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Smooth solutions to the axisymmetric Navier–Stokes equations obey the following maximum principle:

$$\sup_{t\geq 0} \|rv^\theta(t,\cdot)\|_{L^\infty} \leq \|rv^\theta(0,\cdot)\|_{L^\infty}.$$

We prove that all solutions with initial data in $H^{1/2}$ are smooth globally in time if rv^θ satisfies a kind of form boundedness condition (FBC) which is invariant under the natural scaling of the Navier–Stokes equations. In particular, if rv^θ satisfies

$$\sup_{t\geq 0}|rv^\theta(t,r,z)| \leq C_*|\ln r|^{-2}, \quad \text{where} \quad r \leq \delta_0 \in \left(0, \tfrac{1}{2}\right), \quad C_* < \infty,$$

then our FBC is satisfied. Here δ_0 and C_* are independent of neither the profile nor the norm of the initial data. So the gap from regularity is logarithmic in nature. We also prove the global regularity of solutions if $\|rv^\theta(0,\cdot)\|_{L^\infty}$ or $\sup_{t\geq 0}\|rv^\theta(t,\cdot)\|_{L^\infty(r\leq r_0)}$ is small but the smallness depends on a certain dimensionless quantity of the initial data.

1. Introduction

The global regularity problem of three-dimensional incompressible Navier–Stokes equations is commonly considered as supercritical because the a priori estimates based on energy equality become worse when looking into finer and finer scales; see, for instance, [Tao 2007]. Such a "supercriticality" barrier is one of the main reasons why this is such a hard problem.

Recently, the axisymmetric Navier–Stokes equations have attracted tremendous interest from experts. See, for instance, [Burke Loftus and Zhang 2010; Chae and Lee 2002; Chen et al. 2008; 2009; 2015; Hou and Li 2008; Hou et al. 2008; Jiu and Xin 2003; Koch et al. 2009; Lei et al. 2013; Lei and Zhang 2011b; 2011a; Leonardi et al. 1999; Neustupa and Pokorný 2000; 2001; Pan 2016; Seregin and Šverák 2009; Tian and Xin 1998; Zhang and Zhang 2014]. These results heavily depend on the maximum principle of the dimensionless quantity $\Gamma = rv^\theta$, which

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makes the axisymmetric Navier–Stokes equations partially critical (only the swirl component v^θ of the velocity field satisfies a dimensionless a priori estimate). Although the axially symmetric Navier–Stokes equation is a special case of the full three-dimensional one, our level of understanding had been roughly the same, with the previously mentioned difficulty unresolved because the effective a priori bound available is still the energy estimate, which has positive dimension $\frac{1}{2}$.

The aim of this article is to show that the axisymmetric Navier–Stokes equation is, in fact, fully critical. More precisely, we prove all solutions with initial data in $H^{1/2}$ are smooth globally in time if rv^θ satisfies a kind of FBC which is invariant under the natural scaling of the Navier–Stokes equations. In particular, if rv^θ satisfies $\sup_{t\ge 0}|rv^\theta(t,r,z)| \le C_*|\ln r|^{-2}$, where $r \le \delta_0 \in \left(0, \frac{1}{2}\right)$, $C_* < \infty$, then our FBC is satisfied. Here δ_0 and C_* are independent of neither the profile nor the norm of the initial data. The proof is based on the observation that the vorticity equations can be transformed into a system such that the vortex-stretching terms are critical. This means that the potentials in front of unknown functions scale as $1/|x|^2$. For example, in (1-8) below, the function J is regarded as unknown and the potential in front of it is $-2v^\theta/r$ which scales as $1/|x|^2$.

We also prove the global regularity of solutions if $\sup_{t\ge 0}\|rv^\theta(t,\cdot)\|_{L^\infty(r\le r_0)}$ or $\|rv^\theta(0,\cdot)\|_{L^\infty}$ is small but the smallness depends on a certain dimensionless quantity of the initial data. Our work is inspired by the recent interesting result of Chen, Fang and Zhang in [Chen et al. 2015] where, among other things, global regularity is obtained if $rv^\theta(t,\cdot,z)$ is Hölder continuous in the r variable.

To state our result more precisely, let us recall that in cylindrical coordinates r, θ, z with $(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$, axially symmetric solutions of the Navier–Stokes equations are of the following form:

$$\begin{cases} v(t,x) = v^r(t,r,z)e_r + v^\theta(t,r,z)e_\theta + v^z(t,r,z)e_z, \\ p(t,x) = p(t,r,z). \end{cases}$$

The components v^r, v^θ, v^z are all independent of the angle of rotation θ. Here e_r, e_θ, e_z are the basis vectors for $\mathbb{R}^3$ given by

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^\top, \quad e_\theta = \left(\frac{-x_2}{r}, \frac{x_1}{r}, 0\right)^\top, \quad e_z = (0,0,1)^\top.$$

In terms of (v^r, v^θ, v^z, p), the axisymmetric Navier–Stokes equations are

$$\begin{cases} \partial_t v^r + (v^r e_r + v^z e_z)\cdot\nabla v^r - \dfrac{(v^\theta)^2}{r} + \partial_r p = \left(\Delta - \dfrac{1}{r^2}\right)v^r, \\ \partial_t v^\theta + (v^r e_r + v^z e_z)\cdot\nabla v^\theta + \dfrac{v^r v^\theta}{r} = \left(\Delta - \dfrac{1}{r^2}\right)v^\theta, \\ \partial_t v^z + (v^r e_r + v^z e_z)\cdot\nabla v^z + \partial_z p = \Delta v^z, \\ \partial_r v^r + \dfrac{v^r}{r} + \partial_z v^z = 0. \end{cases} \tag{1-1}$$

It is well known that finite energy smooth solutions of the Navier–Stokes equations satisfy the following energy identity

$$\|v(t,\cdot)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s,\cdot)\|_{L^2}^2\,ds = \|v_0\|_{L^2}^2, \quad \forall\, t \geq 0. \tag{1-2}$$

Set

$$\Gamma = r v^\theta.$$

One can easily check that

$$\partial_t \Gamma + (v^r e_r + v^z e_z)\cdot\nabla\Gamma = \Big(\Delta - \frac{2}{r}\partial_r\Big)\Gamma. \tag{1-3}$$

A significant consequence of (1-3) is that smooth solutions of the axisymmetric Navier–Stokes equations satisfy the following maximum principle; see, for instance, [Chae and Lee 2002; Hou and Li 2008; Chen et al. 2008; Neustupa and Pokorný 2000; 2001]:

$$\sup_t \|\Gamma(t,\cdot)\|_{L^\infty} \leq \|\Gamma_0\|_{L^\infty}. \tag{1-4}$$

We emphasize that $\|\Gamma(t,\cdot)\|_{L^\infty}$ is a dimensionless quantity with respect to the natural scaling of the Navier–Stokes equations. From this point of view, the axisymmetric Navier–Stokes equations can be seen as partially critical, while the general Navier–Stokes equations are known to be supercritical; see [Tao 2007].

