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REMARKS ON GJMS OPERATOR OF ORDER SIX

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We study analysis aspects of the sixth-order GJMS operator P_g^6. Under conformal normal coordinates around a point, we present the expansions of Green's function of P_g^6 with pole at this point. As a starting point of the study of P_g^6, we manage to give some existence results of the prescribed Q-curvature problem on Einstein manifolds. One among them is that for $n \geq 10$, let (M^n, g) be a closed Einstein manifold of positive scalar curvature and f a smooth positive function in M. If the Weyl tensor is nonzero at a maximum point of f and f satisfies a vanishing order condition at this maximum point, then there exists a conformal metric $\tilde{g}$ of g such that its Q-curvature $Q_{\tilde{g}}^6$ equals f.

1. Introduction

Recently, some remarkable developments have been achieved in the existence theory of the positive constant Q-curvature problem associated to the Paneitz–Branson operator. One key ingredient in such works is that a strong maximum principle for the fourth-order Paneitz–Branson operator is discovered under a hypothesis on the positivity of some conformal invariants or Q-curvature of the background metric. The readers are referred to [Gursky et al. 2016; Gursky and Malchiodi 2015; Hang and Yang 2016; Li and Xiong 2015] and the references therein. This naturally stimulates us to study the GJMS operator of order six and its associated Q-curvature problem, the analogue to the Yamabe problem and Q-curvature problem for the Paneitz–Branson operator. Except for the aforementioned cases, due to the lack of a maximum principle for higher order elliptic equations in general, the existence theory of such problems needs to be developed. Until an analogue of Aubin's result [1976] for the Yamabe problem is verified in Proposition 3.2 below, by adapting some ideas for the Paneitz–Branson operator from [Esposito and Robert 2002; Djadli et al. 2000], we establish some existence results of the prescribed Q-curvature problem on Einstein manifolds, in which case the sixth-order GJMS operator has constant coefficients.

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The conformally covariant GJMS operators with principle part $(-\Delta_g)^k$, $k \in \mathbb{N}$ were discovered by Graham, Jenne, Mason and Sparling [Graham et al. 1992]. In particular, the GJMS operator of order six and the associated Q-curvature are given as follows (see [Juhl 2013; Wünsch 1986]): on manifolds (M^n, g) of dimension $n \geq 3$ and $n \neq 4$, denote by $\sigma_k(A_g)$ the k-th elementary symmetric function of the Schouten tensor

$$A_{ij} = \frac{1}{n-2}\Big(R_{ij} - \frac{R_g}{2(n-1)} g_{ij}\Big).$$

Denote by

$$C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik}, \quad B_{ij} = \Delta_g A_{ij} - \nabla^k \nabla_j A_{ik} - A^{kl} W_{kijl} = \nabla^k C_{ijk} - A^{kl} W_{kijl}$$

the Cotton tensor and Bach tensor, respectively. Let

$$\begin{aligned} T_2 =&(n-2)\sigma_1(A_g)g - 8A_g = -\frac{8}{n-2}\operatorname{Ric}_g + \frac{n^2-4n+12}{2(n-1)(n-2)} R_g g;\\ T_4 =& -\frac{3n^2-12n-4}{4}\sigma_1(A_g)^2 g + 4(n-4)|A|_g^2 g + 8(n-2)\sigma_1(A_g)A_g\\ &+ (n-6)\Delta_g\sigma_1(A_g)g - 48A_g^2 - \frac{16}{n-4}B_g;\\ v_6 =& -\frac{1}{8}\sigma_3(A_g) - \frac{1}{24(n-4)}\langle B, A\rangle_g. \end{aligned}$$

Then, the Q-curvature Q_g^6 is defined by

$$\begin{aligned} Q_g^6 = &-3!\,2^6 v_6 - \frac{n+2}{2}\Delta_g(\sigma_1(A_g)^2) + 4\Delta_g|A|_g^2\\ &- 8\delta(A_g d\sigma_1(A_g)) + \Delta_g^2\sigma_1(A_g) - \frac{n-6}{2}\sigma_1(A_g)\Delta_g\sigma_1(A_g)\\ &- 4(n-6)\sigma_1(A_g)|A|_g^2 + \frac{(n-6)(n+6)}{4}\sigma_1(A_g)^3, \end{aligned} \tag{1-1}$$

and the GJMS operator of sixth-order P_g^6 is given by[1]

$$-P_g^6 = \Delta_g^3 + \Delta_g\delta T_2 d + \delta T_2 d\Delta_g + \frac{n-2}{2}\Delta_g(\sigma_1(A_g)\Delta_g) + \delta T_4 d - \frac{n-6}{2}Q_g^6, \tag{1-2}$$

where $-\delta d = \Delta_g$. The operator P_g^6 is conformally covariant in the sense that if $\tilde{g} = u^{4/(n-6)}g$, $0 < u \in C^\infty(M)$ with $n \geq 3$ and $n \neq 4, 6$,

$$u^{\frac{n+6}{n-6}} P_{\tilde{g}}^6 \varphi = P_g^6(u\varphi), \tag{1-3}$$

and in dimension 6,

$$P_{e^{2u}g}^6 \varphi = e^{-6u} P_g^6 \varphi$$

[1] The definition of P_g^6 differs from the formula (10.15) in [Juhl 2013] by a minus sign.

for all $\varphi \in C^\infty(M)$. When (M, g) is Einstein, P_g^6 has constant coefficients; explicitly,

$$Q_g^6 = \frac{n^4-20n^2+64}{32n^2(n-1)^3} R_g^3,$$

$$-P_g^6 = \Delta_g^3 + \frac{-3n^2+6n+32}{4n(n-1)} R_g \Delta_g^2 + \frac{3n^4-12n^3-52n^2+128n+192}{16n^2(n-1)^2} R_g^2 \Delta_g - \frac{n-6}{2} Q_g^6.$$

Obviously, when $n \geq 7$, Q_g^6 is a positive constant whenever the scalar curvature R_g is positive. Through a direct computation, the GJMS operator P_g^6 has the following factorization:

$$P_g^6 = \Big(-\Delta_g + \frac{(n-6)(n+4)}{4n(n-1)} R_g\Big)\Big(-\Delta_g + \frac{(n-4)(n+2)}{4n(n-1)} R_g\Big)\Big(-\Delta_g + \frac{n-2}{4(n-1)} R_g\Big). \tag{1-4}$$

In general, as shown in [Fefferman and Graham 2012] and [Gover 2006], on Einstein manifolds the GJMS operator of order $2k$ for all positive integers k satisfies the above property as

$$P_g^{2k} = \prod_{i=1}^{k} \Big(-\Delta_g + \frac{R_g}{4n(n-1)}(n+2i-2)(n-2i)\Big).$$

In particular, choose $M^n = S^n,\ g = g_{S^n}$, then

$$Q_{S^n}^6 = \frac{n(n^4-20n^2+64)}{32},$$

$$\begin{aligned} P_{S^n}^6 &= -\Delta_{S^n}^3 - \frac{-3n^2+6n+32}{4} \Delta_{S^n}^2 - \frac{3n^4-12n^3-52n^2+128n+192}{16} \Delta_{S^n} + \frac{n-6}{2} Q_{S^n}^6 \\ &= \Big(-\Delta_{S^n} + \frac{(n-6)(n+4)}{4}\Big)\Big(-\Delta_{S^n} + \frac{(n-4)(n+2)}{4}\Big)\Big(-\Delta_{S^n} + \frac{n(n-2)}{4}\Big). \end{aligned}$$

From now on, we set $P_g = P_g^6$ and $Q_g = Q_g^6$ unless stated otherwise. Then, for any $\varphi \in H^3(M, g)$, we get

$$\begin{aligned} &\int_M \varphi P_g \varphi \, d\mu_g \\ &= \int_M \Big(|\nabla\Delta\varphi|_g^2 - 2T_2(\nabla\Delta\varphi, \nabla\varphi) - \frac{n-2}{2}\sigma_1(A)(\Delta_g\varphi)^2 - T_4(\nabla\varphi, \nabla\varphi) + \frac{n-6}{2} Q_g \varphi^2\Big) \, d\mu_g. \end{aligned}$$

As a starting point of the study on the sixth-order GJMS operator, we obtain some existence results of conformal metrics with positive Q-curvature candidates on closed Einstein manifolds under some additional natural assumptions.

Theorem 1.1. *Suppose (M^n, g) is a closed Einstein manifold of dimension $n \geq 10$ and has positive scalar curvature. Let f be a smooth positive function on M.*

Assume the Weyl tensor W_g is nonzero at a maximum point p of f and f satisfies the vanishing order condition at p:

$$\begin{cases} \Delta_g f(p) = 0 & \text{if } n = 10, \\ \nabla^k f(p) = 0,\ k = 2, 3, 4 & \text{if } n \geq 11. \end{cases} \tag{1-5}$$

Then there exists a smooth solution to the Q-curvature equation

$$P_g u = f u^{\frac{n+6}{n-6}}, \quad u > 0 \quad \text{in } M.$$

We remark that the condition (1-5) imposed on the Q-curvature candidates f is conformally invariant. The condition that (M, g) is Einstein is only used to seek a *positive* solution. Theorem 1.1 is a special case of a generalized Theorem 3.3.

This paper is organized as follows. In Section 2, the expansions of Green's function for P_g when $n \geq 7$ are presented under conformal normal coordinates around a point. The technique used here is basically inspired by Lee and Parker [1987]; see also [Hang and Yang 2016]. The complicated computations of the term $P_g(r^{6-n})$ are left to the Appendix, where r is the geodesic distance from this point. In Section 3, we prove an analogue (cf., Proposition 3.2) of Aubin's result for any closed manifold of dimension $n \geq 10$, which is not locally conformally flat. Based on this result, using the mountain pass lemma we state in Theorem 3.3 some results of the prescribed Q-curvature problem associated to the sixth-order GJMS operator on Einstein manifolds. Then our main Theorem 1.1 directly follows from Theorem 3.3.

2. Expansion of Green's function of P_g

Based on the survey paper by Lee and Parker [1987] on the Yamabe problem, the method of deriving expansions of Green's function of P_g is more or less standard except for careful computations on some lower-order terms involved in P_g. One may also refer to [Hang and Yang 2016] for the Paneitz–Branson operator case. Green's functions of conformally covariant operators play an important role in the solvability of the constant curvature problems, for instance, the Yamabe problem (see [Lee and Parker 1987] etc.) and the constant Q-curvature problem for the Paneitz–Branson operator (see [Djadli et al. 2000; Esposito and Robert 2002; Gursky et al. 2016; Hang and Yang 2016], etc.). In particular, F. Hang and P. Yang [2016] set up a dual variational method of the minimization for the Paneitz–Branson functional to seek a positive maximizer of the dual functional; such a scheme heavily relies on the positivity and expansion of its Green's function. We expect that the expansion of Green's function for P_g^6 will be useful to some possible future applications.

Throughout, we use the following notation: $2^\sharp = 2n/(n-6)$, $\omega_n = \mathrm{vol}(S^n, g_{S^n})$ and when $n > 6$, $c_n = 1/(8(n-2)(n-4)(n-6)\omega_{n-1})$. For $m \in \mathbb{Z}_+$, let

$$\mathcal{P}_m := \{\text{homogeneous polynomials in } \mathbb{R}^n \text{ of degree } m\}$$

and

$$\mathcal{H}_m := \{\text{harmonic polynomials in } \mathbb{R}^n \text{ of degree } m\}.$$

Then $\mathcal{P}_m$ has the following decomposition (see [Stein 1970], p. 68–70):

$$\mathcal{P}_m = \bigoplus_{k=0}^{[m/2]} (r^{2k} \mathcal{H}_{m-2k}).$$

Proposition 2.1. *Assume $n > 6$ and $\ker P_g = 0$. Let $G_p(x)$ be the Green's function of the sixth-order GJMS operator at the pole $p \in M^n$ with the property that $P_g G_p = c_n \delta_p$ in the sense of distributions. Then, under the conformal normal coordinates around p with conformal metric g, $G_p(x)$ has the following expansions:*

(a) *If n is odd, then*

$$G_p(x) = r^{6-n}\left(1 + \sum_{k=1}^{n} \psi_k\right) + A + O(r),$$

where A is a constant and $\psi_k \in \mathcal{P}_k$.

(b) *If n is even, then*

$$G_p(x) = r^{6-n}\left(1 + \sum_{k=1}^{n} \psi_k\right) + r^{6-n}\left(\sum_{k=n-4}^{n} \varphi_k\right)\log r + r^{6-n}\left(\sum_{k=n-4}^{n} \varphi_k'\right)\log^2 r + r^{6-n}\left(\sum_{k=n-2}^{n} \varphi_k''\right)\log^3 r + \varphi_n''' \log^4 r + A + O(r),$$

where A is a constant and $\psi_k, \varphi_k, \varphi_k', \varphi_k'', \varphi_k''' \in \mathcal{P}_k$.

Moreover, we may restate some of the above results in another way.

(c) *If $n = 7, 8, 9$ or M is conformally flat near p, then*

$$G_p(x) = c_n r^{6-n} + A + O(r),$$

where A is a constant.

