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GABJIN YUN, JINSEOK CO AND SEUNGSU HWANG

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On an n -dimensional complete manifold M , consider an h -almost gradient Ricci soliton, which is a generalization of a gradient Ricci soliton. We prove that if the manifold is Bach-flat and $dh/du > 0$, then the manifold M is either Einstein or rigid. In particular, such a manifold has harmonic Weyl curvature. Moreover, if the dimension of M is four, the metric g is locally conformally flat.

1. Introduction

The notion of an h -almost Ricci soliton was introduced by Gomes, Wang, and Xia [Gomes et al. 2015]. Such a soliton is a generalization of an almost Ricci soliton presented in [Barros and Ribeiro 2012; Pigola et al. 2011]. An h -almost Ricci soliton is a complete Riemannian manifold (M^n, g) with a vector field X on M , a soliton function $\lambda : M \rightarrow \mathbb{R}$ and a signal function $h : M \rightarrow \mathbb{R}^+$ satisfying the equation

$$r_g + \frac{1}{2}h \mathcal{L}_X g = \lambda g,$$

where r_g is the Ricci curvature of g . A function is called signal if it has only one sign; in other words, it is either positive or negative on M . Let (M, g, X, h, λ) denote an h -almost Ricci soliton. In particular, $(M, g, \nabla u, h, \lambda)$ for some smooth function $u : M \rightarrow \mathbb{R}$ is called an h -almost gradient Ricci soliton with potential function u . In this case, we have

$$(1-1) \quad r_g + h D_g du = \lambda g.$$

Here, $D_g du$ denotes the Hessian of u . Note that if we take $u = e^{-f/m}$ and $h = -m/u$, then (1-1) becomes

$$\text{Ric}_f^m = r_g + D_g df - \frac{1}{m} df \otimes df = \lambda g.$$

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In other words, the $(\lambda, n + m)$ -Einstein equation is a special case of (1-1). Here, Ric_f^m is called the m -Bakry–Emery tensor. For further details about h -almost Ricci solitons, see [Gomes et al. 2015].

In this paper we consider Bach-flat h -almost gradient Ricci solitons. The Bach tensor was introduced by R. Bach, and this notion plays an important role in conformal relativity. On any n -dimensional Riemannian manifold (M, g) , $n \geq 4$, the Bach tensor is defined by

$$B = \frac{1}{n - 3} \delta^D \delta \mathcal{W} + \frac{1}{n - 2} \mathring{W}z,$$

where \mathcal{W} is the Weyl tensor, z is the traceless Ricci tensor, and $\mathring{W}z$ is defined by

$$\mathring{W}z(X, Y) = \sum_{i=1}^n z(\mathcal{W}(X, E_i)Y, E_i)$$

for some orthonormal basis $\{E_i\}_{i=1}^n$. It is easy to see that if (M, g) is either locally conformally flat or Einstein, then it is Bach-flat: $B = 0$. When $n = 4$, it is well known that Bach-flat metrics on a compact manifold M are critical points of the functional

$$g \mapsto \int_M |\mathcal{W}|^2 dv_g.$$

It is clear that when $h = 1$ and λ is a positive constant, an h -almost gradient Ricci soliton reduces to a gradient shrinking Ricci soliton. Cao and Chen [2013] proved that a complete Bach-flat gradient shrinking Ricci soliton is either Einstein or rigid. On the other hand, Qing and Yuan [2013] classified Bach-flat static spaces.

Our main result is as follows, which can be considered as a generalization of [Cao and Chen 2013].

Theorem 1.1. *Let $(M, g, \nabla u, h, \lambda)$ be an n -dimensional Bach-flat h -almost gradient Ricci soliton with potential function u . Assume that each level set of u is compact and h is a function of u only. Then $(M, g, \nabla u, h, \lambda)$ is either*

- (1) *Einstein with constant functions u and h , or*
- (2) *locally isometric to a warped product with $(n - 1)$ -dimensional Einstein fibers if $dh/du > 0$ on M .*

For example, when $m > 0$, $h = -m/u < 0$ satisfies the condition of Theorem 1.1, since

$$\frac{dh}{du} = \frac{m}{u^2} > 0.$$

This recovers the result of [Chen and He 2013]. It will be interesting if one can weaken the condition of Theorem 1.1.

In the case of (2) in [Theorem 1.1](#), a warped product metric has vanishing Cotton tensor (see (2-4) below) since its fiber is Einstein. Thus, as a consequence of [Theorem 1.1](#), we have the following.

