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ON THE ASYMPTOTIC BEHAVIOR OF BERGMAN KERNELS FOR POSITIVE LINE BUNDLES

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Let L be a positive line bundle on a projective complex manifold. We study the asymptotic behavior of Bergman kernels associated with the tensor powers L^p of L as p tends to infinity. The emphasis is the dependence of the uniform estimates on the positivity of the Chern form of the metric on L. This situation appears naturally when we approximate a semipositive singular metric by smooth positively curved metrics.

1. Introduction

Let L be an ample holomorphic line bundle over a projective manifold X of dimension n. Fix a (reference) smooth Hermitian metric h_0 on L whose first Chern form ω_0 is a Kähler form. Recall that $\omega_0 = (\sqrt{-1}/2\pi)R_0^L$, where R_0^L is the curvature of the Chern connection on (L, h_0).

Let h^L be a semipositive singular metric on L. For various applications, one needs to understand the asymptotic behavior of the Bergman kernel associated with L^p and h^L when p tends to infinity. A natural approach is to approximate the considered metric by smooth positively curved metrics, and therefore, it is necessary to understand the dependence of the Bergman kernels in terms of the positivity of the curvature of the metric. See [Błocki and Kołodziej 2007; Demailly 1992; Dinh et al. 2015] for the regularization of metrics. This method was already used in our previous work on the speed of convergence of Fekete points, see [Berman et al. 2011; Dinh et al. 2015]. In §2.3 of the latter, inspired by [Berndtsson 2003], an L^1-estimate for Bergman kernels was obtained. Here, we investigate the uniform estimate which can be useful for applications in geometry.

Fix a smooth Kähler form θ on X (one can take $\theta = \omega_0$). Consider a metric $h = e^{-2\phi}h_0$ on L with weight ϕ of class $\mathcal{C}^{n+6}$ whose first Chern form $\omega := dd^c\phi + \omega_0$ (here $d^c := (\sqrt{-1}/2\pi)(\bar\partial - \partial)$) satisfies

$$\omega \geq \zeta\theta \quad \text{for some constant} \quad 0 < \zeta \leq 1. \tag{1-1}$$

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Consider the natural metric on the space of smooth sections of L^p, induced by the metric h on L and the volume form $\theta^n/n!$ on X, which is defined by

$$\|s\|^2_{L^2(p\phi)} := \int_X |s(x)|^2_{p\phi}\theta^n/n!\,. \tag{1-2}$$

Here, $|s(x)|_{p\phi}$ stands for the norm of $s(x)$ with respect to the metric $h^{\otimes p}$ on L^p. Let $\langle\cdot,\cdot\rangle_{p\phi}$ be the associated Hermitian product on $\mathcal{C}^\infty(X, L^p)$, the space of smooth sections of L^p. Let P_p be the orthogonal projection from $(\mathcal{C}^\infty(X, L^p), \langle\cdot,\cdot\rangle_{p\phi})$ onto the subspace of holomorphic sections $H^0(X, L^p)$. The Bergman kernel associated with the above data is the kernel associated with the last projection where we use the volume form $\theta^n/n!$ to integrate functions on X. This kernel is denoted by $P_p(x, x')$, with $x, x' \in X$. It is a section of the line bundle over $X \times X$ which is the tensor product of two line bundles: the first one is the pullback to $X \times X$ of the line bundle L^p over the first factor, and the second one is the pullback of the dual line bundle $(L^*)^p$ of L^p over the second factor. In particular, its restriction to the diagonal of $X \times X$, i.e., $P_p(x, x)$, can be identified to a positive-valued function on X. See [Ma and Marinescu 2007] for details. In fact, if $\{s_j\}_j$ is an orthonormal basis of $(H^0(X, L^p), \langle\cdot,\cdot\rangle)$, then

$$P_p(x,x)=\sum_j |s_j(x)|^2_{p\phi}=\sup\{|s(x)|^2_{p\phi}, s\in H^0(X,L^p) \quad \text{with } \|s\|_{L^2(p\phi)}=1\}. \tag{1-3}$$

Here is the main result in this paper which gives us a uniform estimate of the Bergman kernel in terms of ϕ, ω, p and ζ. This is a version of Tian's theorem [1990]. See [Berndtsson 2003; Boutet de Monvel and Sjöstrand 1976; Catlin 1999; Coman and Marinescu 2016; Dai et al. 2006; Hsiao and Marinescu 2014; Ma and Marinescu 2015; Xu 2012; Zelditch 1998] for various generalizations. We also refer to [Ma and Marinescu 2007] for a comprehensive study of several analytic and geometric aspects of Bergman kernels. The last reference is inspired by the analytic localization technique in [Bismut and Lebeau 1991].

Theorem 1.1. *Under the above assumptions, there exist $\delta > 0$, $c > 0$ satisfying the following condition: for any $l \in \mathbb{N}^*$, there is a constant $c_l > 0$ such that for $p \in \mathbb{N}^*$, $p\zeta > \delta$, and $x \in X$, we have*

$$\Big|p^{-n}P_p(x,x) - \frac{\omega(x)^n}{\theta(x)^n}\Big| \leqslant c|d\phi|^{2n+8}_{n+5}|\omega|^{4n+20}_0|d\phi|^{2n+2}_{n+2}\zeta^{-2n-10}p^{-1} + c_l|\omega|^{2n+2}_n(|d\phi|_2\zeta^{-1})^{6n+6+3l}p^{-l}. \tag{1-4}$$

Note that $|\cdot|_k$ stands for $1 + \|\cdot\|_{\mathcal{C}^k}$. As a direct consequence, we infer the following result by taking $l = 1$.

Corollary 1.2. *There exist* $\delta > 0$, $c > 0$ *such that for any* $0 < \zeta \leq 1$, *any weight* ϕ *of class* $\mathcal{C}^{n+6}$ *with* $dd^c\phi + \omega_0 \geq \zeta\theta$, *and any* $p \in \mathbb{N}^*$ *with* $\zeta p > \delta$, *we have*

$$\left| p^{-n} P_p(x,x) - \frac{\omega(x)^n}{\theta(x)^n} \right| \leq c\zeta^{-6n-9} |d\phi|_{n+5}^{8n+30} p^{-1}. \tag{1-5}$$

If $\phi \in \mathcal{C}^{n+2k+6}$, we can adapt easily the proof of Theorem 1.1 to get the estimate for $\mathcal{C}^k$-norm of the left-hand side of (1-4). Cf., Remark 3.9.

The article is organized as follows. In Section 2, we reduce the problem to the local setting. In Section 3, we establish Theorem 1.1. We need an approach different from previous ones which use the normal coordinates and the extension of connections on L; see [Dai et al. 2006, §4.2] and [Ma and Marinescu 2007, §4.1.3]. Note that throughout the paper, the constants $c, c', c_l, \dots$ may change from line to line.

2. Localization of the problem

Recall that the complex structure on X is given by a smooth section J of the vector bundle $\mathrm{End}(TX)$ such that $-J^2$ is the identity section. Here, TX denotes the real tangent bundle of X. Denote also by $T^{(1,0)}X$ and $T^{(0,1)}X$ the holomorphic and antiholomorphic tangent bundles of X. They are complex vector subbundles of $TX \otimes_{\mathbb{R}} \mathbb{C}$. The Kähler form θ induces a Riemannian metric g^{TX} on X defined by $g^{TX} := \theta(\cdot, J\cdot)$.

Let $\bar\partial^{L^p}$ be the $\bar\partial$-operator acting on L^p and $\bar\partial^{L^p,*}$ its dual operator with respect to the metric $h = e^{-2\phi}h_0$ on L and the Kähler form θ. Consider the Dirac and Laplacian-type operators

$$D_p := \sqrt{2}\big(\bar\partial^{L^p} + \bar\partial^{L^p,*}\big) \quad \text{and} \quad \square_p := \tfrac{1}{2} D_p^2 = \bar\partial^{L^p}\bar\partial^{L^p,*} + \bar\partial^{L^p,*}\bar\partial^{L^p}. \tag{2-1}$$

They act on $\Omega^{0,\bullet}(X, L^p)$, the space of the forms of bidegree $(0,\cdot)$ with values in L^p.

