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A matchbox manifold is a foliated space with totally disconnected transversals, and an equicontinuous matchbox manifold is the generalization of Riemannian foliations for smooth manifolds in this context. We develop the Molino theory for all equicontinuous matchbox manifolds. Our work extends the Molino theory developed by Álvarez López and Moreira Galicia, which required the hypothesis that the holonomy actions for these spaces satisfy the strong quasianalyticity condition. The methods of this paper are based on the authors' previous work on the structure of weak solenoids, and provide many new properties of the Molino theory for the case of totally disconnected transversals, and examples to illustrate these properties. In particular, we show that the Molino space need not be uniquely well defined, unless the global holonomy dynamical system is stable, a notion defined in this work. We show that examples in the literature for the theory of weak solenoids provide examples for which the strong quasianalytic condition fails. Of particular interest is a new class of examples of equicontinuous minimal Cantor actions by finitely generated groups, whose construction relies on a result of Lubotzky. These examples have nontrivial Molino sequences, and other interesting properties.

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1. Introduction

A smooth foliation $\mathcal{F}$ of a connected compact manifold is a smooth decomposition of M into leaves, which are connected submanifolds of M with constant leaf dimension n and codimension q, where $m = n + q$ is the dimension of M. This structure is defined by a finite covering of M by coordinate charts whose image is the product space

$$(-1, 1)^n \times (-1, 1)^q \subset \mathbb{R}^m,$$

such that the leaves are mapped into linear planes of dimension n, and the transition functions between charts preserve these planes. The space $(-1, 1)^q$ is called the local transverse model for $\mathcal{F}$. A smooth foliation $\mathcal{F}$ is said to be *Riemannian*, or *bundle-like*, if there exists a Riemannian metric on the normal bundle $Q \to M$ which is invariant under the transverse holonomy transport along the leaves of $\mathcal{F}$. This condition was introduced by Reinhart [1959], and is a very strong assumption to impose on a foliation. The Molino theory for Riemannian foliations gives a complete structure theory for the geometry and dynamics of this class of foliations on compact smooth manifolds [Haefliger 1989; Moerdijk and Mrčun 2003; Molino 1982; 1988].

An *n-dimensional foliated space* $\mathfrak{M}$, as introduced by Moore and Schochet [2006], is a continuum — a compact connected metrizable space — with a continuous decomposition of $\mathfrak{M}$ into leaves, which are connected manifolds with constant leaf dimension n. Moreover, the decomposition has a local product structure analogous to that for smooth foliations [Candel and Conlon 2000; Moore and Schochet 2006]; that is, every point of $\mathfrak{M}$ has an open neighborhood homeomorphic to the open subset $(-1, 1)^n \subset \mathbb{R}^n$ times an open subset of a Polish space $\mathfrak{X}$, which is said to be the *local transverse model*. Thus, $\mathfrak{M}$ has a foliation denoted by $\mathcal{F}_{\mathfrak{M}}$ whose leaves are the maximal path-connected components, with respect to the fine topology on $\mathfrak{M}$ induced by the plaques of the local product structure.

An *equicontinuous foliated space* is the topological analog of a Riemannian foliation. In this case, the transverse holonomy pseudogroup associated to the foliation is assumed to act via an equicontinuous collection of local homeomorphisms on the transverse model spaces. The transverse holonomy maps are not assumed to be differentiable, so there is no natural normal bundle associated to a foliated space, and the standard methods for showing an analog of the Molino theory do not apply. In a series of papers, Álvarez López and Candel [2009; 2010] and Álvarez López and Moreira Galicia [2016] formulated a *topological Molino theory* for equicontinuous foliated spaces, which is a partial generalization of the Molino theory for smooth Riemannian foliations. They formulated the notion of *strongly quasianalytic* “regularity” for a foliated space, which is a condition on the pseudogroup associated to the foliation, as discussed in Section 9. The topological

Molino theory in [Álvarez López and Moreira Galicia 2016] applies to foliated spaces which satisfy the strongly quasianalytic condition.

The topological Molino theory for an equicontinuous foliated space $\mathfrak{M}$ with *connected* transversals essentially reduces to the smooth theory, by [Álvarez López and Candel 2010; Álvarez López and Moreira Galicia 2016; Álvarez López and Barral Lijó 2016]. In contrast, when the transversals to $\mathcal{F}_{\mathfrak{M}}$ are *totally disconnected*, and we then say that $\mathfrak{M}$ is a matchbox manifold, the development of a Molino theory in [Álvarez López and Moreira Galicia 2016] does not address several key issues, which can be seen as the result of using techniques developed for the smooth theory in the context of totally disconnected spaces. In this work, we apply a completely different approach to developing a topological Molino theory for the case of totally disconnected transversals. The techniques we use were developed in the authors' works [Dyer 2015; Dyer et al. 2016; 2017]. They are used here to develop a topological Molino theory for matchbox manifolds in full generality, and to reveal the far greater complexity of the theory in this case. In particular, we show by our results and examples that the classification of equicontinuous matchbox manifolds via Molino theory is far from complete.

We recall in Section 2 the definitions of a foliated space $\mathfrak{M}$, and of a *matchbox manifold*, which is a foliated space whose local transverse models for the foliation $\mathcal{F}_{\mathfrak{M}}$ are totally disconnected. The terminology "matchbox manifold" follows the usage introduced in continua theory [Aarts and Oversteegen 1991; 1995; Aarts and Martens 1988]. A matchbox manifold with 2-dimensional leaves is a lamination by surfaces, as defined in [Ghys 1999; Lyubich and Minsky 1997]. If all leaves of $\mathfrak{M}$ are dense, then it is called a *minimal matchbox manifold*. A compact minimal set $\mathfrak{M} \subset M$ for a foliation $\mathcal{F}$ on a manifold M yields a foliated space with foliation $\mathcal{F}_{\mathfrak{M}} = \mathcal{F} | \mathfrak{M}$. If the minimal set is exceptional, then $\mathfrak{M}$ is a minimal matchbox manifold. It is an open problem to determine which minimal matchbox manifolds are homeomorphic to exceptional minimal sets of C^r-foliations of compact smooth manifolds, for $r \geq 1$. For example, the issues associated with this problem are discussed in [Cass 1985; Clark and Hurder 2011; Hurder 2013].

It was shown in [Clark and Hurder 2013, Theorem 4.12] that an equicontinuous matchbox manifold $\mathfrak{M}$ is minimal; that is, every leaf is dense in $\mathfrak{M}$. This result generalized a result of Joe Auslander [1988] for equicontinuous group actions. Examples of equicontinuous matchbox manifolds are given by *weak solenoids*, which are discussed in Section 3. Briefly, a weak solenoid $\mathcal{S}_{\mathcal{P}}$ is the inverse limit of a sequence of covering maps

$$\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\},$$

called a *presentation* for $\mathcal{S}_{\mathcal{P}}$, where M_ℓ is a compact connected manifold without boundary and $p_{\ell+1}$ is a finite-to-one covering space. The results of [Clark and

Hurder 2013] reduce the study of equicontinuous matchbox manifolds to the study of weak solenoids:

Theorem 1.1 [Clark and Hurder 2013, Theorem 1.4]. *An equicontinuous matchbox manifold $\mathfrak{M}$ is homeomorphic to a weak solenoid.*

The idea of the proof of this result is to choose a clopen transversal $V_0 \subset \mathfrak{M}$; then associated to the induced holonomy action of $\mathcal{F}_{\mathfrak{M}}$ on V_0, one defines (see Proposition 3.4) a chain of subgroups of finite index, $\mathcal{G} = \{G_0 \supset G_1 \supset \cdots\}$, where G_0 is the fundamental group of the first shape approximation M_0 to $\mathfrak{M}$, where M_0 is a compact manifold without boundary. Then $\mathfrak{M}$ is shown to be homeomorphic to the inverse limit of the infinite chain of coverings of M_0 associated to the subgroup chain $\mathcal{G}$.

The theory of inverse limits for covering spaces, as developed for example in [Fokkink and Oversteegen 2002; McCord 1965; Rogers 1970; Rogers and Tollefson 1971a; 1971b; Schori 1966], reduces many questions about the classification of weak solenoids to questions about properties of the group chain $\mathcal{G}$ associated with the presentation $\mathcal{P}$. Thus, every equicontinuous matchbox manifold $\mathfrak{M}$ admits a presentation which determines its homeomorphism type. In Section 3A, the notion of a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with presentation $\mathcal{P}$ is recalled, and the notion of a dynamical partition of the transversal space V_0 is introduced in Section 3B. As discussed in Section 3C, the homeomorphism constructed in the proof of Theorem 1.1 is well defined up to return equivalence for the action of the respective holonomy pseudogroups [Clark et al. 2013a, Section 4]. Thus, we are interested in invariants for group chains that are independent of the choice of the chain, up to the corresponding notion of return equivalence for group chains. This is the approach we use in this work to formulate and study "Molino theory" for weak solenoids.

Section 4 introduces the group chain model for the holonomy action of weak solenoids, following the approach in [Dyer 2015; Dyer et al. 2016; 2017]. Section 5 then recalls results in the literature about homogeneous matchbox manifolds and the associated group chain models for their holonomy actions, which are fundamental for developing the notion of a "Molino space". Section 6 introduces the notion of the Ellis group associated to the holonomy action of a weak solenoid. Ellis semigroups were developed in [Auslander 1988; Ellis and Gottschalk 1960; Ellis 1960; 1969; Ellis and Ellis 2014], and also appeared in [Álvarez López and Candel 2010]. A key point of our approach is to use this concept as the foundation of our development of a topological Molino theory.

A key aspect of the Molino space for a foliation is that it is *foliated homogeneous*. A continuum $\mathfrak{M}$ is said to be *homogeneous* if given any pair of points $x, y \in \mathfrak{M}$, then there exists a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$. A homeomorphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ preserves the path-connected components, hence a homeomorphism of a matchbox manifold preserves the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. It follows that if $\mathfrak{M}$ is homogeneous, then it is also foliated homogeneous. Our first result is that every equicontinuous matchbox manifold admits a foliated homogeneous Molino space.

Theorem 1.2. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and let $\mathcal{P}$ be a presentation of $\mathfrak{M}$, such that $\mathfrak{M}$ is homeomorphic to a solenoid $\mathcal{S}_{\mathcal{P}}$. Then there exists a homogeneous matchbox manifold $\widehat{\mathfrak{M}}$ with foliation $\widehat{\mathcal{F}}$, called a Molino space of $\mathfrak{M}$, and a compact totally disconnected group $\mathcal{D}$ (the discriminant group for $\mathcal{P}$ as defined in Section 6C) such that there exists a fibration*

$$\mathcal{D} \longrightarrow \widehat{\mathfrak{M}} \xrightarrow{\hat{q}} \mathfrak{M}, \tag{1}$$

*where the restriction of $\hat{q}$ to each leaf in $\widehat{\mathfrak{M}}$ is a covering map of some leaf in $\mathfrak{M}$. We say that (1) is a **Molino sequence** for $\mathfrak{M}$.*

The construction of the spaces in (1) is given in Section 7. The homeomorphism type of the fibration (1) depends on the choice of a homeomorphism of $\mathfrak{M}$ with a weak solenoid $\mathcal{S}_{\mathcal{P}}$, and this in turn depends on the choice of the presentation $\mathcal{P}$ associated to $\mathfrak{M}$ and a section $V_0 \subset \mathfrak{M}$, as discussed in Section 3C. Examples show that the topological isomorphism type of $\mathcal{D}$ may depend on the choice of the section V_0, and the sequence (1) need not be an invariant of the homeomorphism type of $\mathfrak{M}$. This motivates the introduction of the following definition.

Definition 1.3. A matchbox manifold $\mathfrak{M}$ is said to be *stable* if the topological type of the sequence (1) is well defined by choosing a sufficiently small transversal V_0 to the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. A matchbox manifold $\mathfrak{M}$ is said to be *wild* if it is not stable.

In Section 7D we discuss the relation between the above definition and the notion of a stable group chain as given in Definition 7.5. Our next result concerns the existence of stable matchbox manifolds.

Proposition 1.4. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and suppose $\mathfrak{M}$ admits a transverse section V_0 with presentation $\mathcal{P}$, such that the group $\mathcal{D}$ in the Molino sequence (1) is finite. Then $\mathfrak{M}$ is stable.*

Proposition 1.4 is proved in Section 7. Theorem 10.8 shows that every separable Cantor group $\mathcal{D}$ can be realized as the discriminant of a stable weak solenoid, but we do not know of a general criterion for when a weak solenoid whose discriminant is a Cantor group must be stable.

The Molino space $\widehat{\mathfrak{M}}$ is always a homogeneous matchbox manifold. By the results in [Dyer et al. 2016], $\mathfrak{M}$ is homogeneous if and only if for some section V_0, the fibration (1) has trivial fiber $\mathcal{D}$. Each leaf of a homogeneous foliated space has trivial germinal holonomy, and thus the properties of holonomy for a matchbox manifold $\mathfrak{M}$ are closely related to its nonhomogeneity. Section 8 considers the germinal holonomy groups associated to the global holonomy action for a matchbox manifold.

Of special importance is the notion of *locally trivial germinal holonomy*, introduced by Sacksteder and Schwartz [1965], and used by Inaba [1977; 1983] in his study of Reeb stability for noncompact leaves in smooth foliations. A leaf L_x in a matchbox manifold $\mathfrak{M}$, which intersects a transversal section V_0 at a point x, has locally trivial germinal holonomy if there is an open neighborhood $U \subset V_0$ of x such that the holonomy pseudogroup acts trivially on U. A leaf with locally trivial germinal holonomy has trivial germinal holonomy, but the converse need not be true. In particular, we prove the following result in Section 8. We say that a leaf L_x has finite π_1-type if its fundamental group is finitely generated. A matchbox manifold $\mathfrak{M}$ has finite π_1-type if all leaves in the foliation $\mathcal{F}_{\mathfrak{M}}$ have finite π_1-type.

Lemma 1.5. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold with finite π_1-type. Let L_x be a leaf with trivial germinal holonomy. Then L_x has locally trivial germinal holonomy.*

The statement of Lemma 1.5 is implicit in the authors' work [Dyer et al. 2017]. The notion of locally trivial germinal holonomy and the germinal holonomy properties of equicontinuous matchbox manifolds turn out to be important in the study of topological Molino theory. Since a weak solenoid is a foliated space, by a fundamental result of Epstein, Millett and Tischler [Epstein et al. 1977] it contains leaves with trivial germinal holonomy. A *Schori solenoid* is an example of a weak solenoid, and was first constructed in [Schori 1966]. Each leaf in the foliation of a Schori solenoid is a surface of infinite genus.

Proposition 1.6. *The Schori solenoid contains leaves which have trivial germinal holonomy, but do not have locally trivial germinal holonomy.*

Proposition 1.6 is proved in Section 9. Proposition 1.6 shows that the condition of finite generation of the fundamental group is essential for the conclusion of Lemma 1.5. Another result, proved in Section 8, relates the existence of leaves with nontrivial holonomy with nontriviality of the fiber $\mathcal{D}$ in the Molino sequence (1).

Theorem 1.7. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. If $\mathfrak{M}$ has a leaf with nontrivial holonomy, then the Molino sequence (1) is nontrivial for any choice of section $V_0 \subset \mathfrak{M}$.*

The example in [Fokkink and Oversteegen 2002] and new examples in Section 10 show that nontrivial holonomy is not a necessary condition for (1) to be nontrivial, as one can construct nonhomogeneous equicontinuous matchbox manifolds with simply connected leaves.

Álvarez López and Moreira Galicia [2016] investigated Molino theory in the case when the closure of the pseudogroup of an equicontinuous foliated space (in the compact-open topology) satisfies the condition of strong quasianalyticity (SQA). Geometrically, this means that the pseudogroup action is *locally determined*; that is, if a holonomy map acts trivially on an open subset of its domain, then it is trivial everywhere on its domain. A natural problem is to determine which classes of equicontinuous matchbox manifolds are SQA. This question is studied in Section 9.

Note that for equicontinuous actions on Cantor sets the compact-open topology, the uniform topology and the topology of pointwise convergence coincide. The following result is proved in Section 9. The set V_n in the statement below is a partition set of $V_0 \subset \mathfrak{T}$ as defined in Proposition 3.4.

Theorem 1.8. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold which has finite π_1-type. Then there exists a transverse section V_0 such that the action of the holonomy pseudogroup on this section is SQA. In addition, if V_0 can be chosen so that the fiber $\mathcal{D}$ in the Molino sequence (1) is finite, then there exists a section $V_n \subset V_0$ such that the **closure** of the pseudogroup action on V_n is SQA as well.*

On the other hand, there are equicontinuous matchbox manifolds which do not satisfy SQA condition.

Theorem 1.9. *For every transverse section V_0 in the Schori solenoid, the holonomy pseudogroup associated to the section is not SQA.*

Theorem 1.2 proves that the Molino space exists for any matchbox manifold $\mathfrak{M}$, including those that do not admit a section with the SQA holonomy pseudogroup. Thus, for equicontinuous matchbox manifolds, our results are more general than those in [Álvarez López and Moreira Galicia 2016].

Analyzing the results of Lemma 1.5 and Theorem 1.8, one concludes that the condition of finite π_1-type, imposed on a matchbox manifold $\mathfrak{M}$, and the condition of finiteness of the fiber $\mathcal{D}$ in the Molino sequence (1), are quite strong and force the holonomy pseudogroup to possess various nice properties, such as locally trivial germinal holonomy and the SQA condition.

It is natural to ask, how diverse is the class of examples with finite fiber $\mathcal{D}$ in the Molino sequence? The authors' work [Dyer et al. 2016] constructed new examples of equicontinuous matchbox manifolds with finite fiber $\mathcal{D}$, which are weakly normal, that is, restricting to a smaller transverse section one can arrange that the Molino sequence (1) has a trivial fiber. One of these examples is also described in Example 8.6 in this paper. Rogers and Tollefson [1971c] constructed an example of a weak solenoid which turns out to be stable and have finite fiber $\mathcal{D}$, where the nontriviality of $\mathcal{D}$ is due to the presence of a leaf with nontrivial holonomy. This example illustrates Proposition 1.4 and Theorem 1.7.

The concluding section (Section 10) gives the construction of a variety of new classes of examples which illustrate the concepts and results of this work. We first give in Section 10A a reformulation of the constructions of the discriminant groups in Section 6, in terms of closed subgroups of inverse limit groups, which is analogous to a construction attributed to Lenstra in [Fokkink and Oversteegen 2002]. This alternate formulation is of strong interest in itself, as it gives a deeper understanding of the Molino spaces introduced in this work. This construction can be applied to the examples constructed by Lubotzky [1993] showing the existence of various products of torsion groups in the profinite completion of torsion-free groups, as recalled in Section 10D. We then give three applications of these results, which are included in Section 10E. The first construction is based on the conclusions of Theorem 10.4.

Theorem 1.10. *Fix an integer $n \geq 3$. Then there exists a finite index, torsion-free subgroup $G \subset \mathrm{SL}_n(\mathbb{Z})$ of the $n \times n$ integer matrices such that given any finite group F of cardinality $|F|$ which satisfies $4(|F|+2) \leq n$, there exists an irregular group chain $\mathcal{G}_F$ in G with the properties that*

(1) *the discriminant group of $\mathcal{G}_F$ is isomorphic to F;*

(2) *the group chain $\mathcal{G}_F$ is stable, with constant discriminant group isomorphic to F;*

(3) *the kernel $K(\mathcal{G}_F^{\hat{g}})$ of each conjugate $\mathcal{G}_F^{\hat{g}}$ of this group chain is trivial.*

The terminology used in Theorem 1.10 will be explained in later sections, where we will show that given such a group chain, one can construct matchbox manifolds with the following properties:

Corollary 1.11. *Let F be a finite group. Then there exists a nonhomogeneous matchbox manifold $\mathfrak{M}$ such that every leaf of $\mathcal{F}_{\mathfrak{M}}$ has trivial germinal holonomy, and for any sufficiently small transverse section in $\mathfrak{M}$, its Molino sequence is nontrivial with fiber group $\mathcal{D} \cong F$.*

Note that it follows by Theorem 1.8 that for the examples constructed in the proof of Corollary 1.11, there is a section $V \subset \mathfrak{M}$ such that the closure of the pseudogroup action on V satisfies the SQA condition of Álvarez López and Moreira Galicia [2016].

The next two constructions are based on the conclusions of Theorem 10.5, due to Lubotzky. Again, the terminology used in the statements will be explained in later sections.

Theorem 1.12. *There exists a finite index, torsion-free finitely generated group G such that given any separable profinite group K, there exists an irregular group chain $\mathcal{G}_K$ in G such that*

(1) *the discriminant group of $\mathcal{G}_K$ is isomorphic to K;*

(2) *the group chain $\mathcal{G}_F$ is stable, with constant discriminant group isomorphic to K.*

Corollary 1.13. *Let K be a Cantor group. Then there exists a nonhomogeneous matchbox manifold $\mathfrak{M}$ such that, for any sufficiently small transverse section in $\mathfrak{M}$, its Molino sequence is nontrivial with fiber group $\mathcal{D} \cong K$.*

Finally, Theorem 10.10 gives the first examples of equicontinuous matchbox manifolds which are not virtually regular. The *virtually regular* condition was introduced in [Dyer et al. 2017], and is defined in Definition 10.9. As the terminology suggests, this notion is related to the homogeneity properties of finite-to-one coverings of a matchbox manifold $\mathfrak{M}$.

The concluding section (Section 10F) lists some open problems.

2. Equicontinuous Cantor foliated spaces

In this section, we recall background concepts about foliated spaces, and introduce the group chains associated to their equicontinuous Cantor holonomy actions.

2A. ***Equicontinuous Cantor foliated spaces.*** Recall that an *n-dimensional matchbox manifold* $\mathfrak{M}$ is a compact connected metrizable topological space such that every point $x \in \mathfrak{M}$ has an open neighborhood $U \subset \mathfrak{M}$ such that there is a homeomorphism

$$\varphi_x : \overline{U}_x \to [-1, 1]^n \times \mathfrak{T}_x, \tag{2}$$

where $\mathfrak{T}_x$ is a totally disconnected space. The homeomorphism φ_x is called a *local foliation chart*, and the space $\mathfrak{T}_x$ is called a *local transverse model*. As usual in foliation theory, one can choose a finite atlas $\mathcal{U} = \{(\varphi_i, U_i)\}_{1 \le i \le \nu}$ of local charts such that the intersections of the path-connected components in $U_x \cap U_y$ are connected and simply connected, and the images $\mathcal{T}_i = \varphi_i^{-1}(\{0\} \times \mathfrak{T}_i)$ are disjoint. The leaves of the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$ are defined to be the path-connected components of $\mathfrak{M}$, which are then a union of the path-connected components (the plaques) in the open sets U_i. A matchbox manifold is (*topologically*) *minimal* if each leaf $L \subset \mathfrak{M}$ is dense in $\mathfrak{M}$.

We require the matchbox manifold $\mathfrak{M}$ to be *smooth*; that is, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i^{-1}(\overline{U}_i \cap \overline{U}_j) \to \varphi_j(\overline{U}_i \cap \overline{U}_j)$$

are C^∞-maps in the first coordinate $x \in [-1, 1]^n$, and the restrictions to plaques depend continuously on $y \in \mathfrak{T}_i$, in the C^∞-topology on leaves, for $1 \le i, j \le \nu$.

Let $\mathrm{pr}_2 : [-1, 1]^n \times \mathfrak{T}_i \to \mathfrak{T}_i$ be the projection onto the second factor. Then $\pi_i = \mathrm{pr}_2 \circ \varphi_i : \overline{U}_i \to \mathfrak{T}_i$ for $1 \le i \le \nu$ are the local defining maps for the foliation $\mathcal{F}_{\mathfrak{M}}$. Set $\mathfrak{T}_{i,j} = \pi_i(U_i \cap U_j)$ for $1 \le i, j \le \nu$. Since the path-connected components of the charts are either disjoint or have a connected intersection, there is a well-defined change-of-coordinates homeomorphism

$$h_{i,j} = \pi_j \circ \pi_i^{-1} : \mathfrak{T}_{i,j} \to \mathfrak{T}_{j,i} \tag{3}$$

with domain $\mathfrak{T}_{i,j}$ and range $\mathfrak{T}_{j,i}$. Let $\mathcal{G}^1_{\mathcal{F}} = \{(h_{i,j}, \mathfrak{T}_{i,j}) \mid 1 \le i, j \le \nu\}$. Set $\mathfrak{T} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_\nu$. Then the collection of maps $\mathcal{G}^1_{\mathcal{F}}$ generates the *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ acting on the transverse space $\mathfrak{T}$. The construction and properties of $\mathcal{G}_{\mathcal{F}}$ are described in full detail in [Clark and Hurder 2013, Section 3].

For the study of the dynamical properties of $\mathcal{F}_{\mathfrak{M}}$, it is useful to introduce also the collection of maps $\mathcal{G}^*_{\mathcal{F}} \subset \mathcal{G}_{\mathcal{F}}$, defined as follows. Let $\mathcal{G}_0 \subset \mathcal{G}_{\mathcal{F}}$ denote the collection consisting of all possible compositions of homeomorphisms in $\mathcal{G}^1_{\mathcal{F}}$. Then $\mathcal{G}^*_{\mathcal{F}}$ consists of all possible restrictions of homeomorphisms in $\mathcal{G}_0$ to open subsets of their domains. The collection of maps $\mathcal{G}^*_{\mathcal{F}}$ is closed under the operations of compositions, taking inverses, and restrictions to open sets, and is called a *pseudo⋆group* in [Álvarez López and Moreira Galicia 2016; Matsumoto 2010], while $\mathcal{G}^*_{\mathcal{F}}$ is called a *localization* of $\mathcal{G}_0$ in [Álvarez López and Moreira Galicia 2016].

