Pacific Journal of Mathematics

BACH-FLAT h-ALMOST GRADIENT RICCI SOLITONS

GABJIN YUN, JINSEOK CO AND SEUNGSU HWANG

Volume 288 No. 2 June 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 288, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.288.475

BACH-FLAT h-ALMOST GRADIENT RICCI SOLITONS

GABJIN YUN, JINSEOK CO AND SEUNGSU HWANG

On an n-dimensional complete manifold M, consider an h-almost gradient Ricci soliton, which is a generalization of a gradient Ricci soliton. We prove that if the manifold is Bach-flat and $dh/du > 0$, then the manifold M is either Einstein or rigid. In particular, such a manifold has harmonic Weyl curvature. Moreover, if the dimension of M is four, the metric g is locally conformally flat.

1. Introduction

The notion of an h-almost Ricci soliton was introduced by Gomes, Wang, and Xia [Gomes et al. 2015]. Such a soliton is a generalization of an almost Ricci soliton presented in [Barros and Ribeiro 2012; Pigola et al. 2011]. An h-almost Ricci soliton is a complete Riemannian manifold (M^n, g) with a vector field X on M, a soliton function $\lambda : M \to \mathbb{R}$ and a signal function $h : M \to \mathbb{R}^+$ satisfying the equation

$$r_g + \tfrac{1}{2} h \mathcal{L}_X g = \lambda g,$$

where r_g is the Ricci curvature of g. A function is called signal if it has only one sign; in other words, it is either positive or negative on M. Let (M, g, X, h, λ) denote an h-almost Ricci soliton. In particular, $(M, g, \nabla u, h, \lambda)$ for some smooth function $u : M \to \mathbb{R}$ is called an h-almost gradient Ricci soliton with potential function u. In this case, we have

$$r_g + h\, D_g\, du = \lambda g. \tag{1-1}$$

Here, $D_g\, du$ denotes the Hessian of u. Note that if we take $u = e^{-f/m}$ and $h = -m/u$, then (1-1) becomes

$$\mathrm{Ric}_f^m = r_g + D_g\, df - \frac{1}{m}\, df \otimes df = \lambda g.$$

Yun was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0007465). Hwang (corresponding author) was supported by the Ministry of Education (NRF-2015R1D1A1A01057661).
MSC2010: 53C25, 58E11.
Keywords: h-almost gradient Ricci soliton, Bach-flat, Einstein metric.

In other words, the $(\lambda, n+m)$-Einstein equation is a special case of (1-1). Here, Ric_f^m is called the m-Bakry–Emery tensor. For further details about h-almost Ricci solitons, see [Gomes et al. 2015].

In this paper we consider Bach-flat h-almost gradient Ricci solitons. The Bach tensor was introduced by R. Bach, and this notion plays an important role in conformal relativity. On any n-dimensional Riemannian manifold (M, g), $n \geq 4$, the Bach tensor is defined by

$$B = \frac{1}{n-3}\,\delta^D \delta\mathcal{W} + \frac{1}{n-2}\,\mathring{\mathcal{W}}z,$$

where $\mathcal{W}$ is the Weyl tensor, z is the traceless Ricci tensor, and $\mathring{\mathcal{W}}z$ is defined by

$$\mathring{\mathcal{W}}z(X, Y) = \sum_{i=1}^{n} z(\mathcal{W}(X, E_i)Y, E_i)$$

for some orthonormal basis $\{E_i\}_{i=1}^n$. It is easy to see that if (M, g) is either locally conformally flat or Einstein, then it is Bach-flat: $B = 0$. When $n = 4$, it is well known that Bach-flat metrics on a compact manifold M are critical points of the functional

$$g \mapsto \int_M |\mathcal{W}|^2\, dv_g.$$

It is clear that when $h = 1$ and λ is a positive constant, an h-almost gradient Ricci soliton reduces to a gradient shrinking Ricci soliton. Cao and Chen [2013] proved that a complete Bach-flat gradient shrinking Ricci soliton is either Einstein or rigid. On the other hand, Qing and Yuan [2013] classified Bach-flat static spaces.

Our main result is as follows, which can be considered as a generalization of [Cao and Chen 2013].

