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A SHARP HEIGHT ESTIMATE FOR THE SPACELIKE CONSTANT MEAN CURVATURE GRAPH IN THE LORENTZ–MINKOWSKI SPACE

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Based on the local comparison principle of Chen and Huang (1982), we study the local behavior of the difference of two spacelike graphs in a neighborhood of a second contact point. Then we apply it to the spacelike constant mean curvature graph in 3-dimensional Lorentz–Minkowski space $\mathbb{L}^3$, which can be viewed as a solution to the constant mean curvature equation over a convex domain $\Omega \subset \mathbb{R}^2$. We get the uniqueness of critical points for such a solution, which is an analogue of a result of Sakaguchi (1988). Last, by this uniqueness, we obtain a minimum principle for a functional depending on the solution and its gradient. This gives us a sharp gradient estimate for the solution, which leads to a sharp height estimate.

1. Introduction

Spacelike hypersurfaces of constant mean curvature (CMC) and CMC foliations play an important role in general relativity. Such surfaces are important because they provide Riemannian submanifolds with properties reflecting those of the spacetime. For example, if the weak energy condition is satisfied, a maximal hypersurface has positive scalar curvature. So the geometric properties of such hypersurfaces are worth researching, and finding conditions for their existence is a fundamental problem. Under the graph setting and some assumptions, Robert Bartnik and Leon Simon [1982] got a sufficient and necessary condition for the existence of a solution to

$$(1\text{-}1)\qquad \begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1-|Du|^2}} = H(x,u), \quad |Du|<1 & \text{in } \Omega\subset\mathbb{R}^n,\\ u=\phi & \text{on } \partial\Omega, \end{cases}$$

where div stands for the divergence operator in the Euclidean plane $\mathbb{R}^n$ and

$$(1\text{-}2)\qquad Du=(u_1,\dots,u_n),\quad u_i=\frac{\partial u}{\partial x_i}.$$

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In particular, the Theorem 3.6 in [Bartnik and Simon 1982] gives us a solution $u \in C^\infty(\overline{\Omega})$ to

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1-|Du|^2}} = nH, \quad |Du| < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-3}$$

over a bounded $C^{2,\alpha}$ domain Ω with H being a positive constant. In this case, they pointed out that $\nu_{n+1} = 1/\sqrt{1-|Du|^2}$ satisfies following elliptic equation

$$\Delta_M \nu_{n+1} = \nu_{n+1}\|A\|^2, \tag{1-4}$$

where Δ_M and A denote the Laplace operator and the second fundamental form of the graph $M = \{(x, u(x)) : x \in \mathbb{R}^n, u \in C^\infty(\mathbb{R}^n)\}$, respectively. The boundary gradient estimate is the most important step leading to the existence of u. To do so, they used the following spherically symmetric barrier functions:

$$w^\pm = w^\pm(\xi) \pm \int_0^{|x-\xi|} \frac{K - Ht^n}{\sqrt{t^{2n-2} + (K - Ht^n)^2}}\, dt, \tag{1-5}$$

where K is a positive constant. From the proof of their Proposition 3.1, one can get following boundary gradient estimate:

$$\max_{\partial\Omega} |Du| \le \frac{1 - H\varepsilon^{n+1}}{\sqrt{\varepsilon^{2n} + (1 - H\varepsilon^{n+1})^2}}, \tag{1-6}$$

where $\varepsilon = \varepsilon(\Omega)$ is a sufficiently small constant. Obviously, this bound is not sharp. Also, the dependence of ε on Ω is not specific. Since the graph is spacelike, they roughly used the diameter of the domain Ω to control the C^0 norm of the solution u. So the question is, can we give a sharp C^0 or C^1 estimate for the solution in terms of the boundary geometry?

Early in 1979, Lawrence E. Payne and Gérard A. Philippin [1979] have used so-called P-functions to derive sharp C^0 and C^1 upper bounds for the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} = -2H & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-7}$$

over a strictly convex domain $\Omega \subset \mathbb{R}^2$ with H being a positive constant. The key is a maximum principle for the P-function

$$\Phi(x, \alpha) = \int_0^{q^2} \frac{g(\xi + 2\xi g'(\xi))}{\rho}\, d\xi + \alpha \int_0^u f(\eta)\, d\eta, \tag{1-8}$$

where u, g, ρ, f, q satisfy

$$(g(q^2)u_i)_i + \rho(q^2) f(u) = 0, \quad g(\xi) + 2\xi g'(\xi) > 0, \qquad \text{for all } \xi \geq 0,$$
$$\rho > 0, \quad g > 0, \qquad q^2 = |Du|^2 = \sum u_i^2. \tag{1-9}$$

In the same year, by the uniqueness of critical points for a solution and the strict convexity of the domain, G. A. Philippin [1979] also got a minimum principle for $\Phi(x, \alpha)$ provided $\alpha > 1$ and used it to derive lower bounds for C^0 and C^1 norms of the solution. But he did not assert the sharpness of the estimates, since he did not have a similar minimum principle for $\Phi(x, 1)$ at that time. In 2000, Xi-Nan Ma [2000] solved this issue through uniqueness of critical points and analyticity of the solution. He did a long computation to show that all the derivatives of $\Phi(x, 1)$ vanish at the unique critical point if $\Phi(x, 1)$ takes its minimum value at that point. By the strong unique continuation of analytic function, $\Phi(x, 1)$ is a constant. Once one has this minimum principle, the sharpness is easy to derive.

For our question, the maximum principle in [Payne and Philippin 1979] still works. So the upper bound of the gradient estimate and the lower bound of the minimum value are easy to derive, which we will do later in this paper. However, the minimum principle is not available any more. In this paper, we want to prove a minimum principle for $\Phi(x, 1)$ when u is a spacelike CMC graph solving

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1-|Du|^2}} = 2H, \quad |Du| < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-10}$$

and use it to derive sharp C^0 and C^1 bounds for the solution to (1-10).

Not only is the uniqueness of critical point the important ingredient to get the sharpness of the a priori estimate, but is itself worth study. Together with the convexity [Caffarelli and Friedman 1985; Guan and Ma 2003; Chen 2014] and curvature estimates [Ma and Zhang 2013] for level sets, they are the most important geometric properties of solutions to elliptic or parabolic equations. G. A. Philippin [1979] showed that the solution to (1-7) has only one critical point when Ω is strictly convex. His method of proof is based on an idea of L. E. Payne [1973]. Jin-Tzu Chen [1984] proved the uniqueness of the critical point for a solution to

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} = 2H & \text{in } \Omega, \\ \dfrac{Du}{\sqrt{1+|Du|^2}} \cdot \nu = 1 & \text{on } \partial\Omega, \end{cases} \tag{1-11}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with outer normal ν on the boundary $\partial\Omega$ and H is a positive constant. His proof is based on a nice comparison technique and the result in [Chen and Huang 1982] and the method of continuity with respect to the contact angle. Later, Shigeru Sakaguchi [1989] showed that the solution to

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} = 2H & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1-12}$$

or

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} = 2H & & \text{in } \Omega, \\ \dfrac{Du}{\sqrt{1+|Du|^2}} \cdot \nu = \cos\gamma, & \gamma \in \left(0, \frac{\pi}{2}\right) & \text{on } \partial\Omega, \end{cases} \tag{1-13}$$

has only one critical point under the hypothesis of the existence of the solution over a bounded convex domain $\Omega \subset \mathbb{R}^2$.

