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GAP THEOREMS FOR COMPLETE λ-HYPERSURFACES

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An n-dimensional λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ is the critical point of the weighted area functional $\int_M e^{-\frac{1}{4}|X|^2}\, d\mu$ for weighted volume-preserving variations, which is also a generalization of the self-shrinking solution of the mean curvature flow. We first prove that if the L^n-norm of the second fundamental form of the λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ with $n \geq 3$ is less than an explicit positive constant $K(n, \lambda)$, then M is a hyperplane. Secondly, we show that if the L^n-norm of the trace-free second fundamental form of M with $n \geq 3$ is less than an explicit positive constant $D(n, \lambda)$ and the mean curvature is suitably bounded, then M is a hyperplane. We also obtain similar results for λ-surfaces in $\mathbb{R}^3$ under L^4-curvature pinching conditions.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional immersed smooth hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We call the hypersurface a λ-hypersurface if it satisfies

$$H + \tfrac{1}{2}\langle X, N\rangle = \lambda,$$

where λ is a constant, H is the mean curvature and N is the unit inward normal vector of $X : M \to \mathbb{R}^{n+1}$.

McGonagle and Ross [2015] studied λ-hypersurfaces from the viewpoint of variation. Let $A_\mu(M)$ be the functional defined by $A_\mu(M) = \int_M e^{-\frac{1}{4}|X|^2}\, d\mu$. They showed that the critical points of $\delta A_\mu(u) = 0$ for $u \in C_0^\infty$ satisfying

$$\int_M e^{-\frac{1}{4}|X|^2} u \, d\mu = 0$$

are λ-hypersurfaces. Cheng and Wei [2014a] also introduced λ-hypersurfaces in a different way by investigating the weighted volume-preserving mean curvature flow. Obviously, when $\lambda = 0$, a λ-hypersurface is a self-shrinker of the mean curvature flow. It is well known that self-shrinkers play an important role in the study of mean

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curvature flow because they describe the singularity models of the mean curvature flow and they arise as tangent flows of mean curvature flow at singularities; see, for example, [Colding and Minicozzi 2012; Huisken 1990; Ilmanen 1995; White 1997].

The rigidity phenomena of self-shrinkers has been studied extensively [Cheng and Peng 2015; Cheng and Wei 2015; Colding et al. 2015; Colding and Minicozzi 2012; Ding and Xin 2013; 2014; Huisken 1990; Le and Sesum 2011]. For example, Le and Sesum [2011] proved that a smooth self-shrinker with polynomial volume growth and satisfying $|A|^2 < \frac{1}{2}$ is a hyperplane. Here A denotes the second fundamental form of the immersion. Cao and Li [2013] generalized this result to arbitrary codimension and proved that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \leq \frac{1}{2}$ is a generalized cylinder. On the other hand, Ding and Xin [2014] showed that a smooth complete self-shrinker satisfying $\left(\int_M |A|^n \, \mathrm{d}\mu\right)^{1/n} < C$ for a certain positive constant C is a linear space. For more curvature pinching theorems for self-shrinkers, see [Cao et al. 2014; Li and Wei 2014; Lin 2016].

The geometric properties of λ-hypersurfaces were recently investigated by Cheng, Wei, Ogata, Guang [Cheng and Wei 2014a; Cheng et al. 2016; Guang 2014]. As generalizations of self-shrinkers of the mean curvature flow, complete λ-hypersurfaces with polynomial area growth and $H - \lambda \geq 0$ were classified by Cheng and Wei [2014a]. They also defined an $\mathcal{F}$-functional and studied $\mathcal{F}$-stability of λ-hypersurfaces. Cheng, Ogata and Wei [Cheng et al. 2016] proved some gap and rigidity theorems for complete λ-hypersurfaces. See [Cheng and Wei 2014b; Guang 2014; Ogata 2015] for more results on the rigidity of λ-hypersurfaces.

We study the integral curvature pinching theorems for λ-hypersurfaces. We first prove the following L^n-pinching theorem of the second fundamental form.

Theorem 1. *Let $X : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be an n-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If*

$$\left(\int_M |A|^n \, \mathrm{d}\mu\right)^{1/n} < K(n, \lambda),$$

where $K(n, \lambda)$ is an explicit positive constant depending only on n and λ, then $|A| \equiv 0$ and M is a hyperplane.

Remark. It is easy to see from the expression of $K(n, \lambda)$ that $\lim_{\lambda\to 0} K(n, \lambda) = K_n$ for a positive constant K_n depending only on n. Hence if $\lambda = 0$, Theorem 1 reduces to the L^n-pinching theorem for self-shrinkers due to Ding and Xin [2014].

Let $\mathring{A}$ denote the trace-free second fundamental form, which is defined by $\mathring{A} = A - (H/n)g$ with g denoting the induced metric on M. We prove an L^n-pinching theorem of the trace-free second fundamental form for λ-hypersurfaces provided that the mean curvature is suitably bounded.

Theorem 2. *Let $X : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be an n-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Suppose the mean curvature satisfies*

$$|H| \leq \sqrt{\tfrac{1}{3}n + \lambda^2} - |\lambda|.$$

If

$$\left(\int_M |\mathring{A}|^n \, \mathrm{d}\mu\right)^{1/n} < D(n, \lambda),$$

where $D(n, \lambda)$ is an explicit positive constant depending on n and λ, then M is a hyperplane.

For the case $n = 2$, we obtain the following results.

Theorem 3. *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^3$. If*

$$\left(\int_M |A|^4 \, \mathrm{d}\mu\right)^{1/2} < K(\lambda),$$

where $K(\lambda)$ is an explicit positive constant depending only on λ, then $|A| \equiv 0$ and M is a hyperplane.

Theorem 4. *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^3$. Suppose the mean curvature satisfies*

$$|H| \leq \sqrt{\tfrac{2}{3} + \lambda^2} - |\lambda|.$$

If

$$\left(\int_M |\mathring{A}|^4 \, \mathrm{d}\mu\right)^{1/2} < D(\lambda),$$

where $D(\lambda)$ is an explicit positive constant depending on λ, then M is a hyperplane.

The rest of our paper is organized as follows. Some notation and several lemmas are prepared in Section 2. In Section 3, we prove Theorems 1 and 2. Theorems 3 and 4 will be proved in Section 4.

2. Preliminaries

Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional connected hypersurface. Denote by g and $\mathrm{d}\mu$ the induced metric and the volume form on M, respectively. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+1, \quad 1 \leq i, j, k, \ldots \leq n.$$

Choose local orthonormal frame fields $\{e_A\}$ in $\mathbb{R}^{n+1}$ such that, restricted to M, the e_i are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame fields and the connection 1-forms of $\mathbb{R}^{n+1}$, respectively. Then we have the following structure equations:

$$dX = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij} \omega_j e_{n+1},$$

and

$$de_{n+1} = -\sum_{i,j} h_{ij} \omega_j e_i.$$

Restricting these forms to M, we have

$$\omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

where h_{ij} denotes the components of the second fundamental form of M. $H = \sum_i h_{ii}$ is the mean curvature and $A = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$ is the second fundamental form of $X : M^n \to \mathbb{R}^{n+1}$. The trace-free second fundamental form is defined by $\mathring{A} = A - (H/n)g$.