Theorem 1.1. *Let $(M, g, \nabla u, h, \lambda)$ be an n-dimensional Bach-flat h-almost gradient Ricci soliton with potential function u. Assume that each level set of u is compact and h is a function of u only. Then $(M, g, \nabla u, h, \lambda)$ is either*

(1) *Einstein with constant functions u and h, or*

(2) *locally isometric to a warped product with $(n-1)$-dimensional Einstein fibers if $dh/du > 0$ on M.*

For example, when $m > 0$, $h = -m/u < 0$ satisfies the condition of Theorem 1.1, since

$$\frac{dh}{du} = \frac{m}{u^2} > 0.$$

This recovers the result of [Chen and He 2013]. It will be interesting if one can weaken the condition of Theorem 1.1.

In the case of (2) in Theorem 1.1, a warped product metric has vanishing Cotton tensor (see (2-4) below) since its fiber is Einstein. Thus, as a consequence of Theorem 1.1, we have the following.

Corollary 1.2. *Let $(M, g, \nabla u, h, \lambda)$ be an n-dimensional Bach-flat h-almost gradient Ricci soliton with potential function u. Assume that each level set of u is compact and h is a function of u only. If $dh/du > 0$ on M, then (M, g) has harmonic Weyl curvature.*

In particular, when $n = 4$, the Einstein fibers in Theorem 1.1 have constant curvature. A computation shows that such a metric is locally conformally flat, which proves the following theorem.

Theorem 1.3. *Let $(M, g, \nabla u, h, \lambda)$ be a 4-dimensional Bach-flat h-almost gradient Ricci soliton with potential function u. Assume each level set of u is compact and h is a function of u only with $dh/du > 0$. Then (M, g) is locally conformally flat.*

As in [Chen and He 2013], Theorem 1.1, Corollary 1.2, and Theorem 1.3 can be extended to the case in which M has a nonempty boundary.

2. Preliminaries

In this section, we derive several useful identities containing various curvatures and the Cotton tensor.

We start with basic definitions of differential operators acting on tensors. Let us denote by $C^\infty(S^2M)$ the space of sections of symmetric 2-tensors on a Riemannian manifold M. Let D be the Levi-Civita connection of (M, g). Then the differential operator d^D from $C^\infty(S^2M)$ into $C^\infty(\Lambda^2M \otimes T^*M)$ is defined as

$$d^D\omega(X, Y, Z) = (D_X\omega)(Y, Z) - (D_Y\omega)(X, Z)$$

for $\omega \in C^\infty(S^2M)$ and vectors X, Y, and Z. Let us denote by δ^D the formal adjoint operator of d^D.

For a function $f \in C^\infty(M)$ and $\omega \in C^\infty(S^2M)$, $df \wedge \omega$ is defined as

$$(df \wedge \omega)(X, Y, Z) = df(X)\omega(Y, Z) - df(Y)\omega(X, Z).$$

Here, df denotes the usual total differential of f. We also denote by δ the negative divergence operator such that $\Delta f = -\delta\, df$.

Taking the trace of (1-1) gives

$$s_g + h\,\Delta u = n\lambda.$$

Thus,

$$ds_g + \Delta u\, dh + h\, d\Delta u = n\, d\lambda.$$

By taking the divergence of (1-1), we have

$$-\tfrac{1}{2}\,ds_g - D_g\,du(\nabla h,\cdot) - h r_g(\nabla u,\cdot) - h\,d\Delta u = -d\lambda.$$

By adding the previous two equations, we have

$$\tfrac{1}{2}\,ds_g - D_g\,du(\nabla h,\cdot) - h r_g(\nabla u,\cdot) + \Delta u\,dh = (n-1)\,d\lambda. \tag{2-1}$$

Note that

$$\delta(h r_g(\nabla u,\cdot)) = -r_g(\nabla u,\nabla h) - \tfrac{1}{2}h\langle\nabla s_g,\nabla u\rangle + |r_g|^2 - \lambda s_g. \tag{2-2}$$

Therefore, we have the following equality.