Another motivation for studying uniqueness of critical points for solutions to (1-10) is from a recent paper [Albujer et al. 2015]. As we know, CMC spacelike hypersurfaces are very different from those in Euclidean space. For example, Corollary 12.1.8 in [López 2013] tells us any compact spacelike surface immersed in $\mathbb{L}^3$ spanning a plane simple closed curve is a graph over a spacelike plane, which is not true in $\mathbb{R}^3$. Therefore, up to an isometry, we only need to consider the solution to the Dirichlet problem (1-10). Recently, Alma L. Albujer, Magdalena Caballero and Rafael López proved the following interesting theorem on the convexity of the solutions to (1-10):

Theorem A [Albujer et al. 2015]. *Let Σ be a spacelike compact surface in $\mathbb{L}^3$ with constant mean curvature $H \neq 0$ (H-surface for short), such that its boundary is a planar curve which is pseudoelliptic. Then Σ has negative Gaussian curvature at all its interior points. In particular, Σ is a convex surface.*

In their paper, they also proved that pseudoelliptic curves are convex and provided an example that shows the assumption on the boundary can not be replaced by convex curves, but they did not show whether there is a critical point of the solution to (1-10) with nonnegative Gaussian curvature over a convex domain, which is a so-called saddle point. In this paper, we will show that the nonexistence of such saddle points is equivalent to the uniqueness of the critical point. Notice that the Gaussian curvature in [Sakaguchi 1989] is different from that in the Theorem A, which is defined in the next section.

Theorem 1.1. *Any solution to (1-10) in a convex domain for $H \neq 0$ has only one critical point.*

The proof of this theorem is based on the idea of [Sakaguchi 1989], which mainly relies on the comparison of a cylinder with the given surface and the continuity method. In the present result, our comparison surface is a connected component of a hyperbolic cylinder, which is an entire graph over $\mathbb{R}^2$ and, in contrast with the

Euclidean case, the existence of the solution for any bounded domain is assured by the necessary and sufficient conditions given in [Bartnik and Simon 1982].

As we said before, Theorem 1.1 can be used to derive sharp C^0 and C^1 bounds for the solution to (1-10).

Theorem 1.2. *Let $u \in C^\infty(\bar{\Omega})$ be a solution to (1-10) over a strictly convex domain Ω for $H > 0$ and K be the curvature of the boundary $\partial\Omega$ with respect to the inner normal direction. Then*

$$\max_{\bar{\Omega}} |Du|^2 = \max_{\partial\Omega} |Du|^2 \leq \frac{H^2}{H^2 + K_{\min}^2},$$

$$-\frac{1}{H}\left(\frac{\sqrt{H^2 + K_{\min}^2}}{K_{\min}} - 1\right) \leq \min_{\Omega} u \leq -\frac{1}{H}\left(\frac{\sqrt{H^2 + K_{\max}^2}}{K_{\max}} - 1\right) \tag{1-14}$$

where $K_{\min} = \min_{\partial\Omega} K$, $K_{\max} = \max_{\partial\Omega} K$, and one of the equality signs holds if and only if the boundary $\partial\Omega$ is a circle.

At this point, we should give a remark. When $H \neq 0$ and Ω is a round disc of radius R (which is centered at the origin), then

$$u(x, y) = \sqrt{x^2 + y^2 + \frac{1}{H^2}} - \sqrt{R^2 + \frac{1}{H^2}}, \tag{1-15}$$

whose graph is a so-called hyperbolic cap [López 2013].

This article is organized as follows. In Section 3, we will investigate the local behavior of the difference of two spacelike graphs in a neighborhood of a second contact point. In Section 4, we will prove a necessary and sufficient condition for the uniqueness of the minimal point of the solution to (1-10), which is a key step in the proof of Theorem 1.1 in Section 5. In the end, based on the uniqueness of the critical point, we will prove a minimum principle and use it to get the sharp estimates in Theorem 1.2.

2. Notions and local comparison technique

For easier reading, let us recall some background knowledge of Lorentzian geometry. More details can be found in [López 2013]. Let $\mathbb{L}^3$ be the 3-dimensional Lorentz–Minkowski space, that is $\mathbb{R}^3$ endowed with the flat Lorentzian metric

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) are the canonical coordinates in $\mathbb{R}^n$. The nondegenerate metric of index one classifies the vectors of $\mathbb{R}^3$ into three types.

Definition 2.1 [López 2013]. A vector $v \in \mathbb{L}^3$ is said to be:

(1) spacelike if $\langle v, v\rangle > 0$ or $v = 0$;

(2) timelike if $\langle v, v\rangle < 0$;

(3) lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$.

The modulus of v is $|v| = \sqrt{|\langle v, v\rangle|}$.

Definition 2.2 [López 2013]. An immersed surface Σ in $\mathbb{L}^3$ is called spacelike if the induced metric on Σ is positive definite.

Given a spacelike immersed surface Σ, by Proposition 12.1.5 in [López 2013], Σ is orientable. We can choose Σ to be future-oriented, which means the unit normal vector field N satisfies $\langle N, e_3\rangle > 0$. Here $e_3 = (0, 0, 1)$. Let $\bar{\nabla}$ and ∇ denote the Levi-Civita connection in $\mathbb{L}^3$ and Σ, respectively. If $X, Y \in \mathfrak{X}(\Sigma)$, the Gauss and Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) = \nabla_X Y - \langle AX, Y\rangle N \quad \text{and} \quad AX = -\bar{\nabla}_X N, \tag{2-1}$$

respectively, where σ is the second fundamental form and $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator of Σ with respect to N. The mean curvature and the Gaussian curvature are defined by

$$H = -\tfrac{1}{2}\,\mathrm{trace}(A) = -\tfrac{1}{2}(\kappa_1 + \kappa_2) \quad \text{and} \quad K = -\det(A) = -\kappa_1\kappa_2. \tag{2-2}$$

Let $u \in C^2(\Omega)$ be a function defined on a domain $\Omega \in \mathbb{R}^2$ and consider the surface $\Sigma_u = (x, y, u(x, y))$. The coefficients of the first fundamental form are

$$E = 1 - u_x^2, \quad F = -u_x u_y \quad \text{and} \quad G = 1 - u_y^2. \tag{2-3}$$

Thus $EG - F^2 = 1 - u_x^2 - u_y^2 = 1 - |\nabla u|^2$ and since the immersion is spacelike, $|\nabla u|^2 < 1$ on Ω. The future-directed normal is given by

$$N(x, y, u(x, y)) = \frac{(u_x, u_y, 1)}{\sqrt{1 - |\nabla u|^2}} = \frac{(\nabla u, 1)}{\sqrt{1 - |\nabla u|^2}}. \tag{2-4}$$

With this normal, the mean curvature H and Gaussian curvature K satisfy

$$\mathrm{div}\,\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} = 2H \quad \text{and} \quad K = -\frac{u_{xx}u_{yy} - u_{xy}^2}{(1 - |\nabla u|^2)^2}, \tag{2-5}$$

respectively, where div is the Euclidean divergence in $\mathbb{R}^2$.