Now let us introduce the function class where v^θ lives. It is defined in an integral way which is usually called the form boundedness condition (FBC), which is similar to a condition that a certain Hardy-type inequality holds.

Definition 1.1. We say that the angular velocity $v^\theta(t,r,z)$ is in a (δ_*, C_*)-critical class if

$$\int \frac{|v^\theta|}{r}|f|^2\,dx \leq C_* \int |\partial_r f|^2\,dx + C_0 \int_{r\geq r_0} |f|^2\,dx, \tag{1-5}$$

$$\int |v^\theta|^2 |f|^2\,dx \leq \delta_* \int |\partial_r f|^2\,dx + C_0 \int_{r\geq r_0} |f|^2\,dx, \tag{1-6}$$

hold for some $r_0 > 0$, some $C_0 > 0$ and for all $t \geq 0$ and all axisymmetric scalar and vector functions $f \in H^1$.

Clearly, under the natural scaling of the Navier–Stokes equations, namely,

$$v^\lambda(t,x) = \lambda v(\lambda^2 t, \lambda x), \qquad p^\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x),$$

the above definition of FBC is invariant: $(v^\lambda)^\theta$ satisfies (1-5)–(1-6) if v^θ also does.

We now state the first result of this article.

Theorem 1.2. *For any $C_* > 1$, there exists a constant $\delta_* > 0$ depending on C_* such that the following conclusion holds for all local strong solutions to the axially symmetric Navier–Stokes equations with initial data $\|v_0\|_{H^{1/2}} < \infty$ and $\|\Gamma_0\|_{L^\infty} < \infty$. If the angular velocity field v^θ is in the (δ_*, C_*)-critical class, i.e., v^θ satisfies the critical form boundedness condition in* (1-5)–(1-6), *then v is regular globally in time.*

An important corollary of Theorem 1.2 is this:

Corollary 1.3. *Let $\delta_0 \in \left(0, \frac{1}{2}\right)$ and $C_1 > 1$. Let v be the local strong solution of the axially symmetric Navier–Stokes equations with initial data $v_0 \in H^{1/2}$ and $\|\Gamma_0\|_{L^\infty} < \infty$. If*

$$\sup_{0\le t<T} |\Gamma(t, r, z)| \le C_1 |\ln r|^{-2}, \quad r \le \delta_0, \tag{1-7}$$

then v is regular globally in time.

We emphasize that C_* in Theorem 1.2 and C_1 in Corollary 1.3 are independent of neither the profile nor the norm of the given initial data. The proof of this corollary will be given at the end of Section 2. The point is that if (1-7) is satisfied, then the FBC of (1-5)–(1-6) is true. Then one can apply Theorem 1.2 to get the desired conclusion.

Our work is inspired by a recent very interesting work by Chen, Fang and Zhang [2015] where, among other things, the authors proved that v is regular if Γ is Hölder continuous. Let

$$\Omega = \frac{\omega^\theta}{r}, \qquad J = -\frac{\partial_z v^\theta}{r}.$$

We emphasize that J was introduced in [Chen et al. 2015], while Ω appeared much earlier and can be at least tracked back to the book of Majda and Bertozzi [2002]. Both of the two new variables are of great importance in our work. Following [Majda and Bertozzi 2002; Hou and Li 2008; Chen et al. 2015], we also study the equations for J and Ω:

$$\begin{cases} \partial_t J + (b\cdot\nabla) J = \left(\Delta + \frac{2}{r}\partial_r\right) J + (\omega^r \partial_r + \omega^z \partial_z)\dfrac{v^r}{r}, \\ \partial_t \Omega + (b\cdot\nabla)\Omega = \left(\Delta + \frac{2}{r}\partial_r\right)\Omega - 2\dfrac{v^\theta}{r} J. \end{cases} \tag{1-8}$$

Here ω^θ is the angular component of the vorticity $\omega = \nabla\times v$, which reads

$$\omega(t, x) = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z,$$

with

$$\omega^r = -\partial_z v^\theta, \quad \omega^\theta = \partial_z v^r - \partial_r v^z, \quad \omega^z = \partial_r v_\theta + \frac{v^\theta}{r}.$$

Our new observation is that the axisymmetric Navier–Stokes equations exhibit certain critical nature when being formulated in terms of a new set of unknowns,

J and Ω. Our second observation is that, with the FBC assumptions (1-5)–(1-6), the stretching term $(\omega^r \partial_r + \omega^z \partial_z) v^r / r$ in the equation for J could be arbitrarily small, by using the relation of v^r, v^z and Ω in Lemma 2.1 (which was originally proved by Hou, Lei and Li [Hou et al. 2008] in the periodic case and later on extended to the general case by Lei [2015]. Alternatively, one may also use the magic formula given by Miao and Zheng [2013] to prove it). Then we can derive a closed a priori estimate for J and Ω using the first two observations and the structure of the stretching term in the equation for Ω.

Our second goal is to prove that the smallness of $\sup_{t\ge 0} \|\Gamma(t,\cdot)\|_{L^\infty(r\le r_0)}$ or $\|\Gamma_0\|_{L^\infty}$ implies the global regularity of the solutions. Recently, Chen, Fang and Zhang [2015] proved that, among many other interesting results, if $\Gamma(t,\cdot,z)$ is Hölder continuous in the r variable, then the solution of the axisymmetric Navier–Stokes equations is smooth. Both of the results depend on given initial data. More precisely, the smallness of $\sup_{t\ge 0} \|\Gamma(t,\cdot)\|_{L^\infty(r\le r_0)}$ or $\|\Gamma_0\|_{L^\infty}$ in our Theorem 1.4 depends on other dimensionless norms of the initial data. From this point of view, our result improves the one in [Chen et al. 2015].

We define
$$V = \frac{v^\theta}{\sqrt{r}}.$$
Here is the second main result:

Theorem 1.4. *Let $r_0 > 0$. Suppose that $v_0 \in H^{1/2}$ such that $\Omega_0 \in L^2$, $V_0^2 \in L^2$ and $\Gamma_0 \in L^2 \cap L^\infty$. Denote*
$$\big(\|\Omega_0\|_{L^2} + \|V_0^2\|_{L^2}\big)\|\Gamma_0\|_{L^2} = M_0$$
and
$$\big(\|V_0^2\|_{L^2} + \|\Omega_0\|_{L^2} + r_0^{-2}\|v_0\|_{L^2}\|\Gamma_0\|_{L^\infty}^{3/2}\big)\|\Gamma_0\|_{L^2} = M_1.$$
There exists an absolute positive (small) constant $\delta > 0$ such that if either
$$\|\Gamma_0\|_{L^\infty} \le \delta M_0^{-1},$$
or
$$\sup\nolimits_{t\ge 0} \|\Gamma(t,\cdot)\|_{L^\infty(r\le r_0)} \le \delta M_1^{-1},$$
then the axially symmetric Navier–Stokes equations are globally well-posed.