Proposition 2.1. *On M we have*

$$(n-1)\Delta\lambda = \tfrac{1}{2}\,\Delta s_g + |r_g|^2 - \lambda s_g - \tfrac{1}{2}h\langle\nabla s_g,\nabla u\rangle + \left(\Delta u - \frac{\lambda}{h}\right)\Delta h + \frac{1}{h}\langle r_g, D_g\,dh\rangle - 2r_g(\nabla u,\nabla h).$$

On the other hand, by applying d^D to (1-1), we have

$$d^D r_g - \frac{1}{h}\,dh\wedge r_g + h\tilde{i}_{\nabla u}R = d\lambda\wedge g - \frac{\lambda}{h}\,dh\wedge g. \tag{2-3}$$

Here, an interior product $\tilde{i}$ of the final factor is defined by

$$\tilde{i}_\xi R(X,Y,Z) = R(X,Y,Z,\xi),$$

and we used the identity

$$d^D D\,du = \tilde{i}_{\nabla u}R.$$

Hereafter, we denote s_g, r_g, and $D_g\,du$ by s, r, and $D\,du$, respectively. From the curvature decomposition, we can compute that

$$\tilde{i}_{\nabla u}R = \tilde{i}_{\nabla u}\mathcal{W} - \frac{1}{n-2}i_{\nabla u}r\wedge g + \frac{s}{(n-1)(n-2)}\,du\wedge g - \frac{1}{n-2}\,du\wedge r,$$

where $i_{\nabla u}r$ denotes the interior product defined by

$$i_{\nabla u}r(X) = r(\nabla u, X).$$

The Cotton tensor C is defined by

$$C = d^D r - \frac{1}{2(n-1)}\,ds\wedge g. \tag{2-4}$$

Then, by (2-1) and (2-3) as well as the fact that

$$s + h\,\Delta u = n\lambda,$$

we have

$$C + h\tilde{\imath}_{\nabla u}\mathcal{W} = h\,D + \frac{h}{n-1} i_{\nabla u} r \wedge g + d\lambda \wedge g - \frac{1}{2(n-1)}\,ds \wedge g + \frac{1}{h}\,dh \wedge r - \frac{\lambda}{h}\,dh \wedge g = h\,D + H, \tag{2-5}$$

where D is defined (as usual) by

$$(n-2)\,D = du \wedge r + \frac{1}{n-1} i_{\nabla u} r \wedge g - \frac{s}{n-1}\,du \wedge g, \tag{2-6}$$

and H is defined by

$$\begin{aligned} H &= -\frac{1}{n-1} i_{\nabla h} D\,du \wedge g + dh \wedge \left(\frac{1}{h} r + \frac{\Delta u}{n-1} g - \frac{\lambda}{h} g\right) \\ &= db \wedge r + \frac{1}{n-1} i_{\nabla b} r \wedge g - \frac{s}{n-1}\,db \wedge g. \end{aligned}$$

Here, $b = \log|h|$ with $\nabla b = \nabla h/h$. In particular, $g^{ik} H_{ijk} = -g^{ik} H_{jik} = 0$.

Proposition 2.2. *Let $(M, g, \nabla u, h, \lambda)$ be an h-almost gradient Ricci soliton with potential function u. Then*

$$C + h\tilde{\imath}_{\nabla u}\mathcal{W} = h\,D + H.$$

In particular, if h is constant or $dh/du = 0$, then $H \equiv 0$.

3. Bach-flat metrics

In this section, we assume that g is Bach-flat. Note that

$$\delta\mathcal{W} = -\frac{n-3}{n-2} C.$$

Recall that the Bach tensor is given by

$$B = \frac{1}{n-3}\,\delta^D \delta\mathcal{W} + \frac{1}{n-2}\mathring{\mathcal{W}}z = \frac{1}{n-2}(-\delta C + \mathring{\mathcal{W}}z).$$

