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RESTRICTED POISSON ALGEBRAS

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We reformulate Bezrukavnikov and Kaledin's definition of a restricted Poisson algebra, provide some natural and interesting examples, and discuss connections with other research topics.

Introduction

The Poisson bracket was introduced by Poisson [1809] as a tool for classical dynamics. Poisson geometry has become an active research field during the past 50 years. The study of Poisson algebras over $\mathbb{R}$ or a field of characteristic zero [Laurent-Gengoux et al. 2013] also has a long history, and is closely related to noncommutative algebra, differential geometry, deformation quantization, number theory, and other areas. The notion of a *restricted Poisson algebra* was introduced about ten years ago in an important paper of Bezrukavnikov and Kaledin [2008] in the study of deformation quantization in positive characteristic. The project in that paper is a natural extension of the classical deformation quantization of symplectic (or Poisson) manifolds.

Our first goal is to better understand Bezrukavnikov and Kaledin's definition via a Lie-algebraic approach. We reinterpret their definition in the following way.

Throughout the paper let $\Bbbk$ be a base field of characteristic $p \geq 3$. All vector spaces and algebras are over $\Bbbk$.

Definition 0.1. Let $(A, \{-,-\})$ be a Poisson algebra over $\Bbbk$.

(1) We call A a *weakly restricted Poisson algebra* if there is a p-map operation $x \mapsto x^{\{p\}}$ such that $(A, \{-,-\}, (-)^{\{p\}})$ is a restricted Lie algebra.

(2) We call A a *restricted Poisson algebra* if A is a weakly restricted Poisson algebra and the p-map $(-)^{\{p\}}$ satisfies

$$(x^2)^{\{p\}} = 2x^p x^{\{p\}} \tag{E0.1.1}$$

for all $x \in A$.

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The formulation in (E0.1.1) is slightly simpler than the original definition. We will show that Definition 0.1(2) is equivalent to [Bezrukavnikov and Kaledin 2008, Definition 1.8] in Lemma 3.7. Generally it is not easy to prove basic properties for restricted Poisson algebras. For example, it is not straightforward to show that the tensor product preserves the restricted Poisson structure. Different formulations are helpful in understanding and proving some elementary properties.

Since there are several structures on a restricted Poisson algebra, it is delicate to verify all of the compatibility conditions. There are not many examples given in the literature. Our second goal is to provide several canonical examples from different research subjects. Restricted Poisson algebras can be viewed as a Poisson version of restricted Lie algebras, so the first few examples come from restricted (or modular) Lie theory. Let L be a restricted Lie algebra over $\Bbbk$. Then the trivial extension algebra $\Bbbk \oplus L$ (with $L^2 = 0$) is a restricted Poisson algebra. More naturally we have the following.

Theorem 0.2 (Theorem 6.5). *Let L be a restricted Lie algebra over $\Bbbk$ and let $s(L)$ be the p-truncated symmetric algebra. Then $s(L)$ admits a natural restricted Poisson structure induced by the restricted Lie structure of L.*

To use ideas from Poisson geometry, it is a good idea to extend the restricted Poisson structure to the symmetric algebra of a restricted Lie algebra (Example 6.2). The following result is slightly more general and useful in other settings.

Theorem 0.3 (Theorem 6.1). *Let T be an index set and $A = \Bbbk[x_i \mid i \in T]$ be a polynomial Poisson algebra. If, for each $i \in T$, there exists $\gamma(x_i) \in A$ such that $\mathrm{ad}_{x_i}^p = \mathrm{ad}_{\gamma(x_i)}$, then A admits a restricted Poisson structure $(-)^{\{p\}}\colon A \to A$ such that $x_i^{\{p\}} = \gamma(x_i)$ for all $i \in T$.*

The next example comes from deformation theory, which is also considered in [Bezrukavnikov and Kaledin 2008]. See (E7.0.1) for the definition of $M_n^p(f)$.

Proposition 0.4 (Proposition 7.1). *Let $(A, \cdot, \{-,-\})$ be a Poisson algebra over $\Bbbk$ and let $(A[\![t]\!], *)$ be a deformation quantization of A. If $M_n^p(f) = 0$ for $1 \le n \le p-2$ and f^p is central in $A[\![t]\!]$ for all $f \in A$, then A admits a restricted Poisson structure.*

A Lie–Rinehart algebra is an algebraic counterpart of a Lie algebroid, and appears naturally in the study of Gerstenhaber algebras, Batalin–Vilkovisky algebras and Maurer–Cartan algebras [Huebschmann 1990; 2005]. In this paper, we also study the relationship between restricted Poisson algebras and restricted Lie–Rinehart algebras. To save space we refer to [Dokas 2012] for the definition and some properties of restricted Lie–Rinehart algebras.

Theorem 0.5 (Theorem 8.2). *Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a (weakly) restricted Poisson algebra. If the module of Kähler differentials $\Omega_{A/\Bbbk}$ is free over A, then*

$(A, \Omega_{A/\Bbbk}, (-)^{[p]})$ *is a restricted Lie–Rinehart algebra, where the* p*-map of* $\Omega_{A/\Bbbk}$ *is determined by*

$$(x\,\mathrm{d}u)^{[p]} = x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u$$

for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$.

The category of restricted Poisson algebras is a symmetric monoidal category. In particular, the tensor product of two restricted Poisson algebras is again a restricted Poisson algebra (Proposition 9.2). Advances in algebra benefit tremendously from a geometric viewpoint and methods and vice versa. Restricted Poisson algebras are, to some extent, the algebraic counterpart of symplectic differential geometry in positive characteristic. Following this idea, restricted Poisson–Lie groups should correspond to restricted Poisson Hopf algebras which connect both Poisson geometry in positive characteristic and quantum groups at the root of unity. Hence, it is meaningful to introduce the notion of a restricted Poisson Hopf algebra; see Definition 9.3. One natural example of such an algebra is given in Example 9.4.

The paper is organized as follows. Sections 1 and 2 contain basic definitions about restricted Lie algebras and Poisson algebras. In Section 3, we reintroduce the notion of a restricted Poisson algebra. In Sections 4 to 7, we give several natural examples. In Section 8, we prove Theorem 0.5. The notion of a restricted Poisson Hopf algebra is introduced in Section 9.

1. Restricted Lie algebras

We give a short review of restricted Lie algebras.

Lie algebras over a field of positive characteristic often admit an additional structure involving a so-called p-map. The Lie algebra together with a p-map is called a *restricted Lie algebra*, which was first introduced and systematically studied by Jacobson [1941; 1962]. Let $L := (L, [-,-])$ be a Lie algebra over $\Bbbk$. For convenience, for each $x \in L$ we denote by $\mathrm{ad}_x\colon L \to L$ the adjoint representation given by $\mathrm{ad}_x(y) = [x, y]$ for all $y \in L$. We recall the definition of a restricted Lie algebra from [Jacobson 1941, Section 1]. As always, we assume that $\Bbbk$ is of positive characteristic $p \geq 3$.

Definition 1.1 [Jacobson 1941]. A *restricted Lie algebra* $(L, (-)^{[p]})$ over $\Bbbk$ is a Lie algebra L over $\Bbbk$ together with a *p-map* $(-)^{[p]} : x \mapsto x^{[p]}$ such that

(1) $\mathrm{ad}_x^p = \mathrm{ad}_{x^{[p]}}$ for all $x \in L$;

(2) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for all $\lambda \in \Bbbk$ and $x \in L$;

(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \Lambda_p(x, y)$, where $\Lambda_p(x, y) = \sum_{i=1}^{p-1} s_i(x, y)/i$ for all $x, y \in L$ and $s_i(x, y)$ is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}_{tx+y}^{p-1}(x)$.

For simplicity, we write all multiple Lie brackets with the notation

$$[x_1,[x_2,\dots,[x_{n-1},x_n]\dots]] =: [x_1,x_2,\dots,x_{n-1},x_n] \tag{E1.1.1}$$

for $x_1,\dots,x_n \in L$. Clearly,

$$\mathrm{ad}_x^i(y) = [\underbrace{x,\dots,x}_{i \text{ copies}},y]$$

for every i. With this notation, we have

$$s_i(x,y) = \sum_{\substack{x_k = x \text{ or } y \\ \#\{k \mid x_k = x\} = i-1}} [x_1,\dots,x_{p-2},y,x], \tag{E1.1.2}$$

and hence

$$\Lambda_p(x,y) = \sum_{\substack{x_k = x \text{ or } y \\ x_{p-1}=y,\, x_p = x}} \frac{1}{\#(x)}[x_1,\dots,x_{p-1},x_p]. \tag{E1.1.3}$$

Note that $\Lambda_p(x,y)$ is denoted by $L(x,y)$ in [Bezrukavnikov and Kaledin 2008] and denoted by $\sigma(x,y)$ in [Hochschild 1954]. Another way of understanding $\Lambda_p(x,y)$ is to use the universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebra L. By [Hochschild 1954, Condition (3) on p. 559],

$$\Lambda_p(x,y) = (x+y)^p - x^p - y^p \tag{E1.1.4}$$

for all $x, y \in L \subset \mathcal{U}(L)$, where $(-)^p$ is the multiplicative p-th power in $\mathcal{U}(L)$.

We give a well-known example which will be used later.

Example 1.2. Let A be an associative algebra over $\Bbbk$. We denote by A_L the induced Lie algebra with the bracket given by $[x,y] := xy - yx$ for all $x, y \in A$. Then $(A_L, (-)^p)$ is a restricted Lie algebra, where $(-)^p$ is the Frobenius map given by $x \mapsto x^p$.

Jacobson gave a necessary and sufficient condition in which an ordinary Lie algebra over $\Bbbk$ is restricted:

Lemma 1.3 [Jacobson 1962, Theorem 11]. *Let L be a Lie algebra with a $\Bbbk$-basis $\{x_i\}_{i\in I}$ for some index set I. Suppose that there exists an element $\gamma(x_i) \in L$ for each $i \in I$ such that*

$$\mathrm{ad}_{x_i}^p = \mathrm{ad}_{\gamma(x_i)}.$$

Then there exists a unique restricted structure on L such that $x_i^{[p]} = \gamma(x_i)$ for all $i \in I$.

2. Poisson algebras and their enveloping algebras

In this section we recall some definitions. We begin with some basics concerning Poisson algebras.

Definition 2.1 [Laurent-Gengoux et al. 2013, Definition 1.1]. Let A be a commutative algebra over $\Bbbk$. A *Poisson structure* on A is a Lie bracket $\{-,-\}\colon A \otimes A \to A$ such that the following Leibniz rule holds:

(E2.1.1) $$\{xy, z\} = x\{y, z\} + y\{x, z\} \quad \forall\, x, y, z \in A.$$

The algebra A together with a Poisson structure is called a *Poisson algebra*.

The Lie bracket $\{-,-\}$ (which replaces $[-,-]$ in the previous section) is called the *Poisson bracket*, and the associative multiplication of A is sometimes denoted by $\cdot$.

Recall that the module of Kähler differentials, denoted by $\Omega_{A/\Bbbk}$, of a commutative algebra A over $\Bbbk$ is an A-module generated by elements (or symbols) $\mathrm{d}x$ for all $x \in A$, and subject to the relations

$$\mathrm{d}(x+y) = \mathrm{d}x + \mathrm{d}y, \quad \mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x, \quad \mathrm{d}\lambda = 0,$$

where $x, y \in A$ and $\lambda \in \Bbbk \subseteq A$. When $(A, \{-,-\})$ is a Poisson algebra, the module of Kähler differentials $\Omega_{A/\Bbbk}$ admits a Lie algebra structure with Lie bracket given by

$$[x\,\mathrm{d}u, y\,\mathrm{d}v] = x\{u, y\}\,\mathrm{d}v + y\{x, v\}\,\mathrm{d}u + xy\,\mathrm{d}\{u, v\}$$

for all $x\,\mathrm{d}u, y\,\mathrm{d}v \in \Omega_{A/\Bbbk}$. Moreover, A is also a Lie module over $\Omega_{A/\Bbbk}$ with the action given by $(x\,\mathrm{d}u).a = x\{u, a\}$ for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$ and $a \in A$. In fact, the pair $(A, \Omega_{A/\Bbbk})$ is a Lie–Rinehart algebra in the following sense.

Definition 2.2 [Dokas 2012, Definition 1.5]. A *Lie–Rinehart algebra* over A is a pair (A, L), where A is a commutative associative algebra over $\Bbbk$ and L is a Lie algebra equipped with the structure of an A-module together with an *anchor map*

$$\alpha\colon L \to \mathrm{Der}_{\Bbbk}(A)$$

which is both an A-module and a Lie algebra homomorphism such that

(E2.2.1) $$[X, aY] = a[X, Y] + \alpha(X)(a)Y$$

for all $a \in A$ and $X, Y \in L$.