Let ∇^L be the Chern connection on $(L, h = e^{-2\phi}h_0)$ and $R^L = (\nabla^L)^2$ its curvature which is related to the first Chern form ω by

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{2-2}$$

Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}). It preserves $T^{(1,0)}X$, $T^{(0,1)}X$, and its restriction to $T^{(1,0)}X$ is the Chern connection $\nabla^{T^{(1,0)}X}$. Let $\nabla^{\Lambda^{0,\bullet}}$ be the connection on $\Lambda(T^{*(0,1)}X)$ induced by $\nabla^{T^{(1,0)}X}$, and $\nabla^{\Lambda^{0,\bullet}\otimes L^p}$ the connection on $\Lambda(T^{*(0,1)}X) \otimes L^p$ induced by $\nabla^{\Lambda^{0,\bullet}}$ and ∇^L. For $u \in T^{(1,0)}X$ and $v \in T^{(0,1)}X$, let $u^* \in T^{*(0,1)}X$ be the metric dual of u with respect to g^{TX}, define the operator $c(\cdot)$ depending linearly on a vector in $T^{(1,0)}X \oplus T^{(0,1)}X$ by setting

$$c(u) := \sqrt{2}u^* \wedge \quad \text{and} \quad c(v) := \sqrt{2} i_v, \tag{2-3}$$

where i denotes, as usual, the contraction operator. Then by [Ma and Marinescu 2007, p.31], for $\{e_j\}$ an orthonormal frame of (TX, g^{TX}), we have

$$D_p = \sum_j c(e_j)\nabla_{e_j}^{\Lambda^{0,\bullet}\otimes L^p}. \tag{2-4}$$

Denote by K_X^* the anticanonical bundle of X. The curvature of K_X^* with respect to the above Riemannian metric is denoted by $R^{K_X^*}$. Then $\sqrt{-1}R^{K_X^*}$ is the Ricci curvature of (X, g^{TX}). Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$ with dual frame $\{w^j\}_{j=1}^n$. Set

$$\omega_d := -\sum_{l,m} R^L(w_l, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l}. \tag{2-5}$$

Recall that $(\sqrt{-1}/2\pi)R^L = \omega \geq \zeta\theta$. Then ω_d is a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X))$ and R^L acts as the derivative ω_d on $\Lambda(T^{*(0,1)}X)$. By [Ma and Marinescu 2007, (1.4.63)] and using that $\langle\Delta^{0,\bullet}s, s\rangle_{p\phi} \geq 0$, where $\Delta^{0,\bullet}$ is a holomorphic Kodaira type Laplacian, we obtain for $s \in \Omega^{0,\bullet}(X, L^p)$ that

$$\begin{aligned}\|D_p s\|^2_{L^2(p\phi)} &= 2\langle\Box_p s, s\rangle_{p\phi}\\ &\geq -2p\langle\omega_d s, s\rangle_{p\phi} + 2\sum_{l,m}\big\langle R^{K_X^*}(w_l, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l}s, s\big\rangle_{p\phi}.\end{aligned} \tag{2-6}$$

Now by (1-1), (2-2), (2-5) and some standard arguments (see the proof of [Ma and Marinescu 2007, Theorem 1.5.5]) there exists $\delta > 0$, depending only on the Ricci curvature $R^{K_X^*}$, such that if $\zeta p > \delta$, then the spectrum of D_p^2 satisfies

$$\mathrm{Spec}(D_p^2) \subset \{0\} \cup [2\pi\zeta p, +\infty[. \tag{2-7}$$

Let a_X denote the injectivity radius of (X,θ). For $0 < \epsilon_0 < a_X/4$, let $f_{\epsilon_0}: \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f_{\epsilon_0}(v) = \begin{cases} 1 & \text{for } |v| \leqslant \epsilon_0/2,\\ 0 & \text{for } |v| \geqslant \epsilon_0.\end{cases} \tag{2-8}$$

Set

$$\begin{aligned}F_{\epsilon_0}(a) :=& \Big(\int_{-\infty}^{+\infty} f_{\epsilon_0}(v)\,dv\Big)^{-1}\int_{-\infty}^{+\infty} e^{iv\zeta a} f_{\epsilon_0}(v)\,dv\\ =& \Big(\int_{-\infty}^{+\infty} f_{\epsilon_0}(\zeta^{-1}v)\,dv\Big)^{-1}\int_{-\infty}^{+\infty} e^{iva} f_{\epsilon_0}(\zeta^{-1}v)\,dv.\end{aligned} \tag{2-9}$$

Then $F_{\epsilon_0}(a)$ lies in Schwartz space $\mathcal{S}(\mathbb{R})$ and $F_{\epsilon_0}(0) = 1$.

Proposition 2.1. *Let $\delta > 0$ satisfy* (2-7). *Then, for all $l \in \mathbb{N}$, $0 < \epsilon_0 < a_X/4$ and F_{ϵ_0} as above, there exists $c > 0$ such that for $p \geq 1$, $\delta/p < \zeta \leq 1$, $x, x' \in X$*

$$\|F_{\epsilon_0}(D_p)(x,x') - P_p(x,x')\|_{L^\infty(p\phi)} \leq c|\omega|_n^{2n+2}\zeta^{-6n-3l-6}p^{-l}. \tag{2-10}$$

Proof. For $a \in \mathbb{R}$, set

$$\phi_p(a) := 1_{[\sqrt{\zeta p}, +\infty[}(|a|)F_{\epsilon_0}(a). \tag{2-11}$$

By (2-7) and (2-11), for $\zeta p > \delta$, we get

$$F_{\epsilon_0}(D_p) - P_p = \phi_p(D_p). \tag{2-12}$$

By (2-9), for any $m \in \mathbb{N}$ there exists $c > 0$ such that for all $\zeta \in]0, 1[$,

$$\sup_{a \in \mathbb{R}} |a|^m |F_{\epsilon_0}(a)| \leq c\zeta^{-m}. \tag{2-13}$$

Thus, for any $m \in \mathbb{N}$ and $\zeta p > \delta$, we have

$$\|(D_p)^m F_{\epsilon_0}(D_p)\|^{0,0} := \sup_{s \in \Omega^{0,\bullet}(X,L^p)\setminus\{0\}} \frac{\|(D_p)^m F_{\epsilon_0}(D_p)s\|_{L^2(p\phi)}}{\|s\|_{L^2(p\phi)}} \leq c\zeta^{-m}. \tag{2-14}$$

As X is compact, there exists a finite set of points a_i, $1 \leq i \leq r$, such that the family of balls $U_i := B^X(a_i, \epsilon_0)$ of center a_i and radius ϵ_0, is a covering of X. We identify the ball $B^{T_{a_i}X}(0, \epsilon_0)$ in the tangent space of X at a_i with the ball $B^X(a_i, \epsilon_0)$ using the exponential map. We then identify $(TX)_Z$, $\Lambda(T^{*(0,1)}X)_Z$, L^p_Z for $Z \in B^{T_{a_i}X}(0, \epsilon_0)$ with $T_{a_i}X$, $\Lambda(T^{*(0,1)}X)_{a_i}$, $L^p_{a_i}$ by parallel transport with respect to the connections ∇^{TX}, $\nabla^{\Lambda^{0,\bullet}}$, ∇^{L^p} along the curve $\gamma_Z : [0, 1] \ni u \mapsto \exp^X_{a_i}(uZ)$. Then $(L, h)|_{U_i}$ is identified as the trivial bundle (L_{a_i}, h_{a_i}).

Let $\{e_j\}_j$ be an orthonormal basis of $T_{a_i}X \simeq \mathbb{R}^{2n}$. Let $\tilde{e}_j(Z)$ be the parallel transport of e_j with respect to ∇^{TX} along the above curve. Let Γ^L, $\Gamma^{\Lambda^{0,\bullet}}$ be the corresponding connection forms of ∇^L and $\nabla^{\Lambda^{0,\bullet}}$ with respect to any fixed frame for L and $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve γ_Z under the trivialization on U_i. Denote by ∇_v the ordinary differentiation operator on $T_{a_i}X$ in the direction v. As we are working in the Kähler case, by [Ma and Marinescu 2007, Proposition 1.2.6, Theorem 1.4.5, Remark 1.4.8], we can write on U_i

$$D_p = \sum_j c(\tilde{e}_j)\big(\nabla_{\tilde{e}_j} + p\Gamma^L(\tilde{e}_j) + \Gamma^{\Lambda^{0,\bullet}}(\tilde{e}_j))\big). \tag{2-15}$$

In fact, the last identity is a consequence of (2-4). Consider the radial vector field $\mathcal{R} = \sum_j Z_j e_j$. By [Ma and Marinescu 2007, (1.2.32)], the Lie derivative $L_{\mathcal{R}}\Gamma^L$ is equal to $i_{\mathcal{R}}R^L$. Therefore, we get the identity

$$\Gamma^L_Z = \int_0^1 (i_{\mathcal{R}}R^L)_{tZ}dt, \tag{2-16}$$

which allows us to bound Γ^L.

Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$. For $m \in \mathbb{N}$, we define a Sobolev norm on the m-th Sobolev space $H^m(X, \Lambda(T^{*(0,1)}X) \otimes L^p)$ by

$$\|s\|_{H^m}^2 = \sum_{i=1}^{r} \sum_{k=0}^{m} \sum_{j_1,\dots,j_k=1}^{2n} \|\nabla_{e_{j_1}} \cdots \nabla_{e_{j_k}} (\varphi_i s)\|_{L^2}^2. \tag{2-17}$$

Note that here we trivialize the line bundle L using a unitary section; so the section s above is identified with a function. Therefore, we drop the subscript $p\phi$ since this weight is already taken into account.

By (2-15), (2-16) and [Ma and Marinescu 2007, (1.6.9)], for P a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_i, we get

$$\begin{aligned} \|Ps\|_{H^1} &\leqslant c(\|D_p P s\|_{L^2} + p|\omega|_0 \|Ps\|_{L^2}) \\ &\leqslant c'\bigg(\|P D_p s\|_{L^2} + p \sum_{k=0}^{m} |\omega|_k \|s\|_{H^{m-k}}\bigg), \end{aligned} \tag{2-18}$$

for some constants $c, c' > 0$. From (2-18), we get by induction for (other) suitable constants $c, c' > 0$

$$\begin{aligned} \|s\|_{H^{m+1}} &\le c \sum_{k=0}^{m+1} p^{m+1-k} \|D_p^k s\|_{L^2} \prod_{\sum(k_r+1)=m-k+1} |\omega|_{k_r} \\ &\le c' \sum_{k=0}^{m+1} p^{m+1-k} \|D_p^k s\|_{L^2} |\omega|_{m-k}^{m-k+1}. \end{aligned} \tag{2-19}$$

Note that for $k = m+1$ we set $|\omega|_{m-k}^{m-k+1} = 1$.

Let Q be a differential operator of order $m' \in \mathbb{N}$ with scalar principal symbol and with compact support in U_j. We deduce from (2-19) with suitable sections instead of s that

$$\begin{aligned} \|Q^* \phi_p(D_p) D^k s\|_{L^2} &\le c \sum_{k'=0}^{m'} p^{m'-k'} \|D_p^{k'} \phi_p(D_p) D_p^k s\|_{L^2} |\omega|_{m'-k'-1}^{m'-k'} \\ &= c \sum_{k'=0}^{m'} p^{m'-k'} \|D_p^{k+k'} \phi_p(D_p) s\|_{L^2} |\omega|_{m'-k'-1}^{m'-k'}. \end{aligned} \tag{2-20}$$

Note that the operators, considered in the last two lines, commute. Thanks to (2-7), (2-11), (2-12) and then (2-14), if $0 < \zeta \le 1$ and $\zeta p \ge \delta$, for any $q \in \mathbb{N}$, the main factor in the last line can be bounded using

$$\begin{aligned}\|D_p^{k+k'}\phi_p(D_p)s\|_{L^2} &\leq (\zeta p)^{-q/2}\|D_p^{k+k'+q}\phi_p(D_p)s\|_{L^2}\\ &\leq c(\zeta p)^{-q/2}\zeta^{-k-k'-q}\|s\|_{L^2}.\end{aligned} \tag{2-21}$$

Take any $l>0$ and choose $q:=2(m+m'-k-k'+l)$ in (2-21). Using the identity

$$\langle D_p^k\phi_p(D_p)Qs,\,s'\rangle=\langle s,\,Q^*\phi_p(D_p)D_p^k s'\rangle,$$

then by (2-19)-(2-21), there exists $c_l>0$ such that for $0<\zeta\leq 1,\ \zeta p\geq\delta$, we have

$$\begin{aligned}&\|P\phi_p(D_p)Qs\|_{L^2}\\ &\quad\leq c\sum_{k=0}^{m}\sum_{k'=0}^{m'}p^{m+m'-k-k'}(\zeta p)^{-q/2}\zeta^{-k-q-k'}|\omega|^{m-k}_{m-k-1}|\omega|^{m'-k'}_{m'-k'-1}\|s\|_{L^2}\\ &\quad\leq c_l\zeta^{-3m-3m'-3l}p^{-l}|\omega|^{m}_{m-1}|\omega|^{m'}_{m'-1}\|s\|_{L^2}.\end{aligned} \tag{2-22}$$

Finally, on $U_i\times U_j$, by using the standard Sobolev's inequality and (2-12), we get (2-10). Proposition 2.1 follows. □

Remark 2.2. By (2-9) and the finite propagation speed of solutions of hyperbolic equations [Ma and Marinescu 2007, Theorem D.2.1], $F_{\epsilon_0}(D_p)(x,x')$ only depends on the restriction of D_p to $B^X(x,\epsilon_0\zeta)$, and

$$F_{\epsilon_0}(D_p)(x,x')=0 \quad \text{when} \quad \mathrm{dist}(x,x')\geqslant\epsilon_0\zeta. \tag{2-23}$$

To get the uniform estimate of the Bergman kernels in terms of ζ, p, we need an approach different from the use of the normal coordinates and the extension of connections on L in [Dai et al. 2006, §4.2] and [Ma and Marinescu 2007, §4.1.3]. Let $\psi: X\supset U\to V\subset\mathbb{C}^n$ be a holomorphic local chart such that $0\in V$ and V is convex (by abuse of notation, we sometimes identify U with V and x with $\psi(x)$). Then, for any $x\in\frac{1}{2}V:=\{y\in\mathbb{C}^n:2y\in V\}$, we will use the holomorphic coordinates induced by ψ and let $0<\epsilon_0\leq 1$ be such that $B(x,4\epsilon_0)\subset V$ for any $x\in\frac12 V$. We choose ϵ_0 smaller than $a_X/4$ in order to use the estimates given in the proof of Proposition 2.1. Consider the holomorphic family of holomorphic local coordinates $\psi_x:\psi^{-1}(B(x,4\epsilon_0))\to B(0,4\epsilon_0)$ for $x\in\frac12 V$ given by $\psi_x(y):=\psi(y)-x$.

Let σ be a holomorphic frame of L on U and define the function $\varphi(Z)$ on U by $|\sigma|^2_\phi(Z)=:e^{-2\varphi(Z)}$. Consider the holomorphic family of holomorphic trivializations of L associated with the coordinates ψ_x and the frame σ. These trivializations are given by $\Psi_x: L|_{\psi^{-1}(B(x,4\epsilon_0))}\to B(0,4\epsilon_0)\times\mathbb{C}$ with $\Psi_x(y,v):=(\psi_x(y),v/\sigma(y))$ for v a vector in the fiber of L over the point y.

Consider a point $x_0\in\frac12 V$. Denote by $\varphi_{x_0}:=\varphi\circ\psi_{x_0}^{-1}$ the function φ in local coordinates ψ_{x_0}. Denote also by $\varphi^{[1]}_{x_0}$ and $\varphi^{[2]}_{x_0}$ the first and second order Taylor expansions of φ_{x_0}, i.e.,

$$\begin{aligned}\varphi^{[1]}_{x_0}(Z) &:= \sum_{j=1}^{n}\Big(\frac{\partial\varphi}{\partial z_j}(x_0)z_j + \frac{\partial\varphi}{\partial \bar z_j}(x_0)\bar z_j\Big),\\ \varphi^{[2]}_{x_0}(Z) &:= \operatorname{Re}\sum_{j,k=1}^{n}\Big(\frac{\partial^2\varphi}{\partial z_j\partial z_k}(x_0)z_jz_k + \frac{\partial^2\varphi}{\partial z_j\partial \bar z_k}(x_0)z_j\bar z_k\Big),\end{aligned} \tag{2-24}$$

where we write $z=(z_1,\dots,z_n)$ the complex coordinates of Z.