Since

$$\begin{aligned} &\delta(h\tilde{\imath}_{\nabla u}\mathcal{W})(X, Y) \\ &\quad = -\mathcal{W}(\nabla h, X, Y, \nabla u) + h\,\delta\mathcal{W}(X, Y, \nabla u) + h\,\mathcal{W}(X, E_i, Y, D_{E_i}\,du) \\ &\quad = l\mathcal{W}(X, \nabla h, Y, \nabla u) - \frac{n-3}{n-2} h\,C(Y, \nabla u, X) - \mathring{\mathcal{W}}z, \end{aligned}$$

by taking the divergence of (2-5) we have

$$
\begin{aligned}
-(n-2)B(X,Y) = -\mathcal{W}(X,\nabla h, Y, \nabla u) + \frac{n-3}{n-2} h C(Y,\nabla u, X) - i_{\nabla h} D(X,Y) \\
+ h\delta D(X,Y) + \delta H(X,Y).
\end{aligned}
$$

Hence,

$$
-(n-2)B(\nabla u, \nabla u) = -D(\nabla h, \nabla u, \nabla u) + h\delta D(\nabla u, \nabla u) + \delta H(\nabla u, \nabla u).
$$

As a result, from the assumption that $B = 0$ and h is a function of u only,

$$
0 = \frac{1}{h} D(\nabla h, \nabla u, \nabla u) = \delta D(\nabla u, \nabla u) + \frac{1}{h} \delta H(\nabla u, \nabla u).
$$

Let $\{E_i\}_{i=1}^n$ be a normal geodesic frame. Note that, since

$$
hD(E_i, D_{E_i} du, \nabla u) = -D(E_i, E_k, \nabla u) r_{ik} = 0,
$$

we have

$$
\operatorname{div}(D(\,\cdot\,, \nabla u, \nabla u)) = -\delta D(\nabla u, \nabla u) + D(E_i, \nabla u, D_{E_i} du).
$$

Furthermore,

$$
\begin{aligned}
|D|^2 &= \frac{1}{n-2}\big(du(E_i) r(E_j, E_k) - du(E_j) r(E_i, E_k)\big) D_{ijk} \\
&= -\frac{2}{n-2} D(E_i, \nabla u, E_k) r_{ik} \\
&= \frac{2h}{n-2} D(E_i, \nabla u, D_{E_i} du).
\end{aligned}
$$

Similarly, since

$$
hH(E_i, D_{E_i} du, \nabla u) = -H(E_i, E_k, \nabla u) r_{ik} = 0
$$

and h is a function of u only, we have

$$
\operatorname{div}\left(\frac{1}{h} H(\,\cdot\,, \nabla u, \nabla u)\right) = -\frac{1}{h} \delta H(\nabla u, \nabla u) + \frac{1}{h} H(E_i, \nabla u, D_{E_i} du).
$$

Moreover,

$$
\begin{aligned}
|H|^2 &= -\frac{2}{h} H(E_i, \nabla h, E_k) r_{ik} \\
&= -\frac{2}{h}\frac{dh}{du} H(E_i, \nabla u, E_k) r_{ik} \\
&= 2\frac{dh}{du} H(E_i, \nabla u, D_{E_i} du).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 &= \int_{t_1 \le u \le t_2} \delta D(\nabla u, \nabla u) + \frac{1}{h}\delta H(\nabla u, \nabla u) \\
&= \frac{n-2}{2}\int_{t_1 \le u \le t_2} \frac{|D|^2}{h} + \frac{1}{2}\int_{t_1 \le u \le t_2} \frac{|H|^2}{h\,\frac{dh}{du}}.
\end{aligned}
$$

Since h is signal, h is either positive or negative. For each case, we derive $D = H = 0$ when $dh/du > 0$. Therefore we have the following result.

Lemma 3.1. *Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat h-almost gradient Ricci soliton with potential function u. Assume that each level set of u is compact and h is a function of u only. If $dh/du > 0$ on M, then on M we have*

$$D = H = 0.$$

Now, since $D = H = 0$, by (2-4) and (2-5)

$$C = -h\tilde{\imath}_{\nabla u}\mathcal{W}. \tag{3-1}$$

By taking the divergence of (3-1), we have

$$\mathcal{W}(X, \nabla h, Y, \nabla u) = \frac{n-3}{n-2} hC(Y, \nabla u, X).$$

By combining these equations,

$$\frac{n-3}{n-2} h^2 C(Y, \nabla u, X) = -C(X, \nabla h, Y),$$

and

$$\mathcal{W}(X, \nabla h, Y, \nabla u) = -\frac{n-3}{n-2} h^2 \mathcal{W}(X, \nabla u, Y, \nabla u).$$

Therefore, we have the following.