Corollary 1.2. *Let $(M, g, \nabla u, h, \lambda)$ be an n -dimensional Bach-flat h -almost gradient Ricci soliton with potential function u . Assume that each level set of u is compact and h is a function of u only. If $dh/du > 0$ on M , then (M, g) has harmonic Weyl curvature.*

In particular, when $n = 4$, the Einstein fibers in [Theorem 1.1](#) have constant curvature. A computation shows that such a metric is locally conformally flat, which proves the following theorem.

Theorem 1.3. *Let $(M, g, \nabla u, h, \lambda)$ be a 4-dimensional Bach-flat h -almost gradient Ricci soliton with potential function u . Assume each level set of u is compact and h is a function of u only with $dh/du > 0$. Then (M, g) is locally conformally flat.*

As in [[Chen and He 2013](#)], [Theorem 1.1](#), [Corollary 1.2](#), and [Theorem 1.3](#) can be extended to the case in which M has a nonempty boundary.

2. Preliminaries

In this section, we derive several useful identities containing various curvatures and the Cotton tensor.

We start with basic definitions of differential operators acting on tensors. Let us denote by $C^\infty(S^2M)$ the space of sections of symmetric 2-tensors on a Riemannian manifold M . Let D be the Levi-Civita connection of (M, g) . Then the differential operator d^D from $C^\infty(S^2M)$ into $C^\infty(\Lambda^2M \otimes T^*M)$ is defined as

$$d^D\omega(X, Y, Z) = (D_X\omega)(Y, Z) - (D_Y\omega)(X, Z)$$

for $\omega \in C^\infty(S^2M)$ and vectors X, Y , and Z . Let us denote by δ^D the formal adjoint operator of d^D .

For a function $f \in C^\infty(M)$ and $\omega \in C^\infty(S^2M)$, $df \wedge \omega$ is defined as

$$(df \wedge \omega)(X, Y, Z) = df(X)\omega(Y, Z) - df(Y)\omega(X, Z).$$

Here, df denotes the usual total differential of f . We also denote by δ the negative divergence operator such that $\Delta f = -\delta df$.

Taking the trace of (1-1) gives

$$s_g + h \Delta u = n \lambda.$$

Thus,

$$ds_g + \Delta u dh + h d\Delta u = n d\lambda.$$

By taking the divergence of (1-1), we have

$$-\frac{1}{2} ds_g - D_g du(\nabla h, \cdot) - hr_g(\nabla u, \cdot) - h d\Delta u = -d\lambda.$$

By adding the previous two equations, we have

$$(2-1) \quad \frac{1}{2} ds_g - D_g du(\nabla h, \cdot) - hr_g(\nabla u, \cdot) + \Delta u dh = (n - 1) d\lambda.$$

Note that

$$(2-2) \quad \delta(hr_g(\nabla u, \cdot)) = -r_g(\nabla u, \nabla h) - \frac{1}{2}h\langle \nabla s_g, \nabla u \rangle + |r_g|^2 - \lambda s_g.$$

Therefore, we have the following equality.

Proposition 2.1. *On M we have*

$$(n - 1)\Delta\lambda = \frac{1}{2} \Delta s_g + |r_g|^2 - \lambda s_g - \frac{1}{2}h\langle \nabla s_g, \nabla u \rangle + \left(\Delta u - \frac{\lambda}{h}\right)\Delta h + \frac{1}{h}\langle r_g, D_g dh \rangle - 2r_g(\nabla u, \nabla h).$$

On the other hand, by applying d^D to (1-1), we have

$$(2-3) \quad d^D r_g - \frac{1}{h} dh \wedge r_g + h\tilde{i}_{\nabla u} R = d\lambda \wedge g - \frac{\lambda}{h} dh \wedge g.$$