Note that, in the case of a Poisson algebra, the anchor map $\alpha : \Omega_{A/\Bbbk} \to \mathrm{Der}(A)$ is given by

(E2.2.2) $$\alpha(x\,\mathrm{d}u)(z) = x\{u, z\}$$

for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$ and $z \in A$.

Let (A, L) be a Lie–Rinehart algebra. Rinehart [1963] introduced the notion of the universal enveloping algebra $\mathcal{U}(A, L)$ of (A, L), which is an associative $\Bbbk$-algebra satisfying the appropriate universal property; see [Huebschmann 1990] for more details. We recall the definition next.

Denote by $A \rtimes L$ the semidirect product of the Lie algebra L and the L-module A. More precisely, $A \rtimes L$ is the direct sum of A and L as a vector space, and the Lie bracket is given by

$$[(a, X), (b, Y)] = (X(b) - Y(a), [X, Y])$$

for all $(a, X), (b, Y) \in A \rtimes L$. Let $(\mathcal{U}(A \rtimes L), \iota)$ be the universal enveloping algebra of the Lie algebra $A \rtimes L$, where $\iota\colon A \rtimes L \to \mathcal{U}(A \rtimes L)$ is the canonical embedding. We consider the subalgebra $\mathcal{U}^+(A \rtimes L)$ (without unit) generated by $A \rtimes L$. Moreover, $A \rtimes L$ has the structure of an A-module via $a(a', X) = (aa', aX)$ for all $a, a' \in A$ and $X \in L$. The (universal) enveloping algebra $\mathcal{U}(A, L)$ associated to the Lie–Rinehart algebra (A, L) is defined to be the quotient

$$\mathcal{U}(A, L) = \frac{\mathcal{U}^+(A \rtimes L)}{\big(\iota((a, 0))\iota((a', X)) - \iota(a(a', X))\big)}.$$

Note that $(1_A, 0)$ becomes the algebra identity of $\mathcal{U}(A, L)$. There are two canonical maps

$$\iota_1\colon A \to \mathcal{U}(A, L),\ \ a \mapsto (a, 0), \qquad \text{and} \qquad \iota_2\colon L \to \mathcal{U}(A, L),\ \ X \mapsto (0, X).$$

Observe that ι_1 is an algebra homomorphism and ι_2 is a Lie algebra homomorphism. Moreover, we have the relations

$$\iota_1(a)\iota_2(X) = \iota_2(aX) \quad \text{and} \quad [\iota_2(X), \iota_1(a)] = \iota_1(X(a))$$

for all $a \in A$ and $X \in L$.

As a consequence of [Rinehart 1963, Theorem 3.1], we have the following.

Lemma 2.3. *Let (A, L) be a Lie–Rinehart algebra and $\mathcal{U}(A, L)$ the enveloping algebra of (A, L). If L is a projective A-module, then the Lie algebra homomorphism $\iota_2\colon L \to \mathcal{U}(A, L)$ is injective.*

It is worth restating the above construction for Poisson algebras since it is needed later. Denote by $A \rtimes \Omega_{A/\Bbbk}$ the semidirect product of A and $\Omega_{A/\Bbbk}$ with the Lie bracket given by

$$[(a, x\,\mathrm{d}u), (b, y\,\mathrm{d}v)] = \big(x\{u, b\} - y\{v, a\}, x\{u, y\}\,\mathrm{d}v + y\{x, v\}\,\mathrm{d}u + xy\,\mathrm{d}\{u, v\}\big)$$

for $(a, x\,\mathrm{d}u), (b, y\,\mathrm{d}v) \in A \rtimes \Omega_{A/\Bbbk}$. The Poisson enveloping algebra of A, denoted by $\mathcal{P}(A)$ (which is a new notation), is defined to be the enveloping algebra of the

Lie–Rinehart algebra $(A, \Omega_{A/\Bbbk})$, which can be realized as an associated algebra

$$\mathcal{P}(A) := \mathcal{U}(A, \Omega_{A/\Bbbk}) = \mathcal{U}^{+}(A \rtimes \Omega_{A/\Bbbk})/J,$$

where $\mathcal{U}(A \rtimes \Omega_{A/\Bbbk})$ is the universal enveloping algebra of the Lie algebra $A \rtimes \Omega_{A/\Bbbk}$, and J is the ideal generated by

$$(a, 0)(b, x\,\mathrm{d}u) - (ab, ax\,\mathrm{d}u) \tag{E2.3.1}$$

for all $a, b \in A$ and $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$ [Moerdijk and Mrčun 2010; Rinehart 1963]. Here we have two maps

$$\iota_1 \colon A \to A \rtimes \Omega_{A/\Bbbk} \to \mathcal{P}(A), \quad \iota_1(a) = (a, 0),$$

and

$$\iota_2 \colon \Omega_{A/\Bbbk} \to A \rtimes \Omega_{A/\Bbbk} \to \mathcal{P}(A), \quad \iota_2(x\,\mathrm{d}u) = (0, x\,\mathrm{d}u).$$

Then ι_1 and ι_2 are homomorphisms of associative algebras and Lie algebras, respectively. Moreover, we have

$$\iota_1(\{x, y\}) = [\iota_2(\mathrm{d}x), \iota_1(y)], \tag{E2.3.2}$$

$$\iota_2(\mathrm{d}(xy)) = \iota_1(x)\iota_2(\mathrm{d}y) + \iota_1(y)\iota_2(\mathrm{d}x) \tag{E2.3.3}$$

for all $x, y \in A$.

If $\Omega_{A/\Bbbk}$ is a projective A-module, then the canonical map $\iota_2 \colon \Omega_{A/\Bbbk} \to \mathcal{P}(A)$ is injective (Lemma 2.3). It follows that $\Omega_{A/\Bbbk}$ can be seen as a Lie subalgebra of $\mathcal{P}(A)$.

We now recall the definition of a free Poisson algebra; see [Shestakov 1993, Section 3]. Let V be $\Bbbk$-vector space. Let $\mathrm{Lie}(V)$ be the free Lie algebra generated by V. The *free Poisson algebra generated by* V, denoted by $\mathrm{FP}(V)$, is the symmetric algebra over $\mathrm{Lie}(V)$, namely

$$\mathrm{FP}(V) = \Bbbk[\mathrm{Lie}(V)]. \tag{E2.3.4}$$

The following universal property is well known [Shestakov 1993, Lemma 1, p. 312].

Lemma 2.4. *Let A be a Poisson algebra and V be a vector space. Every $\Bbbk$-linear map $g : V \to A$ extends uniquely to a Poisson algebra morphism $G : \mathrm{FP}(V) \to A$ such that g factors through G.*

Shestakov [1993, Section 3] defined the notion of a free Poisson algebra by the universal property stated in Lemma 2.4, and then proved that the free Poisson algebra can be constructed by using (E2.3.4) [Shestakov 1993, Lemma 1, p. 312]. In the same paper, Shestakov also considered the super (or $\mathbb{Z}_2$-graded) version of Poisson algebras.

For each associative commutative algebra A over a base field $\Bbbk$ of characteristic p, let A^p denote the subalgebra generated by $\{ f^p \mid f \in A\}$. The free Poisson algebras have the following special property.

Lemma 2.5. *Let A be a free Poisson algebra* $\mathrm{FP}(V)$.

(1) $\Omega_{A/\Bbbk}$ *is a free module over A. As a consequence, the Lie algebra map* $\iota_2\colon \Omega_{A/\Bbbk} \to \mathcal{P}(A)$ *is injective.*

(2) *The kernel of* $\mathrm{d} : A \to \Omega_{A/\Bbbk}$ *is A^p.*

Proof. (1) Since A is a commutative polynomial ring, $\Omega_{A/\Bbbk}$ is free over A. (The proof is omitted). The consequence follows from Lemma 2.3.

(2) Check directly. □

Let V be a $\Bbbk$-vector space. There are two gradings that can naturally be assigned to $\mathrm{FP}(V)$. The first one is determined by

$$\deg_1(x) = 1 \quad \forall\, 0 \neq x \in \mathrm{Lie}(V).$$

Since $\mathrm{FP}(V)$ is the symmetric algebra associated to $\mathrm{Lie}(V)$, the above extends to an $\mathbb{N}$-grading on $\mathrm{FP}(V)$. Since the Lie bracket $\{-,-\}$ has degree -1, the Poisson bracket on $\mathrm{FP}(V)$ has degree -1. Note that the multiplication on $\mathrm{FP}(V)$ is homogeneous with respect to $\deg_1$.

For the second grading, we assume that

$$\deg_2(x) = 1 \quad \forall\, 0 \neq x \in V$$

and make the free Lie algebra $\mathrm{Lie}(V)$ $\mathbb{N}$-graded (namely, $[-,-]$ is homogeneous of degree zero). Then we extend the $\mathbb{N}$-grading to $\mathrm{FP}(V)$ so that both the Poisson bracket and the multiplication are homogeneous of degree zero.

Let $\{v_i\}_{i\in I}$ be a $\Bbbk$-basis of V and $\{x_j\}_{j\in J}$ a $\Bbbk$-basis of $\mathrm{Lie}(V)$. Let A be the free Poisson algebra $\mathrm{FP}(V)$ and let A^c be the A^p-submodule of A generated by monomials $x_1^{i_1}\cdots x_n^{i_n}$, for $x_1,\dots,x_n \in \mathrm{Lie}(V)$, which are not in A^p.

Recall that

$$\{f_1, f_2, \dots, f_n\} := \{f_1, \{f_2, \dots, \{f_{n-1}, f_n\}\dots\}\} \tag{E2.5.1}$$

for all $f_i \in A$.

Lemma 2.6. *Let A be a free Poisson algebra* $\mathrm{FP}(V)$.

(1) *Let $f_1,\dots,f_n$ be polynomials in v_i (not x_i). If p does not divide $n-1$, then* $\{f_1, f_2, \dots, f_n\} \in A^c$.

(2) *Let f, g be polynomials in v_i. Then $\Lambda_p(f,g) \in A^c$.*

(3) *The following elements are in A^c for any polynomials in f, g, h in v_i:*

 (a) $\Lambda_p(f,g),\ \Lambda_p(f^2,g^2),\ \Lambda_p(f^2+g^2, 2fg)$.

(b) $\Lambda_p(fg,h)$, $\Lambda_p((fg)^2,h^2)$, $\Lambda_p((fg)^2+h^2,2fgh)$.
(c) $\Lambda_p(fg,fh)$.

Proof. (1) By linearity, we may assume that all f_s are monomials in $\{v_i\} \subseteq V$. Then $\deg_1 f_s = \deg_2 f_s$ for $s = 1, \dots, n$. Let $F := \{f_1, f_2, \dots, f_n\}$. Then

$$\deg_1 F = -n + 1 + \deg_2 F.$$

Since p does not divide $n-1$, p cannot divide both $\deg_1 F$ and $\deg_2 F$. This implies that $F \in A^c$.

(2) Note that $\Lambda_p(f,g)$ is a linear combination of terms of the form (E2.5.1) when $n = p$ and $f_i = f$ or g. By part (1), $\Lambda_p(f,g) \in A^c$.

(3) This is a special case of part (2) for different choices of f, g. □

3. Restricted Poisson algebras, definition

In this section we present a formulation of a restricted Poisson algebra that is equivalent to [Bezrukavnikov and Kaledin 2008, Definition 1.8].

Inspired by the notion of a restricted Lie algebra, we first introduce the definition of a weakly restricted Poisson structure over a field $\Bbbk$ of characteristic $p \geq 3$.

Definition 3.1. Let $(A, \cdot, \{-,-\})$ be a Poisson algebra. If there exists a p-map $(-)^{\{p\}}\colon A \to A$ such that $(A, \{-,-\}, (-)^{\{p\}})$ is a restricted Lie algebra, then A is called a *weakly restricted Poisson algebra*.

This definition requires no compatibility condition between the p-map $(-)^{\{p\}}$ and the multiplication $\cdot$. We will see that an additional requirement is very natural from a Lie-algebraic point of view.

Lemma 3.2. *Let $(A, \cdot, \{-,-\})$ be a Poisson algebra and let $x, y \in A$.*

(1) *If there exist $\widetilde{x}$ and $\widetilde{y}$ in A such that $\mathrm{ad}_x^p = \mathrm{ad}_{\widetilde{x}}$ and $\mathrm{ad}_y^p = \mathrm{ad}_{\widetilde{y}}$, then*

$$\mathrm{ad}_{xy}^p = \mathrm{ad}_{x^p\widetilde{y} + y^p\widetilde{x} + \Phi_p(x,y)},$$

where

(E3.2.1) $$\Phi_p(x,y) = (x^p + y^p)\Lambda_p(x,y) - \tfrac{1}{2}\big(\Lambda_p(x^2,y^2) + \Lambda_p(x^2+y^2, 2xy)\big).$$

In particular, $\mathrm{ad}_{x^2}^p = \mathrm{ad}_{2x^p\widetilde{x}}$.