Remark 2.1. The standard definition of a pseudogroup [Candel and Conlon 2000] requires the pseudogroup to be closed under the operations of composition, taking inverses, restriction to open subsets, and combination of maps. A combination of two local homeomorphisms h_1 and h_2, with possibly disjoint domains $D(h_1)$ and $D(h_2)$ and with disjoint ranges, is a homeomorphism h defined on $D(h_1) \cup D(h_2)$ where $h|D(h_1) = h_1$ and $h|D(h_2) = h_2$. However, allowing such arbitrary gluings of maps is unnatural. For example, a composition $h_{j,k} \circ h_{i,j}$ can be associated with the existence of a leafwise path $\gamma_x : [0, 1] \to L_x \in \mathfrak{M}$ with $\gamma_x(0) \in U_i$ and $\gamma_y(1) \in U_k$, where L_x is a leaf such that $\pi_i(x) \in D(h_{j,k} \circ h_{i,j})$. If $\pi_i(y) \in D(h_{j,k} \circ h_{i,j})$, then the path γ_x can be lifted to a nearby leaf L_y to a "parallel" path γ_y with $\gamma_y(0) \in U_i$ and $\gamma_y(1) \in U_k$. Thus a holonomy transformation $h_{j,k} \circ h_{i,j}$ has a geometric meaning as the transverse transport in leaves along a leafwise path. Therefore, in the definitions of $\mathcal{G}_0$ and $\mathcal{G}^*_{\mathcal{F}}$ (and of a pseudo⋆group in [Matsumoto 2010]), one does not allow combinations of local homeomorphisms, unless such homeomorphisms can be obtained by restrictions to open subsets of maximal domains of elements in $\mathcal{G}_0$.

Let $d_{\mathfrak{M}}$ be a metric on $\mathfrak{M}$, and denote by $d_{\mathcal{T}_i}$ the restriction of $d_{\mathfrak{M}}$ to the embedded image $\mathcal{T}_i$ of the transversal $\mathfrak{T}_i$, $1 \le i \le \nu$. For each $1 \le i \le \nu$, consider the pullback $d_{\mathfrak{T}_i}$ of $d_{\mathcal{T}_i}$ along the embedding. Then define a metric $d_{\mathfrak{T}}$ on $\mathfrak{T}$ by the formula

$$d_{\mathfrak{T}}(x, y) = \begin{cases} d_{\mathfrak{T}_i}(x, y) & \text{if } x, y \in \mathfrak{T}_i \text{ for some } i, \\ \infty & \text{otherwise.} \end{cases}$$

For a homeomorphism $\gamma \in \mathcal{G}^*_{\mathcal{F}}$, denote by $D(\gamma)$ and $R(\gamma)$ the domain and the range of γ, respectively.

Definition 2.2. The action of the pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on the transversal $\mathfrak{T}$ is *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\gamma \in \mathcal{G}^*_{\mathcal{F}}$, if $x, x' \in D(\gamma)$ and $d_{\mathfrak{T}}(x, x') < \delta$, then $d_{\mathfrak{T}}(\gamma(x), \gamma(x')) < \varepsilon$.

The following notion is used in the statement of various results in this work.

Definition 2.3. A path-connected topological space X is said to have finite π_1-type if the fundamental group $\pi_1(X, x)$ is a finitely generated group for the choice of some basepoint $x \in X$. A matchbox manifold $\mathfrak{M}$ is said to have finite π_1-type if each leaf $L \subset \mathfrak{M}$ is a space of finite π_1-type.

2B. ***Suspensions.*** There is a well-known construction which yields a foliated space from a group action, called the *suspension construction*, as discussed in [Candel and Conlon 2000, Chapter 3], for example. We state this construction in the restricted context which we use in this work.

Let X be a Cantor space and H a finitely generated group, and assume there is given an action $\varphi : H \to \mathrm{Homeo}(X)$. Suppose that H admits a generating set $\{g_1, \dots, g_k\}$; then there is a homomorphism $\alpha_k : \mathbb{Z} * \cdots * \mathbb{Z} \twoheadrightarrow H$ of the free group on k generators onto H, given by mapping generators to generators. Of course, the map α_k will have nontrivial kernel, unless H happens to be a free group. Next, let Σ_k be a compact surface without boundary of genus k. Then for a choice of basepoint $x_0 \in \Sigma_k$ set $G = \pi_1(\Sigma_k, x_0)$. Then there is a homomorphism $\beta_k : G \to \mathbb{Z} * \cdots * \mathbb{Z}$ onto the free group of k generators. Denote the composition of these maps by $\Phi = \varphi \circ \alpha_k \circ \beta_k$ to obtain the homomorphism $\Phi : G = \pi_1(\Sigma_k, x_0) \to \mathbb{Z} * \cdots * \mathbb{Z} \to H \to \mathrm{Homeo}(X)$.

Now, let $\widetilde{\Sigma}_k$ denote the universal covering space of Σ_k, equipped with the right action of G by covering transformations. Form the product space $\widetilde{\Sigma}_k \times X$ which has a foliation $\widetilde{\mathcal{F}}$ whose leaves are the slices $\widetilde{\Sigma}_k \times \{x\}$ for each $x \in X$. Define a right action of G on $\widetilde{\Sigma}_k \times X$, which for $g \in G$ is given by $(y, x) \cdot g = (y \cdot g, \Phi(g^{-1})(x))$. For each g, this action preserves the foliation $\widetilde{\mathcal{F}}$, so we obtain a foliation $\mathcal{F}_{\mathfrak{M}}$ on the quotient space $\mathfrak{M} = (\widetilde{\Sigma}_k \times X)/G$. Note that all leaves of $\mathcal{F}_{\mathfrak{M}}$ are surfaces, which are in general noncompact.

Note that $\mathfrak{M}$ is a foliated Cantor bundle over Σ_k, and the holonomy of this bundle $\pi : \mathfrak{M} \to \Sigma_k$ acting on the fiber $V_0 = \pi^{-1}(x_0)$ is canonically identified with the action $\Phi : G \to \mathrm{Homeo}(X)$. Consequently, if the action Φ is minimal in the sense of topological dynamics [Auslander 1988], then the foliation $\mathcal{F}_{\mathfrak{M}}$ is minimal. If the action Φ is equicontinuous in the sense of topological dynamics [Auslander 1988], then $\mathcal{F}_{\mathfrak{M}}$ is an equicontinuous foliation in the sense of Definition 2.2.

There is a variation of the above construction, where we assume that G is a *finitely presented* group, and there is given a homomorphism $\Phi : G \to \mathrm{Homeo}(X)$. In this case, it is a well-known folklore result (for example, see [Massey 1991]) that there exists a closed connected 4-manifold B such that for a choice of basepoint $b_0 \in B$, $\pi_1(B, b_0)$ is homeomorphic to G. Then the suspension construction can be applied to the homomorphism $\Phi : \pi_1(B, b_0) \to \mathrm{Homeo}(X)$, where we replace Σ_k above with B, and the space $\widetilde{\Sigma}_k$ with the universal covering $\widetilde{B}$ of B. The resulting foliated space $\mathfrak{M}$ will have holonomy given by the map Φ.

In summary, the suspension construction translates results about equicontinuous minimal Cantor actions to results about equicontinuous matchbox manifolds.

3. Weak solenoids

In this section, we first recall the construction procedure for (*weak*) *solenoids*, and describe some of their properties. In Section 3B, we discuss the construction from [Clark and Hurder 2013] which associates a group chain to an equicontinuous matchbox manifold, which leads to a more precise statement of Theorem 1.1. Then in Section 3C, we make some observations about the conclusion of Theorem 1.1 which are important when considering the definition of the Molino space for matchbox manifolds.

3A. *Weak solenoids.* Let $n \geq 1$. Then for each $\ell \geq 0$, let M_ℓ be a compact connected simplicial complex of dimension n. A *presentation* is a collection $\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$, where each map $p_{\ell+1}$ is a proper surjective map of simplicial complexes with discrete fibers, which is called a *bonding* map. For $\ell \geq 0$ and $x \in M_\ell$, the preimage $\{p_{\ell+1}^{-1}(x)\} \subset M_{\ell+1}$ is compact and discrete, so the cardinality $\#\{p_{\ell+1}^{-1}(x)\}$ is finite. For a presentation $\mathcal{P}$ defined in this generality, the cardinality of the fibers of the maps $p_{\ell+1}$ need not be constant in either ℓ or x.

Associated to a presentation $\mathcal{P}$ is an inverse limit space,

$$\begin{aligned}
(4)\qquad \mathcal{S}_\mathcal{P} &\equiv \varprojlim \{p_{\ell+1} : M_{\ell+1} \to M_\ell\} \\
&= \{(x_0, x_1, \dots) \in \mathcal{S}_\mathcal{P} \mid p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0\} \subset \prod_{\ell \geq 0} M_\ell.
\end{aligned}$$

The set $\mathcal{S}_\mathcal{P}$ is given the relative topology, induced from the product (Tychonoff) topology, so that $\mathcal{S}_\mathcal{P}$ is itself compact and connected.

Definition 3.1. The inverse limit space $\mathcal{S}_\mathcal{P}$ in (4) is called a (*weak*) *solenoid* if for each $\ell \geq 0$ the space M_ℓ is a compact connected manifold without boundary, and $p_{\ell+1}$ is a proper covering map of degree $m_{\ell+1} > 1$.

Weak solenoids are a generalization of 1-dimensional (Vietoris) solenoids, described in Example 3.2 below. Weak solenoids were originally considered by McCord [1965], Rogers and Tollefson [1971a; 1971c] and Schori [1966], and later by Fokkink and Oversteegen [2002].

Example 3.2. Let $M_\ell = \mathbb{S}^1$ for each $\ell \geq 0$, and let the map $p_{\ell+1}$ be a proper covering map of degree $m_{\ell+1} > 1$ for $\ell \geq 0$. Then $\mathcal{S}_\mathcal{P}$ is an example of a classic 1-dimensional solenoid, discovered independently by van Dantzig [1930] and Vietoris [1927]. If $m_{\ell+1} = 2$ for $\ell \geq 0$, then $\mathcal{S}_\mathcal{P}$ is called the *dyadic* solenoid.

Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid as in Definition 3.1. For each $\ell \geq 1$, the composition

(5) $$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell : M_\ell \to M_0$$

is a finite-to-one covering map of the base manifold M_0. For each $\ell \geq 0$, projection onto the ℓ-th factor in the product $\prod_{\ell \geq 0} M_\ell$ in (4) yields a fibration map denoted by $\Pi_\ell : \mathcal{S}_{\mathcal{P}} \to M_\ell$. For $\ell = 0$ this yields the fibration $\Pi_0 : \mathcal{S}_{\mathcal{P}} \to M_0$, and for $\ell \geq 1$ we have

(6) $$\Pi_0 = q_\ell \circ \Pi_\ell : \mathcal{S}_{\mathcal{P}} \to M_0.$$

A choice of basepoint $x_0 \in M_0$ fixes a fiber $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$, which is a Cantor set by the assumption that the fibers of each map $p_{\ell+1}$ have cardinality at least 2. McCord [1965] showed that (6) is a fiber bundle over M_0 with a Cantor set fiber, and the solenoid $\mathcal{S}_{\mathcal{P}}$ has a local product structure as in (2). The path-connected components of $\mathcal{S}_{\mathcal{P}}$ thus define a foliation denoted by $\mathcal{F}_{\mathcal{P}}$. We then have:

Proposition 3.3. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid whose base space M_0 is a compact manifold of dimension $n \geq 1$. Then $\mathcal{S}_{\mathcal{P}}$ is a minimal matchbox manifold of dimension n with foliation $\mathcal{F}_{\mathcal{P}}$.*

Denote by $G_0 = \pi_1(M_0, x_0)$ the fundamental group of M_0 with basepoint x_0, and choose a point $x \in \mathfrak{X}_0$ in the fiber over x_0. This defines basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$ for $\ell \geq 1$.

Let $y \in \mathfrak{X}_0$ be another point, set $y_\ell = \Pi_\ell(y) \in M_\ell$, and note that $y_0 = x_0$ by construction. We will interchangeably write $y = (y_\ell)$ to denote a point in $\mathfrak{X}_0$ or $\mathcal{S}_{\mathcal{P}}$. Let L_y denote the leaf of $\mathcal{F}_{\mathcal{P}}$ containing y. Then the restriction $\Pi_0|L_y : L_y \to M_0$ of the bundle projection to each path-connected component L_y is a covering map. For $g = [\gamma_0] \in G_0$, let $\gamma_\ell : [0,1] \to M_\ell$ be a lift of γ_0 with the starting point $\gamma_\ell(0) = y_\ell$. Define a homeomorphism $h_g : \mathfrak{X}_0 \to \mathfrak{X}_0$ by $h_g(y_\ell) = (\gamma_\ell(1))$. Thus there is a representation

(7) $$\Phi_0 : G_0 \to \mathrm{Homeo}(\mathfrak{X}_0) : \gamma \to h_g,$$

called the *global holonomy map* of the solenoid $\mathcal{S}_{\mathcal{P}}$.

3B. *Dynamical partitions.* It was shown in [Clark and Hurder 2013, Theorem 4.12] that an equicontinuous matchbox manifold $\mathfrak{M}$ is minimal, that is, every leaf is dense in $\mathfrak{M}$. This result generalizes to pseudogroups by a corresponding result of Auslander [1988] for equicontinuous group actions. It follows that for any clopen subset $V_0 \subset \mathfrak{T}$, the restricted pseudo⋆group $\mathcal{G}^*_{V_0} = \mathcal{G}^*_{\mathcal{F}}|V_0$ is *return equivalent* to the pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on $\mathfrak{T}$, where return equivalence is defined and studied in [Clark et al. 2013a, Section 4]. Thus, for the study of the dynamical properties of $\mathcal{F}_{\mathfrak{M}}$ one can restrict to the study of $\mathcal{G}^*_{V_0}$. The following result is based on the constructions in [Clark and Hurder 2013].

Proposition 3.4. *Let $\mathfrak{M}$ be a matchbox manifold with totally disconnected transversal $\mathfrak{T}$ and equicontinuous holonomy pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on $\mathfrak{T}$, let $x \in \mathfrak{T}$ be a point, and let $W \subset \mathfrak{T}$ be a clopen (closed and open) neighborhood of x. Then there exists a clopen subset $x \in V_0 \subset W$ and a descending chain of clopen sets $V_0 \supset V_1 \supset \cdots$ of $\mathfrak{T}$ with $\{x\} = \bigcap_\ell V_\ell$ such that:*

(1) *The restriction $\mathcal{G}^*_{\mathcal{F}}|V_0$ is generated by a group G_0 of transformations of V_0.*

(2) *For each $\ell \geq 1$ the collection $\mathcal{Q}_\ell = \{g \cdot V_\ell\}_{g \in G_0}$ is a finite partition of V_0 into clopen sets.*

(3) *We have* $\operatorname{diam}(g \cdot V_\ell) < 2^{-\ell}$ *for all $g \in G_0$ and all $\ell \geq 0$.*

(4) *The collection of elements which fix V_ℓ, that is,*

$$G^x_\ell = \{g \in G_0 \mid g \cdot V_\ell = V_\ell\},$$

is a subgroup of finite index in G_0. More precisely, $|G_0 : G^x_\ell| = \operatorname{card} \mathcal{Q}_\ell$.

There are many choices involved in the construction of the partitions $\mathcal{Q}_\ell$ and consequently the stabilizer groups G^x_ℓ:

(1) The choice of a transverse section $V_0 \subset \mathfrak{T}$, which results in the choice of the group G_0.

(2) The choice of a basepoint $x \in V_0$.

(3) Given V_0, x and G_0, there is freedom to choose clopen sets $V_1 \supset V_2 \supset \cdots$, which results in the choice of the sequence of groups $G_0 = G^x_0 \supset G^x_1 \supset G^x_2 \supset \cdots$.

Thus, the algebraic and geometric data encoded by these choices must be considered up to suitable notions of equivalence, which will be introduced in Section 4A.

3C. ***Homeomorphisms.*** Let $\mathfrak{M}$ be a matchbox manifold with totally disconnected transversal $\mathfrak{T}$ and equicontinuous holonomy pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ acting on $\mathfrak{T}$, let $x \in \mathfrak{T}$ be a point, and let $\{V_{\ell+1} \subset V_\ell \mid \ell \geq 0\}$ be a descending chain of clopen subsets of $\mathfrak{T}$ with $x \in V_\ell$ for all $\ell \geq 0$, as introduced in Proposition 3.4, where G_0 is a group of transformations of V_0, and G_ℓ denotes the stabilizer subgroup of G_0 of the set V_ℓ.

The basic idea of the proof of Theorem 1.1 is that if we choose the section $V_0 \subset \mathfrak{M}$ appropriately and it is sufficiently small, then there is a compact manifold M_0 and a fibration $\Pi'_0 : \mathfrak{M} \to M_0$ for which the inverse image $(\Pi'_0)^{-1}(x_0)$ equals V_0, where $x_0 = \Pi'_0(x)$. Moreover, the restrictions of the map Π'_0 to the leaves of $\mathcal{F}_{\mathfrak{M}}$ are coverings of M_0. The definition of the map Π'_0 requires the highly technical results of [Clark et al. 2013b] to define a transverse Cantor foliation $\mathcal{H}_0$ to $\mathcal{F}_{\mathfrak{M}}$, so that the quotient space $M_0 = \mathfrak{M}/\mathcal{H}_0$ is a compact manifold, and then Π'_0 is the projection along the leaves of the transverse foliation $\mathcal{H}_0$, or better said the equivalence classes defined by the leaves of $\mathcal{H}_0$. Then V_0 is the $\mathcal{H}_0$-equivalence class of the point $x \in V_0 \subset \mathfrak{M}$.

Let $V_\ell \subset V_0$ be the clopen set in Proposition 3.4 and $G_\ell^x = \{g \in G_0 \mid g \cdot V_\ell = V_\ell\}$ the isotropy subgroup of V_ℓ. Then there is a Cantor subfoliation $\mathcal{H}_\ell$ of $\mathcal{H}_0$ such that V_ℓ is the $\mathcal{H}_\ell$-equivalence class of x. Moreover, there is a quotient map $\Pi'_\ell : \mathfrak{M} \to \mathfrak{M}/\mathcal{H}_\ell \equiv M_\ell$, where is M_ℓ is identified with the covering of M_0 associated to the subgroup $G_\ell \subset G_0 = \pi_1(M_0, x_0)$. Note that the fiber $(\Pi'_\ell)^{-1}(x_0)$ equals V_ℓ and the monodromy action of G_0 on V_0 partitions V_0 into the translates of V_ℓ. There is then a quotient covering map $q_\ell : M_\ell \to M_0$, and as in (6), we have

$$\Pi'_0 = q_\ell \circ \Pi'_\ell : \mathfrak{M} \to M_0. \tag{8}$$

For each $\ell \geq 0$ let $p_{\ell+1} : M_{\ell+1} \to M_\ell$ be the quotient map defined by expanding the equivalence classes of $\mathfrak{M}$ defined by $\mathcal{H}_{\ell+1}$ to the equivalence classes defined by $\mathcal{H}_\ell$. Then we obtain a collection of covering maps $\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$ which defines a weak solenoid $\mathcal{S}_\mathcal{P}$. As the diameters of the clopen partition sets V_ℓ tend to 0 as ℓ increases, it is then standard that the collection of maps $\{\Pi'_\ell : \mathfrak{M} \to M_\ell \mid \ell \geq 0\}$ induces a foliated homeomorphism $\Pi_0^* : \mathfrak{M} \to \mathcal{S}_\mathcal{P}$.

In later sections, we will also consider the presentations $\mathcal{P}_n$ obtained by truncating the initial n terms in the presentation $\mathcal{P}$. That is, for $n \geq 0$ we have

$$\mathcal{P}_n = \{p'_{\ell+1} : M'_{\ell+1} \to M'_\ell \mid \ell \geq 0\}, \quad \text{where } M'_\ell = M_{\ell+n} \text{ and } p'_{\ell+1} = p_{\ell+n+1}. \tag{9}$$

It is a basic property of inverse limit spaces [McCord 1965; Rogers 1970] that for $n \geq 1$ and $m \geq 0$, there is a homeomorphism $\sigma_n : \mathcal{S}_{\mathcal{P}_{m+n}} \cong \mathcal{S}_{\mathcal{P}_n}$, where the homeomorphism is given by the "shift in coordinates" map σ_n in the inverse sequences defining these spaces. Also, by the same reasoning as above, there is a foliated homeomorphism $\Pi_n^* : \mathfrak{M} \to \mathcal{S}_{\mathcal{P}_n}$ and we have a commutative diagram of fibrations:

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "="] \arrow[d, "\Pi^*_{n+m}"'] & \mathfrak{M} \arrow[d, "\Pi^*_m"] \\ \mathcal{S}_{\mathcal{P}_{n+m}} \arrow[r, "\sigma_n"] & \mathcal{S}_{\mathcal{P}_m} \end{tikzcd} \tag{10}$$

Note that if the presentation $\mathcal{P}$ is constructed using the holonomy of $\mathcal{F}_\mathfrak{M}$ acting on the transversal $V_0 \subset \mathfrak{M}$, then for $n > 0$ and $m \geq 0$, the map $\sigma_n : \mathcal{S}_{\mathcal{P}_{n+m}} \to \mathcal{S}_{\mathcal{P}_m}$ satisfies $\sigma_n(V_{m+n}) \subset V_n$. That is, the induced map on $\mathfrak{M}$ sends the transversal $(\Pi^*_{m+n})^{-1}(V_{m+n}) \subset \mathfrak{M}$ into the transversal $(\Pi^*_n)^{-1}(V_n) \subset \mathfrak{M}$. On the other hand, given a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ there is no reason it should map the transversal V_0 into itself. In particular, the induced map

$$(\Pi^*_n) \circ h \circ (\Pi^*_{m+n})^{-1} : \mathcal{S}_{\mathcal{P}_{m+n}} \to \mathcal{S}_{\mathcal{P}_n} \tag{11}$$

on weak solenoids need not be fiber preserving. On the other hand, as discussed in [Fokkink and Oversteegen 2002], there is always a map $h' : \mathfrak{M} \to \mathfrak{M}$ which is homotopic to h such that the induced map as in (11) maps a clopen subset of V_{m+n} into a clopen subset of V_n. Thus, by allowing sufficiently large values of n and m and choice of basepoints in the range and domain, we can always ensure that a given homeomorphism of $\mathfrak{M}$ induces a fiber-preserving map between the weak solenoids $\mathcal{S}_{\mathcal{P}_{m+n}}$ and $\mathcal{S}_{\mathcal{P}_n}$.

4. Group chain models

Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid defined by a presentation $\mathcal{P}$, with basepoint $x \in \mathfrak{X}_0 \equiv \Pi_0^{-1}(x_0) \subset \mathcal{S}_{\mathcal{P}}$. For $G_0 = \pi_1(M_0, x_0)$, let $\Phi_0 : G_0 \to \mathrm{Homeo}(\mathfrak{X}_0)$ be the holonomy action in (7).

The following "combinatorial model" for the action (7) allows for a deeper analysis of the relation between the action Φ_0 and the algebraic structure of G_0. For each $\ell \geq 1$, recall that

$$G_\ell^x = \mathrm{image}\{(q_\ell)_\# : \pi_1(M_\ell, x_\ell) \to G_0\} \tag{12}$$

denotes the image of the induced map $(q_\ell)_\#$ on fundamental groups. In this way, associated to the presentation $\mathcal{P}$ and basepoint $x \in \mathfrak{X}_0$, we obtain a descending chain of subgroups of finite index

$$\mathcal{G}^x : G_0 \supset G_1^x \supset G_2^x \supset \cdots \supset G_\ell^x \supset \cdots . \tag{13}$$

Each quotient $X_\ell^x = G_0/G_\ell^x$ is a finite set equipped with a left G_0-action, and there are surjections $X_{\ell+1}^x \to X_\ell^x$ which commute with the action of G_0. The inverse limit

$$X_\infty^x = \varprojlim\{p_{\ell+1} : X_{\ell+1}^x \to X_\ell^x\} = \{(eG_0, g_1G_1^x, \ldots) \mid g_\ell G_\ell^x = g_{\ell+1}G_\ell^x\} \subset \prod_{\ell \geq 0} X_\ell^x \tag{14}$$

is then a totally disconnected compact perfect set, so is a Cantor set. The fundamental group G_0 acts on the left on X_∞^x via the coordinatewise multiplication on the product in (14). We denote this Cantor action by $(X_\infty^x, G_0, \Phi_x)$.

Lemma 4.1. *There is a homeomorphism $\tau_x : \mathfrak{X}_0 \to X_\infty^x$ equivariant with respect to the action (7) of G_0 on $\mathfrak{X}_0$ and Φ_x on X_∞; that is, $\tau_x \circ h_g(y) = \Phi_x(g) \circ \tau_x(y)$ for all $y \in \mathfrak{X}_0$.*

In particular, this allows us to conclude that the action Φ_0 of G_0 on the fiber of the solenoid $\mathcal{S}_{\mathcal{P}}$ is minimal. Indeed, the left action of G_0 on each quotient space X_ℓ^x is transitive, so the orbits are dense in the product topology on X_∞^x.