Let $\rho:\mathbb{R}\to[0,1]$ be a smooth even function such that

$$\rho(t)=1 \quad \text{if } |t|<2; \qquad \rho(t)=0 \quad \text{if } |t|>4. \tag{2-25}$$

We denote in the sequel $X_0=\mathbb{R}^{2n}\simeq T_{x_0}X$ and equip X_0 with the metric $g^{TX_0}(Z):= g^{TX}(\rho(\epsilon_0^{-1}|Z|)Z)$. Now let $0<\epsilon<\epsilon_0$ and define

$$\varphi_\epsilon(Z) := \rho(\epsilon^{-1}|Z|)\varphi_{x_0}(Z) + \big(1-\rho(\epsilon^{-1}|Z|)\big)\big(\varphi(x_0)+\varphi^{[1]}_{x_0}(Z)+\varphi^{[2]}_{x_0}(Z)\big). \tag{2-26}$$

Let $h^{L_0}_\epsilon$ be the metric on $L_0=X_0\times\mathbb{C}$ defined by

$$|1|^2_{h^{L_0}_\epsilon}(Z) := e^{-2\varphi_\epsilon(Z)}. \tag{2-27}$$

Here, as above, subscript ϵ implies the use of the weight φ_ϵ. Let $\nabla^{L_0}_\epsilon$ be the Chern connection on $(L_0,h^{L_0}_\epsilon)$ and $R^{L_0}_\epsilon$ be the curvature of $\nabla^{L_0}_\epsilon$.

Then there exists a constant A with $c|d\phi|_2^{-1}<A<1$ for $c>0$ such that when $\epsilon\le A\zeta$, the following estimate holds for every $x_0\in U$:

$$\inf\big\{\sqrt{-1}R^{L_0}_{\epsilon,Z}(u,Ju)/|u|^2_{g^{TX_0}} : u\in T_ZX_0 \quad\text{and}\quad Z\in X_0\big\} \geqslant \tfrac{4}{5}\zeta; \tag{2-28}$$

because there exists $C>0$ such that for $|Z|\le 4\epsilon,\ 0\le j\le 2$, we have

$$\big|\varphi_{x_0}(Z) - \big(\varphi(x_0)+\varphi^{[1]}_{x_0}(Z)+\varphi^{[2]}_{x_0}(Z)\big)\big|_{C^j} \le C|d\phi|_2|Z|^{3-j}. \tag{2-29}$$

From now on, we take

$$\epsilon := \epsilon_0 A\zeta. \tag{2-30}$$

Let S_{x_0} be the unitary section of $(L_0,h^{L_0}_\epsilon)$ which is parallel with respect to $\nabla^{L_0}_\epsilon$ along the curve $[0,1]\ni u\to uZ$ for any $Z\in X_0$. We can write it as $S_{x_0}=e^{-\tau}1$ with $\tau(x_0)=\varphi(x_0)$, then

$$\nabla^{L_0}_Z S_{x_0} = i_Z(-d\tau - 2\partial\varphi_\epsilon)S_{x_0} = 0, \tag{2-31}$$

and hence the function τ is given by

$$\tau(Z) = \varphi(x_0) - 2\int_0^1 (i_Z\partial\varphi_\epsilon)_{tZ}\,dt. \tag{2-32}$$

Let

$$D_p^{X_0} = \sqrt{2}(\bar{\partial}^{L_0^p} + \bar{\partial}^{L_0^p *}_{p\varphi_\epsilon}) \tag{2-33}$$

be the Dolbeault operator on X_0 associated with the above data, i.e., $\bar{\partial}^{L_0^p *}_{p\varphi_\epsilon}$ is the adjoint of $\bar{\partial}^{L_0^p}$ with respect to the metrics g^{TX_0} and $h_\epsilon^{L_0}$. Over the ball $B(x_0, 2\epsilon)$, D_p is just the restriction of $D_p^{X_0}$. Now by [Ma and Marinescu 2007, Theorem 1.4.7], and the observation that the tensors associated with g^{TX_0} do not depend on ζ and ϵ, as in (2-7), we get from (2-28) the existence of a constant $\delta > 0$ such that for $\zeta p > \delta$,

$$\operatorname{Spec}(D_p^{X_0})^2 \subset \{0\} \cup [\zeta p, +\infty[. \tag{2-34}$$

Using S_{x_0}, we get an isometry $L_0^p \simeq \mathbb{C}$. Let P_p^0 be the orthogonal projection from $\mathcal{C}^\infty(X_0, L_0^p) \simeq \mathcal{C}^\infty(X_0, \mathbb{C})$ on $\operatorname{Ker}(D_p^{X_0})$. Let $P_p^0(x, x')$ be the smooth kernel of P_p^0 with respect to the volume form $dv_{X_0}(x')$ induced by the metric g^{TX_0}. We have the following result:

Proposition 2.3. *For all $l \in \mathbb{N}$, there exists $c > 0$ such that for $\zeta p > \delta$, $x, x' \in B(x_0, \epsilon)$,*

$$\|(P_p^0 - P_p)(x, x')\|_{\mathcal{C}^0} \le c(|d\phi|_2^{-1}\zeta)^{-6n-3l-6} p^{-l} |\omega|_n^{2n+2}. \tag{2-35}$$

Proof. First, we replace $f_{\epsilon_0}(v)$ in (2-8) by $f_{\epsilon_0}(v/A)$. By Remark 2.2 and (2-30), for $x, x' \in B(x_0, \epsilon)$, we have $F_\epsilon(D_p)(x, x') = F_\epsilon(D_p^0)(x, x')$. Now we have a version of Proposition 2.1 for P_p^0 with $A\zeta$ instead of ζ. Estimate (2-35) follows. □

3. Uniform estimate of the Bergman kernels

We continue to use the notations introduced at the end of the last section. By Proposition 2.3, in order to study the kernel P_p, it suffices to study the kernel P_p^0. For this purpose, we will rescale the operator $(D_p^{X_0})^2$. Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z), \tag{3-1}$$

with $\kappa(0) = 1$.

Let $\{e_j\}_{j=1}^{2n}$ be an oriented orthonormal basis of $T_{x_0}X$, and let $\{e^j\}_{j=1}^{2n}$ be its dual basis. They allow us to identify $X_0 = \mathbb{C}^n$ with $\mathbb{R}^{2n}$ and we write $Z = (Z_1, \dots, Z_{2n})$. If $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is a multi-index, set $Z^\alpha := Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U, and set $\partial_j := \nabla_{e_j}$. Set

$t := p^{-1/2}$. For $s \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})$ and $Z \in \mathbb{R}^{2n}$, define

$$\begin{aligned} (S_t s)(Z) &:= s(Z/t), \qquad \nabla_t := t S_t^{-1} \kappa^{1/2} \nabla^{L_0^p} \kappa^{-1/2} S_t, \\ \mathscr{L}_t &:= S_t^{-1} t^2 \kappa^{1/2} (D_p^{X_0})^2 \kappa^{-1/2} S_t. \end{aligned} \tag{3-2}$$

Once we have done the trivialization of L_0 on X_0, (3-2) is well defined for any $p \in \mathbb{R},\ p \geq 1$.

The notations $\langle \cdot, \cdot \rangle_0$ and $\|\cdot\|_0$ mean respectively the inner product and the L^2-norm on $\mathcal{C}^\infty(X_0, \mathbb{C})$ induced by g^{TX_0}. For $s \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$, set

$$\begin{aligned} \|s\|_{t,0}^2 &:= \|s\|_0^2 = \int_{\mathbb{R}^{2n}} |s(Z)|^2 \, dv_{TX}(Z), \\ \|s\|_{t,m}^2 &:= \sum_{l=0}^{m} \sum_{j_1,\cdots,j_l=1}^{2n} \|\nabla_{t,e_{j_1}} \cdots \nabla_{t,e_{j_l}} s\|_{t,0}^2. \end{aligned} \tag{3-3}$$

We then, for convenience, denote by $\langle s, s' \rangle_{t,0}$ the inner product on $\mathcal{C}^\infty(X_0, L_{x_0}^{\otimes p})$ corresponding to the norm $\|\cdot\|_{t,0}$. Let H_t^m be the Sobolev space of order m with norm $\|\cdot\|_{t,m}$. Let H_t^{-1} be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on H_t^{-1} defined by $\|s\|_{t,-1} := \sup_{0 \neq s' \in H_t^1} |\langle s, s' \rangle_{t,0}| / \|s'\|_{t,1}$. If $B : H_t^m \to H_t^{m'}$ is a bounded linear operator for $m, m' \in \mathbb{Z}$, denote by $\|B\|_t^{m,m'}$ the norm of B with respect to the norms $\|\cdot\|_{t,m}$ and $\|\cdot\|_{t,m'}$.