The proof of Theorem 1.4 is based on a new formulation of the axisymmetric Navier–Stokes equations (1-1) in terms of V and $\Omega = \omega^\theta / r$, and also on the estimate of v^r / r in terms of Ω and its derivative (see Lemma 2.1).

Now let us recall some highlights on the study of the axisymmetric Navier–Stokes equations. It has been known since the late 1960s (see [Ladyzhenskaya 1968; Ukhovskii and Iudovich 1968]) that if the swirl $v_\theta = 0$, then finite energy solutions to (1-1) are smooth for all time. See also [Leonardi et al. 1999], by

Leonardi, Málek, Nečas and Pokorný. In the presence of swirl, it is not known in general if finite energy solutions blow up or not in finite time. Hou and Li [2008] constructed a family of large solutions based on some deep insights on a one-dimensional model. See also some extended results in [Hou et al. 2008] by Hou, Lei and Li. We also mention various a priori estimates of smooth solutions by Chae and Lee [2002] and Burke Loftus and Zhang [2010]. To the best of our knowledge, the best a priori bound of the velocity field is given in [Lei et al. 2013]:

$$|v(t,x)| \le C_* r^{-2} |\ln r|^{1/2}.$$

In [Chen et al. 2008], Chen, Strain, Tsai and Yau obtained a lower bound for the possible blow up rate of singularities: if

$$|v(t,x)| \le \frac{C_*}{r},$$

then v is regular. This seems to be the first time that people have been able to exclude possible singularities in the presence of assumptions on $|x|^{-1}$-type nonsmallness quantities. Soon afterward, Chen, Strain, Yau and Tsai [Chen et al. 2009] and Koch, Nadirashvili, Seregin and Šverák [Koch et al. 2009] extended the result of [Chen et al. 2008] and in particular, excluded the possibility of type I singularities of v. See also a local version by Seregin and Šverák [2009] and various extensions by Pan [2016]. We also mention that Lei and Zhang [2011b] excluded the possibility of singularities under

$$v^r e_r + v^z e_z \in L^\infty([0,T], \mathrm{BMO}^{-1})$$

based on an observation in [Lei and Zhang 2011a]. This solves the regularity problem of $L^\infty\big([0,T], \mathrm{BMO}^{-1}\big)$ solutions of Navier–Stokes equations in the axisymmetric case. Moreover, it extends the result of [Chen et al. 2008] and [Koch et al. 2009] since the assumptions on the axial component of velocity $|v^z| \le C_* r^{-1}$ itself imply $v^r e_r + v^z e_z \in L^\infty([0,T], \mathrm{BMO}^{-1})$ (see [Lei et al. 2013] for details).

Let us also mention that Neustupa and Pokorný [2000] proved that the regularity of one component (either v^r or v^θ) implies regularity of the other components of the solution. The work of Jiu and Xin [2003] also proves regularity under an assumption of sufficiently small zero-dimension scaled norms. See more refined results in [Neustupa and Pokorný 2001] and the work of Ping Zhang and Ting Zhang [Zhang and Zhang 2014]. Chae and Lee [2002] also proved regularity results assuming finiteness of another certain zero-dimensional integral. Tian and Xin [1998] constructed a family of singular axially symmetric solutions with singular initial data.

The remainder of the paper is simply organized as follows. In Section 2 we recall two basic lemmas and prove Corollary 1.3 by assuming the validity of Theorem 1.2. In Section 3 we prove Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.4.

Remark. This paper was posted on the arXiv in May 2015. In August 2015, Dongyi Wei [2016] improved one of our regularity conditions by a factor of $\sqrt{|\ln r|}$.

2. Notations and lemmas

For abbreviation, we denote

$$b(t,x) = v^r e_r + v^z e_z.$$

The last equation in (1-1) shows that b is divergence-free. The Laplacian operator Δ and the gradient operator ∇ in the cylindrical coordinate are

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2, \quad \nabla = e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z.$$

For the scalar axisymmetric function $f(t,r,z)$, we often use the commutation property:

$$\nabla\partial_r f(r,z) = \partial_r \nabla f(r,z).$$

Throughout the proof, we will denote

$$\|f\|_{L^2}^2 = \int |f|^2 r\,dr\,dz, \quad dx = r\,dr\,dz.$$

The estimate in Lemma 2.1 will be used often. It was originally proved in [Hou et al. 2008] in the periodic case and then extended to the general case in [Lei 2015] (see (4.5)–(4.6) there, noting the relation $v^r = -\partial_z\psi^\theta$). Alternatively, one may also use the magic formula given by Miao and Zheng [2013] to prove it.

Lemma 2.1. *Let v^r be the radial component of the velocity field and $\Omega = \omega^\theta/r$. Then there exists an absolute positive constant $K_0 > 0$ such that*

$$\Big\|\nabla\frac{v^r}{r}\Big\|_{L^2} \le K_0\|\Omega\|_{L^2}, \quad \Big\|\nabla^2\frac{v^r}{r}\Big\|_{L^2} \le K_0\|\partial_z\Omega\|_{L^2}.$$

Lemma 2.2 gives the uniform decay estimate for the angular component of vorticity in the r direction for large r. We point out that a weaker estimate for ω^θ has appeared in [Chae and Lee 2002]. Even though we don't need to use the estimate for ω^r and ω^z in this paper, we will include them below for possible future use.

Lemma 2.2. *Suppose that $v_0 \in L^2$ is an axially symmetric divergence-free vector and $(r\omega_0^r, r^2\omega_0^\theta, r\omega_0^z) \in L^2$. Then the smooth solution of the Navier–Stokes equation with initial data v_0 satisfies the following a priori estimates*:

$$\begin{aligned}(2\text{-}1)\quad \sup_{0\le t<T}&\big(\|r\omega^r(t,\cdot)\|_{L^2}^2, \|r\omega^z(t,\cdot)\|_{L^2}^2\big)\\ &+\int_0^T \big(\|\nabla[r\omega^r(t,\cdot)]\|_{L^2}^2, \|\nabla[r\omega^z(t,\cdot)]\|_{L^2}^2\big)\,dt\\ &\le \|r\omega_0^r\|_{L^2}^2 + \|r\omega_0^z\|_{L^2}^2 + 4(\|\Gamma_0\|_{L^\infty}^2+1)\|v_0\|_{L^2}^2,\end{aligned}$$

and

$$\sup_{0\le t<T} \|r^2\omega^\theta(t,\cdot)\|_{L^2}^2 + \int_0^T \|\nabla(r^2\omega^\theta)\|_{L^2}^2\,dt \\ \le C_0\Big(\|r^2\omega_0^\theta\|_{L^2}^2 + \big(\|v_0\|_{L^2}^4 + \|\Gamma_0\|_{L^3}^2\big)\|v_0\|_{L^2}^2\Big)\exp\Big\{\frac{T}{\|\Gamma_0\|_{L^3}^2 + \|v_0\|_{L^2}^4}\Big\}, \tag{2-2}$$

where C_0 is a generic positive constant.