(d) *If $n = 10$, then*

$$G_p(x) = c_n r^{-4} + \frac{1}{17280}|W(p)|^2 \log r + O(1).$$

(e) *If $n \geq 11$, then*

$$G_p(x) = c_n r^{6-n} + \psi_4 r^{6-n} + O(r^{11-n}),$$

where $\psi_4 \in \mathcal{P}_4$ *and*

$$\psi_4(x) = \frac{1}{135(n-2)}\Big[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{r^2}{n+4}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 + \frac{3}{2(n+4)(n+2)}|W(p)|^2r^4\Big] + \frac{3n-20}{270(n+4)(n-4)(n-8)}r^2\Big[\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n}|W(p)|^2r^2\Big] - \frac{5n^2-66n+224}{120(n-8)(n-4)}r^2\Big[\sigma_1(A)_{,ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2\Big] + \frac{3n^4-16n^3-164n^2+400n+2432}{576(n+4)(n+2)n(n-1)}|W(p)|^2r^4.$$

Before starting to derive the expansion of Green's function of P_g, we first need to introduce some notation. For $\alpha \in \mathbb{R}$, set

$$A_\alpha = r^2\Delta_0 + 2\alpha r\partial_r + \alpha(\alpha+n-2), \quad A_{\alpha,g} = r^2\Delta_g + 2\alpha r\partial_r + \alpha(\alpha+n-2),$$

where Δ_0 denotes the Euclidean Laplacian, and

$$B_\alpha = \frac{\partial}{\partial\alpha}A_\alpha = 2r\partial_r + 2\alpha + n - 2.$$

For $k \in \mathbb{Z}_+$, a straightforward computation yields (also see [Hang and Yang 2016, Lemma 2.4])

$$A_\alpha(\varphi\log^k r) = A_\alpha\varphi\log^k r + kB_\alpha\varphi\log^{k-1} r + k(k-1)\varphi\log^{k-2} r.$$

From this, for α, β, $\gamma \in \mathbb{R}$ we get

$$\begin{aligned}(2\text{-}1)\quad & A_\gamma A_\beta A_\alpha(\varphi\log^k r)\\ &= A_\gamma A_\beta A_\alpha\varphi\log^k r + k(B_\gamma A_\beta A_\alpha + A_\gamma B_\beta A_\alpha + A_\gamma A_\beta B_\alpha)\varphi\log^{k-1}r\\ &\quad + k(k-1)(A_\beta A_\alpha + B_\gamma B_\beta A_\alpha + B_\gamma A_\beta B_\alpha + A_\gamma B_\beta B_\alpha + A_\gamma A_\alpha + A_\gamma A_\beta)\varphi\log^{k-2}r\\ &\quad + k(k-1)(k-2)\\ &\qquad (B_\beta A_\alpha + A_\beta B_\alpha + B_\gamma A_\alpha + B_\gamma B_\beta B_\alpha + B_\gamma A_\beta + A_\gamma B_\alpha + A_\gamma B_\beta)\varphi\log^{k-3}r\\ &\quad + k(k-1)(k-2)(k-3)(A_\alpha + A_\beta + A_\gamma + B_\gamma B_\beta + B_\gamma B_\alpha + B_\beta B_\alpha)\varphi\log^{k-4}r\\ &\quad + k(k-1)(k-2)(k-3)(k-4)(B_\alpha + B_\beta + B_\gamma)\varphi\log^{k-5}r\\ &\quad + k(k-1)(k-2)(k-3)(k-4)(k-5)\varphi\log^{k-6}r.\end{aligned}$$

A direct computation yields

$$\Delta_0(r^\alpha\varphi) = r^{\alpha-2}A_\alpha\varphi, \qquad \Delta_0^2(r^\alpha\varphi) = \Delta_0(r^{\alpha-2}A_\alpha\varphi) = r^{\alpha-4}A_{\alpha-2}A_\alpha\varphi,$$
$$\Delta_0^3(r^\alpha\varphi) = r^{\alpha-6}A_{\alpha-4}A_{\alpha-2}A_\alpha\varphi.$$

In particular,

$$\Delta_0^3(r^{6-n}\varphi) = r^{-n}A_{2-n}A_{4-n}A_{6-n}\varphi.$$

Define

$$M_g := \Delta_g\delta T_2 d + \delta T_2 d\Delta_g + \frac{n-2}{2}\Delta_g(\sigma_1(A)\Delta_g) + \delta T_4 d,$$

then rewrite (1-2) as $-P_g = (\Delta_g)^3 + M_g - (n-6)/2Q_g$. Notice that

$$\begin{aligned} A_{\alpha,g} =& A_\alpha + r^2(\Delta_g - \Delta_0) = A_\alpha + r^2\partial_i((g^{ij} - \delta^{ij})\partial_j), \\ -P_g(r^\alpha\varphi) =& r^{\alpha-6}(A_{\alpha-4}A_{\alpha-2}A_\alpha\varphi + K_\alpha\varphi), \end{aligned}$$

where

$$\begin{aligned} K_\alpha\varphi = r^2(\Delta_g - \Delta_0)A_{\alpha-2}A_\alpha\varphi + A_{\alpha-4}(r^2(\Delta_g - \Delta_0))A_\alpha\varphi \\ + A_{\alpha-4}A_{\alpha-2}(r^2(\Delta_g - \Delta_0))\varphi + r^{6-\alpha}M_g(r^\alpha\varphi) - \frac{n-6}{2}r^6Q_g\varphi. \end{aligned} \tag{2-2}$$

We first state the expression of $P_g(r^{6-n})$ and leave the complicated computations to the Appendix.

Lemma 2.2. *Under conformal normal coordinates around p with metric g, we have*

$$\begin{aligned} &-P_g(r^{6-n}) \\ &= -c_n\delta_p + (n-6)r^{-n}\Bigg\{\frac{64(n-4)}{9} \\ &\Bigg[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{r^2}{n+4}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 + \frac{3}{2(n+4)(n+2)}|W(p)|^2r^4\Bigg] \\ &\quad + \frac{16(3n-20)}{9(n+4)}r^2\Bigg[\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n}|W(p)|^2r^2\Bigg] \\ &\quad - 4(5n^2-66n+224)r^2\Bigg[\sigma_1(A)_{,ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2\Bigg] \\ &\quad + \frac{3n^4-16n^3-164n^2+400n+2432}{3(n+4)(n+2)n(n-1)}|W(p)|^2r^4\Bigg\} + O(r^{5-n}), \end{aligned}$$

where W_{ijkl} is the Weyl tensor of metric g and each term in square brackets on the right-hand side of the identity is a harmonic polynomial.

Consequently, we rewrite the above equation in Lemma 2.2 as

$$P_g(r^{6-n}) = c_n\delta_p + r^{-n}f,$$

with $f = O(r^4)$.

Observe that for $i = 0, 1, \dots, [m/2]$,

$$A_\alpha|_{r^{2i}\mathcal{H}_{m-2i}} = (\alpha+2i)(2m-2i+\alpha+n-2)$$

and

$$B_\alpha|_{r^{2i}\mathcal{H}_{m-2i}} = 2m+2\alpha+n-2.$$

Then

$$\begin{aligned}(2\text{-}3)\quad & A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}\mathcal{H}_{m-2i}} \\ &= (6-n+2i)(4-n+2i)(2-n+2i)(2m+4-2i)(2m+2-2i)(2m-2i).\end{aligned}$$

We start to find a formal asymptotic solution like $G_p(x) = r^{6-n}\big(1+\sum_{k=1}^n \psi_k\big)+\varphi$ with $\psi_k \in \mathcal{P}_k$. If we can find $\bar{\psi} = \sum_{k=1}^n \psi_k$ such that

$$A_{2-n}A_{4-n}A_{6-n}\bar{\psi} + K_{6-n}\bar{\psi} + f = O(r^{n+1}), \tag{2-4}$$

the regularity theory for elliptic equations gives that there exists a solution $\varphi \in C^{6,\alpha}_{\mathrm{loc}}$ for any $0 < \alpha < 1$ to

$$P_g(\varphi) = -r^{-n}(A_{2-n}A_{4-n}A_{6-n}\bar{\psi} + K_{6-n}\bar{\psi} + f) \in C^{\alpha}_{\mathrm{loc}}.$$

Thus it only remains to seek $\bar{\psi}$ satisfying (2-4) via induction. For any nonnegative integer k, it is not hard to see from the definition (2-2) of K_{6-n} that $K_{6-n}\varphi \in \mathcal{P}_{k+2}$ when $\varphi \in \mathcal{P}_k$. We first set $\psi_1 = \psi_2 = \psi_3 = 0$ by (2-4) and define

$$f_3 = f = O(r^4).$$

Case 1. n is odd.

If we have found $\psi_1, \dots, \psi_k$ for $3 \le k \le n-1$ with $\psi_k \in \mathcal{P}_k$ and

$$f_k = A_{2-n}A_{4-n}A_{6-n}\bigg(\sum_{i=1}^k \psi_i\bigg) + K_{6-n}\bigg(\sum_{i=1}^k \psi_i\bigg) + f := b_{k+1} + O(r^{k+2}),$$

then it follows from (2-3) that $A_{2-n}A_{4-n}A_{6-n}$ is invertible on $\mathcal{P}_{k+1}$ for $0 \le k \le n-1$. Thus there exists a unique $\psi_{k+1} \in \mathcal{P}_{k+1}$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi_{k+1} + b_{k+1} = 0.$$

This implies that

$$\begin{aligned}f_{k+1} =& A_{2-n}A_{4-n}A_{6-n}\bigg(\sum_{i=1}^{k+1} \psi_i\bigg) + K_{6-n}\bigg(\sum_{i=1}^{k+1} \psi_i\bigg) + f \\ =& f_k + A_{2-n}A_{4-n}A_{6-n}\psi_{k+1} + K_{6-n}\psi_{k+1} \\ =& O(r^{k+2}).\end{aligned}$$

This finishes the induction and assertion (a) follows.

Case 2. n is even and not less than 10.

Since $A_{2-n}A_{4-n}A_{6-n}$ is invertible on $\mathcal{P}_k$ for $0 \le k \le n-7$, by the same induction in Case 1, we may find $\psi_1, \ldots, \psi_{n-7}$ such that

$$f_{n-7} = A_{2-n}A_{4-n}A_{6-n}\left(\sum_{k=1}^{n-7}\psi_k\right) + K_{6-n}\left(\sum_{k=1}^{n-7}\psi_k\right) + f = O(r^{n-6}) := b_{n-6} + O(r^{n-5}).$$

Let $\psi^{(0)}_{n-6} = \alpha^{(0)}_{n-6}(x) + \beta^{(0)}_{n-6}(x)\log r$, where $\alpha^{(0)}_{n-6}(x) \in \mathcal{P}_{n-6} \setminus r^{n-6}\mathcal{H}_0$ and $\beta^{(0)}_{n-6}(x) \in r^{n-6}\mathcal{H}_0$, then it follows from (2-1) that

$$\begin{aligned}&A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-6}\\ &= A_{2-n}A_{4-n}A_{6-n}\alpha^{(0)}_{n-6} + (B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})\beta^{(0)}_{n-6}.\end{aligned}$$

Notice that for $0 \le i \le (n-8)/2$, we have

$$A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}\mathcal{H}_{m-2i}} \neq 0$$

by (2-3) and

$$\begin{aligned}(B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})|_{r^{n-6}\mathcal{H}_0} &= 8(n-2)(n-4)(n-6)\\ &\neq 0.\end{aligned}$$

Hence there exists a unique $\psi^{(0)}_{n-6} \in \mathcal{P}_{n-6} + \mathcal{P}_{n-6}\log r$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-6} + b_{n-6} = 0.$$

This indicates that

$$\begin{aligned}f_{n-6} &= f_{n-7} + A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-6} + K_{6-n}\psi^{(0)}_{n-6}\\ &= O(r^{n-5}) + (K_{6-n}\beta^{(0)}_{n-6})\log r\\ &:= b_{n-5} + O(r^{n-4})\log r + O(r^{n-4}).\end{aligned}$$

Let $\psi^{(0)}_{n-5} = \alpha^{(0)}_{n-5} + \beta^{(0)}_{n-5}\log r$, where $\alpha^{(0)}_{n-5} \in \mathcal{P}_{n-5} \setminus r^{n-6}\mathcal{H}_1$ and $\beta^{(0)}_{n-5} \in r^{n-6}\mathcal{H}_1$. Then we have

$$\begin{aligned}&A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-5}\\ &= A_{2-n}A_{4-n}A_{6-n}\alpha^{(0)}_{n-5} + (B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})\beta^{(0)}_{n-5}.\end{aligned}$$

By similar arguments, there exists a unique $\psi^{(0)}_{n-5} \in \mathcal{P}_{n-5} + r^{n-6}\mathcal{H}_1\log r$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-5} + b_{n-5} = 0.$$

This implies that

$$\begin{aligned}f_{n-5} =&f_{n-6} + A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-5} + K_{6-n}\psi^{(0)}_{n-5}\\ =&O(r^{n-4})\log r + O(r^{n-4})\\ :=&b^{(1)}_{n-4}\log r + O(r^{n-4}) + O(r^{n-3})\log r.\end{aligned}$$

Choose $\psi^{(1)}_{n-4} = \alpha^{(1)}_{n-4}\log r + \beta^{(1)}_{n-4}\log^2 r \in \mathcal{P}_{n-4}\log r + (r^{n-6}\mathcal{H}_2 + r^{n-4}\mathcal{H}_0)\log^2 r$. Then (2-1) gives

$$\begin{aligned}&A_{2-n}A_{4-n}A_{6-n}\psi^{(1)}_{n-4}\\ &= [A_{2-n}A_{4-n}A_{6-n}\alpha^{(1)}_{n-4}\\ &\qquad + 2(B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})\beta^{(1)}_{n-4}]\log r\\ &\qquad + A_{2-n}A_{4-n}A_{6-n}\beta^{(1)}_{n-4}\log^2 r + O(r^{n-4}).\end{aligned}$$