Theorems 3.1, 3.2, 3.4 and Proposition 3.3 below are the analogues of [Ma and Marinescu 2007, Theorem 4.1.9–4.1.14] (cf., also [Dai et al. 2006, Theorem 4.7–4.10]). The emphasis here is the precise dependence of the involved constants on the curvature form ω.

Theorem 3.1. *There exist $c_1, c_2, c_3 > 0$ such that for $t \in]0, 1],\ \zeta \in]0, 1]$, and $s, s' \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbb{C})$,*

$$\begin{aligned} \langle \mathscr{L}_t s, s \rangle_{t,0} &\geqslant c_1 \|s\|_{t,1}^2 - c_2 |\omega|_0 \|s\|_{t,0}^2, \\ |\langle \mathscr{L}_t s, s' \rangle_{t,0}| &\leqslant c_3 |\omega|_0 \|s\|_{t,1} \|s'\|_{t,1}. \end{aligned} \tag{3-4}$$

Proof. By using the Lichnerowicz formula [Ma and Marinescu 2007, (4.1.33)], the same arguments as in (4.1.38)–(4.1.39) of the same work give the result. □

Let δ_ζ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\zeta/2$.

Theorem 3.2. *There exists $\delta > 0$ such that the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for all $\lambda \in \delta_\zeta$ and $t \in]0, \sqrt{\zeta/\delta}]$. There exists $c > 0$ such that for all $t \in]0, \sqrt{\zeta/\delta}],\ \lambda \in \delta_\zeta$, we have*

$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{0,0} \leqslant 2\zeta^{-1}, \quad \|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant c|\omega|_0^2 \zeta^{-1}. \tag{3-5}$$

Proof. By (2-34) and (3-2), we have

$$\mathrm{Spec}(\mathscr{L}_t) \subset \{0\} \cup [\zeta, +\infty[. \tag{3-6}$$

Thus, the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \delta_\zeta$ and $t \in]0, \sqrt{\zeta/\delta}]$, and we get the first inequality of (3-5).

By (3-4), $(\lambda_0 - \mathscr{L}_t)^{-1}$ exists for $\lambda_0 \in \mathbb{R},\ \lambda_0 \leqslant -2c_2|\omega|_0$. Moreover, as

$$c_1\|s\|_{t,1}^2 \leq -\langle(\lambda_0 - \mathscr{L}_t)s, s\rangle_{t,0} \leq \|(\lambda_0 - \mathscr{L}_t)s\|_{t,-1}\|s\|_{t,1},$$

we have

$$\|(\lambda_0 - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{c_1}\cdot \tag{3-7}$$

On the other hand, we have

$$(\lambda - \mathscr{L}_t)^{-1} = (\lambda_0 - \mathscr{L}_t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_t)^{-1}(\lambda_0 - \mathscr{L}_t)^{-1}. \tag{3-8}$$

Therefore, for $\lambda \in \delta_\zeta$, from the first estimate in (3-5) and (3-8), we get

$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,0} \leq \frac{1}{c_1}(1 + 2|\lambda - \lambda_0|\zeta^{-1}). \tag{3-9}$$

In (3-8), we can interchange the last two factors. Then, applying (3-7) and (3-9) gives

$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{c_1} + \frac{|\lambda - \lambda_0|}{c_1{}^2}(1 + 2|\lambda - \lambda_0|\zeta^{-1}) \leqslant c|\omega|_0^2\zeta^{-1}. \tag{3-10}$$

The theorem follows. □

Proposition 3.3. *Take $m \in \mathbb{N}^*$. There is a $c > 0$ such that for $t \in]0, 1]$, $Q_1, \dots, Q_m \in \{\nabla_{t,e_j}, Z_j\}_{j=1}^{2n}$ and $s, s' \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$,*

$$\Big|\big\langle[Q_1, [Q_2, \dots [Q_m, \mathscr{L}_t] \dots]]s, s'\big\rangle_{t,0}\Big| \leqslant c|d\phi|_{m+1}^{\min(2,m)}\|s\|_{t,1}\|s'\|_{t,1}. \tag{3-11}$$

Proof. By [Ma and Marinescu 2007, (1.6.31)] and as in the proof of Proposition 1.6.9 of the same work, we know that $[Q_1, [Q_2, \dots [Q_m, \mathscr{L}_t] \dots]]$ has the same structure as $\mathscr{L}_t$ for $t \in [0, 1]$. More precisely, it has the form

$$\sum_{i,j} a_{ij}(t, tZ)\nabla_{t,e_i}\nabla_{t,e_j} + \sum_j d_j(t, tZ)\nabla_{t,e_j} + c(t, tZ), \tag{3-12}$$

where $a_{ij}(t, Z)$ and its derivatives in Z are uniformly bounded, $d_j(t, Z)$, $c(t, Z)$ and their first derivatives in Z are bounded by $c|d\phi|_{m+1}^{\min(2,m)}$ for $Z \in \mathbb{R}^{2n}$ and $t \in [0, 1]$ and a constant $c > 0$. We then get estimate (3-11). □

Theorem 3.4. *For $Q_1, \dots, Q_m \in \{\nabla_{t,e_j}, Z_j\}_{j=1}^{2n}$, there exists $c > 0$ such that we have for $t \in]0, \sqrt{\zeta/\delta}]$, $\lambda \in \delta_\zeta$ and $s \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$,*

$$\|Q_1 \cdots Q_m(\lambda - \mathscr{L}_t)^{-1} s\|_{t,1} \leqslant c \sum_{k=0}^{m} \sum_{1\leq j_1<\cdots<j_k\leq m} |d\phi|_{m-k+1}^{m-k} (|\omega|_0^2 \zeta^{-1})^{m-k+1} \|Q_{j_1} \cdots Q_{j_k} s\|_{t,0}. \tag{3-13}$$

Proof. For $Q_1, \dots, Q_m \in \{\nabla_{t,e_j}, Z_j\}_{j=1}^{2n}$, we can express $Q_1 \cdots\ Q_m(\lambda - \mathscr{L}_t)^{-1}$ as the sum of $(\lambda - \mathscr{L}_t)^{-1} Q_1 \cdots\ Q_m$ with a linear combination of operators of the type

$$[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_{m_1}}, (\lambda - \mathscr{L}_t)^{-1}] \dots]] Q_{j_{m_1+1}} \cdots Q_{j_m}, \tag{3-14}$$

with $j_1 < j_2 \cdots < j_{m_1},\ j_{m_1+1} < \cdots < j_m$. The coefficients of this combination are bounded when m is bounded. Let $\mathcal{S}_t$ be the family of operators

$$[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \mathscr{L}_t] \dots]] = -[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \lambda - \mathscr{L}_t] \dots]].$$

Note that

$$[Q,(\lambda-\mathscr{L}_t)^{-1}] = -(\lambda-\mathscr{L}_t)^{-1}[Q,\lambda-\mathscr{L}_t](\lambda-\mathscr{L}_t)^{-1} = (\lambda-\mathscr{L}_t)^{-1}[Q,\mathscr{L}_t](\lambda-\mathscr{L}_t)^{-1},$$

thus by the recurrence on m_1 we know that every commutator

$$[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_{m_1}}, (\lambda - \mathscr{L}_t)^{-1}] \dots]]$$

is a linear combination of operators of the form

$$(\lambda - \mathscr{L}_t)^{-1} S_1 (\lambda - \mathscr{L}_t)^{-1} S_2 \cdots S_{m_2} (\lambda - \mathscr{L}_t)^{-1} \tag{3-15}$$

with $S_1, \dots, S_{m_2} \in \mathcal{S}_t$ and $m_2 \leq m_1$. The coefficients of this combination are bounded when m_1 is bounded.