Here, an interior product \tilde{i} of the final factor is defined by

$$\tilde{i}_\xi R(X, Y, Z) = R(X, Y, Z, \xi),$$

and we used the identity

$$d^D D du = \tilde{i}_{\nabla u} R.$$

Hereafter, we denote $s_g, r_g,$ and $D_g du$ by $s, r,$ and $D du,$ respectively. From the curvature decomposition, we can compute that

$$\tilde{i}_{\nabla u} R = \tilde{i}_{\nabla u} \mathcal{W} - \frac{1}{n - 2} i_{\nabla u} r \wedge g + \frac{s}{(n - 1)(n - 2)} du \wedge g - \frac{1}{n - 2} du \wedge r,$$

where $i_{\nabla u} r$ denotes the interior product defined by

$$i_{\nabla u} r(X) = r(\nabla u, X).$$

The Cotton tensor C is defined by

$$(2-4) \quad C = d^D r - \frac{1}{2(n - 1)} ds \wedge g.$$

Then, by (2-1) and (2-3) as well as the fact that

$$s + h \Delta u = n\lambda,$$

we have

$$\begin{aligned}
 (2-5) \quad C + h\tilde{\nabla}_u \mathcal{W} &= hD + \frac{h}{n-1} i_{\nabla u} r \wedge g + d\lambda \wedge g - \frac{1}{2(n-1)} ds \wedge g \\
 &\quad + \frac{1}{h} dh \wedge r - \frac{\lambda}{h} dh \wedge g \\
 &= hD + H,
 \end{aligned}$$

where D is defined (as usual) by

$$(2-6) \quad (n-2)D = du \wedge r + \frac{1}{n-1} i_{\nabla u} r \wedge g - \frac{s}{n-1} du \wedge g,$$

and H is defined by

$$\begin{aligned}
 H &= -\frac{1}{n-1} i_{\nabla h} D du \wedge g + dh \wedge \left(\frac{1}{h} r + \frac{\Delta u}{n-1} g - \frac{\lambda}{h} g \right) \\
 &= db \wedge r + \frac{1}{n-1} i_{\nabla b} r \wedge g - \frac{s}{n-1} db \wedge g.
 \end{aligned}$$

Here, $b = \log |h|$ with $\nabla b = \nabla h / h$. In particular, $g^{ik} H_{ijk} = -g^{ik} H_{jik} = 0$.

Proposition 2.2. *Let $(M, g, \nabla u, h, \lambda)$ be an h -almost gradient Ricci soliton with potential function u . Then*

$$C + h\tilde{\nabla}_u \mathcal{W} = hD + H.$$

In particular, if h is constant or $dh/du = 0$, then $H \equiv 0$.

3. Bach-flat metrics

In this section, we assume that g is Bach-flat. Note that

$$\delta \mathcal{W} = -\frac{n-3}{n-2} C.$$

Recall that the Bach tensor is given by

$$B = \frac{1}{n-3} \delta^D \delta \mathcal{W} + \frac{1}{n-2} \mathring{\mathcal{W}}z = \frac{1}{n-2} (-\delta C + \mathring{\mathcal{W}}z).$$

Since

$$\begin{aligned}
 \delta(h\tilde{\nabla}_u \mathcal{W})(X, Y) &= -\mathcal{W}(\nabla h, X, Y, \nabla u) + h\delta \mathcal{W}(X, Y, \nabla u) + h\mathcal{W}(X, E_i, Y, D_{E_i} du) \\
 &= l\mathcal{W}(X, \nabla h, Y, \nabla u) - \frac{n-3}{n-2} hC(Y, \nabla u, X) - \mathring{\mathcal{W}}z,
 \end{aligned}$$

by taking the divergence of (2-5) we have

$$-(n-2)B(X, Y) = -\mathcal{W}(X, \nabla h, Y, \nabla u) + \frac{n-3}{n-2}hC(Y, \nabla u, X) - i_{\nabla h}D(X, Y) + h\delta D(X, Y) + \delta H(X, Y).$$

Hence,

$$-(n-2)B(\nabla u, \nabla u) = -D(\nabla h, \nabla u, \nabla u) + h\delta D(\nabla u, \nabla u) + \delta H(\nabla u, \nabla u).$$

As a result, from the assumption that $B = 0$ and h is a function of u only,

$$0 = \frac{1}{h}D(\nabla h, \nabla u, \nabla u) = \delta D(\nabla u, \nabla u) + \frac{1}{h}\delta H(\nabla u, \nabla u).$$

Let $\{E_i\}_{i=1}^n$ be a normal geodesic frame. Note that, since

$$hD(E_i, D_{E_i} du, \nabla u) = -D(E_i, E_k, \nabla u)r_{ik} = 0,$$

we have

$$\operatorname{div}(D(\cdot, \nabla u, \nabla u)) = -\delta D(\nabla u, \nabla u) + D(E_i, \nabla u, D_{E_i} du).$$

