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LUKE WILLIAMS

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It is known that for coprime integers $p > q \geq 1$, the lens space $L(p^2, pq-1)$ bounds a rational ball, $B_{p,q}$, arising as the 2-fold branched cover of a (smooth) surface in B^4 bounding the associated 2-bridge knot or link. Lekili and Maydanskiy give handle decompositions for each $B_{p,q}$; whereas, Yamada gives an alternative definition of rational balls, $A_{m,n}$, bounding $L(p^2, pq-1)$ by their handlebody decompositions alone. We show that these two families coincide, answering a question of Kadokami and Yamada. To that end, we show that each $A_{m,n}$ admits a Stein filling of the universally tight contact structure, $\bar{\xi}_{st}$, on $L(p^2, pq-1)$ investigated by Lisca. Furthermore, we construct boundary diffeomorphisms between these families. Using the carving process, pioneered by Akbulut, we show that these boundary maps can be extended to diffeomorphisms between the spaces $B_{p,q}$ and $A_{m,n}$.

1. Introduction

For $p > q \geq 1$ relatively prime, let $B_{p,q}$ be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing $pq-1$ to B^4 by wrapping the attaching circle of the 2-handle p-times around the 1-handle with a q/p-twist; see Figure 1.

From this description, it is immediate that $B_{p,q}$ is always a rational homology ball. Lekili and Maydanskiy [2014] show that each such $B_{p,q}$ arises as the 2-fold branched cover of B^4 branched over a properly embedded surface bounding the 2-bridge link associated to the fraction $-p^2/(pq-1)$. That is, the family $B_{p,q}$ represents handle decompositions of the rational balls introduced by Casson and Harer [1981]. As such, $\partial B_{p,q} \approx L(p^2, pq-1)$, where $\approx$ denotes diffeomorphism of two manifolds throughout. Lekili and Maydanskiy go on to prove that each $B_{p,q}$ supports a Stein structure (see Figure 7) filling the universally tight contact structure on $L(p^2, pq-1)$ [Lekili and Maydanskiy 2014].

In a similar direction, Yamada [2007] defines a family of $X \,||\, Y$ rational balls bounding $L(p^2, pq-1)$ via their handle decompositions: For $n, m \geq 1$ relatively prime, let $A_{m,n}$ be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing mn to B^4 by attaching the 2-handle along a simple closed

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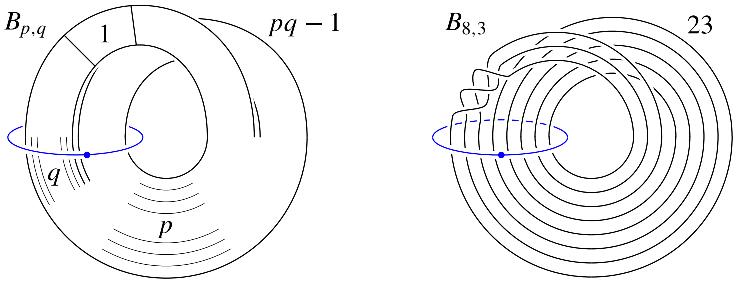

Figure 1. The rational ball $B_{p,q}$ (left); e.g., $B_{8,3}$ (right).

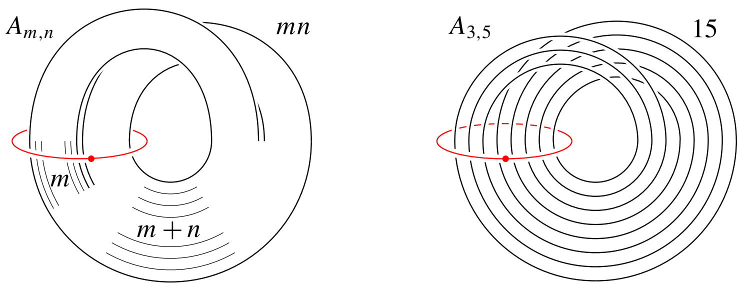

Figure 2. The rational ball $A_{m,n}$ (left); e.g., $A_{3,5}$ (right).

curve embedded on a once-punctured torus viewed in $S^1 \times S^2$ so that the attaching circle traverses the two 1-handles of the torus m and n times respectively (Figure 2).

Yamada goes on to define an involutive symmetric function A on the set of coprime pairs of positive integers such that if $A(p-q, q) = (m, n)$ then $\partial A_{m,n} \approx L(p^2, pq-1)$. Here $m+n = p$ and $mq = \pm 1 \mod p$; Remark 2.4 gives a definition of A.

Given these two constructions of rational balls with coincident boundaries, one arrives at a natural question posed by Kadokami and Yamada:

Question 1.1 [Kadokami and Yamada 2014, Problem 1.9]. Are $A_{m,n}$ and $B_{p,q}$ diffeomorphic, homeomorphic, or even homotopic relative to their boundaries as 4-manifolds?

The goal herein is to provide a complete answer to this question by proving the following theorem.

Theorem 1.2. *For each pair of relatively prime positive integers* (m, n), $A_{m,n}$ *carries a Stein structure* $\tilde{J}_{m,n}$ *filling the universally tight contact structure on the lens space* $\partial A_{m,n}$. *In particular, each* $A_{m,n} \approx B_{p,q}$ *if and only if* $\partial A_{m,n} \approx \partial B_{p,q}$.

The proof of Theorem 1.2 follows by first explicitly writing down a Stein structure on $A_{m,n}$ using Eliashberg's characterization of handle decompositions of Stein

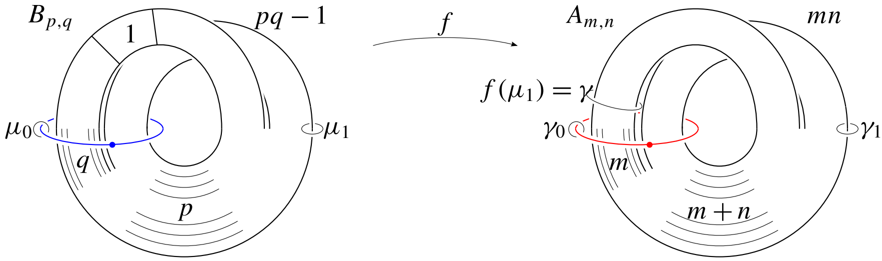

Figure 3. The spaces $B_{p,q}$ and $A_{m,n}$.

domains [Eliashberg 1990; Gompf 1998]. As the homotopy invariants of the induced contact structures on the boundary agree with those of $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$, the two structures are homotopic as 2-plane fields. Work of Honda [2000] and independently Giroux [2000] proves that this is sufficient to conclude that these two contact structures are contactomorphic. Lisca's classification [2008] of the diffeomorphism types of symplectic fillings of $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$ then gives that $A_{m,n} \approx B_{p,q}$. To provide insight into the aforementioned diffeomorphisms, we construct boundary diffeomorphisms which can be extended to explicit diffeomorphisms between $B_{p,q}$ and $A_{m,n}$ through the carving process introduced by Akbulut [1977]. In fact, we have the following result:

Theorem 1.3. *Let $(m,n) = A(p-q, q)$ for some $p > q > 0$ relatively prime. Then there exists a diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$ such that f carries the belt sphere, μ_1, of the single 2-handle in $B_{p,q}$ to a slice knot in $\partial A_{m,n}$ (see Figure 3). Moreover, carving $A_{m,n}$ along $f(\mu_1)$ gives $S^1 \times B^3$.*

Corollary 1.4. *f extends to a diffeomorphism $\tilde{f} : B_{p,q} \to A_{m,n}$.*

Further motivation. Fintushel and Stern [1997] define a smooth operation, the rational blow-down, on 4-manifolds containing certain configurations of spheres by removing a neighborhood of those spheres and replacing them by the rational ball $B_{p,1}$. Park [1997] generalized the operation to a larger set of configurations at the expense of having to glue in $B_{p,q}$ for q other than 1. In the presence of a symplectic structure and a symplectic configuration of spheres, both operations can be performed symplectically [Symington 1998; 2001]. Moreover, under mild assumptions (see [Fintushel and Stern 1997; Park 1997] for details), nontrivial solutions to the Seiberg–Witten equations on the original 4-manifold induce nontrivial solutions on the surgered manifold and vice versa.

Therefore, having well understood handle decompositions for $B_{p,q}$ allows one to construct explicit examples of rationally blown-down 4-manifolds. For instance, Stipsicz and Szabó [2005] take advantage of such decompositions to construct an exotic $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$. Corollary 1.4 and Theorem 1.2 are then useful, since either the decomposition $B_{p,q}$ or $A_{m,n}$ can conceivably be used interchangeably.

Organization. The paper is organized as follows: In Section 2, we dispense with notation and necessary calculations involving lens spaces. Then, in Section 3, we bring in the relevant symplectic topology and construct Stein handle decompositions on each $A_{m,n}$, proving Theorem 1.2. Finally, in Section 4, we recall the carving procedure and construct boundary diffeomorphisms from $\partial B_{p,q}$ and $\partial A_{m,n}$ to their lens space boundaries, proving Theorem 1.3.

2. Preliminaries

Conventions and assumptions. Unless specifically stated to the contrary, throughout the paper, we assume $p-q > q \geq 1$, $n > m \geq 1$, and that both pairs are relatively prime. As $B_{p,q} \approx B_{p,p-q}$ and $A_{m,n} \approx A_{n,m}$, this assumption does not represent a restriction. We adopt the standard orientation convention that $L(p,q)$ is the result of $-p/q$-surgery on the unknot in S^3. It is well known that $L(p,q)$ is also given as the boundary of a linear plumbing of D^2-bundles over S^2 with Euler classes chosen according to a continued fraction associated to $-p/q$:

$$[c_1, \dots, c_n] \doteq c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_n}}} = -\frac{p}{q}$$

where the c_i are uniquely determined provided each $c_i \leq -2$ (see Figure 4). Where convenient, we will use weighted trees to describe these plumbings. We will often forgo the uniqueness of the c_i in favor of shorter continued fraction expansions and thus smaller bounding 4-manifolds. In spite of this, we make the following definition.

Definition 2.1. Given $p > 0$ and q coprime, let $C_{p,q}$ be the 4-manifold bounding $L(p,q)$ obtained by plumbing D^2-bundles over S^2 according to a linear graph with weights $c_i \leq -2$ chosen so that $[c_1, \dots, c_n] = -p/q$ (see Figure 4). For conciseness, we denote $C_{p^2,pq-1}$ by $\mathcal{C}_{p,q}$.