From Proposition 3.3 we deduce that the $\|\cdot\|_t^{1,-1}$ norms of the operators $[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \mathscr{L}_t] \dots]]$ are uniformly bounded from above by a constant times $|d\phi|_{l+1}^{l}$. Hence, by Theorem 3.2, the $\|\cdot\|_t^{0,1}$ norm of the operator (3-15) is bounded by a constant times

$$\zeta^{-m_2-1} |\omega|_0^{2m_2+2} \sum_{\substack{l_1+\cdots+l_{m_2}=m_1 \\ l_1,\dots,l_{m_2}\geq 1}} \prod_{j=1}^{m_2} |d\phi|_{l_j+1}^{l_j}.$$

The theorem follows. ◻

Let $\mathcal{P}_t : (\mathcal{C}^\infty(X_0, \mathbb{C}), \|\cdot\|_0) \to \mathrm{Ker}(\mathscr{L}_t)$ be the orthogonal projection corresponding to the norm $\|\cdot\|_{t,0}$ given in (3-3). Let $\mathcal{P}_t(Z, Z')$, (with $Z, Z' \in X_0$) be the smooth kernel of $\mathcal{P}_t$ with respect to $dv_{TX}(Z')$. Note that $\mathscr{L}_t$ is a family of differential operators on $T_{x_0}X$ with coefficients in $\mathbb{C}$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX with itself on X. We can view $\mathcal{P}_t(Z, Z')$ as smooth functions over $TX \times_X TX$ by identifying a section $F \in \mathcal{C}^\infty(TX \times_X TX, \mathbb{C})$ with the family $(F_{x_0})_{x_0 \in X}$, where $F_{x_0} := F|_{\pi^{-1}(x_0)}$. In the following result we adapt [Ma and Marinescu 2007, Theorem 4.1.24] to the present situation.

Theorem 3.5. *For any $r \in \mathbb{N}$, $\sigma > 0$, there exists $c > 0$, such that for $t \in]0, \sqrt{\zeta/\delta}]$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leqslant \sigma$,*

$$\left\| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z, Z') \right\|_{\mathcal{C}^0(X)} \leqslant c\zeta^{-2n-4r-2} |d\phi|^{4r+2n}_{2r+n+1} |\omega|_0^{8r+4n+4} |d\phi|^{2n+2}_{n+2}. \tag{3-16}$$

Proof. By (3-6), for every $k \in \mathbb{N}^*$,

$$\mathcal{P}_t = \frac{1}{2\pi\sqrt{-1}} \int_{\delta_\zeta} \lambda^{k-1} (\lambda - \mathscr{L}_t)^{-k} d\lambda. \tag{3-17}$$

For $m \in \mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\nabla_{t,e_{i_1}} \cdots \nabla_{t,e_{i_j}}$ with $j \leqslant m$. We apply Theorem 3.4 to $m-1$ operators $Q_2, \ldots, Q_m$ instead of m operators. We deduce that for $l, m \in \mathbb{N}^*$ with $l \geq m$, and $Q = Q_1 \cdots Q_m \in \mathcal{Q}^m$, there are $c, c' > 0$ such that for $t \in]0, \sqrt{\zeta/\delta}]$, $\zeta \in [0, 1]$, $s \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$ and $\lambda \in \delta_\zeta$

$$\begin{aligned} &\|Q_1 \cdots Q_m (\lambda - \mathscr{L}_t)^{-l} s\|_{t,0} \\ &\leqslant c \|Q_2 \cdots Q_m (\lambda - \mathscr{L}_t)^{-l} s\|_{t,1} \\ &\leqslant c' \sum_{k=0}^{m-1} \sum_{1 < i_1 < \cdots < i_k \leq m} |d\phi|^{m-k-1}_{m-k} (|\omega|_0^2 \zeta^{-1})^{m-k} \|Q_{i_1} \cdots Q_{i_k} (\lambda - \mathscr{L}_t)^{-l+1} s\|_{t,0}. \end{aligned} \tag{3-18}$$

Then, by induction and using (3-5), we get

$$\|Q_1 \cdots Q_m (\lambda - \mathscr{L}_t)^{-l} s\|_{t,0} \leqslant c\zeta^{-m-l+1} |d\phi|^{m-1}_m |\omega|_0^{2m} \|s\|_{t,0}. \tag{3-19}$$

As $\mathscr{L}_t$ is symmetric, we can consider the adjoint of the operator in (3-19) and get for $Q' = Q'_1 \cdots Q'_{m'} \in \mathcal{Q}^{m'}$,

$$\|(\lambda - \mathscr{L}_t)^{-l} Q'_1 \cdots Q'_{m'} s\|_{t,0} \leqslant c\zeta^{-m'-l+1} |d\phi|^{m'-1}_{m'} |\omega|_0^{2m'} \|s\|_{t,0}. \tag{3-20}$$

Note that for $m = 0$ and $l \in \mathbb{N}$ we also have $\|(\lambda - \mathscr{L}_t)^{-l} s\|_{t,0} \leq c\zeta^{-l} \|s\|_{t,0}$. Thus, for $Q \in \mathcal{Q}^m$, $Q' \in \mathcal{Q}^{m'}$ with $m, m' > 0$, by taking $k = m + m'$, we get

$$\begin{aligned} \|Q\mathcal{P}_t Q'\|_t^{0,0} &\leqslant \frac{1}{2\pi} \int_{\delta_\zeta} |\lambda|^{m+m'-1} \|Q(\lambda - \mathscr{L}_t)^{-m-m'} Q'\|_t^{0,0} |d\lambda| \\ &\leqslant c |d\phi|^{m-1}_m |\omega|_0^{2m} |d\phi|^{m'-1}_{m'} |\omega|_0^{2m'} \zeta^{-2m-2m'+2} \zeta^{m+m'} \\ &= c |d\phi|^{m-1}_m |\omega|_0^{2m} |d\phi|^{m'-1}_{m'} |\omega|_0^{2m'} \zeta^{-m-m'+2}. \end{aligned} \tag{3-21}$$

By [Ma and Marinescu 2007, Lemma 1.2.4], (2-31), (2-32) and (3-2), on $B^{T_{x_0}X}(0, \epsilon/t)$,

$$\nabla_{t,e_i}|_Z = \nabla_{e_i} + \tfrac{1}{2} R^L_{x_0}(Z, e_i) + O(t|Z|^2)|d\phi|_2. \tag{3-22}$$

Let $|\cdot|_{(\sigma),m}$ denote the usual Sobolev norm on $\mathcal{C}^\infty(B^{T_{x_0}X}(0, \sigma+1), \mathbb{C})$ induced by the volume form $dv_{TX}(Z)$ as in (3-3). Observe that by (3-3), (3-22), for $m > 0$, there exists $c > 0$ such that for $s \in \mathcal{C}^\infty(X_0, \mathbb{C})$ with $\operatorname{supp}(s) \subset B(0, \sigma+1)$,

$$\frac{1}{c|d\phi|^m_{m+1}} \|s\|_{t,m} \leqslant |s|_{(\sigma),m} \leqslant c|d\phi|^m_{m+1} \|s\|_{t,m}. \tag{3-23}$$

Now, we want to estimate $Q_Z Q'_{Z'} \mathcal{P}_t(Z, Z')$ using the standard Sobolev's inequality for $Q \in \mathcal{Q}^m$ and $Q' \in \mathcal{Q}^{m'}$. If we define $S := Q\mathcal{P}_t Q'$ then we have for $|Z|, |Z'| \le \sigma$

$$|Q_Z Q'_{Z'} \mathcal{P}_t(Z, Z')| \le c \sup\Big\{ \Big\| \frac{\partial^{|\alpha|}}{\partial Z^\alpha} S \frac{\partial^{|\alpha'|} s}{\partial Z'^{\alpha'}} \Big\|_{(\sigma),n+1}, \|s\|_{L^2} = 1, \operatorname{supp}(s) \subset B(0, \sigma+1), |\alpha|, |\alpha'| \le n+1 \Big\}. \tag{3-24}$$

Hence, by (3-23), applied twice to $n+1$ instead of m, and also (3-21), applied to $m+n+1, m'+n+1$ instead of m, m', we get

$$\begin{aligned} \sup_{|Z|,|Z'| \leqslant \sigma} &|Q_Z Q'_{Z'} \mathcal{P}_t(Z, Z')| \\ &\leqslant c' |d\phi|^{m+n}_{m+n+1} |\omega|_0^{2m+2n+2} |d\phi|^{m'+n}_{m'+n+1} |\omega|_0^{2m'+2n+2} |d\phi|^{2n+2}_{n+2} \zeta^{-m-m'-2n}. \end{aligned} \tag{3-25}$$

By (3-22) and (3-25) for $m = m' = 0$, estimate (3-16) holds for $r = 0$.