Proof. First of all, let us recall that

$$\begin{cases} \partial_t\omega^r + b\cdot\nabla\omega^r - \partial_r v^r\omega^r = \Big(\Delta - \frac{1}{r^2}\Big)\omega^r + \partial_z v^r\omega^z, \\ \partial_t\omega^\theta + b\cdot\nabla\omega^\theta - \frac{v^r}{r}\omega^\theta = \Big(\Delta - \frac{1}{r^2}\Big)\omega^\theta + \partial_z\frac{(v^\theta)^2}{r}, \\ \partial_t\omega^z + b\cdot\nabla\omega^z - \partial_z v^z\omega^z = \Delta\omega^z + \partial_r v^z\omega^r. \end{cases} \tag{2-3}$$

Let us first prove (2-1). Taking the L^2 inner product of the first equation of (2-3) with $r^2\omega^r$, and of the third equation with $r^2\omega^z$, we have

$$\frac{1}{2}\frac{d}{dt}\int(\omega^r)^2r^2\,dx - \int r^2\omega^r\Big(\Delta - \frac{1}{r^2}\Big)\omega^r\,dx \\ = -\int r^2\omega^r b\cdot\nabla\omega^r\,dx + \int r^2\partial_r v^r(\omega^r)^2\,dx + \int r^2\omega^r\partial_z v^r\omega^z\,dx$$

and

$$\frac{1}{2}\frac{d}{dt}\int(\omega^z)^2r^2\,dx - \int r^2\omega^z\Delta\omega^z\,dx \\ = -\int r^2\omega^z b\cdot\nabla\omega^z\,dx + \int r^2\partial_z v^z(\omega^z)^2\,dx + \int r^2\omega^z\partial_r v^z\omega^r\,dx.$$

Using integration by parts, we have

$$-\int r^2\omega^r\Big(\Delta - \frac{1}{r^2}\Big)\omega^r\,dx = \int|\nabla(r\omega^r)|^2\,dx$$

and

$$-\int r^2\omega^z\Delta\omega^z\,dx = \int|\nabla(r\omega^z)|^2\,dx - 2\int|\omega^z|^2\,dx.$$

Using integration by parts and the incompressibility constraint, one has

$$-\int r^2\omega^r b\cdot\nabla\omega^r\,dx - \int r^2\omega^z b\cdot\nabla\omega^z\,dx + \int r^2\partial_r v^r(\omega^r)^2dx + \int r^2\partial_z v^z(\omega^z)^2dx \\ = \int(rv^r + r^2\partial_r v^r)(\omega^r)^2 + (rv^r + r^2\partial_z v^z)(\omega^z)^2\,dx \\ = -\int\big[\partial_z v^z(r\omega^r)^2 + \partial_r v^r(r\omega^z)^2\big]\,dx.$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\big[(\omega^r)^2+(\omega^z)^2\big]r^2\,dx+\int\big(|\nabla(r\omega^r)|^2+|\nabla(r\omega^z)|^2\big)\,dx\\
&\quad=\int|\omega^z|^2\,dx-\int\big[\partial_z v^z(r\omega^r)^2+\partial_r v^r(r\omega^z)^2\big]\,dx+\int\big(r^2\omega^r\partial_z v^r\omega^z+r^2\omega^z\partial_r v^z\omega^r\big)\,dx\\
&\quad\le 2\int|\omega^z|^2\,dx+\|\nabla b\|_{L^2}\big(\|r\omega^r\|_{L^4}^2+\|r\omega^z\|_{L^4}^2\big).
\end{aligned}$$

Note that by Gagliardo–Nirenberg's inequality and the maximum principle $\|\Gamma\|_{L^\infty}\le\|\Gamma_0\|_{L^\infty}$, one has

$$\begin{aligned}
\|r\omega^r\|_{L^4}^2+\|r\omega^z\|_{L^4}^2&=\|\nabla\Gamma\|_{L^4}^2\\
&=\Big(\int-\Gamma\nabla\cdot(\nabla\Gamma|\nabla\Gamma|^2)\,dx\Big)^{1/2}\le 3\|\Gamma\|_{L^\infty}\|\Delta\Gamma\|_{L^2}\\
&\le 3\|\Gamma_0\|_{L^\infty}\big(\|\partial_r(r\omega^z)\|_{L^2}+\|\partial_z(r\omega^r)\|_{L^2}+\|\omega^z\|_{L^2}\big).
\end{aligned}$$

Hence, by Hölder's inequality, we have

$$\frac{d}{dt}\int\big[(\omega^r)^2+(\omega^z)^2\big]r^2\,dx+\int\big(|\nabla(r\omega^r)|^2+|\nabla(r\omega^z)|^2\big)\,dx\le 4\big(\|\Gamma_0\|_{L^\infty}^2+1\big)\|\nabla b\|_{L^2}^2.$$

Integrating the above differential inequality with respect to time and recalling the basic energy estimate, one gets (2-1).

Next, let us prove (2-2). Let us first write the second equation of (2-3) as:

$$\partial_t(r^2\omega^\theta)+b\cdot\nabla(r^2\omega^\theta)-3rv^r\omega^\theta=\frac{\partial_z\Gamma^2}{r}+\Delta(r^2\omega^\theta)-\frac{4}{r}\partial_r(r^2\omega^\theta)+3\omega^\theta.$$

The standard energy estimate gives that

$$\frac{1}{2}\frac{d}{dt}\|r^2\omega^\theta\|_{L^2}^2+\|\nabla(r^2\omega^\theta)\|_{L^2}^2=3\int rv^r\omega^\theta r^2\omega^\theta\,dx+\int\partial_z\Gamma^2 r\omega^\theta\,dx+3\int\omega^\theta r^2\omega^\theta\,dx.$$