Remark 4.2. The group chain (14) and the homeomorphism in Lemma 4.1 depend on the choice of a point $x \in \mathfrak{X}_0$. For a different basepoint $y \in \mathfrak{X}_0$ in the fiber over x_0, let $\tau_x(y) = (g_i G_\ell^x) \in X_\infty^x$; then the group chain $\mathcal{G}^y$ associated to y is given by a chain of conjugate subgroups in G_0, where $G^y_\ell = g_\ell G^x_\ell g_\ell^{-1}$ for $\ell \geq 0$. The group chains $\mathcal{G}^y$ and $\mathcal{G}^x$ are said to be *conjugate chains*. The composition $\tau_y \circ \tau_x^{-1} : X^x_\infty \to X^y_\infty$ gives a topological conjugacy between the minimal Cantor actions $(X^x_\infty, G_0, \Phi_x)$ and $(X^y_\infty, G_0, \Phi_y)$. The map $\tau_x : \mathfrak{X}_0 \to X^x_\infty$ can be viewed as "coordinates" on the inverse limit space $\mathfrak{X}_0$, and the composition $\tau_y \circ \tau_x^{-1}$ as a "change of coordinates". Properties of the minimal Cantor action $(X^x_\infty, G_0, \Phi_x)$ which are independent of the choice of these coordinates are thus properties of the topological type of $\mathcal{S}_\mathcal{P}$.

4A. ***Equivalence of group chains.*** Fokkink and Oversteegen [2002] and the authors [Dyer et al. 2016] studied equivalences of group chains associated to a given equicontinuous minimal Cantor system (V_0, G_0, Φ). We now briefly recall the key results.

Denote by $\mathfrak{G}$ the collection of all possible subgroup chains in G_0. Then there are two equivalence relations on $\mathfrak{G}$. The first was introduced by Rogers and Tollefson [1971b]:

Definition 4.3. In a finitely generated group G_0, two group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ with $G_0 = H_0$ are *equivalent* if and only if there is a group chain $\{K_\ell\}_{\ell\geq 0}$ and infinite subsequences $\{G_{\ell_k}\}_{k\geq 0}$ and $\{H_{j_k}\}_{k\geq 0}$ such that $K_{2k} = G_{\ell_k}$ and $K_{2k+1} = H_{j_k}$ for $k \geq 0$.

The next definition was introduced by Fokkink and Oversteegen [2002].

Definition 4.4. Two group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ in $\mathfrak{G}$ are *conjugate equivalent* if and only if there exists a sequence $(g_\ell) \subset G_0$ for which the compatibility condition $g_\ell G_\ell = g_{\ell+1} G_\ell$ for all $\ell \geq 0$ is satisfied, and such that the group chains $\{g_\ell G_\ell g_\ell^{-1}\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ are equivalent.

The dynamical meaning of the equivalences in Definitions 4.3 and 4.4 is given by the following theorem, which follows from results in [Fokkink and Oversteegen 2002]; see also [Dyer et al. 2016].

Theorem 4.5. *Let $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ be group chains in G_0, with $H_0 = G_0$, and let*

$$G_\infty = \varprojlim\{G_0/G_{\ell+1} \to G_0/G_\ell\},$$
$$H_\infty = \varprojlim\{G_0/H_{\ell+1} \to G_0/H_\ell\}.$$

Then

(1) *the group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ are* ***equivalent*** *if and only if there exists a homeomorphism $\tau : G_\infty \to H_\infty$ equivariant with respect to the G_0-actions on G_∞ and H_∞, and such that $\varphi(eG_\ell) = (eH_\ell)$;*

(2) *the group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ are* ***conjugate equivalent*** *if and only if there exists a homeomorphism $\tau : G_\infty \to H_\infty$ equivariant with respect to the G_0-actions on G_∞ and H_∞.*

That is, an equivalence of two group chains corresponds to the existence of a *basepoint-preserving* equivariant homeomorphism between their inverse limit systems, while a conjugate equivalence of two group chains corresponds to the existence of a equivariant conjugacy between their inverse limit systems, which need not preserve the basepoint.

Let $\mathfrak{G}(\Phi_0)$ denote the class of group chains in G_0 which are *conjugate equivalent* to the group chain $\{G_\ell^x\}_{\ell\geq 0}$ with basepoint x. The following result gives a geometric interpretation of the conjugate equivalence class $\mathfrak{G}(\Phi_0)$ of a group chain $\{G_\ell^x\}_{\ell\geq 0}$.

Proposition 4.6. *Given an equicontinuous minimal Cantor action (V_0, G_0, Φ_0), let $\{G_\ell^x\}_{\ell\geq 0}$ be a group chain with partitions $\{\mathcal{Q}_\ell\}_{\ell\geq 0}$ and basepoint x, as in Proposition 3.4. Then a group chain $\{H_\ell\}_{\ell\geq 0}$ is in $\mathfrak{G}(\Phi_0)$ if and only if there exists a collection of G_0-invariant partitions $\mathcal{S}_\ell = \{g\cdot U_\ell\}_{g\in G_0}$ of V_0, where $U_\ell\subset V_0$ is a clopen set, and $\bigcap_\ell U_\ell = \{y\}\subset V_0$, such that $H_\ell = H_\ell^y$ is the isotropy group at U_ℓ of the action of G_0 on the partition $\mathcal{S}_\ell$, for all $\ell\geq 0$.*

4B. ***Kernels of group chains.*** The following notion is important for the study of group chains.

Definition 4.7. The *kernel* of a group chain $\mathcal{G} = \{G_\ell\}_{\ell\geq 0}$ is the subgroup of G_0 given by

$$K(\mathcal{G}) = \bigcap_{\ell\geq 0} G_\ell. \tag{15}$$

The following property is immediate from the definitions.

Lemma 4.8. *Suppose that the group chains $\mathcal{G} = \{G_\ell\}_{\ell\geq 0}$ and $\mathcal{H} = \{H_\ell\}_{\ell\geq 0}$ with $G_0 = H_0$ are equivalent. Then $K(\mathcal{G}) = K(\mathcal{H})\subset G_0$.*

If the chains $\mathcal{G}$ and $\mathcal{H}$ are only conjugate equivalent, then the kernels need not be equal.

An infinite group G_0 which admits a group chain $\mathcal{C} = \{C_\ell\}_{\ell\geq 0}$ where each C_ℓ is a *normal* subgroup of G_0, and such that $\bigcap C_\ell = \{e\}$, where e denotes the identity element in G_0, is said to be *residually finite*. It is an elementary fact that given any group chain $\mathcal{G} = \{G_\ell\}_{\ell\geq 0}$ in G_0, there is an associated core group chain $\mathcal{G}$ for which $C_\ell\subset G_\ell$ with C_ℓ normal in G_0 for all $\ell > 0$, as will be discussed in Section 6B below. Thus, if the group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell\geq 0}$ introduced above has $K(\mathcal{G}^x)$ the trivial group, then G_0 must be a residually finite group. On the other hand, there are many classes of groups which are not residually finite, and thus any group chain for these groups must have nontrivial kernels. For example, many of the types of Baumslag–Solitar groups are not residually finite [Levitt 2015a; 2015b; Meskin 1972], so every equicontinuous minimal Cantor system defined by an action of one of these groups will have nontrivial kernels.

The kernel $K(\mathcal{G}^x)$ has an interpretation in terms of the topology of the leaves of the foliation $\mathcal{F}_{\mathcal{P}}$ of a weak solenoid. Let (V_0, G_0, Φ_0) be the holonomy action for a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with presentation $\mathcal{P}$ and basepoint $x \in V_0$, and let $\mathcal{G}^x = \{G_i^x\}_{i \geq 0}$ be the group chain at x. Recall that the restriction of the bundle projection $\Pi_0 | L_x : L_x \to M_0$ to the leaf L_x containing x is a covering map. Let $\widetilde{M}_0$ be the universal cover of M_0. Then by standard arguments of covering space theory (see [McCord 1965]) there is a homeomorphism

$$\widetilde{M}_0 / K(\mathcal{G}^x) \to L_x. \tag{16}$$

Now let $y \in \mathfrak{X}_0$ be another point. Then by Remark 4.2, the group chain associated to y is given by $\mathcal{G}^y = \{g_i G_i^x g_i^{-1}\}_{i \geq 0}$ where $\tau_x(y) = (g_i G_i^x)$. If y is in the orbit of x under the G_0-action, then we can take $g_i = g$ for some $g \in G_0$, and thus $K(\mathcal{G}^y) = g K(\mathcal{G}^x) g^{-1}$; that is, the kernels of $\mathcal{G}^x$ and $\mathcal{G}^y$ are conjugate, reflecting the fact that the fundamental group of the leaf L_x is replaced by a conjugate as x changes. If y is not in the orbit of x, then the relationship between $K(\mathcal{G}^x)$ and $K(\mathcal{G}^y)$ depends on the dynamical properties of the solenoid.

In particular, in Section 8 we relate the algebraic properties of the kernels $K(\mathcal{G}^y)$ with the germinal holonomy groups of the foliation $\mathcal{F}_{\mathcal{P}}$. Recall from Section 1 that a manifold L has π_1-finite type if its fundamental group is finitely generated. A matchbox manifold $\mathfrak{M}$ has finite π_1-type if all leaves in $\mathcal{F}_{\mathfrak{M}}$ have finite π_1-type. The following statement is immediate from the above discussion.

Lemma 4.9. *An equicontinuous matchbox manifold $\mathfrak{M}$ has finite π_1-type if and only if, for the associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$, for all $\mathcal{G}^y \in \mathfrak{G}(\Phi)$, the kernel $K(\mathcal{G}^y)$ is a finitely generated subgroup of G_0.*

We next give two examples to illustrate the above concepts.

Example 4.10. Let $\mathcal{S}_{\mathcal{P}}$ be a Vietoris solenoid, as in Example 3.2, where $m_\ell > 1$ is the degree of p_ℓ. Choose $x \in \mathcal{S}_{\mathcal{P}}$ so that $\Pi_\ell(x) = 0$ for $\ell \geq 0$. Then $G_0 = \mathbb{Z}$, and $G_\ell^x = \widetilde{m}_\ell \mathbb{Z}$, where $\widetilde{m}_\ell = m_1 m_2 \cdots m_\ell$ is the product of the degrees of the coverings. Then the kernel $K(\mathcal{G}^x)$ is $\{0\}$, and the path-connected component L_x is homeomorphic to the real line. Let $y \in \mathfrak{X}_0$ be any other point in the fiber. Since $\mathbb{Z}$ is abelian, any subgroup conjugate to $G_\ell^x = \widetilde{m}_\ell \mathbb{Z}$ is equal to it. It follows that $K(\mathcal{G}^y) = \{0\}$, and L_y is homeomorphic to the real line for any $y \in \mathfrak{X}_0$.

More generally, suppose $\mathcal{S}_{\mathcal{P}}$ is an n-dimensional solenoid and G_ℓ^x is a normal subgroup of G_0 for all $\ell \geq 1$. Then for any $y \in \mathfrak{X}_0$ we have $\mathcal{G}^y = \mathcal{G}^x$, and so $K(\mathcal{G}^y) = K(\mathcal{G}^x)$. It follows that all leaves in $\mathcal{S}_{\mathcal{P}}$ are homeomorphic. The Vietoris solenoid $\mathcal{S}_{\mathcal{P}}$ is of finite π_1-type.

Example 4.11. This example is due to Rogers and Tollefson [1971c]. Consider a map of the plane given by a translation by $\frac{1}{2}$ in the first component, and by reflection in the second component, i.e.,

$$r \times i : \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{where } (x, y) \mapsto \left(x + \tfrac{1}{2}, -y\right).$$

This map commutes with translations by the elements in the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, and so induces the map $r \times i : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$ of the torus. This map is an involution, and the quotient space $K = \mathbb{T}^2/(x, y) \sim r \times i(x, y)$ is homeomorphic to the Klein bottle.

Consider the double covering map $L : \mathbb{T}^2 \to \mathbb{T}^2$ given by $L(x, y) = (x, 2y)$. The inverse limit $\mathbb{T}_\infty = \varprojlim\{L : \mathbb{T}^2 \to \mathbb{T}^2\}$ is a solenoid with 2-dimensional leaves. Let $x_0 = (0, 0) \in M_0 = \mathbb{T}^2$. The fundamental group $G_0 = \mathbb{Z}^2$ is abelian, so for any $x, y \in \mathfrak{X}_0$ the kernels $K(\mathcal{G}^x) = K(\mathcal{G}^y)$ are isomorphic to $\mathbb{Z}$, and every leaf is homeomorphic to an open two-ended cylinder.

The involution $r \times i$ is compatible with the covering maps L, and so it induces an involution $(r \times i)_\infty : \mathbb{T}_\infty \to \mathbb{T}_\infty$, which is seen to have a single fixed point $(0, 0, \dots) \in \mathbb{T}_\infty$ and permute other path-connected components. Let $p : K \to K$ be the double covering of the Klein bottle by itself, given by $p(x, y) = (x, 2y)$, and consider the inverse limit space $K_\infty = \varprojlim\{p : K \to K\}$. Note that taking the quotient by the involution $r \times i$ is compatible with the covering maps L and p; that is, $p \circ (r \times i) = L$, and so induces the map $i_\infty : \mathbb{T}_\infty \to K_\infty$ of the inverse limit spaces. Under this map, the path-connected component of the fixed point $(0, 0, \dots)$ is identified so as to become a nonorientable one-ended cylinder. The image of any other path-connected component is an orientable two-ended cylinder.

Let $x = (x_\ell) \in K_\infty$ for $x_\ell \in K$. Then $G_0 = \pi_1(K, x_0) = \langle a, b \mid bab^{-1} = a^{-1}\rangle$. Fokkink and Oversteegen [2002] computed the kernel $K(\mathcal{G}^x) = \langle b\rangle$ of the group chain $\mathcal{G}^x$. They also computed kernels for group chains at any other basepoint $y \in \mathfrak{X}_0$ and found that either $K(\mathcal{G}^y)$ is conjugate to $\langle b\rangle$, or $K(\mathcal{G}^y)$ is equal to $\langle b^2\rangle$. This example has finite π_1-type.

5. Homogeneous solenoids and actions

In this section, we review the results from various works about the criteria for homogeneity of matchbox manifolds. These data will be of use later, when we give the proof Theorem 1.2.

A continuum $\mathfrak{M}$ is said to be *homogeneous* if given any pair of points $x, y \in \mathfrak{M}$, there exists a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$. A homeomorphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ preserves the path-connected components, hence preserves the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. It follows that if $\mathfrak{M}$ is homogeneous, then it is also foliated homogeneous.

By [Clark and Hurder 2013, Theorem 5.2] a homogeneous matchbox manifold $\mathfrak{M}$ is equicontinuous. Hence by Theorem 1.1 above, which is proved in [Clark and Hurder 2013, Theorem 1.4], the foliated space $\mathfrak{M}$ is homeomorphic to a weak solenoid $\mathcal{S}_{\mathcal{P}}$. We restrict our attention to equicontinuous foliated spaces, so consider the problem of giving conditions for when a weak solenoid $\mathcal{S}_{\mathcal{P}}$ is homogeneous, which is thus equivalent to asking when an equicontinuous matchbox manifold is homogeneous. This is one of the original motivating problems in the study of solenoids, to obtain necessary and sufficient conditions for when the solenoid $\mathcal{S}_{\mathcal{P}}$ is homogeneous [Fokkink and Oversteegen 2002; Rogers 1970; Rogers and Tollefson 1971a; Schori 1966]. In this section, we recall the relevant results of these previous works, and of [Dyer 2015; Dyer et al. 2016; 2017].

5A. *Regular actions.* An *automorphism* of (V_0, G_0, Φ_0) is a homeomorphism $h : V_0 \to V_0$ which commutes with the G_0-action on V_0. Denote by $\mathrm{Aut}(V_0, G_0, \Phi_0)$ the group of automorphisms of the action (V_0, G_0, Φ_0). Note that $\mathrm{Aut}(V_0, G_0, \Phi_0)$ is a topological group for the compact-open topology on maps, and is a closed subgroup of $\mathrm{Homeo}(V_0)$.

Definition 5.1. The equicontinuous minimal Cantor action (V_0, G_0, Φ_0) is

(1) *regular* if the action of $\mathrm{Aut}(V_0, G_0, \Phi_0)$ on V_0 has a single orbit;

(2) *weakly normal* if the action of $\mathrm{Aut}(V_0, G_0, \Phi_0)$ decomposes V_0 into a finite collection of orbits;

(3) *irregular* if the action of $\mathrm{Aut}(V_0, G_0, \Phi_0)$ decomposes V_0 into an infinite collection of orbits.

The terminology in Definition 5.1 is chosen to be consistent with the terminology in [Dyer et al. 2016; Fokkink and Oversteegen 2002].

Recall that $\mathfrak{G}$ denotes the collection of all possible subgroup chains in G_0, and let $\mathfrak{G}(\Phi_0) \subset \mathfrak{G}$ denote the collection of all group chains in $\mathfrak{G}$ which are conjugate equivalent to a given group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$. Theorem 4.5 states that a group chain $\{G^x_\ell\}_{\ell \geq 0}$ is equivalent to the group chain $\{H^y_\ell\}_{\ell \geq 0}$ if and only if there exists a conjugacy $h : V_0 \to V_0$ of the G_0-action on V_0 such that $h(x) = y$. Such an h is an automorphism of (V_0, G_0, Φ_0), which gives the following result.

Theorem 5.2. *Let* (V_0, G_0, Φ_0) *be an equicontinuous minimal Cantor action, and* $\{G^x_\ell\}_{\ell \geq 0} \in \mathfrak{G}$ *be a group chain associated to the action. Then* (V_0, G_0, Φ_0) *is*

(1) ***regular*** *if all group chains in* $\mathfrak{G}(\Phi_0)$ *are equivalent*;

(2) ***weakly normal*** *if* $\mathfrak{G}(\Phi_0)$ *contains a finite number of classes of equivalent group chains*;

(3) ***irregular*** *if* $\mathfrak{G}(\Phi_0)$ *contains an infinite number of classes of equivalent group chains.*

McCord [1965] studied the case when the chain $\{G^x_\ell\}_{\ell \geq 0}$ consists of normal subgroups of G_0. In this case, every quotient $X^x_\ell = G_0/G^x_\ell$ is a finite group, and the inverse limit X_∞^x, defined by (14), is then a profinite group. The group X_∞^x is identified with V_0 as a topological space, and it acts transitively on V_0 on the right. The right action of X_∞^x commutes with the left action of G_0 on X_∞^x, and thus $X_\infty^x \subset \mathrm{Aut}(V_0, G_0, \Phi_0)$, and so the automorphism group acts transitively on H_∞. McCord [1965] used this observation to show that the group $\mathrm{Homeo}(\mathcal{S}_\mathcal{P})$ acts transitively on $\mathcal{S}_\mathcal{P}$, proving the following theorem.

Theorem 5.3. *Let $\mathcal{S}_\mathcal{P}$ be a solenoid with a group chain $\{G_\ell^x\}_{\ell \geq 0}$ such that G_ℓ^x is a normal subgroup of G_0 for all $\ell \geq 0$. Then $\mathcal{S}_\mathcal{P}$ is homogeneous.*

For example, if G_0 is abelian, then every group chain $\{G_\ell^x\}_{\ell \geq 0}$ consists of normal subgroups, and the solenoid $\mathcal{S}_\mathcal{P}$ is homogeneous.

5B. *Weakly normal actions.* We next consider the problem of giving necessary and sufficient conditions for when a solenoid $\mathcal{S}_\mathcal{P}$ is homogeneous.

The converse to Theorem 5.3 is not true. Indeed, Rogers and Tollefson [1971b] gave an example of a weak solenoid for which the presentation yields a chain of subgroups which are not normal in G_0, yet the inverse limit is a profinite group, and so the solenoid is homogeneous. This example was the motivation for the work of Fokkink and Oversteegen [2002], where they gave a necessary and sufficient condition on the chain $\{G_\ell^x\}_{\ell \geq 0}$ for the weak solenoid to be homogeneous. In particular, they proved the following result. Let $N_{G_0}(G_\ell)$ denote the normalizer of the subgroup G_ℓ in G_0; that is, $N_{G_0}(G_\ell) = \{g \in G_0 \mid g\, G_\ell\, g^{-1} = G_\ell\}$.

Theorem 5.4 [Fokkink and Oversteegen 2002]. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor action, $x \in V_0$ be a point, and $\{G_\ell^x\}_{\ell \geq 0}$ be an associated group chain with conjugate equivalence class $\mathfrak{G}(\Phi_0)$. Then*

(1) *(V_0, G_0, Φ_0) is regular if and only if there exists a group chain $\{N_\ell\}_{\ell \geq 0} \in \mathfrak{G}(\Phi_0)$ such that N_ℓ is a normal subgroup of G_0 for each $\ell \geq 0$;*

(2) *(V_0, G_0, Φ_0) is weakly normal if and only if there exists $\{G_\ell^{x'}\}_{i \geq 0} \in \mathfrak{G}(\Phi_0)$ and an $n > 0$ such that $G_\ell^{x'} \subset G_n^x \subseteq N_{G_0}(G_\ell^{x'})$ for all $\ell \geq n$.*

In Theorem 5.4, the set $\mathfrak{G}(\Phi_0)$ contains group chains which are conjugate equivalent to the given chain $\{G_\ell^x\}_{\ell \geq 0}$. The condition that the group chain $\{N_\ell\}_{\ell \geq 0}$ consists of normal subgroups implies that every chain in $\mathfrak{G}(\Phi_0)$ is equivalent to $\{N_\ell\}_{\ell \geq 0}$, and so $\{G_\ell^x\}_{\ell \geq 0}$ is equivalent to $\{N_\ell\}_{\ell \geq 0}$. In statement (2), the condition $G_\ell^{x'} \subset G_n^x \subseteq N_{G_0}(G_\ell^{x'})$ implies that the group chain $\{G_\ell^{x'}\}_{\ell \geq 0}$ is equivalent to $\{G_\ell^x\}_{\ell \geq 0}$. Indeed, suppose that $G_\ell^{x'} \subset G_m^{x\,\prime} \subseteq N_{G_0}(G_\ell^{x'})$ for some m. Then for $n \leq m$ and $\ell \leq n$ we have $G_\ell^{x'} \subset G_n^{x'} \subseteq N_{G_0}(G_\ell^{x'})$. If $\{G_\ell^{x'}\}_{i \geq 0}$ is equivalent to $\{G_\ell^x\}_{i \geq 0}$, then for some $n \leq m$ we have $G_n^{x'} \subset G_n^x \subset G_m^{x\,\prime}$, which yields the statement.

Recall that Proposition 3.4 introduced the descending chain of clopen sets $\{V_{\ell+1} \subset V_\ell \mid \ell \geq 0\}$ of V_0 such that V_ℓ is stabilized by the action of G_ℓ. Thus, the weak normality condition in Theorem 5.4 implies that if we restrict the G_0 action to the clopen set $V_n \subset V_0$, then the restricted action (V_n, G_n, Φ_n) with associated group chain $\mathcal{G}^x_n = \{G^x_\ell\}_{\ell \geq n}$ is regular. In the case where the group chain $\{G_\ell\}_{\ell\geq 0}$ is associated to a weak solenoid $\mathcal{S}_\mathcal{P}$, restricting to the action (V_n, G_n, Φ_n) amounts to discarding the initial manifolds $\{M_0, \dots, M_{n-1}\}$ in the presentation $\mathcal{P}$, to obtain the presentation $\mathcal{P}_n$ defined in (9). Then as discussed in Section 3C, there is a homeomorphism $\mathcal{S}_{\mathcal{P}_n} \cong \mathcal{S}_\mathcal{P}$, where the homeomorphism is given by the "shift" map σ_n. Thus, $\mathcal{S}_\mathcal{P}$ is homogeneous if and only if $\mathcal{S}_{\mathcal{P}_n}$ is homogeneous, and so by Theorem 5.3 a weak solenoid whose associated group chain is weakly normal is homogeneous. We thus obtain the following result of Fokkink and Oversteegen [2002], giving a criterion for when a weak solenoid is homogeneous.

Proposition 5.5. *Let $\mathcal{S}_\mathcal{P}$ be a weak solenoid, defined by a presentation $\mathcal{P}$ with associated group chain $\{G^x_\ell\}_{\ell\geq 0}$. Then $\mathcal{S}_\mathcal{P}$ is homogeneous if and only if $\{G^x_\ell\}_{\ell\geq 0}$ is weakly normal.*

We also have the following property of presentations of homogeneous solenoids.

Proposition 5.6 [Fokkink and Oversteegen 2002]. *Let $\mathcal{S}_\mathcal{P}$ be a weak solenoid, defined by a presentation $\mathcal{P}$ with associated group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell\geq 0}$. If $\mathcal{S}_\mathcal{P}$ is homogeneous, then the kernel $K(\mathcal{G}^x) \subset G_0$ has a finite number of conjugacy classes in G_0.*

Proof. Suppose that $\mathcal{S}_\mathcal{P}$ is homogeneous. Then by Theorem 5.4, there exists $\mathcal{G}^{x\prime} = \{G^{x\prime}_\ell\}_{\ell\geq 0} \in \mathfrak{G}(\Phi_0)$ and an $n > 0$ such that $G^{x\prime}_\ell \subset G^x_n \subseteq N_{G_0}(G^{x\prime}_\ell)$ for all $\ell \geq n$. Then $G^{x\prime}_n \subseteq N_{G_0}(G^{x\prime}_\ell)$ for all $\ell \geq n$, which implies that $G^{x\prime}_n \subset N_{G_0}(K(\mathcal{G}'_x))$. Indeed, the chain $\{G^{x\prime}_\ell\}_{\ell\geq n}$ contains subgroups normal in $G^{x\prime}_n$, and its intersection is then again normal in $G^{x\prime}_n$. Then for any $h \in G^{x\prime}_n$ we have

$$h \cdot K(\mathcal{G}^{x\prime}) \cdot h^{-1} = K(\mathcal{G}^{x\prime}), \tag{17}$$

and $K(\mathcal{G}^{x\prime})$ has only a finite number of conjugacy classes, at most $[G_0 : G^{x\prime}_n]$. Since $\mathcal{G}^x$ is equivalent to $\mathcal{G}^{x\prime}$, we have that $G^x_0 = G^{x\prime}_0 \supset G^x_1 \supset G^{x\prime}_1 \supset G^x_2 \supset G^{x\prime}_2 \supset \cdots$, and so $K(\mathcal{G}^x) = K(\mathcal{G}^{x\prime})$, which yields the statement. □

6. Ellis group of equicontinuous minimal systems

In [Ellis and Gottschalk 1960; Ellis 1960], the *Ellis* (*enveloping*) *semigroup* associated to a continuous group action $\Phi : G \times X \to X$ was introduced, and it is treated in [Auslander 1988; Ellis 1969; Ellis and Ellis 2014]. The construction of $\widehat{E}(X, G, \Phi)$ is abstract, and it can be difficult to calculate this group exactly. A key problem is to understand the relation between the algebraic properties of $\widehat{E}(X, G, \Phi)$ and the dynamics of the action. In this section, we briefly recall some basic properties of $\widehat{E}(X, G, \Phi)$, then consider the results for the special case of equicontinuous minimal systems.