Corollary 3.2. *When $D = H = 0$, we have*

$$\mathcal{W}(\,\cdot\,, \nabla u, \,\cdot\,, \nabla u) = C(\,\cdot\,, \nabla u, \,\cdot\,) = 0, \tag{3-2}$$

unless

$$\frac{dh}{du} = -\left(\frac{n-3}{n-2}\right) h^2.$$

For example, when $h = -m/u$, (3-2) holds if $m \neq 0$ or $-(n-2)/(n-3)$. Note that (3-2) also holds if h is constant.

Moreover, we have the following result.

Lemma 3.3. *Suppose that $dh/du > 0$. Then, for X orthogonal to ∇u,*

$$r(X, \nabla u) = 0. \tag{3-3}$$

In particular,

$$i_{\nabla u} r = \alpha\, du,$$

where $\alpha = r(N, N)$ *with* $N = \nabla u/|\nabla u|$.

Proof. By Lemma 3.1, $D = H = 0$. From (2-3), if X is orthogonal to ∇u,

$$d^D r(X, Y, \nabla u) = -\frac{1}{h}\, dh(Y)\, r(X, \nabla u) + d\lambda(X)\, du(Y).$$

Since $C(X, Y, \nabla u) = -h\mathcal{W}(X, Y, \nabla u, \nabla u) = 0$ by (3-1), by (2-4) we have

$$d^D r(X, Y, \nabla u) = \frac{1}{2(n-1)}\, ds(X)\, du(Y).$$

Thus, by (2-1)

$$\begin{aligned} \frac{1}{h}\frac{dh}{du} r(X, \nabla u) &= d\lambda(X) - \frac{1}{2(n-1)}\, ds(X) \\ &= \frac{1}{(n-1)h}\left(\frac{dh}{du} - h^2\right) r(X, \nabla u), \end{aligned}$$

which implies that

$$\left((n-2)\frac{dh}{du} + h^2\right) r(X, \nabla u) = 0. \qquad \square$$

Note that Lemma 3.3 holds with the assumptions that $D = H = 0$ and

$$\frac{dh}{du} \neq -\frac{1}{n-2} h^2 \tag{3-4}$$

without $dh/du > 0$. For example, in the case of the m-Bakry–Emery tensor, $h = -m/u$ satisfies (3-4) if $m \neq 2 - n$.

4. Level sets of *u*

In this section, we investigate the structure of regular level sets of the potential function u. For a regular value c, we denote the level set $u^{-1}(u)$ by L_c. On L_c, let $\{E_i\}$, $1 \le i \le n$, be an orthonormal frame with $E_n = N = \nabla u/|\nabla u|$.

Furthermore, throughout the section we assume that $D = H = 0$ with

$$\frac{dh}{du} \neq -\left(\frac{n-3}{n-2}\right) h^2 \quad \text{and} \quad \frac{dh}{du} \neq -\frac{1}{n-2} h^2.$$

Then, by Corollary 3.2, (3-2) and (3-3) hold. Furthermore, for X orthogonal to ∇u, by the proof of Lemma 3.3,

$$d\lambda(X) = \frac{1}{2(n-1)}\, ds(X).$$

Thus, $s+2(1-n)\lambda$ is constant on each level set of u. Furthermore,

$$\tfrac{1}{2}X(|\nabla u|^2)=\langle D_X\,du,\nabla u\rangle=\frac{1}{h}\big(\lambda\,du(X)-r(X,\nabla u)\big)=0,$$

which implies that $|\nabla u|^2$ is constant on each level set of u. Therefore, we have the following.

Lemma 4.1. *$|\nabla u|^2$ and $s+2(1-n)\lambda$ are constant on each regular level set of u.*

For further investigation, we need the following key lemma.