Since

$$\begin{aligned}(B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})|_{r^{n-6}\mathcal{H}_2} &= 8(n+2)n(n-2)\\ &\neq 0;\\ (B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})|_{r^{n-4}\mathcal{H}_0} &= -4n(n-2)(n-4)\\ &\neq 0\end{aligned}$$

and $A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}\mathcal{H}_{n-4-2i}} \neq 0$ for $0 \leq i \leq (n-8)/2$, there exists a unique $\psi^{(1)}_{n-4}$ such that

$$\begin{aligned}&A_{2-n}A_{4-n}A_{6-n}\alpha^{(1)}_{n-4}\\ &\qquad + 2(B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})\beta^{(1)}_{n-4} + b^{(1)}_{n-4} = 0\end{aligned}$$

and

$$\begin{aligned}f^{(1)}_{n-4} =&f_{n-5} + A_{2-n}A_{4-n}A_{6-n}\psi^{(1)}_{n-4} + K_{6-n}\psi^{(1)}_{n-4}\\ =&O(r^{n-4}) + O(r^{n-3})\log r + O(r^{n-2})\log^2 r\\ :=&b^{(0)}_{n-4} + O(r^{n-3})\log r + O(r^{n-3}) + O(r^{n-2})\log^2 r.\end{aligned}$$

Choose $\psi^{(0)}_{n-4} \in \mathcal{P}_{n-4} + (r^{n-6}\mathcal{H}_2 + r^{n-4}\mathcal{H}_0)\log r$ to remove the term $b^{(0)}_{n-4}$ and set

$$\begin{aligned}f^{(0)}_{n-4} &= f^{(1)}_{n-4} + A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-4} + K_{6-n}\psi^{(0)}_{n-4}\\ &= O(r^{n-3})\log r + O(r^{n-3}) + O(r^{n-2})\log^2 r.\end{aligned}$$

By similar arguments and (2-1), we get

$$\begin{aligned}
&\psi^{(1)}_{n-3} \in \mathcal{P}_{n-3}\log r + (r^{n-6}\mathcal{H}_3 + r^{n-4}\mathcal{H}_1)\log^2 r;\\
&\psi^{(0)}_{n-3} \in \mathcal{P}_{n-3} + (r^{n-6}\mathcal{H}_3 + r^{n-4}\mathcal{H}_1)\log r;\\
&\psi^{(i)}_{n-2} \in \mathcal{P}_{n-2}\log^i r + (r^{n-6}\mathcal{H}_4 + r^{n-4}\mathcal{H}_2 + r^{n-2}\mathcal{H}_0)\log^{i+1} r, \quad \text{for } i = 0, 1, 2;\\
&\psi^{(i)}_{n-1} \in \mathcal{P}_{n-1}\log^i r + (r^{n-6}\mathcal{H}_5 + r^{n-4}\mathcal{H}_3 + r^{n-2}\mathcal{H}_1)\log^{i+1} r, \quad \text{for } i = 0, 1, 2;\\
&\psi^{(i)}_{n} \in \mathcal{P}_{n}\log^i r + (r^{n-6}\mathcal{H}_6 + r^{n-4}\mathcal{H}_4 + r^{n-2}\mathcal{H}_2)\log^{i+1} r, \quad \text{for } i = 0, 1, 2, 3.
\end{aligned}$$

Now we set

$$\psi_{n-6} = \psi^{(0)}_{n-6},\ \psi_{n-5} = \psi^{(0)}_{n-5},\ \psi_{n-4} = \psi^{(0)}_{n-4} + \psi^{(1)}_{n-4},\ \psi_{n-3} = \psi^{(0)}_{n-3} + \psi^{(1)}_{n-3}$$

and

$$\psi_{n-2} = \sum_{i=0}^{2} \psi^{(i)}_{n-2}, \quad \psi_{n-1} = \sum_{i=0}^{2} \psi^{(i)}_{n-1}, \quad \psi_{n} = \sum_{i=0}^{3} \psi^{(i)}_{n}.$$

Eventually, we obtain

$$\begin{aligned}
f_n &= A_{2-n}A_{4-n}A_{6-n}\left(\sum_{k=1}^{n}\psi_k\right) + K_{6-n}\left(\sum_{k=1}^{n}\psi_k\right) + f\\
&= O(r^{n+1})(\log^3 r + \log^2 r + \log r + 1) + O(r^{n+2})\log^4 r.
\end{aligned}$$

Hence, $r^{-n} f_n \in C^{\alpha}$ for any $0 < \alpha < 1$. This finishes the induction and we obtain assertion (b) as desired.

Case 3. $n = 8$.

Notice that

$$P_g(G_p(x) - c_n r^{-2}) = O(r^{-4}) \in L^p,$$

for some $\frac{8}{5} < p < 2$. Then it follows from the regularity theory of elliptic equations that $G_p(x) - c_n r^{-2} \in C^{6-8/p}_{\mathrm{loc}}$. From this, we have $G_p(x) = c_n r^{-2} + A + O(r)$.

Case 4. M is locally conformally flat.

One may choose g flat near p and $P_g = -\Delta_0^3$. Hence, $P_g(G(x) - c_n r^{6-n}) = 0$ and then $G_p(x) - c_n r^{6-n}$ is smooth near p.

Therefore, the assertion (c) follows from cases 1,3,4. In some special cases, the leading term ψ_4 can be computed with the help of Lemma 2.2. The proof of Proposition 2.1 is complete.

3. n ≥ 10 and not locally conformally flat

Similar to the Yamabe constant, for $n \geq 3$ and $n \neq 4, 6$, we define

$$Y_6^+(M, g) = \inf_{0<u\in H^3(M,g)} \frac{\int_M u P_g u \, d\mu_g}{\left(\int_M u^{\frac{2n}{n-6}} \, d\mu_g\right)^{\frac{n-6}{n}}}.$$

It follows from (1-3) that $Y_6^+(M, g)$ is a conformal invariant. However, due to the lack of a maximum principle for higher order elliptic equations in general, we first study another conformally invariant quantity,

$$Y_6(M, g) = \inf_{u\in H^3(M,g)\setminus\{0\}} \frac{\int_M u P_g u \, d\mu_g}{\left(\int_M |u|^{\frac{2n}{n-6}} \, d\mu_g\right)^{\frac{n-6}{n}}}.$$

In particular, we have $Y_6(S^n) = Y_6^+(S^n) = (n-6)/2 Q_{S^n} \omega_n^{6/n}$. For $w \in C_c^\infty(\mathbb{R}^n)$, let

$$\|w\|_{\mathcal{D}^{3,2}} := \sum_{|\beta|=3} \|D^\beta w\|_{L^2(\mathbb{R}^n)} \approx \|\nabla\Delta w\|_{L^2(\mathbb{R}^n)},$$

and let $\mathcal{D}^{3,2}(\mathbb{R}^n)$ denote the completion of $C_c^\infty(\mathbb{R}^n)$ under this norm. The equivalence of the above last two norms can be easily deduced by the formula (3-4) below. We first recall an optimal Euclidean Sobolev inequality (see [Lions 1985, p.154–165], [Lieb 1983]).

Lemma 3.1. *For $n \geq 7$, the following sharp Sobolev embedding inequality holds*:

$$Y_6(S^n)\left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-6}} dy\right)^{\frac{n-6}{n}} \leq \int_{\mathbb{R}^n} |\nabla\Delta w|^2 \, dy \quad \text{for all} \quad w \in \mathcal{D}^{3,2}(\mathbb{R}^n).$$

The equality holds if and only if $w(y) = (2/(1+|y|^2))^{(n-6)/2}$ up to any nonzero constant multiple, as well as all translations and dilations.

Proposition 3.2. *On a closed Riemannian manifold (M^n, g) of dimension $n \geq 10$, if there exists $p \in M^n$ such that the Weyl tensor $W_g(p) \neq 0$, then $Y_6(M^n) < Y_6(S^n)$.*

Proof. Recall the definition of P_g:

$$-P_g = \Delta_g^3 + \Delta_g \delta T_2 d + \delta T_2 d \Delta_g + \frac{n-2}{2} \Delta_g(\sigma_1(A)\Delta_g) + \delta T_4 d - \frac{n-6}{2} Q_g.$$

Then for all $\varphi \in H^3(M, g)$,

$$\int_M \varphi P_g \varphi d\mu_g = \int_M |\nabla\Delta\varphi|_g^2 d\mu_g - 2\int_M T_2(\nabla\varphi, \nabla\Delta\varphi) d\mu_g - \frac{n-2}{2}\int_M \sigma_1(A)(\Delta\varphi)^2 d\mu_g - \int_M T_4(\nabla\varphi, \nabla\varphi) d\mu_g + \frac{n-6}{2}\int_M Q_g \varphi^2 d\mu_g.$$

Fix $\rho > 0$ small and choose test functions

$$\varphi(x) = \eta_\rho(x) u_\epsilon(x), \quad u_\epsilon(x) = \Big(\frac{2\epsilon}{\epsilon^2+|x|^2}\Big)^{\frac{n-6}{2}}, \quad \epsilon > 0,$$

where $r = |x| = d_g(x, p)$ and

$$\eta_\rho \in C_c^\infty, \quad 0 \le \eta_\rho \le 1, \quad \eta_\rho \equiv 1 \quad \text{in} \quad B_\rho \quad \text{and} \quad \eta_\rho \equiv 0 \quad \text{in} \quad B_{2\rho}^c.$$

It is known from Lee and Parker [1987] that up to a conformal factor, under conformal normal coordinates around p with metric g, for all $N \ge 5$, we have

$$\sigma_1(A_g)(p) = 0, \quad \sigma_1(A_g)_{,i}(p) = 0, \quad \Delta_g \sigma_1(A_g)(p) = -\frac{|W(p)|_g^2}{12(n-1)}$$

and $\sqrt{\det g} = 1 + O(r^N)$.

Our purpose is to estimate $\int_M \varphi P_g \varphi \, d\mu_g$ and $\int_M \varphi^{2n/(n-6)} \, d\mu_g$. A direct computation shows

$$u_\epsilon' = -(n-6) u_\epsilon \frac{r}{\epsilon^2 + r^2}, \quad u_\epsilon'' = -(n-6) u_\epsilon \frac{\epsilon^2 - (n-5) r^2}{(\epsilon^2 + r^2)^2}$$

and

$$\Delta_0 u_\epsilon = -(n-6) \frac{u_\epsilon}{(\epsilon^2 + r^2)^2} (n\epsilon^2 + 4r^2),$$

$$(\Delta_0 u_\epsilon)' = (n-6)(n-4) \frac{u_\epsilon r}{(\epsilon^2 + r^2)^3} [(n+2)\epsilon^2 + 4r^2].$$

We start with $\int_M |\nabla\Delta\varphi|_g^2 \, d\mu_g$ and divide its integral into two parts: $\int_M = \int_{B_\rho} + \int_{M \setminus \overline{B}_\rho}$. Compute

$$\begin{aligned}
&\int_{B_\rho} |\nabla\Delta\varphi|_g^2 \, d\mu_g \\
&\quad = \int_{B_\rho} g^{ij} (\Delta\varphi)_{,i} (\Delta\varphi)_{,j} \, d\mu_g \\
&\quad = \int_{B_\rho} (\delta^{ij} + O(r^2))(\Delta_0\varphi + O(r^{N-1})\varphi')_{,i} (\Delta_0\varphi + O(r^{N-1})\varphi')_{,j} (1 + O(r^N)) \, dx \\
&\quad = \int_{B_\rho} |(\nabla\Delta)_0 \varphi|^2 \, dx + \int_{B_\rho} (\Delta_0\varphi)' (O(r^{N-2})\varphi' + O(r^{N-1})\varphi'') \, dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus \overline{B}_\rho} |(\nabla\Delta)_0 \varphi|^2 \, dx &= (n-6)^2 (n-4)^2 \int_{\mathbb{R}^n \setminus \overline{B}_\rho} \frac{u_\epsilon^2 r^2}{(\epsilon^2 + r^2)^6} [(n+2)\epsilon^2 + 4r^2]^2 \, dx \\
&\le C \int_{\rho/\epsilon}^\infty \sigma^{5-n} d\sigma = O(\epsilon^{n-6}).
\end{aligned}$$

Similarly, we estimate $\int_{M\setminus\overline{B}_\rho}|\nabla\Delta\varphi|_g^2\,d\mu_g=O(\epsilon^{n-6})$. Thus, we obtain

$$\int_M|\nabla\Delta\varphi|_g^2\,d\mu_g=\int_{\mathbb{R}^n}|\nabla\Delta_0u_\epsilon|^2dx+O(\epsilon^{n-6}).$$