In Section 3, we need to perform calculations in the group $H_1(L(p^2, pq-1; \mathbb{Z})$. The following lemma will prove useful.

Lemma 2.2. *Suppose that $L(p,q)$ is given by the linear plumbing of Figure 4 where the η_i are meridians spanning $H_1(L(p,q), \mathbb{Z})$. Then*

$$H_1(L(p,q), \mathbb{Z}) = \langle \eta_1 : (\det C_n)\eta_1 = 0 \rangle$$

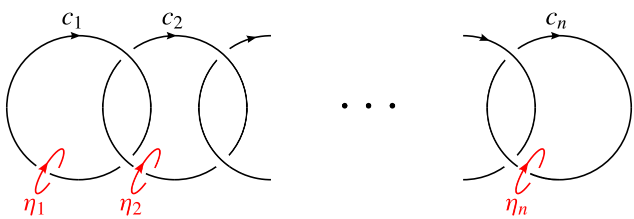

Figure 4. A linear plumbing bounding $L(p,q)$. Elements spanning $H_1(L(p,q))$ are shown in red.

where $C_i \doteq \begin{pmatrix} c_1 & 1 & & \\ 1 & c_2 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & c_i \end{pmatrix}$ *and* $\eta_i = (-1)^{i-1}(\det C_{i-1})\eta_1$ *for* $i \in \{2,\dots,n\}$.

Proof. Given a Dehn surgery description of a 3-manifold, one obtains a presentation for the first homology in terms of the right handed meridians of the (oriented) framed link; see [Gompf and Stipsicz 1999]. In the above case, we find that

$$H_1(L(p,q),\mathbb{Z}) = \langle \eta_1,\dots,\eta_n : \eta_2 = -c_1\eta_1,\ \{\eta_{i+1} = -c_i\eta_i - \eta_{i-1}\}_{i=2}^{n-1},\ c_n\eta_n = -\eta_{n-1}\rangle$$

As $\eta_2 = -c_1\eta_1 = (-1)^{2-1}(\det C_{2-1})\eta_1$, the result follows by induction using that

$$\det C_k = c_k \det C_{k-1} - \det C_{k-2}. \qquad \square$$

Determining $\mathcal{C}_{p,q}$. The continued fraction associated to $-p^2/(pq-1)$ involves the Euclidean algorithm; see [Casson and Harer 1981; Yamada 2007] as well as Proposition 2.5 below. Therefore, we use the Euclidean algorithm to define sequences of remainders and divisors of p and q as follows:

Definition 2.3. For $p > q \geq 1$, relatively prime, let $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ be defined recursively by setting $r_{-1} \doteq p$, $r_0 \doteq q$. and

$$r_{i+1} = r_{i-1} \mod r_i, \qquad r_{i-1} = r_i s_i + r_{i+1}.$$

Let ℓ be the last index where $r_\ell > 1$ so that $r_{\ell+1} = 1$ and $r_{\ell+2} \doteq 0$.

Remark 2.4. For bookkeeping purposes, we will differentiate between the above sequences for p and q and the analogously defined sequences $\{\rho_i\}_{i=-1}^{\ell+2}$ and $\{\sigma_i\}_{i=0}^{\ell+1}$ associated to $n > m \geq 1$. Furthermore, provided that $p-q > q$, and that $A(p-q,q)$ either equals (m,n) or (n,m), the four sequences are related by the following recursive dictionary:

$$\begin{array}{rcr}
r_{-1} = p & s_0 \longleftrightarrow \rho_\ell & 1 = \rho_{\ell+1} \\
r_0 = q & s_1 \longleftrightarrow \sigma_\ell & \\
& & \\
r_{i+1} = r_{i-1} - r_i s_i & s_j \longleftrightarrow \sigma_{\ell-j+1} & \rho_i\sigma_i + \rho_{i+1} = \rho_{i-1} \\
& & \\
& s_\ell \longleftrightarrow \sigma_1 & m = \rho_0 \\
r_{\ell+1} = 1 & r_\ell - 1 \longleftrightarrow \sigma_0 & n = \rho_{-1}
\end{array}$$

That is, given the sequences associated to p and q, we get the associated sequences for m and n by declaring $\rho_{\ell+1} = 1$, $\rho_\ell = s_0$ and making the indicated identifications for the σ_j in order to recursively recover each ρ_j; ultimately determining $m = \rho_0$ and $n = \rho_{-1}$. Similarly, we may start from m and n to recover p and q. In fact, we will take this correspondence as our definition of the function A defined by Yamada [2007]. It is straightforward to verify that formulation is equivalent to Yamada's definition. As we will independently see in Section 4, this correspondence ensures that $\partial B_{p,q} \approx \partial A_{m,n}$ (see Remark 4.8); so, nothing is lost.

We can explicitly write down $\mathcal{C}_{p,q}$ in terms of these Euclidean sequences. The following is proved in Section 4 as Corollary 4.3.

Proposition 2.5. *For $p > q > 0$ coprime, the lens space $L(p^2, pq-1)$ bounds the linear plumbing $X(\Gamma)$ where Γ is the weighted graph of Figure 5 and where $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ are as in Definition 2.3.*

$X(\Gamma)$ defined in Proposition 2.5 has spheres of positive self-intersection and is therefore *not* $\mathcal{C}_{p,q}$. Given a sphere in $X(\Gamma)$ with self-intersection $s > 0$, by blowing up $s-1$ of these intersections we get a sphere with one positive self-intersection — which can be blown-down. This allows the exchange of each positive Euler-class disk bundle for, possibly many negative Euler-class bundles without altering the boundary. By applying this process at each sphere with positive self-intersection we arrive at $\mathcal{C}_{p,q}$.

Corollary 2.6. *For $p > q \geq 1$, coprime, let $\{s_i\}_{i=0}^{\ell}$ and $\{r_i\}_{i=-1}^{\ell+1}$ be as defined in Definition 2.3, the space $\mathcal{C}_{p,q}$ is given by one of the linear plumbings of Figure 6 (depending upon the parity of ℓ).*

Remark 2.7. By Definition 2.1, Figure 6 specifies $\mathcal{C}_{p,q}$. This follows since each s_i is at least 1, ensuring that each weight in the graphs of Figure 6 is less than or equal to -2. The meridians (in red) of Figure 6 are used in homological calculations in

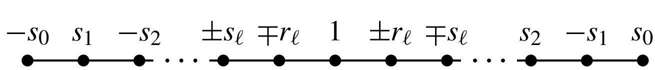

Figure 5. A linear plumbing bounding $L(p^2, pq-1)$.

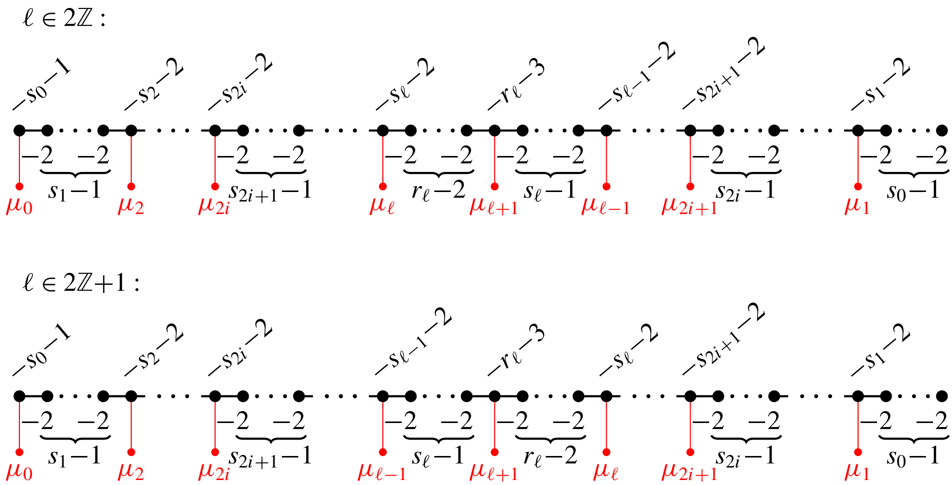

Figure 6. $\mathcal{C}_{p,q}$ when $\ell \in 2\mathbb{Z}$ and when $\ell \in 2\mathbb{Z}+1$ with relevant meridians used in homology calculations (in red).

Section 3. It is worth noting that combining Lemma 2.2 with the following lemma, we find that $\mu_i = (-1)^i \rho_{\ell-i+1}\mu_0 \in H_1(L(p^2, pq-1); \mathbb{Z})$.

Lemma 2.8. *Let* $\{\rho_i\}_{i=-1}^{\ell+2}$ *and* $\{\sigma_i\}_{i=0}^{\ell+1}$ *be as defined in Definition 2.3 (associated to* n *and* m*). Then for each* $i \leq \ell+1$,

$$\det\begin{pmatrix} -\rho_\ell & 1 & & \\ 1 & \sigma_\ell & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & (-1)^{\ell+1-i}\sigma_{\ell+1-i} \end{pmatrix} = -\Big(\sin\Big(\frac{\pi}{2}i\Big)+\cos\Big(\frac{\pi}{2}i\Big)\Big)\rho_{\ell-i}.$$

Proof. Induct on i, using that $\rho_{\ell+1}=1$ and that $\rho_{\ell-i} = \rho_{\ell-i+1}\sigma_{\ell-i+1} + \rho_{\ell-i+2}$. □

3. Stein structures on $A_{m,n}$

We are now ready to show that $A_{m,n}$ admits a Stein structure. To accomplish this, we use Eliashberg's handle characterization of Stein surfaces [Eliashberg 1990; Gompf 1998]. The reader should consult [Gompf and Stipsicz 1999] as well as [Ozbagci and Stipsicz 2004] for thoughtful treatments of the subject. Such a Stein structure induces a (tight) contact structure on $\partial A_{m,n}$. Tight contact structures on lens spaces are well understood; Honda [2000], and independently Giroux [2000], completely classify them. Moreover, Lisca classifies the diffeomorphism types of symplectic fillings of $(L(p,q), \bar{\xi}_{st})$ where $\bar{\xi}_{st}$ is the universally tight contact structure $L(p,q)$ inherits from the unique tight contact structure on S^3 via the cyclic group action. In particular, Lisca defines collections of 4-manifolds $W_{p,q}(\boldsymbol{n})$, such that

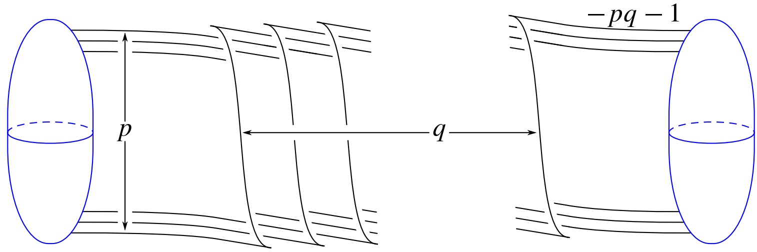

Figure 7. $(B_{p,q}, J_{p,q})$.