6A. *Ellis (enveloping) group.* Let X be a compact Hausdorff topological space and G be a finitely generated group. Consider the space $X^X = \mathrm{Maps}(X, X)$ with the topology of *pointwise convergence on maps*. With this topology, X^X is a compact Hausdorff space. Each $g \in G$ defines an element $\hat{g} \in \mathrm{Homeo}(X) \subset X^X = \mathrm{Maps}(X, X)$. Denote by $\widehat{G}$ the set of all such elements. Ellis [1960] showed that the closure $\overline{\widehat{G}} \subset X^X$ has the structure of a right topological semigroup. Moreover, if the action (X, G, Φ) is equicontinuous, then the semigroup $\overline{\widehat{G}}$ is a group naturally identified with the closure $\overline{\Phi(G)}$ of $\Phi(G) \subset \mathrm{Homeo}(X)$ in the *uniform topology on maps*. Each element of $\overline{\Phi(G)}$ is the limit of a sequence of points in $\widehat{G}$, and we use the notation (g_i) to denote a sequence $\{g_i \mid i \geq 1\} \subset G$ such that the sequence $\{\hat{g}_i = \Phi(g_i) \mid i \geq 1\} \subset \mathrm{Homeo}(X)$ converges in the uniform topology.

Assume the action of G on X is minimal, that is, the orbit $\Phi(G)(x)$ is dense in X for any $x \in X$. It then follows that the orbit of the Ellis group $\overline{\Phi(G)}(x)$ equals X for any $x \in X$. That is, the group $\overline{\Phi(G)}$ acts transitively on X. Then for the isotropy group of the action at x,

$$\overline{\Phi(G)}_x = \{(g_i) \in \overline{\Phi(G)} \mid (g_i) \cdot x = x\}, \tag{18}$$

we have the natural identification $X \cong \overline{\Phi(G)}/\overline{\Phi(G)}_x$ of left G-spaces.

Given an equicontinuous minimal Cantor system (X, G, Φ), the Ellis group $\overline{\Phi(G)}$ depends only on the image $\Phi(G) \subset \mathrm{Homeo}(X)$. On the other hand, the isotropy group $\overline{\Phi(G)}_x$ may depend on the point $x \in X$. Since the action of $\overline{\Phi(G)}$ is transitive on X, given any $y \in X$, there exists $(g_i) \in \overline{\Phi(G)}$ such that $(g_i) \cdot x = y$. It follows that

$$\overline{\Phi(G)}_y = (g_i) \cdot \overline{\Phi(G)}_x \cdot (g_i)^{-1}. \tag{19}$$

Thus, the *cardinality* of the isotropy group $\overline{\Phi(G)}_x$ is independent of the point $x \in X$, and so the Ellis group $\overline{\Phi(G)}$ and the cardinality of $\overline{\Phi(G)}_x$ are invariants of (X, G, Φ).

6B. *Ellis group for group chains.* We consider the Ellis group for an equicontinuous minimal Cantor action (V_0, G_0, Φ), in terms of an associated group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ for $x \in V_0$. For each subgroup G^x_ℓ consider the maximal normal subgroup of G^x_ℓ which is given by

$$C_\ell \equiv \mathrm{core}_{G_0}\, G^x_\ell \equiv \bigcap_{g \in G_0} g G^x_\ell g^{-1} \subseteq G^x_\ell. \tag{20}$$

The group C_ℓ is called the *core* of G_ℓ in G_0. Since C_ℓ is normal in G_0, the quotient G_0/C_ℓ is a finite group, and the collection $\mathcal{C} = \{C_\ell\}_{\ell \geq 0}$ forms a descending chain of normal subgroups of G_0. The inclusions of coset spaces define bonding maps

$\delta_\ell^{\ell+1}$ for the inverse sequence of quotients G_0/C_ℓ, and the inverse limit space

$$C_\infty = \{(eG_0, g_1C_1, \dots) \mid g_\ell C_\ell = g_{\ell+1}C_\ell\} \subset \prod_{\ell \geq 0} G_0/C_\ell \tag{21}$$

$$\cong \varprojlim \{\delta_\ell^{\ell+1} : G_0/C_{\ell+1} \to G_0/C_\ell\} \tag{22}$$

is a profinite group. Let $\hat{\iota} : G_0 \to C_\infty$ be the homomorphism defined by $\hat{\iota}(g) = (gC_\ell)$ for $g \in G_0$. Then the induced left action of G_0 on C_∞ yields a minimal Cantor system, denoted by $(C_\infty, G_0, \widehat{\Phi}_0)$.

Also, introduce the descending chain of clopen neighborhoods of the identity $(eC_\ell) \in C_\infty$, which for $n \geq 0$ defines a neighborhood system for C_∞:

$$C_{n,\infty} = \{(g_\ell C_\ell) \in C_\infty \mid g_n \in C_n\}, \tag{23}$$

$$\cong \varprojlim \{\delta_\ell^{\ell+1} : C_n/C_{\ell+1} \to C_n/C_\ell \mid \ell \geq n\}. \tag{24}$$

6C. *The discriminant.* Observe that for each $\ell \geq 0$, the quotient group $D_\ell^x = G_\ell^x/C_\ell \subset G_0/C_\ell$. It follows that the inverse limit space

$$\mathcal{D}_x = \varprojlim \{\delta_\ell^{\ell+1} : D_{\ell+1}^x \to D_\ell^x\} \tag{25}$$

is a closed subgroup of C_∞. The group $\mathcal{D}_x$ is called the *discriminant group* of the action (V_0, G_0, Φ_0).

The relationship between C_∞ and the Ellis group of (V_0, G_0, Φ_0) is given by the following result.

Theorem 6.1 [Dyer et al. 2016, Theorem 4.4]. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor action, let $x \in V_0$, and let $\mathcal{G}^x \equiv \{G_\ell^x\}_{i \geq 0}$ be the associated group chain at x. Then there is a natural isomorphism of topological groups $\widehat{\Theta} : \overline{\Phi(G_0)} \cong C_\infty$ such that the restriction $\widehat{\Theta} : \overline{\Phi(G_0)}_x \cong \mathcal{D}_x$.*

Moreover, the discriminant subgroup is simple by the next result.

Proposition 6.2 [Dyer et al. 2016, Proposition 5.3]. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor system, $x \in V_0$ a basepoint, and $\overline{\Phi_0(G_0)}_x$ the isotropy group of x. Then*

$$\mathrm{core}_{G_0} \overline{\Phi_0(G_0)}_x = \bigcap_{k \in G_0} k \, \overline{\Phi_0(G_0)}_x \, k^{-1} \tag{26}$$

is the trivial group. Thus, the maximal normal subgroup of $\overline{\Phi_0(G_0)}_x$ in $\overline{\Phi_0(G_0)}$ is also trivial.

We next consider the homogeneity properties of a solenoid $\mathcal{S}_\mathcal{P}$ in terms of $\mathcal{D}_x$ (see [Dyer et al. 2016]). It follows from Proposition 6.2 that if $\mathcal{D}_x$ is nontrivial, then it is not normal in C_∞, and therefore the quotient $X_\infty^x = C_\infty/\mathcal{D}_x$ is not a group. We thus conclude:

Proposition 6.3 [Dyer et al. 2016]. *The action* (V_0, G_0, Φ_0) *is regular if and only if* $\mathcal{D}_x$ *is trivial.*

Note that Proposition 6.3 does not take into account the possibility that the action of a subgroup G^x_ℓ on a smaller section V_ℓ is regular. The general formulation is then as follows.

Corollary 6.4. *An equicontinuous matchbox manifold* $\mathfrak{M}$ *is homogeneous if and only if it admits a transverse section* V_0 *and a presentation* $\mathcal{P}$ *with associated group chain* $\{G^x_\ell\}_{\ell\geq 0}$ *such that the discriminant group* $\mathcal{D}_x$ *is trivial.*

7. Molino theory for weak solenoids

In this section, we obtain a Molino theory for weak solenoids, and hence for all equicontinuous matchbox manifolds, including those for which the hypotheses of [Álvarez López and Moreira Galicia 2016] are not satisfied. There are often subtle, and not so subtle, differences between the theory for matchbox manifolds and for smooth Riemannian foliations, as will be discussed further in the following sections.

7A. ***Molino overview.*** Molino theory for Riemannian foliations gives a structure theory for the geometry and dynamics of this class of foliations on compact smooth manifolds. The Séminaire Bourbaki article by Haefliger [1989] gives a concise overview of the theory and its applications, and Molino's book [1988] and its multiple appendices give a more detailed treatment of this theory and its applications. The book [Moerdijk and Mrčun 2003] is also an excellent reference about the essentials of Molino theory. We give a very brief summary below of some key properties of the Molino space $\widehat{M}$ associated to a smooth Riemannian foliation $\mathcal{F}$ of a compact connected manifold M.

Given a Riemannian foliation $\mathcal{F}$ of a compact connected manifold M, the associated *Molino space* $\widehat{M}$ is a compact connected manifold with a Riemannian foliation $\widehat{\mathcal{F}}$ whose leaves have the same dimension as those of $\mathcal{F}$. In the case where $\mathcal{F}$ is a minimal foliation, in the sense that each leaf of $\mathcal{F}$ is dense in M, then we can assume that the foliation $\widehat{\mathcal{F}}$ is also minimal.

Associated to a minimal Riemannian foliation $\mathcal{F}$ is the *structural Lie algebra* $\mathfrak{h}$, given by the algebra of holonomy-invariant vector fields normal to $\mathcal{F}$, and which is well defined up to isomorphism.

There is a fibration $\hat{\pi} : \widehat{M} \to M$ equipped with a fiber-preserving right action of a connected Lie group H whose Lie algebra is $\mathfrak{h}$, and for which the foliation $\widehat{\mathcal{F}}$ is invariant under the action of H. Moreover, for each leaf $\widehat{L} \subset \widehat{M}$, there is a leaf $L \subset M$ such that the restriction $\hat{\pi} : \widehat{L} \to L$ is the holonomy covering of L. We say

$$H \longrightarrow \widehat{M} \xrightarrow{\hat{\pi}} M \tag{27}$$

is a *Molino sequence* for M, and H is the structural Lie group for $\widehat{\mathcal{F}}$.

A key property of the Molino space $\widehat{M}$ of $\mathcal{F}$ is that it is *transversally parallelizable*, or TP. This condition states that there are nonvanishing vector fields $\{\vec{v}_1, \dots, \vec{v}_q\}$ on M which span the normal bundle to $\mathcal{F}$ at each $x \in M$, and the vector fields are locally projectable. As a consequence, given any pair of points $x, y \in \widehat{M}$ there exists a diffeomorphism $h : \widehat{M} \to \widehat{M}$ which maps leaves of $\widehat{\mathcal{F}}$ to leaves of $\widehat{\mathcal{F}}$, and satisfies $h(x) = y$. A foliation $\widehat{\mathcal{F}}$ satisfying this condition is said to be *foliated homogeneous*.

7B. ***Molino sequences for weak solenoids.*** For a matchbox manifold, the TP condition cannot be defined, as the transversal space to the foliation is totally disconnected. Thus, we need an alternative approach to defining the Molino fibration (27) in the case where the transversal space to the foliation is a Cantor set. The basic observation is that the foliated homogeneous condition for $\widehat{M}$ admits a natural generalization to all foliated spaces, as discussed for weak solenoids in Section 5. For weak solenoids, we will see below that the structural Lie group H is replaced by the discriminant subgroup $\mathcal{D}_x \subset C_\infty$ of Section 6C, and the foliated homogeneous condition is a consequence of the Ellis group construction. We now restate and prove Theorem 1.2.

Theorem 7.1. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and let $\mathcal{P}$ be a presentation of $\mathfrak{M}$, such that $\mathfrak{M}$ is homeomorphic to a solenoid $\mathcal{S}_{\mathcal{P}}$. Then there exists a homogeneous matchbox manifold $\widehat{\mathfrak{M}}$ with foliation $\widehat{\mathcal{F}}$, called a Molino space of $\mathfrak{M}$, a compact totally disconnected group $\mathcal{D}$, and a fibration*

$$\mathcal{D} \longrightarrow \widehat{\mathfrak{M}} \xrightarrow{\hat{q}} \mathfrak{M}, \tag{28}$$

where the restriction of $\hat{q}$ to each leaf in $\widehat{\mathfrak{M}}$ is a covering map of some leaf in $\mathfrak{M}$. We say that (28) is a ***Molino sequence*** *for $\mathfrak{M}$.*

Proof. Let $V_0 \subset \mathfrak{M}$ be a transverse section to the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$, as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let G_0 be the restricted holonomy group acting on V_0. Let $\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$ be a presentation at x such that there is a homeomorphism $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$, and for $x \in V_0$ let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$. Let $\Pi_0 : \mathcal{S}_{\mathcal{P}} \to M_0$ and set $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$. Let $\tau : V_0 \to \mathfrak{X}_0$ with $\tau_x(x) = (eG^x_\ell)$ be the homeomorphism defined in Lemma 4.1.

Recall that the covering map $q_\ell : M_\ell \to M_0$ defined in (5) is associated to the subgroup $G^x_\ell \subset G_0 = \pi_1(M_0, x_0)$. Recall that the core subgroup $C_\ell \subset G^x_\ell$ is the maximal normal subgroup of G_0 contained in G^x_ℓ, and has finite index in G^x_ℓ. For each $\ell > 0$, let $\hat{q}_\ell : \widehat{M}_\ell \to M_0$ be the proper covering space associated to the normal subgroup C_ℓ. Each inclusion $C_{\ell+1} \subset C_\ell$ induces a normal covering map $\hat{p}_{\ell+1} : \widehat{M}_{\ell+1} \to \widehat{M}_\ell$, and so yields a presentation $\widehat{\mathcal{P}} = \{\hat{p}_{\ell+1} : \widehat{M}_{\ell+1} \to \widehat{M}_\ell \mid \ell \geq 0\}$.

Definition 7.2. The *Molino space* associated to a weak solenoid $\mathcal{S}_{\mathcal{P}}$ defined by a presentation $\mathcal{P}$ is the inverse limit space associated to the presentation $\widehat{\mathcal{P}}$,

$$\widehat{\mathcal{S}}_{\mathcal{P}} \equiv \varprojlim \{\hat{p}_{\ell+1} : \widehat{M}_{\ell+1} \to \widehat{M}_{\ell}\}. \tag{29}$$

Let $\widehat{\Pi}_0 : \widehat{\mathcal{S}}_{\mathcal{P}} \to M_0$ be the projection map, with fiber $\widehat{\mathfrak{X}}_0 = \widehat{\Pi}_0^{-1}(x_0)$.

We state some of the basic properties of the space $\widehat{\mathcal{S}}_{\mathcal{P}}$. The proofs of the following statements are omitted, as they follow by arguments analogous to the corresponding statements for $\mathcal{S}_{\mathcal{P}}$.

Proposition 7.3. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid defined by a presentation $\mathcal{P}$, and let $\widehat{\mathcal{S}}_{\mathcal{P}}$ be the solenoid defined by (29). Then*

(1) *there is a natural isomorphism $\widehat{\mathfrak{X}}_0 \cong C_\infty$, where C_∞ is the profinite group defined by (22);*

(2) *there is a natural map of fibrations $\hat{q} : \widehat{\mathcal{S}}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, whose fiber over $x \in \mathfrak{X}_0$ is the discriminant group $\mathcal{D}_x$;*

(3) *the global holonomy of the fibration $\widehat{\Pi}_0 : \widehat{\mathcal{S}}_{\mathcal{P}} \to M_0$ is naturally conjugate as G_0-actions with the minimal Cantor system $(C_\infty, G_0, \widehat{\Phi}_0)$.*

Definition 7.4. The *Molino sequence* for the weak solenoid $\mathcal{S}_{\mathcal{P}}$ is the principal fibration

$$\mathcal{D}_x \longrightarrow \widehat{\mathcal{S}}_{\mathcal{P}} \xrightarrow{\hat{q}} \mathcal{S}_{\mathcal{P}}. \tag{30}$$

Proposition 7.3(3) implies that the foliation $\widehat{\mathcal{F}}_{\mathcal{P}}$ on $\widehat{\mathcal{S}}_{\mathcal{P}}$ is minimal, and the restrictions of $\hat{q}$ to the leaves of $\widehat{\mathcal{F}}_{\mathcal{P}}$ are covering maps by construction, as there is a covering map $\widehat{M}_\ell \to M_\ell$ for each $\ell \geq 1$ which induces $\hat{q}$. Finally, the space $\widehat{\mathcal{S}}_{\mathcal{P}}$ is homogeneous by Proposition 5.5, as it is defined using the normal group chain $\{C_\ell\}_{\ell\geq 0}$.

Set $\widehat{\mathfrak{M}} = \widehat{\mathcal{S}}_{\mathcal{P}}$ and $\mathcal{D} = \mathcal{D}_x$. Then we have established Theorem 7.1. □

The construction of the sequence in (30) may depend on the various choices made, and this is a fundamental aspect of the Molino theory for weak solenoids. We consider in Section 7C the dependence of the discriminant group on the partition sets $V_n \subset V_0$. Then in Section 8, we consider the dependence of the sequence (30) on the choice of the basepoint $x \in V_0$ and the role of the holonomy of the leaf L_x in the properties of $\mathcal{D}_x$.

7C. *Stability of the Molino sequence.* We next consider the stability of the discriminant group for an equicontinuous Cantor minimal system (V_0, G_0, Φ_0) when one restricts to a section $V_n \subset V_0$.

We start with an example that highlights the importance of the "asymptotic algebraic structure" of the group chain $\mathcal{G}^x$ for the definition of the Molino space. Consider a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell\geq 0}$ defined by the holonomy action (V_0, G_0, Φ_0) for a clopen subset $V_0 \subset \mathfrak{X}_0$, and suppose that $\mathcal{G}^x$ is not regular. Then by Proposition 6.3, the discriminant group $\mathcal{D}_x$ is nontrivial, and thus the sequence (30) has nontrivial fiber. Now suppose that, in addition, the group chain $\mathcal{G}^x$ is weakly normal. Then by Theorem 5.4, there exists some $n > 0$ such that the restricted action (V_n, G_n, Φ_n) is regular; hence the discriminant group $\mathcal{D}_x^n$ for the truncated chain $\mathcal{G}_n^x = \{G_\ell^x\}_{\ell \geq n}$ associated to the restricted action is trivial. For the truncated presentation $\mathcal{P}_n$ defined by (9), we have $\widehat{\mathcal{S}}_{\mathcal{P}_n} = \mathcal{S}_{\mathcal{P}_n}$ as $\mathcal{D}_x^n$ is the trivial group, and $\mathcal{S}_{\mathcal{P}_n} \cong \mathcal{S}_{\mathcal{P}}$ as remarked in Section 3C; hence we can consider $\widehat{\mathcal{S}}_{\mathcal{P}_n}$ as a Molino space for $\mathcal{S}_{\mathcal{P}}$ as well. That is, for this choice of V_n as a section, the Molino sequence (30) has trivial fiber.

We next develop a comparison, for $n \geq 0$, of the discriminant groups $\mathcal{D}_x^n$ for the group chain $\mathcal{G}_n^x$ associated to the truncated presentation $\mathcal{P}_n$ defined by (9). We work with the group chain model $(X_\infty^x, G_0, \Phi_x)$ of Lemma 4.1 for the holonomy action $\Phi_0 : G_0 \to \mathrm{Homeo}(\mathfrak{X}_0)$. By definition (25) of the discriminant group, it suffices to consider this invariant in sufficiently small clopen neighborhoods of the identity in the core group associated with the group chains. For $n \geq 0$, we have the clopen neighborhoods of $\{e\} \in X_\infty$:

$$U_n = \{(g_\ell G_\ell) \in X_\infty \mid g_n \in G_n^x\} \subset X_\infty \tag{31}$$

$$\cong \varprojlim\{\delta_\ell^{\ell+1} : G_n^x / G_{\ell+1}^n \to G_n^x / G_\ell \mid \ell \geq n\}. \tag{32}$$

Note that U_n is just the inverse limit group defined by the truncated group chain $\mathcal{G}_n^x$. Next, we introduce the core groups of $\mathcal{G}_n^x$ for arbitrarily small neighborhoods of $\{e\} \in U_n$. For $\ell \geq n \geq 0$, set

$$E_{n,\ell} \equiv \mathrm{core}_{G_n^x}\, G_\ell^x \equiv \bigcap_{g \in G_n^x} g\, G_\ell^x g^{-1}. \tag{33}$$

Note that $E_{0,\ell} = C_\ell$, and that for all $m \geq n \geq 0$ and $\ell > m$, we have $E_{n,\ell} \subset E_{m,\ell} \subset G_\ell^x$.

For $k \geq n \geq 0$, define the clopen neighborhood $V_{n,k}$ of $\{e\}$ for the core group of $\mathcal{G}_n^x$ by

$$V_{n,k} = \{(g_\ell E_{n,\ell}) \mid \ell \geq k,\ g_k \in G_k^x,\ g_{\ell+1} E_{n,\ell} = g_\ell E_{n,\ell}\} \tag{34}$$

$$\cong \varprojlim\{\delta_\ell^{\ell+1} : G_k^x / E_{n,\ell} \to G_k^x / E_{n,\ell+1} \mid \ell \geq k\}. \tag{35}$$

Then $V_{n,n}$ is the core limit group, or the Ellis group, for the truncated group chain $\mathcal{G}_n^x$, and $\{e\} \in V_{n,k} \subset V_{n,n}$ for all $k \geq n$. Note also that $V_{0,0} = C_\infty$ is the Ellis group for $\mathcal{G}^x$.

For each $\ell \geq k \geq m \geq n$, the inclusions $E_{n,\ell} \subset E_{m,\ell}$ induce group surjections

$$G_k^x / E_{n,\ell} \xrightarrow{\varphi_{k,n,m}^\ell} G_k^x / E_{m,\ell}, \tag{36}$$

so we obtain surjective homomorphisms of profinite groups $\varphi_{n,m} : V_{n,k} \to V_{m,k}$ for each $m > n \geq 0$. In particular, for $k = m$, this states that the clopen neighborhood $V_{n,m}$ of $\{e\}$ in the limit core group for $\mathcal{G}^x_n$ maps onto the limit core group $V_{m,m}$ of $\mathcal{G}^x_m$.

We consider next the discriminant groups associated to the group chains $\mathcal{G}^x_n$ for $n \geq 0$, $\mathcal{D}^n_x \subset V_{n,n}$:

$$\mathcal{D}^n_x = \varprojlim\{\delta^{\ell+1}_{\ell} : G^x_{\ell+1}/E_{n,\ell+1} \to G^x_{\ell}/E_{n,\ell} \mid \ell \geq n\} \tag{37}$$

$$\cong \varprojlim\{\delta^{\ell+1}_{\ell} : G^x_{\ell+1}/E_{n,\ell+1} \to G^x_{\ell}/E_{n,\ell} \mid \ell \geq m\} \quad \text{for } m \geq n. \tag{38}$$

It follows from (36) and (38) that for $m > n$, there are surjective homomorphisms:

$$\mathcal{D}_x \xrightarrow{\psi_{0,n}} \mathcal{D}^n_x \xrightarrow{\psi_{n,m}} \mathcal{D}^m_x. \tag{39}$$

Definition 7.5. A group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ is said to be *stable* if there exists $n_0 \geq 0$ such that the maps $\psi_{n,m} : \mathcal{D}^n_x \to \mathcal{D}^m_x$ defined in (39) are isomorphisms for all $m \geq n \geq n_0$. Otherwise, the group chain is said to be *wild*.

Theorem 5.4 implies that if the group chain $\{G^x_\ell\}_{\ell \geq 0}$ is weakly normal, then it is stable, as there exists some $n_0 \geq 0$ such that $\mathcal{D}^n_x$ is the trivial group for all $n \geq n_0$. This discussion and Lemma 7.6 yield Proposition 1.4 of the Introduction.

Lemma 7.6. *If the discriminant group $\mathcal{D}_x$ for $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ is finite, then $\mathcal{G}^x$ is stable.*

Proof. The map $\psi_{0,n} : \mathcal{D}_x \to \mathcal{D}^n_x$ is surjective for all $n \geq 0$, so the assumption that the cardinality $\#\mathcal{D}_x$ is finite implies that the cardinality $\#\mathcal{D}^n_x$ of the group $\mathcal{D}^n_x$ is decreasing with n, and thus there exists $n_0 \geq 0$ such that the cardinality of its image must stabilize for $n \geq n_0$. Then for $n \geq n_0$, the homomorphism $\psi_{n,m} : \mathcal{D}^{n_0}_x \to \mathcal{D}^n_x$ is an isomorphism. □

7D. ***Stable matchbox manifolds.*** We next consider the relationship between the notion of stable for a matchbox manifold as given in Definition 1.3, and stable for a group chain as given in Definition 7.5.