(2) *If $(A, \cdot, \{-,-\})$ is a weakly restricted Poisson algebra, then*

(E3.2.2) $$\mathrm{ad}_{(xy)^{\{p\}}} = \mathrm{ad}_{x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x,y)}.$$

In particular,

(E3.2.3) $$\mathrm{ad}_{(x^2)^{\{p\}}} = \mathrm{ad}_{2x^p x^{\{p\}}}.$$

Proof. (1) We first prove the assertion when $x = y$. By the Leibniz rule, we have $\mathrm{ad}_{(fg)} = f\,\mathrm{ad}_g + g\,\mathrm{ad}_f$ for any $f, g \in A$. Clearly,

$$\mathrm{ad}^p_{x^2} = (2x\,\mathrm{ad}_x)^p = (2x)^p(\mathrm{ad}_x)^p = 2x^p\,\mathrm{ad}^p_x = 2x^p\,\mathrm{ad}_{\widetilde{x}} = \mathrm{ad}_{2x^p\widetilde{x}}.$$

In the general case, considering the universal enveloping algebra of the Lie algebra $(A, \{-,-\})$ and using (E1.1.4), we get $\mathrm{ad}_{\Lambda_p(f,g)} = \mathrm{ad}^p_{f+g} - \mathrm{ad}^p_f - \mathrm{ad}^p_g$ for any $f, g \in A$. Therefore,

$$\begin{aligned}\mathrm{ad}_{x^p\widetilde{y}+y^p\widetilde{x}+\Phi(x,y)} &= \mathrm{ad}_{x^p\widetilde{y}+y^p\widetilde{x}+(x^p+y^p)\Lambda_p(x,y)-\frac{1}{2}\left(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\right)}\\ &= x^p\,\mathrm{ad}^p_y + y^p\,\mathrm{ad}^p_x + (x^p+y^p)(\mathrm{ad}^p_{x+y} - \mathrm{ad}^p_x - \mathrm{ad}^p_y)\\ &\qquad + \tfrac{1}{2}(\mathrm{ad}^p_{x^2} + \mathrm{ad}^p_{y^2} + \mathrm{ad}^p_{2xy} - \mathrm{ad}^p_{(x+y)^2})\\ &= x^p\,\mathrm{ad}^p_y + y^p\,\mathrm{ad}^p_x + (x^p+y^p)(\mathrm{ad}^p_{x+y} - \mathrm{ad}^p_x - \mathrm{ad}^p_y)\\ &\qquad + x^p\,\mathrm{ad}^p_x + y^p\,\mathrm{ad}^p_y + \mathrm{ad}^p_{xy} - (x+y)^p\,\mathrm{ad}^p_{x+y}\\ &= \mathrm{ad}^p_{xy},\end{aligned}$$

which completes the proof.

(2) This is an immediate consequence of (1). □

Concerning the notation Φ_p in (E3.2.1), we also have the following characterization by considering the Poisson enveloping algebra.

Proposition 3.3. *Let A be a Poisson algebra and $\mathcal{P}(A)$ the Poisson enveloping algebra of A. Then, for all $x, y \in A$, we have*

(E3.3.1) $\iota_2(\mathrm{d}\Phi_p(x,y)) = (\iota_2(\mathrm{d}(xy)))^p - \iota_1(x^p)(\iota_2(\mathrm{d}y))^p - \iota_1(y^p)(\iota_2(\mathrm{d}x))^p.$

Proof. By the definition of $\mathcal{P}(A)$, we have

$$(0, \mathrm{d}x^2)^p = (0, 2x\,\mathrm{d}x)^p = ((2x,0)(0,\mathrm{d}x))^p = (2x,0)^p(0,\mathrm{d}x)^p = 2(x^p,0)(0,\mathrm{d}x)^p$$

and hence

(E3.3.2) $$(\iota_2(\mathrm{d}x^2))^p = 2\iota_1(x^p)(\iota_2(\mathrm{d}x))^p$$

for any $x \in A$. It follows that (E3.3.1) holds when $x = y$.

Considering the Frobenius map of $\mathcal{P}(A)$, we have

$$\begin{aligned}(\iota_2(\mathrm{d}(x+y)))^p &= (0, \mathrm{d}(x+y))^p = ((0,\mathrm{d}x) + (0,\mathrm{d}y))^p\\ &= (0,\mathrm{d}x)^p + (0,\mathrm{d}y)^p + \Lambda_p((0,\mathrm{d}x),(0,\mathrm{d}y))\\ &= (\iota_2(\mathrm{d}x))^p + (\iota_2(\mathrm{d}y))^p + \iota_2(\mathrm{d}\Lambda_p(x,y))\end{aligned}$$

since ι_2 is a homomorphism of Lie algebras. By the above computation and (E3.3.2),

$$\begin{aligned}(\iota_2(\mathrm{d}(x+y)^2))^p &= 2\iota_1((x+y)^p)(\iota_2(\mathrm{d}(x+y)))^p\\ &= 2\iota_1(x^p+y^p)\big((\iota_2(\mathrm{d}x))^p+(\iota_2(\mathrm{d}y))^p+\iota_2(\mathrm{d}\Lambda_p(x,y))\big).\end{aligned}$$

By a direct calculation and (E3.3.2),

$$\begin{aligned}(\iota_2(\mathrm{d}(x+y)^2))^p &= (\iota_2(\mathrm{d}x^2+\mathrm{d}y^2+2\,\mathrm{d}(xy)))^p\\ &= (\iota_2(\mathrm{d}x^2+\mathrm{d}y^2))^p+(\iota_2(2\,\mathrm{d}(xy)))^p+\iota_2(\mathrm{d}\Lambda_p(x^2+y^2,2xy))\\ &= (\iota_2(\mathrm{d}x^2))^p+(\iota_2(\mathrm{d}y^2))^p+\iota_2(\mathrm{d}\Lambda_p(x^2,y^2))\\ &\qquad +2(\iota_2(\mathrm{d}(xy)))^p+\iota_2(\mathrm{d}\Lambda_p(x^2+y^2,2xy))\\ &= 2\iota_1(x^p)(\iota_2(\mathrm{d}x))^p+2\iota_1(y^p)(\iota_2(\mathrm{d}y))^p+\iota_2(\mathrm{d}\Lambda_p(x^2,y^2))\\ &\qquad +2(\iota_2(\mathrm{d}(xy)))^p+\iota_2(\mathrm{d}\Lambda_p(x^2+y^2,2xy)).\end{aligned}$$

Comparing the above two equations, we get

$$\begin{aligned}&(\iota_2(\mathrm{d}(xy)))^p+\tfrac{1}{2}\big(\iota_2\big(\mathrm{d}\big(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\big)\big)\big)\\ &\qquad = \iota_1(x^p)(\iota_2(\mathrm{d}y))^p+\iota_1(y^p)(\iota_2(\mathrm{d}x))^p+\iota_1(x^p+y^p)\iota_2(\mathrm{d}\Lambda_p(x,y))\\ &\qquad = \iota_1(x^p)(\iota_2(\mathrm{d}y))^p+\iota_1(y^p)(\iota_2(\mathrm{d}x))^p+\iota_2\big(\mathrm{d}((x^p+y^p)\Lambda_p(x,y))\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\iota_2(\mathrm{d}\Phi_p(x,y))\\ &\qquad = \iota_2\big(\mathrm{d}\big((x^p+y^p)\Lambda_p(x,y)-\tfrac{1}{2}\big(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\big)\big)\big)\\ &\qquad = (\iota_2(\mathrm{d}(xy)))^p-\iota_1(x^p)(\iota_2(\mathrm{d}y))^p-\iota_1(y^p)(\iota_2(\mathrm{d}x))^p. \qquad \square\end{aligned}$$

For a weakly restricted Poisson algebra, it is desirable to consider the compatibility between the p-map and the associative multiplication. By removing ad from (E3.2.3) (which can be done in some cases), we obtain (E3.4.1) below. Similarly, if we remove ad from (E3.2.2), we obtain (E3.5.1) below. Both Lemma 3.2 and Proposition 3.3 suggest the following definition. Following Lemma 3.2(2), condition (E3.4.1) is forced.

Definition 3.4. Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a weakly restricted Poisson algebra over $\Bbbk$. We call A a *restricted Poisson algebra* if, for every $x \in A$,

$$(x^2)^{\{p\}} = 2x^p x^{\{p\}}. \tag{E3.4.1}$$

In this case, the p-map $(-)^{\{p\}}$ is a *restricted Poisson structure* on A.

Next we give another description of condition (E3.4.1) which is convenient for some computations.

Proposition 3.5. *Let A be a weakly restricted Poisson algebra.*

(1) *Suppose* (E3.4.1) *holds. Then* $(\lambda 1_A)^{\{p\}} = 0$ *for all* $\lambda \in \Bbbk$.

(2) *Equation* (E3.4.1) *holds for all* $x \in A$ *if and only if every pair of elements* (x, y) *in* A *satisfies*

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x, y). \tag{E3.5.1}$$

Consequently, A is a restricted Poisson algebra if and only if (E3.5.1) *holds.*

(3) *Suppose* (E3.5.1) *holds. Then*

$$(x^n)^{\{p\}} = n x^{(n-1)p} x^{\{p\}} \tag{E3.5.2}$$

for all n. As a consequence, $(x^p)^{\{p\}} = 0$ *for all* $x \in A$.

(4) *If* $(1_A)^{\{p\}} = 0$, *then* (E3.5.1) *holds for pairs* $(x, \lambda 1_A)$ *and* $(\lambda 1_A, x)$ *for all* $x \in A$ *and all* $\lambda \in \Bbbk$.

Proof. (1) Clearly, $1_A^{\{p\}} = 2 \cdot 1_A^p 1_A^{\{p\}}$ and hence $1_A^{\{p\}} = 0$. For every $\lambda \in \Bbbk$, $(\lambda 1_A)^{\{p\}} = \lambda^p 1_A^{\{p\}} = 0$.

(2) The "if" part is trivial since $\Phi_p(x, x) = 0$ for any $x \in A$. Next, we show the "only if" part. By (E3.4.1) and Definition 1.1(3), we have

$$((x+y)^2)^{\{p\}} = 2(x+y)^p(x+y)^{\{p\}} = 2(x^p + y^p)(x^{\{p\}} + y^{\{p\}} + \Lambda_p(x, y)).$$

Since $(A, \{-,-\}, (-)^{\{p\}})$ is a restricted Lie algebra, by Definition 1.1(2,3) we have

$$\begin{aligned}
&((x+y)^2)^{\{p\}}\\
&\quad = (x^2 + y^2 + 2xy)^{\{p\}}\\
&\quad = (x^2 + y^2)^{\{p\}} + 2^p (xy)^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy)\\
&\quad = (x^2)^{\{p\}} + (y^2)^{\{p\}} + \Lambda_p(x^2, y^2) + 2^p (xy)^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy)\\
&\quad = 2x^p x^{\{p\}} + 2y^p y^{\{p\}} + \Lambda_p(x^2, y^2) + 2(xy)^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy).
\end{aligned}$$

Comparing the above two equations and using $2 \neq 0$, we obtain (E3.5.1).

(3) This follows by induction.

(4) First of all, $(\lambda 1_A)^{\{p\}} = \lambda^p 1_A^{\{p\}} = 0$ for all $\lambda \in \Bbbk$. The assertion follows by the fact $\Phi_p(\lambda 1_A, x) = \Phi_p(x, \lambda 1_A) = 0$. □

Remark 3.6. Several remarks are collected below.

(1) As in [Bezrukavnikov and Kaledin 2008], we assume that $p \geq 3$. So the polynomial $\Phi_p(x, y)$ in (E3.2.1) is well defined. When $p = 3$, we have

$$\Phi_3(x, y) = x^2 y\{y, y, x\} + xy^2\{x, x, y\} + xy\{x, y\}^2.$$

For $p > 3$, it is too long to write out all the terms as above.

(2) Considering $\Phi_p(x, y)$ as an element in $\mathrm{FP}(V)$, where $V = \Bbbk x \oplus \Bbbk y$, it is homogeneous of degree $p+1$ with respect to $\deg_2$ and homogeneous of degree $2p$ with respect to $\deg_1$.

(3) Bezrukavnikov and Kaledin [2008, Definition 1.8] defined a *restricted Poisson algebra* as a weakly restricted Poisson algebra $(A, \{-,-\}, (-)^{\{p\}})$ such that the p-map satisfies

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + P(x, y) \tag{E3.6.1}$$

for all $x, y \in A$. Here $P(x, y)$ is a canonical quantized polynomial determined by [Bezrukavnikov and Kaledin 2008, (1.3)]. We will show that (E3.6.1) is equivalent to (E3.5.1).

(4) $P(x, y)$ is defined implicitly, but it follows from [Bezrukavnikov and Kaledin 2008, (1.3)] that $P(x, x) = 0$. Therefore a restricted Poisson algebra in the sense of [Bezrukavnikov and Kaledin 2008, Definition 1.8] is a restricted Poisson algebra in the sense of Definition 3.4.