Secondly, we compute

$$\begin{aligned}
&\int_{B_\rho}\sigma_1(A)(\Delta\varphi)^2\,d\mu_g\\
&\quad=\int_{B_\rho}\biggl(\frac{1}{2}\sigma_1(A)_{,ij}(p)x^ix^j+O(r^3)\biggr)(\Delta_0\varphi+O(r^{N-1})\varphi')^2(1+O(r^N))\,dx\\
&\quad=\int_{B_\rho}\frac{1}{2n}\Delta\sigma_1(A)(p)|x|^2(\Delta_0\varphi)^2\,dx+\int_{B_\rho}O(r^3)\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}(n\epsilon^2+4r^2)^2\,dx\\
&\quad=-\frac{(n-6)^2|W(p)|^2}{24n(n-1)}\omega_{n-1}\int_0^\rho\frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2r^{n+1}dr+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx,
\end{aligned}$$

and for some large enough N

$$\begin{aligned}
\int_{B_{2\rho}\setminus\overline{B}_\rho}\sigma_1(A)(\Delta\varphi)^2\,d\mu_g&\le C\int_{B_{2\rho}\setminus\overline{B}_\rho}|\Delta_0\varphi+O(r^{N-1})\varphi'|^2(1+O(r^N))\,dx\\
&\le C\int_{B_{2\rho}\setminus\overline{B}_\rho}[(\Delta_0\varphi)^2+O(r^{2(N-1)})|\varphi'|^2]\,dx\\
&\le C\int_{B_{2\rho}\setminus\overline{B}_\rho}(u_\epsilon\Delta_0\eta_\rho+2\nabla u_\epsilon\cdot\nabla\eta_\rho+\eta_\rho\Delta_0u_\epsilon)^2\,dx+O(\epsilon^{n-6})\\
&\le C\int_\rho^{2\rho}\frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2r^{n-1}\,dr+O(\epsilon^{n-6})\\
&\overset{\sigma=r/\epsilon}{\le}C\epsilon^2\int_{\rho/\epsilon}^{2\rho/\epsilon}\frac{(n+4\sigma^2)^2\sigma^{n-1}}{(1+\sigma^2)^{n-2}}\,d\sigma+O(\epsilon^{n-6})\\
&\le C\epsilon^2\biggl(\frac{\rho}{\epsilon}\biggr)^{8-n}+O(\epsilon^{n-6})=O(\epsilon^{n-6}).
\end{aligned}$$

Observe that

$$\int_{B_\rho}\frac{r^3u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx=\begin{cases}O(\epsilon^4)&\text{if } n=10,\\ O(\epsilon^5|\log\epsilon|)&\text{if } n=11,\\ O(\epsilon^5)&\text{if } n\ge 12.\end{cases}\tag{3-1}$$

Hence,

$$
\begin{aligned}
-\frac{n-2}{2}\int_M \sigma_1(A)(\Delta\varphi)^2\,d\mu_g
&= \frac{(n-6)^2(n-2)|W(p)|^2}{48n(n-1)}\omega_{n-1}\int_0^\rho \frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2 r^{n+1}\,dr \\
&\quad + \begin{cases} O(\epsilon^4) & \text{if } n=10, \\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11, \\ O(\epsilon^5) & \text{if } n\geq 12. \end{cases}
\end{aligned}
$$

Thirdly, we compute $\int_M T_2(\nabla\varphi,\nabla\Delta\varphi)\,d\mu_g$.

$$
\int_{B_\rho} T_2(\nabla\varphi,\nabla\Delta\varphi)\,d\mu_g = \int_{B_\rho}[(n-2)\sigma_1(A)\langle\nabla\varphi,\nabla\Delta\varphi\rangle - 8A_{ij}\varphi_{,i}(\Delta\varphi)_{,j}]\,d\mu_g.
$$

Observe that $u_{\epsilon,i} = (x^i/r)u_\epsilon'$ and $(\Delta_0 u_\epsilon)_{,i} = (x^i/r)(\Delta_0 u_\epsilon)'$. Then we get

$$
\begin{aligned}
&(n-2)\int_{B_\rho}\sigma_1(A)\langle\nabla\varphi,\nabla\Delta\varphi\rangle\,d\mu_g \\
=&(n-2)\int_{B_\rho}\left(\frac{1}{2}\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\right)g^{kl}\varphi_{,k}(\Delta\varphi)_{,l}\,d\mu_g \\
=&(n-2)\int_{B_\rho}\left(\frac{1}{2}\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\right)(\delta^{kl}+O(r^2))\varphi_{,k}(\Delta_0\varphi+O(r^{N-1})\varphi')_{,l}\,d\mu_g \\
=&\frac{n-2}{2}\int_{B_\rho}\frac{1}{n}\Delta\sigma_1(A)(p)|x|^2\varphi_{,i}(\Delta_0\varphi)_{,i}\,dx + \int_{B_\rho}O(r^3)|\varphi'||(\Delta_0\varphi)'|\,dx \\
=&-\frac{(n-2)|W(p)|^2}{24n(n-1)}\int_{B_\rho}\left\{-(n-6)^2(n-4)\frac{u_\epsilon^2 r^4}{(\epsilon^2+r^2)^4}\Big[(n+2)\epsilon^2+4r^2\Big]\right\}dx \\
&\quad + \int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx \\
=&\frac{(n-2)(n-4)(n-6)^2}{24n(n-1)}|W(p)|^2\int_{B_\rho}\frac{r^4}{(\epsilon^2+r^2)^4}u_\epsilon^2[(n+2)\epsilon^2+4r^2]\,dx \\
&\quad + \int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&-8\int_{B_\rho} A_{ij}\varphi_{,i}(\Delta\varphi)_{,j}\,d\mu_g\\
=&-8\int_{B_\rho}\left(A_{ij,k}(p)x^k+\frac{1}{2}A_{ij,kl}(p)x^kx^l+O(r^3)\right)\varphi_{,i}(\Delta_0\varphi+O(r^{N-1})\varphi')_{,j}\,d\mu_g\\
=&-4\int_{B_\rho} A_{ij,kl}(p)x^kx^lx^ix^j\Big[-(n-4)(n-6)^2\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\Big]\,dx\\
&\qquad+\int_{B_\rho} O(r^3)|\varphi'||(\Delta_0\varphi)'|\,dx\\
=&\,4(n-4)(n-6)^2\int_{B_\rho}\Big[-\frac{2}{9}\frac{1}{n-2}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2-\frac{\sigma_1(A)_{,ij}(p)x^ix^jr^2}{n-2}\Big]\\
&\qquad\times\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dx+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx\\
=&-\frac{8(n-4)(n-6)^2}{9(n-2)}\int_{B_\rho}\sum_{k,l}(W_{iklj}(p)W_{sklt}(p)x^ix^jx^sx^t)\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dx\\
&-\frac{4(n-4)(n-6)^2}{n(n-2)}\int_{B_\rho}\frac{\Delta\sigma_1(A)(p)r^4}{(\epsilon^2+r^2)^4}u_\epsilon^2[(n+2)\epsilon^2+4r^2]\,dx+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx\\
=&-\frac{(n-4)(n-6)^2}{(n-1)n(n+2)}\omega_{n-1}|W(p)|^2\int_0^\rho\frac{r^{n+3}u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx,
\end{aligned}
$$

where the last identity follows from

$$
\begin{aligned}
&\sum_{k,l}W_{iklj}(p)W_{sklt}(p)\int_{B_\rho}x^ix^jx^sx^t\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dx\\
=&\sum_{k,l}W_{iklj}(p)W_{sklt}(p)\int_{\mathbb{S}^{n-1}}\xi^i\xi^j\xi^s\xi^t\,d\mu_{\mathbb{S}^{n-1}}\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\\
=&\frac{\omega_{n-1}}{n(n+2)}\sum_{k,l}W_{iklj}(p)W_{sklt}(p)[\delta_{ij}\delta_{st}+\delta_{is}\delta_{jt}+\delta_{it}\delta_{js}]\int_0^\rho\frac{r^{n+3}u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\\
=&\frac{\omega_{n-1}}{n(n+2)}[|W(p)|^2+W_{iklj}(p)W_{jkli}(p)]\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\\
=&\frac{3}{2}\frac{\omega_{n-1}}{n(n+2)}|W(p)|^2\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&-2\int_{B_\rho} T_2(\nabla\varphi, \nabla\Delta\varphi)\, d\mu_g \\
&= -\frac{(n^2-28)(n-4)(n-6)^2}{12n(n-1)(n+2)} |W(p)|^2 \omega_{n-1} \int_0^\rho r^{n+3} \frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\, dr \\
&\qquad + \int_{B_\rho} \frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx.
\end{aligned}
$$

By a similar argument, one has

$$
\begin{aligned}
\left| \int_{B_{2\rho}\setminus \overline{B_\rho}} T_2(\nabla\varphi, \nabla\Delta\varphi) \right| &\le C \int_{B_{2\rho}\setminus \overline{B_\rho}} |\nabla\varphi||\nabla\Delta\varphi|\, d\mu_g \\
&\le C \int_{B_{2\rho}\setminus \overline{B_\rho}} |u_\epsilon'||(\Delta u_\epsilon)'|\, dx + O(\epsilon^{n-6}) = O(\epsilon^{n-6}).
\end{aligned}
$$

Fourthly, we compute $\int_M T_4(\nabla\varphi, \nabla\varphi)\, d\mu_g$.

$$
\begin{aligned}
(n-6)\int_{B_\rho} \Delta\sigma_1(A)|\nabla\varphi|_g^2\, d\mu_g &= (n-6)\int_{B_\rho} (\Delta\sigma_1(A)(p) + O(r))(|\varphi'|^2 + O(r^2)|\varphi|^2)\, dx \\
&= -(n-6)^3 \frac{|W(p)|^2}{12(n-1)} \int_{B_\rho} \frac{u_\epsilon^2 r^2}{(\epsilon^2+r^2)^2}\, dx + \int_{B_\rho} \frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx.
\end{aligned}
$$

Using (A-5), we get

$$
\begin{aligned}
&-\frac{16}{n-4}\int_{B_\rho} B_{ij}\varphi_{,i}\varphi_{,j}\, d\mu_g \\
&= -\frac{16}{n-4}\int_{B_\rho} (n-6)^2 u_\epsilon^2 \frac{B_{ij}x^i x^j}{(\epsilon^2+r^2)^2}\, dx \\
&= -\frac{16(n-6)^2}{n-4}\int_{B_\rho} \Big[-\frac{2}{9}\frac{1}{n-2}\sum_{k,l,s}[(W_{ikls}(p)+W_{ilks}(p))x^i]^2 \\
&\quad + \frac{1}{12(n-2)(n-1)}|W(p)|^2 r^2 - \frac{7n-8}{n-2}\sigma_1(A)_{,ij}(p)x^i x^j + O(r^3)\Big] \frac{u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx \\
&= -\frac{16(n-6)^2}{n-4}\Big[-\frac{2}{3n(n-2)} + \frac{1}{12(n-2)(n-1)} + \frac{7n-8}{12(n-2)(n-1)n} \Big] \\
&\qquad |W(p)|^2 \int_{B_\rho} \frac{r^2 u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx + \int_{B_\rho} \frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx \\
&= \int_{B_\rho} \frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx,
\end{aligned}
$$

where the second identity follows from

$$\sum_{i,k,l,s}(W_{ikls}(p)+W_{ilks}(p))^2=2|W(p)|^2+2\sum_{i,k,l,s}W_{ikls}(p)W_{ilks}(p)=3|W(p)|^2,$$

in view of

$$0=W_{ikls}(W_{ilks}+W_{iksl}+W_{islk})=W_{ikls}W_{ilks}+W_{ikls}W_{iksl}+W_{ikls}W_{islk}=2W_{ikls}W_{ilks}-|W|^2$$

at p. Also we have

$$\int_{B_{2\rho}\setminus\overline{B_\rho}}T_4(\nabla\varphi,\nabla\varphi)\,d\mu_g\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}|\nabla\varphi|_g^2\,d\mu_g=O(\epsilon^{n-6}).$$

Hence, collecting the above terms together with (3-1), we obtain

$$\begin{aligned}-\int_M &T_4(\nabla\varphi,\nabla\varphi)\,d\mu_g\\ &=-(n-6)\int_{B_\rho}\Delta\sigma_1(A)|\nabla\varphi|_g^2\,d\mu_g+\frac{16}{n-4}\int_{B_\rho}B_{ij}\varphi_{,i}\varphi_{,j}\,d\mu_g+O(\epsilon^{n-6})\\ &=(n-6)^3\frac{|W(p)|^2}{12(n-1)}\int_{B_\rho}\frac{u_\epsilon^2r^2}{(\epsilon^2+r^2)^2}\,dx+\begin{cases}O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\ge 12.\end{cases}\end{aligned}$$

Finally, we compute $((n-6)/2)\int_M Q_g\varphi^2\,d\mu_g$. By the definition (1-1) of Q_g, integration by parts gives

$$\begin{aligned}\frac{n-6}{2}\int_M Q_g\varphi^2\,d\mu_g=&\frac{n-6}{2}\int_M\Delta^2\sigma_1(A)\varphi^2\,d\mu_g+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx+O(\epsilon^{n-6})\\ =&\frac{n-6}{2}\int_M\Delta\sigma_1(A)\Delta\varphi^2\,d\mu_g+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx+O(\epsilon^{n-6})\\ =&-\frac{(n-6)^2|W(p)|^2}{12(n-1)}\omega_{n-1}\int_0^\rho\frac{u_\epsilon^2r^{n-1}}{(\epsilon^2+r^2)^2}[(n-10)r^2-n\epsilon^2]\,dr\\ &+\begin{cases}O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\ge 12,\end{cases}\end{aligned}$$

by (3-1), where the last identity follows from

$$\begin{aligned}
&\frac{n-6}{2}\int_{B_\rho}\Delta\sigma_1(A)\Delta\varphi^2\,d\mu_g\\
=&\frac{n-6}{2}\int_{B_\rho}(\Delta\sigma_1(A)(p)+O(r))(\Delta_0\varphi^2+O(r^{N-1})(\varphi^2)')\,dx\\
=&\frac{n-6}{2}\Delta\sigma_1(A)(p)\int_{B_\rho}2(\varphi\Delta_0\varphi+|\nabla\varphi|_0^2)\,dx+\int_{B_\rho}\frac{O(r)u_\epsilon^2}{\epsilon^2+r^2}\,dx\\
=&-\frac{(n-6)^2|W(p)|^2}{12(n-1)}\omega_{n-1}\int_0^\rho\frac{u_\epsilon^2r^{n-1}}{(\epsilon^2+r^2)^2}[(n-10)r^2-n\epsilon^2]dr+\int_{B_\rho}\frac{O(r)u_\epsilon^2}{\epsilon^2+r^2}\,dx
\end{aligned}$$

and the first identity follows from

$$\left|\int_{B_{2\rho}\setminus\overline{B_\rho}}Q_g\varphi^2\,d\mu_g\right|\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}u_\epsilon^2\,dx=O(\epsilon^{n-6}).$$