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Let G_0 be the group of transformations of V_0 which induces the restricted holonomy group acting on V_0, and let $G^x_\ell \subset G_0$ be the stabilizer group of the set V_ℓ. Let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$, let $\mathcal{P}_n$ be the presentation (9) associated to the truncated group chain $\mathcal{G}^x_n = \{G^x_\ell\}_{\ell \geq n}$, and let $\mathcal{S}_{\mathcal{P}_n}$ be the inverse limit solenoid. For each $n \geq 0$, let $\widehat{\mathcal{S}}_{\mathcal{P}_n}$ be the homogeneous solenoid associated to the normal group chain $\{E_{n,\ell}\}_{\ell \geq n}$ defined by (33).

Assume that the group chain $\mathcal{G}^x$ is stable in the sense of Definition 7.5. That is, there exists an index n_0 such that for any $m > n \geq n_0$ restricting to the smaller sections $V_m \subset V_n \subset V_0$ with induced presentations $\mathcal{P}_m$ and $\mathcal{P}_n$, then the induced map $\psi_{n,m} : \mathcal{D}_x^n \to \mathcal{D}_x^m$ in (39) is a topological isomorphism. Then we have a commutative diagram of fibrations:

$$
\begin{tikzcd}
\mathcal{D}_x^n \arrow[r, "\psi_{n,m}"] \arrow[d] & \mathcal{D}_x^m \arrow[d] \\
\widehat{\mathcal{S}}_{\mathcal{P}_n} \arrow[r, "\hat{\sigma}_{m-n}"] \arrow[d] & \widehat{\mathcal{S}}_{\mathcal{P}_m} \arrow[d] \\
\mathcal{S}_{\mathcal{P}_n} \arrow[r, "\sigma_{m-n}"] & \mathcal{S}_{\mathcal{P}_m}
\end{tikzcd}
\tag{40}
$$

By the discussion in Section 3C, the shift map σ_{m-n} is a homeomorphism, and by assumption, the map $\psi_{n,m} : \mathcal{D}_n \cong \mathcal{D}_m$ is a topological isomorphism. Thus the map $\hat{\sigma}_{m-n} : \widehat{\mathcal{S}}_{\mathcal{P}_n} \to \widehat{\mathcal{S}}_{\mathcal{P}_m}$ is a homeomorphism. Hence, the Molino sequences for the presentations $\mathcal{P}_n$ and $\mathcal{P}_m$ yield isomorphic topological fibrations. Conversely, if the topological type of the Molino sequence

$$\mathcal{D}_x^n \longrightarrow \widehat{\mathcal{S}}_{\mathcal{P}_n} \longrightarrow \mathcal{S}_{\mathcal{P}_n} \tag{41}$$

is well defined up to homeomorphism of fibrations, for given V_0 and $n \geq 0$ sufficiently large, then there exists $n_0 \geq 0$ such that $m > n \geq n_0$ implies that $\mathcal{D}_x^n \xrightarrow{\psi_{n,m}} \mathcal{D}_x^m$ is a topological isomorphism. Thus, the map of fibers $\psi_{n,m} : \mathcal{D}_x^n \to \mathcal{D}_x^m$ is a topological isomorphism, and hence $\mathcal{G}^x$ is stable.

The following statement summarizes these conclusions.

Theorem 7.7. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Let G_0 be the restricted holonomy group acting on V_0, and let $G_\ell^x \subset G_0$ be the stabilizer group of the set V_ℓ. Let $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$, let $\mathcal{P}_n$ be the presentation (9) associated to the truncated group chain $\mathcal{G}_n^x = \{G_\ell^x\}_{\ell \geq n}$, and let $\mathcal{S}_{\mathcal{P}_n}$ be the inverse limit solenoid. For each $n \geq 0$, let $\widehat{\mathcal{S}}_{\mathcal{P}_n}$ be the homogeneous solenoid associated to the normal group chain $\{E_\ell^n\}_{\ell \geq n}$ defined by (33).*

(1) *If $\mathcal{G}^x$ is stable, then there exists $n_0 \geq 0$ such that for all $n \geq n_0$ the fibration (41) is a Molino sequence for $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}_n}$, and the fiber group $\mathcal{D}_x^n$ is well defined up to topological isomorphism.*

(2) *If $\mathcal{G}^x$ is wild, then the topological isomorphism type of the fiber in the sequence (41) does not stabilize as n tends to infinity.*

Theorem 7.7 implies that the Molino sequence of a matchbox manifold $\mathfrak{M}$ need not be well defined, though if the associated group chain $\mathcal{G}^x$ is stable, then $\mathfrak{M}$ does have a well-defined Molino sequence.

8. Germinal holonomy in solenoids

In this section, we investigate the relationship between the germinal holonomy groups of leaves in a solenoid, the kernels of the associated group chains, and the discriminant group of the action.

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold with transverse section V_0, let $x \in V_0$ be a point, and let $\mathcal{P} = \{f_i^{\ell+1} : M_{\ell+1} \to M_\ell\}$ be a presentation with associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ in $G_0 = \pi_1(M_0, x_0)$. Then by Theorem 1.1, there is a foliated homeomorphism $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$.

Let $C_\infty = \varprojlim\{G_0/C_{\ell+1} \to G_0/C_\ell\}$, where C_ℓ is the maximal normal subgroup of G_ℓ^x, $\ell \geq 0$, and let $\mathcal{D}_x$ be the discriminant group at x. Denote by $L_x \subset \mathcal{S}_{\mathcal{P}}$ the leaf of $\mathcal{F}_{\mathcal{P}}$ through x. Recall that the kernel of $\mathcal{G}^x$ is the subgroup $K(\mathcal{G}^x) \subset G_0$ as defined in Definition 4.7, and is the isotropy subgroup of the action (V_0, G_0, Φ_0) at x.

8A. ***Locally trivial germinal holonomy.*** The following properties of pseudogroup actions are basic for understanding their dynamical properties.

Definition 8.1. Given $g_1, g_2 \in K(\mathcal{G}^x)$, we say g_1 and g_2 have the same *germinal holonomy* at x if there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that the restrictions $\Phi_0(g_1)|U_x$ and $\Phi_0(g_2)|U_x$ agree on U_x. In particular, we say that $g \in K(\mathcal{G}^x)$ has *trivial germinal holonomy* at x if there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that the restriction $\Phi_0(g)|U_x$ is the trivial map.

By straightforward checking of definitions, one can see that the notion "germinal holonomy at x" defines an equivalence relation on the image of the isotropy subgroup $K(\mathcal{G}^x)$ under the global holonomy map $\Phi_0 : G_0 \to \mathrm{Homeo}(V_0)$. Denote by $\mathrm{Germ}(\Phi_0, x)$ the quotient of $\Phi_0(K(\mathcal{G}^x))$ by this equivalence relation. Thus the composition of $\Phi_0 : K(\mathcal{G}^x) \to \mathrm{Homeo}(V_0)$ with the quotient map gives us a surjective map $K(\mathcal{G}^x) \to \mathrm{Germ}(\Phi_0, x)$. A standard argument shows that if $\mathrm{Germ}(\Phi_0, x)$ is trivial, and y is in the same G_0-orbit of x, then $\mathrm{Germ}(\Phi_0, y)$ is trivial. This leads to the following definition.

Definition 8.2. We say that a leaf L_x is *without holonomy*, or that L_x has *trivial holonomy*, if $\mathrm{Germ}(\Phi_0, x)$ is trivial. We say that $\mathrm{Germ}(\Phi_0, x)$ is *locally trivial* if there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that for *every* $g \in K(\mathcal{G}_x)$ the restriction $\Phi_0(g)|U_x$ is the trivial map.

The distinction between the holonomy group $\mathrm{Germ}(\Phi_0, x)$ being trivial and it being locally trivial may seem technical, but this distinction is related to fundamental dynamical properties of the foliation $\mathcal{F}_{\mathcal{P}}$ of $\mathcal{S}_{\mathcal{P}}$. For example, it is a key concept in the generalizations of the Reeb stability theorem from compact leaves to the noncompact case for codimension-one foliations, as discussed in [Sacksteder and Schwartz 1965; Inaba 1977; 1983]. The nomenclature "locally trivial" was introduced by Inaba [1977; 1983]. As we see below, this distinction is also important for the study of the dynamics of weak solenoids. First, we make an elementary observation, which implies Lemma 1.5 of the Introduction.

Lemma 8.3. *Suppose that* $K(\mathcal{G}^x)$ *is finitely generated. If* $\mathrm{Germ}(\Phi_0, x)$ *is trivial, then* $\mathrm{Germ}(\Phi_0, x)$ *is locally trivial.*

Proof. Let $\{g_1, \dots, g_k\} \subset K(\mathcal{G}^x)$ be a set of generators. Then $\mathrm{Germ}(\Phi_0, x)$ being trivial implies that for each $1 \leq i \leq k$ there exists an open $U_i \subset V_0$ with $x \in U_i$ such that the restriction $\Phi_0(g_i)|U_i$ is the trivial map. Then let $U_x = U_1 \cap \cdots \cap U_k$, which is an open neighborhood of x, and the restriction $\Phi_0(g)|U_x$ is then trivial for all $g \in K(\mathcal{G}^x)$. □

We also recall a basic result, which is a version of the fundamental result of Epstein, Millett and Tischler [Epstein et al. 1977] in the language of group actions on Cantor sets.

Theorem 8.4. *Let* (V_0, G_0, Φ_0) *be a given action, and suppose that* V_0 *is a Baire space. Then the union of all* $x \in V_0$ *such that* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group forms a* G_δ *subset of* V_0*. In particular, there exists at least one* $x \in V_0$ *such that* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group.*

The following is an immediate consequence of this result and Definition 5.1.

Corollary 8.5. *Let* (V_0, G_0, Φ_0) *be a* ***regular*** *equicontinuous minimal Cantor system. Then* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group for all* $x \in V_0$*. Consequently, if* $\mathfrak{M}$ *is a homogeneous matchbox manifold, then all leaves of* $\mathcal{F}_{\mathfrak{M}}$ *are without germinal holonomy.*

8B. ***Algebraic conditions.*** Next, we explore the relation between the structure of a group chain $\mathcal{G}^x$ and the germinal holonomy group at x. First, note that for a given section V_0 and the holonomy action (V_0, G_0, Φ_0), the assumption that the germinal holonomy group $\mathrm{Germ}(\Phi_0, x)$ is trivial need not imply that $K(\mathcal{G}^x)$ is trivial, or even that it is a normal subgroup of G_0, as the following example shows.

Example 8.6. Let Γ be a finitely presented group and $\{\Gamma_\ell\}_{\ell \geq 0}$ be a chain of normal subgroups in Γ with kernel $\Gamma_x = \bigcap_\ell \Gamma_\ell$. Let H be a finite simple group, and let $K \subset H$ be a nontrivial subgroup. Since H is simple, K is not normal in H.

Let $G_0 = H \times \Gamma$ and $G_\ell = K \times \Gamma_\ell$, $\ell \geq 0$. Note that G_ℓ is a normal subgroup of $G_1 = K \times \Gamma_1$ for all $\ell \geq 1$, but G_ℓ is not normal in G_0. Thus, the group chain $\{G_\ell\}_{\ell \geq 0}$ is weakly normal. Let M_0 be a compact connected manifold without boundary such that $\pi_1(M_0, x_0) = G_0$, where $x_0 \in M_0$ is some basepoint. Then the group chain $\mathcal{G}^x = \{G_\ell\}_{\ell \geq 0}$ yields a presentation $\mathcal{P} = \{f_\ell^{\ell+1} : M_{\ell+1} \to M_\ell\}$, and the corresponding solenoid $\mathcal{S}_{\mathcal{P}}$ is homogeneous by Proposition 5.5.

By Theorem 8.4, $\mathcal{S}_{\mathcal{P}}$ has a leaf L_y without holonomy. By Remark 4.2, a group chain with basepoint y is given by $\mathcal{G}^y = \{g_i G_i g_i^{-1}\}_{i \geq 0}$, where $g_i = (c_i, \gamma_i)$. Since the projection $G_0/G_{\ell+1} \to G_0/G_\ell$ restricts to the identity map on the factor H/K, for all $\ell \geq 0$, one can write $g_i = (c, \gamma_i)$ for some $c \in H$. Since each Γ_ℓ is a normal subgroup, we have that $g_i G_i g_i^{-1} = cKc^{-1} \times \Gamma_i$. Thus, $K(\mathcal{G}^y) = cKc^{-1} \times \Gamma_x$ is not a normal subgroup of G_0, since H is simple.

Next, we consider the holonomy action of the elements in $K(\mathcal{G}^x)$ on V_0 in more detail, using the inverse limit model $\tau_x : V_0 \cong X_\infty^x = \{G_0/G_{\ell+1}^x \to G_0/G_\ell^x\}$. For each $n \geq 0$, set

$$U(x,n) = \{(g_\ell G_\ell^x) \in X_\infty^x \mid g_\ell = e \text{ if } \ell \leq n;\ g_\ell G_\ell^x = g_{\ell+1} G_\ell^x \text{ for all } \ell \geq n\}, \tag{42}$$

which is a "cylinder neighborhood" of $(eG_\ell^x) \in V_0$. Note that $\tau_x(V_n) = U(x,n)$ for $n \geq 0$, where V_n is a generating set in the partition introduced in Proposition 3.4.

Since $K(\mathcal{G}^x)$ is a subgroup of G_0, for each $n \geq 1$ one can consider its left action on the cosets in G_0/G_n^x. Such an action fixes the coset eG_n^x; thus the action of $g \in K(\mathcal{G}_n^x)$ fixes the neighborhood of the identity as a set, $\Phi_0(g) : U(x,n) \to U(x,n)$ for $g \in G_n^x$, and permutes the points in $U(x,n)$.

Now observe that the action of g has trivial germinal holonomy at x if for some $n_g > 0$, g acts trivially on the clopen neighborhood $U(x, n_g)$ of x; that is, $\Phi_0(g)|U(x,n_g)$ is the trivial map. The following algebraic characterization of elements without holonomy was obtained in [Dyer et al. 2017, Lemma 5.3].

Lemma 8.7. *The action of $g \in K(\mathcal{G}^x)$ has* ***trivial germinal holonomy*** *at x if and only if there exists some index $i_g \geq 0$ such that multiplication by g satisfies $g \cdot hK(\mathcal{G}^x) = hK(\mathcal{G}^x)$ for all $h \in G_{i_g}$. That is, $h^{-1}gh \in K(\mathcal{G}^x)$ for all $h \in G_{i_g}$.*

In the case where the kernel $K(\mathcal{G}^x)$ is finitely generated, we have the following consequence of Lemma 8.7, whose proof can be compared with that of Lemma 8.3.

Proposition 8.8. *Let $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ be a group chain, and suppose the kernel $K(\mathcal{G}^x)$ is finitely generated. Suppose that* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group. Then there is an index $\ell_x \geq 0$ such that $K(\mathcal{G}^x)$ is a normal subgroup of $G_{\ell_x}^x$.*

Proof. Let $\{g_1, \ldots, g_k\} \subset K(\mathcal{G}^x)$ be a set of generators. Then for each $1 \leq \ell \leq k$, there exists $i_\ell \geq 0$ such that $h^{-1}gh \in K(\mathcal{G}^x)$ for all $h \in G_{i_\ell}$. Let $\ell_x = \max\{i_1, \ldots, i_k\}$. Then this implies that $h^{-1}gh \in K(\mathcal{G}^x)$ for all $g \in K(\mathcal{G}^x)$ and $h \in G_{\ell_x}$; that is, $K(\mathcal{G}^x)$ is a normal subgroup of $G_{\ell_x}^x$. □

Remark 8.9. The condition that the kernel $K(\mathcal{G}^x)$ of the group chain $\mathcal{G}^x$ is finitely generated is essential. Example 9.7 gives a group chain whose kernel at x is infinitely generated, and the germinal holonomy group $\mathrm{Germ}(\Phi_0, x)$ is not locally trivial.

Proposition 8.8 implies the following result, which is an algebraic analog of Reeb stability.

Proposition 8.10. *Let (V_0, G_0, Φ_0) be a minimal equicontinuous Cantor group action. Let $x, y \in V_0$ be such that both germinal holonomy groups $\mathrm{Germ}(\Phi_0, x)$ and $\mathrm{Germ}(\Phi_0, y)$ are* ***locally trivial****. Then for associated group chains $\mathcal{G}^x$ and $\mathcal{G}^y$, the kernels $K(\mathcal{G}^x)$ and $K(\mathcal{G}^y)$ are conjugate subgroups of G_0.*

Proof. Let $\mathcal{G}^x$ and $\mathcal{G}^y$ be group chains at x and y, respectively, for the action (V_0, G_0, Φ_0). Let $\tau_x : \mathfrak{X}_0 \to X^x_\infty$ and $\tau_y : \mathfrak{X}_0 \to X^y_\infty$ be the corresponding homeomorphisms defined in Lemma 4.1, each of which is equivariant with respect to the action (7) of G_0.

By the assumption that $\mathrm{Germ}(\Phi_0, x)$ is locally trivial, there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that for every $g \in K(\mathcal{G}^x)$ the restriction $\Phi_0(g)|U_x$ is the trivial map. As the image $\tau_x(U_x) \subset X^x_\infty$ is open and contains $(eG^x_i) = \tau_x(x)$, there exists an index $\ell_x > 0$ such that $U((eG^x_i), \ell_x) \subset \tau_x(U_x)$, where $U((eG^x_i), \ell_x)$ is defined in (42). Note that $G^x_{\ell_x}$ is the stabilizer of $U((eG^x_i), \ell_x)$ for the action of G_0. Then $K(\mathcal{G}^x)$ acts trivially on $U((eG^x_i), \ell_x)$, so $K(\mathcal{G}^x)$ is a normal subgroup of $G^x_{\ell_x}$ by Lemma 8.7.

Set $V_1 = \tau_x^{-1}(U((eG^x_\ell), \ell_x)) \subset U_x$ and let $z \in V_1$ with $z \neq x$. Then the image $\tau_x(z)$ is $(h_i G^x_i)$, where $h_i \in G^x_{\ell_x}$ for $i \geq \ell_x$ and $h_i = e$ for $i \leq \ell_x$. As usual, the sequence (h_i) also satisfies the compatibility condition $h_i G^x_i = h_j G^x_i$ for all $i \geq 0$ and $j > i$. By Remark 4.2, we have that $\mathcal{G}^z = \{h_i G^x_i h_i^{-1}\}_{i \geq 0}$.

Note that $h_i K(\mathcal{G}^x) h_i^{-1} = K(\mathcal{G}^x)$ for $i \geq 0$, since $K(\mathcal{G}^x)$ is normal in $G^x_{\ell_x}$, so we have

$$K(\mathcal{G}^x) = \bigcap_{i \geq 0} G^x_i = \bigcap_{i \geq 0} h_i K(\mathcal{G}^x) h_i^{-1} \subseteq \bigcap_{i \geq 0} h_i \, G^x_i \, h_i^{-1} = K(\mathcal{G}^z). \tag{43}$$

In general, this inclusion may be proper, as illustrated in Example 9.6.

Now assume that $\mathrm{Germ}(\Phi_0, z)$ is locally trivial. We show that $K(\mathcal{G}^z) \subseteq K(\mathcal{G}^x)$. First, note that there exists an open set $U_z \subset V_0$ with $z \in U_z$ such that for every $g \in K(\mathcal{G}^z)$ the restriction $\Phi_0(g)|U_z$ is the trivial map. Recall that $\tau_x(z) = (h_i G^x_i) \in U((eG^x_\ell), \ell_x)$. Then there exists $\ell_z \geq \ell_x$ such that

$$U((h_i G^x_i), \ell_z) = \{(g_i G^x_i) \in X^x_\infty \mid g_i = h_i \text{ for } i \leq \ell\} \subset \tau_x(U_z). \tag{44}$$

That is, $g \in K(\mathcal{G}^z)$ acts trivially on the cylinder set $U((h_i G^x_i), \ell_z)$ in X^x_∞. Let $h = h_{\ell_z} \in G^x_{\ell_x}$, so we obtain an element $(hG^x_i) \in X^x_\infty$. By choice of h and (44) we have $(hG^x_i) \in U((h_i G^x_i), \ell_z)$. Now let $g \in K(\mathcal{G}^z)$. Then the restricted map $\Phi_0(g)|U_z$ is the identity, so we have $g \cdot (hG^x_i) = (hG^x_i)$. But this means that $h^{-1} g h G^x_i = G^x_i$

for all $i \geq 0$, and thus $h^{-1}gh \in K(\mathcal{G}^x)$, or $g \in hK(\mathcal{G}^x)h^{-1}$. Since $h \in G^x_{\ell_z} \subset G^x_{\ell_x}$, and $K(\mathcal{G}^x)$ is a normal subgroup of $G^x_{\ell_x}$, this implies that $K(\mathcal{G}^z) \subseteq K(\mathcal{G}^x)$.

Now suppose that $y \in V_0$ is such that $\mathrm{Germ}(\Phi_0, y)$ is locally trivial. The action of G_0 on V_0 is assumed to be minimal, so there exists $g \in G_0$ such that $z = \Phi_0(g)(y) \in V_1$. Then the holonomy at z is also locally trivial, so $K(\mathcal{G}^z) = K(\mathcal{G}^x)$ by the argument above. On the other hand, we have $K(\mathcal{G}^y) = g^{-1}K(\mathcal{G}^z)g$ as $K(\mathcal{G}^y)$ is the isotropy subgroup of y. The claim of the proposition then follows. □

8C. *Kernels and discriminants.* We give two results concerning the relation between the kernel of a group chain and its discriminant.

Proposition 8.11. *Let* (V_0, G_0, Φ_0) *be an equicontinuous minimal Cantor system, $x \in V_0$ be a choice of basepoint, and $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be a group chain associated to (V_0, G_0, Φ_0) at x. Let $\mathcal{L}_0 = \mathrm{Ker}(\Phi_0)$ denote the kernel of $\Phi_0 : G_0 \to \mathrm{Homeo}(V_0)$. Then $K(\mathcal{G}^x) \subset \mathcal{L}_0$ if and only if the intersection $\Phi_0(G_0) \cap \overline{\Phi_0(G_0)}_x$ is the trivial group.*

Proof. By Theorem 6.1, we can identify $\overline{\Phi_0(G_0)} \cong C_\infty$ and $\overline{\Phi_0(G_0)}_x \cong \mathcal{D}_x$, where the image $\Phi_0(G_0)$ is identified with the elements $(g_\ell C_\ell) \in C_\infty$ such that $g_\ell C_\ell = gC_\ell$ for all $\ell \geq 0$, for some $g \in G_0$.

First, suppose that $g \in G_0$ satisfies $\Phi(g) \in \overline{\Phi_0(G_0)}_x$ and $\Phi(g)$ is not the trivial element. Then $\hat{g} = (gC_\ell) \in \mathcal{D}_x$ and $(gC_\ell) \neq (eC_\ell)$, so there exists $\ell_0 > 0$ such that $g \notin C_{\ell_0}$. By the definition of $\mathcal{D}_x$ in (25), we have that $\hat{g}$ is in the image of the map $\delta^{\ell+1}_\ell : D^x_{\ell+1} \to D^x_\ell$ for all $\ell > 0$ where $D^x_\ell = G^x_\ell / C_\ell$. This implies that $gC_\ell \subset G^x_\ell$, and hence $g \in G^x_\ell$ for all $\ell \geq 0$, and so $g \in K(\mathcal{G}^x)$. We claim that $\Phi_0(g)$ is not the trivial action, so that $g \notin \mathcal{L}_0$. It is given that $g \notin C_{\ell_0}$; hence $gC_{\ell_0} \neq C_{\ell_0}$. Then for all $\ell \geq \ell_0$, we have $gC_\ell \neq C_\ell$, so $g \cdot (eC_\ell) \neq (eC_\ell)$, which implies $g \notin \mathcal{L}_0$. It follows that $K(\mathcal{G}^x) \not\subset \mathcal{L}_0$, as was to be shown.

Conversely, let $g \in K(\mathcal{G}^x)$ and suppose that $g \notin \mathcal{L}_0$. First note that $g \in G^x_\ell$ for all $\ell \geq 0$, and so we have $\hat{g} = (gC_\ell) \in \mathcal{D}_x$. The assumption that $g \notin \mathcal{L}_0$ implies there exists some $(h_\ell C_\ell) \in C_\infty$ such that $g \cdot (h_\ell C_\ell) \neq (h_\ell C_\ell)$. Thus, there exists $\ell_0 > 0$ such that for all $\ell \geq \ell_0$ we have $gh_\ell C_\ell \neq h_\ell C_\ell$, which implies that $h_\ell^{-1} g h_\ell \notin C_\ell$ and so $g \notin C_\ell$ as C_ℓ is a normal subgroup of G_0. Thus, $(eC_\ell) \neq (gC_\ell)$ for all $\ell \geq \ell_0$, and so $(gC_\ell) \in \mathcal{D}_x$ is nontrivial. That is, $\Phi(g) \in \overline{\Phi_0(G_0)}_x$ is a nontrivial element, as was to be shown. □

Compare the following application of Proposition 8.11 with the conclusions of Theorem 7.7.