(5) There are other interpretations of $\Phi_p(x, y)$. Using

$$xy = \tfrac{1}{4}[(x+y)^2 - (x-y)^2],$$

we obtain that

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi'_p(x, y), \tag{E3.6.2}$$

where

$$\begin{aligned}\Phi'_p(x, y) = \tfrac{1}{4}\Lambda_p\big((x+y)^2, -(x-y)^2\big) \\ + \tfrac{1}{2}\big((x^p + y^p)\Lambda_p(x, y) - (x^p - y^p)\Lambda_p(x, -y)\big).\end{aligned} \tag{E3.6.3}$$

One can show that $\Phi_p(x, y) = \Phi'_p(x, y)$ in the free Poisson algebra generated by x and y.

(6) The following are clear by definition.

(a) $\Lambda_p(x, y) = \Lambda_p(y, x)$ for all $x, y \in A$.

(b) If $\{x, y\} = 0$, then $\Lambda_p(x, y) = 0$.

(c) $\Phi_p(x, y) = \Phi_p(y, x)$ for all $x, y \in A$.

(d) If $\{x, y\} = 0$, then $\Phi_p(x, y) = 0$.

Lemma 3.7. *The definitions of restricted Poisson algebras in Definition 3.4 and* [Bezrukavnikov and Kaledin 2008, Definition 1.8] *are equivalent.*

Proof. Let $P(x, y)$ be the polynomial defined in [Bezrukavnikov and Kaledin 2008, (1.3)]. By Proposition 3.5(2), it remains to show that $P(x, y) = \Phi_p(x, y)$. Let $\mathrm{Lie}(V)$ be the free Lie algebra over a vector space V and consider the tensor (free) algebra $T(V)$ as a universal enveloping algebra over Lie(V). Then we have a Poincaré–Birkhoff–Witt filtration on $T(V)$. The free quantized algebra $Q^\bullet(V)$ is the Rees algebra associated to this filtration. By definition, for each n,

$$\mathcal{F}_n := F_nT(V) = \Bbbk \oplus L^\bullet(V) \oplus (L^\bullet(V))^2 \oplus \cdots \oplus (L^\bullet(V))^n.$$

We are omitting the symbol h which represents the natural embedding $h: \mathcal{F}_\bullet \to \mathcal{F}_{\bullet+1}$ in the Rees ring. Taking $V = \Bbbk x \oplus \Bbbk y$, inside the Rees ring we have

$$\begin{aligned}
(x^p + y^p)^2 &+ (\Lambda_p(x,y))^2 + \Lambda_p(x,y)(x^p+y^p) + (x^p+y^p)\Lambda_p(x,y)\\
&= (x^p + y^p + \Lambda_p(x,y))^2\\
&= (x+y)^{2p}\\
&= (x^2+y^2+xy+yx)^p\\
&= (x^2+y^2)^p + (xy+yx)^p + \Lambda_p(x^2+y^2, xy+yx)\\
&= x^{2p} + y^{2p} + \Lambda_p(x^2,y^2) + (xy)^p + (yx)^p + \Lambda_p(xy,yx)\\
&\qquad + \Lambda_p(x^2+y^2, xy+yx),
\end{aligned}$$

and hence

$$\begin{aligned}
(xy)^p &+ (yx)^p - x^py^p - y^px^p\\
&= \Lambda_p(x,y)(x^p+y^p) + (x^p+y^p)\Lambda_p(x,y) + (\Lambda_p(x,y))^2\\
&\qquad - \Lambda_p(x^2,y^2) - \Lambda_p(xy,yx) - \Lambda_p(x^2+y^2, xy+yx).
\end{aligned}$$

On the other hand,

$$[x,y]^p = (xy-yx)^p = (xy)^p - (yx)^p + \Lambda_p(xy, -yx).$$

So we have

$$\begin{aligned}
2P(x,y) &= 2((xy)^p - x^py^p)\\
&= \Lambda_p(x,y)(x^p+y^p) + (x^p+y^p)\Lambda_p(x,y) - \Lambda_p(x^2,y^2)\\
&\qquad - \Lambda_p(x^2+y^2, xy+yx) + (\Lambda_p(x,y))^2 - \Lambda_p(xy,yx)\\
&\qquad - \Lambda_p(xy,-yx) + [x,y]^p - [x^p,y^p].
\end{aligned}$$

In fact, it is easily seen that $(\Lambda_p(x,y))^2 \in \mathcal{F}_2$, $[x,y]^p \in \mathcal{F}_p$. On the other hand,

$$[x^p, y^p] = \mathrm{ad}_x^p(y^p) = -\,\mathrm{ad}_x^{p-1}(\mathrm{ad}_y^p(x)) \in \mathcal{F}_1,$$

where $\mathrm{ad}_x(y) = [x,y]$. By (E1.1.3), we have

$$\Lambda_p(xy,yx) = \sum_{x_k = xy \text{ or } yx} \frac{1}{\#(xy)}\, \mathrm{ad}_{x_1}\cdots \mathrm{ad}_{x_{p-2}}([yx,xy]),$$

where $\#(xy)$ is the number of xy in the collection of possibly repeated elements $\{x_1, x_2, \dots, x_{p-2}, yx, yx\}$. Since

$$[yx, xy] = [yx, yx + [x, y]] = [yx, [x, y]] \in \mathcal{F}_2,$$

we have $\Lambda_p(xy, yx) \in \mathcal{F}_p$. Similarly, $\Lambda_p(xy, -yx) \in \mathcal{F}_p$. By definition (see [Bezrukavnikov and Kaledin 2008, (1.3)]), $P(x, y)$ is homogeneous of degree $p+1$. Therefore, after removing lower-degree components,

$$\begin{aligned}2P(x, y) = \Lambda_p(x, y)(x^p + y^p) + (x^p + y^p)\Lambda_p(x, y)\\ - \Lambda_p(x^2, y^2) - \Lambda_p(x^2 + y^2, xy + yx).\end{aligned}$$

Since the multiplication is commutative in a Poisson algebra, we have

$$P(x, y) = (x^p + y^p)\Lambda_p(x, y) - \tfrac{1}{2}\big(\Lambda_p(x^2, y^2) + \Lambda_p(x^2 + y^2, 2xy)\big) = \Phi_p(x, y). \quad \square$$

4. Elementary properties and examples

We start with something obvious.

Definition 4.1. Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a restricted Poisson algebra. A Poisson ideal I of A is said to be *restricted* if $x^{\{p\}} \in I$ for any $x \in I$.

The proofs of the following three assertions are easy and omitted.

Lemma 4.2. *Let A be a restricted Poisson algebra. Suppose that I is a Poisson ideal of A that is generated by $\{x_i \mid i \in S\}$ as an ideal of the commutative ring A. If $x_i^{\{p\}} \in I$ for any $i \in S$, then I is a restricted Poisson ideal.*

Proposition 4.3. *Let A be a restricted Poisson algebra and I a restricted Poisson ideal of A. Then the quotient Poisson algebra A/I is a restricted Poisson algebra.*

Clearly, we have the following fact.

Proposition 4.4. *Let $f\colon A \to A'$ be a homomorphism of restricted Poisson algebras. Then* $\operatorname{Ker} f$ *is a restricted Poisson ideal of A.*

Let A^p be the subalgebra of A generated by $\{f^p \mid f \in A\}$ — the image of the Frobenius map.

Lemma 4.5. *Let A be a Poisson algebra and $f, g, h \in A$. Then the following hold*:

(1) $f^p\Phi_p(g, h) - \Phi_p(fg, h) + \Phi_p(f, gh) - h^p\Phi_p(f, g) = 0$.

(2) *If f is in the Poisson center of A, then $f^p\Phi_p(g, h) = \Phi_p(fg, h) = \Phi_p(g, fh)$.*

(3) $\Phi_p(f, g + h) - \Phi_p(f, g) - \Phi_p(f, h) = \Lambda_p(fg, fh) - f^p\Lambda_p(g, h)$.

Proof. It is clear that (2) is a consequence of (1). It suffices to show assertions (1) and (3) for the free Poisson algebra $\mathrm{FP}(A)$ since there is a surjective Poisson algebra map $\mathrm{FP}(A) \to A$ (Lemma 2.4). So the hypothesis becomes that f, g, h are in a $\Bbbk$-space V sitting inside a free Poisson algebra $\mathrm{FP}(V)$.

When A is a free Poisson algebra $\mathrm{FP}(V)$, by Lemma 2.5(1), ι_2 is injective. It follows from Lemma 2.5(2) that

(a) the kernel of the map

$$A \xrightarrow{\mathrm{d}} \Omega_{A/\Bbbk} \xrightarrow{\iota_2} \mathcal{P}(A)$$

is A^p.

Let $\{v_i\}_{i\in S}$ be a basis of the V. Let A^c be the A^p-submodule of $A = \mathrm{FP}(V)$ defined before Lemma 2.6. Then

(b) $A^c \cap A^p = \{0\}$ and $\Lambda_p(x, y) \in A^c$ for all $x, y \in \Bbbk[V]$ by Lemma 2.6(2).

Now we prove (1) and (3) under conditions (a) and (b).

(1) For all $f, u \in A$, we have $\mathrm{d}(f^p u) = f^p\,\mathrm{d}u$ and $\iota_2(\mathrm{d}(f^p u)) = (f^p, 0)(0, \mathrm{d}u) \in \mathcal{P}(A)$. By Proposition 3.3,

$$\begin{aligned}
\iota_2(\mathrm{d}(f^p\Phi_p(g,h))) &= (f^p,0)(0,\mathrm{d}(gh))^p - (f^pg^p,0)(0,\mathrm{d}h)^p - (f^ph^p,0)(0,\mathrm{d}g)^p,\\
\iota_2(\mathrm{d}\Phi_p(fg,h)) &= (0,\mathrm{d}(fgh))^p - ((fg)^p,0)(0,\mathrm{d}h)^p - (h^p,0)(0,\mathrm{d}(fg))^p,\\
\iota_2(\mathrm{d}\Phi_p(f,gh)) &= (0,\mathrm{d}(fgh))^p - (f^p,0)(0,\mathrm{d}(gh))^p - ((gh)^p,0)(0,\mathrm{d}f)^p,\\
\iota_2(\mathrm{d}(h^p\Phi_p(f,g))) &= (h^p,0)(0,\mathrm{d}(fg))^p - (h^pf^p,0)(0,\mathrm{d}g)^p - (h^pg^p,0)(0,\mathrm{d}f)^p
\end{aligned}$$

for all $f, g, h \in V$. It follows that

$$\iota_2(\mathrm{d}(f^p\Phi_p(g,h) - \Phi_p(fg,h) + \Phi_p(f,gh) - \Phi_p(f,g)h^p)) = 0.$$

By condition (a), we get

$$X := f^p\Phi_p(g,h) - \Phi_p(fg,h) + \Phi_p(f,gh) - h^p\Phi_p(f,g) \in A^p.$$

By definition, X is in the A^p-submodule generated by $\Lambda_p(x, y)$ for all $x, y \in A$, or in A^c as given in condition (b). But since $A^p \cap A^c = \{0\}$ by condition (b), we obtain that $X = 0$ and that the desired identity holds.

(3) The proof is similar to that of (1) and is omitted. □

Proposition 4.6. *Let A be a weakly restricted Poisson algebra.*

(1) *If (x, y) satisfies* (E3.5.1), *then so do $(x, \lambda y)$ and $(\lambda x, y)$ for all $\lambda \in \Bbbk$.*

(2) *Let $f, g, h \in A$. Suppose that (f, g) and (g, h) satisfy* (E3.5.1)*. Then (fg, h) satisfies* (E3.5.1) *if and only if (f, gh) does.*

(3) *If (f, g) and (f, h) satisfy* (E3.5.1), *then so does $(f, g + h)$.*

(3′) *If* (g, f) *and* (h, f) *satisfy* (E3.5.1), *then so does* $(g+h, f)$.

(4) *Fix an* $x \in A$ *and let* R_x *be the set of* $y \in A$ *such that* (x, y) *satisfies* (E3.5.1). *Then* R_x *is a* $\Bbbk$*-subspace of* A.

(4′) *Fix an* $x \in A$ *and let* L_x *be the set of* $y \in A$ *such that* (y, x) *satisfies* (E3.5.1). *Then* L_x *is a* $\Bbbk$*-subspace of* A.

Proof. (1) Assuming (E3.5.1) for (x, y), we have

$$\begin{aligned}(x\lambda y)^{\{p\}} &= (\lambda(xy))^{\{p\}} = \lambda^p (xy)^{\{p\}}\\ &= \lambda^p (x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x, y))\\ &= x^p\big((\lambda y)^{\{p\}} + (\lambda y)^p x^{\{p\}} + \lambda^p \Phi_p(x, y)\big)\\ &= x^p\big((\lambda y)^{\{p\}} + (\lambda y)^p x^{\{p\}} + \Phi_p(x, \lambda y)\big),\end{aligned}$$

where the last equation is Lemma 4.5(2). So $(x, \lambda y)$ satisfies (E3.5.1). Similarly for $(\lambda x, y)$.