It is easy to estimate that

$$\begin{aligned}
\int\partial_z\Gamma^2 r\omega^\theta\,dx&\le 2\|\Gamma\|_{L^3}\,\|\nabla v^\theta\|_{L^2}\,\|r^2\omega^\theta\|_{L^6}\\
&\le 4\|\Gamma_0\|_{L^3}^2\,\|\nabla v^\theta\|_{L^2}^2+\tfrac{1}{4}\|\nabla(r^2\omega^\theta)\|_{L^2}^2.
\end{aligned}$$

Next, one also has

$$\begin{aligned}
\int\omega^\theta r^2\omega^\theta\,dx&\le\|\omega^\theta\|_{L^2(r\le R(t))}^2R^2(t)+\|r^2\omega^\theta\|_{L^2(r>R(t))}^2R^{-2}(t)\\
&\le\|\omega^\theta\|_{L^2}^2R^2(t)+\|r^2\omega^\theta\|_{L^2}^2R^{-2}(t).
\end{aligned}$$

Finally, we estimate that

$$\begin{aligned}\int r v^r \omega^\theta r^2 \omega^\theta \, dx &\le \|v^r\|_{L^2} \, \|(r^2\omega^\theta)^{3/2}\|_{L^4} \, \|(\omega^\theta)^{1/2}\|_{L^4} \\ &\le \|v_0\|_{L^2}^4 \, \|\omega^\theta\|_{L^2}^2 + \tfrac{1}{4}\|\nabla(r^2\omega^\theta)\|_{L^2}^2.\end{aligned}$$

By taking $R(t) = \|\Gamma_0\|_{L^3} + \|u_0\|_{L^2}^2$, we arrive at

$$\begin{aligned}&\frac{d}{dt}\|r^2\omega^\theta\|_{L^2}^2 + \|\nabla(r^2\omega^\theta)\|_{L^2}^2 \\ &\qquad \lesssim \big(\|\Gamma_0\|_{L^3}^2 + \|v_0\|_{L^2}^4\big)\|\nabla v\|_{L^2}^2 + \big(\|\Gamma_0\|_{L^3}^2 + \|u_0\|_{L^2}^4\big)^{-1}\|r^2\omega^\theta\|_{L^2}^2.\end{aligned}$$

Clearly, (2-2) follows by the basic energy estimate and applying Gronwall's inequality to the above differential inequality. □

Finally, let us prove Corollary 1.3 by using Theorem 1.2.

Proof. It suffices to check the validity of FBC in (1-5)–(1-6) under the assumptions in Corollary 1.3. Let $\delta_0 \in \big(0, \frac{1}{2}\big)$ and $C_1 > 1$ be arbitrarily large. Noting $\|\Gamma_0\|_{L^\infty} < \infty$ and using the maximum principle, we have

$$\|\Gamma(t,\cdot)\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty}.$$

Take a smooth cut-off function of r such that

$$\phi \equiv 1 \text{ if } 0 \le r \le 1, \quad \phi \equiv 0 \text{ if } r \ge 2.$$

For all $\delta < \delta_0/2$, using (1-7), one has

$$\int \frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \le \int \frac{C_1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz.$$

Using integration by parts, one has

$$\begin{aligned}&\int \frac{1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \\ &\qquad = \int \Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 d|\ln r|^{-1}\,dz = \int |\ln r|^{-1}\phi\Big(\frac{r}{\delta}\Big)f\,\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\,dr\,dz \\ &\qquad \le \frac{1}{2}\int \frac{1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz + \frac{1}{2}\int \Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz.\end{aligned}$$

Hence, we have

$$\int \frac{1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \le \int \Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz,$$

which further gives that

$$\int \frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \le C_1\int \Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz.$$

On the other hand, it is easy to see that

$$\int \frac{|v^\theta|}{r}\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz \le \|\Gamma_0\|_{L^\infty}\delta^{-2}\int\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz.$$

Consequently, we have

$$\begin{aligned}(2\text{-}4)\quad &\int \frac{|v^\theta|}{r}|f|^2 r\,dr\,dz\\ &\le 2C_1\int\Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz+2\|\Gamma_0\|_{L^\infty}\delta^{-2}\int\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz\\ &\le 4C_1\int|\partial_r f|^2 r\,dr\,dz+C\delta^{-2}\int_{r\ge\delta}|f|^2 r\,dr\,dz.\end{aligned}$$

Here and in the next inequality we use C to denote some generic positive constant whose value may change from line to line and which may depend on $\|\Gamma_0\|_{L^\infty}$ and C_1.

Next, using (2-4), we have

$$\begin{aligned}(2\text{-}5)\quad &\int |v^\theta|^2|f|^2 r\,dr\,dz\\ &\le 2\int|rv^\theta|\frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz+2\int|rv^\theta|^2r^{-2}\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz\\ &\le 2C_1|\ln\delta|^{-2}\int\frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz+2\|\Gamma_0\|_{L^\infty}^2\delta^{-2}\int_{r\ge\delta}|f|^2 r\,dr\,dz\\ &\le C\delta^{-2}\int_{r\ge\delta}|f|^2 r\,dr\,dz+8C_1^2|\ln\delta|^{-2}\int|\partial_r f|^2 r\,dr\,dz.\end{aligned}$$

Hence, one may choose δ small enough so that $16C_1^2|\ln\delta|^{-2}\le\delta_*$ and choose $C_*=4C_1$. Then it is clear from (2-4) and (2-5) that the assumptions in equations (1-5)–(1-6) are satisfied. Using Theorem 1.2, one concludes that v is smooth for all $t>0$. □

3. Criticality of axisymmetric Navier–Stokes equations

Proof of Theorem 1.2. First of all, for initial data $v_0\in H^{1/2}$, by the classical results of Leray [1934] and Fujita and Kato [1964], there exists a unique local strong solution v to the Navier–Stokes equations (1-1). Moreover, $v(t,\cdot)\in H^s$ for any $s\ge 0$, at least on a short time interval $[\epsilon,2\epsilon]$. In particular, $\nabla\omega(t,\cdot)\in L^2$, at least on a short time interval $[\epsilon,2\epsilon]$. A consequence is that $\nabla\omega^r$, $\nabla\omega^\theta$, $\nabla\omega^z$, ω^r/r, ω^θ/r are all L^2-functions. In particular, recalling that

$$J=\frac{\omega^r}{r}\quad\text{and}\quad\Omega=\frac{\omega^\theta}{r},$$

one has $J(t,\cdot)\in L^2$ and $\Omega(t,\cdot)\in L^2$ for $t\in[\epsilon,2\epsilon]$. Inductively, one also has $J(t,\cdot)\in H^2$ and $\Omega(t,\cdot)\in H^2$. Without loss of generality, we may assume that

$$J_0\in H^2 \quad\text{and}\quad \Omega_0\in H^2.$$

Otherwise we may start from $t=\epsilon$. As long as the solution is still smooth, one has

$$\|J(t,\cdot)\|_{L^2}+\|\Omega(t,\cdot)\|_{L^2}<\infty, \qquad \|\nabla J(t,\cdot)\|_{L^2}^2+\|\nabla\Omega(t,\cdot)\|_{L^2}^2<\infty$$

and

$$\int_{-\infty}^{\infty}\big(|J(t,0,z)|^2+|\Omega(t,0,z)|^2\big)dz\lesssim\|J(t,\cdot)\|_{H^2}^2+\|\Omega(t,\cdot)\|_{H^2}^2<\infty.$$

So all calculations below are legal as long as the solution is still smooth. Our task is to derive a certain sufficiently strong a priori estimate.