Let $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_l \nabla_k h_{ij}$, where ∇ is the Levi-Civita connection on M. Gauss equations, Codazzi equations and Ricci formulas are given by

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad h_{ijk} = h_{ikj},$$

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im} R_{mjkl} + \sum_{m=1}^{n} h_{mj} R_{mikl}.$$

For λ-hypersurfaces, an elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L} = \Delta - \tfrac{1}{2}\langle X, \nabla(\cdot)\rangle = e^{\frac{1}{4}|X|^2} \operatorname{div}\big(e^{-\frac{1}{4}|X|^2} \nabla(\cdot)\big),$$

where Δ and div denote the Laplacian and divergence on the λ-hypersurface, respectively. The $\mathcal{L}$ operator was introduced by Colding and Minicozzi [2012] when they investigated self-shrinkers. They showed that $\mathcal{L}$ is self-adjoint with respect to the measure $e^{-\frac{1}{4}|X|^2}\,d\mu$. We set $\rho = e^{-\frac{1}{4}|X|^2}$ and the volume form $d\mu$ might be omitted in the integrations for notational simplicity.

The following lemma, which was proved in [Cheng and Wei 2014a], is needed in order to prove our results. For convenience, we also include the proof here.

Lemma 5. *Let $X : M \to \mathbb{R}^{n+1}$ be a λ-hypersurface satisfying $H + \frac{1}{2}\langle X, N\rangle = \lambda$. Then*

$$\tfrac{1}{2}\mathcal{L}H^2 = |\nabla H|^2 + \tfrac{1}{2}H^2 + |A|^2(\lambda - H)H, \tag{1}$$

$$\tfrac{1}{2}\mathcal{L}|A|^2 = |\nabla A|^2 + \big(\tfrac{1}{2} - |A|^2\big)|A|^2 + \lambda f_3, \tag{2}$$

where $f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}$.

Proof. Since $H + \frac{1}{2}\langle X, N\rangle = \lambda$, one has

$$\nabla_i H = \frac{1}{2}\sum_j h_{ij}\langle X, e_j\rangle,$$

and

$$\nabla_k\nabla_i H = \frac{1}{2}\sum_j h_{ijk}\langle X, e_j\rangle + \tfrac{1}{2}h_{ik} + \sum_j h_{ij}h_{jk}(\lambda - H).$$

Hence,

$$\Delta H = \sum_i \nabla_i\nabla_i H = \frac{1}{2}\sum_i \nabla_i H\langle X, e_i\rangle + \tfrac{1}{2}H + |A|^2(\lambda - H),$$

and

$$\mathcal{L}H = \Delta H - \frac{1}{2}\sum_i \nabla_i H\langle X, e_i\rangle = \tfrac{1}{2}H + |A|^2(\lambda - H).$$

Therefore, we obtain

$$\tfrac{1}{2}\mathcal{L}H^2 = \tfrac{1}{2}\Delta H^2 - \frac{1}{4}\sum_i \nabla_i H^2\langle X, e_i\rangle = |\nabla H|^2 + \tfrac{1}{2}H^2 + |A|^2(\lambda - H)H.$$

By using the Ricci formulas, the Gauss equations and the Codazzi equations, we have

$$\begin{aligned}\mathcal{L}h_{ij} &= \Delta h_{ij} - \frac{1}{2}\sum_k \langle X, e_k\rangle h_{ijk}\\ &= \sum_k h_{ijkk} - \frac{1}{2}\sum_k \langle X, e_k\rangle h_{ijk}\\ &= \left(\tfrac{1}{2} - |A|^2\right)h_{ij} + \lambda\sum_k h_{ik}h_{kj}.\end{aligned}$$

Then it follows that

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}|A|^2 &= \tfrac{1}{2}\Delta\left(\sum_{ij} h_{ij}^2\right) - \frac{1}{4}\sum_k \langle X, e_k\rangle\nabla_k\left(\sum_{ij} h_{ij}^2\right)\\ &= \sum_{i,j,k} h_{ijk}^2 + \left(\tfrac{1}{2} - |A|^2\right)\sum_{ij} h_{ij}^2 + \lambda\sum_{i,j,k} h_{ik}h_{kj}h_{ji}\\ &= |\nabla A|^2 + \left(\tfrac{1}{2} - |A|^2\right)|A|^2 + \lambda f_3,\end{aligned}$$

where $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$. □

We need the following Sobolev inequality for submanifolds in the Euclidean space.

Lemma 6 [Xu and Gu 2007a; Hoffman and Spruck 1974]. *Let M^n $(n \geq 3)$ be an n-dimensional complete submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Let f be a* *nonnegative C^1 function with compact support. Then we have*

$$\|f\|_{2n/(n-2)}^2 \le D^2(n)\left[\frac{4(n-1)^2(1+s)}{(n-2)^2}\|\nabla f\|_2^2 + \left(1+\frac{1}{s}\right)\frac{1}{n^2}\big\||H|f\big\|_2^2\right],$$

where

$$D(n) = 2^n(1+n)^{(n+1)/n}(n-1)^{-1}\sigma_n^{-1/n},$$

and σ_n denotes the volume of the unit ball in $\mathbb{R}^n$.

3. Gap theorems for λ-hypersurfaces

Proof of Theorem 1. It follows from (2) and the inequality $|\nabla A|^2 \ge \big|\nabla|A|\big|^2$, which is an easy consequence of the Schwartz inequality, that

$$\begin{aligned}\mathcal{L}|A|^2 &= 2|\nabla A|^2 + 2\big(\tfrac{1}{2} - |A|^2\big)|A|^2 + 2\lambda f_3 \\ &\ge 2\big|\nabla|A|\big|^2 + 2\big(\tfrac{1}{2} - |A|^2\big)|A|^2 - 2|\lambda||A|^3.\end{aligned}$$

Let η be a smooth function with compact support on M. Multiplying $\eta^2|A|^{n-2}$ on both sides of the inequality above and integrating by parts with respect to the measure $\rho\,\mathrm{d}\mu$ on M yields that for any $\tau > 0$

$$\begin{aligned}0 &\ge 2\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\int_M |A|^{n+1}\eta^2\rho - \int_M \eta^2|A|^{n-2}\rho\mathcal{L}|A|^2 \\ &= 2\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\int_M |A|^{n+1}\eta^2\rho + 2\int_M \rho|A|\nabla|A|\cdot\nabla(|A|^{n-2}\eta^2) \\ &= 2(n-1)\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\int_M |A|^{n+1}\eta^2\rho + 4\int_M (\nabla|A|\cdot\nabla\eta)|A|^{n-1}\eta\rho \\ &\ge 2(n-1)\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\left(\frac{\tau}{2}\int_M |A|^n\eta^2\rho + \frac{1}{2\tau}\int_M |A|^{n+2}\eta^2\rho\right) + 4\int_M (\nabla|A|\cdot\nabla\eta)|A|^{n-1}\eta\rho.\end{aligned}$$

By the Cauchy inequality, for any $\varepsilon > 0$, we have

$$\left(\frac{|\lambda|}{\tau}+2\right)\int_M |A|^{n+2}\eta^2\rho + (|\lambda|\tau - 1)\int_M |A|^n\eta^2\rho + \frac{2}{\varepsilon}\int_M |A|^n|\nabla\eta|^2\rho \geq 2(n-1-\varepsilon)\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho. \tag{3}$$