As mentioned previously, every compact spacelike surface Σ in $\mathbb{L}^3$ with simple closed boundary contained in a hyperplane can be regarded as the graph of a solution $u(x, y)$ to (1-10). There are more interesting facts on compact spacelike surfaces in $\mathbb{L}^3$ with constant mean curvature spanning a given boundary curve (see [López 2013]).

From now on, we assume u to be a solution to (1-10) with $H > 0$ in a convex domain Ω. For $H < 0$, we can consider $-u$ and our theorem still holds. By the maximum principle, u has a interior minimal point, which is a point of nonpositive Gaussian curvature.

In the rest of this section, based on the local comparison technique found in [Chen and Huang 1982], we will investigate the local behavior of the difference of two spacelike graphs in a neighborhood of the point where they have the second contact.

Lemma 2.3. *Let $u(x, y)$, $v(x, y)$ satisfy the same spacelike constant mean curvature equation (the first equations in (1-10) or (2-5)). Without loss of generality, we assume that u, v have a second order contact at $P_0 = (x_0, y_0, u(x_0, y_0))$ with $(x_0, y_0) = (0, 0)$. Then by changing coordinates from (x, y) to (ξ, η) linearly, the difference $u - v$ around $(\xi, \eta) = (0, 0) = (x, y)$ is given by*

$$u - v = \mathbb{R}\Diamond(\lambda \cdot (\xi + \eta i)^n + o(\xi^2 + \eta^2)^{\frac{n}{2}}), \tag{2-6}$$

where $n \geq 3$, λ is a complex number and $\xi + \eta i$ is the complex coordinate.

Proof. Let $w = u - v$. Since u and v solve the same constant mean curvature equation, we have

$$\begin{aligned} 0 &= (1 - u_x^2 - u_y^2)(u_{xx} + u_{yy}) + (u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}) \\ &\quad - 2H(\sqrt{1 - |Du|^2})^3 \\ &= (1 - u_y^2)u_{xx} + (1 - u_x^2)u_{yy} + 2u_x u_y u_{xy} - 2H\big(\sqrt{1 - u_x^2 - u_y^2}\big)^3, \end{aligned} \tag{2-7}$$

$$0 = (1 - v_y^2)v_{xx} + (1 - v_x^2)v_{yy} + 2v_x v_y v_{xy} - 2H\big(\sqrt{1 - v_x^2 - v_y^2}\big)^3. \tag{2-8}$$

Define $r(\tau)$, $s(\tau)$, $t(\tau)$, $p(\tau)$, $q(\tau)$ for $0 \leq \tau \leq 1$ by

$$\begin{aligned} &r(\tau) = (1 - \tau)v_{xx} + \tau u_{xx}, \quad s(\tau) = (1 - \tau)v_{xy} + \tau u_{xy}, \\ &t(\tau) = (1 - \tau)v_{yy} + \tau u_{yy}, \quad p(\tau) = (1 - \tau)v_x + \tau u_x, \\ &q(\tau) = (1 - \tau)v_y + \tau u_y, \end{aligned} \tag{2-9}$$

and consider the function

$$F = F(\tau) = (1 - q^2)r + 2pqs + (1 - p^2)t - 2H\big(\sqrt{1 - p^2 - q^2}\big)^3. \tag{2-10}$$

Then we get

$$\begin{aligned} 0 = F(1) - F(0) &= \int_0^1 \frac{\partial F}{\partial \tau}\, d\tau \\ &= a_{11}w_{xx} + 2a_{12}w_{xy} + a_{22}w_{yy} + b_1 w_x + b_2 w_y, \end{aligned} \tag{2-11}$$

with

$$a_{11}=\int_0^1(1-q^2)\,d\tau,\quad a_{12}=\int_0^1 pq\,d\tau,\quad a_{22}=\int_0^1(1-p^2)\,d\tau,\tag{2-12}$$
$$b_1=-2\int_0^1\big[(pt-qs)-3H\sqrt{1-p^2-q^2}\,p\big]\,d\tau,$$
$$b_2=-2\int_0^1\big[(qr-ps)-3H\sqrt{1-p^2-q^2}\,q\big]\,d\tau.$$

Since $Dw=0$ at $(0,0)$, there exists a neighborhood, say $O(0,0)$, such that (p,q) stays in the unit ball, i.e., $p^2+q^2<1$ over $O(0,0)$. Therefore, we have

$$\begin{aligned}a_{12}^2=\Big(\int_0^1 pq\,d\tau\Big)^2&\le\int_0^1(p^2)\,d\tau\int_0^1(q^2)\,d\tau\\&<\int_0^1(p^2)\,d\tau\int_0^1(1-p^2)\,d\tau\\&<\int_0^1(1-q^2)\,d\tau\int_0^1(1-p^2)\,d\tau=a_{11}a_{22}.\end{aligned}\tag{2-13}$$

Hence, w satisfies a homogeneous elliptic equation

$$Lw=a_{11}w_{xx}+2a_{12}w_{xy}+a_{22}w_{yy}+b_1w_x+b_2w_y,\tag{2-14}$$

in $O(0,0)$.

Now, we transform (x,y) into (ξ,η) such that $\xi(0,0)=0$ and $\eta(0,0)=0$ and at $(0,0)$

$$Lw=\Big(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+b_1'\frac{\partial}{\partial\xi}+b_2'\frac{\partial}{\partial\eta}\Big)w.\tag{2-15}$$

Since the coefficient of Lw and w itself are analytic in (x,y) as well as in (ξ,η), we have the expansion around $(\xi,\eta)=(0,0)$ as follows,

$$\begin{aligned}Lw=\Big\{&(1+\alpha_{11}\xi+\beta_{11}\eta+O(\xi^2+\eta^2))\frac{\partial^2}{\partial\xi^2}+2(\alpha_{12}\xi+\beta_{12}\eta+O(\xi^2+\eta^2))\frac{\partial^2}{\partial\xi\partial\eta}\\&+(1+\alpha_{22}\xi+\beta_{22}\eta+O(\xi^2+\eta^2))\frac{\partial^2}{\partial\eta^2}+(\gamma_1+\delta_1\xi+\lambda_1\eta+O(\xi^2+\eta^2))\frac{\partial}{\partial\xi}\\&+(\gamma_2+\delta_2\xi+\lambda_2\eta+O(\xi^2+\eta^2))\frac{\partial}{\partial\eta}\Big\}w.\end{aligned}$$

By Theorem I in [Bers 1955], we know

$$w=w(\xi,\eta)=P_n(\xi,\eta)+o(\xi^2+\eta^2)^{n/2},\tag{2-16}$$

where $P_n(\xi,\eta)$ is a nonzero harmonic homogeneous polynomial in (ξ,η) of degree n. We know $n\ge 3$, as u and v have a second contact at $(0,0)$. Thus the argument in page 82 of [Axler et al. 2001] tells us

$$P_n(\xi, \eta) = \mathrm{Re}(\lambda \cdot (\xi + \eta i)^n), \tag{2-17}$$

where λ is a complex number. This, together with the expansion above, completes the proof. □