By applying the standard energy estimate to the first equation in (1-8), we have

$$\frac{1}{2}\frac{d}{dt}\|J\|_{L^2}^2=-\int J(b\cdot\nabla)Jr\,dr\,dz+\int J\Big(\Delta+\frac{2}{r}\partial_r\Big)J \\ +\int J(\omega^r\partial_r+\omega^z\partial_z)\frac{v^r}{r}r\,dr\,dz.$$

Using the incompressibility constraint, one has

$$-\int J(b\cdot\nabla)Jr\,dr\,dz=\frac{1}{2}\int J^2\nabla\cdot br\,dr\,dz=0.$$

On the other hand, by direct calculations, one has

$$\int J\Big(\Delta+\frac{2}{r}\partial_r\Big)J=-\|\nabla J\|_{L^2}^2-\int|J(t,0,z)|^2\,dz.$$

Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\|J\|_{L^2}^2+\|\nabla J\|_{L^2}^2+\int_{-\infty}^{\infty}|J(t,0,z)|^2\,dz=\int J(\omega^r\partial_r+\omega^z\partial_z)\frac{v^r}{r}r\,dr\,dz. \tag{3-1}$$

Similarly, by applying the energy estimate to the second equation in (1-8), one obtains that

$$\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2+\int_{-\infty}^{\infty}|\Omega(t,0,z)|^2\,dz=-2\int\frac{v^\theta}{r}J\Omega r\,dr\,dz. \tag{3-2}$$

In the remaining part of the proof of Theorem 1.2, we will use C to denote a generic positive constant whose value may change from line to line and which may depend on $\|\Gamma_0\|_{L^\infty}$, C_0, C_* and r_0. Using $\|\Gamma\|_{L^\infty}\le\|\Gamma_0\|_{L^\infty}$ and the form boundedness condition in (1-5), one has

$$
\begin{aligned}
(3\text{-}3)\quad &\left|\int \frac{v^\theta}{r} J\Omega r\,dr\,dz\right| \\
&\le \frac{1}{4C_*}\int\left|\frac{v^\theta}{r}\right|\Omega^2 r\,dr\,dz + C_*\int\left|\frac{v^\theta}{r}\right| J^2 r\,dr\,dz \\
&\le \frac{1}{4}\int |\partial_r\Omega|^2 r\,dr\,dz + C_*^2\int |\partial_r J|^2 r\,dr\,dz + C\int_{r\ge r_0}\big(|J|^2+|\Omega|^2\big) r\,dr\,dz.
\end{aligned}
$$

Inserting (3-3) into (3-2), one has

$$
(3\text{-}4)\qquad \frac{d}{dt}\|\Omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 + 2\int_{-\infty}^{\infty} |\Omega(t,0,z)|^2\,dz \le 2C_*^2\|\nabla J\|_{L^2}^2 + C\|\omega\|_{L^2}^2.
$$

Next, we estimate that

$$
\begin{aligned}
\left|\int J(\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r}\,dx\right| &= \left|\int [\nabla\times(v^\theta e_\theta)]\cdot\Big(J\nabla\frac{v^r}{r}\Big)\,dx\right| \\
&\le \|\nabla J\|_{L^2}\left\|v^\theta\nabla\frac{v^r}{r}\right\|_{L^2}.
\end{aligned}
$$

Again, using the form boundedness condition in (1-6), one has

$$
\left\|v^\theta\nabla\frac{v^r}{r}\right\|_{L^2}^2 \le \delta_*\left\|\partial_r\nabla\frac{v^r}{r}\right\|_{L^2}^2 + C_0\int_{r\ge r_0}\left|\nabla\frac{v^r}{r}\right|^2 dr\,dr\,dz.
$$

Using Lemma 2.1 and the identity

$$
\nabla\frac{v^r}{r} = \frac{\nabla v^r}{r} - e_r\frac{v^r}{r^2} = \frac{\nabla v^r}{r} + e_r\frac{\partial_r v^r + \partial_z v^z}{r},
$$

we have

$$
\begin{aligned}
(3\text{-}5)\quad &\left|\int J(\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r} r\,dr\,dz\right| \\
&\le \tfrac{1}{2}\|\nabla J\|_{L^2}^2 + \frac{\delta_*}{2}\left\|\partial_r\nabla\frac{v^r}{r}\right\|_{L^2}^2 + \frac{C_0}{2}\int_{r\ge r_0}\left|\nabla\frac{v^r}{r}\right|^2 dr\,dr\,dz \\
&\le \tfrac{1}{2}\|\nabla J\|_{L^2}^2 + \frac{K_0\delta_*}{2}\|\partial_z\Omega\|_{L^2}^2 + C\int|\nabla v|^2\,dr\,dr\,dz.
\end{aligned}
$$

Inserting (3-5) into (3-1), we have

$$
\begin{aligned}
(3\text{-}6)\quad &\frac{d}{dt}\|J\|_{L^2}^2 + \|\nabla J\|_{L^2}^2 + 2\int_{-\infty}^{\infty}|J(t,0,z)|^2\,dz \\
&\le K_0\delta_*\|\partial_z\Omega\|_{L^2}^2 + C\int|\nabla v|^2\,dr\,dr\,dz.
\end{aligned}
$$

Multiplying (3-6) by $3C_*^2$ and then adding it to (3-4), we have

$$\frac{d}{dt}\big(3C_*^2\|J\|_{L^2}^2+\|\Omega\|_{L^2}^2\big)+\big(C_*^2\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\big)\\ +\int_{-\infty}^{\infty}\big(6C_*^2|J(t,0,z)|^2+2|\Omega(t,0,z)|^2\big)\,dz\\ \le 3C_*^2K_0\delta_*\|\nabla\Omega\|_{L^2}^2+C\int|\nabla v|^2r\,dr\,dz.$$