Therefore collecting all the above terms together, we obtain

$$\int_M\varphi P_g\varphi\,d\mu_g=\int_{\mathbb{R}^n}|\nabla\Delta_0u_\epsilon|^2\,dx+A_{n,\rho,\epsilon}|W(p)|^2\omega_{n-1}+O(\epsilon^{\min\{n-6,5\}}),$$

where $A_{n,\rho,\epsilon}$ is a constant given by

$$\begin{aligned}
&(n-6)^2\bigg(\frac{n-2}{48n(n-1)}\int_0^\rho\frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2r^{n+1}\,dr+\frac{n-6}{12(n-1)}\int_0^\rho\frac{u_\epsilon^2r^{n+1}}{(\epsilon^2+r^2)^2}\,dr\\
&\qquad-\frac{1}{12(n-1)}\int_0^\rho\frac{u_\epsilon^2r^{n-1}}{(\epsilon^2+r^2)^2}[(n-10)r^2-n\epsilon^2]\,dr\\
&\qquad-\frac{(n^2-28)(n-4)}{12n(n-1)(n+2)}\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\bigg)\\
&=2^{n-6}\frac{(n-6)^2}{12(n-1)}\epsilon^4\bigg(\frac{n-2}{4n}\int_0^{\rho/\epsilon}\frac{(n+4\sigma^2)^2}{(1+\sigma^2)^4}(1+\sigma^2)^{-(n-6)}\sigma^{n+1}\,d\sigma\\
&\qquad+(n-6)\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^2}(1+\sigma^2)^{-(n-6)}\sigma^{n+1}\,d\sigma\\
&\qquad-\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^2}(1+\sigma^2)^{-(n-6)}\sigma^{n-1}[(n-10)\sigma^2-n]\,d\sigma\\
&\qquad-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^{\rho/\epsilon}\frac{\sigma^{n+3}}{(1+\sigma^2)^4}(1+\sigma^2)^{-(n-6)}[(n+2)+4\sigma^2]\,d\sigma\bigg),
\end{aligned}$$

where $r=\epsilon\sigma$. When $n=10$, we claim that the leading term of the constant in the parentheses on the right-hand side of the above identity:

$$\frac{1}{5}\int_0^{\rho/\epsilon}\frac{(4\sigma^2+10)^2}{(1+\sigma^2)^4}(1+\sigma^2)^{-4}\sigma^{11}\,d\sigma+\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^2}(1+\sigma^2)^{-4}(4\sigma^2+10)\sigma^9\,d\sigma$$
$$-\frac{18}{5}\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^4}(1+\sigma^2)^{-4}(4\sigma^2+12)\sigma^{13}\,d\sigma$$

is a negative constant multiple of $|\log\epsilon|$. To see this, notice it is obviously true for the third term, and the first two terms equal

$$\frac{1}{5}\int_0^{\rho/\epsilon}\{\sigma^2[(4\sigma^2+10)^2-18\sigma^2(4\sigma^2+12)]+5(4\sigma^2+10)(1+\sigma^2)^2\}(1+\sigma^2)^{-8}\sigma^9\,d\sigma$$
$$=\frac{1}{5}\int_0^{\rho/\epsilon}(-36\sigma^6-46\sigma^4+220\sigma^2+50)(1+\sigma^2)^{-8}\sigma^9\,d\sigma,$$

whose leading term is also a negative constant multiple of $|\log\epsilon|$. For $n\geq 11$, let $t=\sigma^2$. The limit of the coefficient of $|W(p)|^2\omega_{n-1}$ as $\epsilon\to 0$ is

$$2^{n-7}\frac{(n-6)^2}{12(n-1)}\epsilon^4\left\{\frac{n-2}{4n}\int_0^\infty\frac{(n+4t)^2}{(1+t)^{n-2}}t^{\frac{n}{2}}\,dt\right.$$
$$+(n-6)\int_0^\infty\frac{1}{(1+t)^{n-4}}t^{\frac{n}{2}}\,dt-\int_0^\infty\frac{(n-10)t-n}{(1+t)^{n-4}}t^{\frac{n}{2}-1}\,dt$$
$$\left.-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^\infty\frac{(n+2)+4t}{(1+t)^{n-2}}t^{\frac{n}{2}+1}\,dt\right\}.$$

With the help of the Beta function:

$$\int_0^\infty\frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,dx=B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

for $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, we have

$$\frac{n-2}{4n}\int_0^\infty\frac{(n+4t)^2}{(1+t)^{n-2}}t^{\frac{n}{2}}dt$$
$$=\frac{n-2}{4n}\int_0^\infty\frac{(n-4)^2+8(n-4)(1+t)+16(1+t)^2}{(1+t)^{n-2}}t^{\frac{n}{2}}dt$$
$$=\frac{n-2}{4n}\left[(n-4)^2B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big)+8(n-4)B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)+16B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)\right],$$
$$(n-6)\int_0^\infty\frac{1}{(1+t)^{n-4}}t^{\frac{n}{2}}dt=(n-6)B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big),$$
$$-\int_0^\infty\frac{(n-10)t-n}{(1+t)^{n-4}}t^{\frac{n}{2}-1}dt=-(n-10)B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)+nB\Big(\frac{n}{2},\frac{n}{2}-4\Big),$$

and

$$
\begin{aligned}
&-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^\infty \frac{(n+2)+4t}{(1+t)^{n-2}}t^{\frac{n}{2}+1}\,dt\\
&=-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^\infty \frac{4(1+t)^2+(n-6)(1+t)-(n-2)}{(1+t)^{n-2}}t^{\frac{n}{2}}\,dt\\
&=-\frac{4(n^2-28)(n-4)}{n(n+2)}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)-\frac{(n^2-28)(n-4)(n-6)}{n(n+2)}B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)\\
&\quad+\frac{(n^2-28)(n-4)(n-2)}{n(n+2)}B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big).
\end{aligned}
$$

Hence, the above limit of the coefficient of $|W(p)|^2\omega_{n-1}$ is rewritten as

$$
\begin{aligned}
&2^{n-7}\frac{(n-6)^2}{12(n-1)}\epsilon^4\Big\{nB\Big(\frac{n}{2},\frac{n}{2}-4\Big)\\
&\quad+B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big)\Big[\frac{n-2}{4n}(n-4)^2+\frac{(n^2-28)(n-4)(n-2)}{n(n+2)}\Big]\\
&\quad+B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)\Big[\frac{2(n-2)(n-4)}{n}-\frac{(n^2-28)(n-4)(n-6)}{n(n+2)}\Big]\\
&\quad+B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)\Big[\frac{4(n-2)}{n}-n+10+n-6-\frac{4(n^2-28)(n-4)}{n(n+2)}\Big]\Big\}\\
&=2^{n-7}\frac{(n-6)^2}{12(n-1)}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)\\
&\qquad\epsilon^4\Big\{(n-10)+\frac{(n-2)(\frac{n}{2}-4)(\frac{n}{2}-5)}{4n(n+2)(n-3)}(5n^2-2n-120)\\
&\qquad\qquad+\frac{\frac{n}{2}-5}{n(n+2)}(-n^3+8n^2+28n-176)\\
&\qquad\qquad\qquad+\frac{4}{n(n+2)}(-n^3+6n^2+30n-116)\Big\},
\end{aligned}
\tag{3-2}
$$

where we have used some elementary identities

$$
\begin{aligned}
B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big)&=\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2}-3)}{\Gamma(n-2)}&&=\frac{(\frac{n}{2}-4)(\frac{n}{2}-5)}{(n-3)(n-4)}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big),\\
B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)&=\frac{\frac{n}{2}-5}{n-4}B(\tfrac{n}{2}+1,\tfrac{n}{2}-5),\\
B\Big(\frac{n}{2},\frac{n}{2}-4\Big)&=\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}-4)}{\Gamma(n-4)}&&=\frac{n-10}{n}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big).
\end{aligned}
$$

The constant in the last brace of (3-2) when $n \geq 11$ is

$$n-10+\frac{1}{16n(n+2)(n-3)}\{(n-2)(n-8)(n-10)(5n^2-2n-120)$$
$$+8(n-3)[(n-10)(-n^3+8n^2+28n-176)+8(-n^3+6n^2+30n-116)]\}$$
$$=n-10+\frac{1}{16n(n+2)(n-3)}[-3n^5+2n^4+228n^3-264n^2-1760n-768]$$
$$=\frac{-3n^5+18n^4+52n^3-200n^2-800n-768}{16n(n+2)(n-3)}<0.$$

On the other hand, we have

$$\int_M \varphi^{\frac{2n}{n-6}}\,d\mu_g=\int_{B_\rho} u_\epsilon^{\frac{2n}{n-6}}\,d\mu_g+\int_{B_{2\rho}\setminus\overline{B}_\rho}\varphi^{\frac{2n}{n-6}}\,d\mu_g=\int_{\mathbb{R}^n} u_\epsilon^{\frac{2n}{n-6}}\,dx+O(\epsilon^n).$$

Therefore, putting these facts together, we conclude by Lemma 3.1 that

$$\frac{\int_M \varphi P_g\varphi\,d\mu_g}{\left(\int_M\varphi^{\frac{2n}{n-6}}\,d\mu_g\right)^{\frac{n-6}{n}}}=Y_6(S^n)+A_{n,\rho,\epsilon}|W(p)|^2\omega_{n-1}+\begin{cases}O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\geq 12,\end{cases}$$
$$=\begin{cases}Y_6(S^n)-C_n|W(p)|^2\epsilon^4|\log\epsilon|+O(\epsilon^4) & \text{if } n=10,\\ Y_6(S^n)-C_n|W(p)|^2\epsilon^4+o(\epsilon^4) & \text{if } n\geq 11,\end{cases}$$

for some positive constant $C_n>0$. Consequently, choosing ϵ sufficiently small, we obtain $Y_6(M^n)<Y_6(S^n)$. This finishes the proof. □

Given a smooth positive function f on M^n, we define a "free" energy functional by

$$E_f[u]=\frac{1}{2}\int_M uP_gu\,d\mu_g-\frac{1}{2^\sharp}\int_M f|u|^{2^\sharp}\,d\mu_g.$$

Let $u_{,i}$ or $\nabla_i u$ denote the covariant derivatives of u with respect to the metric g and R_{ijk}^l be the Riemannian curvature tensor of metric g. Notice that

$$\nabla_j\nabla_i\nabla^i u=\nabla_i\nabla_j\nabla^i u+R_{iij}^k\nabla_k u=\nabla_i\nabla^i\nabla_j u-R_j^k\nabla_k u.$$

We have

$$\int_M|\nabla\Delta u|_g^2\,d\mu_g=\int_M|\Delta\nabla_j u-R_j^k\nabla_k u|_g^2\,d\mu_g. \tag{3-3}$$

Under g-normal coordinates around a point, one gets

$$\tfrac{1}{2}\Delta_g|\nabla^2u|_g^2$$
$$=|\nabla^3u|_g^2+\langle\nabla\Delta\nabla_i u,\nabla\nabla^i u\rangle_g+u_{,ij}(R_{ijk}^l u_{,lk}+R_j^l u_{,il}+R_{ijk,k}^l u_{,l}+R_{ijk}^l u_{,lk}).$$

Integrating the above identity over M gives

$$\text{(3-4)}\quad \int_M |\Delta\nabla u|_g^2\, d\mu_g = \int_M |\nabla^3 u|_g^2\, d\mu_g + \int_M O\big(|\mathrm{Rm}||\nabla^2 u|_g + |\nabla \mathrm{Rm}||\nabla u|_g\big)|\nabla^2 u|_g\, d\mu_g.$$

From (3-3) and (3-4), it yields that the following two norms are equivalent:

$$\|u\|_{H^3} := \left(\int_M (|\nabla\Delta u|_g^2\, d\mu_g + |\nabla^2 u|_g^2 + |\nabla u|_g^2 + u^2)\, d\mu_g\right)^{1/2} \approx \left(\int_M (|\nabla^3 u|_g^2\, d\mu_g + |\nabla^2 u|_g^2 + |\nabla u|_g^2 + u^2)\, d\mu_g\right)^{1/2}, \quad u \in H^3(M, g).$$

Let $\|\cdot\|_p$ denote the norm of $L^p(M, g)$ for $1 \le p \le \infty$.

A sequence $\{u_k\}$ in $H^3(M, g)$ is called a Palais–Smale (P–S)$_\beta$ sequence for E_f if $E_f[u_k] \to \beta \in \mathbb{R}$ and $DE_f[u_k] \to 0$ as $k \to \infty$. The energy E_f satisfies the (P–S)$_\beta$ condition if any Palais–Smale sequence of E_f has a strongly convergent subsequence. We call P_g is coercive if there exists a constant $\mu(g) > 0$ such that

$$\int_M \psi P_g \psi\, d\mu_g \ge \mu(g) \int_M \psi^2\, d\mu_g, \quad \text{for all} \quad \psi \in H^3(M, g).$$

Remark. If (M, g) is Einstein and of positive constant scalar curvature, from the factorization (1-4) of P_g, the coercivity of P_g is automatically satisfied.

As an application, we adapt some arguments in Esposito and Robert [2002] to show some existence results of the prescribed Q-curvature equation, whose solution may change signs due to the lack of maximum principles (in general).