Let $u - v$ to be defined on $D \in \mathbb{R}^2$ and Z be the zero set of $u - v$ extended to the closure $\bar{D}$ of D. By Lemma 2.3, Z divides a neighborhood U of $(0, 0)$ into at least six components on which the sign of $u - v$ alternate. However, Lemma 2.3 does not tell us that $Z \cap U$ is a union of smooth arcs intersecting at $(0, 0)$. We do not know if Z may contain cusps at $(0, 0)$. To exclude such irregular possibilities, we need a lemma from Chen and Huang:

Lemma 2.4 [Chen and Huang 1982, Lemma 2]. *Let $f = f(x, y)$ be a nonconstant solution of a homogeneous quasilinear elliptic equation of the form*

$$Lf = a_{11} f_{xx} + 2a_{12} f_{xy} + a_{22} f_{yy} + b_1 f_x + b_2 f_y = 0 \tag{2-18}$$

in Ω having analytic coefficients the a_{ij} and b_k in x, y and involving no zero order term. Then every interior critical point of f is an isolated critical point.

Using the previous two lemmas as well as the implicit function theorem, we see that the zero set $Z \cap U$ of $u - v$ consists of at least three smooth arcs intersecting at $(0, 0)$ and dividing U into at least six sectors. Furthermore, the zero set Z is globally a union of smooth arcs.

3. Nonuniqueness of the minimal point

In this section, by using Lemmas 2.3 and 2.4, we will prove a sufficient and necessary condition for the nonuniqueness of minimal points of the solutions v_t ($t \in [0, 1]$) to

$$\begin{cases} \operatorname{div} \dfrac{Dv}{\sqrt{1 - t^2|Dv|^2}} = 2H, \quad t|Dv| < 1 & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

Let $u_t = t v_t$ for $t > 0$. Then u_t satisfies

$$\begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1 - |Du|^2}} = 2tH, \quad |Du| < 1 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3-2}$$

Proposition 3.1. *There always exists a unique solution v_t to (3-1) satisfying*

$$t|Dv_t| \le 1 - \theta_0 < 1, \quad \textit{in } \bar{\Omega}, \quad \|v_t\|_{C^{2,\alpha}(\bar{\Omega})} \le C, \quad \textit{for all } t \in [0, 1], \tag{3-3}$$

where C, θ_0, α are positive constants independent of t.

Proof. By Theorem 3.6 in [Bartnik and Simon 1982], Theorem 13.8 in [Gilbarg and Trudinger 1983] and Theorem 12.2.2 in [López 2013], there is a unique solution $u_t \in C^{2,\alpha}(\overline{\Omega})$ to the problem (3-2) with

$$|Du_t| < 1-\theta_0 < 1 \quad \text{in } \overline{\Omega} \qquad \text{and} \qquad \|u_t\|_{C^{2,\alpha}(\overline{\Omega})} \le C, \tag{3-4}$$

where C, θ_0, α are positive constants independent of t.

Put $v_t = t^{-1}u_t$. Then v_t satisfies (3-1). By putting

$$b(x) = (1-|Du_t|^2)^{-1/2}, \tag{3-5}$$

we regard v_t as a unique solution to the linear elliptic Dirichlet problem:

$$\begin{cases} \operatorname{div}(b(x)Dv_t) = 2H & \text{in } \Omega, \\ v_t = 0 & \text{on } \partial\Omega. \end{cases} \tag{3-6}$$

In view of (3-4), using the Schauder global estimate (see Theorem 6.6 in [Gilbarg and Trudinger 1983]), we get

$$\|v_t\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\sup_\Omega |v_t| + 2H). \tag{3-7}$$

Also, it follows from their Theorem 3.7 that

$$\sup_\Omega |v_t| \le C. \tag{3-8}$$

Therefore, we get (3-3) for $t \in (0,1]$. When $t = 0$, (3-1) is a linear problem. Hence there exists a unique solution $v_0 \in C^\infty(\overline{\Omega})$ to (3-1). This completes the proof. $\square$

Before proving the sufficient and necessary condition for nonuniqueness of the minimal point of v_t, we need the following lemmas.

Lemma 3.2. *Let t belong to* $(0,1]$. *If $Dv_t = 0$ at some point $p \in \Omega$, then the Gaussian curvature $K_t(p)$ of the graph $\Sigma_{v_t} = (x, y, v_t(x, y))$ at p does not vanish.*

Proof. Since t is positive, it suffices to show this for $u_t = tv_t$. Recall that graph of u_t has constant mean curvature tH. Let p be a critical point of u_t with $K_t(p) = 0$.

Consider the upper connected component of a hyperbolic cylinder in $\mathbb{L}^3$, S, with radius $r = 1/(2tH)$, tangent to Σ_{u_t} at p and such that the line generators are parallel to the zero principal curvature direction of Σ_{u_t} at p. Recall that each connected component of a hyperbolic cylinder is an entire graph over $\mathbb{R}^2$ with constant mean curvature $|H| = 1/(2r)$ and zero Gaussian curvature.

In general, the intersection of S and $\mathbb{R}^2$ should be a branch of a hyperbola or two parallel lines. In our case, it should be the latter one, as S touches u_t at its critical point p. Hence, $S \cap \mathbb{R}^2$ divides $\mathbb{R}^2$ into three domains, and suppose that the piece of S with negative height is the graph of a function $v \in C^\infty(\Omega')$, $v < 0$.

Define $D = \Omega \cap \Omega'$. On the one hand, by the convexity of Ω, we see $\partial(\Omega \cap \Omega')$ consists of at most four arcs, each of which belongs to $\partial\Omega$ or $\partial\Omega'$ alternatively. Consider $A = \{(x, y) \in \Omega \cap \Omega' \mid u_t(x, y) > v(x, y)\}$. Since $u_t = 0$ on $\partial\Omega$ and $v = 0$ on $\partial\Omega'$, there are at most two components of A, each of which meets the boundary $\Omega \cap \Omega'$. On the other hand, by the previous construction, u_t and v have a second order contact at p. Lemma 2.3 and Lemma 2.4 tell us A has at least three components, each of which meets $\Omega \cap \Omega'$. Thus we get a contradiction. This completes the proof. □

Now, we see that there is no critical point of v_t with Gaussian curvature vanishing for $t \in (0, 1]$. What about the case of $t = 0$?