Integrating the above inequality with respect to time, we have

$$3C_*^2\|J(t,\cdot)\|_{L^2}^2+\|\Omega(t,\cdot)\|_{L^2}^2+\int_0^t\big(C_*^2\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\big)\,ds\\ +\int_0^t\int_{-\infty}^{\infty}\big(6C_*^2|J(t,0,z)|^2+2|\Omega(t,0,z)|^2\big)\,dz\,ds\\ \le CC_*^2\big(\|J_0\|_{L^2}^2+\|\Omega_0\|_{L^2}^2\big)+3C_*^2K_0\delta_*\int_0^t\|\nabla\Omega\|_{L^2}^2\,ds+C\|v_0\|_{L^2}.$$

Here we used the basic energy identity (1-2). Recall that K_0 is an absolute positive constant determined in Lemma 2.1. Hence, we may take δ_* so that $3C_*^2K_0\delta_*<\frac{1}{2}$. Consequently, we have

$$\sup_{0\le t<T}\big(\|J(t)\|_{L^2}^2+\|\Omega(t)\|_{L^2}^2\big)+\int_0^T\big(\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\big)dt\\ +\int_0^T\int_{-\infty}^{\infty}\big(|J(t,0,z)|^2+|\Omega(t,0,z)|^2\big)dz\,dt<\infty \tag{3-7}$$

for all $T<\infty$.

Clearly, the a priori estimate (3-7) and Sobolev imbedding theorem imply that

$$\sup\nolimits_{0\le t<T}\|b(t,\cdot)\|_{L^p(r\le 1)}<\infty,\qquad 3\le p\le 6,\quad T<\infty. \tag{3-8}$$

Remark 3.1. There are several ways to prove the regularity of v from here. For instance, the easiest way is just to use the result in [Neustupa and Pokorný 2000]. If one would like to assume further decay properties on the initial data so that the conditions in Lemma 2.2 are satisfied, then one can easily derive an $L^\infty\big([0,T],L^p\big)$ estimate for b when $r\ge 1$ and $3\le p\le 6$, by using the basic energy estimate and Sobolev imbedding theorem. This, combined with the local L^p estimate of v in (3-8), immediately gives that b is in $L^\infty\big([0,T],L^3\big)$ for any $T<\infty$. Using the Sobolev imbedding theorem once more, one has $b\in L^\infty\big([0,T],\mathrm{BMO}^{-1}\big)$. Then the result in [Lei and Zhang 2011b] implies that v is regular up to time T.

Let us give an alternative and self-contained proof. We first derive an L^4 a priori estimate for v^θ without using the result of [Lei and Zhang 2011b] and Lemma 2.2. Using the equation of v^θ in (1-1) and the standard energy estimate, one has

$$\begin{aligned}\frac{d}{dt}\|v^\theta\|_{L^4}^4+\|\nabla(v^\theta)^2\|_{L^2}^2+\|r^{-1}(v^\theta)^2\|_{L^2}^2 &\le C\left|\int\frac{v^r(v^\theta)^4}{r}r\,dr\,dz\right|\\ &\le C\|r^{-1}v^r\|_{L^\infty}\|v^\theta\|_{L^4}^4.\end{aligned}$$

Hence, by using Lemma 2.1 and the three-dimensional interpolation inequality $\|f\|_{L^\infty}^2\lesssim\|\nabla f\|_{L^2}\|\nabla^2 f\|_{L^2}$, one has

$$\left\|\frac{v^r}{r}\right\|_{L^\infty}\lesssim\left\|\nabla\partial_z\frac{\psi^\theta}{r}\right\|_{L^2}^{1/2}\left\|\nabla^2\partial_z\frac{\psi^\theta}{r}\right\|_{L^2}^{1/2}=\left\|\nabla\frac{v^r}{r}\right\|_{L^2}^{1/2}\left\|\nabla^2\frac{v^r}{r}\right\|_{L^2}^{1/2}\lesssim\|\Omega\|_{L^2}^{1/2}\|\partial_z\Omega\|_{L^2}^{1/2},$$

and using (3-7), one concludes from Gronwall's inequality that

$$\|v^\theta\|_{L^4}<\infty,\quad\text{and}\quad\int_0^T\|r^{-1}(v^\theta)^2\|_{L^2}^2\,dt<\infty,\quad 0\le t\le T.$$

Then we use the second equation of (2-3) to derive that

$$\begin{aligned}&\frac{d}{dt}\|\omega^\theta\|_{L^2}^2+2\|\nabla\omega^\theta\|_{L^2}^2+2\|r^{-1}\omega^\theta\|_{L^2}^2\\ &\qquad=-\int\frac{v^r}{r}(\omega^\theta)^2r\,dr\,dz+\int\omega^\theta\partial_z\frac{(v^\theta)^2}{r}r\,dr\,dz\\ &\qquad\le C\|r^{-1}v^r\|_{L^\infty}\|\omega^\theta\|_{L^2}^2+\|\partial_z\omega^\theta\|_{L^2}^2+\tfrac14\|r^{-1}(v^\theta)^2\|_{L^2}^2.\end{aligned}$$

Hence, Gronwall's inequality similarly gives that

$$\omega^\theta\in L^2,\qquad T<\infty.$$

By the basic energy identity (1-2) and Sobolev imbedding, one has

$$b\in L^p,\qquad 2\le p\le 6.$$

Hence, $v\in L_T^\infty(L_x^4)$. So the Serrin-type criterion implies v is regular up to time T. □

4. Small $\|\Gamma_0\|_{L^\infty}$ or $\|\Gamma(t,\cdot)\|_{L^\infty(r\le r_0)}$ global regularity

This section is devoted to proving Theorem 1.4.

Proof of Theorem 1.4. Recall that

$$V=\frac{v^\theta}{\sqrt r},\qquad\Omega=\frac{\omega^\theta}{r}.$$

Let us first formulate the axisymmetric Navier–Stokes equations (1-1) in terms of V and Ω as follows:

$$\begin{cases}\partial_t V+b\cdot\nabla V+\dfrac{3v^r}{2r}V=\Big(\Delta+\dfrac1r\partial_r-\dfrac{3}{4r^2}\Big)V,\\ \partial_t\Omega+b\cdot\nabla\Omega=\Big(\Delta+\dfrac2r\partial_r\Big)\Omega+\dfrac{2\partial_z V^2}{r}.\end{cases}\tag{4-1}$$

By the energy estimate, one has

$$\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 \le \tfrac{1}{2}\|\partial_z\Omega\|_{L^2}^2 + \tfrac{1}{2}\big\|r^{-1}|V|^2\big\|_{L^2}^2. \tag{4-2}$$

Let us first prove the global regularity under

$$\|\Gamma_0\|_{L^\infty} \le \delta M_0^{-1}.$$

Using Lemma 2.1, one has

$$\Big\|\frac{v^r}{r}\Big\|_{L^\infty} \lesssim \|\Omega\|_{L^2}^{1/2}\|\partial_z\Omega\|_{L^2}^{1/2}.$$