(2) By symmetry, we only prove one implication and assume that (fg, h) satisfies (E3.5.1). We show next that (f, gh) satisfies (E3.5.1):

$$\begin{aligned}(f(gh))^{\{p\}} &= ((fg)h)^{\{p\}} = (fg)^p h^{\{p\}} + h^p (fg)^{\{p\}} + \Phi_p(fg, h)\\ &= (fg)^p h^{\{p\}} + h^p\big(f^p g^{\{p\}} + g^p f^{\{p\}} + \Phi_p(f, g)\big) + \Phi_p(fg, h)\\ &= f^p g^p h^{\{p\}} + f^p h^p g^{\{p\}} + g^p h^p f^{\{p\}} + \Phi_p(fg, h) + h^p \Phi_p(f, g)\\ &= f^p g^p h^{\{p\}} + f^p h^p g^{\{p\}} + g^p h^p f^{\{p\}}\\ &\qquad + f^p \Phi_p(g, h) + \Phi_p(f, gh) \qquad \text{by Lemma 4.5(1)}\\ &= f^p\big(g^p h^{\{p\}} + h^p g^{\{p\}} + \Phi_p(g, h)\big) + (gh)^p f^{\{p\}} + \Phi_p(f, gh)\\ &= f^p (gh)^{\{p\}} + (gh)^p f^{\{p\}} + \Phi_p(f, gh).\end{aligned}$$

(3) Assume (f, g) and (f, h) satisfy (E3.5.1). Then

$$\begin{aligned}(f(g+h))^{\{p\}} &= (fg + fh)^{\{p\}}\\ &= (fg)^{\{p\}} + (fh)^{\{p\}} + \Lambda_p(fg, fh)\\ &= f^p g^{\{p\}} + g^p x^{\{p\}} + \Phi_p(f, g) + x^p h^{\{p\}} + h^p f^{\{p\}}\\ &\qquad + \Phi_p(f, h) + \Lambda_p(fg, fh)\\ &= f^p\big(g^{\{p\}} + h^{\{p\}} + \Lambda_p(g, h)\big) + (g+h)^p f^{\{p\}} + \Phi_p(f, g+h)\\ &= f^p (g+h)^{\{p\}} + (g+h)^p f^{\{p\}} + \Phi_p(f, g+h),\end{aligned}$$

where the second-to-last equality is deduced from Lemma 4.5(3). So $(f, g+h)$ satisfies (E3.5.1).

(3′) is equivalent to (3).

(4) Let

$$R_x = \{ y \in A \mid \text{(E3.5.1) holds for the pair } (x, y)\}.$$

By Proposition 4.6(1), we have

(i) if $y \in R_x$, then so is λy for all $\lambda \in \Bbbk$.

By Proposition 4.6(3),

(ii) if $g, h \in R_x$, then so is $g + h$.

By (i) and (ii) above, R_x is a $\Bbbk$-subspace of A.

(4′) This is true because $L_x = R_x$. □

The following result will be used several times.

Theorem 4.7. *Let A be a weakly restricted Poisson algebra. Let $\boldsymbol{b} := \{b_i\}_{i \in S}$ be a $\Bbbk$-basis of A. If* (E3.5.1) *holds for every pair $(x, y) \subseteq \boldsymbol{b}$, then A is a restricted Poisson algebra.*

Proof. We need to show that (E3.5.1) holds for all $x, y \in A$. First we fix any $x \in \boldsymbol{b}$ and let

$$R_x = \{ y \in A \mid \text{(E3.5.1) holds for the pair } (x, y)\}.$$

By Proposition 4.6(4), R_x is a $\Bbbk$-subspace of A. By hypothesis, we see that $\boldsymbol{b} \subseteq R_x$. Since $\boldsymbol{b}$ is a basis of A, $R_x = A$.

Next we fix $y \in A$ and consider

$$L_y = \{x \in A \mid \text{(E3.5.1) holds for the pairs } (x, y)\}.$$

Similarly, by Proposition 4.6(4′), L_y is a $\Bbbk$-subspace. It contains $\boldsymbol{b}$ because $R_x = A$ for all $x \in \boldsymbol{b}$ (see the first paragraph). Hence, $L_y = A$. This means that (E3.5.1) holds for all pairs (x, y) in A. Therefore A is a restricted Poisson algebra. □

One of the main goals of this paper is to provide some interesting examples of restricted Poisson algebras. In the rest of this section we give some elementary (but nontrivial) examples. We would like to give a gentle warning before the examples. We have checked that all p-maps given below satisfy (E3.5.1); however, our proofs are tedious computations and therefore omitted. On the other hand, since the p-maps are explicitly expressed by partial derivatives, one can verify the assertions with enough patience. More-sophisticated examples are given in later sections.

Example 4.8. Let $A = \Bbbk[x, y]$ be a polynomial algebra in two variables x, y, where the (classical) Poisson bracket is given by

$$\{f, g\} = f_x g_y - f_y g_x \tag{E4.8.1}$$

for all $f, g \in A$, and f_x and f_y are the partial derivatives of f with respect to the variables x and y, respectively. (The bracket defined in (E4.8.1) was the original Poisson bracket studied by many people including Poisson [1809] when $\Bbbk = \mathbb{R}$.)

(1) Let $\Bbbk$ be a base field of characteristic 3. For every $f \in A$, we define

$$f^{\{3\}} = f_x^2 f_{yy} + f_y^2 f_{xx} + f_x f_y f_{xy}, \tag{E4.8.2}$$

where f_{xx}, f_{yy} and f_{xy} are the second order partial derivatives of f. Then $(A, \cdot, \{-,-\}, (-)^{\{3\}})$ is a restricted Poisson algebra.

(2) Let $\Bbbk$ be a base field of characteristic 5. For every $f \in A$, define

$$\begin{aligned} f^{\{5\}} = {} & f_1^4 f_{2222} + f_1^3 f_2 f_{1222} + f_1^2 f_2^2 f_{1122} + f_1 f_2^3 f_{1112} + f_2^4 f_{1111} \\ & + f_{12}(f_1^3 f_{222} - f_1^2 f_2 f_{122} - f_1 f_2^2 f_{112} + f_2^3 f_{111}) \\ & - f_1 f_{22}(f_1^2 f_{122} - 2 f_1 f_2 f_{112} + f_2^2 f_{111}) \\ & - f_2 f_{11}(f_2^2 f_{112} - 2 f_2 f_1 f_{122} + f_1^2 f_{222}) \\ & + 2(f_{12}^2 - f_{11} f_{22})(f_1^2 f_{22} - 2 f_1 f_2 f_{12} + f_2^2 f_{11}), \end{aligned} \tag{E4.8.3}$$

where $f_{i_1 i_2 \cdots i_k}$ denotes the k-th order partial derivative of f with respect to the variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$. Then $(A, \cdot, \{-,-\}, (-)^{\{5\}})$ is a restricted Poisson algebra.

See Example 7.3 for general p. It would be interesting to understand the meaning of (E4.8.2) and (E4.8.3) and to find its connection with other subjects.

The next two are slight generalizations of the previous example.

Example 4.9. Suppose $\operatorname{char} \Bbbk = 3$ and let $A = \Bbbk[x, y]$ be a polynomial Poisson algebra in two variables x, y, where the Poisson bracket is given by

$$\{f, g\} = \varphi(f_x g_y - f_y g_x),$$

and $\varphi = \lambda x + \mu y + \nu$, $\lambda, \mu, \nu \in \Bbbk$. For every $f \in A$, we define

$$\begin{aligned} f^{\{3\}} = {} & \lambda \varphi f_x f_y^2 + \mu \varphi f_x^2 f_y \\ & + \varphi^2 (f_x^2 f_{yy} + f_y^2 f_{xx} + f_x f_y f_{xy}) + \lambda^2 y f_y^3 + \mu^2 x f_x^3. \end{aligned} \tag{E4.9.1}$$

Then $(A, \cdot, \{-,-\}, (-)^{\{3\}})$ is a restricted Poisson algebra.

Example 4.10. Suppose $\operatorname{char} \Bbbk = 3$ and let $A = \Bbbk[x_1, x_2, \ldots, x_n]$ be a Poisson algebra, where the Lie bracket is given by $\{x_i, x_j\} = 2c_{ij} \in \Bbbk$ with $c_{ij} + c_{ji} = 0$ for $1 \le i, j \le n$. Clearly, $\{f, g\} = \sum_{1 \le i,j \le n} c_{ij}(f_i g_j - f_j g_i)$ for $f, g \in A$, where f_i denotes the partial derivative of f with respect to the variable x_i for $i = 1, 2, \ldots, n$.

Then A is a restricted Poisson algebra with the p-map given by

$$f^{\{3\}} = \sum_{1\leq i,j,k,l\leq n} c_{ij}c_{kl} f_i f_k f_{jl}$$

for any $f \in A$, where f_{jl} is the second partial derivative of f with respect to the variables x_j and x_l.

5. Existence and uniqueness of restricted structures

By Lemma 3.2(2), a weakly restricted Poisson structure on a Poisson algebra is very close to a restricted Poisson structure (up to a factor in the Poisson center). In this section, we study the existence and uniqueness of (weakly) restricted Poisson structures. First we consider the trivial extension.

Lemma 5.1. *Let A be a Poisson algebra, and let $A = \Bbbk 1_A \oplus \mathfrak{m}$ be its decomposition as a Lie algebra.*

(1) *If $x \mapsto x^{\{p\}}$ is a restriction p-map of the Lie algebra $\mathfrak{m}$, it can naturally be extended to A by defining $1_A^{\{p\}} = 0$. As a consequence, A is a weakly restricted Poisson algebra.*

(2) *If, further, the p-map on $\mathfrak{m}$ satisfies* (E3.4.1), *then so does the extended p-map on A. In this case, A is a restricted Poisson algebra.*

Proof. (1) This follows from Lemma 1.3. For all $\lambda \in \Bbbk$ and $x \in \mathfrak{m}$, the p-map is defined by $(\lambda 1_A + x)^{\{p\}} = x^{\{p\}}$.

(2) We check (E3.4.1) for elements in A as follows:

$$\begin{aligned}((\lambda 1_A + x)^2)^{\{p\}} &= (\lambda^2 1_A + 2\lambda x + x^2)^{\{p\}} \\ &= (2\lambda x + x^2)^{\{p\}} = (2\lambda x)^{\{p\}} + (x^2)^{\{p\}} \\ &= 2\lambda^p x^{\{p\}} + 2x^p x^{\{p\}} = 2(\lambda 1_A + x)^p x^{\{p\}} \\ &= 2(\lambda 1_A + x)^p (\lambda 1_A + x)^{\{p\}}.\end{aligned}$$

Therefore A is a restricted Poisson algebra. □

The following example is immediate.

Example 5.2. (1) Let L be a restricted Lie algebra and let $A = \Bbbk 1_A \oplus L$, where the associate product on L is 0. Then A is a Poisson algebra in the obvious way. Both sides of (E3.4.1) are zero for elements in L (since $L \cdot L = 0$). By Lemma 5.1(2), A is a restricted Poisson algebra.

(2) Consider the special case when $L = \Bbbk x + \Bbbk y$ is a solvable Lie algebra with $[x, y] = x$. For $f = \lambda_1 x + \lambda_2 y \in L$, we define the p-map by

$$f^{\{p\}} = \lambda_2^{p-1}(\lambda_1 x + \lambda_2 y).$$

It is straightforward to check that $(L, (-)^{\{p\}})$ is a restricted Lie algebra. Let $A = \Bbbk 1_A \oplus L$. Then, by part (1), A is a restricted Poisson algebra. As a commutative algebra, $A = \Bbbk[x, y]/(x^2, xy, y^2)$ with $\Bbbk$-linear basis $\{1, x, y\}$. The Poisson bracket is given by $\{x, y\} = x$.

Let L be a restricted Lie algebra. It is well known that the p-map of L is unique up to a semilinear map from L to $Z(L)$, where $Z(L)$ is the center of L. Recall that a map $\gamma\colon L \to Z(L)$ being semilinear means that for any $x, y \in A$ and $\lambda \in \Bbbk$,

$$\gamma(x + y) = \gamma(x) + \gamma(y),$$
$$\gamma(\lambda x) = \lambda^p \gamma(x).$$

The following lemma is well known and easy to prove.