Lemma 3.3. *Every critical point p of v_0 is a minimal point, i.e., the Gaussian curvature $K_0(p)$ of the graph Σ_{v_0} is negative at p.*

Proof. Let p be a critical point of v_0. Then $K_0(p) = -((v_0)_{xx}(v_0)_{yy} - (v_0)_{xy}^2)$ by the second equation of (2-5). Suppose that $K_0(p) \geq 0$. For simplicity, by translation and rotation of the coordinates, we may assume that $p = (0, 0)$ and $[D_{ij}v_0] = \operatorname{diag}[\lambda_1, \lambda_2]$, where $\lambda_1 + \lambda_2 = 2H > 0$, $\lambda_1 > 0$ and $\lambda_2 \leq 0$. Then $v_0(x, y) = w(x, y) + P(x, y)$, where $w(x, y) = v_0(0, 0) + \frac{1}{2}\lambda_1 x^2 + \frac{1}{2}\lambda_2 y^2$ and $P(x, y)$ is a harmonic function in Ω. Consider

$$A = \{(x, y) \in \Omega \mid P(x, y) > 0\}, \quad B = \{(x, y) \in \Omega \mid P(x, y) < 0\}. \tag{3-9}$$

Since $P(x, y)$ vanishes up to second order derivatives at $(0, 0)$ and $P(x, y)$ is real analytic, it follows from Lemma 2.3 and Lemma 2.4 that both A and B have at least three components, each of which meets the boundary $\partial\Omega$. Put

$$\Omega' = \{(x, y) \in \mathbb{R}^2 \mid w(x, y) < 0\}. \tag{3-10}$$

Since Ω is convex and w is a quadratic function with $\lambda_1 > 0$ and $\lambda_2 \leq 0$, we see that $\partial(\Omega \cap \Omega')$ consists of at most four arcs each of which belongs to $\partial\Omega$ or $\partial\Omega'$ alternatively. Let $A' = \{(x, y) \in \Omega \cap \Omega' \mid P(x, y) > 0\}$. Since $v_0 = 0$ on $\partial\Omega$ and $w = 0$ on $\partial\Omega'$, there are at most two components of A' each of which meets the boundary $\partial(\Omega \cap \Omega')$. This contradicts the fact that both A and B have at least three components which meet the boundary of $\partial\Omega$. This completes the proof. □

Now, we can prove the sufficient and necessary condition for nonuniqueness of the minimal point of v_t.

Theorem 3.4. *Let t belong to* $[0, 1]$. *The solution v_t has more than two minimal points if and only if there exists a saddle point $p \in \Omega$, i.e., $Dv_t(p) = 0$ and $K_t(p) > 0$.*

Proof. It follows from Hopf's boundary point lemma that $Dv_t \cdot \nu$ is positive on $\partial\Omega$. There v_t does not have minimal point on the boundary $\partial\Omega$.

“If” part: Let $p \in \Omega$ be a point with $Dv_t(p) = 0$ and $K_t(p) > 0$. Then there exists an open neighborhood U of p in which the zero set of $\tilde{v}_t = v_t - v_t(p)$ consists of two smooth arcs intersecting at p and divides U into four sections. Consider the open set $E = \{(x, y) \in \Omega \mid \tilde{v}_t > 0\}$. It follows from the maximum principle that each component of E has to meet the boundary $\partial\Omega$. Accordingly, we see that the open set $G = \{(x, y) \in \Omega \mid \tilde{v}_t < 0\}$ has more than two components. This shows that v_t has more than two minimal points.

“Only if” part: Suppose that v_t has more than two minimal points and there is no point p with $Dv_t(p) = 0$ and $K_t(p) > 0$. By Lemma 3.2 and Lemma 3.3, we see that each critical point of v_t is a minimal point. Since Dv_t does not vanish on $\partial\Omega$, then Lemma 3.2 and Lemma 3.3 imply that every critical point of v_t is isolated and the number of critical (minimal) points is finite, say $\{P_1, \dots, P_N\}$. Hence, we have

$$Dv_t(x, y) \neq 0, \quad \text{for all } (x, y) \in \Omega - \{P_1, \dots, P_N\}. \tag{3-11}$$

Put $m_0 = \max\{v_t(P_j) \mid 1 \leq j \leq N\}$. Consider the level set $L_m = \{(x, y) \in \Omega \mid v_t(x, y) < m\}$ for $m_0 < m < 0$. It follows from (3-11) and Theorem 3.1 in [Milnor 1963] that the boundary ∂L_m is a smooth manifold for $m_0 < m < 0$ and $\{\partial L_m\}$ are diffeomorphic to each other. Since $K_t(P_j)$ is negative, if m is near m_0, L_m has more than two components. On the other hand, if m is near 0, ∂L_m is diffeomorphic to $\partial\Omega$ and L_m is connected. This is a contradiction, so the proof is complete. □

Now, Lemma 3.2, Lemma 3.3 and Theorem 3.4 tell us the nonexistence of the critical point described in the first question of the first section is equivalent to the uniqueness for the critical point of the solution to (1-10), which will be proved in the next section.

4. Proof of Theorem 1.1

In view of Lemma 3.2, Lemma 3.3 and Theorem 3.4, it suffices to show that the set of minimal points of the solution consists of only one point. Put $I = [0, 1]$. Divide I into two sets I_1 and I_2 as follows:

$$\begin{aligned} I_1 &= \{t \in I \mid v_t \text{ has only one minimal point in } \Omega\}, \\ I_2 &= \{t \in I \mid v_t \text{ has more than two minimal points in } \Omega\}. \end{aligned} \tag{4-1}$$

Then $I = I_1 + I_2$ and $I_1 \cap I_2 = \varnothing$. Lemma 3.3 and Theorem 3.4 imply that $0 \in I_1$, so I_1 is not empty.

On the one hand, I_2 is open in I. That is, for any $t_0 \in I_2$, there exists a constant $\varepsilon > 0$ such that $(t_0 - \varepsilon, t_0 + \varepsilon) \subset I_2$. If it were not so, we can assume that there exists a sequence of solutions $\{v_{t_n}\}$ with only one minimal point and $t_n \in (t_0 - 1/n, t_0 + 1/n)$ for some positive $t_0 \in I_2$. By Lemma 3.2 and Theorem 3.4, v_{t_n} has only one critical point. By compactness and Lemma 3.2, we can take a subsequence of v_{t_n} such that

$$p_n \to p, \quad Dv_{t_n}(p_n) = 0, \quad K_{t_n}(p_n) < 0, \quad Dv_{t_0}(p) = 0, \quad K_{t_0}(p) < 0. \tag{4-2}$$

Since $t_0 \in I_2$, there exists another point $q \in U(q) \subseteq \Omega$ such that

$$q_n \to q, \quad Dv_{t_n}(q_n) \to Dv_{t_0}(q) = 0. \tag{4-3}$$

By uniqueness of the critical point of v_{t_n}, we can take a subsequence of $\{v_{t_n}\}$ such that v_{t_n} are all monotone in the line $l(p_n, q_n)$. Then there exists a sequence of points $\{s_n \mid s_n \in l(p_n, q_n)\}$ such that

$$|Dv_{t_n}(s_n)| \le |Dv_{t_n}(q_n)| \to 0, \quad |K_{t_n}(s_n)| = \frac{|Dv_{t_n}(q_n)|}{|p_n - q_n|} \to 0. \tag{4-4}$$

Therefore, there should be a point $s \in l(p, q)$ which satisfies

$$Dv_{t_0}(s) = 0, \quad K_{t_0}(s) = 0. \tag{4-5}$$

This is a contradiction with Lemma 3.2.