Lemma 4.2.

$$\begin{aligned}0=\frac{ns-(n-1)^2\lambda-\alpha}{(n-1)h}r-D_{\nabla u}r-\frac{r\circ r}{h}+\frac{n-3}{2(n-1)}\,du\otimes ds\\ +\frac{1}{n-1}\big(ds(u)-\langle\nabla u,\nabla\alpha\rangle\big)g+\frac{s+(1-n)\lambda}{(n-1)h}(\alpha-s)g+\frac{1}{n-1}\,du\otimes d\alpha.\end{aligned}$$

Proof. To find δD, by (2-6), we first compute

$$\delta(du\wedge r)=\frac{s-(n-1)\lambda}{h}r-D_{\nabla u}r-\frac{r\circ r}{h}+\tfrac{1}{2}\,du\otimes ds.$$

By Lemma 3.3, $i_{\nabla u}r=\alpha\,du$. Thus,

$$\delta(i_{\nabla u}r\wedge g)=-\langle\nabla u,\nabla\alpha\rangle g+\frac{s+(1-n)\lambda}{h}\alpha g+du\otimes d\alpha-\frac{\alpha}{h}r.$$

Similarly,

$$-\delta(s\,du\wedge g)=ds(u)g-\frac{s^2+(1-n)s\lambda}{h}g-du\otimes ds+\frac{s}{h}r.$$

Hence, by (2-6) together with (3-3), we have

$$\begin{aligned}(n-2)\,\delta D=&\frac{ns-(n-1)^2\lambda-\alpha}{(n-1)h}r-D_{\nabla u}r-\frac{r\circ r}{h}\\ &+\frac{n-3}{2(n-1)}\,du\otimes ds+\frac{1}{n-1}\,du\otimes d\alpha\\ &+\frac{1}{n-1}\left(ds(u)-\langle\nabla u,\nabla\alpha\rangle+\frac{s+(1-n)\lambda}{h}(\alpha-s)\right)g.\end{aligned}$$

Since $D=\delta D=0$, the proof follows. □

Thus, we have the following.

Corollary 4.3. *$(n-3)s+2\alpha$ is constant on each regular level set of u.*

Proof. Let X be a vector orthogonal to ∇u. By putting $(X,\nabla u)$ in the equation in Lemma 4.2,

$$D_{\nabla u}r(X,\nabla u)=0. \tag{4-1}$$

Now, by putting $(\nabla u, X)$ in the equation in Lemma 4.2 again, we have

$$0 = \frac{n-3}{2(n-1)} |\nabla u|^2 \, ds(X) + \frac{1}{n-1} |\nabla u|^2 \, d\alpha(X),$$

since $r(X, \nabla u) = 0$ and

$$D_{\nabla u} r(\nabla u, X) = D_{\nabla u} r(X, \nabla u). \qquad \square$$

Lemma 4.4. *$s_g + 2(1-n)\alpha$ is constant on each regular level set of u.*

Proof. For X orthogonal to ∇u, by (3-2) and (4-1)

$$\begin{aligned} 0 &= C(X, \nabla u, \nabla u) \\ &= D_X r(\nabla u, \nabla u) - \frac{1}{2(n-1)} \, ds(X) \, |\nabla u|^2. \end{aligned}$$

Thus,

$$\begin{aligned} X(\alpha) &= \frac{1}{|\nabla u|^2} X(r(\nabla u, \nabla u)) \\ &= \frac{1}{|\nabla u|^2} \big(D_X r(\nabla u, \nabla u) + 2r(D_X \, du, \nabla u) \big) \\ &= \frac{1}{2(n-1)} \, ds(X), \end{aligned}$$

since

$$r(D_X \, du, \nabla u) = \frac{1}{h} (\lambda r(X, \nabla u) - r \circ r(X, \nabla u)) = 0. \qquad \square$$

By combining Lemma 4.1, Corollary 4.3, and Lemma 4.4, we have the following.

Theorem 4.5. *Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat h-almost gradient Ricci soliton with potential function u. Assume that each level set of u is compact and h is a function of u only with $dh/du > 0$. Then s_g, α, and λ are constant on each regular level set of u. In particular, if h is constant, the condition on dh/du is not necessary.*

When $D = 0$, the Ricci tensor has the following characterization.