Proposition 8.12. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Let G_0 be the restricted holonomy group acting on V_0, and let $G^x_\ell \subset G_0$ be the* *stabilizer group of the set V_ℓ. Let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$, and let $\mathcal{G}^x_n = \{G^x_\ell\}_{\ell \geq n}$ be the associated truncated group chain. Assume that the leaf L_x containing x has nontrivial germinal holonomy. Then the discriminant $\mathcal{D}^x_n$ for the chain $\mathcal{G}^x_n$ is nontrivial for all $n \geq 0$.*

Proof. Let $n \geq 0$, and let $\mathcal{L}_n \subset G^x_n$ be the kernel of the restricted action $\Phi_n : G^x_n \to \mathrm{Homeo}(V_n)$. Then by Lemma 8.7, the kernel $K(\mathcal{G}^x_n) \subset G^x_n$ is not a normal subgroup, so $\mathcal{L}_n \subset K(\mathcal{G}^x_n)$ is a proper inclusion. Then by Proposition 8.11, the discriminant group $\mathcal{G}^x_n$ is nontrivial also. □

This yields the proof of Theorem 1.7 of the Introduction, which we restate now.

Theorem 8.13. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. If there exists a leaf with nontrivial holonomy for $\mathcal{F}_{\mathfrak{M}}$, then for any choice of transversal $V_0 \subset \mathfrak{M}$, the resulting Molino sequence (28) has nontrivial fiber $\mathcal{D}$.*

The converse to Theorem 8.13 is not true. Fokkink and Oversteegen [2002, Theorem 35] constructed an example of a solenoid with simply connected leaves which is nonhomogeneous. Since the leaves are simply connected, they have trivial holonomy. In Section 10 we construct further examples of actions with nontrivial Molino fiber and simply connected leaves.

9. Strongly quasianalytic actions

In this section, we study the condition of *strong quasianalyticity*, abbreviated as the SQA condition, for equicontinuous matchbox manifolds, as defined in Definition 9.2 below. We identify classes of matchbox manifolds for which this condition holds, and also give examples for which it does not. The generalization of Molino theory in [Álvarez López and Moreira Galicia 2016] applies to equicontinuous foliated spaces such that the closure of their holonomy pseudo⋆groups satisfies the SQA condition. Thus, it is important to characterize the weak solenoids with this property.

9A. ***The strong quasianalyticity condition.*** The precise notion of the SQA condition has evolved in the literature, motivated by the search for a condition equivalent to the quasianalyticity condition for the pseudo⋆groups of smooth foliations as introduced by Haefliger [1985]. Álvarez López and Candel [2009] introduced the notion of a *quasieffective* pseudo⋆group as part of their study of equicontinuous foliated spaces. This terminology was replaced by the notion of a *strongly quasianalytic* pseudo⋆group in [Álvarez López and Moreira Galicia 2016].

Definition 9.1. [Haefliger 1985] A pseudo⋆group $\mathcal{G}^*$ acting on a locally compact locally connected space $\mathfrak{T}$ is *quasianalytic* if for every $h \in \mathcal{G}^*$ the following holds: Let $U \subset \mathrm{Dom}(h) \subset \mathfrak{T}$ be an open set, and suppose $x \in \mathfrak{T}$ is in the closure of U. Suppose the restriction $h|U$ is the identity map. Then there is an open neighborhood V of x such that the restriction $h|V$ is the identity map.

Definition 9.1 describes the properties of pseudo⋆groups, which were discussed in Remark 2.1, where the action of an element is *locally determined*; that is, if h is the identity on an open set, then it is the identity on a larger set. For the case where the space $\mathfrak{T}$ is not locally connected, Álvarez López and Candel [2009] introduced the following modification of this notion.

Definition 9.2. A pseudo⋆group $\mathcal{G}^*$ acting on a locally compact space $\mathfrak{T}$ is *strongly quasianalytic*, or SQA, if for every $h \in \mathcal{G}^*$ the following holds: Let $U \subset \mathrm{Dom}(h)$ be a nonempty open set, and suppose the restriction $h|U$ is the identity map. Then h is the identity map on its domain $\mathrm{Dom}(h)$. A matchbox manifold $\mathfrak{M}$ satisfies the SQA condition if there exists a traversal $V_0 \subset \mathfrak{M}$ such that the induced pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on V_0 satisfies the SQA condition.

Definition 9.2 says that the action of an equicontinuous strongly quasianalytic pseudo⋆group $\mathcal{G}$ is locally determined. That is, if h is the identity on a nonempty open subset of its domain, then it is the identity on $\mathrm{Dom}(h)$. In the case where the transversal $\mathfrak{T}$ is locally compact and locally connected, this condition is equivalent to quasianalyticity by [Álvarez López and Candel 2009, Lemma 9.8]. However, when $\mathfrak{T}$ is totally disconnected, the SQA condition becomes a statement about the algebraic properties of the group chain associated to the action, as we next discuss.

Recall from Proposition 3.4(1) that if $\mathfrak{M}$ is an equicontinuous matchbox manifold, then we can assume that the pseudo⋆group action on the transversal is given by an equicontinuous minimal Cantor action (V_0, G_0, Φ_0). Thus, for each $h \in G_0$ we have $\mathrm{Dom}(h) = V_0$. Moreover, the assumption that the restriction $h|U$ is the identity in the statement of Definition 9.2 means that the SQA condition need only be checked for $h \in G_0$ such that there exists $x \in V_0$ for which $\Phi_0(h)(x) = x$, that is, those elements whose action fixes at least a point.

Recall from Section 6A that the closure $\overline{\Phi_0(G_0)} \subset \mathrm{Homeo}(V_0)$ in the uniform topology of the image $\Phi_0(G_0) \subset \mathrm{Homeo}(V_0)$ is called the Ellis group of the Cantor system (V_0, G_0, Φ_0), which yields a Cantor system $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$, where $\widehat{\Phi}_0 : \overline{\Phi_0(G_0)} \to \mathrm{Homeo}(V_0)$. Given $x \in V_0$ then $\overline{\Phi_0(G_0)}_x \subset \overline{\Phi_0(G_0)}$ denotes the isotropy subgroup at x for the action, and then the SQA condition must be checked for all elements of $\overline{\Phi_0(G_0)}_x$. We set $\widehat{\Phi}_0(G_0) = \{\Phi_0(g) \mid g \in G_0\}$, which is a dense subgroup of $\overline{\Phi_0(G_0)}$. The following result follows from the definitions.

Lemma 9.3. *If $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ satisfies the SQA condition, then (V_0, G_0, Φ_0) also satisfies the SQA condition. Conversely, suppose that $\overline{\Phi_0(G_0)}_x \subset \widehat{\Phi}_0(G_0)$. Then (V_0, G_0, Φ_0) satisfying the SQA condition implies that $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ satisfies the SQA condition.*

Proof. Let $g \in G_0$ and set $\hat{g} = \Phi_0(g) \in \mathrm{Homeo}(V_0)$. Then $\hat{g} \in \overline{\Phi_0(G_0)}$, so that if $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ satisfies the SQA condition then so must the action of $\hat{g}$. Conversely, suppose (V_0, G_0, Φ_0) satisfies the SQA condition. As noted above, the

SQA property need only be checked for $h \in \overline{\Phi_0(G_0)}_x$. By assumption, such an h belongs to $\Phi_0(G_0)$ and so satisfies the SQA condition. □

Note that the assumption that $\overline{\Phi_0(G_0)}_x \subset \widehat{\Phi}_0(G_0)$ implies that the compact set $\overline{\Phi_0(G_0)}_x$ is contained in a countable set, hence it must be finite. Thus, by Theorem 6.1, this implies that the discriminant group $\mathcal{D}_x$ of the action is finite. The converse need not be true. That is, if the discriminant $\overline{\Phi_0(G_0)}_x$ is finite, then it may be possible to choose a point $y \in V_0$ such that $\overline{\Phi_0(G_0)}_y$ has trivial intersection with $\widehat{\Phi}_0(G_0)$; for instance, this is the case for Example 9.6. Examples in Section 10 show that it is possible to construct actions (V_0, G_0, Φ_0) such that $\overline{\Phi_0(G_0)}_x$ has trivial intersection with $\widehat{\Phi}_0(G_0)$ for any choice of $x \in V_0$.

We next consider the SQA property for an equicontinuous minimal Cantor system (V_0, G_0, Φ_0) and its associated Ellis system $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$. This condition for the system (V_0, G_0, Φ_0) can be formulated in terms of the group chain model developed in Sections 4 and 6B, in which case Lemma 8.7 and Proposition 8.8 imply that the condition is a statement about the holonomy action of the kernel $K(\mathcal{G}^x)$ of the chain $\mathcal{G}^x$ for each $x \in V_0$. Examples 9.6 and 9.7 below and the discussion in Section 10 illustrate the possibilities.

The SQA property for the system $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ can be much more subtle to check, as now it is a condition on the action of the isotropy group $\overline{\Phi_0(G_0)}_x \cong \mathcal{D}_x$ which depends on the algebraic properties of the closed subgroup $\mathcal{D}_x \subset C_\infty$. Note that in this case, for any $x, y \in V_0$ the isotropy groups $\mathcal{D}_x$ and $\mathcal{D}_y$ are conjugate in C_∞, so it suffices to consider the condition for a fixed choice of basepoint $x \in V_0$.

9B. ***Sufficient conditions for the SQA property.*** We next indicate a few classes of solenoids which satisfy the quasianalyticity condition.

Lemma 9.4. *If a matchbox manifold $\mathfrak{M}$ is homogeneous, then there exists a section V_0 with associated presentation $\mathcal{P}$ such that the actions (V_0, G_0, Φ_0) and $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ are SQA.*

Proof. By Corollary 6.4 one can assume that V_0 and $\mathcal{P}$ are chosen so that the associated group chain $\{G_\ell^x\}_{\ell \geq 0}$ consists of normal subgroups. Then $K(\mathcal{G}^x)$ is a normal subgroup of G_0, so by Lemma 8.7, each $g \in K(\mathcal{G}^x)$ defines a trivial holonomy action on V_0. Hence the action of G_0 on V_0 is SQA.

Since $\{G_\ell^x\}_{\ell \geq 0}$ is a chain of normal subgroups, the isotropy group $\overline{\Phi_0(G_0)}_x$ is trivial by Proposition 6.3, and so the condition $\overline{\Phi_0(G_0)}_x \subset \widehat{\Phi}_0(G_0)$ is trivially satisfied. Then by Lemma 9.3 the action $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ is SQA. □

Note that the holonomy pseudogroups associated to homogeneous solenoids, as in Lemma 9.4, satisfy a stronger condition than SQA. Recall from [Álvarez López and Moreira Galicia 2016, Definition 2.22] that the action of G_0 on V_0 is *strongly locally free* if for all $h \in G_0$, if $h(x) = x$, then $h(y) = y$ for all $y \in V_0$. If $\mathfrak{M}$ is homogeneous, then the action on a local section V_0, as given by Lemma 9.4, is strongly locally free. The actions in Lemma 9.4 are the actions in [Álvarez López and Moreira Galicia 2016, Example 2.35].

The following result gives a class of equicontinuous matchbox manifolds which satisfy the SQA condition. This is Theorem 1.8 of the Introduction.

Theorem 9.5. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold of finite π_1-type. Then there exists a section V_0 with a presentation $\mathcal{P}$ such that the action (V_0, G_0, Φ_0) is SQA. Further, if V_0 can be chosen so that the discriminant group $\mathcal{D}_x = \overline{\Phi_0(G_0)}_x$ is finite, then there exists an $n \geq 0$ such that the restricted action (V_n, G_n^x, Φ_n) and the action $(V_n, \overline{\Phi_n(G_n^x)}, \widehat{\Phi}_n)$ are both SQA.*

Proof. Let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. By Theorem 8.4 we can assume that x is chosen so that L_x is a leaf without holonomy. As the leaves of $\mathcal{F}_{\mathfrak{M}}$ are assumed to have finite π_1-type, by Lemma 8.7 and Proposition 8.8, and restricting to a smaller section is necessary, we can assume that V_0 and $\{\mathcal{G}_\ell^x\}_{\ell \geq 0}$ are chosen so that $K(\mathcal{G}^x)$ is a normal subgroup of G_0. Then by Proposition 8.10, $K(\mathcal{G}^x) \subseteq K(\mathcal{G}^y)$ for all $y \in V_0$, and, if $\mathrm{Germ}(y, \Phi_0)$ is trivial, then $K(\mathcal{G}^x) = K(\mathcal{G}^y)$.

Since the G_0-orbit of x is dense in V_0, any $g \in G_0$ which is the identity on a nonempty open set in V_0 must be contained in $K(\mathcal{G}^x)$, and so it is the identity on V_0. Thus, (V_0, G_0, Φ_0) is SQA.

Now let C_∞ be the Ellis group, associated to (V_0, G_0, Φ_0), and suppose the discriminant group $\mathcal{D}_x \cong \overline{\Phi_0(G_0)}_x$ is finite. Suppose there exists a nontrivial element $\hat{g} \in \overline{\Phi_0(G_0)}_x$ which fixes an open subset U of V_0 around x.

Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Choose an index $n \geq 0$ large enough so that $V_n \subset U$. Let $y \in V_n$. Then

$$\hat{g} = (g_i C_i) \in \overline{\Phi_0(G_0)}_y,$$

and it follows that

$$\hat{g} \in \bigcap_{y \in V_n} \overline{\Phi_0(G_0)}_y,$$

that is, the intersection $\bigcap_{y \in V_n} \mathcal{D}_y$ is nontrivial.

Consider the truncated chain $\{G_\ell^x\}_{\ell \geq n}$ and the corresponding action (V_n, G_n^x, Φ_n). Recall from Section 7C that $E_\ell^n = \mathrm{core}_{G_n^x} G_\ell^x$ is a maximal normal subgroup of G_ℓ^x in G_n^x, and there is an inclusion

$$C_\ell \subset E_\ell^n \subset G_\ell^x, \tag{45}$$

where C_ℓ is the maximal normal subgroup of G_ℓ^x in G_0. The Ellis group E_∞^n of the restricted action (V_n, G_n^x, Φ_n) is defined by (34) as the inverse limit of coset spaces

G_n^x/E_ℓ^n. The inclusions (45) yield a commutative diagram

$$\begin{tikzcd} G_n^x/C_\ell \arrow[rr, "\varphi_{n,\ell}"] \arrow[dr] & & G_n^x/E_\ell^n \arrow[dl] \\ & G_n^x/G_\ell^x & \end{tikzcd} \tag{46}$$

which is equivariant with respect to the natural action of G_n^x on its coset spaces. Taking the inverse limits, we obtain the commutative diagram

$$\begin{tikzcd} C_\infty^n \arrow[rr, "\varphi_{n,\infty}"] \arrow[dr] & & E_\infty^n \arrow[dl] \\ & G_\infty^n \cong V_n & \end{tikzcd} \tag{47}$$

where C_∞^n is the profinite subgroup of C_∞, defined by (31), which is again equivariant with respect to the action of G_n^x on the inverse limits, and $\varphi_{n,\infty}$ is a surjective group homomorphism.

Let $\hat{g}_n = \varphi_{n,\infty}(\hat{g})$. We will show that $\hat{g}_n$ acts trivially on V_n. Indeed, let $\hat{g} = (g_\ell C_\ell)$, where $g_\ell \in G_n^x$. Then $\hat{g}_n = (g_\ell E_\ell^n)$ for $\ell \geq n$. Since C_ℓ and E_ℓ^n are normal subgroups of G_n^x, the actions of $g_\ell C_\ell$ and $g_\ell E_\ell^n$ on G_n^x/G_ℓ are well defined; for example, for any $h \in G_n^x$ we have

$$g_\ell C_\ell h G_\ell^x = g_\ell h C_\ell h^{-1} h G_\ell^x = g h G_\ell^x,$$

and similarly for $g_\ell E_\ell^n$. Since diagram (46) is a commutative diagram of equivariant maps, we obtain that

$$g_\ell C_\ell h G_\ell^x = h G_\ell^x \implies g_\ell E_\ell^n h G_\ell^x = h G_\ell^x,$$

and it follows that if $\hat{g}$ acts trivially on $y = (h_i G_\ell^x) \in V_n$, then $\hat{g}_n$ acts trivially on y as well.

Then by an argument similar to the one at the beginning of this proof, we obtain that $\hat{g}_n \in \bigcap_{y \in V_n} \mathcal{D}_y^n$, where $\mathcal{D}_y^n$ is the discriminant group of the truncated action (V_0, G_n^x, Φ_n) at $y \in V_n$. We note that $\bigcap_{y \in V_n} \mathcal{D}_y^n$ is the maximal normal subgroup of $\mathcal{D}_x^n$, and so by Proposition 6.2 it must be trivial. Therefore, $\hat{g}_n = \varphi_{n,\infty}(\hat{g})$ is the identity in E_∞^n.

We note that the restricted group action (V_n, G_n^x, Φ_n) is SQA since (V_0, G_0, Φ_0) is SQA. By restricting to a smaller section and applying the above argument a finite number of times we may assume that no element of the discriminant group $\mathcal{D}_x^n$ fixes an open subset of V_n. It follows that the action $(V_n, \overline{\Phi_n(G_n^x)}, \widehat{\Phi}_n)$ of the closure is SQA. □

9C. ***SQA counterexamples.*** We give two classes of examples to illustrate the above results.

Example 9.6. We first give an example of a group action, corresponding to the holonomy of a solenoid with leaves of finite π_1-type, that is not strongly locally free.

Let K be the Klein bottle, with fundamental group $G_0 = \langle a, b \mid bab^{-1} = a^{-1} \rangle$, and let $K_\infty = \varprojlim \{p : K \to K\}$ be the inverse limit space, as described in Example 4.11. The solenoid K_∞ contains one nonorientable leaf with one end, and every other leaf is an open two-ended cylinder. Thus, each leaf is homotopic to a circle, and thus has finite π_1-type.

The group chain $\mathcal{G}^x$, associated to the choice of basepoint as in Example 4.11, consists of subgroups $G^x_\ell = \langle a^{2^\ell}, b \rangle$, $\ell \geq 0$, and $K(\mathcal{G}^x) = \langle b \rangle$. This leaf has nontrivial holonomy, with $\mathrm{Germ}(x, \Phi) \cong \mathbb{Z}_2$. Fokkink and Oversteegen [2002] computed that the kernel of a group chain based at any point which is not in the orbit of x is $K(\mathcal{G}^y) = \langle b^2 \rangle$, which is easily seen to be a normal subgroup of G_0. Thus for the chosen section V_0, for every point y with trivial $\mathrm{Germ}(y, \Phi)$ the kernel $K(\mathcal{G}^y)$ is a normal subgroup of G^y_ℓ, $\ell \geq 0$, and the section satisfies Proposition 8.10. So the action (V_0, G_0, Φ) satisfies the SQA condition.

This action is not strongly locally free. Indeed, the action of the element b fixes x, but it does not fix any y with trivial $\mathrm{Germ}(y, \Phi)$. The nontrivial element in $\overline{\Phi(G_0)}_x$ acts nontrivially on any open subset of V_0, and so the action $(V_0, \overline{\Phi(G_0)}_x, \widehat{\Phi})$ satisfies the SQA condition.

Example 9.7. We next give an example of a solenoid for which the action of the holonomy group on the fiber is not SQA for any choice of a transverse section V_0. This example is the Schori solenoid [1966]. We now recall its construction, as described in [Clark et al. 2014].

Let X_0 be a genus-2 surface. Recall that a 1-handle is a 2-torus without an open disc, and note that the genus-2 surface X_0 can be seen as the union of two 1-handles H_0 and F_0 intersecting along the boundaries of the open discs taken out. Let x_0 be a point in the intersection of the handles. Recall that the fundamental group of the genus-2 surface can be presented as

$$\pi_1(X_0, x_0) = \langle a, b, c, d \mid aca^{-1}c^{-1}bdb^{-1}d^{-1} = 1 \rangle,$$

where a and b are longitudinal loops in X_0.

Cut the handle H_0 (resp. F_0) along a closed curve C_0 (resp. D_0), as shown in Figure 1, top left. Pull the cut handles apart to obtain the surface with boundary $\overline{X}_0$ (see Figure 1, top center). Take three copies of $\overline{X}_0$, denoted by $\overline{X}^1_0$, $\overline{X}^2_0$, $\overline{X}^3_0$, and identify their boundaries as shown in Figure 1, top right. The resulting surface X_1 (see Figure 1, bottom) has genus 4, and there is an obvious 3-to-1 covering map $f^1_0 : X_1 \to X_0$. Let x_1 be the preimage of x_0 in the second copy of the handle. We

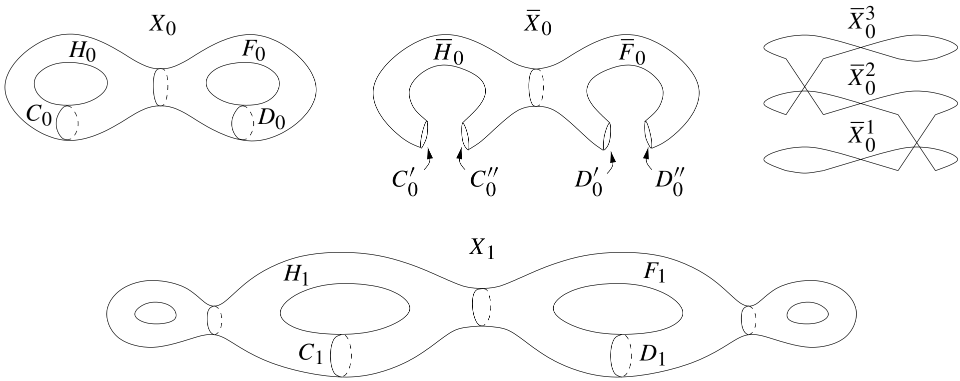

Figure 1. Construction of the Schori example. Top left: choice of the handles H_0 and F_0 and closed curves C_0 and D_0 in X_0. Top center: the cut surface $\overline{X}_0$. Top right: identifications between $\overline{X}_0^\ell$, $\ell = 1, 2, 3$. Each $\overline{X}_0^i$ is represented by a cut copy of a figure 8, and identifications are depicted with straight lines. Bottom: the surface X_1 and the choice of the handles H_1 and F_1 and closed curves C_1 and D_1.

note that the covering f_0^1 is not regular; that is, the image $(f_0^1)_*\pi_1(X_1, x_1)$ of the fundamental group of X_1 is not a normal subgroup of $\pi_1(X_0, x_0)$. Geometrically, we can see that f_0^1 is irregular as follows: Take a longitudinal loop γ in X_0, which represents an equivalence class of loops in $\pi_1(X_0, x_0)$. The fiber of f_0^1 consists of three points, and we see from Figure 1, top right, that depending on the initial point of the lift, γ may lift to a loop or to a nonclosed curve [Schori 1966].

Proceed inductively to obtain a collection of 3-to-1 coverings $f_\ell^{\ell+1} : X_{\ell+1} \to X_\ell$. That is, we can see X_ℓ as the union of two handles H_ℓ and F_ℓ, intersecting along their boundaries (see Figure 1, bottom) for $\ell = 1$. We cut the handle H_ℓ (resp. F_ℓ) along a closed curve C_ℓ (resp. D_ℓ), pull the handles apart to obtain the surface with boundary $\overline{X}_\ell$, take three copies of $\overline{X}_\ell$, denoted by $\overline{X}_\ell^1$, $\overline{X}_\ell^2$, $\overline{X}_\ell^3$, and identify their boundaries in a way similar to Figure 1, top right. The resulting surface $X_{\ell+1}$ is a 3-to-1 nonregular cover of X_ℓ. This defines a presentation $\mathcal{P} = \{f_\ell^{\ell+1} : X_{\ell+1} \to X_\ell, \ell \geq 0\}$ of the Schori solenoid $\mathcal{S}_\mathcal{P}$. Let $\mathfrak{X}_0$ be the fiber of $\mathcal{S}_\mathcal{P}$ at x_0.

For each $\ell \geq 0$, we choose $x_{\ell+1}$ to be a preimage of x_ℓ under the covering map $f_\ell^{\ell+1}$ in the second copy of X_ℓ. Denote by $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ the corresponding group chain, and recall that there is a conjugacy

$$\varphi : \mathfrak{X}_0 \to X_\infty^x = \varprojlim \{G_0/G_{\ell+1}^x \to G_0/G_\ell^x, \ell \geq 0\}.$$

As before, we denote by $U(x, \ell)$ the cylinder set in X_∞^x containing (eG_ℓ^x). If $y \in \mathfrak{X}_0$ is a point with $\varphi(y) = (g_\ell G_\ell^x)$, then $g_\ell \cdot U(x, \ell) = \Phi(g_\ell)(U(x, \ell))$ is a cylinder set containing $\varphi(y)$. Set $U_\ell^y = \varphi^{-1}(g_\ell \cdot U(x,\ell))$. The group chain $\mathcal{G}^y = \{G_\ell^y = g_\ell G_\ell^x g_\ell^{-1}\}_{\ell \geq 0}$ corresponds to a presentation $\mathcal{P}'$ of the Schori solenoid with basepoint y.

The following result is Theorem 1.9 of the Introduction.

Theorem 9.8. *In the Schori solenoid, for any choice of basepoint $y \in \mathfrak{X}_0$, and any choice of section U_n^y, $n \geq 0$, the holonomy action (U_n^y, G_n^y, Φ_n) is not SQA.*

Proof. Let $y \in \mathfrak{X}_0$, and let (U_n^y, G_n^y, Φ_n) be the holonomy action. At the end of Section 2 we described the procedure of restricting to a smaller section, which gives us a presentation $\mathcal{P}_n' = \{f_\ell^{\ell+1} : X_{\ell+1} \to X_\ell,\ \ell \geq n\}$. By a slight abuse of notation, we now set $G_n^y = \pi_1(X_n, y_n)$ and $G_\ell^y = (f_n^\ell)_* \pi_1(X_\ell, y_\ell)$ (these groups are isomorphic to the groups $(f_0^\ell)_* \pi_1(X_\ell, y_\ell)$, which we denoted by G_ℓ^y earlier). Thus we have a homeomorphism

$$\varphi_n' : U_n^y \to X_{\infty,n}^y = \varprojlim\{G_n^y / G_{\ell+1}^y \to G_n^y / G_\ell^y\},$$

which commutes with the action of G_n^y on U_y^n and $X_{\infty,n}^y$. Denote by $U(y,\ell)$ the cylinder neighborhoods of (eG_ℓ^y) in $X_{\infty,n}^y$. In particular, $U(y,n) = X_{\infty,n}$.