On the other hand, I_2 is closed in I. In fact, let $\{t_j\}$ be a sequence of points in I_2 such that t_j converges to t_0 as j goes to ∞. Theorem 3.4 and the compactness imply that there exists a subsequence $\{t_k\}$, a sequence $\{p_k\}$ and a point $p \in \Omega$ such that

$$p_k \to p \quad \text{as } k \to \infty, \quad Dv_{t_k}(p_k) = 0, \quad \text{and} \quad K_{t_k}(p_k) > 0. \tag{4-6}$$

By continuity, we have

$$Dv_{t_0}(p) = 0, \quad \text{and} \quad K_{t_0}(p) \ge 0. \tag{4-7}$$

Since $Dv_{t_0} \ne 0$ on $\partial\Omega$, $p \in \Omega$. Therefore it follows from Lemma 3.2 and Lemma 3.3, Theorem 3.4 and (4-7) that $t_0 \in I_2$. This shows that I_2 is closed in I.

Hence, I_2 must be $\varnothing$ or I. Since I_1 is not $\varnothing$, $I_1 = I$. This completes the proof.

5. Sharp C^0 and C^1 estimates

In [Payne and Philippin 1979], the authors derived a maximum principle for a function $\Phi(x; \alpha)$ defined by

$$\Phi(x; \alpha) = \int_0^{q^2} \frac{g(\xi) + 2\xi g'(\xi)}{\rho(\xi)}\, d\xi + \alpha \int_0^u f(\eta)\, d\eta, \tag{5-1}$$

where $g > 0$, $\rho > 0$, f are functions and u satisfies the following elliptic equation:

$$\sum_i (g(q^2) u_i)_i + \rho(q^2) f(u) = 0, \quad q^2 = \sum_i u_i u_i = |Du|^2. \tag{5-2}$$

In our case, we can take $g(\xi)=(1-\xi)^{-1/2}$, $\rho=1$, $f=-2H$. Then

$$\Phi(x;\alpha)=2\left(\frac{1}{\sqrt{1-|Du|^2}}-1-\alpha Hu\right). \tag{5-3}$$

In particular, $\Phi := \Phi(x;1)=2(1/\sqrt{1-|Du|^2}-1-Hu)$.

Theorem 4 in [Payne and Philippin 1979] gives us

$$\sum_{i,j}\left(\delta_{ij}+\frac{u_iu_j}{1-|Du|^2}\right)\Phi_{ij}+\sum_k W_k\Phi_k\geq 0, \tag{5-4}$$

where $W_k s$ are just the components of a vector function uniformly bounded in Ω. It follows that $\Phi(x;1)$ takes its maximum value on $\partial\Omega$. Together with (5-1), we know $\Phi(x;1)$ takes its maximum value where $|Du|^2=\max_{\partial\Omega}|Du|^2$. It follows that, at any point $x\in\Omega$, we have

$$-Hu\leq\frac{1}{\sqrt{1-\max_{\partial\Omega}|Du|^2}}-\frac{1}{\sqrt{1-|Du|^2}}. \tag{5-5}$$

So, at the critical point, we get

$$-Hu_{\min}\leq\frac{1}{\sqrt{1-q_{\max}^2}}-1, \tag{5-6}$$

where $u_{\min}=\min_\Omega u$ and $q_{\max}=\max_{\partial\Omega}|Du|$.

Now, we want to derive the upper bound for $|Du|^2_{\max}$. Suppose $\Phi(x;\alpha)$ attains its maximum at $p\in\partial\Omega$. Then $|Du|(p)=q_{\max}$. On the one hand, by the strong maximum principle, we have at p,

$$\frac{\partial\Phi(x;\alpha)}{\partial\nu}=2\frac{g+2q^2g'}{\rho}u_\nu u_{\nu\nu}+fu_\nu\geq 0, \tag{5-7}$$

where $\partial/\partial\nu$ or a subscript ν denotes the outward directed normal derivative on $\partial\Omega$ and the equality holds if and only if $\Phi(x;\alpha)=\text{constant}$. On the other hand, making use of (5-2) evaluated on $\partial\Omega$, we have

$$(g+2q^2g')u_{\nu\nu}+gKu_\nu+\rho f=0. \tag{5-8}$$

Together with (5-7), this leads to

$$\frac{\partial\Phi(x;\alpha)}{\partial\nu}=-(2Kgu_\nu^2+fu_\nu)\geq 0. \tag{5-9}$$

Applying to our case, we get

$$\frac{q_{\max}}{\sqrt{1-q_{\max}^2}}\leq\frac{H}{K(p)}\leq\frac{H}{K_{\min}}. \tag{5-10}$$

So

$$q_{\max}^2 \leq \frac{H^2}{H^2+K_{\min}^2}. \tag{5-11}$$

Therefore, the left inequality in (1-14) follows from (5-6) and (5-11). And the equality holds if and only if the boundary is a circle. In fact, if the equality holds, then $\Phi(x;1) = \text{constant}$ on $\partial\Omega$ from the strong maximum principle. From (5-1), $u_\nu = \text{constant}$ on $\partial\Omega$. So $\partial\Omega$ is a circle according to Theorem 2 and Remark 1 in [Serrin 1971]. Conversely, if $\partial\Omega$ is a circle, then the solution u is radially symmetric. So $u_\nu = \text{constant}$ on $\partial\Omega$, and then the equality in (5-11) follows from the divergence theorem.

To derive the upper bound of $u_{\min}$ in the same way above, we need a minimum principle for $\Phi(x;1)$. First, we need the following lemma.

Lemma 5.1 [Payne and Philippin 1979].

$$\sum_{i,j}\Bigl(\delta_{ij}+\frac{u_iu_j}{1-|Du|^2}\Bigr)\Phi_{ij}(x,\alpha)+\sum_k \widehat{W_k}\Phi_k(x,\alpha) = 4H^2(\alpha-1)(\alpha-2)\frac{1}{\sqrt{1-|Du|^2}}, \tag{5-12}$$

where $\widehat{W_k}$s are the components of a vector function which is singular at the critical point of u.

From Lemma 5.1 and the Hopf maximum principle, we conclude that $\Phi(x;\alpha)$ takes its minimum value either on the boundary $\partial\Omega$, or at the unique critical point of u in Ω when $\alpha \in [1,2]$. What if the second alternative happens? We answer this in the following theorem whose Euclidean version was proved by Xi-Nan Ma [2000]:

Theorem 5.2. *Let $u \in C^\infty(\overline{\Omega})$ be a solution to (1-10). If $\Phi(x;1)$ attains its minimum at the unique critical point in Ω, then $\Phi(x;1)$ is a constant on $\overline{\Omega}$.*

By Theorem 5.2, we assume $\Phi(x;1)$ takes its minimum at $p' \in \partial\Omega$, then $|Du|(p') = q_{\min} = \min_{\partial\Omega}|Du|$ and

$$-Hu_{\min} \geq \frac{1}{\sqrt{1-q_{\min}^2}}-1, \tag{5-13}$$

and

$$\frac{\partial\Phi}{\partial\nu}(p';1) \leq 0, \tag{5-14}$$

where the equality holds if and only if $\Phi(x;1) = \text{constant}$. As before, one can also get

$$\frac{q_{\min}}{\sqrt{1-q_{\min}^2}} \geq \frac{H}{K(p')} \geq \frac{H}{K_{\max}}. \tag{5-15}$$

So

$$q_{\min}^2 \geq \frac{H^2}{H^2+K_{\max}^2}, \tag{5-16}$$

where the equality holds if and only if the boundary is a circle. Therefore, the right inequality in (1-14) follows from (5-13) and (5-16).