Theorem 3.1 [Lisca 2008, Theorem 1.1]. *Let $p > q \geq 1$ be relatively prime. Then each symplectic filling (W, ω) of $(L(p, q), \bar{\xi}_{\mathrm{st}})$ is orientation preserving diffeomorphic to a smooth blowup of $W_{p,q}(\boldsymbol{n})$ for some $\boldsymbol{n} \in \mathbf{Z}_{p,q}$. Moreover, if $b_2(W) = 0$, then W is unique.*

In light of Theorem 3.1, if we show that not only does $A_{m,n}$ admit a Stein structure, but that such a structure gives a symplectic filling of $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$, then we immediately have that $A_{m,n} \approx B_{p,q}$ since it is known that $B_{p,q}$ admits a Stein structure giving such a filling. Indeed, by sliding the 2-handle of $B_{p,q}$ under the 1-handle q-times one arrives at the Stein domain, $(B_{p,q}, J_{p,q})$, investigated by Lekili and Maydanskiy [2014] given in Figure 7. There, they prove that $(B_{p,q}, J_{p,q})$ fills the standard contact structure on $L(p^2, pq-1)$.

Tight contact structures on lens spaces. Before we explicitly construct a Stein handle decomposition for $A_{m,n}$, we note that *any* Stein structure on $A_{m,n}$ necessarily induces a tight contact structure which is contactomorphic to $\bar{\xi}_{\mathrm{st}}$ (see Proposition 3.4). When identifying tight contact structures on lens spaces, it is enough to know that the two contact structures in question are homotopic up to contactomorphism.

Theorem 3.2 [Honda 2000, Proposition 4.24; Giroux 2000, Theorem 1.1]. *The homotopy classes of the tight contact structures of $L(p, q)$ are all distinct. Moreover, if $q < p-1$, then all but exactly two tight contact structures on $L(p, q)$ are virtually overtwisted.*

The two universally tight contact structures are both contactomorphic to $\bar{\xi}_{\mathrm{st}}$. Furthermore, the problem of determining the homotopy type of the underlying 2-plane field of a given tight contact structure is completely solved by Gompf [1998].

In fact, for contact structures with c_1 torsion (which is always satisfied for 3-manifolds with $b_1 = 0$; e.g., lens spaces) two homotopy invariants d_3 and Γ completely determine their homotopy classes as 2-plane fields.

Theorem 3.3 [Gompf 1998, Theorem 4.16]. *If (Y^3, ξ_i) for $i = 1, 2$, satisfies that $c_1(\xi_1)$ is torsion and $\Gamma(\xi_1, \boldsymbol{s}) = \Gamma(\xi_2, \boldsymbol{s})$ for some spin structure $\boldsymbol{s}$, then ξ_1 is homotopic to ξ_2 if and only if their d_3 invariants coincide.*

According to Theorem 3.3, two 2-plane fields (with torsion c_1) are homotopic if and only if they have the same Γ *and* d_3 invariants. Lisca [2001] proves that in the case of tight contact structures on a lens space, the Γ invariant alone is enough — that is, if $\Gamma(\xi_x, \boldsymbol{s}) = \Gamma(\xi_y, \boldsymbol{s})$, then ξ_x is homotopic to ξ_y (and their d_3 invariants necessarily coincide). One cannot expect the same result to hold with d_3 in place of Γ. However, the d_3-invariant does detect the universally tight structures on $L(p^2, pq-1)$. In fact by using the "correction terms" from Heegaard Floer homology to determine which spin$^{\mathbb{C}}$-structures on $L(p^2, pq-1)$ induced from a tight contact structure therein can extend across a rational ball bounding the lens space we arrive at the following proposition known to experts:

Proposition 3.4. *Every tight contact structure ξ on $L(p^2, pq-1)$ with $d_3(\xi) = -\frac{1}{2}$ is universally tight.*

For completeness, we include a proof of Proposition 3.4 below. Before dispatching with that, we first recall the definitions of d_3 and Γ. For the three-dimensional invariant, d_3, we use the normalized definition [Ozbagci and Stipsicz 2004] — but note that it is equivalent to the definition of θ originally defined by Gompf [1998] which relies on the fact that each contact 3-manifold can be realized as the J-convex boundary of an almost complex 4-manifold as well as the fact that for (X^4, J), a closed almost complex 4-manifold, the quantity $c_1^2(X, J) - 3\sigma(X) - 2\chi(X) = 0$ where $\sigma(X)$ and $\chi(X)$ are the signature and Euler characteristic of X respectively.

Definition 3.5 [Gompf 1998, Definition 4.2]. For a contact 3-manifold (Y, ξ) with $c_1(\xi)$ torsion, the three-dimensional invariant

$$d_3(\xi) = \tfrac{1}{4}\big(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)\big) \in \mathbb{Q}$$

for any almost complex 4-manifold (X, J) with $\partial X = Y$ satisfying $TY \cap JTY = \xi$.

The function Γ associates to each spin structure on (Y, ξ) an element of $H_1(Y; \mathbb{Z})$. This is accomplished by noting that Spin$^{\mathbb{C}}(Y)$ is an $H^2(Y; \mathbb{Z})$-torsor. So any two $\mathfrak{t}_0, \mathfrak{t}_1 \in$ Spin$^{\mathbb{C}}(Y)$, satisfy that their difference $\mathfrak{t}_1 - \mathfrak{t}_0$ is a well defined element of $H^2(Y; \mathbb{Z})$. A spin structure on Y can be canonically viewed as a spin$^{\mathbb{C}}$-structure. Then $\Gamma(\xi, \boldsymbol{s})$ is Poincaré dual to the difference $\mathfrak{t}_\xi - \boldsymbol{s}$. Furthermore, if (Y, ξ) is the boundary of a Stein 4-manifold (X, J), Gompf provides a combinatorial formula for Γ (we state it only in the case when X lacks 1-handles; we also suppress the definition of a characteristic sublink associated to $\boldsymbol{s} \in$ Spin(Y) as we will not make use of it herein — the interested reader can refer to [Gompf 1998; Kaplan 1979] for details).

Proposition 3.6 [Gompf 1998, Theorem 4.12]. *Let (X, J) be obtained from B^4 by attaching Stein 2-handles along Legendrian knots $K_1, \ldots, K_k$ such that $\partial X = Y$ and $\xi = TY \cap JTY$. Orient $K_1 \cup \cdots \cup K_k$ to obtain a spanning set for $H_2(X; \mathbb{Z})$. Then*

$\Gamma(\xi, \boldsymbol{s}) \in H_1(\partial X; \mathbb{Z})$ is Poincaré dual to the restriction of the class $\rho \in H^2(X; \mathbb{Z})$ whose value on each $[K_i]$ is given by

$$\rho([K_i]) = \tfrac{1}{2}\big(\mathrm{rot}(K_i) + \ell k(K_i, L)\big) \in \mathbb{Z}$$

where L is the characteristic sublink associated to $\boldsymbol{s}$.

Honda [2000] and Giroux [2000] prove that each tight contact structure on $L(p, q)$ is induced by a Stein filling of $C_{p,q}$. In general, $\mathcal{C}_{p,q}$ admits numerous Stein fillings. Each is obtained by attaching the 2-handles of $\mathcal{C}_{p,q}$ along Legendrian unknots whose Seifert framings are one less than the their respective Thurston–Bennequin framings. For each $n < -1$, by stabilizing the standard Legendrian unknot positively and or negatively as needed, there are exactly $|n| - 1$ distinct rotation numbers for Legendrian unknots with Thurston–Bennequin framing equal to $n + 1$: namely $n + 2, n + 4, \ldots, -n - 2$. In particular, each unknot in the handle decomposition of $\mathcal{C}_{p,q}$ with Seifert framing -2 necessarily has rotation number zero for any Stein handle attachment. Therefore, if we let K_i denote the attaching circle of the 2-handle in $\mathcal{C}_{p,q}$ whose belt-sphere is the meridian given by μ_i as labeled in Figure 6, we see that specifying rotation numbers only for K_i fixes a Stein structure on $\mathcal{C}_{p,q}$. With this in mind, for each $x = (x_0, \ldots, x_{\ell+1})$ chosen so that

$$\begin{aligned} x_0 &\in \{1 - s_0, 3 - s_0, \ldots, s_0 - 1\}, \\ x_i &\in \{-s_i, 2 - s_i, \ldots, s_i\}, \quad \text{for } i \in \{1, \ldots, \ell\} \\ x_{\ell+1} &\in \{-1 - r_\ell, 1 - r_\ell, \ldots, r_\ell + 1\}, \end{aligned}$$

we get a unique Stein structure on $\mathcal{C}_{p,q}$ inducing a distinct (up to isotopy) tight contact structure on $L(p^2, pq - 1)$. In an abuse of notation, we ignore the obvious dependence on p and q and choose to call this structure J_x.