Theorem 3.3. *Let (M^n, g) be a smooth closed manifold of dimension $n \ge 10$ and f be a smooth positive function in M^n. Suppose the Weyl tensor W_g is nonzero at a maximum point of f and f satisfies the vanishing order condition* (1-5) *at this maximum point. If P_g is coercive, then there exists a nontrivial $C^{6,\mu}$ $(0 < \mu < 1)$ solution to*

$$\text{(3-5)}\qquad P_g u = f|u|^{2^\sharp - 2}u \quad \text{in } M.$$

In addition, if (M, g) is Einstein and of positive scalar curvature, then there exists a smooth solution to the Q-curvature equation

$$\text{(3-6)}\qquad P_g u = f u^{\frac{n+6}{n-6}}, u > 0 \quad \text{in } M.$$

Proof. By the assumptions, there exists $p \in M$ such that $f(p) = \max_{x \in M^n} f(x)$, $W_g(p) \neq 0$ and the vanishing order condition (1-5) of f is true at p. Let

$$\gamma_\epsilon(t) = t \frac{\varphi}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}},$$

where $\varphi = \eta_\rho u_\epsilon$ is the test function chosen in Proposition 3.2. By choosing t_0 large enough, we get $E[\gamma_\epsilon(t_0)] < 0$. Let

$$\Gamma = \left\{ \gamma(t) \in C([0, t_0], H^3(M, g)); \gamma(0) = 0, \gamma(t_0) = t_0 \frac{\varphi}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}} \right\}.$$

From the coercivity of P_g and the Sobolev embedding theorem, we have

$$E_f\left[\frac{\varphi}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}}\right] = \frac{1}{2}\frac{\int_M \varphi P_g \varphi \, d\mu_g}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}^2} - \frac{1}{2^\sharp} \geq \frac{1}{2}C - \frac{1}{2^\sharp}.$$

It suffices to only estimate the term:

$$\begin{aligned}\int_M f\varphi^{\frac{2n}{n-6}} d\mu_g &= \int_{B_\rho}\left[f(p) + \sum_{k=2}^{4}\frac{1}{k!}\partial_{i_1\cdots i_k} f(p) x^{i_1}\cdots x^{i_k} + O(|x|^5)\right] u_\epsilon^{2^\sharp} dx + O(\epsilon^n) \\ &= f(p)\int_{\mathbb{R}^n} u_\epsilon^{\frac{2n}{n-6}} dx + \begin{cases} O(\epsilon^4) & \text{if } n = 10, \\ o(\epsilon^4) & \text{if } n \geq 11, \end{cases}\end{aligned}$$

where the second equality follows from the vanishing order condition (1-5) of f at p. From this and some existing estimates in the proof of Proposition 3.2, we conclude that there exist some sufficiently small $\epsilon > 0$ and a constant $C_n' > 0$ such that

$$\begin{aligned}\sup_{t \geq 0} E_f[\gamma_\epsilon(t)] &= E_f[\gamma_\epsilon(t^*)] \\ &= \frac{3}{n}\left(\frac{\int_M \varphi P_g \varphi d\mu_g}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}^2}\right)^{2^\sharp/(2^\sharp-2)} \\ &\leq \begin{cases} \frac{3}{n}(\max_M f)^{\frac{6-n}{6}} Y_6(S^n)^{\frac{n}{6}} - C_n'|W(p)|^2\epsilon^4|\log\epsilon| + O(\epsilon^4) & \text{if } n = 10, \\ \frac{3}{n}(\max_M f)^{\frac{6-n}{6}} Y_6(S^n)^{\frac{n}{6}} - C_n'|W(p)|^2\epsilon^4 + o(\epsilon^4) & \text{if } n \geq 11, \end{cases}\end{aligned}$$

where $t^* = \left(\int_M \varphi P_g \varphi \, d\mu_g / \|f^{1/2^\sharp}\varphi\|_{2^\sharp}^2\right)^{1/(2^\sharp-2)}$. Then it follows from the mountain pass lemma (see [Ambrosetti and Rabinowitz 1973] or [Esposito and Robert 2002, Proposition 1]) that

$$\beta = \inf_{\gamma \in \Gamma} \sup_{0 \leq t \leq t_0} E_f[\gamma(t)] \leq \sup_{t \geq 0} E_f[\gamma_\epsilon(t)] < \frac{3}{n} Y_6(S^n)^{\frac{n}{6}} (\max_M f)^{\frac{6-n}{6}}$$

is a critical value of E_f and there exists a (P–S)$_\beta$ sequence $\{u_k\}$ of E_f in $H^3(M, g)$.

Next we claim that E_f satisfies the (P–S)$_\beta$ condition. For the above (P–S)$_\beta$ sequence $\{u_k\}$ satisfying $E_f[u_k] \to \beta$ and $DE_f[u_k] \to 0$ as $k \to \infty$, we have

$$2\beta + o(\|u_k\|_{H^3}) = 2E_f[u_k] - \langle DE_f[u_k], u_k\rangle = \frac{6}{n}\int_M f|u_k|^{2^\sharp}\, d\mu_g.$$

Together with the coercivity of P_g, one has

$$\mu(g)\|u_k\|_{H^3} \le 2E_f[u_k] + \frac{2}{2^\sharp}\int_M f|u_k|^{2^\sharp}\, d\mu_g \le C + o(\|u_k\|_{H^3}).$$

From this, we get $\{u_k\}$ is bounded in $H^3(M, g)$. Then up to a subsequence, as $k \to \infty$, $u_k \rightharpoonup u$ in $H^3(M, g)$ and $u_k \to u$ in $L^p(M, g)$ for $1 \le p < 2^\sharp$. It is easy to verify that u is a weak solution to (3-5), that is, for all $\psi \in H^3(M, g)$,

$$\int_M \psi P_g u\, d\mu_g = \int_M f|u|^{2^\sharp-2}u\psi\, d\mu_g.$$

Choosing $\psi = u$, one has

$$\int_M u P_g u\, d\mu_g = \int_M f|u|^{2^\sharp}\, d\mu_g,$$

whence

$$E_f[u] = \frac{3}{n}\int_M f|u|^{2^\sharp}\, d\mu_g \ge 0.$$

Applying the Brezis–Lieb lemma to

$$\begin{aligned}\int_M |\nabla\Delta u_k|_g^2\, d\mu_g &= \int_M |\nabla\Delta u|_g^2\, d\mu_g + \int_M |\nabla\Delta(u-u_k)|_g^2\, d\mu_g + o(1),\\ \int_M f|u_k|^{2^\sharp}\, d\mu_g &= \int_M f|u|^{2^\sharp}\, d\mu_g + \int_M f|u-u_k|^{2^\sharp}\, d\mu_g + o(1),\end{aligned}$$

we have

$$\begin{aligned}E_f[u_k] - E_f[u] &= \frac{1}{2}\int_M |\nabla\Delta(u-u_k)|_g^2 - \frac{1}{2^\sharp}\int_M f|u-u_k|^{2^\sharp}\, d\mu_g + o(1)\\ &= E_f[u-u_k] + o(1).\end{aligned}$$

Since $DE_f[u_k] \to 0$ in $(H^3(M, g))'$, we have

$$\begin{aligned}o(1) &= \langle u_k - u, DE_f[u_k]\rangle\\ &= \langle u_k - u, DE_f[u_k] - DE_f[u]\rangle\\ &= \int_M |\nabla\Delta(u-u_k)|_g^2\, d\mu_g - \int_M f|u-u_k|^{2^\sharp}\, d\mu_g + o(1).\end{aligned}$$

Thus, we obtain

$$\frac{3}{n}\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g + o(1) = E_f[u_k - u]$$
$$= E_f[u_k] - E_f[u] + o(1) \le E_f[u_k] + o(1) \to \beta,$$

as $k\to\infty$, which yields

$$\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g \le \frac{n}{3}\beta + o(1). \tag{3-7}$$

Mimicking a cut-and-paste argument as in [Djadli et al. 2000], we obtain that given $\epsilon > 0$, there exists a constant $B_\epsilon > 0$ such that

$$\left(\int_M |\psi|^{2^\sharp}\,d\mu_g\right)^{2/2^\sharp} \le (1+\epsilon)Y_6(S^n)^{-1}\int_M (|\nabla\Delta\psi|_g^2 + |\nabla^2\psi|_g^2 + |\nabla\psi|_g^2)\,d\mu_g + B_\epsilon\int_M \psi^2\,d\mu_g,$$

for all $\psi\in H^3(M,g)$. Choosing $\psi = u_k - u$ and k sufficiently large, we get

$$\left(\int_M |u-u_k|^{2^\sharp}\,d\mu_g\right)^{2/2^\sharp} \le (1+\epsilon)Y_6(S^n)^{-1}\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g + o(1).$$

Hence we have

$$\begin{aligned} o(1) &= \int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g - \int_M f|u-u_k|^{2^\sharp}\,d\mu_g \\ &\ge \int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g \\ &\qquad \left[1 - \left(\max_M f\right)(1+\epsilon)^{\frac{2^\sharp}{2}} Y_6(S^n)^{-\frac{2^\sharp}{2}} \left(\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g\right)^{\frac{6}{n-6}}\right]. \end{aligned}$$

From (3-7) and $\beta < (3/n)Y_6(S^n)^{n/6}(\max_M f)^{(6-n)/6}$, choosing ϵ sufficiently small, we get

$$o(1) \ge C\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g.$$

Combining the above inequality and the coercivity of P_g to show that $u_k\to u$ in $H^3(M,g)$. Using the regularity result in Lemma 3.4 below, we know that $u\in C^{6,\mu}(M)$ for any $0<\mu<1$.

In addition, assume (M,g) is Einstein and has positive constant scalar curvature. We define the modified energy in $H^3(M,g)$ by

$$E_f^+[u] = \frac{1}{2}\int_M uP_g u\,d\mu_g - \frac{1}{2^\sharp}\int_M f u_+^{2^\sharp}\,d\mu_g,$$

where $u_+ = \max\{u, 0\}$. Using the above similar arguments associated with the mountain pass lemma and mimicking what we did in Lemma 3.4 below for E_f^+, we get that there exists a nontrivial C^6-solution u to

$$P_g u = f u_+^{\frac{n+6}{n-6}} \quad \text{in} \quad M. \tag{3-8}$$

Since P_g is coercive by the remark on page 57, testing equation (3-8) with $u_- = \min\{u, 0\}$ we conclude that $u \geq 0$ in M. Together with R_g being a positive constant and the factorization (1-4) of GJMS operator:

$$\Big(-\Delta_g + \frac{(n-6)(n+4)}{4n(n-1)} R_g\Big)\Big(-\Delta_g + \frac{(n-4)(n+2)}{4n(n-1)} R_g\Big)\Big(-\Delta_g + \frac{n-2}{4(n-1)} R_g\Big) u \geq 0$$

and $u \not\equiv 0$ in M, we employ the maximum principle twice and strong maximum principle once for elliptic equations of second-order to show that u is a positive solution to the equation (3-6). From this and Schauder estimates for elliptic equations, we conclude that $u \in C^\infty(M)$. This completes the proof. □

We are now concerned with the regularity of mountain pass critical points for E.

Lemma 3.4. *Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 7$. Assume $u \in H^3(M, g)$ is a weak solution of equation* (3-5). *Then $u \in C^{6,\mu}(M)$ for any $0 < \mu < 1$.*

Proof. Rewrite $P_g = (-\Delta_g)^3 - M_g + (n-6)/2 Q_g$ by (1-2). Let $u \in H^3(M, g)$ be a weak solution of equation (3-5) and rewrite this equation as

$$\begin{aligned} (-\Delta_g + 1)^3 u =& M_g u + 3\Delta_g^2 u - 3\Delta_g u + (1 - \tfrac{n-6}{2} Q_g) u + f|u|^{2^\sharp - 2} u \\ :=& b + f|u|^{2^\sharp - 2} u, \end{aligned} \tag{3-9}$$

where $b \in H^{-1}(M, g)$. By the Sobolev embedding theorem we have $u \in L^{2^\sharp}(M, g)$ and $|u|^{2^\sharp - 2} \in L^{n/6}(M, g)$. Given $\epsilon > 0$, there exist a $K_\epsilon > 0$ and a decomposition of $f|u|^{2^\sharp - 2} = h_\epsilon + \eta_\epsilon$ with $\|h_\epsilon\|_{n/6} \leq \epsilon$, $\|\eta_\epsilon\|_\infty \leq K_\epsilon$. Inspired by the arguments in [Esposito and Robert 2002, Proposition 3], for $s > 1$ we define an operator

$$H_\epsilon : v \in L^s(M, g) \to (-\Delta_g + 1)^{-3}(h_\epsilon v) \in L^s(M, g).$$