Set $f = |A|^{n/2}\rho^{1/2}\eta$. Integrating by parts, we obtain

$$\begin{aligned}\int_M |\nabla f|^2 &= \int_M \big|\nabla(|A|^{n/2}\eta)\big|^2\rho + \int_M |A|^n\eta^2|\nabla\rho^{1/2}|^2 + \frac{1}{2}\int_M \nabla(|A|^n\eta^2)\nabla\rho \\ &= \int_M \big|\nabla(|A|^{n/2}\eta)\big|^2\rho + \frac{1}{16}\int_M |A|^n\eta^2|X^T|^2\rho - \frac{1}{2}\int_M |A|^n\eta^2\Delta\rho.\end{aligned} \tag{4}$$

Since

$$\Delta|X|^2 = 2|\nabla X|^2 + 2\langle X, \Delta X\rangle = 2n + 2H\langle X, N\rangle = 2n + 2\lambda\langle X, N\rangle - |X^N|^2,$$

where X^N is the normal part of X, we have

$$\begin{aligned}\Delta\rho &= -\tfrac{1}{4}\rho\Delta|X|^2 + \tfrac{1}{16}\rho\big|\nabla|X|^2\big|^2 = -\tfrac{1}{4}\rho\big(2n + 2\lambda\langle X, N\rangle - |X^N|^2\big) + \tfrac{1}{4}\rho|X^T|^2 \\ &= -\tfrac{1}{2}n\rho - \tfrac{1}{2}\lambda\rho\langle X, N\rangle + \tfrac{1}{4}\rho|X|^2.\end{aligned}$$

From (4), we get

$$\begin{aligned}\int_M |\nabla f|^2 = \int_M \big|\nabla(|A|^{n/2}\eta)\big|^2\rho - \frac{1}{16}\int_M |A|^n\eta^2|X^T|^2\rho - \frac{1}{8}\int_M |A|^n\eta^2|X^N|^2\rho \\ + \frac{n}{4}\int_M |A|^n\eta^2\rho + \frac{1}{4}\int_M |A|^n\eta^2\lambda\langle X, N\rangle\rho.\end{aligned} \tag{5}$$

Combining the Sobolev inequality in Lemma 6 and (5), we have

$$\begin{aligned}&\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\quad\leq D^2(n)\cdot\left[\frac{4(n-1)^2(1+s)}{(n-2)^2}\int_M |\nabla f|^2 + \left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M H^2 f^2\right] \\ &\quad= \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2}\left[\int_M \big|\nabla(|A|^{n/2}\eta)\big|^2\rho - \frac{1}{16}\int_M |A|^n\eta^2|X^T|^2\rho\right. \\ &\qquad\qquad - \frac{1}{8}\int_M |A|^n\eta^2|X^N|^2\rho + \frac{n}{4}\int_M |A|^n\eta^2\rho \\ &\qquad\qquad \left. + \frac{1}{4}\int_M |A|^n\eta^2\lambda\langle X, N\rangle\rho\right] \\ &\qquad + D^2(n)\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M |A|^n\eta^2\big(\lambda - \tfrac{1}{2}\langle X, N\rangle\big)^2\rho.\end{aligned}$$

We choose

$$s=\frac{(n-2)^2}{2n^2(n-1)^2}\in\mathbb{R}^+$$

such that

$$\frac{4(n-1)^2(1+s)}{(n-2)^2}\cdot\frac{1}{8}=\frac{1}{4}\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}.$$

Hence

$$\begin{aligned}&\left(\int_M|f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\ &\quad\le\frac{2D^2(n)[(n-2)^2+2n^2(n-1)^2]}{n^2(n-2)^2}\biggl[\int_M\bigl|\nabla(|A|^{n/2}\eta)\bigr|^2\rho\\ &\qquad\qquad+\frac{n}{4}\int_M|A|^n\eta^2\rho+\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\biggr]\\ &\qquad+\frac{D^2(n)[(n-2)^2+2n^2(n-1)^2]}{n^2(n-2)^2}\biggl[\int_M\lambda^2|A|^n\eta^2\rho-\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\biggr].\end{aligned}$$

Now we put

$$\kappa=\frac{2D^2(n)[(n-2)^2+2n^2(n-1)^2]}{n^2(n-2)^2}.$$

It follows from the inequality above that

$$\begin{aligned}(6)\quad&\kappa^{-1}\left(\int_M|f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\ &\le\int_M\bigl|\nabla(|A|^{n/2}\eta)\bigr|^2\rho+\frac{n}{4}\int_M|A|^n\eta^2\rho+\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\\ &\qquad+\frac{1}{2}\left(\int_M\lambda^2|A|^n\eta^2\rho-\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\right)\\ &=\int_M\bigl|\nabla(|A|^{n/2}\eta)\bigr|^2\rho+\left(\frac{n+2\lambda^2}{4}\right)\int_M|A|^n\eta^2\rho-\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\\ &=\int_M\left(\frac{n^2}{4}\bigl|\nabla|A|\bigr|^2|A|^{n-2}\eta^2+n|A|^{n-1}\eta\nabla|A|\cdot\nabla\eta+|A|^n|\nabla\eta|^2\right)\rho\\ &\qquad+\left(\frac{n+2\lambda^2}{4}\right)\int_M|A|^n\eta^2\rho-\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho.\end{aligned}$$

On the other hand, for any $\theta > 0$, we have

$$
\begin{aligned}
(7)\quad -\frac{1}{2}\int_M |A|^n\eta^2\lambda\langle X, N\rangle\rho &= -\int_M |A|^n\eta^2\lambda(\lambda - H)\rho \\
&= -\int_M |A|^n\eta^2\lambda^2\rho + \int_M |A|^n\eta^2\lambda H\rho \\
&\le -\lambda^2\int_M |A|^n\eta^2\rho + |\lambda|\int_M |A|^n\eta^2\left(\frac{\theta}{2}H^2 + \frac{1}{2\theta}\right)\rho \\
&\le \left(\frac{|\lambda|}{2\theta} - \lambda^2\right)\int_M |A|^n\eta^2\rho + \frac{|\lambda|\theta}{2}\int_M |A|^n\eta^2 H^2\rho \\
&\le \left(\frac{|\lambda|}{2\theta} - \lambda^2\right)\int_M |A|^n\eta^2\rho + \frac{n|\lambda|\theta}{2}\int_M |A|^{n+2}\eta^2\rho.
\end{aligned}
$$

Combining (6) and (7), we get

$$
\begin{aligned}
(8)\quad & \kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\le \int_M \left(\frac{n^2}{4}\big|\nabla|A|\big|^2|A|^{n-2}\eta^2 + n|A|^{n-1}\eta\nabla|A|\cdot\nabla\eta + |A|^n|\nabla\eta|^2\right)\rho \\
&\qquad + \left(\frac{|\lambda|}{4\theta} + \frac{n}{4}\right)\int_M |A|^n\eta^2\rho + \frac{n\theta|\lambda|}{4}\int_M |A|^{n+2}\eta^2\rho.
\end{aligned}
$$