Lemma 5.3. *Let $(L, (-)^{[p]})$ be a restricted Lie algebra.*

(1) *Let $(-)^{\{p\}}$ be another restricted Lie structure on L. Then there is a map $\gamma : L \to Z(L)$ such that $(-)^{\{p\}} = (-)^{[p]} + \gamma$.*
(2) *Let γ be a map from L to $Z(L)$. Then $(-)^{[p]} + \gamma$ is a restricted Lie structure on L if and only if γ is a semilinear map from L to $Z(L)$.*

Let A be a Poisson algebra over $\Bbbk$ and $Z(A)$ the center of A. Observe that $Z(A)$ is a left A-module with the action given by

$$A \times Z(A) \to Z(A), \quad (a, z) \mapsto a^p z.$$

A semilinear map $\psi\colon A \to Z(A)$ is called a *Frobenius derivation* of A with the values in $Z(A)$ provided that $\psi(ab) = a^p \psi(b) + b^p \psi(a)$ for any $a, b \in A$. For example, if $\psi_0\colon A \to A$ is a derivation, then $\psi\colon A \to Z(A)$, defined by $\psi(a) = (\psi_0(a))^p$ for all $a \in A$, is a Frobenius derivation of A with the values in $Z(A)$.

By Lemma 5.3(1), any two restricted Poisson structures on A differ by a semilinear map γ which appears in the next proposition, which was mentioned in [Bezrukavnikov and Kaledin 2008, p. 414].

Proposition 5.4. *Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a restricted Poisson algebra and γ a map from A to itself. Then the map $(-)^{\{p\}} + \gamma$ is a restricted Poisson structure if and only if γ is a Frobenius derivation of A with values in $Z(A)$.*

Proof. Let $(-)^{\{p\}_1}\colon A \to A$ be another p-map such that $(A, \cdot, \{-,-\}, (-)^{\{p\}_1})$ is also a restricted Poisson algebra. Since $(-)^{\{p\}_1}$ and $(-)^{\{p\}}$ are restricted structures on Lie algebra $(A, \{-,-\})$, $\gamma = (-)^{\{p\}_1} - (-)^{\{p\}}$ is a semilinear map from A to $Z(A)$ by Lemma 5.3. Moreover, for any $x, y \in A$, $(xy)^{\{p\}_1} = x^p y^{\{p\}_1} + y^p x^{\{p\}_1} + \Phi_p(x, y)$, and

$$\begin{aligned}\gamma(xy) &= (xy)^{\{p\}_1} - (xy)^{\{p\}}\\ &= x^p(y^{\{p\}_1} - y^{\{p\}}) + y^p(x^{\{p\}_1} - x^{\{p\}})\\ &= x^p\gamma(y) + y^p\gamma(x).\end{aligned}$$

It follows that γ is a Frobenius derivation of A with values in $Z(A)$.

Conversely, it follows from Lemma 5.3 that the map $(-)^{\{p\}}+\gamma$ is also a restricted Lie structure on $(A,\{-,-\})$, since γ is a semilinear map from A to $Z(A)$ and $(-)^{\{p\}}$ is a p-map of Lie algebra $(A,\{-,-\})$. Moreover, for any $x,y\in A$,

$$(xy)^{\{p\}}+\gamma(xy)=x^p(y^{\{p\}}+\gamma(y))+y^p(x^{\{p\}}+\gamma(x))+\Phi_p(x,y).$$

It follows that the Poisson algebra A together with the map $(-)^{\{p\}}+\gamma$ is a restricted structure. $\square$

By Proposition 5.4, the p-map of a restricted Poisson algebra is unique up to Frobenius derivations.

Remark 5.5. Let $(A,\cdot,\{-,-\},(-)^{\{p\}})$ be a restricted Poisson algebra and let $\gamma: A\to Z(A)$ be a semilinear map. Suppose that γ is not a Frobenius derivation (which is possible for many A) and defines a new p-map $(-)'^{\{p\}}=(-)^{\{p\}}+\gamma$. Then by Proposition 5.4, $(A,\cdot,\{-,-\},(-)'^{\{p\}})$ is not a restricted Poisson algebra, but it is still a weakly restricted Poisson algebra by Lemma 5.3(2).

6. Restricted Poisson algebras from restricted Lie algebras

We start with a general result.

Theorem 6.1. *Let $A=\Bbbk[x_i\mid i\in T]$ be a polynomial Poisson algebra with an index set T. If for each $i\in T$, there exists $\gamma(x_i)\in A$ such that $\mathrm{ad}^p_{x_i}=\mathrm{ad}_{\gamma(x_i)}$, then A admits a restricted Poisson structure $(-)^{\{p\}}$ such that $x_i^{\{p\}}=\gamma(x_i)$ for all $i\in T$.*

Proof. First we show that A has a weakly restricted Poisson structure, and then verify that the weakly restricted Poisson structure satisfies (E3.5.1).

For the sake of simplicity, we assume that $T=\{1,2,\dots,n\}$. To apply Lemma 1.3, we choose a canonical monomial $\Bbbk$-basis of A, which is

$$\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\mid i_1,i_2,\dots,i_n\geq 0\}.$$

We define $(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})^{\{p\}}$ inductively on the degree $i_1+i_2+\cdots+i_n$ such that

$$\mathrm{ad}^p_{(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})}=\mathrm{ad}_{(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})^{\{p\}}},$$

and therefore get the restricted Lie structure on $(A,\{-,-\})$ by Lemma 1.3. For convenience, we write $x^I=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ and $|I|=i_1+\cdots+i_n$ for $I=(i_1,\dots,i_n)$.

If $|I|=0$, then $x^I=1$ and we define $1^{\{p\}}=0$, and if $|I|=1$, then $x^I=x_i$ for some $1\leq i\leq n$. We define $x_i^{\{p\}}=\gamma(x_i)$ for each $1\leq i\leq n$. By hypothesis, $\mathrm{ad}^p_{x^I}=\mathrm{ad}_{(x^I)^{\{p\}}}$ for any I with $|I|=0,1$.

We proceed by induction and assume that $(x^I)^{\{p\}}$ has been defined such that $\mathrm{ad}^p_{x^I}=\mathrm{ad}_{(x^I)^{\{p\}}}$ for any x^I with $|I|\leq m$. For each monomial x^I of degree

$m+1$, we assume that k is the smallest subscript such that $i_k \geq 1$ in I, i.e., $I=(0,\ldots,0,i_k,\ldots,i_n)$, and define

$$\text{(E6.1.1)}\quad (x^I)^{\{p\}} = x_k^p(x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})^{\{p\}} + (x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})^p x_k^{\{p\}} + \Phi_p(x_k, x_k^{i_1-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n}).$$

By Lemma 3.2(1) for $(x,y)=(x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})$ and the above definition, we have $\mathrm{ad}_{x^I}^p = \mathrm{ad}_{(x^I)^{\{p\}}}$ for any $|I|=m+1$, which completes the induction. By Lemma 1.3, A has a weakly restricted Poisson structure.

Now let $\boldsymbol{b}$ be the set of all monomials, which is a $\Bbbk$-basis of A. We prove that (E3.5.1) holds for any pair of elements (x,y) in $\boldsymbol{b}$ by induction on $\deg x + \deg y$. If x or y is 1, then (E3.5.1) holds trivially, which also takes care of the case when $m := \deg x + \deg y \leq 1$. Suppose that the assertion holds for m and now assume that $\deg x + \deg y = m+1$. Let

$$xy = x_k^{i_k}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n}, \quad \text{where } i_k > 0.$$

By (E6.1.1), the pair $(x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})$ satisfies (E3.5.1). By symmetry, we may assume that $x = x_k g$. Then the above says that the pair (x_k, gy) satisfies (E3.5.1). By the induction hypothesis, the pairs (x_k, g) and (g,y) satisfy (E3.5.1). By Proposition 4.6(2), $(x,y)=(x_k g, y)$ satisfies (E3.5.1). By induction, (E3.5.1) holds for any two elements in $\boldsymbol{b}$. Finally the main statement follows from Theorem 4.7. □

As a consequence, we have the following.

Example 6.2. Let L be a restricted Lie algebra. We claim that the polynomial Poisson algebra $A := \Bbbk[L]$ (also denoted by $S(L)$) is a restricted Poisson algebra. Let $\{x_i\}_{i\in I}$ be a basis of L. Then, for each i, there is an $\gamma(x_i) := x_i^{[p]} \in L$ such that $\mathrm{ad}_{x_i}^p = \mathrm{ad}_{\gamma(x_i)}$ when restricted to L. Since A is a polynomial ring over L, both $\mathrm{ad}_{x_i}^p$ and $\mathrm{ad}_{\gamma(x_i)}$ extend uniquely to derivations of A. Thus $\mathrm{ad}_{x_i}^p = \mathrm{ad}_{\gamma(x_i)}$ holds when applied to A. The claim follows from Theorem 6.1 and there is a unique restricted structure $(-)^{\{p\}}$ on A such that

$$x^{\{p\}} = x^{[p]} \quad \forall\, x \in L.$$

Let V be a vector space. Then the free restricted Lie algebra $\mathrm{RLie}(V)$ can be defined by using the universal property or by taking the restricted Lie subalgebra of the free associative algebra generated by V with the p-map being the p-powering map. Now we can define the free restricted Poisson algebra generated by V.

Definition 6.3. Let V be a $\Bbbk$-space. The free restricted Poisson algebra generated by V is defined to be

$$\mathrm{FRP}(V) = \Bbbk[\mathrm{RLie}(V)].$$

The following universal property is standard [Shestakov 1993, Lemma 1, p. 312].

Lemma 6.4. *Let A be a restricted Poisson algebra and V be a vector space. Every $\Bbbk$-linear map $g : V \to A$ extends uniquely to a restricted Poisson algebra morphism $G : \mathrm{FRP}(V) \to A$ such that g factors through G.*

Continuing Example 6.2, when L is a restricted Lie algebra over $\Bbbk$ and $S(L) := \Bbbk[L]$ the symmetric algebra on L, then $S(L)$ admits an induced restricted Poisson structure. One natural setting in positive characteristic is to replace the symmetric algebra $S(L)$ by the truncated (or small) symmetric algebra $s(L)$. By definition, when L has a $\Bbbk$-basis $\{x_i\}_{i\in I}$,

$$s(L) = \Bbbk[x_i \mid i \in I]/(x_i^p,\ \forall\, i \in I). \tag{E6.4.1}$$

It is easily seen that $s(L)$ admits a Poisson structure with the bracket

$$\{f, g\} = \sum_{i,j} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) \{x_i, x_j\}$$

for any $f, g \in s(L)$. Next we show that $s(L)$ has a natural restricted Poisson structure.

Theorem 6.5. *Let L be a restricted Lie algebra over $\Bbbk$ of characteristic p and let $s(L)$ be the Poisson algebra with the bracket induced by L. Then $s(L)$ admits a natural restricted Poisson structure induced by the p-map of L.*

Proof. By Example 6.2, $S(L)$ has an induced restricted Poisson algebra structure. By (E6.4.1),

$$s(L) = S(L)/J,$$

where J is the Poisson ideal generated by x_i^p for all $i \in I$. By Proposition 3.5(3), $(x_i^p)^{\{p\}} = 0$. By Lemma 4.2, J is a restricted Poisson ideal as desired. □

7. Restricted Poisson algebras from deformation quantization

Bezrukavnikov and Kaledin [2008, Section 1.2] showed that a Frobenius-constant quantization automatically gives a restricted Poisson algebra. In this section, we consider a special deformation quantization of a Poisson algebra to produce more examples under a weaker condition.

Let A be a commutative (associative) algebra. Let $\Bbbk[\![t]\!]$ be the formal power series ring in one variable t. A *formal deformation* of A means an associative algebra $A[\![t]\!]$ over $\Bbbk[\![t]\!]$ with multiplication, denoted by m_t, satisfying

$$m_t(a \otimes b) = a * b = ab + m_1(a, b)t + \cdots + m_n(a, b)t^n + \cdots$$

for all $a, b \in A \subset A[\![t]\!]$. We should view $A[\![t]\!]$ as the power series ring in one variable t with coefficients in A where the associative multiplication m_t (or the star product $*$) is induced by a family of $\Bbbk$-bilinear maps $\{m_i\colon A\otimes A\to A\}_{i\geq 0}$ with $m_0(a,b)=ab$.

Define a bilinear map $\{-,-\}\colon A\otimes A\to A$ by setting $\{a,b\}=m_1(a,b)-m_1(b,a)$. It is easy to check that A together with the bracket $\{-,-\}$ is a Poisson algebra. Then $(A,\{-,-\})$ is called the *classical limit* of $(A[\![t]\!],m_t)$, and $(A[\![t]\!],m_t)$ is called a *deformation quantization* of the Poisson algebra $(A,\{-,-\})$.

For every $f\in A$, we write the p-power of f as

$$f^{*p}=\sum_{n=0}^{\infty}M_n^p(f)t^n=f^p+M_1^p(f)t+M_2^p(f)t^2+\cdots\in A[\![t]\!], \tag{E7.0.1}$$

where $M_i^p(f)\in A$ for all $i=0,1,2,\ldots$.