Consider now $r \ge 1$. Set

$$I_{k,r} := \Big\{ (\boldsymbol{k}, \boldsymbol{r}) = \{(k_i, r_i)\}_{i=0}^j : \sum_{i=0}^j k_i = k + j, \quad \sum_{i=1}^j r_i = r, \quad k_i, r_i \in \mathbb{N}^* \Big\}. \tag{3-26}$$

Then there exist $a^{\boldsymbol{k}}_{\boldsymbol{r}} \in \mathbb{R}$ such that

$$\begin{aligned} A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t) &= (\lambda - \mathscr{L}_t)^{-k_0} \frac{\partial^{r_1} \mathscr{L}_t}{\partial t^{r_1}} (\lambda - \mathscr{L}_t)^{-k_1} \cdots \frac{\partial^{r_j} \mathscr{L}_t}{\partial t^{r_j}} (\lambda - \mathscr{L}_t)^{-k_j}, \\ \frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_t)^{-k} &= \sum_{(\boldsymbol{k},\boldsymbol{r}) \in I_{k,r}} a^{\boldsymbol{k}}_{\boldsymbol{r}} A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t). \end{aligned} \tag{3-27}$$

Set $g_{ij}(Z) := \langle \partial/\partial Z_i, \partial/\partial Z_j \rangle_Z$, and (g^{ij}) the inverse matrix of (g_{ij}). Note that $(\partial^u/\partial t^u)(g^{ij}(tZ))$, $(\partial^u/\partial t^u)(\nabla_{t,e_i} - (1/t)\Gamma^{L_0}_\epsilon(tZ))$ are functions which do not depend on ζ, and $(\partial^u/\partial t^u) R^{L_0}_\epsilon(tZ)$, $(\partial^u/\partial t^u)\Gamma^{L_0}_\epsilon(tZ)$ are functions of type $d'(tZ)Z^\beta$, and $\nabla_{e_{j_1}} \cdots \nabla_{e_{j_l}} d'(tZ)$ is uniformly controlled by $|d\phi|_{l+u+1}$.

We handle now the operator $A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)Q'$. We will move first all the terms Z^β in $d'(tZ)Z^\beta$ (defined above) to the right-hand side of this operator. To do so, we always use the commutator trick as in the proof of [Ma and Marinescu 2007, Theorem 1.6.10], i.e., each time, we perform only the commutation with Z_i (not directly with Z^β with $|\beta| > 1$). Then $A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)Q'$ is as the form $\sum_{|\beta|\leqslant 2r} \mathscr{L}_{\beta,t} Q''_\beta Z^\beta$, and Q''_β is obtained from Q' and its commutation with Z^β. Observe that $[Z_i, \mathscr{L}_t]$ is a first order differential operator and $[Z_{j_1}, [Z_{j_2}, \mathscr{L}_t]] = g^{j_1 j_2}(tZ)$ is a bounded function. Therefore, $\mathscr{L}_{\beta,t}$ is a linear combination of operators of the form

$$(\lambda - \mathscr{L}_t)^{-k'_0} S_1 (\lambda - \mathscr{L}_t)^{-k'_1} S_2 \cdots S_{l'} (\lambda - \mathscr{L}_t)^{-k'_{l'}}, \tag{3-28}$$

with $S_i \in \{a(tZ)\nabla_{t,e_{j_1}}\nabla_{t,e_{j_2}}, d_{j_1}(tZ)\nabla_{t,e_{j_1}}, d'(tZ)\}$ and the number of $\nabla_{t,e_{j_1}}$ in all $\{S_i\}_i$ is less than $\sum_i r_i + 2j = r + 2j$. As $k > 2(r+1) + m + m'$, we can split the above operator into two parts as in [Ma and Marinescu 2007, (4.1.51)] and use the fact that the term $\nabla_{t,e_j}(\lambda - \mathscr{L}_t)^{-l_1}$ will contribute ζ^{-l_1}. Similarly to (3-18), we get that $A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)$ is well defined and for $m, m' \in \mathbb{N}$, $k > 2(r+1) + m + m'$, $Q \in \mathcal{Q}^m$, $Q' \in \mathcal{Q}^{m'}$, there exists $c > 0$ such that for $\lambda \in \delta_\zeta$ and $t \in \,]0, \sqrt{\zeta/\delta}]$,

$$\begin{aligned}\|QA^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)Q's\|_{t,0} &\leqslant c|d\phi|^{m+2r-1}_{m+2r}|\omega|^{2m+4r}_0 |d\phi|^{m'+2r-1}_{m'+2r}|\omega|^{2m'+4r}_0 \zeta^{-\sum_{i=0}^{j}k_i - m - m' - 3r} \sum_{|\beta|\leqslant 2r} \|Z^\beta s\|_{t,0} \\ &\leq c|d\phi|^{m+2r-1}_{m+2r}|\omega|^{2m+4r}_0 |d\phi|^{m'+2r-1}_{m'+2r}|\omega|^{2m'+4r}_0 \zeta^{-k-m-m'-4r} \sum_{|\beta|\leqslant 2r} \|Z^\beta s\|_{t,0}.\end{aligned} \tag{3-29}$$

By (3-17), (3-27) and (3-29), as in (3-21), for $m, r \in \mathbb{N}$, $Q \in \mathcal{Q}^m$ and $Q' \in \mathcal{Q}^{m'}$, there exists $c > 0$ such that for $t \in \,]0, \sqrt{\zeta/\delta}]$ and $s \in \mathscr{C}^\infty_0(X_0, \mathbb{C})$,

$$\begin{aligned}\Big\| Q \frac{\partial^r}{\partial t^r} \mathcal{P}_t Q' s \Big\|_{t,0} &\leqslant c|d\phi|^{m+2r-1}_{m+2r} |d\phi|^{m'+2r-1}_{m'+2r} |\omega|^{2m+2m'+8r}_0 \zeta^{-m-m'-4r} \sum_{|\beta|\leqslant 2r} \|Z^\beta s\|_{t,0}.\end{aligned} \tag{3-30}$$

Finally, equations (3-23) and (3-30) together with Sobolev's inequalities imply for $|Z|, |Z'| \leqslant \sigma$,

$$\sup_{|Z|,|Z'|\leqslant\sigma} \Big| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z, Z') \Big| \leqslant c|d\phi|^{2n+4r}_{2r+n+1} |\omega|^{2(2n+2+4r)}_0 |d\phi|^{2n+2}_{n+2} \zeta^{-2n-4r-2}. \tag{3-31}$$

This ends the proof of the theorem. □

Note that by (3-22), the operator $\mathscr{L}_t$ has a limit when $t \to 0$ which we denote by $\mathscr{L}_0$. For k big enough, set

$$F_r := \frac{1}{2\pi\sqrt{-1}\, r!} \int_{\delta_\zeta} \lambda^{k-1} \sum_{(\boldsymbol{k},\boldsymbol{r}) \in I_{k,r}} a_{\boldsymbol{r}}^{\boldsymbol{k}} A_{\boldsymbol{r}}^{\boldsymbol{k}}(\lambda, 0) d\lambda. \tag{3-32}$$

Let $F_r(Z, Z') \in \mathcal{C}^\infty(TX \times_X TX, \mathbb{C})$ be the smooth kernel of F_r with respect to $dv_{TX}(Z')$.