Combining the Cauchy inequality, (3) and (8), we have for any $\delta > 0$

$$
\begin{aligned}
&\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\le (1+\delta)\frac{n^2}{4}\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \left(1+\frac{1}{\delta}\right)\int_M |A|^n|\nabla\eta|^2\rho \\
&\quad + \left(\frac{|\lambda|}{4\theta} + \frac{n}{4}\right)\int_M |A|^n\eta^2\rho + \frac{n\theta|\lambda|}{4}\int_M |A|^{n+2}\eta^2\rho \\
&\le \frac{(1+\delta)n^2}{8(n-1-\varepsilon)}\bigg[\left(\frac{|\lambda|}{\tau}+2\right)\int_M |A|^{n+2}\eta^2\rho \\
&\qquad + (|\lambda|\tau - 1)\int_M |A|^n\eta^2\rho + \frac{2}{\varepsilon}\int_M |A|^n|\nabla\eta|^2\rho\bigg] \\
&\quad + \left(1+\frac{1}{\delta}\right)\int_M |A|^n|\nabla\eta|^2\rho + \left(\frac{|\lambda|}{4\theta} + \frac{n}{4}\right)\int_M |A|^n\eta^2\rho + \frac{n\theta|\lambda|}{4}\int_M |A|^{n+2}\eta^2\rho.
\end{aligned}
$$

Put

$$
\delta = \frac{2(|\lambda| + n\theta)(n-1+\varepsilon)}{(1-|\lambda|\tau)\theta n^2} - 1 > 0,
$$

where $\varepsilon, \theta, \tau$ are positive constants such that $|\lambda|\tau - 1 < 0$. Then

$$\begin{aligned}
(9)\qquad & \kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\
&\le \Big[\frac{n\theta+|\lambda|}{4\theta(1-|\lambda|\tau)}\cdot\Big(\frac{|\lambda|}{\tau}+2\Big)\frac{n-1+\varepsilon}{n-1-\varepsilon}+\frac{n\theta|\lambda|}{4}\Big]\int_M |A|^{n+2}\eta^2\rho \\
&\quad + \Big[\frac{n\theta+|\lambda|}{2\theta\varepsilon(1-|\lambda|\tau)}\frac{n-1+\varepsilon}{n-1-\varepsilon}+1+\frac{1}{\delta}\Big]\int_M |A|^n|\nabla\eta|^2\rho \\
&\le \frac{(n\theta+|\lambda|)(|\lambda|+2\tau)+n\tau\theta^2(1-|\lambda|\tau)|\lambda|}{4\tau\theta(1-|\lambda|\tau)}\cdot\frac{n-1+\varepsilon}{n-1-\varepsilon} \\
&\qquad\qquad \times\Big(\int_M |A|^{2\cdot\frac{n}{2}}\Big)^{\frac{2}{n}}\cdot\Big(\int_M (|A|^n\eta^2\rho)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \\
&\quad + \Big(\frac{n\theta+|\lambda|}{2\theta\varepsilon(1-|\lambda|\tau)}\cdot\frac{n-1+\varepsilon}{n-1-\varepsilon}+1+\frac{1}{\delta}\Big)\int_M |A|^n|\nabla\eta|^2\rho.
\end{aligned}$$

Set

$$K(n,\lambda,\theta,\tau)=\sqrt{\frac{4\tau\theta(1-|\lambda|\tau)}{\big[(n\theta+|\lambda|)(|\lambda|+2\tau)+n\tau\theta^2(1-|\lambda|\tau)|\lambda|\big]\kappa}}.$$

By a direct computation, $K(n,\lambda,\theta,\tau)$ achieves its maximum

$$K(n,\lambda)=\sqrt{\frac{2(\sqrt{\lambda^2+2}-|\lambda|)}{\big(n|\lambda|+2\sqrt{n}|\lambda|+n\sqrt{\lambda^2+2}\big)\kappa}}$$

when

$$\tau=\tfrac{1}{2}\big(\sqrt{\lambda^2+2}-|\lambda|\big),\qquad \theta=\sqrt{\frac{|\lambda|+2\tau}{n\tau-n|\lambda|\tau^2}}=\frac{2}{\sqrt{n}\big(\sqrt{\lambda^2+2}-|\lambda|\big)}=\frac{1}{\sqrt{n}\tau}.$$

Since

$$\Big(\int_M |A|^n\,\mathrm{d}\mu\Big)^{1/n}<K(n,\lambda),$$

we have from (9) that there exists $0<\varepsilon_0<1$ such that

$$\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\le\frac{n-1+\varepsilon}{n-1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\kappa}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+C(\varepsilon,\lambda)\int_M |A|^n|\nabla\eta|^2\rho,$$

namely,

$$(10)\qquad \frac{(n-1+\varepsilon)\varepsilon_0-2\varepsilon}{(n-1-\varepsilon)\kappa}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\le C(\varepsilon,\lambda)\int_M |A|^n|\nabla\eta|^2\rho.$$

Let $\eta(X) = \eta_r(X) = \phi(|X|/r)$ for any $r > 0$, where ϕ is a nonnegative function on $[0, +\infty)$ satisfying

$$\phi(x) = \begin{cases} 1 & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [2, +\infty), \end{cases}$$

and $|\phi'| \leq C$ for some absolute constant. Let $\varepsilon = \frac{1}{2}\varepsilon_0$. Since $\int_M |A|^n \,\mathrm{d}\mu$ is bounded, the right-hand side of (10) approaches zero as $r \to +\infty$, which implies $|A| \equiv 0$. Hence M is a hyperplane of $\mathbb{R}^{n+1}$. This completes the proof of Theorem 1. □

Setting $\mathring{A} = \sum_{i,j} \mathring{h}_{ij}\omega_i \otimes \omega_j$, we have $\mathring{h}_{ij} = h_{ij} - (H/n)g_{ij}$. Choose $\{e_i\}$ such that $h_{ij} = \lambda_i\delta_{ij}$ at a point p. Then $\mathring{h}_{ij} = \mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i = \lambda_i - H/n$, and

$$f_3 = \sum_i \lambda_i^3 = \sum_i \left(\mathring{\lambda}_i + \frac{H}{n}\right)^3 = B_3 + \frac{3}{n}H|\mathring{A}|^2 + \frac{1}{n^2}H^3,$$

where $|\mathring{A}|^2 = \sum_i \mathring{\lambda}_i^2 = |A|^2 - H^2/n$ and $B_3 = \sum_i \mathring{\lambda}_i^3$. Thus, from (1) and (2) we have

$$\begin{aligned}
\tfrac{1}{2}\mathcal{L}|\mathring{A}|^2 &= \tfrac{1}{2}\mathcal{L}|A|^2 - \tfrac{1}{2}\mathcal{L}\left(\frac{H^2}{n}\right) \\
&= |\nabla A|^2 - \frac{1}{n}|\nabla H|^2 + \left(\tfrac{1}{2} - |A|^2\right)|A|^2 + \lambda f_3 - \frac{H^2}{2n} - |A|^2(\lambda - H)\frac{H}{n} \\
&= |\nabla \mathring{A}|^2 + \left(\tfrac{1}{2} - |\mathring{A}|^2\right)|\mathring{A}|^2 - \frac{1}{n}H^2|\mathring{A}|^2 + \lambda B_3 + \frac{2}{n}\lambda H|\mathring{A}|^2.
\end{aligned}$$