Proposition 7.1. *Let $(A,\cdot,\{-,-\})$ be a Poisson algebra over $\Bbbk$ and let $(A[\![t]\!],*)$ be a deformation quantization of A. If $M_n^p(f)=0$ for $1\leq n\leq p-2$ and f^p is central in $A[\![t]\!]$ for all $f\in A$, then A admits a restricted Poisson structure.*

Proof. Recall that $f*g=\sum_{n=0}^{\infty}m_n(f,g)t^n\in A[\![t]\!]$ for all $f,g\in A$, where $m_n(f,g)\in A$ for all n. By the definition of the deformation quantization,

$$\{f,g\}=m_1(f,g)-m_1(g,f)$$

for all $f,g\in A$. Considering the Frobenius map $f\mapsto f^{*p}$ in $A[\![t]\!]$, we get

$$[f^{*p},g]_*=[\underbrace{f,\ldots,f}_{p\text{ copies}},g]_* \tag{E7.1.1}$$

for all $f,g\in A$.

Since $[f,g]_*=\{f,g\}t \pmod{t^2}$ and $[-,-]_*$ is $\Bbbk[\![t]\!]$-bilinear, we have

$$[\underbrace{f,\ldots,f}_{p\text{ copies}},g]_*\equiv\{\underbrace{f,\ldots,f}_{p\text{ copies}},g\}t^p \pmod{t^{p+1}}.$$

By assumption, $M_n^p(f)=0$ for $1\leq n\leq p-2$ and f^p is central in $A[\![t]\!]$. Using the fact that (E7.1.1) or $\mathrm{ad}_{f^{*p}}(g)=(\mathrm{ad}_f)^p(g)$, it follows that

$$\{M_{p-1}^p(f),g\}t^p=\{\underbrace{f,\ldots,f}_{p\text{ copies}},g\}t^p \pmod{t^{p+1}}$$

or

$$\{M_{p-1}^p(f),g\}=m_1(M_{p-1}^p(f),g)-m_1(g,M_{p-1}^p(f))=\{\underbrace{f,\ldots,f}_{p\text{ copies}},g\} \tag{E7.1.2}$$

for all $g\in A$. We define $f^{\{p\}}=M_{p-1}^p(f)$ for any $f\in A$, and prove that the map $f\mapsto M_{p-1}^p(f)$ gives rise to a restricted Poisson structure on A.

Note that Definition 1.1(1) follows from (E7.1.2). Definition 1.1(2) follows from the fact that $(\lambda f)^{*p} = \lambda^p f^{*p}$. For the condition in Definition 1.1(3), we consider the Frobenius map of $A[\![t]\!]$, and get a restricted Lie structure of $(A[\![t]\!], [-,-]_*)$. It follows from Example 1.2 that

$$(f+g)^{*p} - f^{*p} - g^{*p} = \Lambda_p^*(f,g).$$

Computing the coefficients of t^{p-1} in the above equation, we get

$$(f+g)^{\{p\}} - f^{\{p\}} - g^{\{p\}} = \Lambda_p(f,g) \tag{E7.1.3}$$

as desired.

Finally it remains to show (E3.4.1). By assumption, $M_n^p(f) = 0$ for all $1 \le n \le p-2$. We compute the coefficient of t^{p-1} in the expression of $f^{*,2p}$ as follows:

$$\begin{aligned} f^{*,2p} &= f^{*p} * f^{*p} \\ &= (f^p + t^{p-1} M_{p-1}^p(f) + \cdots) * (f^p + t^{p-1} M_{p-1}^p(f) + \cdots) \\ &\equiv f^{2p} + 2 f^p M_{p-1}^p(f) t^{p-1} \pmod{t^p}. \end{aligned}$$

Assume that $f * f = f^2 + tW$, where

$$W = m_1(f,f) + m_2(f,f)t + \cdots .$$

It follows that

$$\begin{aligned} f^{*,2p} &= (f^{*2})^{*p} = (f^2 + tW)^{*p} \\ &= (f^2)^{*p} + (tW)^{*p} + \Lambda_p^*(f^2, tW) \\ &\equiv f^{2p} + M_{p-1}^p(f^2) t^{p-1} \pmod{t^p}. \end{aligned}$$

Therefore, for all $f \in A$, $f^{2^{\{p\}}} = 2 f^p f^{\{p\}}$, which is (E3.4.1). □

Before giving some explicit examples, we recall a result.

Lemma 7.2 [Bezrukavnikov and Kaledin 2008, Lemma 1.3]. *Let B be an associative algebra over a base field k of characteristic $p > 0$, and let $B_{(k)}$, $B_{(1)} = B$, $B_{(k)} = [B, B_{(k-1)}]$ be its central series with respect to the commutator. If $B_{(p)} = 0$ and $B_{(2)}^p = 0$, then the Frobenius map $x \mapsto x^p$ preserves the addition and the multiplication.*

Example 7.3. Let $A := \Bbbk[x,y]$ be a polynomial Poisson algebra over a field $\Bbbk$ of characteristic $p \ge 3$ with the bracket given by $\{x,y\} = 1$.

By a direct calculation, the Poisson algebra A admits a deformation quantization $(A[\![t]\!], *)$ with the star product given by

$$f * g = \sum_{0 \le n \le p-1} m_n(f,g) t^n = \sum_{0 \le n \le p-1} \frac{t^n}{n!} (\partial_1^n f)(\partial_2^n g) \tag{E7.3.1}$$

for all $f, g \in A$, where ∂_1 and ∂_2 are the partial derivatives of f with respect to the variables x and y, respectively.

Clearly, $f^p * g = f^p g = g * f^p$ for any $f, g \in A$, whence f^p is central in $A[\![t]\!]$. Moreover, for every $f \in A$, we claim that

$$M_n^p(f) = 0 \quad \text{for } 1 \leq n \leq p-2.$$

In fact, for any $f, g \in A \subset A[\![t]\!]$, we have

$$[f, g]_* = f * g - g * f \in (t).$$

It follows that

$$[f, g]_*^{*,p} \in (t^p),$$

and

$$[f_1, \ldots, f_p]_* \in (t^{p-1})$$

for all $f_1, \ldots, f_p \in A[\![t]\!]$. Since t is central, we can define the quotient algebra $B := A[\![t]\!]/(t^{p-1})$. The above computation shows that $B_{(p)} = 0$ and $B_{(2)}^p = 0$. By Lemma 7.2, it follows that the Frobenius map $b \mapsto b^{*p}$ of B is additive and multiplicative. By (E7.3.1), an easy computation shows that $x^{*p} = x^p$ and $y^{*p} = y^p$ in $A[\![t]\!]$. For each $f \in A$, let $\bar{f}$ be the corresponding element in B. Then

$$\bar{f}^{*p} = \overline{f^p} \in B$$

since the map $\bar{f} \mapsto \bar{f}^{*p}$ preserves the addition and the multiplication in B. It follows that $f^{*p} - f^p \in (t^{p-1})$ and therefore $M_n^p = 0$ for any $1 \leq n \leq p-2$.

By Proposition 7.1, A admits a restricted Poisson structure with the p-map $f^{\{p\}} = M_{p-1}^p(f)$ for any $f \in A$. The p-map agrees with (E4.8.2) when $p = 3$ and (E4.8.3) when $p = 5$.

The next example is a generalization of the previous one.

Example 7.4. Let $A := \Bbbk[x_1, \ldots, x_m]$ be a polynomial Poisson algebra with Poisson bracket determined by $\{x_i, x_j\} = c_{ij} \in \Bbbk$ for $1 \leq i < j \leq m$. Let μ denote the associative product of A which is extended to the power series ring of A. Let $\partial_i := \partial/\partial x_i$ for all $1 \leq i \leq m$. For each scalar $c \in \Bbbk$, let $\exp(tc\partial_i \otimes \partial_j)$ be the operator

$$\sum_{0 \leq n \leq p-1} \frac{(ct)^n}{n!} \partial_i^n \otimes \partial_j^n \; : \; A[\![t]\!] \otimes A[\![t]\!] \to A[\![t]\!] \otimes A[\![t]\!].$$

By a direct calculation, a deformation quantization $(A[\![t]\!], *)$ of the Poisson algebra A is given by

$$f * g = \mu\Big(\prod_{1 \leq i < j \leq m} \big(\exp(c_{ij} t \, \partial_i \otimes \partial_j)\big)(f \otimes g) \Big)$$

for all $f, g \in A$. Clearly, $f^p \in A \subset A[\![t]\!]$ is central for any $f \in A$. Being similar to the proof of Example 7.3, we have $M_n^p(f) = 0$ for $1 \le n \le p-2$ and all $f \in A$. By Proposition 7.1, A admits a restricted Poisson structure with the p-map $f^{\{p\}} = M_{p-1}^p(f)$ for any $f \in A$. When $p = 3$, the p-map is given in Example 4.10.

Example 7.5. Let $B_{2n} = \Bbbk[x_1, \dots, x_{2n}]/I$ be the p-truncated polynomial Poisson algebra in $2n$ variables over $\Bbbk$, where the Poisson bracket is defined by

$$\{f, g\} = \sum_{i=1}^{n} \bigl(\partial_i(f)\partial_{n+i}(g) - \partial_{n+i}(f)\partial_i(g)\bigr)$$

for all $f, g \in B_{2n}$, and I is generated by x_i^p, $i = 1, \dots, 2n$. Skryabin [2002] introduced the notion of the normalized p-map on $(B_{2n}, \{-,-\})$, say, $1^{\{p\}} = 0$ and $f^{\{p\}} \in \mathfrak{m}^2$ for all $f \in \mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of B_{2n} as an associative algebra.

We consider the Poisson algebra $A = \Bbbk[x_1, \dots, x_{2n}]$ in Example 7.4 with the bracket given by $c_{ij} = \delta_{i+n,j}$ for all $1 \le i < j \le 2n$. Clearly, x_i^p is central and I is a Poisson ideal of A. By Proposition 3.5(3), $(x_i^p)^{\{p\}} = 0$ for all $i \in I$, and by Lemma 4.2, I is a restricted Poisson ideal of A. Therefore, it follows from Proposition 4.3 that the Poisson algebra B_{2n} admits a restricted Poisson structure. Clearly, this p-map is normalized.

8. Connection with restricted Lie–Rinehart algebras

Some definitions concerning Lie–Rinehart algebras were given in Section 2. Let A be a Poisson algebra and $\Omega_{A/\Bbbk}$ its Kähler differentials module. Then the pair $(A, \Omega_{A/\Bbbk})$ is a Lie–Rinehart algebra over $\Bbbk$, where the anchor map $\alpha : \Omega_{A/\Bbbk} \to \mathrm{Der}(A)$ is given in (E2.2.2). Dokas [2012] introduced the notion of a restricted Lie–Rinehart algebra and studied its cohomology theory. The goal of this section is to show that the Lie–Rinehart algebra $(A, \Omega_{A/\Bbbk})$ admits a natural restricted structure if the Poisson algebra A is weakly restricted and $\Omega_{A/\Bbbk}$ is a free module over A.

Let $(L, (-)^{[p]})$ and $(L', (-)^{[p]})$ be restricted Lie algebras. A map $f : (L, (-)^{[p]}) \to (L', (-)^{[p]})$ is called a restricted Lie homomorphism if f is a Lie algebra homomorphism and satisfies $f(x^{[p]}) = f(x)^{[p]}$ for all $x \in L$.

Definition 8.1 [Dokas 2012, Definition 1.7]. A *restricted Lie–Rinehart algebra* $(A, L, (-)^{[p]})$ over a commutative $\Bbbk$-algebra A is a Lie–Rinehart algebra over A such that

(a) $(L, (-)^{[p]})$ is a restricted Lie algebra over $\Bbbk$,

(b) the anchor map is a restricted Lie homomorphism, and

(c) we have
$$(aX)^{[p]} = a^p X^{[p]} + (aX)^{p-1}(a)X$$
for all $a \in A$ and $X \in L$.

We now prove Theorem 0.5.

Theorem 8.2. *Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a weakly restricted Poisson algebra. If the module of Kähler differentials $\Omega_{A/\Bbbk}$ is free, then the Lie–Rinehart algebra $(A, \Omega_{A/\Bbbk}, (-)^{[p]})$ is restricted, where the p-map of $\Omega_{A/\Bbbk}$ is defined by*
$$(x\,\mathrm{d}u)^{[p]} = x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u$$
for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$.

Proof. Since $\Omega_{A/\Bbbk}$ is a free A-module, $\Omega_{A/\Bbbk}$ can be embedded into the universal enveloping algebra $\mathcal{U}(A, \Omega_{A/\Bbbk})$ (Lemma 2.3). By the proof of [Dokas 2012, Proposition 2.2], it suffices to show that
$$\mathrm{ad}^p_{x\,\mathrm{d}u}(y\,\mathrm{d}v) = [x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u,\, y\,\mathrm{d}v]$$
for all $x\,\mathrm{d}u$ and $y\,\mathrm{d}v \in \Omega_{A/\Bbbk}$.