For completeness, we will prove Theorem 5.2 to end this paper. Our proof is similar to that in [Ma 2000] except for the different signs in some places.

Proof of Theorem 5.2. The proof consists of four steps. Assume the unique critical point to be $P \in \Omega$.

Step 1: Derivatives of Φ up to the second order vanish at P. From the proof of Theorem 1.1, we can choose the coordinates at P such that

$$u_1(P) = u_2(P) = 0, \quad u_{11} > 0, \quad u_{22} > 0, \quad u_{12} = 0. \tag{5-17}$$

By direct computation, we have

$$\Phi_1 = 2v^{-\frac{3}{2}} u_i u_{i1} - 2Hu_1 = 0, \tag{5-18}$$

$$\Phi_2 = 2v^{-\frac{3}{2}} u_i u_{i2} - 2Hu_2 = 0, \tag{5-19}$$

$$\begin{aligned}\Phi_{11} &= \tfrac{3}{2}v^{-\frac{5}{2}}(2u_iu_{i1})(2u_ju_{j1}) + 2v^{-\frac{3}{2}}u_{i1}^2 + 2v^{-\frac{3}{2}}u_iu_{i11} - 2Hu_{11} \\ &= 2u_{11}^2 - 2Hu_{11},\end{aligned} \tag{5-20}$$

$$\begin{aligned}\Phi_{12} &= \tfrac{3}{2}v^{-\frac{5}{2}}(2u_iu_{i1})(2u_ju_{j2}) + 2v^{-\frac{3}{2}}u_{i1}u_{i2} + 2v^{-\frac{3}{2}}u_iu_{i12} - 2Hu_{12} \\ &= 0,\end{aligned} \tag{5-21}$$

$$\begin{aligned}\Phi_{22} &= \tfrac{3}{2}v^{-\frac{5}{2}}(2u_iu_{i2})(2u_ju_{j2}) + 2v^{-\frac{3}{2}}u_{i2}^2 + 2v^{-\frac{3}{2}}u_iu_{i22} - 2Hu_{22} \\ &= 2u_{22}^2 - 2Hu_{22},\end{aligned} \tag{5-22}$$

where $v = 1 - |Du|^2$. Since Φ attains its minimum at P, we get

$$\Phi_{11}(P)\Phi_{22}(P) - \Phi_{12}(P) \geq 0. \tag{5-23}$$

Together with (5-17), we know

$$u_{11}(P) = u_{22}(P) = H, \tag{5-24}$$

and

$$\Phi_{11}(P) = \Phi_{22}(P) = 0. \tag{5-25}$$

Step 2: Derivatives of Φ up to the fifth order vanish at P. First we claim

$$\Phi_{x_1^k x_2^{3-k}}(P) = 0, \quad k = 0, 1, 2, 3. \tag{5-26}$$

By (5-17), (5-24) and direct calculations, we have

$$\begin{aligned} \Phi_{x_1^3}(P) &= 4Hu_{x_1^3}, & \Phi_{x_1^2 x_2}(P) &= 4Hu_{x_1^2 x_2}, \\ \Phi_{x_1 x_2^2}(P) &= 4Hu_{x_1 x_2^2}, & \Phi_{x_2^3}(P) &= 4Hu_{x_2^3}. \end{aligned} \tag{5-27}$$

Now, by differentiating (1-10), we obtain

$$u_{x_1^3} = -u_{x_1^2 x_2} \quad \text{and} \quad u_{x_1 x_2^2} = -u_{x_2^3}. \tag{5-28}$$

Together with (5-18), (5-19), (5-21) and (5-25), we can expand Φ in a neighborhood of P:

$$\Phi(x_1, x_2; 1) - \Phi(P; 1) = \frac{r^3}{3!}\big(\Phi_{x_1^3}(P)\cos(3\phi) + \Phi_{x_1^2 x_2}(P)\sin(3\phi)\big) + O(r^4), \tag{5-29}$$

where (r, ϕ) are polar coordinates. Suppose

$$\sqrt{(\Phi_{x_1^3}(P))^2 + (\Phi_{x_1^2 x_2}(P))^2} \neq 0. \tag{5-30}$$

Then (5-29) becomes

$$\Phi(x_1, x_2; 1) - \Phi(P; 1) = A_3(P)\cos[3\phi - \beta_3] r^3 + O(r^4), \tag{5-31}$$

with

$$A_3(P) = \frac{\sqrt{(\Phi_{x_1^3}(P))^2 + (\Phi_{x_1^2 x_2}(P))^2}}{3!}, \quad \cos\beta_3 = \frac{\Phi_{x_1^3}(P)}{\sqrt{(\Phi_{x_1^3}(P))^2 + (\Phi_{x_1^2 x_2}(P))^2}}, \tag{5-32}$$

$$\sin\beta_3 = \frac{\Phi_{x_1^2 x_2}(P)}{\sqrt{(\Phi_{x_1^3}(P))^2 + (\Phi_{x_1^2 x_2}(P))^2}}.$$

From (5-31) we conclude that Φ has at least three nodal lines forming equal angles at P, but Lemma 5.1 tells us that Φ takes its minimum value only on $\partial\Omega$ or at P, which is a contradiction. Thus $A_3(P) = 0$. That is,

$$\Phi_{x_1^k x_2^{3-k}}(P) = 0, \quad k = 0, 1, 2, 3, \tag{5-33}$$

and

$$u_{x_1^k x_2^{3-k}}(P) = 0, \quad k = 0, 1, 2, 3. \tag{5-34}$$

Using a similar argument we can show

$$
\begin{aligned}
0 &= \Phi_{x_1^4}(P) = 6H(u_{x_1^4}(P) + 3H^3) \\
&= -\Phi_{x_1^2x_2^2}(P) = 6H(u_{x_1^2x_2^2}(P) + H^3) \\
&= \Phi_{x_2^4}(P) = 6H(u_{x_2^4}(P) + 3H^3),
\end{aligned} \tag{5-35}
$$

$$
0 = \Phi_{x_1^3x_2}(P) = 6Hu_{x_1^3x_2}(P) = -\Phi_{x_1x_2^3}(P) = 6Hu_{x_1x_2^3}(P), \tag{5-36}
$$

$$
\begin{aligned}
&u_{x_1^4}(P) = u_{x_2^4}(P) = -3H^3, \quad u_{x_1^3x_2}(P) = u_{x_1x_2^3}(P) = 0, \\
&u_{x_1^2x_2^2}(P) = -H^3,
\end{aligned} \tag{5-37}
$$

$$
\Phi_{x_1^kx_2^{5-k}}(P) = u_{x_1^kx_2^{5-k}}(P) = 0, \quad k = 0, 1, 2, 3, \tag{5-38}
$$

and

$$
\Phi_{x_1^5}(P) = -\Phi_{x_1^3x_2^2}(P) = \Phi_{x_1x_2^4}(P), \quad \Phi_{x_1^4x_2}(P) = -\Phi_{x_1^2x_2^3}(P) = \Phi_{x_2^5}(P). \tag{5-39}
$$

Step 3: Now we assume all derivatives of Φ up to the n-th order vanish at P, where $n \geq 5$. Using the same argument as in the previous step, we have the following relations.