Furthermore,

$$\begin{aligned} |D|^2 &= \frac{1}{n-2}(du(E_i)r(E_j, E_k) - du(E_j)r(E_i, E_k))D_{ijk} \\ &= -\frac{2}{n-2}D(E_i, \nabla u, E_k)r_{ik} \\ &= \frac{2h}{n-2}D(E_i, \nabla u, D_{E_i} du). \end{aligned}$$

Similarly, since

$$hH(E_i, D_{E_i} du, \nabla u) = -H(E_i, E_k, \nabla u)r_{ik} = 0$$

and h is a function of u only, we have

$$\operatorname{div}\left(\frac{1}{h}H(\cdot, \nabla u, \nabla u)\right) = -\frac{1}{h}\delta H(\nabla u, \nabla u) + \frac{1}{h}H(E_i, \nabla u, D_{E_i} du).$$

Moreover,

$$\begin{aligned} |H|^2 &= -\frac{2}{h}H(E_i, \nabla h, E_k)r_{ik} \\ &= -\frac{2}{h}\frac{dh}{du}H(E_i, \nabla u, E_k)r_{ik} \\ &= 2\frac{dh}{du}H(E_i, \nabla u, D_{E_i} du). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \int_{t_1 \leq u \leq t_2} \delta D(\nabla u, \nabla u) + \frac{1}{h} \delta H(\nabla u, \nabla u) \\ &= \frac{n-2}{2} \int_{t_1 \leq u \leq t_2} \frac{|D|^2}{h} + \frac{1}{2} \int_{t_1 \leq u \leq t_2} \frac{|H|^2}{h \frac{dh}{du}}. \end{aligned}$$

Since h is signal, h is either positive or negative. For each case, we derive $D = H = 0$ when $dh/du > 0$. Therefore we have the following result.

Lemma 3.1. *Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat h -almost gradient Ricci soliton with potential function u . Assume that each level set of u is compact and h is a function of u only. If $dh/du > 0$ on M , then on M we have*

$$D = H = 0.$$

Now, since $D = H = 0$, by (2-4) and (2-5)

$$(3-1) \quad C = -h \tilde{\tau}_{\nabla u} \mathcal{W}.$$

By taking the divergence of (3-1), we have

$$\mathcal{W}(X, \nabla h, Y, \nabla u) = \frac{n-3}{n-2} h C(Y, \nabla u, X).$$

By combining these equations,

$$\frac{n-3}{n-2} h^2 C(Y, \nabla u, X) = -C(X, \nabla h, Y),$$

and

$$\mathcal{W}(X, \nabla h, Y, \nabla u) = -\frac{n-3}{n-2} h^2 \mathcal{W}(X, \nabla u, Y, \nabla u).$$

Therefore, we have the following.

Corollary 3.2. *When $D = H = 0$, we have*

$$(3-2) \quad \mathcal{W}(\cdot, \nabla u, \cdot, \nabla u) = C(\cdot, \nabla u, \cdot) = 0,$$

unless

$$\frac{dh}{du} = -\left(\frac{n-3}{n-2}\right) h^2.$$

For example, when $h = -m/u$, (3-2) holds if $m \neq 0$ or $-(n-2)/(n-3)$. Note that (3-2) also holds if h is constant.

Moreover, we have the following result.

Lemma 3.3. *Suppose that $dh/du > 0$. Then, for X orthogonal to ∇u ,*

$$(3-3) \quad r(X, \nabla u) = 0.$$

In particular,

$$i_{\nabla u}r = \alpha du,$$

where $\alpha = r(N, N)$ with $N = \nabla u/|\nabla u|$.

Proof. By Lemma 3.1, $D = H = 0$. From (2-3), if X is orthogonal to ∇u ,

$$d^D r(X, Y, \nabla u) = -\frac{1}{h} dh(Y)r(X, \nabla u) + d\lambda(X) du(Y).$$

Since $C(X, Y, \nabla u) = -h\mathcal{W}(X, Y, \nabla u, \nabla u) = 0$ by (3-1), by (2-4) we have

$$d^D r(X, Y, \nabla u) = \frac{1}{2(n-1)} ds(X) du(Y).$$

Thus, by (2-1)

$$\begin{aligned} \frac{1}{h} \frac{dh}{du} r(X, \nabla u) &= d\lambda(X) - \frac{1}{2(n-1)} ds(X) \\ &= \frac{1}{(n-1)h} \left(\frac{dh}{du} - h^2 \right) r(X, \nabla u), \end{aligned}$$

which implies that

$$\left((n-2) \frac{dh}{du} + h^2 \right) r(X, \nabla u) = 0. \quad \square$$

Note that Lemma 3.3 holds with the assumptions that $D = H = 0$ and

$$(3-4) \quad \frac{dh}{du} \neq -\frac{1}{n-2} h^2$$

without $dh/du > 0$. For example, in the case of the m -Bakry–Emery tensor, $h = -m/u$ satisfies (3-4) if $m \neq 2 - n$.