By Hochschild's relation [1955, Lemma 1], we get in $\mathcal{U}(A, L)$ the relation
$$(\iota_2(x\,\mathrm{d}u))^p = \iota_1(x^p)(\iota_2(\mathrm{d}u))^p + \iota_2((x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u)$$
for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$. Considering the Frobenius map of $\mathcal{U}(A, L)$, we have
$$[(\iota_2(\mathrm{d}u))^p, \iota_1(y)] = [\iota_2(\mathrm{d}u), \dots, \iota_2(\mathrm{d}u), \iota_1(y)] = \iota_1((\mathrm{ad}_u)^p(y)),$$
and hence
$$\iota_2(\mathrm{d}u)^p \iota_1(y) = \iota_1(y)\iota_2(\mathrm{d}u)^p + \iota_1((\mathrm{ad}_u)^p(y))$$
for all $\mathrm{d}u \in \Omega_{A/\Bbbk}$ and $y \in A$. Moreover, for $x\,\mathrm{d}u, y\,\mathrm{d}v \in \Omega_{A/\Bbbk} \subset \mathcal{U}(A, L)$,
$$\begin{aligned}
&[\iota_1(x^p)(\iota_2(\mathrm{d}u))^p, \iota_2(y\,\mathrm{d}v)] \\
&\qquad= \iota_1(x^p)(\iota_2(\mathrm{d}u))^p\iota_1(y)\iota_2(\mathrm{d}v) - \iota_1(y)\iota_2(\mathrm{d}v)\iota_1(x^p)(\iota_2(\mathrm{d}u))^p \\
&\qquad= \iota_1(x^p)\big(\iota_1(y)(\iota_2(\mathrm{d}u))^p + \iota_1((\mathrm{ad}_u)^p(y))\big)\iota_2(\mathrm{d}v) \\
&\qquad\qquad\qquad - \iota_1(y)\big(\iota_1(x^p)\iota_2(\mathrm{d}v) + \iota_1(\{v, x^p\})\big)(\iota_2(\mathrm{d}u))^p \\
&\qquad= \iota_1(x^p y)[(\iota_2(\mathrm{d}u))^p, \iota_2(\mathrm{d}v)] + \iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v) \\
&\qquad= \iota_1(x^p y)\iota_2(\mathrm{ad}^p_{\mathrm{d}u}(\mathrm{d}v)) + \iota_2(x^p(\mathrm{d}u)^p(y)\,\mathrm{d}v) \\
&\qquad= \iota_1(x^p y)\iota_2(\mathrm{d}(\mathrm{ad}^p_u(v))) + \iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v),
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\iota_2(\mathrm{ad}^p_{x\,\mathrm{d}u}(y\,\mathrm{d}v)) \\
&\quad = [(\iota_2(x\,\mathrm{d}u))^p, \iota_2(y\,\mathrm{d}v)] \\
&\quad = [\iota_1(x^p)(\iota_2(\mathrm{d}u))^p + \iota_2((x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u), \iota_2(y\,\mathrm{d}v)] \\
&\quad = \iota_1(x^p y)\iota_2(\mathrm{d}(\mathrm{ad}^p_u(v))) + \iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v) + \iota_2([(x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u, y\,\mathrm{d}v]) \\
&\quad = \iota_2(x^p y\,\mathrm{d}(\mathrm{ad}^p_u(v))) + \iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v) + \iota_2([(x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u, y\,\mathrm{d}v]) \\
&\quad = \iota_2([x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u, y\,\mathrm{d}v]),
\end{aligned}$$

and hence

$$\mathrm{ad}^p_{x\,\mathrm{d}u}(y\,\mathrm{d}v) = [x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u,\, y\,\mathrm{d}v]$$

as desired. □

For Poisson algebras A in Examples 4.8–4.10, Example 6.2, Theorem 6.5, and Examples 7.3–7.5, it is automatic that $\Omega_{A/\Bbbk}$ is free over A.

9. Restricted Poisson Hopf algebras

We first recall the definition of Poisson Hopf algebras. The notion of a Poisson Hopf algebra was probably first introduced by Drinfel'd [1985; 1987]; see also [Doebner et al. 1990].

Definition 9.1. Let A be a Poisson algebra. We say that A is a Poisson Hopf algebra if

(1) A is a Hopf algebra with the usual operations Δ, ϵ, S;

(2) $\Delta : A \to A \otimes A$ and $\epsilon : A \to \Bbbk$ are Poisson algebra morphisms and $S : A \to A$ is a Poisson algebra antiautomorphism.

To define restricted Poisson Hopf algebras, we first need to show that the tensor product of two restricted Poisson algebras is again a restricted Poisson algebra.

Proposition 9.2. *Let A and B be two restricted Poisson algebras. Then there is a unique restricted Poisson structure on $A \otimes B$ such that*

(E9.2.1) $$(a \otimes b)^{\{p\}} = a^{\{p\}} \otimes b^p + a^p \otimes b^{\{p\}}$$

for all $a \in A$ and $b \in B$.

Proof. First of all, it is well known that $A \otimes B$ is a Poisson algebra with bracket defined by

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Let $\{a_i\}_{i\in I}$ (respectively, $\{b_j\}_{j\in J}$) be a $\Bbbk$-basis of A (respectively, B) and assume that $1_A \in \{a_i\}_{i\in I}$ and $1_B \in \{b_j\}_{i\in J}$. Then $\{a_i \otimes b_j\}_{i\in I, j\in J}$ is a $\Bbbk$-basis of $A \otimes B$.

For any $a \in A$ and $b \in B$, $\mathrm{ad}^p_{a\otimes b}$ is a derivation. For any $c \otimes d \in A \otimes B$, we have

$$\begin{aligned}
\mathrm{ad}^p_{a\otimes b}(c\otimes d) &= (1\otimes d)\,\mathrm{ad}^p_{a\otimes b}(c\otimes 1) + (c\otimes 1)\,\mathrm{ad}^p_{a\otimes b}(1\otimes d)\\
&= (1\otimes d)(\mathrm{ad}^p_a(c)\otimes b^p) + (c\otimes 1)(a^p\otimes \mathrm{ad}^p_b(d))\\
&= (1\otimes d)(\mathrm{ad}_{a^{\{p\}}}(c)\otimes b^p) + (c\otimes 1)(a^p\otimes \mathrm{ad}_{b^{\{p\}}}(d))\\
&= (1\otimes d)(\mathrm{ad}_{a^{\{p\}}\otimes b^p}(c\otimes 1)) + (c\otimes 1)(\mathrm{ad}_{a^p\otimes b^{\{p\}}}(1\otimes d))\\
&= (1\otimes d)(\mathrm{ad}_{a^{\{p\}}\otimes b^p}(c\otimes 1)) + (c\otimes 1)(\mathrm{ad}_{a^{\{p\}}\otimes b^p}(1\otimes d))\\
&\qquad + (1\otimes d)(\mathrm{ad}_{a^p\otimes b^{\{p\}}}(c\otimes 1)) + (c\otimes 1)(\mathrm{ad}_{a^p\otimes b^{\{p\}}}(1\otimes d))\\
&= \mathrm{ad}_{a^{\{p\}}\otimes b^p}(c\otimes d) + \mathrm{ad}_{a^p\otimes b^{\{p\}}}(c\otimes d)\\
&= \mathrm{ad}_{a^{\{p\}}\otimes b^p + a^p\otimes b^{\{p\}}}(c\otimes d).
\end{aligned}$$

In particular,

$$\mathrm{ad}^p_{a_i\otimes b_j} = \mathrm{ad}_{(a_i^{\{p\}}\otimes b_j^p + a_i^p\otimes b_j^{\{p\}})}$$

for all i and j. Since $\{a_i \otimes b_j\}_{i\in I, j\in J}$ is a $\Bbbk$-basis of $A\otimes B$, by Lemma 1.3, there is a unique weak restricted Poisson structure on $A \otimes B$ such that

$$(a_i\otimes b_j)^{\{p\}} = a_i^{\{p\}}\otimes b_j^p + a_i^p \otimes b_j^{\{p\}} \tag{E9.2.2}$$

for all i, j, which agrees with (E9.2.1). It remains to show that this weak restricted Poisson structure on $A\otimes B$ is indeed a restricted Poisson structure and (E9.2.1) holds.

We first prove (E9.2.1). By (E9.2.2), $(a_i \otimes 1)^{\{p\}} = a_i^{\{p\}} \otimes 1$. It follows from Definition 1.1 that

$$(a\otimes 1)^{\{p\}} = a^{\{p\}}\otimes 1 \tag{E9.2.3}$$

for all $a \in A$. By symmetry, $(1 \otimes b)^{\{p\}} = 1 \otimes b^{\{p\}}$ for all $b \in B$. Since $\{a_i \otimes 1, 1 \otimes b_j\} = 0$, (E9.2.2) implies that the pair $(a_i \otimes 1, 1 \otimes b_j)$ satisfies (E3.5.1). By Proposition 4.6(4), $R_{a_i\otimes 1}$ is a $\Bbbk$-vector space, and by assumption, $\{b_j\}$ is a $\Bbbk$-basis of B, so we have that $R_{a_i\otimes 1} \supseteq B$. Or, for any $b \in B$, the pair $(a_i \otimes 1, 1 \otimes b)$ satisfies (E3.5.1). By switching a and b and applying the same argument, one sees that any pair $(a \otimes 1, 1 \otimes b)$ satisfies (E3.5.1). This means that

$$\begin{aligned}
(a\otimes b)^{\{p\}} &= (a\otimes 1)^{\{p\}}(1\otimes b)^p + (a\otimes 1)^p(1\otimes b)^{\{p\}} + \Phi_p(a\otimes 1, 1\otimes b)\\
&= (a\otimes 1)^{\{p\}}(1\otimes b)^p + (a\otimes 1)^p(1\otimes b)^{\{p\}}\\
&= a^{\{p\}}\otimes b^p + a^p\otimes b^{\{p\}}.
\end{aligned}$$

So we proved (E9.2.1).

For the rest, we claim that for any pair of elements $(a_i \otimes b_j, a_k \otimes b_l)$, (E3.5.1) holds. By using (E9.2.3), (E3.5.1) holds for all pairs of the form $(a \otimes 1, a' \otimes 1)$. By symmetry, (E3.5.1) holds for all pairs of the form $(1 \otimes b, 1 \otimes b')$. By (E9.2.1), (E3.5.1) holds for pairs of the form $(a \otimes 1, 1 \otimes b)$. Set $f = a \otimes 1$, $g = a' \otimes 1$ and $h = 1 \otimes b$ for any $a, a' \in A$ and $b \in B$. Then (f, g), (g, h) and (fg, h) satisfy (E3.5.1). By Proposition 4.6(2), (f, gh) satisfies (E3.5.1). Or equivalently, $(a \otimes 1, a' \otimes b)$ satisfies (E3.5.1). By symmetry, $(1 \otimes b, a \otimes b')$, $(a \otimes b, a' \otimes 1)$ and $(a \otimes b, 1 \otimes b')$ satisfy (E3.5.1). Recycle the letters and let $f = a \otimes b$, $g = a' \otimes 1$ and $h = 1 \otimes b'$. We have that (f, g), (g, h) and (fg, h) all satisfy (E3.5.1). By Proposition 4.6(2), (f, gh) satisfies (E3.5.1). By choosing special a, a', b, b' we have that $(a_i \otimes b_j, a_k \otimes b_l)$ satisfies (E3.5.1) as desired. This says that every pair of elements from the $\Bbbk$-basis $\{a_i \otimes b_j\}_{i \in I, j \in J}$ satisfies (E3.5.1). By Theorem 4.7, the weak restricted Poisson structure on $A \otimes B$ is actually a restricted Poisson structure.

The above proof shows that there is a unique restricted Poisson structure on $A \otimes B$ satisfying (E9.2.2). Since (E9.2.1) is a consequence of (E3.5.1), the assertion follows. □

Now it is reasonable to define a restricted Poisson Hopf algebra.

Definition 9.3. A restricted Poisson algebra H is called a *restricted Poisson Hopf algebra* if there are restricted Poisson algebra maps $\Delta : H \to H \otimes H$ and $\epsilon : H \to \Bbbk$ and a restricted Poisson algebra antiautomorphism $S : H \to H$ such that H together with (Δ, ϵ, S) becomes a Hopf algebra.

One canonical example is the following.

Example 9.4. Let L be a restricted Lie algebra. Then $s(L)$ (given in Theorem 6.5) is a restricted Poisson Hopf algebra with the structure maps determined by

$$\begin{aligned} \Delta &: x \to x \otimes 1 + 1 \otimes x, \\ \epsilon &: x \to 0, \\ S &: x \to -x \end{aligned}$$

for all $x \in L$. It is straightforward to check that $s(L)$ is a restricted Poisson Hopf algebra. Similarly, $S(L)$ (given in Example 6.2) is a restricted Poisson Hopf algebra with structure maps determined as above.

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