Indeed, from the Sobolev embedding theorem, the standard $W^{2,p}$-regularity theory of the elliptic operator $-\Delta_g + 1$ and Hölder's inequality, we have

$$\begin{aligned} \|H_\epsilon v\|_s \leq C\|(-\Delta_g + 1)^{-3}(h_\epsilon v)\|_{W^{6,\frac{ns}{n+6s}}} \leq& C\|h_\epsilon v\|_{\frac{ns}{n+6s}} \\ \leq& C\|h_\epsilon\|_{\frac{n}{6}} \|v\|_s \leq C\epsilon \|v\|_s, \end{aligned}$$

where the constant C is independent of u. If we choose $\epsilon > 0$ small enough, then the norm of H_ϵ on the space $L^s(M, g)$ satisfies

$$\|H_\epsilon\|_{L^s \to L^s} \leq C\epsilon \leq \tfrac{1}{2}.$$

With the help of the operator H_ϵ, we rewrite equation (3-9) as

$$(\mathrm{Id} - H_\epsilon)u = (-\Delta_g + 1)^{-3}(b + \eta_\epsilon u),$$

then it is easy to show $\mathrm{Id} - H_\epsilon : L^s \to L^s$ is bounded and invertible. We intend to prove $u \in H^6(M, g)$. To see this, notice that $(-\Delta_g + 1)^{-3}(b + \eta_\epsilon u) \in H^5(M, g)$ since $b + \eta_\epsilon u \in H^{-1}(M, g)$. In the following, we first show $u \in H^4(M, g)$. Apply the Sobolev embedding theorem and the L^s-boundedness of the operator $(\mathrm{Id} - H_\epsilon)^{-1}$ to show that if $n \le 10$, $u \in L^p(M, g)$ for all $p > 1$, and if $n > 10$, $u \in L^{2n/(n-10)}(M, g)$. In the latter case we have $|u|^{2^\sharp - 2}u \in L^{2n(n-6)/((n+6)(n-10))}(M, g)$. From equation (3-9), we get

$$(-\Delta_g + 1)^2 u = (-\Delta_g + 1)^{-1} b + (-\Delta_g + 1)^{-1}(f|u|^{2^\sharp - 2}u).$$

From $(-\Delta_g + 1)^{-1}(|u|^{2^\sharp - 2}u) \in W^{2,2n(n-6)/((n+6)(n-10))}(M, g) \hookrightarrow L^2(M, g)$ and $(-\Delta_g + 1)^{-1} b \in L^2(M, g)$, we have $u \in H^4(M, g)$ in both cases. Repeat the above step with $u \in H^4(M, g)$ and $b \in L^2(M, g)$ in this situation. Notice that $(-\Delta_g + 1)^{-3}(b + \eta_\epsilon u) \in H^6(M, g)$, similar arguments in the above step show that if $n \le 12$, $u \in L^p(M, g)$ for all $p > 1$ and if $n > 12$, $u \in L^{2n/(n-12)}(M, g)$. In the latter case, we get $|u|^{2^\sharp - 2}u \in L^2(M, g)$ due to $2n(n-6)/((n+6)(n-12)) > 2$. Hence we obtain $u \in H^6(M, g)$.

Finally we start with the classical bootstrap. We now construct a nondecreasing sequence $s_k \in \mathbb{R} \cup \{+\infty\}$ such that $u \in W^{6,s_k}(M, g)$ for all $k \in \mathbb{N}$. Set $s_0 = 2$, and find $k \ge 0$ such that $u \in W^{6,s_k}(M, g)$. Next we will define s_{k+1} by induction. The Sobolev embedding theorem yields

$$b \in L^{\frac{ns_k}{n-2s_k}}(M, g),$$

with the convention that $ns_k/(n - 2s_k) = +\infty$ if $s_k \ge n/2$, and

$$|u|^{2^\sharp - 2}u \in L^{\frac{ns_k(n-6)}{(n-6s_k)(n+6)}}(M, g),$$

with the convention that $ns_k/(n - 6s_k) = +\infty$ if $s_k \ge n/6$. In view of equation (3-9), we have

$$u \in W^{6,s_{k+1}}(M, g) \quad \text{with } s_{k+1} = \min\left\{\frac{ns_k}{n - 2s_k}, \frac{ns_k(n-6)}{(n - 6s_k)(n+6)}\right\}.$$

If $s_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, it must hold that $s_k \to +\infty$. Then we have $u \in W^{6,p}(M, g)$ for all $1 \le p < +\infty$. If $s_k = +\infty$ for all $k \ge k_0 + 1$, then $s_{k_0} \ge n/6$, whence $b \in L^{n/4}(M, g)$ and $|u|^{2^\sharp - 2}u \in L^q(M, g)$ for all $1 \le q < +\infty$. The equation (3-9) leads to $u \in W^{6,n/4}(M)$. Repeating the argument twice, we obtain $u \in W^{6,p}(M, g)$ for all $1 \le p < +\infty$. From this and the Sobolev embedding theorem, we have $u \in C^{5,\nu}(M)$ for all $0 < \nu < 1$. By the regularity theory for the classical solution of the elliptic operator $-\Delta_g + 1$, we get $u \in C^{6,\mu}(M)$ for some $0 < \mu < 1$. This completes the proof. □

Appendix: proof of Lemma 2.2

As in Proposition 3.2, one may employ all computations under conformal normal coordinates of the metric g around a point in M. From Lee and Parker [1987] that up to a conformal factor, under g-conformal normal coordinates around this point, for all $N \geq 5$ we have

$$\sigma_1(A_g) = 0, \quad \sigma_1(A_g)_{,i} = 0, \quad \Delta_g \sigma_1(A_g) = -\frac{|W|_g^2}{12(n-1)}$$

at this point and $\sqrt{\det g} = 1 + O(r^N)$ near this point.

To simplify the notation, we will omit the subscript g. Notice that

$$\begin{aligned} -P_g(r^{6-n}) =& \Big[\Delta^3 + \Delta\delta T_2 d + \delta T_2 d\Delta + \frac{n-2}{2}\Delta(\sigma_1(A)\Delta) + \delta T_4 d - \frac{n-6}{2} Q_g\Big](r^{6-n}) \\ :=& \sum_{k=1}^{6} I_k. \end{aligned}$$

Next, we begin to estimate all terms I_1–I_6.

For I_1, let $u = u(r)$ be a radial function. We have

$$\begin{aligned} \Delta u(r) =& \Delta_0 u(r) + O(r^{N-1})u'; \\ \Delta^2 u(r) =& \Delta_0(\Delta_0 u(r) + O(r^{N-1})u') + O(r^{N-1})(\Delta_0 u(r) + O(r^{N-1})u')' \\ =& \Delta_0^2 u(r) + O(r^{N-1})u''' + O(r^{N-2})u'' + O(r^{N-3})u'; \\ \Delta^3 u(r) =& \Delta_0^3 u(r) + O(r^{N-1})u^{(5)} + O(r^{N-2})u^{(4)} + O(r^{N-3})u''' \\ &+ O(r^{N-4})u'' + O(r^{N-5})u'. \end{aligned}$$

Hence we obtain

$$I_1 = \Delta^3(r^{6-n}) = -c_n\delta_p + O(r^{N-n}).$$

To estimate I_2, notice that

$$I_2 = \Delta\delta T_2 d(r^{6-n}) = -\Delta[(T_2)_{ij}(r^{6-n})_{,j}]_{,i} = -\Delta[(T_2)_{ij,i}(r^{6-n})_{,j} + (T_2)_{ij}(r^{6-n})_{,ji}].$$

Using

$$\begin{aligned} &(r^{6-n})_{,j} = (6-n)r^{4-n}x^j, \\ \text{(A-1)} \qquad &(r^{6-n})_{,ji} = (4-n)(6-n)r^{2-n}x^i x^j + (6-n)r^{4-n}\delta_{ij} + O(r^{6-n}), \end{aligned}$$

one has

$$(T_2)_{ij,i}(r^{6-n})_{,j} = (n-10)\sigma_1(A)_{,j}(6-n)r^{4-n}x^j = (n-10)(6-n)\sigma_1(A)_{,j}x^j r^{4-n}$$

and

$$\begin{aligned}(T_2)_{ij}(r^{6-n})_{,ji} &= [(n-2)\sigma_1(A)g_{ij}-8A_{ij}](6-n)[(4-n)r^{2-n}x^ix^j+r^{4-n}\delta_{ij}+O(r^{6-n})]\\ &= (6-n)[4(n-4)\sigma_1(A)r^{4-n}-8(4-n)A_{ij}x^ix^jr^{2-n}]+O(r^{7-n}).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}I_2 &= -(6-n)\Delta[(n-10)\sigma_1(A)_{,j}x^jr^{4-n}+4(n-4)\sigma_1(A)r^{4-n}-8(4-n)A_{ij}x^ix^jr^{2-n}]\\
&= (n-6)\{(n-10)[4(4-n)\sigma_1(A)_{,j}x^jr^{2-n}+2(4-n)\sigma_1(A)_{,jk}x^jx^kr^{2-n}\\
&\qquad +\sigma_1(A)_{,jkk}x^jr^{2-n}+2\Delta\sigma_1(A)r^{4-n}]+O(r^{5-n})\\
&\qquad +4(n-4)[\Delta\sigma_1(A)r^{4-n}+2(4-n)\sigma_1(A)_{,k}x^kr^{2-n}+2(4-n)\sigma_1(A)r^{2-n}]\\
&\qquad +8(n-4)[4(2-n)A_{ij}x^ix^jr^{-n}+\Delta A_{ij}x^ix^jr^{2-n}\\
&\qquad +4\sigma_1(A)_{,i}x^ir^{2-n}+2\sigma_1(A)r^{2-n}]\}\\
&= (n-6)\big\{-4(n-4)(3n-26)\sigma_1(A)_{,j}x^jr^{2-n}+6(n-6)\Delta\sigma_1(A)r^{4-n}\\
&\qquad -2(n-10)(n-4)\sigma_1(A)_{,jk}x^jx^kr^{2-n}\\
&\qquad +(n-10)\sigma_1(A)_{,jkk}x^jr^{4-n}+O(r^{5-n})-8(n-6)(n-4)\sigma_1(A)r^{2-n}\\
&\qquad -32(n-4)(n-2)A_{ij}x^ix^jr^{-n}+8(n-4)\Delta A_{ij}x^ix^jr^{2-n}\big\}\\
&= (n-6)\Big\{-4(n-4)(3n-26)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}-\frac{n-6}{2(n-1)}|W(p)|^2r^{4-n}\\
&\qquad -2(n-10)(n-4)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\\
&\qquad -4(n-6)(n-4)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\\
&\qquad -16(n-4)(n-2)A_{ij,kl}(p)x^ix^jx^kx^lr^{-n}\\
&\qquad +8(n-4)\Delta A_{ij}x^ix^jr^{2-n}\Big\}+O(r^{5-n})\\
&= (n-6)\Big\{-2(n-4)(9n-74)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}-\frac{n-6}{2(n-1)}|W(p)|^2r^{4-n}\\
&\qquad -16(n-4)(n-2)A_{ij,kl}(p)x^ix^jx^kx^lr^{-n}\\
&\qquad +8(n-4)\Delta A_{ij}x^ix^jr^{2-n}\Big\}+O(r^{5-n}).\end{aligned}$$

To estimate

$$I_3=\delta T_2 d\Delta(r^{6-n})=-[(T_2)_{ij}(\Delta r^{6-n})_{,j}]_{,i}=-(T_2)_{ij,i}(\Delta r^{6-n})_{,j}-(T_2)_{ij}(\Delta r^{6-n})_{,ji}.$$

Recall that $T_2=(n-2)\sigma_1(A)g-8A$. Then

$$(T_2)_{ij,i}=(n-10)\sigma_1(A)_{,j}.$$

Observe that

$$\begin{aligned}\Delta r^{6-n} &= 4(6-n)r^{4-n}+O(r^{N+4-n}),\\ (\Delta r^{6-n})_{,j} &= 4(6-n)(4-n)x^jr^{2-n}+O(r^{N+3-n}),\end{aligned}$$

and

$$(\Delta r^{6-n})_{,ji} = 4(6-n)(4-n)[(2-n)x^i x^j r^{-n} + r^{2-n}\delta_{ij}] + O(r^{4-n}).$$

Then we have

$$\begin{aligned}
&(T_2)_{ij}(\Delta r^{6-n})_{,ji} \\
&= 4(n-6)(n-4)[(n-2)\sigma_1(A)g_{ij} - 8A_{ij}][(2-n)x^i x^j r^{-n} + r^{2-n}\delta_{ij} + O(r^{4-n})] \\
&= 4(n-6)(n-4)[-(n-2)^2\sigma_1(A)r^{2-n} + n(n-2)\sigma_1(A)r^{2-n} \\
&\quad + 8(n-2)r^{-n}A_{ij}x^i x^j - 8\sigma_1(A)r^{2-n}] + O(r^{5-n}) \\
&= 4(n-6)(n-4)[2(n-6)\sigma_1(A)r^{2-n} + 8(n-2)r^{-n}A_{ij}x^i x^j] + O(r^{5-n}) \\
&= 4(n-6)(n-4)[(n-6)\sigma_1(A)_{,ij}(p)x^i x^j r^{2-n} + 4(n-2)r^{-n}(A_{ij,kl}(p)x^i x^j x^k x^l)] \\
&\quad + O(r^{5-n}).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
I_3 =& -4(n-6)(n-4)[(n-6)\sigma_1(A)_{,ij}(p)x^i x^j r^{2-n} + 4(n-2)r^{-n}(A_{ij,kl}(p)x^i x^j x^k x^l)] \\
&- 4(n-6)(n-4)(n-10)r^{2-n}\sigma_1(A)_{,i}x^i + O(r^{5-n}) \\
=& -8(n-8)(n-6)(n-4)\sigma_1(A)_{,ij}(p)x^i x^j r^{2-n} \\
&- 16(n-6)(n-4)(n-2)r^{-n}(A_{ij,kl}(p)x^i x^j x^k x^l) + O(r^{5-n}).
\end{aligned}$$