Lemma 4.6. *Suppose that $D = 0$. Then the Ricci curvature tensor has at most two eigenvalues.*

Proof. Let $\{E_i\}$, $1 \le i \le n$, be an orthonormal frame with $E_n = N = \nabla u / |\nabla u|$. Then

$$II_{ij} = \frac{1}{h |\nabla u|} (\lambda g_{ij} - r_{ij}), \tag{4-2}$$

and

$$m = \operatorname{tr} II = \frac{n-1}{h |\nabla u|} \left(\lambda + \frac{\alpha - s}{n-1} \right).$$

Thus, m is constant on each level set of u, and

$$\begin{aligned}\left|II - \frac{m}{n-1}g\right|^2 &= |II|^2 - \frac{m^2}{n-1}\\ &= \frac{1}{h^2|\nabla u|^2}\left(|r|^2 - \alpha^2 - \frac{(s-\alpha)^2}{n-1}\right)\\ &= \frac{1}{h^2|\nabla u|^2}\left(|r|^2 - \frac{n}{n-1}\alpha^2 + \frac{2s\alpha}{n-1} - \frac{s^2}{n-1}\right).\end{aligned}$$

Since $r \circ r(\nabla u, \nabla u) = \alpha^2|\nabla u|^2$, from the identity

$$\frac{n-2}{2}|D|^2 = |r|^2|\nabla u|^2 - \frac{n}{n-1} r \circ r(\nabla u, \nabla u) + \frac{2s}{n-1} r(\nabla u, \nabla u) - \frac{s^2}{n-1}|\nabla u|^2,$$

we have

$$|D|^2 = \frac{2}{n-2} h^2 |\nabla u|^4 \left|II - \frac{m}{n-1}g\right|^2.$$

Since $D = 0$, we have

$$II_{ij} = \frac{m}{n-1} g_{ij}, \tag{4-3}$$

which implies that

$$r_{ij} = \frac{s-\alpha}{n-1} g_{ij} \tag{4-4}$$

for $i = 1, \dots, n-1$ by (4-2). $\square$

As an immediate consequence, on an open set $\{x \in M \mid \nabla u(x) \neq 0\}$, the Ricci tensor may be written as

$$r_g = \beta\, du \otimes du + \left(\frac{s-\alpha}{n-1}\right) g,$$

where

$$\beta = \frac{n\alpha - s}{(n-1)|\nabla u|^2}.$$

Thus, by (1-1) we have

$$D_g\, du = \frac{1}{h}\left(\lambda + \frac{\alpha - s}{n-1}\right) g - \frac{\beta}{h}\, du \otimes du.$$

Now, we are ready to prove Corollary 1.2, which shows the relationship between Bach-flat metrics and harmonic Weyl metrics.

Proof of Corollary 1.2. Note that, by (3-1) and (3-2)

$$C(\,\cdot\,, \cdot\,, \nabla u) = C(\,\cdot\,, \nabla u, \cdot\,) = 0.$$

On the other hand, by the Codazzi equation,

$$\langle R(X, Y)Z, N\rangle = D_Y II(X, Z) - D_X II(Y, Z).$$

Thus, for $1 \le i, j, k \le n-1$, by (4-3)

$$\begin{aligned}\langle R(E_i, E_j)E_k, N\rangle &= E_j(II(E_i, E_k)) - II(D_{E_j}E_i, E_k) - II(E_i, D_{E_j}E_k)\\ &\quad - E_i(II(E_j, E_k)) + II(D_{E_i}E_j, E_k) + II(E_j, D_{E_i}E_k)\\ &= 0.\end{aligned}$$

Therefore, by (2-3)

$$d^D r(E_i, E_j, E_k) = 0,$$

which implies that

$$C(E_i, E_j, E_k) = d^D r(E_i, E_j, E_k) - \frac{1}{2(n-1)}\, ds \wedge g(E_i, E_j, E_k) = 0.$$

Hence, C is identically zero, and so is $\delta\mathcal{W}$. □

The following is a restatement of Theorem 1.1.

Theorem 4.7. *Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat h-almost gradient Ricci soliton with potential function u. Assume that each level set of u is compact with $dh/du > 0$ on M. Then, either g is Einstein with constant function u or the metric can be written as*

$$g = dt^2 + \psi^2(t)\,\hat{g}_E,$$

where $\hat{g}_E$ is the Einstein metric on the level set $E = L_{c_0}$ for some c_0.