It is known that $J_{x_{\min}}$ and $J_{x^{\max}}$ induce the two universally tight contact structures on $L(p^2, pq - 1)$, where $x^{\max}$ fixes the largest allowable rotation number on each K^i and $x_{\min} = -x^{\max}$. Let ξ_x, $\xi_{\min}$ and $\xi^{\max}$ be the contact structures induced by J_x, $J_{\min}$ and $J^{\max}$ respectively; similarly define the spin$^{\mathbb{C}}$-structures $\mathfrak{t}_x$, $\mathfrak{t}_{\min}$ and $\mathfrak{t}^{\max}$. As shown by Lekili and Maydanskiy [2014], $\xi_{\min}$ and $\xi^{\max}$ are also induced by the Stein structures $(B_{p,q}, J_{p,q})$ and $(B_{p,p-q}, J_{p,p-q})$ specified in Figure 7. Therefore, the spin$^{\mathbb{C}}$-structures $\mathfrak{t}_{\min}$ and $\mathfrak{t}^{\max}$ both extend over $B_{p,q}$ to $\mathfrak{s}_{\min}, \mathfrak{s}^{\max} \in \mathrm{Spin}^{\mathbb{C}}(B_{p,q})$. No other $\mathfrak{t}_x$ has this property:

Proposition 3.7. *Let $\Xi_{p,q}$ denote the set of homotopy classes of 2-plane fields induced by tight contact structures on $L(p^2, pq - 1)$ and let*

$$\mathcal{S} = \big\{\mathfrak{t}_\xi \in \mathrm{Spin}^{\mathbb{C}}(L(p^2, pq - 1)) : \xi \in \Xi_{p,q}\big\},$$

then $\mathcal{S}$ contains exactly two spin$^{\mathbb{C}}$*-structures that extend across the ball $B_{p,q}$, both of which arise from contact structures contactomorphic to $\bar{\xi}_{\mathrm{st}}$.*

Before we prove Proposition 3.7 we recall the obstruction to extending a given spin$^{\mathbb{C}}$-structure $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(L(p^2, pq-1))$ across a rational ball bounding the space $L(p^2, pq-1)$. We can measure this obstruction against any fixed spin$^{\mathbb{C}}$-structure which is known to extend. As every 4-manifold admits a spin$^{\mathbb{C}}$-structure (which extends its restriction to the boundary), we always have such an element to measure against. A standard obstruction theoretic proof gives the following lemma:

Lemma 3.8. *Suppose that $\mathcal{B}$ is a rational ball bounding $L(p^2, pq-1)$. For each pair $\mathfrak{t}_0, \mathfrak{t}_1 \in \mathrm{Spin}^{\mathbb{C}}(\partial\mathcal{B})$ such that $\mathfrak{t}_0$ extends across $\mathcal{B}$ to some $\mathfrak{s}_0 \in \mathrm{Spin}^{\mathbb{C}}(\mathcal{B})$, $\mathfrak{t}_1$ extends across $\mathcal{B}$ if and only if p divides the difference $\mathfrak{t}_0 - \mathfrak{t}_1 \in H^2(\partial\mathcal{B}; \mathbb{Z})$.*

We can use Lemma 3.8 to determine which other spin$^{\mathbb{C}}$-structures induced by some J_x extend over $B_{p,q}$. Note that for any spin-structure $\boldsymbol{s} \in \mathrm{Spin}(L(p^2, pq-1))$ the difference

$$\mathrm{PD}(\Gamma(\xi_y, \boldsymbol{s})) - \mathrm{PD}(\Gamma(\xi_x, \boldsymbol{s})) = (\mathfrak{t}_y - \boldsymbol{s}) - (\mathfrak{t}_x - \boldsymbol{s}) = \mathfrak{t}_y - \mathfrak{t}_x$$

doesn't depend on the choice of spin-structure. Using Proposition 3.6, we calculate

$$\mathrm{PD}(\mathfrak{t}_y - \mathfrak{t}_x) = \sum_{i=0}^{\ell+1} \frac{y_i - x_i}{2}\mu_i = \sum_{i=0}^{\ell+1} (-1)^i \frac{y_i - x_i}{2}\rho_{\ell-i+1}\mu_0$$

where the last equality follows from Remark 2.7.

Proof of Proposition 3.7. Suppose that $\mathfrak{t} \in \mathcal{S}$ extends across $B_{p,q}$. We can assume that $\mathfrak{t} = \mathfrak{t}_x$ for some Stein structure $(\mathcal{C}_{p,q}, J_x)$ on $\mathcal{C}_{p,q}$. Lemma 3.8 gives that $\mathfrak{t}_x$ extends if and only if p divides the difference $\mathrm{PD}(\mathfrak{t}^{\max} - \mathfrak{t}_x)$ in $H_1(L(p^2, pq-1))$. Write $x = x^{\max} - 2c$ where $c = (c_0, c_1, \ldots, c_{\ell+1})$ necessarily satisfies $c_0 \in \{0, 1, \ldots, s_0 - 1\}$, $c_i \in \{0, 1, \ldots, s_i\}$ for each $i \in \{1, 2, \ldots, \ell\}$ and $c_{\ell+1} = \{0, 1, \ldots, r_\ell + 1\}$. Then

$$\mathrm{PD}(\mathfrak{t}^{\max} - \mathfrak{t}_x) = \sum_{i=0}^{\ell+1} (-1)^i \frac{x_i^{\max} - x_i}{2}\rho_{\ell-i+1}\mu_0 = \sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1}\mu_0.$$

Therefore, we investigate solutions to $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \equiv 0 \bmod p$. We will prove in Corollary 3.12 that there are exactly two solutions — namely $c = 0$ and $2c = x^{\max}$ (giving that the only spin$^{\mathbb{C}}$-structures which extend correspond to $x^{\max}$ and $x_{\min} = -x^{\max}$) which are known to induce the universally tight contact structures on $L(p^2, pq-1)$. $\square$

To finish the proof of Proposition 3.4, recall that Ozsváth and Szabó [2004b; 2004a] define relatively $\mathbb{Z}$-graded homology groups $\mathrm{HF}^{\pm}$, HF^{∞} associated to each 3-manifold endowed with a spin$^{\mathbb{C}}$-structure. If the spin$^{\mathbb{C}}$-structure is torsion, they obtain absolute $\mathbb{Q}$-gradings [Ozsváth and Szabó 2006]. Using this grading, they define the correction term $d(Y, \mathfrak{t})$ of any rational homology spin$^{\mathbb{C}}$ 3-sphere $(Y, \mathfrak{t})$ as the minimal degree of the image of a nontorsion element of $\mathrm{HF}^{\infty}(Y, \mathfrak{t})$ in $\mathrm{HF}^{+}(Y, \mathfrak{t})$

[Ozsváth and Szabó 2003]. Of interest to the present problem, is the following result of Ozsváth, Stipsicz and Szabó:

Proposition 3.9 [Ozsváth et al. 2005, Corollary 1.7]. *Suppose (Y, ξ) is a rational homology 3-sphere equipped with a symplectically fillable contact structure ξ supported by a planar open book, then*

$$d_3(\xi) + \tfrac{1}{2} = -d(Y, \mathfrak{t}_\xi).$$

As every tight contact structure on a lens space is supported by a planar open book [Schönenberger 2007], we gain knowledge about the three-dimensional invariant d_3 from the correction term and vice versa. In particular, compare Lemma 3.8 with the following result of Jabuka, Robins and Wang:

Proposition 3.10 [Jabuka et al. 2013]. *Suppose that $\mathfrak{t}_0$ and $\mathfrak{t}_1$ are spin-c structures on $L(p^2, pq-1)$ such that their respective correction terms vanish. Then p divides $\mathfrak{t}_0 - \mathfrak{t}_1 \in H^2(L(p^2, pq-1))$.*

Proof of Proposition 3.4. As ξ is symplectically fillable and supported by a planar open book, Proposition 3.9 gives that

$$d(L(p^2, pq-1), \mathfrak{t}_\xi) = -d_3(\xi) - \tfrac{1}{2} = 0.$$

Proposition 3.10 then gives that p divides $\mathfrak{t}_{\bar{\xi}_{\mathrm{st}}} - \mathfrak{t}_\xi$; and thus $\mathfrak{t}_\xi$ extends across $B_{p,q}$ as $\mathfrak{t}_{\bar{\xi}_{\mathrm{st}}}$ does. Clearly $\xi \in \Xi_{p,q}$, so by Proposition 3.7, ξ is contactomorphic to $\bar{\xi}_{\mathrm{st}}$. □

Finally, Proposition 3.7 relies on the observation that there are exactly two integral solutions to $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \equiv 0 \bmod p$ under the appropriate restrictions of the c_i. The following lemma gives bounds that imply this fact as a corollary.

Lemma 3.11. *Fix integers $c_0 \in [0, s_0-1]$, $c_i \in [0, s_i]$ for all $1 \le i \le \ell$, and $c_{\ell+1} \in [0, r_\ell - 1]$. Then for each $k < \ell+1$,*

$$1 - \rho_{\ell-2\lfloor (k+1)/2\rfloor+1} \le \sum_{i=0}^{k}(-1)^i c_i \rho_{\ell-i+1} \le -1 + \rho_{\ell-\lfloor k/2\rfloor},$$

and

$$-p < 1 - \rho_0 \le (-1)^{\ell+1}\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \le \rho_{-1} + 2\rho_0 - 1 < 2p.$$

Consequently, $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} = 0$ if and only if each $c_i = 0$.