We now compute

$$\begin{aligned}
I_4 &= \frac{n-2}{2}\Delta(\sigma_1(A)\Delta(r^{6-n})) \\
&= 2(n-2)(6-n)\Delta(\sigma_1(A)r^{4-n}) + O(r^{N+4-n}) \\
&= 2(n-2)(6-n)r^{2-n}[\Delta\sigma_1(A)r^2 + 2(4-n)\sigma_1(A)_{,i}x^i + 2(4-n)\sigma_1(A)] \\
&\quad + O(r^{N+2-n}) \\
&= 2(n-2)(n-6)r^{2-n}\left[\frac{1}{12(n-1)}|W(p)|^2 r^2 + 3(n-4)\sigma_1(A)_{,ij}(p)x^i x^j\right] \\
&\quad + O(r^{5-n}).
\end{aligned}$$

For I_5, from (A-1) we have

$$\begin{aligned}
I_5 =& \delta T_4 d(r^{6-n}) \\
=& -((T_4)_{ij}r^{6-n}{}_{,j})_{,i} \\
=& -(T_4)_{ij,i}(r^{6-n})_{,j} - (T_4)_{ij}(r^{6-n})_{,ji} \\
=& (n-6)[r^{4-n}(T_4)_{ij,i}x^j - (n-4)r^{2-n}(T_4)_{ij}x^i x^j + r^{4-n}\operatorname{tr}(T_4)] \\
:=& (n-6)[I_1^{(5)} + I_2^{(5)} + I_3^{(5)}].
\end{aligned}$$

Also from [Lee and Parker 1987], we have

$$\mathrm{Sym}\big(R_{kl,ij} + \tfrac{2}{9} R_{nklm} R_{nijm}\big)(p) = 0 \quad \text{and} \quad R_{ij}(p) = 0,$$

then

$$R_{kl,ij}(p) x^i x^j x^k x^l = -\tfrac{2}{9} W_{nklm}(p) W_{nijm}(p) x^i x^j x^k x^l.$$

Thus we have

$$A_{kl,ij}(p) x^i x^j x^k x^l = -\frac{2}{9}\frac{1}{n-2}\sum_{k,l}(W_{iklj}(p) x^i x^j)^2 - \frac{\sigma_1(A)_{,ij}(p) x^i x^j r^2}{n-2}. \tag{A-2}$$

To estimate $I_3^{(5)}$. From the definition of T_4, one gets

$$\begin{aligned}\mathrm{tr}(T_4) &= -\frac{3n^3 - 12n^2 - 36n + 64}{4}\sigma_1(A)^2 + 4(n^2 - 4n - 12)|A|^2 + n(n-6)\Delta\sigma_1(A)\\ &= -\frac{n(n-6)}{12(n-1)}|W(p)|^2 + O(r).\end{aligned}$$

Thus one obtains

$$I_3^{(5)} = -\frac{n(n-6)}{12(n-1)}|W(p)|^2 r^{4-n} + O(r^{5-n}).$$

For the term $I_1^{(5)}$, it is easy to see

$$I_1^{(5)} = r^{4-n}(T_4)_{ij,i} x^j = O(r^{5-n}).$$

It remains to estimate the term $I_2^{(5)}$. One has

$$(T_4)_{ij} x^i x^j = (n-6)\Delta\sigma_1(A) r^2 - \frac{16}{n-4} B_{ij} x^i x^j + O(r^4). \tag{A-3}$$

Notice that

$$\begin{aligned}B_{ij} x^i x^j &= [C_{ijk,k} - A_{kl} W_{kijl}] x^i x^j = [(A_{ij,k} - A_{ik,j})_{,k} - A_{kl} W_{kijl}] x^i x^j\\ &= [\Delta A_{ij} - A_{ik,jk} + O(r)] x^i x^j\end{aligned}$$

and

$$\begin{aligned}\Delta(A_{ij} x^i x^j) &= (A_{ij,k} x^i x^j + A_{ij}(x^i \delta_{jk} + x^j \delta_{ik}))_{,k}\\ &= (\Delta A_{ij}) x^i x^j + 2A_{ij,k}(x^i \delta_{jk} + x^j \delta_{ik}) + 2\sigma_1(A) + O(r^3)\\ &= (\Delta A_{ij}) x^i x^j + 4\sigma_1(A)_{,i} x^i + 2\sigma_1(A) + O(r^3).\end{aligned}$$

By (A-2), one gets

$$\begin{aligned}(\Delta A_{ij})x^ix^j &= \Delta(A_{ij}x^ix^j) - 4\sigma_1(A)_{,i}x^i - 2\sigma_1(A) + O(r^3)\\ &= \Delta\left[\frac{1}{2}A_{ij,kl}(p)x^ix^jx^kx^l + O(r^5)\right] - 4[\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)]\\ &\quad - \sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\\ &= \Delta\left[-\frac{1}{9}\frac{1}{n-2}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{\sigma_1(A)_{,ij}(p)x^ix^jr^2}{2(n-2)}\right]\\ &\quad - 5\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\\ &= -\frac{2}{9}\frac{1}{n-2}\sum_{k,l,s}[(W_{ikls}(p) + W_{ilks}(p))x^i]^2\\ &\quad + \frac{1}{12(n-2)(n-1)}|W(p)|^2r^2 - 6\frac{n-1}{n-2}\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3), \end{aligned} \tag{A-4}$$

where the last identity follows from the following two estimates:

$$\begin{aligned}&\Delta(\sigma_1(A)_{,ij}(p)x^ix^jr^2)\\ &\quad = \Delta(\sigma_1(A)_{,ij}(p)x^ix^j)r^2 + 2\nabla_s(\sigma_1(A)_{,ij}(p)x^ix^j)\nabla_s r^2 + (\sigma_1(A)_{,ij}(p)x^ix^j)\Delta r^2\\ &\quad = 2\Delta\sigma_1(A)(p)r^2 + 8\sigma_1(A)_{,ij}(p)x^ix^j + 2n\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\\ &\quad = -\frac{1}{6(n-1)}|W(p)|^2r^2 + 2(n+4)\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\end{aligned}$$

and

$$\Delta\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 = 2\sum_{k,l,s}[W_{iklj}(p)(x^i\delta_{js} + x^j\delta_{is})]^2 = 2\sum_{k,l,s}[(W_{ikls}(p) + W_{ilks}(p))x^i]^2,$$

which follows from

$$\Delta\left[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2\right] = 2\sum_{k,l}[(W_{iklj}(p)x^ix^j)\Delta(W_{sklt}(p)x^sx^t) + |\nabla(W_{iklj}(p)x^ix^j)|^2]$$

and $\Delta(W_{sklt}(p)x^sx^t) = (W_{sklt}(p)(x^s\delta_{it} + x^t\delta_{is}))_{,i} = 2W_{sklt}(p)\delta_{st} = 0$. Using $A_{ik,jk} = A_{ik,kj} + R^l_{ijk}A_{lk} + R^l_{kjk}A_{il} = \sigma_1(A)_{,ij} + R_{lijk}A_{lk} + R_{lj}A_{il}$, one has

$$\begin{aligned}A_{ik,jk}x^ix^j &= \sigma_1(A)_{,ij}x^ix^j + R_{lijk}A_{lk}x^ix^j + R_{lj}A_{il}x^ix^j\\ &= (\sigma_1(A)_{,ij}(p) + O(r))x^ix^j\\ &\qquad + (W_{lijk}(p) + O(r))(A_{lk,m}(p)x^m + O(r^2))x^ix^j + O(r^4)\\ &= \sigma_1(A)_{,ij}(p)x^ix^j + O(r^3).\end{aligned}$$

Thus, one obtains

$$B_{ij}x^ix^j=-\frac{2}{9}\frac{1}{n-2}\sum_{k,l,s}[(W_{ikls}(p)+W_{ilks}(p))x^i]^2+\frac{1}{12(n-2)(n-1)}|W(p)|^2r^2 - \frac{7n-8}{n-2}\sigma_1(A)_{,ij}(p)x^ix^j+O(r^3). \tag{A-5}$$

Inserting the above equations into (A-3), one gets

$$(T_4)_{ij}x^ix^j=-\frac{n-6}{12(n-1)}|W(p)|^2r^2+\frac{32}{9(n-4)(n-2)}\sum_{k,l,s}[(W_{ikls}(p)+W_{ilks}(p))x^i]^2 - \frac{4}{3(n-4)(n-2)(n-1)}|W(p)|^2r^2 + \frac{16(7n-8)}{(n-4)(n-2)}\sigma_1(A)_{,ij}(p)x^ix^j+O(r^3),$$

whence

$$I_2^{(5)}=r^{2-n}\Bigg[\frac{(n-6)(n-4)}{12(n-1)}|W(p)|^2r^2-\frac{32}{9(n-2)}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 +\frac{4}{3(n-2)(n-1)}|W(p)|^2r^2-\frac{16(7n-8)}{n-2}\sigma_1(A)_{,ij}(p)x^ix^j\Bigg]+O(r^{5-n}).$$

Combining all the terms together, one has

$$I_5=\Bigg[-\frac{n^2-8n+8}{3(n-1)(n-2)}|W(p)|^2r^{4-n} -\frac{32}{9(n-2)}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2r^{2-n} -\frac{16(7n-8)}{n-2}\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\Bigg](n-6)+O(r^{5-n}).$$

Finally, from the definition of Q_g in (1-1), it is not difficult to show that $I_6=-(n-6)/2Q_gr^{6-n}=O(r^{6-n})$.

Therefore, collecting all the terms I_1–I_6 together with (A-2) and (A-4), we conclude that

$$-P_g(r^{6-n})=-c_n\delta_p+(n-6)\Bigg[-\frac{16}{9}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2r^{2-n} -\frac{2(n-8)}{3(n-1)}|W(p)|^2r^{4-n}+\frac{64(n-4)}{9}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2r^{-n} -4(5n^2-66n+224)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\Bigg]+O(r^{5-n})$$

$$
\begin{aligned}
&= -c_n\delta_p + (n-6)r^{-n}\Bigg\{\frac{64(n-4)}{9}\Bigg[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 \\
&\qquad - \frac{r^2}{n+4}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 \\
&\qquad + \frac{3}{2(n+4)(n+2)}|W(p)|^2r^4\Bigg] \\
&+ \frac{16(3n-20)}{9(n+4)}r^2\Bigg[\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n}|W(p)|^2r^2\Bigg] \\
&- 4(5n^2-66n+224)r^2\Bigg[\sigma_1(A)_{,ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2\Bigg] \\
&+ \frac{3n^4-16n^3-164n^2+400n+2432}{3(n+4)(n+2)n(n-1)}|W(p)|^2r^4\Bigg\} + O(r^{5-n}),
\end{aligned}
$$

where each term in square brackets on the right-hand side of the last identity is a harmonic polynomial. This finishes the proof of Lemma 2.2.

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References

[Ambrosetti and Rabinowitz 1973] A. Ambrosetti and P. H. Rabinowitz, “Dual variational methods in critical point theory and applications”, *J. Functional Analysis* **14**:4 (1973), 349–381. MR Zbl

[Aubin 1976] T. Aubin, “Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire”, *J. Math. Pures Appl.* (9) **55**:3 (1976), 269–296. MR Zbl

[Djadli et al. 2000] Z. Djadli, E. Hebey, and M. Ledoux, “Paneitz-type operators and applications”, *Duke Math. J.* **104**:1 (2000), 129–169. MR Zbl

[Esposito and Robert 2002] P. Esposito and F. Robert, “Mountain pass critical points for Paneitz–Branson operators”, *Calc. Var. Partial Differential Equations* **15**:4 (2002), 493–517. MR Zbl

[Fefferman and Graham 2012] C. Fefferman and C. R. Graham, *The ambient metric*, Annals of Mathematics Studies **178**, Princeton Univ. Press, 2012. MR Zbl

[Gover 2006] A. R. Gover, “Laplacian operators and Q-curvature on conformally Einstein manifolds”, *Math. Ann.* **336**:2 (2006), 311–334. MR Zbl

[Graham et al. 1992] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, “Conformally invariant powers of the Laplacian, I: Existence”, *J. London Math. Soc.* (2) **46**:3 (1992), 557–565. MR Zbl

[Gursky and Malchiodi 2015] M. J. Gursky and A. Malchiodi, "A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature", *J. Eur. Math. Soc.* **17**:9 (2015), 2137–2173. MR Zbl

[Gursky et al. 2016] M. J. Gursky, F. Hang, and Y.-J. Lin, "Riemannian manifolds with positive Yamabe invariant and Paneitz operator", *Int. Math. Res. Not.* **2016**:5 (2016), 1348–1367. MR Zbl

[Hang and Yang 2016] F. Hang and P. C. Yang, "Q-curvature on a class of manifolds with dimension at least 5", *Comm. Pure Appl. Math.* **69**:8 (2016), 1452–1491. MR Zbl

[Juhl 2013] A. Juhl, "Explicit formulas for GJMS-operators and Q-curvatures", *Geom. Funct. Anal.* **23**:4 (2013), 1278–1370. MR Zbl

[Lee and Parker 1987] J. M. Lee and T. H. Parker, "The Yamabe problem", *Bull. Amer. Math. Soc.* (*N.S.*) **17**:1 (1987), 37–91. MR Zbl

[Li and Xiong 2015] Y. Li and J. Xiong, "Compactness of conformal metrics with constant Q-curvature, I", preprint, 2015. arXiv

[Lieb 1983] E. H. Lieb, "Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities", *Ann. of Math.* (2) **118**:2 (1983), 349–374. MR Zbl

[Lions 1985] P.-L. Lions, "The concentration-compactness principle in the calculus of variations: the limit case, I", *Rev. Mat. Iberoamericana* **1**:1 (1985), 145–201. MR Zbl

[Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series **30**, Princeton Univ. Press, 1970. MR Zbl

[Wünsch 1986] V. Wünsch, "On conformally invariant differential operators", *Math. Nachr.* **129** (1986), 269–281. MR Zbl

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