Theorem 3.6. *For all $j \in \mathbb{N}$, $\sigma > 0$, there exists $c > 0$ such that for $t \in]0, \sqrt{\zeta/\delta}]$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$, we have*

$$\begin{aligned} \Big\| \Big(\mathcal{P}_t - \sum_{r=0}^{j} F_r t^r \Big)(Z, Z') \Big\|_{\mathcal{C}^0(X)} \\ \leqslant c |d\phi|_{2j+n+3}^{2(2j+n+2)} |\omega|_0^{2(4j+2n+6)} |d\phi|_{n+2}^{2n+2} \zeta^{-4j-2n-6} t^{j+1}. \end{aligned} \tag{3-33}$$

Proof. By [Ma and Marinescu 2007, (4.1.69)], we have

$$\frac{1}{r!} \frac{\partial^r}{\partial t^r} \mathcal{P}_t \Big|_{t=0} = F_r. \tag{3-34}$$

Recall that the Taylor expansion with integral rest of a function $G \in \mathcal{C}^{j+1}([0, 1])$ is

$$G(t) - \sum_{r=0}^{j} \frac{1}{r!} \frac{\partial^r G}{\partial t^r}(0) t^r = \frac{1}{j!} \int_0^t (t - t_0)^j \frac{\partial^{j+1} G}{\partial t^{j+1}}(t_0) dt_0, \quad t \in [0, 1]. \tag{3-35}$$

Theorem 3.5 and (3-34) show estimate (3-16) holds if we replace $(1/r!)(\partial^r/\partial t^r)\mathcal{P}_t$ with F_r. Using this new estimate together with (3-35) and (3-16), we get (3-33). □

Let $\mathcal{P}$ be the orthogonal projection from $L^2(X_0, \mathbb{C})$ onto $\mathrm{Ker}(\mathscr{L}_0)$, and let $\mathcal{P}(Z, Z')$ be the smooth kernel of $\mathcal{P}$ with respect to $dv_{TX}(Z')$. Then $\mathcal{P}(Z, Z')$ is the Bergman kernel of $\mathscr{L}_0$. By [Ma and Marinescu 2007, (4.1.84)], if we choose $\{w_j\}$ to be an orthonormal basis of $T_{x_0}^{(1,0)}X$ such that $\dot{R}_{x_0}^L = \mathrm{diag}(a_1, \dots, a_n) \in \mathrm{End}(T_{x_0}^{(1,0)}X)$ with $\langle \dot{R}_{x_0}^L W, \overline{Y} \rangle = R^L(W, \overline{Y})$ for $W, Y \in T_{x_0}^{(1,0)}X$, then

$$\mathcal{P}(Z, Z') = \prod_{i=1}^{n} \frac{a_i}{2\pi} \exp\Big(-\frac{1}{4} \sum_i a_i \big(|z_i|^2 + |z_i'|^2 - 2 z_i \bar{z}_i' \big) \Big). \tag{3-36}$$

The following result was established in [Ma and Marinescu 2007, Theorem 4.1.21]:

Theorem 3.7. *There exist polynomials $J_r(Z, Z')$ in Z, Z' with the same parity as r and* $\deg J_r(Z, Z') \leqslant 3r$, *whose coefficients are polynomials in R^{TX} (resp. R^L) and their derivatives of order $\leqslant r - 2$ (resp. $\leqslant r$), and reciprocals of linear combinations of eigenvalues of R^L at x_0, such that*

$$F_r(Z, Z') = J_r(Z, Z') \mathcal{P}(Z, Z'). \tag{3-37}$$

Moreover, we have

$$J_0 = 1 \quad \textit{and} \quad F_0 = \mathcal{P}. \tag{3-38}$$

Owing to (3-1), (3-2), as in [Ma and Marinescu 2007, (4.1.96)], we have

$$P_p^0(Z,Z') = t^{-2n}\kappa^{-1/2}(Z)\mathcal{P}_t(Z/t,Z'/t)\kappa^{-1/2}(Z'), \quad \text{for all } Z,Z' \in \mathbb{R}^{2n}. \tag{3-39}$$

From Theorems 3.6 and 3.7 and (3-39), we get the following near-diagonal expansion of the Bergman kernels. Recall that we are working with $t = p^{-1/2}$.

Theorem 3.8. *For every $j \in \mathbb{N}$, there exists $c > 0$ such that the estimate*

$$\begin{aligned} \left|\left(\frac{1}{p^n}P_p^0(Z,Z') - \sum_{r=0}^{j} F_r(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-1/2}(Z)\kappa^{-1/2}(Z')p^{-r/2}\right)\right| \\ \leqslant c|d\phi|_{2j+n+3}^{2(2j+n+2)}|\omega|_0^{2(2n+4j+6)}|d\phi|_{n+2}^{2n+2}p^{-(j+1)/2}\zeta^{-2n-4j-6} \end{aligned} \tag{3-40}$$

holds for all $0 < \zeta \leq 1$, $\zeta p > \delta$, and all $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leq \sigma/\sqrt{p}$.

End of the proof of Theorem 1.1. We apply Theorem 3.8 to $Z = Z' = 0$ and $j = 1$. Note that $F_1(0,0) = 0$ because the function F_1 is odd. By equation (3-36), $\mathcal{P}(0,0) = \omega(x_0)^n/\theta(x_0)^n$. So from (3-40), we get

$$\left\|\frac{1}{p^n}P_p^0(0,0) - \frac{\omega(x_0)^n}{\theta(x_0)^n}\right\|_{\mathcal{C}^0(X)} \leqslant c|d\phi|_{n+5}^{2n+8}|\omega|_0^{4n+20}|d\phi|_{n+2}^{2n+2}\zeta^{-2n-10}p^{-1}. \tag{3-41}$$

We then deduce the result form Propositions 2.1, 2.3 and (3-41). □

Remark 3.9. Assume now $\phi \in \mathcal{C}^{n+2k+6}$. Then by the usual $\mathcal{C}^k$-norm on each U_j and Sobolev embedding theorem, from (2-22), we get

$$\|F_{\epsilon_0}(D_p)(x,x') - P_p(x,x')\|_{\mathcal{C}^k} \leq c|\omega|_{n+k}^{2n+2+2k}\zeta^{-6n-3l-6-3k}p^{-l}. \tag{3-42}$$

Note that $\nabla^{L^p} = d + p\Gamma^L$ (cf., (2-15)), thus if we use the $\mathcal{C}^k$-norm induced by ∇^{L^p}, then we get

$$\begin{aligned} \|F_{\epsilon_0}(D_p)(x,x') - P_p(x,x')\|_{\mathcal{C}^k(X\times X)} \\ \leq c\sum_{r=0}^{k}|\omega|_{n+r}^{2n+2+2r}\zeta^{-6n-3(k-r+1)-6-3k}p^{-k+r-1}|\omega|_{k-r}^{k-r}p^{k-r} \\ \leq c|\omega|_{n+k}^{2n+2+2k}\zeta^{-6n-9-3k}p^{-1}. \end{aligned} \tag{3-43}$$

In the same way as (2-35) and above, we get

$$\|(P_p^0 - P_p)(x,x')\|_{\mathcal{C}^k(X\times X)} \leq c(|d\phi|_2^{-1}\zeta)^{-6n-3k-9}p^{-1}|\omega|_{n+k}^{2n+2+2k}. \tag{3-44}$$

Combining [Ma and Marinescu 2007, (4.1.64)] and the argument for (3-16), we get

$$\left\| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z,Z') \right\|_{\mathcal{C}^{m'}} \leqslant c\zeta^{-2n-4(r+m')-2} |d\phi|^{4(r+m')+2n}_{2(r+m')+n+1} |\omega|_0^{8(r+m')+4n+4} |d\phi|^{2n+2}_{n+2}; \tag{3-45}$$

here $\mathcal{C}^{m'}$ is the usual $\mathcal{C}^{m'}$-norm for the parameter x_0.

Thus we get an extension of (1-4):

$$\begin{aligned} \left\| p^{-n} P_p(x,x) - \frac{\omega(x)^n}{\theta(x)^n} \right\|_{\mathcal{C}^k} \leqslant & \, c |d\phi|^{2n+4k+8}_{n+2k+5} |\omega|_0^{4n+8k+20} |d\phi|^{2n+2}_{n+2} \zeta^{-2n-4k-10} p^{-1} \\ & + c|\omega|^{2n+2k+2}_{n+k} (|d\phi|_2 \zeta^{-1})^{6n+9+3k} p^{-1}. \end{aligned} \tag{3-46}$$

Remark 3.10. Let ϕ be a function of class $\mathcal{C}^\alpha$, with $0 < \alpha \leq 1$, which is ω_0-plurisubharmonic, i.e., $dd^c\phi + \omega_0 \geq 0$. For each $0 < \zeta \leq 1$, we can find a smooth ω_0-plurisubharmonic function ϕ_ζ such that $\|\phi_\zeta\|_{\mathcal{C}^k} \leq c\zeta^{-k+\alpha}$ and $dd^c\phi_\zeta + \omega_0 \geq \zeta\omega_0$, see [Dinh et al. 2015]. As mentioned in Section 1, we can study ϕ by applying our results to ϕ_ζ. Some steps in the proof of our estimates can be strengthened using $\|\phi_\zeta\|_{\mathcal{C}^k} \leq c\zeta^{-k+\alpha}$ for each $0 \leq k \leq n+6$ instead of using only the $\mathcal{C}^{n+6}$-norm.

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