Noting that

$$\|V\|_{L^4}^4 \lesssim \big\|r^{-1}|V|^2\big\|_{L^2}^{3/2}\|\Gamma\|_{L^4},$$

one can apply the L^4 energy estimate for V to get

$$\begin{aligned}\frac{d}{dt}\big\||V|^2\big\|_{L^2}^2 + \big\|\nabla|V|^2\big\|_{L^2}^2 + \big\|r^{-1}|V|^2\big\|_{L^2}^2 &\\ \lesssim \Big\|\frac{v^r}{r}\Big\|_{L^\infty}\|V\|_{L^4}^4 &\lesssim \|\Omega\|_{L^2}^{1/2}\|\partial_z\Omega\|_{L^2}^{1/2}\big\|r^{-1}|V|^2\big\|_{L^2}^{3/2}\|\Gamma\|_{L^4}\\ &\lesssim \|\Omega\|_{L^2}^{1/2}\|\Gamma\|_{L^2}^{1/2}\|\Gamma\|_{L^\infty}^{1/2}\big(\|\partial_z\Omega\|_{L^2}^2 + \big\|r^{-1}|V|^2\big\|_{L^2}^2\big).\end{aligned} \tag{4-3}$$

Combining (4-2) and (4-3), we arrive at

$$\begin{aligned}\frac{d}{dt}\Big(\big\||V|^2\big\|_{L^2}^2 + \|\Omega\|_{L^2}^2\Big) + \Big(\big\|\nabla|V|^2\big\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2\Big) + \big\|r^{-1}|V|^2\big\|_{L^2}^2&\\ \lesssim \|\Omega\|_{L^2}^{1/2}\|\Gamma\|_{L^2}^{1/2}\|\Gamma\|_{L^\infty}^{1/2}\Big(\|\partial_z\Omega\|_{L^2}^2 + \big\|r^{-1}|V|^2\big\|_{L^2}^2\Big).&\end{aligned} \tag{4-4}$$

Recall that we have the following a priori estimate:

$$\|\Gamma\|_{L^2} \le \|\Gamma_0\|_{L^2}, \qquad \|\Gamma\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty}.$$

Hence, under the condition of the theorem, there exists $T > 0$ such that

$$\big\||V|^2\big\|_{L^2}^2 + \|\Omega\|_{L^2}^2 < 2\big\||V_0|^2\big\|_{L^2}^2 + 2\|\Omega_0\|_{L^2}^2, \quad \forall\, 0 \le t < T.$$

If δ is a suitably small positive constant and

$$\|\Gamma_0\|_{L^\infty} \le \delta M_0^{-1}$$

is satisfied, then we have

$$\|\Omega\|_{L^2}^{1/2}\|\Gamma\|_{L^2}^{1/2}\|\Gamma\|_{L^\infty}^{1/2} \lesssim M_0^{1/2}(\delta M_0^{-1})^{1/2} \lesssim \delta^{1/2}, \quad \forall\, 0 \le t < T.$$

Hence, by (4-4), we derive that

$$\frac{d}{dt}\Big(\big\||V|^2\big\|_{L^2}^2 + \|\Omega\|_{L^2}^2\Big) \le 0, \quad \forall\, 0 \le t < T,$$

which implies that

$$\big\| |V|^2 \big\|_{L^2}^2 + \|\Omega\|_{L^2}^2 \le \big\| |V_0|^2 \big\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2, \quad 0 \le t \le T.$$

The above argument implies, by the standard continuation method that

$$\big\| |V|^2 \big\|_{L^2}^2 + \|\Omega\|_{L^2}^2 \le \big\| |V_0|^2 \big\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2, \quad \forall\, t \ge 0.$$

Hence the proof for first part of the theorem is finished.

On the other hand, if

$$\|\Gamma(t, \cdot)\|_{L^\infty(r \le r_0)} \le \delta M_1^{-1}$$

is satisfied, then one may treat (4-3) as follows:

$$\begin{aligned}
\frac{d}{dt}\big\| |V|^2 \big\|_{L^2}^2 &+ \big\| \nabla |V|^2 \big\|_{L^2}^2 + \big\| r^{-1} |V|^2 \big\|_{L^2}^2 \\
&\lesssim \Big\| \frac{v^r}{r} \Big\|_{L^\infty} \|V\|_{L^4(r \le r_0)}^4 + \int_{r \ge r_0} \Big| \frac{v^r}{r} \frac{(v^\theta)^4}{r^2} \Big| r \, dr \, dz \\
&\lesssim \|\Omega\|_{L^2}^{1/2} \|\Gamma\|_{L^2}^{1/2} \|\Gamma\|_{L^\infty(r \le r_0)}^{1/2} \big(\|\partial_z \Omega\|_{L^2}^2 + \big\| r^{-1} |V|^2 \big\|_{L^2}^2 \big) \\
&\qquad + r_0^{-4} \Big\| \frac{v^r}{r} \Big\|_{L^2} \Big\| \frac{v^\theta}{r} \Big\|_{L^2} \|\Gamma\|_{L^\infty(r \ge r_0)}^3,
\end{aligned}$$

which, combined with (4-2), gives that

$$\begin{aligned}
\big\| |V(t, \cdot)|^2 \big\|_{L^2}^2 &+ \|\Omega(t, \cdot)\|_{L^2}^2 + \int_0^t \Big(\big\| \nabla |V|^2 \big\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 \Big) \, ds \\
&\lesssim \big\| |V_0|^2 \big\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2 + r_0^{-4} \|v_0\|_{L^2}^2 \|\Gamma_0\|_{L^\infty}^3 \\
&\quad + \big(\|\Gamma\|_{L^\infty(r \le r_0)} \|\Gamma_0\|_{L^2} \sup_{0 \le s \le t} \|\Omega(s, \cdot)\|_{L^2} \big)^{1/2} \big(\|\partial_z \Omega\|_{L^2}^2 + \big\| r^{-1} |V|^2 \big\|_{L^2}^2 \big).
\end{aligned}$$

Here $r_0 > 0$ is arbitrary. Then similar continuation arguments as in the proof used in the first part imply that if δ is a suitable small absolute positive constant, then the solution v is regular. Here M_1 is given in the statement of Theorem 1.4. This shows that the smallness of Γ locally in r implies the regularity of the solutions. □

Remark. Since $\big\| |V_0|^2 \big\|_{L^2}$ and $\|\Omega_0\|_{L^2}$ have dimension $-\frac{3}{2}$, and $\|\Gamma_0\|_{L^2}$ has dimension $\frac{3}{2}$, the constant M_0 in Theorem 1.4 has dimension 0. Similarly, one can check that M_1 is also dimensionless if one assigns r_0 dimension 1.

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