4. Level sets of u

In this section, we investigate the structure of regular level sets of the potential function u . For a regular value c , we denote the level set $u^{-1}(c)$ by L_c . On L_c , let $\{E_i\}$, $1 \leq i \leq n$, be an orthonormal frame with $E_n = N = \nabla u/|\nabla u|$.

Furthermore, throughout the section we assume that $D = H = 0$ with

$$\frac{dh}{du} \neq -\left(\frac{n-3}{n-2}\right)h^2 \quad \text{and} \quad \frac{dh}{du} \neq -\frac{1}{n-2}h^2.$$

Then, by Corollary 3.2, (3-2) and (3-3) hold. Furthermore, for X orthogonal to ∇u , by the proof of Lemma 3.3,

$$d\lambda(X) = \frac{1}{2(n-1)} ds(X).$$

Thus, $s + 2(1 - n)\lambda$ is constant on each level set of u . Furthermore,

$$\frac{1}{2} X(|\nabla u|^2) = \langle D_X du, \nabla u \rangle = \frac{1}{h} (\lambda du(X) - r(X, \nabla u)) = 0,$$

which implies that $|\nabla u|^2$ is constant on each level set of u . Therefore, we have the following.

Lemma 4.1. $|\nabla u|^2$ and $s + 2(1 - n)\lambda$ are constant on each regular level set of u .

For further investigation, we need the following key lemma.

Lemma 4.2.

$$\begin{aligned} 0 = & \frac{ns - (n - 1)^2\lambda - \alpha}{(n - 1)h} r - D_{\nabla u} r - \frac{r \circ r}{h} + \frac{n - 3}{2(n - 1)} du \otimes ds \\ & + \frac{1}{n - 1} (ds(u) - \langle \nabla u, \nabla \alpha \rangle) g + \frac{s + (1 - n)\lambda}{(n - 1)h} (\alpha - s) g + \frac{1}{n - 1} du \otimes d\alpha. \end{aligned}$$

Proof. To find δD , by (2-6), we first compute

$$\delta(du \wedge r) = \frac{s - (n - 1)\lambda}{h} r - D_{\nabla u} r - \frac{r \circ r}{h} + \frac{1}{2} du \otimes ds.$$

By Lemma 3.3, $i_{\nabla u} r = \alpha du$. Thus,

$$\delta(i_{\nabla u} r \wedge g) = -\langle \nabla u, \nabla \alpha \rangle g + \frac{s + (1 - n)\lambda}{h} \alpha g + du \otimes d\alpha - \frac{\alpha}{h} r.$$

Similarly,

$$-\delta(s du \wedge g) = ds(u) g - \frac{s^2 + (1 - n)s\lambda}{h} g - du \otimes ds + \frac{s}{h} r.$$

Hence, by (2-6) together with (3-3), we have

$$\begin{aligned} (n - 2)\delta D = & \frac{ns - (n - 1)^2\lambda - \alpha}{(n - 1)h} r - D_{\nabla u} r - \frac{r \circ r}{h} \\ & + \frac{n - 3}{2(n - 1)} du \otimes ds + \frac{1}{n - 1} du \otimes d\alpha \\ & + \frac{1}{n - 1} \left(ds(u) - \langle \nabla u, \nabla \alpha \rangle + \frac{s + (1 - n)\lambda}{h} (\alpha - s) \right) g. \end{aligned}$$

Since $D = \delta D = 0$, the proof follows. □

Thus, we have the following.

Corollary 4.3. $(n - 3)s + 2\alpha$ is constant on each regular level set of u .