The surface X_ℓ in the presentation $\mathcal{P}'$ has genus $m_\ell = 3^\ell + 1$ (see [Clark et al. 2014]), so G_ℓ^y has m_ℓ generators, represented by longitudinal loops. In particular, there are loops γ_n and δ_n which wind around the handles H_n and F_n in X_n, respectively. Denote by g_γ and g_δ the elements represented by γ_n and δ_n in G_n^y, respectively.

Now consider the construction of the surface X_{n+1}. It is obtained by the identification of three copies $\overline{X}_n^{1,2,3}$ of X_n similar to the identification in Figure 1, bottom left. There is a point y_{n+1} in one of the copies which satisfies $f_n^{n+1}(y_{n+1}) = y_n$, and which corresponds to our choice of the basepoint y. Denote by z_{n+1} and v_{n+1} the other two points such that

$$f_n^{n+1}(z_{n+1}) = f_n^{n+1}(v_{n+1}) = y_n.$$

Denote by $\gamma_{y_{n+1}}$, $\gamma_{z_{n+1}}$, and $\gamma_{v_{n+1}}$ the copies of γ_n in $\overline{X}_n^{1,2,3}$ with respective basepoints y_{n+1}, z_{n+1}, and v_{n+1}. Note that these loops are cut when constructing $\overline{X}_n^{1,2,3}$. We now proceed to identify the boundaries of $\overline{X}_n^{1,2,3}$ according to the construction, which would close one of the loops back, and would intertwine the boundaries of the other two loops, so as to create a single loop of twice the length of γ_n.

We have the following alternatives: first, suppose $\gamma_{z_{n+1}}$ is identified into a loop, and $\gamma_{y_{n+1}}$ and $\gamma_{v_{n+1}}$ are identified to make a single loop of twice the length. Then the lift of γ_n with the starting point y_{n+1} is the curve $\gamma_{y_{n+1}}$ which is not closed and has v_{n+1} as its ending point. This means that the action of g_γ on the coset space G_n^y / G_{n+1}^y maps eG_{n+1}^y onto $g_\gamma G_n^y$, and so maps the cylinder neighborhood $U(y, n+1)$ onto the clopen set $g_\gamma(U(y, n+1))$. At the same time, the lift of γ_n

with the starting point z_{n+1} is a closed loop. So the action of g_γ fixes the coset $\gamma_\delta G^y_{n+1}$, and the clopen set $g_\delta(U(y,n+1))$. We note that on the subsequent steps of the construction, when creating X_{n+i}, the lifts of the loop $\gamma_{z_{n+1}}$ are never cut and identified, which means that the action of g_γ is the identity on $g_\delta(U(y,n+1))$.

Another alternative is that $\gamma_{y_{n+1}}$ is identified into a loop, and $\gamma_{z_{n+1}}$ and $\gamma_{v_{n+1}}$ are identified to make a single loop. Arguing similarly, in this case we obtain that the action of g_γ is the identity on $U(y,n+1)$, and it permutes the sets $g_\delta(U(y,n+1))$ and $g_\gamma \circ g_\delta(U(y,n+1))$. Thus in both cases we obtain an element which is the identity on a clopen subset of the section U^y_n, which permutes two other subsets of U^n_y, which means that (U^y_n, G^y_n, Φ) is not SQA. Since the choice of y and n was arbitrary, we conclude that the holonomy pseudogroup for the Schori solenoid is not SQA. □

From the proof of Theorem 9.8 we obtain the following corollary, which shows that the hypotheses of Proposition 8.8 are necessary.

Corollary 9.9. *In the Schori solenoid, for any choice of a transverse section V_0, and any choice of a point x,* $\mathrm{Germ}(\Phi_0, x)$ *is not locally trivial.*

Proof. From the proof of Theorem 9.8 we conclude that, for any choice of basepoint $y \in \mathfrak{X}_0$, and any choice of group chain $\mathcal{G}^y_n = \{G^y_n\}_{i\geq 0}$, the kernel $K(G^y_n)$ is not a normal subgroup of G^x_n. It follows that even if $\mathrm{Germ}(\Phi_0, x)$ is trivial, it is not locally trivial. □

10. A universal construction

In this section, we give a general method of constructing examples of group chains with prescribed discriminant groups. This construction is inspired by the proof of Lemma 37 in Section 8 of [Fokkink and Oversteegen 2002], which they attribute to Hendrik Lenstra. The construction of Lenstra is given in Section 10A, and Section 10B discusses some properties of this construction. Then in Section 10C we give criteria for when the resulting group chains are stable.

Section 10D recalls two basic results of Lubotzky [1993]. The first, given here as Theorem 10.4, realizes any given finite group F embedded into the profinite completion of a finitely generated, torsion-free group G. A second result of Lubotzky, given here as Theorem 10.5, embeds the infinite product $\boldsymbol{H}$ of a collection of finite groups as a subgroup of the profinite completion of a finitely generated, torsion-free group G. Then in Section 10E, these constructions of Lubotzky are used to construct the examples used in the proofs of Theorems 1.10 and 1.12 of the Introduction.

There is an extensive literature on embedding groups into the profinite completion of a given torsion-free, finitely generated group (see [Ribes and Zalesskii 2000] for a discussion of this topic and further references). The methods of this section apply in this generality to yield an enormous range of equicontinuous minimal Cantor actions with infinite, hence Cantor discriminant, groups.

10A. ***A profinite construction.*** We first give a reformulation of the constructions in Sections 6B and 6C, in analogy with the construction of Lenstra in [Fokkink and Oversteegen 2002]. This alternate formulation is of strong interest in itself, as it gives a deeper understanding of the Molino spaces introduced in this work.

Let G_0 be a finitely generated group, $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ be a group chain in G_0, and $\mathcal{C} = \{C_\ell\}_{\ell \geq 0}$ be the core group chain associated to $\mathcal{G}$, with C_∞ the core group associated with $\mathcal{C}$. Assume the kernel $K(\mathcal{G})$ is trivial, so the map $\widehat{\Phi} : G_0 \to C_\infty$ is an injective homomorphism with dense image $\widehat{G}_0 = \widehat{\Phi}(G_0) \subset C_\infty$. Then the discriminant of $\mathcal{G}$ is a compact subgroup $\mathcal{D} \subset C_\infty$, whose rational core as defined in (26) is trivial by Proposition 6.2.

Let $C_{n,\infty} \subset C_\infty$ be the clopen normal subgroup neighborhood of the identity $\{e\}$ defined in (23). As $\bigcap_{n \geq 1} C_{n,\infty} = \{e\}$, the collection $\{C_{n,\infty} \mid n \geq 1\}$ is a clopen neighborhood system about the identity in C_∞. Observe that from the definition (21), we have that $C_\infty / C_{n,\infty} \cong G_0 / C_n$ and $\widehat{G}_0 \cap C_{n,\infty} \cong G_n$. As each subgroup $C_{n,\infty}$ is normal and $\mathcal{D}$ is compact, the product $V_n = \mathcal{D} \cdot C_{n,\infty} \subset C_\infty$ is a clopen subgroup of C_∞ containing $\mathcal{D}$, and we have $\mathcal{D} = \bigcap_{n \geq 1} V_n$. Thus, $\mathcal{D}$ is realized as the countable intersection of clopen subgroups of C_∞. It is an exercise to show that this formulation of $\mathcal{D}$ agrees the definition of $\mathcal{D}$ as an inverse limit in (25).

We now turn the order of the above remarks around to obtain a construction of a group chain with prescribed discriminant group.

Proposition 10.1. *Let C_∞ be a profinite group, and let $G \subset C_\infty$ be a finitely generated dense subgroup. Let $\mathcal{D} \subset C_\infty$ be a compact subgroup of infinite index which has trivial rational core,*

$$\operatorname{core}_G \mathcal{D} = \bigcap_{k \in G} k \mathcal{D} k^{-1} = \{e\}. \tag{48}$$

Then there exists a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ with $G_0 = G$, with discriminant group $\mathcal{D}$.

Proof. By the assumption that C_∞ is a profinite group, there exists a group chain $\{U_\ell \mid \ell \geq 1\}$ that is a clopen neighborhood system about the identity in C_∞, such that

(1) each U_ℓ is normal in C_∞,

(2) for each $\ell \geq 0$ there is a proper inclusion $U_{\ell+1} \subset U_\ell$,

(3) $\bigcap_{\ell \geq 1} U_\ell = \{e\}$.

In particular, each quotient $H^\ell \equiv C_\infty / U_\ell$ is a finite group. Let $\iota_\ell^{\ell+1} : H^{\ell+1} \to H^\ell$ be the map induced by inclusion of cosets. Then there is a natural identification

$$C_\infty \cong \varprojlim \{\iota_\ell^{\ell+1} : H^{\ell+1} \to H^\ell\}. \tag{49}$$

Next, for each $\ell \geq 1$, set $W_\ell = \mathcal{D} \cdot U_\ell$, which is a subgroup of C_∞, as U_ℓ is normal. Moreover, the assumption that $\mathcal{D}$ is compact implies that each W_ℓ is a clopen subset of C_∞. Then set $G_\ell = G \cap W_\ell$, which is a subgroup of finite index in G, and so $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is a subgroup chain in G. Note that

(50) $$K(\mathcal{G}) = \bigcap_{\ell \geq 0} G_\ell = \bigcap_{\ell \geq 0} G \cap W_\ell = G \cap \bigcap_{\ell \geq 0} W_\ell = G \cap \mathcal{D}.$$

We next calculate the discriminant of the chain $\mathcal{G}$. Let $\pi^\ell : C_\infty \to H^\ell$ be the quotient map. As each H^ℓ is finite, the image $\mathcal{D}^\ell \equiv \pi^\ell(\mathcal{D})$ is a finite set. The group G is dense in C_∞ so has nontrivial intersection with each clopen set gU_ℓ. Thus,

(51) $$\mathcal{D}^\ell = \pi^\ell(\mathcal{D}) = \pi^\ell(W_\ell) = \pi^\ell(G \cap W_\ell) = \pi^\ell(G_\ell) \subset H^\ell.$$

The core of the group G_ℓ is the group $C_\ell \equiv \mathrm{core}_G\, G_\ell = \bigcap_{g \in G} g G_\ell g^{-1}$. We have

(52) $$\begin{aligned} \pi^\ell(C_\ell) &= \pi^\ell\Big(\bigcap_{g \in G} g\, G_\ell\, g^{-1}\Big) \\ &= \bigcap_{g \in G} \pi^\ell(g)\pi^\ell(G_\ell)\pi^\ell(g)^{-1} \\ &= \bigcap_{g \in G} \pi^\ell(g)\pi^\ell(\mathcal{D})\pi^\ell(g)^{-1} \\ &= \{e^\ell\}, \end{aligned}$$

where $e^\ell \in H^\ell$ is the identity, and the last equality follows since G is dense in C_∞ and the core of $\mathcal{D}$ is trivial. It follows that $C_\ell = G \cap U_\ell$, and thus we obtain induced maps on the quotients, $\bar{\pi}^\ell : G/C_\ell \to H_\ell$. Then note that $\pi^\ell(G_\ell/C_\ell) = \pi^\ell(\mathcal{D}) = \mathcal{D}^\ell$ for all $\ell \geq 0$.

The map $\iota_\ell^{\ell+1} : H^{\ell+1} \to H^\ell$ induces a map (denoted the same), $\iota_\ell^{\ell+1} : \mathcal{D}^{\ell+1} \to \mathcal{D}^\ell$. Then for the inverse limits we have

(53) $$\varprojlim\{\delta_\ell^{\ell+1} : G_{\ell+1}/C_{\ell+1} \to G_\ell/C_\ell\} = \varprojlim\{\iota_\ell^{\ell+1} : \mathcal{D}^{\ell+1} \to \mathcal{D}^\ell\}.$$

The term on the left-hand side of (53) is by definition the discriminant of the chain $\mathcal{G}$, while the term on the right-hand side of (53) is homeomorphic to the subgroup $\mathcal{D}$, as $\{U_\ell \mid \ell \geq 1\}$ is a clopen neighborhood system about the identity in C_∞. □

10B. ***Properties of the Lenstra construction.*** We make some remarks about the construction in Proposition 10.1. First, note that the proof of [Fokkink and Oversteegen 2002, Lemma 37] defined the chain $\mathcal{G}_n$ using a collection of clopen neighborhoods of $e \in C_\infty$. However, the proof in that paper that the chain $\mathcal{G}_n$ is not weakly regular used Proposition 5.6, that is, the fact that if the number of conjugacy classes of the kernel $K(\mathcal{G}_n)$ is infinite, then $\mathcal{G}_n$ cannot be weakly regular. Our approach is to calculate the discriminant group for the chain directly.

Assume there is given a profinite group C_∞, a compact subgroup $\mathcal{D} \subset C_\infty$, and a dense subgroup $G \subset C_\infty$ satisfying the hypotheses of Proposition 10.1. Set $X = C_\infty/\mathcal{D}$, which is a Cantor space. The left action of G on X defines a map $\Phi : G \to \mathrm{Homeo}(X)$, which is a minimal action as G is dense in C_∞. Thus, the construction yields an equicontinuous minimal Cantor system (X, G, Φ).

Next, given a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in C_∞ which satisfies the conditions in the proof of Proposition 10.1, let $\mathcal{G} \equiv \{G_\ell\}_{\ell \geq 0}$ be the group chain in G constructed with respect to this clopen neighborhood system. Then it is an exercise, using the techniques of the proof of Proposition 10.1, to show that there is a G-equivariant homeomorphism of spaces

$$\tau : X \cong \varprojlim\{\iota_{\ell+1} : G/C_{\ell+1} \to G/G_\ell\} \equiv X_\infty.$$

Now suppose that $\{V_\ell \mid \ell \geq 1\}$ is another clopen neighborhood system about the identity in C_∞ which also satisfies the conditions in the proof of Proposition 10.1, and let $\mathcal{H} \equiv \{H_\ell\}_{\ell \geq 0}$ be the group chain in G constructed with respect to this second clopen neighborhood system. A basic property of neighborhood systems is that given any $\ell \geq 0$ there exists $\ell' \geq 0$ such that $V_{\ell'} \subset U_\ell$, and $\ell'' \geq 0$ such that $U_{\ell''} \subset V_\ell$. It follows from their definitions that the group chains $\mathcal{G}$ and $\mathcal{H}$ are equivalent in the sense of Definition 4.3.

Suppose that $G \cap \mathcal{D} = \{e\}$. Then the calculation (50) shows that the kernel $K(\mathcal{G}) = \{e\}$ is trivial. Moreover, suppose the choice of $\mathcal{D}$ is made so that $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} = \{e\}$ for all $\hat{g} \in C_\infty$. Given $y \in X$ let $\tau(y) = (g_\ell G_\ell) \in X_\infty$, and let $\mathcal{G}^y = \{g_\ell G_\ell g_\ell^{-1}\}_{\ell \geq 0}$ be the conjugate group chain. Choose $\hat{g} \in C_\infty$ such that $\tau(\hat{g}\mathcal{D}) = (g_\ell G_\ell)$. Then

$$K(\mathcal{G}^y) = G \cap (\hat{g}\mathcal{D}\hat{g}^{-1}) = \{e\} \tag{54}$$

so that $\mathcal{G}^y$ also has trivial kernel. Thus, if we choose $\mathcal{D}$ so that $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} = \{e\}$ for all $\hat{g} \in C_\infty$ is satisfied, then the Cantor system (X, G, Φ) has trivial kernel for the group chain $\mathcal{G}^y$ at y for all points $y \in X$. For example, suppose that G is a torsion-free group and $\mathcal{D}$ is a torsion group. Then the condition $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} = \{e\}$ for all $\hat{g} \in C_\infty$ is automatically satisfied, as each nontrivial element of $\mathcal{D}$, and hence $\hat{g}\mathcal{D}\hat{g}^{-1}$, has finite order. We use this observation in Theorem 10.7 below.

On the other hand, given $G \subset C_\infty$ as in Proposition 10.1, suppose that the compact subgroup $\mathcal{D} \subset C_\infty$ is chosen so that $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} \neq \{e\}$ for some $\hat{g} \in C_\infty$. Then by Proposition 8.11 there exists $y \in X$ such that the Cantor system (X, G, Φ) has nontrivial kernel $K(\mathcal{G}^y)$ for the group chain $\mathcal{G}^y$ about y. It then follows that the germinal holonomy group $\mathrm{Germ}(\Phi, y)$ is nontrivial, so this method can also be used to construct examples with nontrivial germinal holonomy groups.

10C. ***Stable actions.*** Recall from Definition 7.5 that a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is said to be *stable* if there exists $n_0 \geq 0$ such that the maps $\psi_{n,m} : \mathcal{D}^n \to \mathcal{D}^m$

defined in (39) are isomorphisms for all $m \geq n \geq n_0$. We consider the problem of when a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ constructed using the method in the proof of Proposition 10.1 is stable.

We assume the hypotheses of Proposition 10.1, and the constructions of its proof. Fix $n > 0$, and consider the truncated group chain $\mathcal{G}_n = \{G_\ell\}_{\ell \geq n}$. Then the calculation of the kernel $K(\mathcal{G}_n) = G \cap \mathcal{D}$ is the same as (50). Also, note that $\mathcal{D} \subset W_\ell$ for all $\ell \geq 0$, so the calculations in (51) also proceed analogously. However, the last equality in (52) requires the additional assumption

$$\operatorname{core}_U \mathcal{D} = \bigcap_{k \in U} k\mathcal{D}k^{-1} = \{e\} \tag{55}$$

for the clopen neighborhoods $U = U_\ell$ of the identity in order to conclude that $\mathcal{D}$ is the discriminant group for $\mathcal{G}_n$. In other words, we require that the subgroup $\mathcal{D}$ is "totally not-normal" for every neighborhood of the identity in $\widehat{G}$. The above remarks yield:

Proposition 10.2. *Let C_∞ be a profinite group, let $G \subset C_\infty$ be a finitely generated dense subgroup, and let $\mathcal{D} \subset C_\infty$ be a compact subgroup of infinite index, such that (55) holds for every clopen neighborhood $\{e\} \in U \subset C_\infty$. Choose a group chain $\{U_\ell \mid \ell \geq 1\}$ which is a clopen neighborhood system about the identity in C_∞. Then the associated group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ with $G_0 = G$ has discriminant group $\mathcal{D}$ and is stable.*

Finally, in the case where $\mathcal{D} \subset C_\infty$ is a compact subgroup of infinite index, but need not satisfy the condition that its core is trivial, then noting that the core is a normal subgroup, we can modify the construction above as follows to obtain a minimal Cantor action.

Corollary 10.3. *Let C'_∞ be a profinite group, $G' \subset C'_\infty$ be a finitely generated dense subgroup, and $\mathcal{D}' \subset C'_\infty$ be a nontrivial compact subgroup of infinite index, and let $\operatorname{core}_{G'} \mathcal{D}'$ denote the rational core of $\mathcal{D}'$ as in (55), which is a normal subgroup of C'_∞ as G' is dense. Set*

$$C_\infty = C'_\infty/(\operatorname{core}_{G'} \mathcal{D}'), \quad G = G'/(G' \cap \operatorname{core}_{G'} \mathcal{D}'), \quad \mathcal{D} = \mathcal{D}'/\operatorname{core}_{G'} \mathcal{D}'.$$

Then there exists a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ with $G_0 = G$ and discriminant group $\mathcal{D}$.

10D. ***Constructing embedded groups.*** We next recall the remarkable constructions of Lubotzky, which when combined with the techniques of Proposition 10.1, make possible the construction of a wide class of equicontinuous minimal Cantor actions by a finitely generated, torsion-free, residually finite group G, with prescribed discriminant group $\mathcal{D}$. There are two cases of the construction.

Theorem 10.4 [Lubotzky 1993, Theorem 2(b)]. *Let F be a nontrivial finite group, and set $\boldsymbol{F}_i = F$ for all integers $i \geq 1$. Let $\boldsymbol{F} = \prod \boldsymbol{F}_i$ denote the infinite cartesian product of F. Then there exists a finitely generated, residually finite, torsion-free group $G \subset \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ sufficiently large whose profinite completion $\widehat{G}$ contains $\boldsymbol{F}$.*

Proof. We give just an outline of the construction used in the proof of Theorem 2(b) in [Lubotzky 1993], with details as required for the constructions of our examples. First recall some basic facts. For $n \geq 3$, let $\Gamma_n = \mathrm{SL}_n(\mathbb{Z})$ denote the $n \times n$ integer matrices. The group Γ_n is finitely generated and residually finite, and hence so are all finite index subgroups of Γ_n. Let $\Gamma_n(m)$ denote the congruence subgroup

$$\Gamma_n(m) \equiv \mathrm{Ker}\{\varphi_m : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})\}.$$

For $m \geq 3$, $\Gamma_n(m)$ is torsion-free. Moreover, by the congruence subgroup property, every finite index subgroup of Γ_n contains $\Gamma_n(m)$ for some nonzero m. Then this implies

$$\widehat{\mathrm{SL}_n(\mathbb{Z})} \cong \varprojlim \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z}) \cong \mathrm{SL}_n(\widehat{\mathbb{Z}}) \cong \prod_p \mathrm{SL}_n(\mathbb{Z}_p), \tag{56}$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and we use that $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers, and the product is taken over all primes. Note that the factors in the cartesian product on the right-hand side of (56) commute with each other.

Let $G \subset \Gamma_n$ be a finite index, torsion-free subgroup, which is then finitely generated, and its profinite completion $\widehat{G}$ is an open subgroup of $\widehat{\mathrm{SL}_n(\mathbb{Z})}$. Then there exists a cofinite subgroup $\mathcal{P}(G)$ of the primes such that

$$\prod_{p \in \mathcal{P}(G)} \mathrm{SL}_n(\mathbb{Z}_p) \subset \widehat{G}. \tag{57}$$

Let $d_F = |F|$ denote the cardinality of F, and let $n \geq |F| + 2$. Then F embeds in the alternating group $\mathrm{Alt}(n)$ on n symbols. Then F being nontrivial implies that $n \geq 4 > 3$. For each $p \in \mathcal{P}(F)$, the group $\mathrm{Alt}(n)$ embeds into $\mathrm{SL}_n(\mathbb{Z}_p)$, and thus we obtain an embedding

$$\iota_\infty : \boldsymbol{F} \cong \prod_{p \in \mathcal{P}(G)} F_p \subset \prod_{p \in \mathcal{P}(G)} \mathrm{Alt}(n) \subset \prod_{p \in \mathcal{P}(G)} \mathrm{SL}_n(\mathbb{Z}_p) \subset \widehat{G}, \tag{58}$$

where $F_p = F$ for each $p \in \mathcal{P}(G)$. This completes the construction. □

Lubotzky [1993, Theorem 1] extended the above construction to obtain an embedding for a group $\mathcal{D}$ which is an infinite product of possibly distinct finite groups $\{\boldsymbol{H}_i \mid i = 1, 2, \ldots\}$. The extension is highly nontrivial, as if all of the groups $\boldsymbol{H}_i$ are distinct, then the degrees $|\boldsymbol{H}_i|$ must tend to infinity, and so the above straightforward strategy for embedding no longer works.

Theorem 10.5 [Lubotzky 1993, Theorem 1]. *Let $\{\boldsymbol{H}_i \mid i = 1, 2, \dots\}$ be an infinite collection of nontrivial finite groups, and let $\boldsymbol{H} = \prod \boldsymbol{H}_i$ denote their cartesian product. Then there exists a finitely generated, residually finite, torsion-free group G whose profinite completion $\widehat{G}$ contains $\boldsymbol{H}$.*

Proof. Again, we only sketch some key aspects of the proof from [Lubotzky 1993]. Let $G \subset \Gamma_n$ be the finitely generated, torsion-free, residually finite group constructed on page 330 of [Lubotzky 1993], and let $\widehat{G}$ be its profinite completion. Lubotzky constructs by induction an increasing sequence of primes $\{p_n \mid n \geq 3\}$ such that

$$\prod_{n=3}^{\infty} \mathrm{SL}_n(\mathbb{Z}_{p_n}) \subset \widehat{G}. \tag{59}$$

For $i \geq 1$, let $d_i = |\boldsymbol{H}_i|$ denote the cardinality of $\boldsymbol{H}_i$. Then each $\boldsymbol{H}_i$ embeds in the alternating group $\mathrm{Alt}(d_i + 2)$ on $d_i + 2$ symbols. Now choose an increasing sequence of integers $\{n_i \mid i \geq 1\}$ such that $n_i \geq d_i + 2$. Then for each $i \geq 1$, the group $\mathrm{Alt}(d_i + 2)$ embeds into the alternating group $\mathrm{Alt}(n_i)$ by taking only the permutations on the first $d_i + 2$ symbols. For each $i \geq 1$ the group $\mathrm{Alt}(n_i)$ embeds into $\mathrm{SL}_{n_i}(\mathbb{Z}_{p_{n_i}})$. Thus, we have embeddings $\boldsymbol{H}_i \subset \mathrm{Alt}(n_i) \subset \mathrm{SL}_{n_i}(\mathbb{Z}_{p_{n_i}})$.