By using an algebraic inequality in [Okumura 1974], we have

$$|B_3| \leq \frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3,$$

and the equality holds if and only if at least $n-1$ of the $\mathring{\lambda}_i$ are equal. Then we get

(11)
$$\begin{aligned}
\tfrac{1}{2}\mathcal{L}|\mathring{A}|^2 &\geq |\nabla \mathring{A}|^2 + \left(\tfrac{1}{2} - |\mathring{A}|^2\right)|\mathring{A}|^2 - \frac{1}{n}H^2|\mathring{A}|^2 - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3 + \frac{2}{n}\lambda H|\mathring{A}|^2 \\
&\geq \big|\nabla|\mathring{A}|\big|^2 + \left(\tfrac{1}{2} - |\mathring{A}|^2\right)|\mathring{A}|^2 - \frac{1}{n}\left(\lambda - \tfrac{1}{2}\langle X, N\rangle\right)^2|\mathring{A}|^2 \\
&\qquad - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3 + \frac{2}{n}\lambda\left(\lambda - \tfrac{1}{2}\langle X, N\rangle\right)|\mathring{A}|^2 \\
&= \big|\nabla|\mathring{A}|\big|^2 + \left(\frac{1}{2} + \frac{\lambda^2}{n}\right)|\mathring{A}|^2 - \frac{1}{4n}|\mathring{A}|^2|X^N|^2 - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3 - |\mathring{A}|^4.
\end{aligned}$$

By using (11), we give the proof of Theorem 2 as follows.

Proof of Theorem 2. Let η be a smooth function with compact support on M. Multiplying $|\mathring{A}|^{n-2}\eta^2$ on both sides of the inequality (11) above and integrating by parts with respect to the measure $\rho\,\mathrm{d}\mu$ on M yields

$$\begin{aligned}
0 &\geq 2\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left(1+\frac{2\lambda^2}{n}\right)\int_M |\mathring{A}|^n\eta^2\rho - \frac{1}{2n}\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho \\
&\quad - 2|\lambda|\frac{n-2}{\sqrt{n(n-1)}}\int_M |\mathring{A}|^{n+1}\eta^2\rho - 2\int_M |\mathring{A}|^{n+2}\eta^2\rho - \int_M |\mathring{A}|^{n-2}\eta^2\mathcal{L}|\mathring{A}|^2\rho \\
&= 2\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left(1+\frac{2\lambda^2}{n}\right)\int_M |\mathring{A}|^n\eta^2\rho - \frac{1}{2n}\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho \\
&\quad - 2|\lambda|\frac{n-2}{\sqrt{n(n-1)}}\int_M |\mathring{A}|^{n+1}\eta^2\rho - 2\int_M |\mathring{A}|^{n+2}\eta^2\rho \\
&\quad + 2\int_M \rho|\mathring{A}|\nabla|\mathring{A}|\cdot\nabla(|\mathring{A}|^{n-2}\eta^2) \\
&\geq 2(n-1)\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left[\left(1+\frac{2\lambda^2}{n}\right) - |\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}}\right]\int_M |\mathring{A}|^n\eta^2\rho \\
&\quad - \left(2+\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho - \frac{1}{2n}\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho \\
&\quad + 4\int_M (\nabla|\mathring{A}|\cdot\nabla\eta)|\mathring{A}|^{n-1}\eta\rho
\end{aligned}$$

with constant $\zeta > 0$.

From the assumption $|H| \leq \sqrt{\tfrac{1}{3}n+\lambda^2} - |\lambda| \triangleq C$, we have

$$\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho = 4\int_M |\mathring{A}|^n(\lambda-H)^2\eta^2\rho \leq 4(\lambda^2+C^2+2C|\lambda|)\int_M |\mathring{A}|^n\eta^2\rho.$$

This implies

$$\begin{aligned}
0 &\geq 2(n-1)\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho \\
&\quad + \left[\left(1+\frac{2\lambda^2}{n}\right) - |\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}} - \frac{2}{n}(\lambda^2+C^2+2C|\lambda|)\right]\int_M |\mathring{A}|^n\eta^2\rho \\
&\quad - \left(2+\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho + 4\int_M (\nabla|\mathring{A}|\cdot\nabla\eta)|\mathring{A}|^{n-1}\eta\rho.
\end{aligned}$$

By using the Cauchy inequality, for any $\varepsilon > 0$ we obtain

$$\begin{aligned}
&\left(\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}+2\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho \\
&\quad + \left[|\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}} + \frac{2}{n}(C^2+2C|\lambda|) - 1\right]\int_M |\mathring{A}|^n\eta^2\rho + \frac{2}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\rho \\
&\qquad \geq 2(n-1-\varepsilon)\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho.
\end{aligned} \tag{12}$$

Set $f = |\mathring{A}|^{n/2}\rho^{1/2}\eta$. Using the same argument as in the proof of Theorem 1, for any $\delta > 0$ we get

$$\begin{aligned}\text{(13)}\quad \kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\le (1+\delta)\frac{n^2}{4}\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left(1+\frac{1}{\delta}\right)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho \\ &\quad + \frac{n+2\lambda^2}{4}\int_M |\mathring{A}|^n\eta^2\rho - \frac{1}{4}\int_M |\mathring{A}|^n\eta^2\lambda\langle X, N\rangle\rho.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\text{(14)}\quad -\int_M |\mathring{A}|^n\eta^2\lambda\langle X, N\rangle\rho &= -2\int_M |\mathring{A}|^n\eta^2\lambda(\lambda - H)\rho \\ &= -2\int_M |\mathring{A}|^n\eta^2\lambda^2\rho + 2\int_M |\mathring{A}|^n\eta^2\lambda H\rho \\ &\le 2(C|\lambda| - \lambda^2)\int_M |\mathring{A}|^n\eta^2\rho.\end{aligned}$$

Combining (12), (13) and (14), we have

$$\begin{aligned}&\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\quad\le (1+\delta)\frac{n^2}{4}\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left(1+\frac{1}{\delta}\right)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho \\ &\qquad + \frac{n+2C|\lambda|}{4}\int_M |\mathring{A}|^n\eta^2\rho \\ &\quad\le \frac{(1+\delta)n^2}{8(n-1-\varepsilon)}\left\{\left(\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}} + 2\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho\right. \\ &\qquad + \left[|\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}} + \frac{2}{n}(C^2 + 2C|\lambda|) - 1\right]\int_M |\mathring{A}|^n\eta^2\rho \\ &\qquad \left. + \frac{2}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\rho\right\} + \left(1+\frac{1}{\delta}\right)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho + \frac{n+2C|\lambda|}{4}\int_M |\mathring{A}|^n\eta^2\rho.\end{aligned}$$

Let

$$|\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}} + \frac{2}{n}(C^2 + 2C|\lambda|) - 1 < 0,$$

i.e.,

$$0 < \zeta < \frac{\big[n - 2(C^2 + 2C|\lambda|)\big]\sqrt{n(n-1)}}{n(n-2)|\lambda|}.$$