Proof. Let X be a vector orthogonal to ∇u . By putting $(X, \nabla u)$ in the equation in Lemma 4.2,

$$(4-1) \quad D_{\nabla u} r(X, \nabla u) = 0.$$

Now, by putting $(\nabla u, X)$ in the equation in Lemma 4.2 again, we have

$$0 = \frac{n-3}{2(n-1)} |\nabla u|^2 ds(X) + \frac{1}{n-1} |\nabla u|^2 d\alpha(X),$$

since $r(X, \nabla u) = 0$ and

$$D_{\nabla u} r(\nabla u, X) = D_{\nabla u} r(X, \nabla u). \quad \square$$

Lemma 4.4. $s_g + 2(1-n)\alpha$ is constant on each regular level set of u .

Proof. For X orthogonal to ∇u , by (3-2) and (4-1)

$$\begin{aligned} 0 &= C(X, \nabla u, \nabla u) \\ &= D_X r(\nabla u, \nabla u) - \frac{1}{2(n-1)} ds(X) |\nabla u|^2. \end{aligned}$$

Thus,

$$\begin{aligned} X(\alpha) &= \frac{1}{|\nabla u|^2} X(r(\nabla u, \nabla u)) \\ &= \frac{1}{|\nabla u|^2} (D_X r(\nabla u, \nabla u) + 2r(D_X du, \nabla u)) \\ &= \frac{1}{2(n-1)} ds(X), \end{aligned}$$

since

$$r(D_X du, \nabla u) = \frac{1}{h} (\lambda r(X, \nabla u) - r \circ r(X, \nabla u)) = 0. \quad \square$$

By combining Lemma 4.1, Corollary 4.3, and Lemma 4.4, we have the following.

Theorem 4.5. Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat h -almost gradient Ricci soliton with potential function u . Assume that each level set of u is compact and h is a function of u only with $dh/du > 0$. Then s_g, α , and λ are constant on each regular level set of u . In particular, if h is constant, the condition on dh/du is not necessary.

When $D = 0$, the Ricci tensor has the following characterization.

Lemma 4.6. Suppose that $D = 0$. Then the Ricci curvature tensor has at most two eigenvalues.

Proof. Let $\{E_i\}$, $1 \leq i \leq n$, be an orthonormal frame with $E_n = N = \nabla u/|\nabla u|$. Then

$$(4-2) \quad II_{ij} = \frac{1}{h|\nabla u|} (\lambda g_{ij} - r_{ij}),$$

and

$$m = \text{tr } II = \frac{n-1}{h|\nabla u|} \left(\lambda + \frac{\alpha - s}{n-1} \right).$$

Thus, m is constant on each level set of u , and

$$\begin{aligned} \left| II - \frac{m}{n-1} g \right|^2 &= |II|^2 - \frac{m^2}{n-1} \\ &= \frac{1}{h^2 |\nabla u|^2} \left(|r|^2 - \alpha^2 - \frac{(s-\alpha)^2}{n-1} \right) \\ &= \frac{1}{h^2 |\nabla u|^2} \left(|r|^2 - \frac{n}{n-1} \alpha^2 + \frac{2s\alpha}{n-1} - \frac{s^2}{n-1} \right). \end{aligned}$$

Since $r \circ r(\nabla u, \nabla u) = \alpha^2 |\nabla u|^2$, from the identity

$$\frac{n-2}{2} |D|^2 = |r|^2 |\nabla u|^2 - \frac{n}{n-1} r \circ r(\nabla u, \nabla u) + \frac{2s}{n-1} r(\nabla u, \nabla u) - \frac{s^2}{n-1} |\nabla u|^2,$$

we have

$$|D|^2 = \frac{2}{n-2} h^2 |\nabla u|^4 \left| II - \frac{m}{n-1} g \right|^2.$$

Since $D = 0$, we have

$$(4-3) \quad II_{ij} = \frac{m}{n-1} g_{ij},$$

which implies that

$$(4-4) \quad r_{ij} = \frac{s-\alpha}{n-1} g_{ij}$$

for $i = 1, \dots, n-1$ by (4-2). □

As an immediate consequence, on an open set $\{x \in M \mid \nabla u(x) \neq 0\}$, the Ricci tensor may be written as

$$r_g = \beta du \otimes du + \left(\frac{s-\alpha}{n-1} \right) g,$$

where

$$\beta = \frac{n\alpha - s}{(n-1)|\nabla u|^2}.$$

Thus, by (1-1) we have

$$D_g du = \frac{1}{h} \left(\lambda + \frac{\alpha - s}{n-1} \right) g - \frac{\beta}{h} du \otimes du.$$

Now, we are ready to prove [Corollary 1.2](#), which shows the relationship between Bach-flat metrics and harmonic Weyl metrics.