The product in (59) is over all $n \geq 3$, while the group $\boldsymbol{H}_n = \boldsymbol{H}_{n_i}$ if $n = n_i$ for some n_i as chosen above. For $n \neq n_i$ for some i, let $\boldsymbol{H}_n$ be the trivial group. Set $\boldsymbol{A}_n = \mathrm{Alt}(n_i)$ if $n = n_i$ for some n_i and let $\boldsymbol{A}_n$ be the trivial group otherwise. Then we obtain an embedding of the infinite product $\boldsymbol{H}$,

$$\iota_\infty : \mathcal{D} \cong \prod_{n \geq 3} \boldsymbol{H}_n \subset \prod_{n \geq 3} \boldsymbol{A}_n \subset \prod_{i \geq 1} \mathrm{SL}_{n_i}(\mathbb{Z}_{p_{n_i}}) \subset \widehat{G}. \tag{60}$$

This completes the construction. □

10E. *Constructing stable actions.* We next use Theorems 10.4 and 10.5, and observations from their proofs in [Lubotzky 1993], to construct examples of stable equicontinuous minimal Cantor group actions.

We first require a simple observation. For $n \geq 2$, the alternating group $\mathrm{Alt}(n)$ on n symbols embeds into the alternating group $\mathrm{Alt}(4n)$ on $4n$ symbols, by considering $\mathrm{Alt}(n)$ as acting on the first n symbols and fixing the remaining $3n$ symbols. We thus consider $\mathrm{Alt}(n) \subset \mathrm{Alt}(4n)$ as a subgroup.

Lemma 10.6. *The core of* $\mathrm{Alt}(n)$ *in* $\mathrm{Alt}(4n)$ *is the trivial group.*

Proof. There exists an element $\sigma \in \mathrm{Alt}(4n)$ which swaps the first $2n$ symbols for the last $2n$ symbols. Then $\sigma^{-1}\mathrm{Alt}(n)\sigma$ is contained in the alternating group which permutes the last $2n$ symbols, and hence is disjoint from the subgroup $\mathrm{Alt}(n)$. □

Lemma 10.6 is used to ensure that the chains constructed below satisfy the conditions of Section 10C.

Theorem 10.7. *Let F be a finite group. Then there exists a finite index, torsion-free group $G \subset \mathrm{SL}_n(\mathbb{Z})$ and an embedding of F into the profinite completion $\widehat{G}$, so that the resulting group chain $\mathcal{G}_K = \{G_\ell\}_{\ell \geq 0}$ constructed as in Section 10A yields an equicontinuous minimal Cantor system (X_∞, G, Φ) whose discriminant group for the truncated group chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to F for all $k \geq 0$. Hence the action is stable and irregular. Moreover, the germinal holonomy group for each $x \in X$ is trivial.*

Proof. As noted in the proof of Theorem 10.4, if F is a nontrivial finite group of order $d_F = |F|$, then F embeds in the alternating group $\mathrm{Alt}(d_F + 2)$. We then embed $\mathrm{Alt}(d_F + 2)$ in the alternating group $\mathrm{Alt}(n)$ for $n \geq 4(d_F + 2)$, by considering $\mathrm{Alt}(n)$ as acting on the first n symbols, as in the proof of Lemma 10.6. We identify F with its image, and then note that the core of F in $\mathrm{Alt}(n)$ is the trivial group. Note that $d_F \geq 2$, so we have that $n \geq 16$. Also note that if F' is any other finite group of order at most d_F, then it also embeds into $\mathrm{Alt}(d_F + 2)$, and hence the following construction is universal for all finite groups F' with $|F'| \leq |F|$.

For $n \geq 4(d_F + 2)$, let $G \subset \Gamma_n = \mathrm{SL}_n(\mathbb{Z})$ be the finite index, torsion-free subgroup constructed in the proof of Theorem 10.4. Set $\boldsymbol{H}_\ell = \mathrm{Alt}(n)$ for all integers $\ell \geq 1$, and let $\boldsymbol{H} = \prod \boldsymbol{H}_\ell$ denote their cartesian product. Then the embedding (58) becomes

$$\iota_\infty : \boldsymbol{H} \cong \prod_{p \in \mathcal{P}(G)} \mathrm{Alt}_p(n) \subset \prod_{p \in \mathcal{P}(G)} \mathrm{SL}_n(\mathbb{Z}_p) \subset \widehat{G}, \tag{61}$$

where $\mathrm{Alt}_p(n) = \mathrm{Alt}(n)$ for each prime p.

For each $i \geq 1$, we have the embedding

$$F \subset \mathrm{Alt}(d_F + 2) \subset \mathrm{Alt}(n) = \boldsymbol{H}_\ell.$$

Let $F \to \boldsymbol{H}$ be the diagonal embedding into the infinite product, which then yields an embedding $\iota_F : F \to \widehat{G}$ into the profinite completion of G, with image denoted by $\mathcal{D} = \iota_F(F)$.

Next, use the method of Section 10A to construct a group chain in G. The group G is residually finite, so there exists a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in $\widehat{G}$, where each U_ℓ is normal in $\widehat{G}$. Note that G is dense in $\widehat{G}$ and each U_ℓ is closed, so the closure of $G \cap U_\ell$ in $\widehat{G}$ is equal to U_ℓ. Set $W_\ell = \mathcal{D} \cdot U_\ell$ for $\ell \geq 1$, and $G_\ell = G \cap W_\ell$. Let $\mathcal{G}_F = \{G_\ell\}_{\ell \geq 0}$ denote the resulting group chain.

Let $\{e\} \in U \subset \widehat{G}$ be a normal clopen neighborhood of the identity, so that $\widehat{G}/U$ is a finite group with cardinality $|\widehat{G}/U|$. We claim $\mathrm{core}_U \, \mathcal{D} = \{e\}$. The normal subgroup U has finite index; hence, as argued in the proof of [Lubotzky 1993, Theorem 2], there exists a cofinite subset of primes $\mathcal{P}(G, U) \subset \mathcal{P}(G)$ of the list in the product in (61) such that

$$\prod_{p \in \mathcal{P}(G,U)} \mathrm{Alt}_p(n) \subset \prod_{p \in \mathcal{P}(G,U)} \mathrm{SL}_n(\mathbb{Z}_p) \subset U \subset \widehat{G}.$$

For $p \in \mathcal{P}(G, U)$, note that for the diagonal embedding of F into $\boldsymbol{H}$, the projection to each factor of $\boldsymbol{H}$ is an isomorphism. For the image of F in the p-th factor, we have

$$F \subset \mathrm{Alt}(d_F + 2) \subset \mathrm{Alt}(n) = \mathrm{Alt}_p(n) \subset \mathrm{SL}_n(\mathbb{Z}_p).$$

The image group has trivial core by Lemma 10.6. The projection of $\mathcal{D}$ to $F \subset \mathrm{Alt}_p(n)$ is an isomorphism, so this implies that $\mathcal{D}$ has trivial core in U as well. Then by Proposition 10.2, for all $k \geq 0$, the discriminant group for the truncated group chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to F. In particular, $\mathcal{G}_F$ is a stable group chain.

Next, observe that $\mathcal{D}$ being compact implies that the closure $\overline{G_\ell}$ of G_ℓ in $\widehat{G}$ equals W_ℓ, and $\mathcal{D} = \bigcap \overline{G_\ell}$. For the kernel of $\mathcal{G}_F$ as defined in Section 4B, we then have

$$K(\mathcal{G}_F) = \bigcap_{\ell \geq 0} G_\ell \subset \bigcap_{\ell \geq 0} \overline{G_\ell} = \mathcal{D}. \tag{62}$$

The group $\mathcal{D}$ is finite, hence every element of $\mathcal{D}$ has finite order, while $K(\mathcal{G}_F)$ is a torsion-free subset of G. Thus, $K(\mathcal{G}_F) \subset \mathcal{D} \cap G = \{e\}$; hence $K(\mathcal{G}_F)$ is the trivial group. Moreover, for each $\hat{g} \in \widehat{G}$ let

$$\mathcal{G}_F^{\hat{g}} = \{\hat{g}\, G_\ell\, \hat{g}^{-1}\}_{\ell \geq 0}$$

denote the conjugate group chain. Then by the same reasoning, we also have $K(\mathcal{G}_F^{\hat{g}}) = \{e\}$, as $\hat{g}^{-1}\mathcal{D}\hat{g} \subset \widehat{G}$ is again a finite subgroup, hence has trivial intersection with G.

Let (X, G, Φ) be the equicontinuous minimal Cantor system with $X = \widehat{G}/\mathcal{D}$ with the associated group chain $\mathcal{G}_F$, as discussed in Section 10B. The discriminant group of $\mathcal{G}_F$ is $\mathcal{D}$, and each nontrivial element $h \in \mathcal{D}$ is torsion, hence its image in $\widehat{G}$ is torsion, and thus any conjugate of it is not contained in the torsion-free subgroup G. Thus, for each $y \in X$, the action Φ has trivial germinal holonomy at y.

The discriminant group of the truncated chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to F for all $k \geq 0$. Thus, $\mathcal{G}_F$ cannot be a weakly regular group chain. This establishes all of the claims of Theorem 10.7. □

Note that the action (X, G, Φ) satisfies the SQA condition by default, as all germinal holonomy groups are trivial. The action of $\widehat{G}$ on $X = \widehat{G}/\mathcal{D}$ satisfies the SQA condition by Theorem 9.5. Corollary 1.11 now follows by using the construction in Section 2B to obtain a matchbox manifold with section $V_0 \cong X$ and induced holonomy action (X, G, Φ).

We remark that it is tempting to use the fact that $G \subset \mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$ is a torsion-free subgroup, and then use the quotient space $M_0 = \mathrm{SL}_n(\mathbb{R})/G$ as the base of a presentation for a weak solenoid $\mathcal{S}_{\mathcal{P}}$. However, this quotient space is not compact, and we do not have a "theory of weak solenoids" over noncompact manifolds.

We next use Theorem 10.5 to construct two types of embeddings of Cantor groups into profinite groups. Theorem 10.8 embeds a profinite group such that the resulting action is stable. Theorem 10.10 embeds a Cantor group such that the resulting action is not virtually regular.

Theorem 10.8. *Let K be a separable profinite group. There exists a finitely generated, residually finite, torsion-free group G, and an embedding of K into its profinite completion $\widehat{G}$, such that the resulting group chain $\mathcal{G}_K = \{G_\ell\}_{\ell\geq 0}$ constructed as in Section 10A yields an equicontinuous minimal Cantor system (X, G, Φ) whose discriminant group for the truncated group chain $\{G_\ell\}_{\ell\geq k}$ is isomorphic to K for all $k \geq 0$. Hence the action is stable and irregular.*

Proof. Let $G \subset \Gamma_n$ be the finitely generated, torsion-free, residually finite group used in the proof of Theorem 10.5, as constructed on page 330 of [Lubotzky 1993], and let $\widehat{G}$ be its profinite completion.

The assumption that K is a separable profinite group implies that K is isomorphic to an inverse system of finite groups

$$K \cong \varprojlim\{\varphi_\ell^{\ell+1} : \boldsymbol{K}_{\ell+1} \to \boldsymbol{K}_\ell \mid \ell \geq 0\} \subset \boldsymbol{K} \cong \prod \boldsymbol{K}_\ell, \tag{63}$$

where each $\boldsymbol{K}_\ell$ is a finite group, and the bonding maps $\varphi_\ell^{\ell+1}$ are epimorphisms for all $\ell \geq 0$, but not isomorphisms. Thus, their cardinalities $\{|\boldsymbol{K}_\ell| \mid \ell \geq 0\}$ form an increasing sequence of integers. Note that we have isomorphisms for all $k > 0$, induced by the shift map σ_i on indices,

$$\sigma_i : K \cong \varprojlim\{\varphi_\ell^{\ell+1} : \boldsymbol{K}_{\ell+1} \to \boldsymbol{K}_\ell \mid \ell \geq k\}. \tag{64}$$

For each $\ell \geq 0$, set $d_\ell = 4(|\boldsymbol{K}_\ell|+2)$. Then as in the construction in Theorem 10.7, there is an embedding of $\boldsymbol{K}_\ell$ into the alternating group, $\boldsymbol{K}_\ell \subset \mathrm{Alt}(|\boldsymbol{K}_\ell|+2) \subset \mathrm{Alt}(d_\ell)$. Choose an increasing sequence of integers $\{n_\ell \mid \ell \geq 1\}$ so that $n_\ell \geq d_\ell$ for all $\ell \geq 1$.

Then as in the proof of Theorem 10.5, we set $\boldsymbol{H}_n = \mathrm{Alt}(d_\ell)$ if $n = n_\ell$ for some n_ℓ as chosen above. If $n \neq n_\ell$ for all ℓ, let $\boldsymbol{H}_n$ be the trivial group. Set $\boldsymbol{A}_n = \mathrm{Alt}(n_\ell)$ if $n = n_\ell$ for some n_ℓ, and let $\boldsymbol{A}_n$ be the trivial group otherwise. Then we obtain an embedding of the infinite product,

$$\boldsymbol{H} \equiv \prod_{n\geq 3} \boldsymbol{H}_n \subset \boldsymbol{A} \equiv \prod_{n\geq 3} \boldsymbol{A}_n \subset \prod_{\ell\geq 1} \mathrm{SL}_{n_\ell}(\mathbb{Z}_{p_{n_\ell}}) \subset \widehat{G}. \tag{65}$$

Now observe that the inverse limit presentation in (63), along with the above embedding (65), gives an embedding

$$\Delta_K : K \subset \prod \boldsymbol{K}_\ell \subset \prod_{n\geq 3} \boldsymbol{H}_n \subset \prod_{n\geq 3} \boldsymbol{A}_n \subset \widehat{G}. \tag{66}$$

Set $\mathcal{D} = \Delta_K(K) \subset \widehat{G}$. Then as in the proof of Theorem 10.7, use the method of Section 10A to construct a group chain in G. The group G is residually finite, so there exists a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in $\widehat{G}$, where each U_ℓ is normal in $\widehat{G}$. Set $W_\ell = \mathcal{D} \cdot U_\ell$ for $\ell \geq 1$, and $G_\ell = G \cap W_\ell$. Let $\mathcal{G}_K = \{G_\ell\}_{\ell \geq 0}$ denote the resulting group chain.

Let $\{e\} \in U \subset \widehat{G}$ be a normal clopen neighborhood of the identity, so that $\widehat{G}/U$ is a finite group with cardinality $|\widehat{G}/U|$. We claim $\mathrm{core}_U \mathcal{D} = \{e\}$. Note that for $m \geq 5$, the alternating group $\mathrm{Alt}(m)$ is simple, and its cardinality $|\mathrm{Alt}(m)| = \frac{1}{2}m!$ tends to infinity as m increases. As the sequence $\{n_\ell\}$ is increasing, for some $\ell_0 > 0$, we have $\ell \geq \ell_0$, and $\boldsymbol{A}_m = \mathrm{Alt}(n_\ell)$ being nontrivial implies that $\boldsymbol{H}_m$ has order $|\boldsymbol{H}_m| = \frac{1}{2}(n_\ell)! > |\widehat{G}/U|$. Thus, the projection $\boldsymbol{A}_m \subset \widehat{G} \to \widehat{G}/U$ cannot be an injection, and as $\boldsymbol{A}_m$ is a simple group, it must be contained in the kernel, so $\boldsymbol{A}_m \subset U$. Let $\pi_m : \boldsymbol{A} \to \boldsymbol{A}_m$ be the projection onto the m-th factor. We have that $\mathcal{D} \subset \boldsymbol{H} \subset \boldsymbol{A}$. Let $\mathcal{D}_m \subset \boldsymbol{A}_m$ denote its image. By the choice of m, and because $n_\ell \geq d_\ell = 4(|\boldsymbol{K}_\ell| + 2)$, Lemma 10.6 implies the subgroup $\mathcal{D}_m$ has trivial core in $\boldsymbol{A}_m$. It follows that $\mathcal{D}$ has trivial core in U.

Then by Proposition 10.2, for all $k \geq 0$, the discriminant group for the truncated group chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to K. In particular, $\mathcal{G}_K$ is a stable group chain and is not weakly normal.

The rest of the proof proceeds as for that of Theorem 10.7. □

Note that in the above proof, we cannot assert that all leaves of the suspended foliation $\mathcal{F}_{\mathfrak{M}}$ have trivial holonomy, as examples show that some conjugate of $\mathcal{D}$ in $\widehat{G}$ may intersect G nontrivially.

Our final example, which is again based on the application of Theorem 10.5, answers a question posed in [Dyer et al. 2016]. In that work, the notion of a virtually regular action (X, G, Φ) with group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ was introduced:

Definition 10.9. [Dyer et al. 2016, Definition 1.12] A group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is said to be *virtually regular* if there exists a normal subgroup $G_0' \subset G_0$ of finite index such that the restricted chain $\mathcal{G}' = \{G_\ell'\}_{\ell \geq 0}$, where $G_\ell' = G_\ell \cap G_0'$, is weakly normal in G_0'.

There is an alternate definition of this concept, which was shown in [Dyer et al. 2016] to be equivalent: a matchbox manifold $\mathfrak{M}$ is *virtually regular* if there exists a homogeneous matchbox manifold $\mathfrak{M}'$ and a finite-to-one normal covering map $h : \mathfrak{M}' \to \mathfrak{M}$. Thus, the notion of virtually regular is a natural property of a matchbox manifold $\mathfrak{M}$, and can be checked by considering a group chain model for the holonomy action of the foliation $\mathcal{F}_{\mathfrak{M}}$.

The following example is the first known to the authors which is not virtually regular, and gives a natural paradigm for the construction of group chains which are not virtually regular.

Theorem 10.10. *There exists a finitely generated, residually finite, torsion-free group G with profinite completion $\widehat{G}$ such that for any infinite collection $\{F_\ell\}_{\ell\geq 1}$ of nontrivial finite simple groups, their cartesian product $\boldsymbol{F}=\prod F_\ell$ can be embedded into $\widehat{G}$, so that the resulting group chain $\mathcal{G}_{\boldsymbol{F}}=\{G_\ell\}_{\ell\geq 0}$ constructed as in Section 10A yields an equicontinuous minimal Cantor system (X, G, Φ) whose discriminant group for the group chain $\mathcal{G}_{\boldsymbol{F}}$ is isomorphic to $\boldsymbol{F}$. Moreover, $\mathcal{G}_{\boldsymbol{F}}$ is not virtually regular.*

Proof. The proof follows the same approach as that used in the proof of Theorem 10.8. Let $G\subset\Gamma_n$ be the finitely generated, torsion-free, residually finite group used in the proof of Theorem 10.5, as constructed on page 330 of [Lubotzky 1993], and let $\widehat{G}$ be its profinite completion.

For each $\ell\geq 0$, set $d_\ell=4(|F_\ell|+2)$. Then there is an embedding of F_ℓ into the alternating groups, $F_\ell\subset\mathrm{Alt}(|F_\ell|+2)\subset\mathrm{Alt}(d_\ell)$, as in the proof of Theorem 10.8. Choose an increasing sequence of integers $\{n_\ell\mid\ell\geq 1\}$ so that $n_\ell\geq d_\ell$ for all $\ell\geq 1$. Let $\mathrm{Alt}(d_\ell)\subset\mathrm{Alt}(n_\ell)$ be the embedding as the permutations on the first d_ℓ symbols. Then we obtain an embedding $\iota_{\boldsymbol{F}}:\boldsymbol{F}\to\widehat{G}$, of the infinite product $\boldsymbol{F}$ into $\widehat{G}$, given by the composition

$$\iota_{\boldsymbol{F}}:\boldsymbol{F}\cong\prod_{\ell\geq 1}F_\ell\subset\prod_{\ell\geq 1}\mathrm{Alt}(d_\ell)\subset\prod_{\ell\geq 1}\mathrm{Alt}(n_\ell)\subset\prod_{\ell\geq 1}\mathrm{SL}_{n_\ell}(\mathbb{Z}_{p_{n_\ell}})\subset\widehat{G}. \tag{67}$$

Set $\mathcal{D}=\iota_{\boldsymbol{F}}(\boldsymbol{F})\subset\widehat{G}$. Use the method of Section 10A to construct a group chain in G. The group G is residually finite, so there exists a clopen neighborhood system $\{U_\ell\mid\ell\geq 1\}$ about the identity in $\widehat{G}$, where each U_ℓ is normal in $\widehat{G}$. Set $W_\ell=\mathcal{D}\cdot U_\ell$ for $\ell\geq 1$, and $G_\ell=G\cap W_\ell$. Let $\mathcal{G}_{\boldsymbol{F}}=\{G_\ell\}_{\ell\geq 0}$ denote the resulting group chain.

Let $U\subset\widehat{G}$ be a normal clopen neighborhood of the identity. For example, given a normal subgroup $G'\subset G$ with finite index, we can take U to be the profinite completion of G' in $\widehat{G}$. Let $\mathcal{G}^U_{\boldsymbol{F}}=\{G'_\ell\}_{\ell\geq 0}$ be the group chain defined by $G'_\ell=G_\ell\cap U$ for $\ell\geq 0$. Then $\mathcal{D}\cap U=\bigcap(U_\ell\cap U)$.

We next show that the normal core, $\mathrm{core}_U\,\mathcal{D}\subset\mathcal{D}$, of $\mathcal{D}\cap U$ in U is a finite subgroup, and then apply Corollary 10.3 to conclude that the discriminant of the action defined by the group chain $\mathcal{G}^U_{\boldsymbol{F}}$ is a nontrivial Cantor group. The following argument is similar to that used in the proof of Theorem 10.8, and uses that the alternating group $\mathrm{Alt}(m)$ is simple for $m\geq 5$ and has order $|\mathrm{Alt}(m)|=\frac{1}{2}m!$. Let $d_U=|\widehat{G}/U|$ be the order of the finite group.

Choose $\ell_U\geq 1$ such that $n_{\ell_U}\geq 5$ and $|\mathrm{Alt}(n_{\ell_U})|=\frac{1}{2}(n_{\ell_U})!>d_U$. Then for all $\ell\geq\ell_U$, the factor $\mathrm{Alt}(n_\ell)$ in the product in (67) is contained in the kernel of the projection $\widehat{G}\to\widehat{G}/U$, and thus, $F_\ell\subset\mathrm{Alt}(n_\ell)\subset U$. Consequently, we have that

$$\mathcal{D}_{\ell_U}\equiv\prod_{\ell\geq\ell_U}F_\ell\subset\boldsymbol{A}_{\ell_U}\equiv\prod_{\ell\geq\ell_U}\mathrm{Alt}(n_\ell)\subset\mathcal{D}\cap U. \tag{68}$$

In particular, this shows that $\mathcal{D} \cap U$ contains a nontrivial Cantor group. Moreover, by applying Lemma 10.6 to each factor of the product in $\mathcal{D}_{\ell_U}$, we see that $\mathcal{D}_{\ell_U}$ has trivial core in U_ℓ as well. Thus, we have

$$\operatorname{core}_U \mathcal{D} \subset \prod_{1 \le \ell < \ell_U} F_\ell, \tag{69}$$

and so $\operatorname{core}_U \mathcal{D}$ is a finite normal subgroup of U.

By Corollary 10.3, the quotient group chain $\{(G_\ell \cap U)/(\operatorname{core}_U \mathcal{D})\}_{\ell \ge 0}$ has a nontrivial discriminant group $\mathcal{D}/(\operatorname{core}_U \mathcal{D})$ which contains a subgroup isomorphic to the nontrivial Cantor group $\mathcal{D}_{\ell_U}$. For $\ell > 0$, apply this to the case $U = U_\ell$ to obtain that the quotient chain $\{(G_\ell \cap U_\ell)/(\operatorname{core}_{U_\ell} \mathcal{D})\}_{\ell \ge 0}$ is not equivalent to a normal chain. Now suppose that the restricted group chain $\mathcal{G}_{\boldsymbol{F}}^{U_\ell}$ is equivalent to a normal chain. Then as $\operatorname{core}_{U_\ell} \mathcal{D}$ is a normal subgroup of U_ℓ, this implies that the quotient group chain $\{(G_\ell \cap U_\ell)/(\operatorname{core}_{U_\ell} \mathcal{D})\}_{\ell \ge 0}$ is equivalent to a normal chain, hence has trivial discriminant, which is a contradiction. Thus, the group chain $\mathcal{G}_{\boldsymbol{F}}$ is not virtually regular. □

10F. ***Open problems.*** There are many variations of the above method that can be considered, and open questions about the resulting minimal Cantor actions. First, it is interesting to understand the answer to the following.

Problem 10.11. Given a separable profinite group $\widehat{H}$ and an embedding into a profinite group $\widehat{G}$ with trivial rational core, constructed using the methods of [Lubotzky 1993], give criteria for when the resulting equicontinuous minimal Cantor system (X, G, Φ) is weakly normal, and whether the action is stable or wild. Furthermore, when do the resulting actions satisfy the SQA condition of Section 9A?

There is also an extensive literature for the construction of embeddings of groups H into the profinite completions of torsion-free, finitely generated nilpotent and solvable groups. For example, [Crawley-Boevey et al. 1988] showed that if G is a finitely generated, torsion-free nilpotent group, then the profinite completion $\widehat{G}$ is torsion-free, so if $D \subset \widehat{G}$ is a closed subgroup, then it must be a Cantor group.

On the other hand, [Evans 1990; Kropholler and Wilson 1993] showed that there exists a countable, torsion-free, residually finite, metabelian group G such that its profinite completion contains a nontrivial torsion subgroup. Quick [2001] studied the profinite topology of nilpotent groups of class two and finitely generated center-by-metabelian groups, and used this to construct embeddings of finite groups into the profinite completions of these classes of groups. However, the embedding obtained in [Quick 2001] is contained in the center of G, so does not satisfy the trivial core condition. We conclude with an open question, suggested by the examples and results of [Dyer 2015; Dyer et al. 2016; 2017; Fokkink and Oversteegen 2002; Rogers and Tollefson 1971b; Schori 1966].

Problem 10.12. Determine which groups H can be embedded as a closed subgroup of $\widehat{G}$ with trivial rational core, where G is a finitely generated, torsion-free amenable group.

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