Putting

$$\delta = \frac{2(n+2C|\lambda|)\sqrt{n-1}\cdot(n-1+\varepsilon)}{n\big[n\sqrt{n-1}-(n-2)\sqrt{n}\,|\lambda|\,\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)\big]}-1>0$$

for some $\varepsilon>0$ to be defined later, we have

$$\begin{aligned}(15)\quad & \kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ & \leq \left\{\frac{\sqrt{n}(n+2C|\lambda|)[(n-2)|\lambda|+2\zeta\sqrt{n(n-1)}]}{4\zeta[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)]}\right\} \\ & \qquad \times\frac{n-1+\varepsilon}{n-1-\varepsilon}\int_M |\mathring{A}|^{n+2}\eta^2\rho \\ & \quad +\left\{\frac{n(n+2C|\lambda|)\sqrt{n-1}\cdot(n-1+\varepsilon)}{2\varepsilon[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)]}+1+\frac{1}{\delta}\right\} \\ & \qquad \times\int_M |\mathring{A}|^n|\nabla\eta|^2\rho \\ & \leq \left\{\frac{\sqrt{n}(n+2C|\lambda|)[(n-2)|\lambda|+2\zeta\sqrt{n(n-1)}]}{4\zeta[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)]}\right\} \\ & \qquad \times\frac{n-1+\varepsilon}{n-1-\varepsilon}\cdot\left(\int_M |\mathring{A}|^{2\cdot\frac{n}{2}}\right)^{\frac{2}{n}}\left(\int_M (|\mathring{A}|^n\eta^2\rho)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \\ & \quad +\widetilde{C}(\varepsilon,\lambda,n,\zeta,C)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho.\end{aligned}$$

Set

$$D(n,\lambda,\zeta,C)=\sqrt{\frac{4\zeta\big[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)\big]}{\sqrt{n}(n+2C|\lambda|)\big[(n-2)|\lambda|+2\zeta\sqrt{n(n-1)}\big]\kappa}}.$$

We choose

$$\zeta=\frac{\sqrt{(n-2)^2\lambda^2+2(n-1)\big[n-2(C^2+2C|\lambda|)\big]}-(n-2)|\lambda|}{2\sqrt{n(n-1)}}$$

such that $D(n,\lambda,\zeta,C)$ achieves its maximum $D(n,\lambda)$ with

$$\begin{aligned}D(n,\lambda)&=\frac{\sqrt{(n-2)^2\lambda^2+2(n-1)\big[n-2(C^2+2C|\lambda|)\big]}-(n-2)|\lambda|}{\sqrt{n(n-1)(n+2C|\lambda|)\kappa}} \\ &=\frac{\sqrt{(n-2)^2\lambda^2+\frac{2}{3}n(n-1)}-(n-2)|\lambda|}{\sqrt{n(n-1)\Big(n+2|\lambda|\sqrt{\frac{1}{3}n+\lambda^2}-2\lambda^2\Big)\kappa}}.\end{aligned}$$

Combining the assumption

$$\left(\int_M |\mathring{A}|^n \, \mathrm{d}\mu\right)^{1/n} < D(n,\lambda)$$

and (15) implies that there exists $0 < \varepsilon_0 < 1$ such that

$$\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ \leq \frac{n-1+\varepsilon}{n-1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\kappa}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \widetilde{C}(\varepsilon,\lambda,n)\int_M |\mathring{A}|^n |\nabla\eta|^2\rho,$$

namely,

$$\frac{(n-1+\varepsilon)\varepsilon_0 - 2\varepsilon}{(n-1-\varepsilon)\kappa}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \widetilde{C}(\varepsilon,\lambda,n)\int_M |\mathring{A}|^n |\nabla\eta|^2\rho.$$

Let $\varepsilon = \frac{1}{2}\varepsilon_0$ and choose η as in the proof of Theorem 1. Since $\int_M |\mathring{A}|^n \, \mathrm{d}\mu$ is bounded, by using a similar argument we obtain $\mathring{A} \equiv 0$. Therefore, M is totally umbilical, i.e., M is $\mathbb{S}^n(\sqrt{\lambda^2+2n}-\lambda)$ or $\mathbb{R}^n$. Since we have assumed that

$$|H| \leq \sqrt{\tfrac{1}{3}n+\lambda^2} - |\lambda|,$$

the first case is excluded. This completes the proof of Theorem 2. □

Remark. In fact, we can prove that if $\sup_M |H| < \sqrt{\frac{1}{2}n+\lambda^2} - |\lambda|$ and if

$$\left(\int_M |\mathring{A}|^n \, \mathrm{d}\mu\right)^{1/n} < D(n,\lambda,\sup_M |H|),$$

then M is a hyperplane. Here $D(n,\lambda,\sup_M |H|)$ is a positive constant depending on n, λ and $\sup_M |H|$.

Remark. In particular, if $\lambda = 0$, Theorem 2 reduces to the rigidity result for self-shrinkers in [Lin 2016]. For the higher codimension case, Cao, Xu and Zhao [Cao et al. 2014] proved some L^n-pinching theorems of $\mathring{A}$ for self-shrinkers.

4. Gap theorems in dimension 2

We need another Sobolev-type inequality in dimension 2, which was proved by Xu and Gu [2007b]:

$$\tilde{c}^{-1}\left(\int f^4 \, \mathrm{d}\mu\right)^{1/2} \leq \frac{1}{t}\int |\nabla f|^2 \, \mathrm{d}\mu + t\int f^2 \, \mathrm{d}\mu + \frac{1}{2}\int |H| f^2 \, \mathrm{d}\mu \tag{16}$$

for all $f \in C_c^\infty(M)$ and for all $t \in \mathbb{R}^+$, where $\tilde{c} = 12\sqrt{3\pi}/\pi$.

Proof of Theorem 3. As in the proof of Theorem 1, for any $0 < \varepsilon < 1$, we have

$$\left(\frac{|\lambda|}{\tau}+2\right)\int_M |A|^4\eta^2\rho + (|\lambda|\tau - 1)\int_M |A|^2\eta^2\rho + \frac{2}{\varepsilon}\int_M |A|^2|\nabla\eta|^2\rho \geq 2(1-\varepsilon)\int_M \big|\nabla|A|\big|^2\eta^2\rho. \tag{17}$$

Setting $f = |A|\eta\rho^{1/2}$, we get

$$\int_M |\nabla f|^2 = \int_M \big|\nabla(|A|\eta)\big|^2\rho - \frac{1}{16}\int_M |A|^2\eta^2|X^T|^2\rho - \frac{1}{8}\int_M |A|^2\eta^2|X^N|^2\rho + \frac{1}{2}\int_M |A|^2\eta^2\rho + \frac{1}{4}\int_M |A|^2\eta^2\lambda\langle X, N\rangle\rho. \tag{18}$$