If $n = 2l,\ l \geq 3$. Then

$$
u_{x_1^mx_2^{k-m}}(P) = u_{x_1^{k-m}x_2^m}(P) \qquad \text{for any } m = 0, 1, 2, \dots, k, \ \text{if } k = 5, 6, 8, \dots, 2l, \tag{5-40}
$$

$$
u_{x_1^mx_2^{k-m}}(P) = 0 \qquad \text{for any } m = 0, 1, 2, \dots, k, \ \text{if } k = 5, 7, 9, \dots, 2l-1, \tag{5-41}
$$

$$
u_{x_1^mx_2^{2p-m}}(P) = 0 \qquad \text{for any } m = 1, 3, 5, \dots, 2p-1, \ \text{if } p = 3, 4, 5, \dots, l, \tag{5-42}
$$

and

$$
u_{x_1^{2p}}(P) = (-1)^{p+1}(2p-1)[(2p-3)(2p-5)\cdots 1]^2 H^{2p-1} \tag{5-43}
$$

for any $p = 3, 4, 5, \dots, l$. When l is even, we have for any $p = 4, 6, 8, \dots, l$

$$
\frac{u_{x_1^{2p}}}{u_{x_1^{2p-2}x_2^2}}(P) = 2p-1, \quad \frac{u_{x_1^{2p-2}x_2^2}}{u_{x_1^{2p-4}x_2^4}}(P) = \frac{2p-3}{3}, \cdots, \frac{u_{x_1^{p+2}x_2^{p-2}}}{u_{x_1^px_2^p}}(P) = \frac{p+1}{p-1}, \tag{5-44}
$$

and for any $p = 3, 5, 7, \dots, l-1$, we have

$$
\frac{u_{x_1^{2p}}}{u_{x_1^{2p-2}x_2^2}}(P) = 2p-1, \quad \frac{u_{x_1^{2p-2}x_2^2}}{u_{x_1^{2p-4}x_2^4}}(P) = \frac{2p-3}{3}, \cdots, \frac{u_{x_1^{p+3}x_2^{p-3}}}{u_{x_1^{p+1}x_2^{p-1}}}(P) = \frac{p+2}{p-2}. \tag{5-45}
$$

When l is odd, we have similar relations to (5-44) and (5-45).

If $n = 2l+1,\ l \geq 2$, by a similar argument we have (5-40)–(5-45) and

$$u_{x_1^m x_2^{2l+1-m}}(P) = 0, \quad \text{for any } m = 0, 1, 2, \ldots, 2l+1. \tag{5-46}$$

Step 4: Derivatives of Φ of order n+1 vanish at P. We divide this step into two parts according to whether n is odd or even.

Case A: If $n = 2l,\ l \geq 3$. By the inductive assumption, we have

$$v_{x_1^m x_2^{k-m}}(P) = 0 \quad \text{for any } m = 0, 1, 2, \ldots, k, \quad \text{if } k = 1, 3, 5, \ldots, n-1. \tag{5-47}$$

Then for any $m = 0, 1, 2, \ldots, n+1$,

$$\begin{aligned}(2v^{-\frac{1}{2}})_{x_1^m x_2^{n+1-m}}(P) &= -v^{\frac{3}{2}} v_{x_1^m x_2^{n+1-m}}(P) \\ &= 2v^{-\frac{3}{2}}((n+1-m)Hu_{x_1^m x_2^{n+1-m}} + mHu_{x_1^m x_2^{n+1-m}}) \\ &= 2(n+1)Hu_{x_1^m x_2^{n+1-m}}.\end{aligned} \tag{5-48}$$

So

$$\Phi_{x_1^m x_2^{n+1-m}}(P) = 2nHu_{x_1^m x_2^{n+1-m}}(P). \tag{5-49}$$

Now, by differentiating (1-10), we obtain

$$u_{x_1^m x_2^{n+1-m}}(P) = -u_{x_1^{m+2} x_2^{n-1-m}}(P), \quad \text{for } m = 0, 1, 2, \ldots, n+1. \tag{5-50}$$

Then

$$\Phi_{x_1^m x_2^{n+1-m}}(P) = -\Phi_{x_1^{m+2} x_2^{n-1-m}}(P), \quad \text{for } m = 0, 1, 2, \ldots, n+1. \tag{5-51}$$

Using Taylor expansion as in Step 2, we can conclude that the derivatives of Φ of order $n+1$ vanish at P.

Case B: If $n = 2l+1,\ l \geq 2$, so $n+1 = 2(l+1)$ is even. We first look for the relations among $\Phi_{x_1^m x_2^{n+1-m}}(P)$, where $m = 0, 2, 4, \ldots, n+1$. Through computations, we have

$$\Phi_{x_1^{n+1}}(P) = 2nH(u_{x_1^{n+1}} + (-1)^{l+1}(2l+1)[(2l-1)(2l-3)\cdots 1]^2 H^{2l+1}), \tag{5-52}$$

and

$$\Phi_{x_1^{n-1} x_2^2}(P) = 2nH(u_{x_1^{n-1} x_2^2} + (-1)^{l+1}[(2l-1)(2l-3)\cdots 1]^2 H^{2l+1}). \tag{5-53}$$

Now, by differentiating (1-10), we get

$$(\Delta u + u_i u_j u_{ij} v^{-1})_{x_1^{n-1}}(P) = (2Hv_{\frac{1}{2}})_{x_1^{n-1}}(P). \tag{5-54}$$

Together with the relations in Step 3, this leads to

$$u_{x_1^{n+1}} + u_{x_1^{n-1} x_2^2} = (n+1)(-1)^l[(2l-1)(2l-3)\cdots 1]^2 H^{2l+1}. \tag{5-55}$$

So

$$\Phi_{x_1^{n+1}}(P) = -\Phi_{x_1^{n-1}x_2^2}(P). \tag{5-56}$$

By a similar argument, it follows that

$$\Phi_{x_1^m x_2^{n+1-m}}(P) = -\Phi_{x_1^{m+2} x_2^{n-1-m}}(P), \quad \text{for } m = 0, 2, 4, \ldots, n+1. \tag{5-57}$$

Then, using the same argument, we have

$$\Phi_{x_1^m x_2^{n+1-m}}(P) = -\Phi_{x_1^{m+2} x_2^{n-1-m}}(P), \quad \text{for } m = 0, 1, 2, \ldots, n+1. \tag{5-58}$$

Now, as in Case A, we can show the derivatives of Φ of order $n+1$ vanish at P.

By the unique continuation of analytic functions, we know if Φ attains its minimum at P, then it must be a constant. This completes the proof. □

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