Proof of Corollary 1.2. Note that, by (3-1) and (3-2)

$$C(\cdot, \cdot, \nabla u) = C(\cdot, \nabla u, \cdot) = 0.$$

On the other hand, by the Codazzi equation,

$$\langle R(X, Y)Z, N \rangle = D_Y II(X, Z) - D_X II(Y, Z).$$

Thus, for $1 \leq i, j, k \leq n - 1$, by (4-3)

$$\begin{aligned} \langle R(E_i, E_j)E_k, N \rangle &= E_j(II(E_i, E_k)) - II(D_{E_j}E_i, E_k) - II(E_i, D_{E_j}E_k) \\ &\quad - E_i(II(E_j, E_k)) + II(D_{E_i}E_j, E_k) + II(E_j, D_{E_i}E_k) \\ &= 0. \end{aligned}$$

Therefore, by (2-3)

$$d^D r(E_i, E_j, E_k) = 0,$$

which implies that

$$C(E_i, E_j, E_k) = d^D r(E_i, E_j, E_k) - \frac{1}{2(n-1)} ds \wedge g(E_i, E_j, E_k) = 0.$$

Hence, C is identically zero, and so is $\delta\mathcal{W}$. □

The following is a restatement of [Theorem 1.1](#).

Theorem 4.7. *Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat h -almost gradient Ricci soliton with potential function u . Assume that each level set of u is compact with $dh/du > 0$ on M . Then, either g is Einstein with constant function u or the metric can be written as*

$$g = dt^2 + \psi^2(t) \hat{g}_E,$$

where \hat{g}_E is the Einstein metric on the level set $E = L_{c_0}$ for some c_0 .

Proof. Assume that u is not constant. By [Lemma 3.1](#), $D = H = 0$. Since $|\nabla u|^2$ depends only on u by [Lemma 4.6](#), as shown in the proof of [Theorem 7.9](#) of [[He et al. 2012](#)] with [Remark 3.2](#) of [[Cao and Chen 2013](#)], the metric can be locally written as

$$g = dt^2 + \hat{g}_c.$$

Here, \hat{g}_c denotes the induced metric on the level set $L_c = u^{-1}(c)$ for each regular value c . Furthermore, (L_c, \hat{g}_c) is necessarily Einstein; by the Gauss equation

$$\hat{R}_{ijij} = R_{ijij} + II_{ii} II_{jj} - II_{ij}^2 = R_{ijij} + \frac{m^2}{(n-1)^2}.$$

Thus,

$$\hat{r}_{ii} = r_{ii} - R(N, E_i, N, E_i) + \frac{m^2}{n-1}.$$

By (3-2) and (4-4), we have

$$R(E_i, N, E_i, N) = \frac{1}{n-2} (r_{ii} + \alpha) - \frac{s}{(n-1)(n-2)} = \frac{\alpha}{n-1}.$$

Hence, it follows that

$$\hat{r}_{ii} = r_{ii} + \frac{m^2 - \alpha}{n-1} = \frac{1}{n-1} (s - 2\alpha + m^2) = \hat{\lambda}_0.$$

Since s , α , and m are constant along L_c , this proves that (L_c, \hat{g}_c) has constant Ricci curvature. As a result, by a suitable change of variable, the metric g can be written as in the statement of [Theorem 4.7](#). \square

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GABJIN YUN
DEPARTMENT OF MATHEMATICS
MYONG JI UNIVERSITY
SAN 38-2 NAMDONG
YONGIN, GYEONGGI 449-728
SOUTH KOREA
gabjin@mju.ac.kr

JINSEOK CO
DEPARTMENT OF MATHEMATICS
CHUNG-ANG UNIVERSITY
84 HEUKSEOK-RO, DONGJAK-GU
SEOUL 06969
SOUTH KOREA
co1010@hanmail.net

SEUNGSU HWANG
DEPARTMENT OF MATHEMATICS
CHUNG-ANG UNIVERSITY
84 HEUKSUK-RO, DONGJAK-GU
SEOUL 06969
SOUTH KOREA
seungsu@cau.ac.kr

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University of California
Los Angeles, CA 90095-1555
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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jlhu@maths.hku.hk

Jie Qing
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Santa Cruz, CA 95064
qing@cats.ucsc.edu

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