Combining the Sobolev-type inequality (16) and (18), we have

$$\begin{aligned}
\tilde{c}^{-1}&\left(\int_M |f|^4\right)^{1/2} \\
&\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho - \frac{1}{16}\int_M |A|^2\eta^2|X^T|^2\rho \\
&\qquad\qquad -\frac{1}{8}\int_M |A|^2\eta^2|X^N|^2\rho + \frac{1}{2}\int_M |A|^2\eta^2\rho + \frac{1}{4}\int_M |A|^2\eta^2\lambda\langle X, N\rangle\rho\bigg] \\
&\qquad + t\int_M |A|^2\eta^2\rho + \frac{1}{2}\int |H||A|^2\eta^2\rho \\
&\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho + \frac{1}{2}\int_M |A|^2\eta^2\rho + \frac{\lambda^2}{8}\int_M |A|^2\eta^2\rho\bigg] \\
&\qquad + t\int_M |A|^2\eta^2\rho + \frac{1}{2}\int |H||A|^2\eta^2\rho.
\end{aligned}$$

By the Cauchy inequality, for any $\theta > 0$, we get

$$\begin{aligned}
\tilde{c}^{-1}\left(\int_M |f|^4\right)^{1/2} &\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho + \frac{1}{2}\int_M |A|^2\eta^2\rho + \frac{\lambda^2}{8}\int_M |A|^2\eta^2\rho\bigg] \\
&\qquad + t\int_M |A|^2\eta^2\rho + \frac{1}{2}\int \left(\frac{\theta}{2}H^2 + \frac{1}{2\theta}\right)|A|^2\eta^2\rho \\
&\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho + \frac{1}{2}\int_M |A|^2\eta^2\rho + \frac{\lambda^2}{8}\int_M |A|^2\eta^2\rho\bigg] \\
&\qquad + t\int_M |A|^2\eta^2\rho + \frac{1}{2}\int \left(\theta|A|^2 + \frac{1}{2\theta}\right)|A|^2\eta^2\rho
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&= \frac{1}{t}\int_M \left(\big|\nabla |A|\big|^2 \eta^2 + 2|A|\eta \nabla |A| \cdot \nabla \eta + |A|^2 |\nabla \eta|^2 \right)\rho \\
&\qquad + \left(t + \frac{1}{4\theta} + \frac{1}{2t} + \frac{\lambda^2}{8t} \right) \int |A|^2 \eta^2 \rho + \frac{\theta}{2} \int |A|^4 \eta^2 \rho \\
&\le \frac{1}{t}\left[(1+\delta)\int_M \big|\nabla |A|\big|^2 \eta^2 \rho + \left(1+\frac{1}{\delta}\right)\int_M |A|^2 |\nabla \eta|^2 \rho \right] \\
&\qquad + \left(t + \frac{1}{4\theta} + \frac{1}{2t} + \frac{\lambda^2}{8t} \right) \int |A|^2 \eta^2 \rho + \frac{\theta}{2} \int |A|^4 \eta^2 \rho.
\end{aligned}
$$

Combining (17) and (19), we have

$$
\begin{aligned}
&\tilde{c}^{-1}\left(\int_M |A|^4 \eta^4 \rho^2 \right)^{1/2} \\
&\le \frac{1}{t}(1+\delta)\cdot \frac{1}{2(1-\varepsilon)} \left[\left(\frac{|\lambda|}{\tau} + 2 \right) \int_M |A|^4 \eta^2 \rho + (|\lambda|\tau - 1)\int_M |A|^2 \eta^2 \rho \right. \\
&\qquad\qquad \left. + \frac{2}{\varepsilon}\int_M |A|^2 |\nabla \eta|^2 \rho \right] \\
&\quad + \frac{1}{t}\left(1+\frac{1}{\delta}\right)\int_M |A|^2 |\nabla \eta|^2 \rho + \left(t + \frac{1}{4\theta} + \frac{1}{2t} + \frac{\lambda^2}{8t} \right) \int |A|^2 \eta^2 \rho + \frac{\theta}{2} \int |A|^4 \eta^2 \rho.
\end{aligned}
$$

Put

$$
\delta = \frac{(4\theta + 2t + 8\theta t^2 + \theta \lambda^2)(1+\varepsilon)}{4\theta(1-|\lambda|\tau)} - 1 > 0,
$$

where $\varepsilon, \theta, \tau, t$ are positive constants such that $|\lambda|\tau - 1 < 0$. Then

$$
\begin{aligned}
(20)\quad &\tilde{c}^{-1}\left(\int_M |A|^4 \eta^4 \rho^2 \right)^{1/2} \\
&\le \left[\frac{1}{t}\cdot \frac{(1+\varepsilon)}{2(1-\varepsilon)} \cdot \frac{(4\theta + 2t + 8\theta t^2 + \theta\lambda^2)}{4\theta(1-|\lambda|\tau)} \cdot \left(\frac{|\lambda|}{\tau} + 2 \right) + \frac{\theta}{2} \right] \int_M |A|^4 \eta^2 \rho \\
&\qquad + \frac{1}{t}\left[\frac{(1+\delta)}{\varepsilon(1-\varepsilon)} + \left(1+\frac{1}{\delta}\right) \right] \int_M |A|^2 |\nabla \eta|^2 \rho \\
&\le \frac{(4\theta + 2t + 8\theta t^2 + \theta\lambda^2)(|\lambda| + 2\tau) + 4\theta^2 t\tau(1-|\lambda|\tau)}{8\theta t\tau(1-|\lambda|\tau)} \cdot \frac{1+\varepsilon}{1-\varepsilon} \\
&\qquad\qquad \times \left(\int_M |A|^4 \eta^4 \rho^2 \right)^{1/2} \cdot \left(\int_M |A|^4 \right)^{1/2} \\
&\qquad + \frac{1}{t}\left[\frac{(1+\delta)}{\varepsilon(1-\varepsilon)} + \left(1+\frac{1}{\delta}\right) \right] \int_M |A|^2 |\nabla \eta|^2 \rho.
\end{aligned}
$$

Set

$$K(t,\lambda,\theta,\tau)=\frac{8\theta t\tau(1-|\lambda|\tau)}{\big[(4\theta+2t+8\theta t^2+\theta\lambda^2)(|\lambda|+2\tau)+4\theta^2 t\tau(1-|\lambda|\tau)\big]\tilde{c}},$$

where $\tilde{c}=12\sqrt{3\pi}/\pi$. By a direct computation, $K(t,\lambda,\theta,\tau)$ achieves its maximum

$$K(\lambda)=\frac{\sqrt{2}\big(\lambda^2+1-\sqrt{\lambda^2+2}|\lambda|\big)}{\big(2\sqrt{4+\lambda^2}+\sqrt{\lambda^2+2}-|\lambda|\big)\tilde{c}}$$

when

$$t=\sqrt{\tfrac{1}{8}(4+\lambda^2)},\quad \tau=\tfrac{1}{2}\big(\sqrt{\lambda^2+2}-|\lambda|\big),\quad \theta=\sqrt{\frac{|\lambda|+2\tau}{2\tau(1-|\lambda|\tau)}}=\frac{1}{\sqrt{2}\tau}.$$

Since

$$\bigg(\int_M |A|^4\bigg)^{1/2}<K(\lambda),$$

we have from (20) that there exists $0<\varepsilon_0<1$ such that

$$\tilde{c}^{-1}\bigg(\int_M |A|^4\eta^4\rho^2\bigg)^{1/2}\le\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\tilde{c}}\bigg(\int_M |A|^4\eta^4\rho^2\bigg)^{1/2}+C(\varepsilon,\lambda)\int_M |A|^2|\nabla\eta|^2\rho.$$

Let $\varepsilon=\frac{1}{2}\varepsilon_0$. Since $\int_M |A|^4\,\mathrm{d}\mu$ is bounded, we choose η as in the proof of Theorem 1 and a similar argument implies $|A|\equiv 0$. □

Using a similar argument, we give the proof of Theorem 4.