Proof. Assume that u is not constant. By Lemma 3.1, $D = H = 0$. Since $|\nabla u|^2$ depends only on u by Lemma 4.6, as shown in the proof of Theorem 7.9 of [He et al. 2012] with Remark 3.2 of [Cao and Chen 2013], the metric can be locally written as

$$g = dt^2 + \hat{g}_c.$$

Here, $\hat{g}_c$ denotes the induced metric on the level set $L_c = u^{-1}(c)$ for each regular value c. Furthermore, $(L_c, \hat{g}_c)$ is necessarily Einstein; by the Gauss equation

$$\hat{R}_{ijij} = R_{ijij} + II_{ii}II_{jj} - II_{ij}^2 = R_{ijij} + \frac{m^2}{(n-1)^2}.$$

Thus,

$$\hat{r}_{ii} = r_{ii} - R(N, E_i, N, E_i) + \frac{m^2}{n-1}.$$

By (3-2) and (4-4), we have

$$R(E_i, N, E_i, N) = \frac{1}{n-2}(r_{ii} + \alpha) - \frac{s}{(n-1)(n-2)} = \frac{\alpha}{n-1}.$$

Hence, it follows that

$$\hat{r}_{ii} = r_{ii} + \frac{m^2 - \alpha}{n-1} = \frac{1}{n-1}(s - 2\alpha + m^2) = \hat{\lambda}_0.$$

Since s, α, and m are constant along L_c, this proves that $(L_c, \hat{g}_c)$ has constant Ricci curvature. As a result, by a suitable change of variable, the metric g can be written as in the statement of Theorem 4.7. □

Acknowledgment

The authors express their gratitude to the referee for several valuable comments.

References

[Barros and Ribeiro 2012] A. Barros and E. Ribeiro, Jr., "Some characterizations for compact almost Ricci solitons", *Proc. Amer. Math. Soc.* **140**:3 (2012), 1033–1040. MR Zbl

[Cao and Chen 2013] H.-D. Cao and Q. Chen, "On Bach-flat gradient shrinking Ricci solitons", *Duke Math. J.* **162**:6 (2013), 1149–1169. MR Zbl

[Chen and He 2013] Q. Chen and C. He, "On Bach flat warped product Einstein manifolds", *Pacific J. Math.* **265**:2 (2013), 313–326. MR Zbl

[Gomes et al. 2015] J. N. Gomes, Q. Wang, and C. Xia, "On the h-almost Ricci soliton", preprint, 2015. arXiv

[He et al. 2012] C. He, P. Petersen, and W. Wylie, "On the classification of warped product Einstein metrics", *Comm. Anal. Geom.* **20**:2 (2012), 271–311. MR Zbl

[Pigola et al. 2011] S. Pigola, M. Rigoli, M. Rimoldi, and A. G. Setti, "Ricci almost solitons", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **10**:4 (2011), 757–799. MR Zbl

[Qing and Yuan 2013] J. Qing and W. Yuan, "A note on static spaces and related problems", *J. Geom. Phys.* **74** (2013), 18–27. MR Zbl

Received April 14, 2016. Revised September 29, 2016.

GABJIN YUN
DEPARTMENT OF MATHEMATICS
MYONG JI UNIVERSITY
SAN 38-2 NAMDONG
YONGIN, GYEONGGI 449-728
SOUTH KOREA
gabjin@mju.ac.kr

JINSEOK CO
DEPARTMENT OF MATHEMATICS
CHUNG-ANG UNIVERSITY
84 HEUKSEOK-RO, DONGJAK-GU
SEOUL 06969
SOUTH KOREA
co1010@hanmail.net

SEUNGSU HWANG
DEPARTMENT OF MATHEMATICS
CHUNG-ANG UNIVERSITY
84 HEUKSUK-RO, DONGJAK-GU
SEOUL 06969
SOUTH KOREA
seungsu@cau.ac.kr

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 288 No. 2 June 2017

0030-8730(201706)288:2;1-R