Proof. First, assume the inequalities; note $c_0\rho_{\ell+1} = 0$ if and only if $c_0 = 0$. By way of induction, suppose the only solution to $\sum_{i=0}^{k}(-1)^i c_i \rho_{\ell-i+1} = 0$ is the trivial solution. Any purported nontrivial solution to $\sum_{i=0}^{k+1}(-1)^i c_i \rho_{\ell-i+1} = 0$, has $c_{k+1} > 0$ by induction; however,

$$c_{k+1}\rho_{\ell-k} > \rho_{\ell-k} - 1 \geq (-1)^k \sum_{i=0}^{k}(-1)^i c_i \rho_{\ell-i+1},$$

contradicting $\sum_{i=0}^{k+1}(-1)^i c_i \rho_{\ell-i+1} = 0$. The lower bounds follow by noting that the sum minimizes by taking the c_i maximal for odd indices and zero otherwise: when $k < \ell+1$,

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i c_i \rho_{\ell-i+1} &\geq \sum_{i=1}^{\lfloor (k+1)/2 \rfloor} -\sigma_{\ell-2i+2}\,\rho_{\ell-2i+2} \\ &= \sum_{i=1}^{\lfloor (k+1)/2 \rfloor} (\rho_{\ell-2i+3} - \rho_{\ell-2i+1}) = \rho_{\ell+1} - \rho_{\ell-2\lfloor (k+1)/2 \rfloor+1}\end{aligned}$$

here we use that $s_i = \sigma_{\ell-i+1}$ and that $\rho_{i+1}\sigma_{i+1} = \rho_i - \rho_{i+2}$. The arguments are similar for the upper bounds and those when $k = \ell+1$. □

Corollary 3.12. *For the c_i as in Lemma 3.11, there are exactly two solutions to*

$$\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \equiv 0 \mod p.$$

Proof. By Lemma 3.11, $\left|\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1}\right| < 2p$, so we only need to consider solutions with

$$\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \in \{0, \pm p\}.$$

The last inequality in Lemma 3.11 implies that if there is a solution summing to $\pm p$ then there is not one summing to $\mp p$. Lemma 3.11 also gives that there is exactly one solution summing to zero. Note that choosing the c_i maximal gives

$$\sum_{i=0}^{\ell+1}(-1)^i c_i^{\max} \rho_{\ell-i+1} = s_0 - 1 + \sum_{i=1}^{\ell}(-1)^i s_i \rho_{\ell-i+1} + (-1)^{\ell+1}(r_\ell - 1)\rho_0 = (-1)^{\ell+1} p.$$

This solution is necessarily unique; whenever $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} = (-1)^{\ell+1} p$,

$$\sum_{i=1}^{\ell+1}(-1)^i (c_i^{\max} - c_i)\rho_{\ell-i+1} = 0,$$

forcing each $c_i = c_i^{\max}$. Thus, there are exactly two solutions: $c_{\min} \equiv 0$ and $c^{\max}$. □

A Stein handle decomposition. Here we prove that each rational ball $A_{m,n}$ admits a Stein structure filling the universally tight contact structure on the lens space $\partial A_{m,n}$, thereby proving Theorem 1.2. By Proposition 3.4, it is sufficient to find *any* Stein handle decomposition giving $A_{m,n}$, as all such Stein structures will induce contact structures with three-dimensional homotopy invariant equal to $-\frac{1}{2}$.

The 2-handle attachment in $A_{m,n}$ defined by Yamada (Figure 2) is Legendrian. However, the 2-handle is attached via the zero framing when measured against the resulting contact framing. We prove that there exists an ambient isotopy within $S^1 \times S^2$ of the attaching circle to a different Legendrian isotopy class satisfying that the 2-handle is attached with framing one less than the contact framing induced from this new Legendrian embedding. To that end, we have the following proposition.

Proposition 3.13. *Each $A_{m,n}$ admits a Stein structure, $\tilde{J}_{m,n}$, specified by the Stein handle decomposition of Figure 8, where we assume $\{\rho_i\}_{i=-1}^{\ell+1}$ and $\{\sigma_i\}_{i=0}^{\ell}$ are as in Definition 2.3.*

This isotopy is performed in two steps. First the 2-handle is slid under the 1-handle (around a hemisphere of a $\{\mathrm{pt}\} \times S^2$) once, then the 2-handle is dragged over the 1-handle (winding in the $S^1 \times \{\mathrm{pt}\}$ direction) repeatedly to arrive at the desired Legendrian knot specified in Figure 8. Proposition 3.13, is proved inductively. To motivate the proof as well as set up the base cases for induction we first slide the 2-handle of $A_{m,n}$ once under the 1-handle as shown in the upper left of Figure 9. Referring to the portion of the attaching circle K passing behind the central plane of the two attaching balls of the 1-handle as the "bad" strand, we can pair off negative crossings in the bad strand with positive crossings in K by "unraveling" the 2-handle. To accomplish this, begin by dragging the bad strand once over the 1-handle (bottom of Figure 9). By dragging the bad strand another $\sigma_0 - 1$ times over the 1-handle we find the bad strand now involves $\rho_1 - 1$ strands rather than the original $\rho_{-1} - 1$ strands (upper right of Figure 9). In fact, if $\rho_1 = 1$, then we immediately have the Stein structure $(A_{m,n}, \tilde{J}_{m,n})$ of Proposition 3.13.

Remark 3.14. We cannot assume $\rho_1 = 1$. That said, the same principle holds far more generally; there exist isotopies of K taking the bad strand from involving $\rho_{2i-1} - 1$ strands to involving only $\rho_{2i+1} - 1$ strands. This is the content of the following proposition.

Proposition 3.15. *For each integer k such that $0 \le 2k \le \ell$, $A_{m,n}$ is specified by attaching a 2-handle with framing $mn + 2(m+n)$ along* (*the closure across the* 1-*handle of*) *the braid B_k defined in Figure 10.*

Proposition 3.15 immediately gives Proposition 3.13 in the case $\ell \in 2\mathbb{Z}$ since $\rho_{\ell+1} - 1 = 0$ and the central band vanishes at the ℓ-th stage. Proposition 3.15 is proved by investigating how long bands of blackboard parallel strands remain

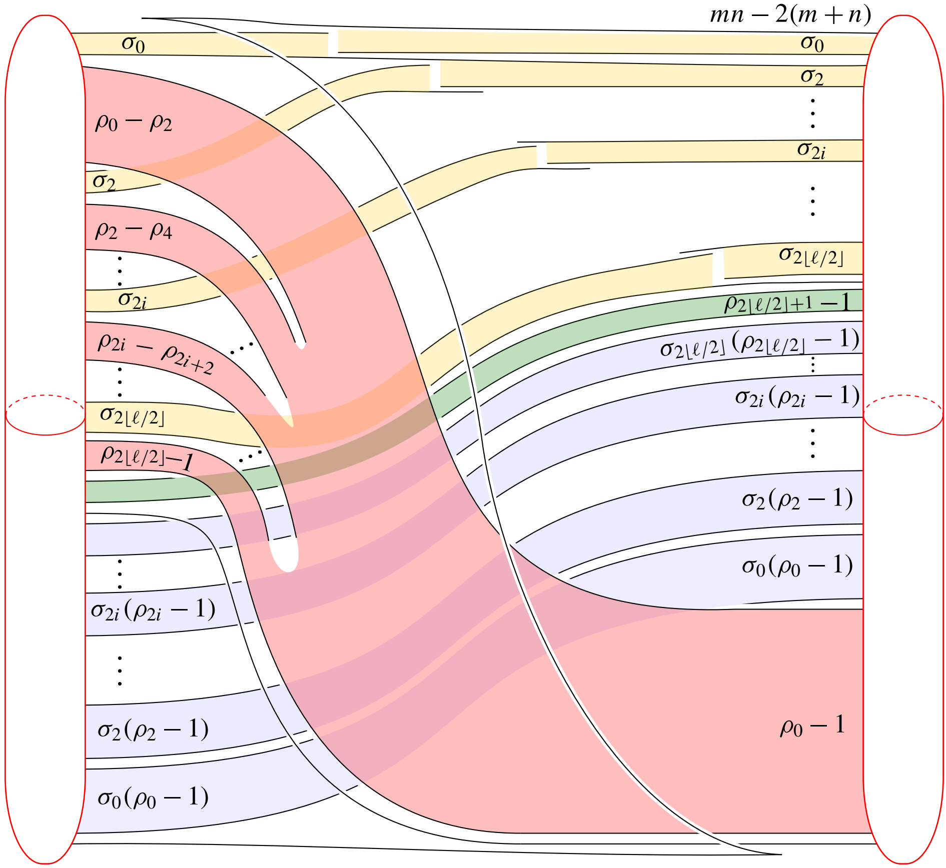

Figure 8. The Legendrian 2-handle attachment specifying the Stein structure $(A_{m,n}, \tilde{J}_{m,n})$. Here an integer superimposed on a given colored band indicates the number of blackboard parallel strands running within the band. *Warning: The vertical scaling is nonlinear and differs between the left and right foot of the* 1*-handle.*

together as they wrap around the braid B_k. To that end, we will denote the bands moving downward in B_k by D_i and those moving upward by U_i (as in Figure 10). Notice that we suppress the dependence on k for these bands since for each $i<k$, D_i (respectively U_i) persists for larger values of k. The only labeled band that changes when passing from B_k to B_{k+1} is D_k, which splits off D_{k+1}. Whereas, U_{k+1} consists of strands coming from the central band in B_k. With this notation in place we have the following lemma:

Lemma 3.16. *In the braid B_k, the D_i band returns to itself shifted down exactly ρ_{2i+1} strands and the U_i band returns to itself shifted up exactly $\rho_{2i}-1$ strands* (*e.g.*, *see Figure 11 for the case when $k=0$*).

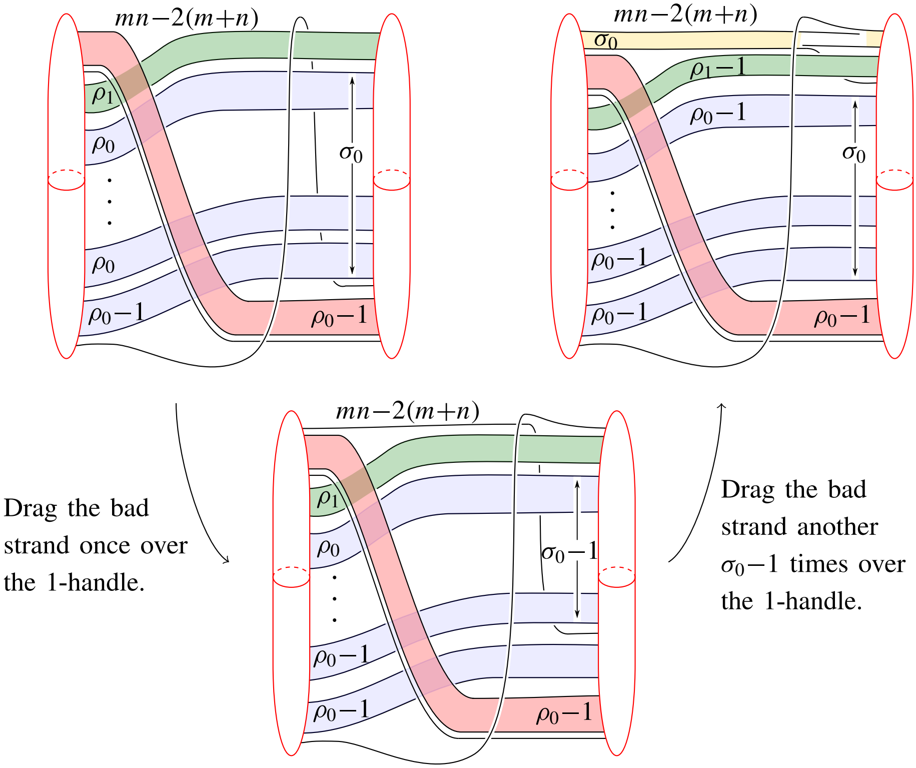

Figure 9. The result of sliding the attaching circle K once under the 1-handle, followed by isotopies of K as described.

