau_i) = 0$ for all i. If $d_{\mathrm{LHO}^+}(w) = \sum_{j=1}^r w_j$, then we recall that

$$d_{\mathrm{Ho}}(\check{w}) = d_{\mathrm{Ho}_+}(\check{w}) + \delta_{\mathrm{Ho}}(\check{w}) = \check{d}_{\mathrm{LHO}^+}(\check{w}) + \delta_{\mathrm{Ho}}(\check{w}).$$

Let w be a chord in $\mathrm{LHO}^+(\Lambda)$. Then, there exists i such that $w \in C_i$ and

$$d_{\mathrm{Ho}}(\check{w}) = \sum_{j=1}^r \check{w}_j + \alpha_{wi}\tau_i,$$

where α_{wi} is the algebraic number of components of the 1-dimensional moduli space of holomorphic disks with one positive and no negative boundary punctures. Thus

$$\epsilon \circ d_{\mathrm{Ho}}(\check{w}) = \sum_{j=1}^r \epsilon'(w_j) + \sum_{i=1}^n \alpha_{wi} = \epsilon' \circ d_{\mathrm{LHO}}(w) = 0,$$

since α_{wi} is exactly the constant term of $d_{\mathrm{LHA}}(w)$, ϵ' is an augmentation of $\mathrm{LHA}(\Lambda)$, $\mathrm{LHO}(\Lambda) \subset \mathrm{LHA}(\Lambda)$, and $d_{\mathrm{LHO}} = d_{\mathrm{LHA}}|_{\mathrm{LHO}}$. If $w \in \mathrm{LHO}^+(\Lambda)$ is a linearly composable monomial which is not a chord, then

$$d_{\mathrm{Ho}}(\check{w}) = \sum_{j=1}^{r} \check{w}_j$$

and so

$$\epsilon \circ d_{\mathrm{Ho}}(\check{w}) = \sum_{j=1}^{r} \epsilon'(w_j) = \epsilon' \circ d_{\mathrm{LHO}}(w) = 0$$

since $d_{\mathrm{LHO}}(w)$ does not have a constant term.

If $v = c_1 \cdots c_\ell \in \mathrm{LHO}^+(\Lambda)$, then we recall that

$$\begin{aligned} d_{\mathrm{Ho}}(\hat{v}) &= d_{\mathrm{Ho}_+}(\hat{v}) + \delta_{\mathrm{Ho}}(\hat{v}) = d_{M\ Ho_+}(\hat{v}) + \hat{d}_{\mathrm{LHO}^+}(\hat{v}) + 0 \\ &= \check{c}_1 c_2 \cdots c_\ell - c_1 \cdots c_{\ell-1}\check{c}_\ell + \hat{d}_{\mathrm{LHO}^+}(\hat{v}) \\ &= \check{c}_1 c_2 \cdots c_\ell - (-1)^{|c_\ell|(|c_1|+\cdots+|c_{\ell-1}|)}\check{c}_\ell c_1 \cdots c_{\ell-1} + \hat{d}_{\mathrm{LHO}^+}(\hat{v}). \end{aligned}$$

Thus

$$\begin{aligned} &\epsilon \circ d_{\mathrm{Ho}}(\hat{v}) \\ &\qquad = \epsilon'(c_1 \cdots c_\ell) - (-1)^{|c_\ell|(|c_1|+\cdots+|c_{\ell-1}|)}\epsilon'(c_\ell c_1 \cdots c_{\ell-1}) + \epsilon' \circ \hat{d}_{\mathrm{LHO}^+}(\hat{v}) \\ &\qquad = \epsilon'(c_1 \cdots c_\ell) - (-1)^{|c_\ell|(|c_1|+\cdots+|c_{\ell-1}|)}\epsilon'(c_\ell c_1 \cdots c_{\ell-1}) + 0 \\ &\qquad = 0 \end{aligned}$$

since ϵ' is a graded augmentation of $\mathrm{LHA}(\Lambda)$ so if $\epsilon(c_1 \cdots c_\ell) \neq 0$, then $\epsilon(c_i) \neq 0$ for all i and thus $|c_i| = 0$ for all i.

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