Proof of Theorem 4. For $n=2$, we have

$$\tfrac{1}{2}\mathcal{L}|\mathring{A}|^2\ge\big|\nabla|\mathring{A}|\big|^2+\frac{1+\lambda^2}{2}|\mathring{A}|^2-\tfrac{1}{8}|\mathring{A}|^2|X^N|^2-|\mathring{A}|^4,$$

and

$$(21)\quad 2\int_M|\mathring{A}|^4\eta^2\rho+(C^2+2C|\lambda|-1)\int_M|\mathring{A}|^2\eta^2\rho+\frac{2}{\varepsilon}\int_M|\mathring{A}|^2|\nabla\eta|^2\rho \ge 2(1-\varepsilon)\int_M\big|\nabla|\mathring{A}|\big|^2\eta^2\rho$$

with $0<\varepsilon<1$.

Set $f = |\mathring{A}|\rho^{1/2}\eta$. By (16) and the hypothesis $|H| \le \sqrt{\frac{2}{3}+\lambda^2} - |\lambda| \triangleq C$, we have

$$\tag{22} \begin{aligned} \tilde{c}^{-1}\left(\int_M |f|^4\right)^{1/2} &\le \frac{1}{t}\left[\int_M \left|\nabla(|\mathring{A}|\eta)\right|^2\rho + \frac{1}{2}\int_M |\mathring{A}|^2\eta^2\rho + \frac{\lambda^2}{8}\int_M |\mathring{A}|^2\eta^2\rho\right] \\ &\quad + t\int_M |\mathring{A}|^2\eta^2\rho + \frac{1}{2}\int |H|\,|\mathring{A}|^2\eta^2\rho \\ &\le \frac{1}{t}\int_M \left(\left|\nabla|\mathring{A}|\right|^2\eta^2 + 2|\mathring{A}|\eta\nabla|\mathring{A}|\cdot\nabla\eta + |\mathring{A}|^2|\nabla\eta|^2\right)\rho \\ &\quad + \left(t + \frac{C}{2} + \frac{1}{2t} + \frac{\lambda^2}{8t}\right)\int_M |\mathring{A}|^2\eta^2\rho. \end{aligned}$$

Combining the Cauchy inequality, (21) and (22), we have for any $\delta > 0$

$$\begin{aligned} \tilde{c}^{-1}\left(\int_M |f|^4\right)^{1/2} &\le \frac{1}{t}\left[(1+\delta)\int_M \left|\nabla|\mathring{A}|\right|^2\eta^2\rho + \left(1+\frac{1}{\delta}\right)\int_M |\mathring{A}|^2|\nabla\eta|^2\rho\right] \\ &\quad + \left(t + \frac{C}{2} + \frac{1}{2t} + \frac{\lambda^2}{8t}\right)\int_M |\mathring{A}|^2\eta^2\rho \\ &\le \frac{1+\delta}{t}\frac{1}{2(1-\varepsilon)}\left[2\int_M |\mathring{A}|^4\eta^2\rho + (C^2+2C|\lambda|-1)\int_M |\mathring{A}|^2\eta^2\rho \right. \\ &\qquad \left. + \frac{2}{\varepsilon}\int_M |\mathring{A}|^2|\nabla\eta|^2\rho\right] \\ &\quad + \frac{1}{t}\left(1+\frac{1}{\delta}\right)\int_M |\mathring{A}|^2|\nabla\eta|^2\rho + \left(t + \frac{C}{2} + \frac{1}{2t} + \frac{\lambda^2}{8t}\right)\int_M |\mathring{A}|^2\eta^2\rho. \end{aligned}$$

Put

$$\delta = \frac{(4+\lambda^2+8t^2+4tC)(1+\varepsilon)}{4[1-(C^2+2C|\lambda|)]} - 1 > 0.$$

Then we get

$$\tag{23} \begin{aligned} &\tilde{c}^{-1}\left(\int_M |f|^4\right)^{1/2} \\ &\le \frac{1}{t}\cdot\frac{4+\lambda^2+8t^2+4tC}{4[1-(C^2+2C|\lambda|)]}\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot\int_M |\mathring{A}|^4\eta^2\rho \\ &\quad + \frac{1}{t}\left[\frac{1+\delta}{\varepsilon(1-\varepsilon)} + 1 + \frac{1}{\delta}\right]\int_M |\mathring{A}|^2|\nabla\eta|^2\rho \\ &\le \frac{1}{t}\cdot\frac{4+\lambda^2+8t^2+4tC}{4[1-(C^2+2C|\lambda|)]}\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot\left(\int_M |f|^4\right)^{1/2}\cdot\left(\int_M |\mathring{A}|^4\right)^{1/2} \\ &\quad + \frac{1}{t}\left[\frac{1+\delta}{\varepsilon(1-\varepsilon)} + 1 + \frac{1}{\delta}\right]\int_M |\mathring{A}|^2|\nabla\eta|^2\rho. \end{aligned}$$

Set

$$D(\lambda, C, t) = \frac{4t[1-(C^2+2C|\lambda|)]}{(4+\lambda^2+8t^2+4tC)\tilde{c}}.$$

We choose $t = \sqrt{\frac{1}{8}(4+\lambda^2)}$ such that $D(\lambda, C, t)$ achieves its maximum

$$D(\lambda) = \frac{1}{3\left(\sqrt{8+2\lambda^2}+\sqrt{\frac{2}{3}+\lambda^2}-|\lambda|\right)\tilde{c}}.$$

Since

$$\left(\int_M |\mathring{A}|^4\right)^{1/2} < D(\lambda),$$

we have from (23) that there exists $0 < \varepsilon_0 < 1$ such that

$$\tilde{c}^{-1}\left(\int_M |f|^4\right)^{1/2} \le \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\tilde{c}}\cdot\left(\int_M |f|^4\right)^{1/2} + C(\varepsilon,\lambda)\int_M |\mathring{A}|^2|\nabla\eta|^2\rho.$$

Let $\varepsilon = \frac{1}{2}\varepsilon_0$. Since $\int_M |\mathring{A}|^4\,d\mu$ is bounded, we choose η as above and a similar argument implies $\mathring{A} \equiv 0$. Therefore, M is totally umbilical, i.e., M is $\mathbb{S}^2(\sqrt{\lambda^2+4}-\lambda)$ or $\mathbb{R}^2$. Since we have assumed that

$$|H| \le \sqrt{\frac{2}{3}+\lambda^2}-|\lambda|,$$

the first case is excluded. This completes the proof of Theorem 4. □

Remark. Similarly, it is seen from the proof of Theorem 4 that we can prove that if $\sup_M |H| < \sqrt{1+\lambda^2}-|\lambda|$ and if $\left(\int_M |\mathring{A}|^4\,d\mu\right)^{1/2} < D(\lambda, \sup_M |H|)$, then M is a hyperplane. Here $D(\lambda, \sup_M |H|)$ is a positive constant depending on λ and $\sup_M |H|$.

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