Proof. We proceed by induction on k. The fact that the U_0 and D_0 bands return to themselves shifted up $\rho_0 - 1$ and down ρ_1 strands respectively is evident when looking at the closure of B_0 shown in Figure 11.

Suppose the result holds for each $0 \le i \le k-1$ in B_{k-1}. It is immediate that these shifts persist in B_k for each of the U_i and D_i bands provided $i < k$. Therefore, we only need to understand how the U_k and D_k bands return to themselves in B_k. We investigate how the U_k band returns first. To do this, we trace the U_k band as it enters and subsequently exits each of the D_i bands.

The key observation here is that the D_i band consists of a multiple of ρ_{2i+1} strands as $\rho_{2i} - \rho_{2i+2} = \sigma_{2i+1}\rho_{2i+1}$. When $i < k$, by induction, this is precisely the number of strands by which D_i shifts down when returning to itself. So the uppermost ρ_{2i+1} strands of D_i remain within D_i for a total of $\sigma_{2i+1} - 1$ returns before exiting directly below the D_i band entirely on the σ_{2i+1}-th return. We prove that the U_k band enters D_i within the uppermost ρ_{2i+1} strands. This is at least feasible since the U_k band has few enough strands to fit into uppermost ρ_{2i+1} strands of D_i as

$$\begin{aligned}\rho_{2i+1} &= \rho_{2k+1} + \sum_{j=i+1}^{k} (\rho_{2j-1} - \rho_{2j+1}) \\ &= \rho_{2k+1} + \sum_{j=i+1}^{k} \rho_{2j}\sigma_{2j} = \rho_{2k+1} + \sum_{j=i+1}^{k} (\sigma_{2j} + \sigma_{2j}(\rho_{2j} - 1)).\end{aligned}$$

When $i = 0$, we find that the U_k band indeed enters the D_0 band entirely within the uppermost ρ_1 strands as shown in the left side of Figure 12 (see also the upper right corner of Figure 10).

From above, we know that after σ_1 returns, these ρ_1 strands will have been shifted directly below the D_0 band. Of these ρ_1 strands, the uppermost σ_2 of them then pair off with those between the D_0 and D_1 bands and U_k is seen to enter the D_1 band within the first ρ_3 strands (e.g., see the center of Figure 12 taking $i = 1$). This process repeats and we find that for each $0 < i < k$, the U_k band enters the D_i

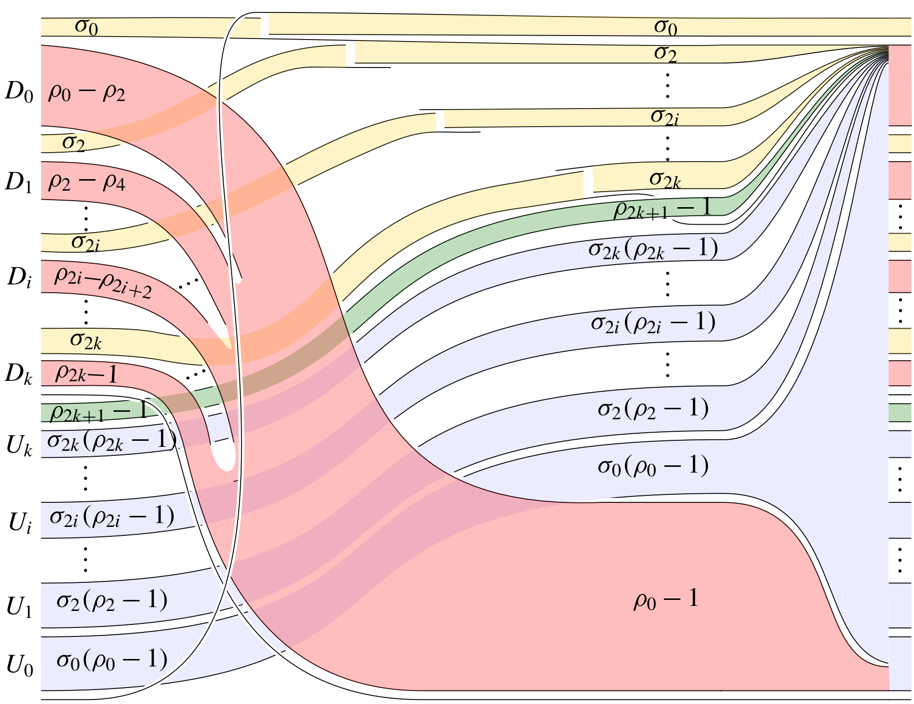

Figure 10. The braid B_k: Isotoping away the “bad strand” of the attaching circle for the 2-handle in $A_{m,n}$. The bands labeled D_i and U_i are those described in Lemma 3.16. *Warning: the* 1*-handle of* $A_{m,n}$ *has been suppressed and the vertical scaling is nonlinear.*

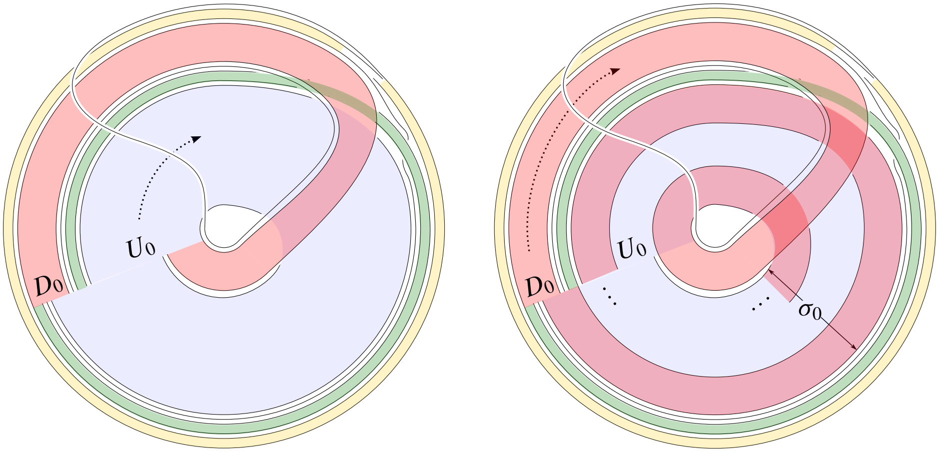

Figure 11. Left: The U_0 band in the braid B_0 returns to itself shifted up by $\rho_0 - 1$ the number of strands in the D_0 band. Right: The D_0 band in the braid B_0 returns to itself shifted down ρ_1 the number of strands in the central band.

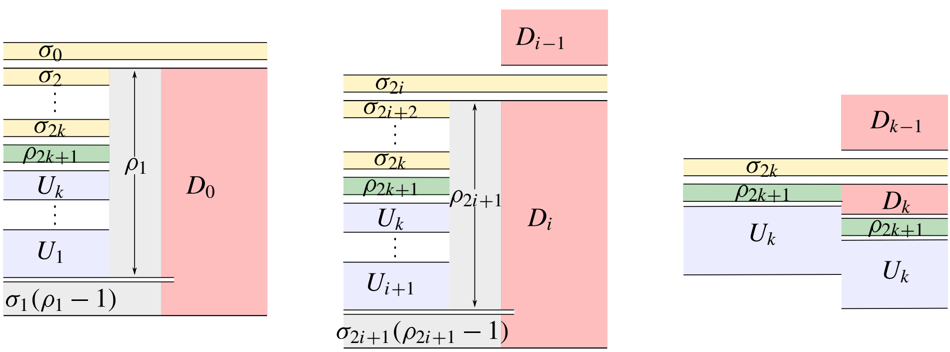

Figure 12. Left: The U_k band entering the D_0 band. Center: The U_k band entering the D_i band for $0 < i < k$. Right: The U_k band meeting the D_k band. Notice that the U_k band has returned to itself shifted up by exactly $\rho_{2k} - 1$ strands — the number of strands in the D_k band.

band as in the center of Figure 12. Therefore, we see that in B_k the strands in the U_k band remain blackboard parallel through each of the D_i bands for $i < k$. When the U_k band exits the D_{k-1} band, the U_k has returned to itself shifted up by the number of strands in the D_k band, that is, up by exactly $\rho_{2k} - 1$ strands, as claimed (see the right side of Figure 12).

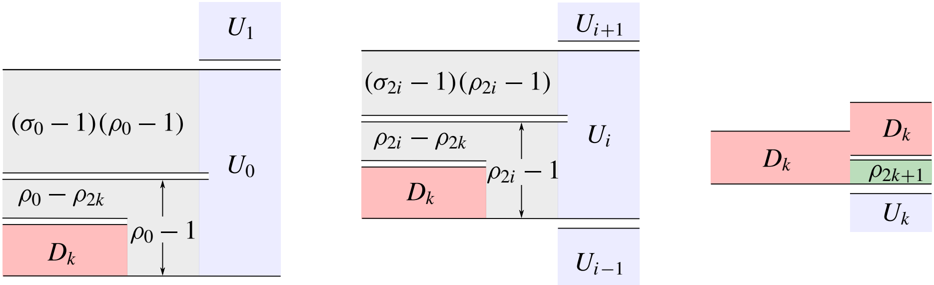

Figure 13. Left: The D_k band entering the U_0 band. Center: The D_k band entering the U_i band for $0 < i < k$. Right: The D_k band returning to the D_k band. Notice that the D_k band has shifted down by exactly ρ_{2k+1} strands.

Knowing that within the braid B_k, each U_i band returns to itself shifted up by exactly $\rho_{2i}-1$ strands, for each i less than or *equal* to k, now allows us to show that D_k returns to itself shifted down ρ_{2k+1} strands. The approach is the same as above; we make use of the fact that the number of strands in the U_i band is a multiple of the number of strands by which the U_i band shifts up when first returning to itself within B_k. Our induction hypothesis then ensures that the lower most $\rho_{2i}-1$ strands in U_i can be shifted up and out of U_i to the $\rho_{2i}-1$ strands above.

We follow the D_k band as it enters and exits each of the U_i bands. First, notice that the D_k band enters the U_0 band as the lowermost $\rho_{2k}-1$ strands as in the right side of Figure 13 (see also the lower right corner of Figure 10).

By induction, we know that when tracing the U_0 band as it returns to itself, the lowermost ρ_0-1 strands are shifted up by ρ_0-1 strands. So D_k enters U_0 a second time shifted up by ρ_0-1 strands. This process repeats a total of σ_0 times before D_k exits U_0 and enters U_1 as the lowermost $\rho_{2k}-1$ strands. Continuing by induction, for each $0 < i \leq k$, we find that D_k enters U_i as in center of Figure 13. From above, we know that the U_k band returns to itself shifted up $\rho_{2k}-1$ strands, so the D_k band continues through the U_k band σ_{2k} times before exiting directly above the U_k band (right side of Figure 13). At this point, D_k has come back to itself shifted down by ρ_{2k+1} strands, giving the result. □

Proof of Proposition 3.15. We proceed by induction on k. Figure 9 gives the case when $k=0$. Suppose K has been isotoped to B_k for some k with $2k < \ell-2$. We view the "bad" strand as a tangle on ρ_{2k+1} strands. We begin to push this tangle over the 1-handle repeatedly. Notice that anytime the bad strand enters D_i, Lemma 3.16 ensures that it can be moved down ρ_{2i+1} strands. The bad strand initially enters the D_0 band within the uppermost ρ_1 strands (see the upper left of Figure 14).

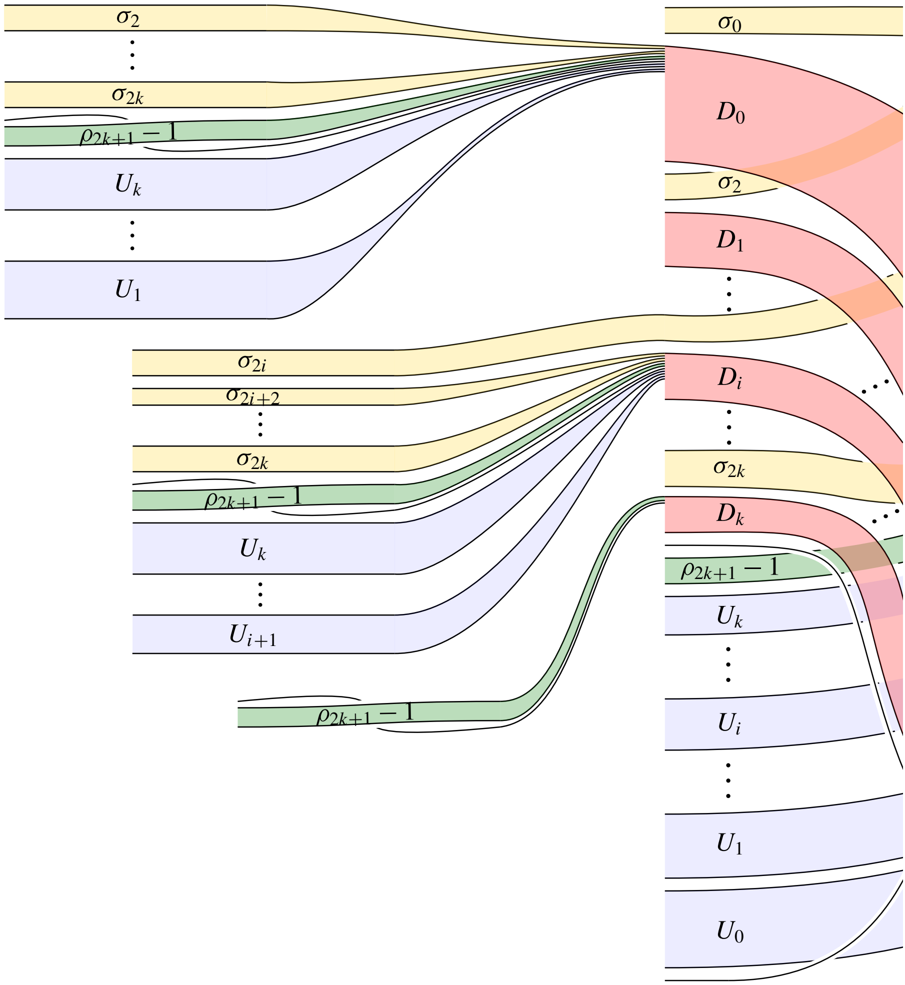

Figure 14. Pushing the bad strand into D_0 (upper left). Repeated application of Lemma 3.16 proves that the bad strand can be pushed into each D_i for $i < k$ (center left) and for $i = k$ (bottom left).

Applying Lemma 3.16, σ_1 times to the D_0 band shows that the bad strand can be isotoped (by pushing it along the blackboard parallel strands of D_0) into the uppermost ρ_3 strands of the D_2 band. By applying Lemma 3.16 to each D_j band, we can position the bad strand within the uppermost ρ_{2i+1} strands of the D_i band (see Figure 14) for each $i \leq k$.

As D_k consists of $\rho_{2k}-1 = \rho_{2k+1}\sigma_{2k+1}+\rho_{2k+2}-1$ strands, applying Lemma 3.16 to D_k, we can move the bad tangle down a total of σ_{2k+1} times before it begins to leave D_k. At this point, we find that the bad strand only involves $\rho_{2k+1}-(\rho_{2k+2}-1)$ strands. This occurs at the expense of splitting the lowermost $\rho_{2k+2}-1$ strands

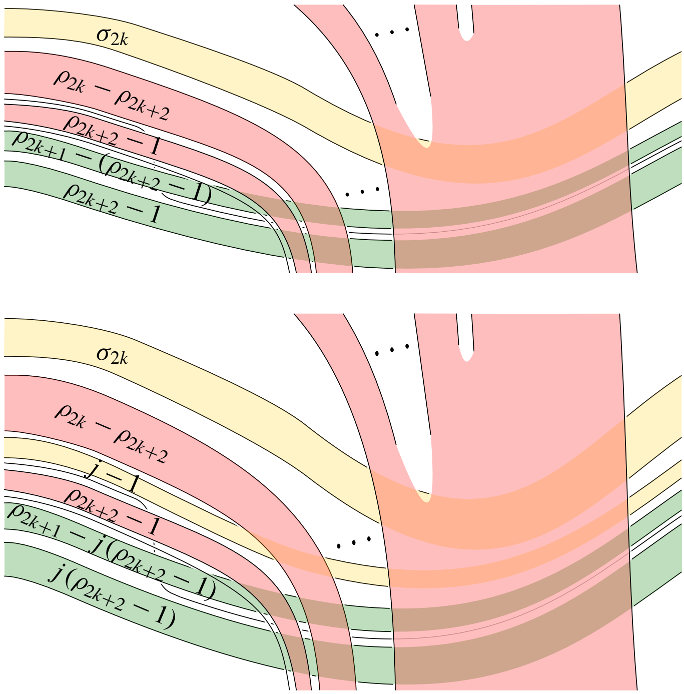

Figure 15. Top: The result of pushing the bad strand once through each of the D_i bands. Bottom: The result of pushing the bad strand j-times through each of the D_i bands. When $j = \sigma_{2k+2}$, we have the completed the isotopy from B_k to B_{k+1} claimed in Proposition 3.15.

from D_k, thereby forming what will be the D_{k+1} band within the braid B_{k+1} (top of Figure 15).

This process is repeated, each time the bad strand involving $\rho_{2k+2} - 1$ fewer strands. Repeating the process j times results in the bottom of Figure 15. Taking $j = \sigma_{2k+2}$ then gives B_{k+1}. □

Proof of Proposition 3.13. By Proposition 3.15, the 2-handle attachment of Figure 8 is isotopic to the 2-handle attachment defined by Yamada (Figure 2). Indeed if $\ell \in 2\mathbb{Z}$ then B_ℓ agrees with Figure 8. When $\ell \in 2\mathbb{Z}+1$, one applies the induction step of Proposition 3.15 a final time to arrive at Figure 8. Therefore, Figure 8 specifies $A_{m,n}$.

Moreover, as each isotopy from B_k to B_{k+1} is writhe preserving. The writhe of B_k is that of B_0 which equals $mn - 2(m+n) + 2$. Therefore, the 2-handle attachment of Theorem 1.2 is Stein since K's induced contact framing is

$$\text{writhe}(K) - \#(\text{left cusps}) = (mn - 2(m+n) + 2) - 1.$$

Eliashberg's characterization of handle decompositions of Stein domains [Eliashberg 1990; Gompf 1998] then gives that $A_{m,n}$ is realized as a Stein domain. □

Proof of Theorem 1.2. The fact that $(\partial A_{m,n}, \xi_{\tilde{J}_{m,n}})$ is contactomorphic to the universally tight lens space $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$ follows by noting that any almost complex structure on the rational ball $A_{m,n}$ (indeed any rational ball) satisfies

$$\frac{c_1^2(A_{m,n}, J) - 2\chi(A_{m,n}) - 3\sigma(A_{m,n})}{4} = -\frac{1}{2},$$

thus $d_3(\xi_{\tilde{J}_{m,n}}) = -\frac{1}{2}$. By Proposition 3.4, $\xi_{\tilde{J}_{m,n}}$ is universally tight. Since $(A_{m,n}, \tilde{J}_{m,n})$ gives a symplectic filling of the space $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$, Lisca's classification then gives that $A_{m,n} \approx B_{p,q}$. □

4. Boundary diffeomorphisms

From here, we pursue a handle-theoretic approach to understanding the diffeomorphisms $B_{p,q} \approx A_{m,n}$ ensured by Theorem 1.2. To that end, we define maps from $\partial B_{p,q}$ and $\partial A_{m,n}$ to the same linear plumbing of S^1-bundles.

It is worth noting that such diffeomorphisms have been known previously. Yamada [2007] produces similar diffeomorphisms from $\partial A_{m,n}$ to $L(p^2, pq-1)$ expressed as the boundary of $\mathcal{C}_{p,q}$. To accomplish this, one must carefully keep track of every stage of the Euclidean algorithm applied to $(p-q, q) = 1$. We perform a courser bookkeeping of the Euclidean algorithm via Definition 2.3, which allows for arguably clearer definitions. However, we do this at the expense of arriving at the plumbing of Proposition 2.5 rather than $\mathcal{C}_{p,q}$. This approach has the added advantage of applying to $\partial B_{p,q}$ in a structurally similar way.

Composing these maps gives a diffeomorphism from $\partial B_{p,q}$ to $\partial A_{m,n}$ that can be seen as a restriction of a diffeomorphism between the 4-manifolds $B_{p,q}$ and $A_{m,n}$ through carving, introduced by Akbulut [1977]; see also [Akbulut 2016]. By doing so, we will prove Theorem 1.3 as well as Corollary 1.4. For convenience we briefly outline the carving procedure.

Carving 4-manifolds. Suppose we have two 4-manifolds X and X' and a diffeomorphism $f : \partial X \to \partial X'$ where X admits a handle decomposition consisting of a single 0-handle, k 1-handles, and N 2-handles, where the i-th 2-handle h_i is attached along a knot K_i in $\#k(S^1 \times S^2)$. Let μ_i denote the belt-sphere of h_i (i.e., a meridian of K_i).

If f extends to a diffeomorphism between X and X', then in particular it extends across a neighborhood of the collection of cocores of the 2-handles in X. Thus, a necessary condition for f to extend is the property that the image of the belt-spheres $f(\mu_1) \cup \cdots \cup f(\mu_N)$ must be a slice link in $\partial X'$. That is, there exists a collection of properly embedded disks $D_i \subset X'$ such that $D_i \cap D_j = \varnothing$ and $\partial D_i = f(\mu_i)$.

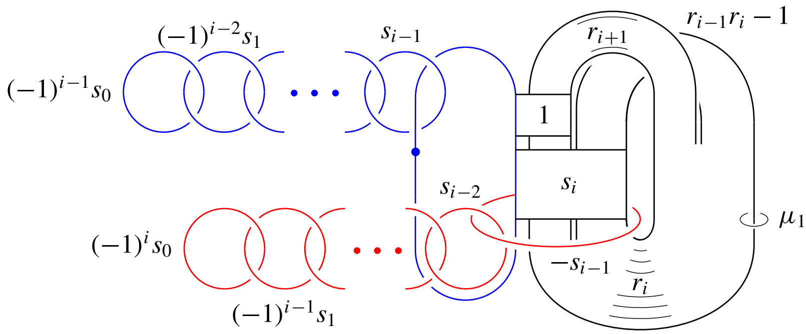

Figure 16. The 4-manifold $B^i_{p,q}$

Assuming this, if f carries the 0-framing of each μ_i (induced by the cocore) to the framing of $f(\mu_i)$ induced by the slice disk, then f extends across the neighborhoods of the cocores of the 2-handles in X. In order to extend f across the rest of X, we are left needing to extend a map $f_0 : \#k(S^1 \times S^2) \to \#k(S^1 \times S^2)$. Laudenbach and Poenaru [1972] prove that every self diffeomorphism of $\partial(\natural k(S^1 \times B^3))$ extends. Therefore, f_0 extends provided that

$$X' - \nu(D_1 \cup \cdots \cup D_N) \approx \natural k(S^1 \times B^3)$$

as obviously removing neighborhoods of the cocores of the 2-handles in X gives $\natural k(S^1 \times B^3)$.

Boundary diffeomorphisms: $\partial B_{p,q}$. The key observation to build such maps is that if $p = qs + r$, then $\partial B_{p,q}$ is obtained from $\partial B_{q,r}$ via integral surgeries on two unknotted circles. The boundary maps that we are after are obtained by iterating this process. As we define these maps, we trace the belt-sphere of the single 2-handle of $B_{p,q}$.

Proposition 4.1. *Let $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ be as defined in Definition 2.3. Then for each $i \in \{0, \ldots, \ell+1\}$, $\partial B_{p,q} \approx \partial B^i_{p,q}$ where $B^i_{p,q}$ is the 4-manifold specified by Figure 16.*

Proof. We induct on i. When $i = 0$, the result is immediate since $B^0_{p,q} \approx B_{p,q}$. Therefore, the proposition holds provided that $\partial B^i_{p,q} \approx \partial B^{i+1}_{p,q}$. Let K^i_1 be the attaching circle of the $r_{i-1}r_i - 1$-framed 2-handle in $B^i_{p,q}$. Suppose the result holds for some $i \leq \ell$. For $i+1$, first, surger the single 1-handle and introduce a canceling pair of 1- and 2-handles to remove the s_i-full twists between K^i_1 and the, now surgered, 1-handle (Figure 17).

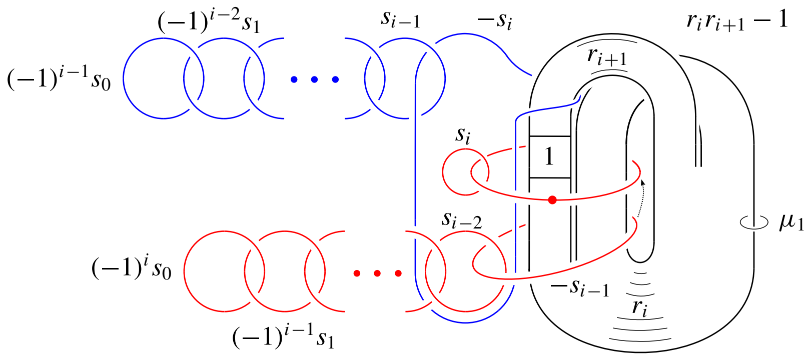

Figure 17. Introducing a canceling pair after surgery.

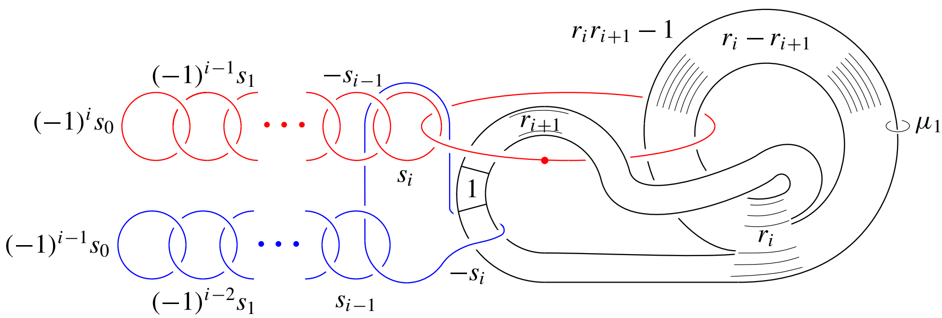

Figure 18. Isotoping K_1^i.

Since K_1^i links the new 1-handle r_i times, the framing on K_1^i decreases by $s_ir_i^2$ and the new framing on K_1^i is

$$r_{i-1}r_i-1-s_ir_i^2=r_i(r_{i-1}-s_ir_i)-1=r_ir_{i+1}-1.$$

Sliding the $-s_{i-1}$-framed 2-handle under the new 1-handle as indicated in Figure 17, and isotoping the r_{i+1}-stranded band (see Figure 18), we find that the r_{i+1}-stranded band traverses the 1-handle (positively) s_{i+1}-times as a complete band, while r_{i+2} strands traverse an additional one time to make up the complete $s_{i+1}r_{i+1}+r_{i+2}=r_i$ linking. With this view in mind, we isotope K_1^i into a closed braid on r_{i+1} strands appropriately linking the carving disk of the 1-handle; see Figure 19. The result holds by induction. □

Remark 4.2. At no point does μ_1, the meridian of K_1^i, get damaged under the boundary diffeomorphisms defined in Proposition 4.1. In particular, for each i, μ_1 bounds a disk in $B_{p,q}^i$ and the image of a collar neighborhood of μ_1 arising from such a disk persists under the boundary diffeomorphisms defined above. So, each diffeomorphism preserves the 0-framing on μ_1.

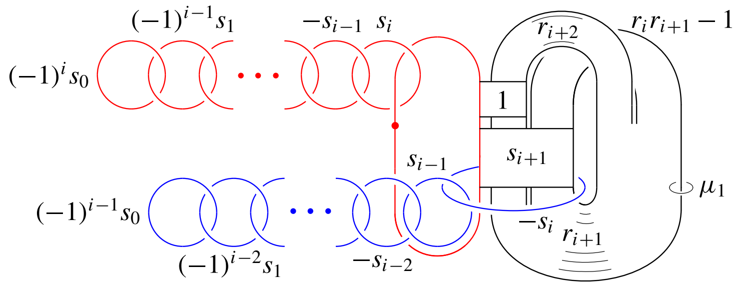

Figure 19. Further isotopy of K_1^i to K_1^{i+1}

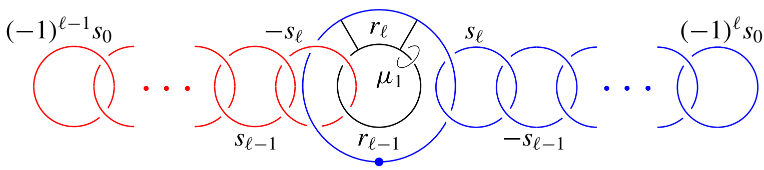

Figure 20. The space $B_{p,q}^{\ell+1}$.

Since $r_{\ell+1} = 1$ and $r_{\ell+2} = 0$, by definition, $s_{\ell+1} = s_{\ell+1}r_{\ell+1} + r_{\ell+2} = r_\ell$. By looking at $B_{p,q}^{\ell+1}$ we arrive at the following result of Casson and Harer [1981].

Corollary 4.3. $\partial B_{p,q} \approx L(p^2, pq-1)$.

Proof. By Proposition 4.1, we have that $\partial B_{p,q} \approx \partial B_{p,q}^{\ell+1}$ (Figure 20). The boundary diffeomorphism from $\partial B_{p,q}^{\ell+1}$ to a linear plumbing of S^1-bundles over S^2 is contained in Figure 21. □

Remark 4.4. It is an easy exercise to verify that the linear plumbing in Figure 21 bounds $L(p^2, pq-1)$. Indeed, one finds that

$$[-s_0, s_1, \ldots, \pm r_\ell, 1, \mp r_\ell, \ldots, -s_1, s_0] = -\frac{p^2}{pq-1}.$$

Boundary Diffeomorphisms: $\partial A_{m,n}$. As in the previous section, we exhibit explicit diffeomorphisms, this time from $\partial A_{m,n}$ to $L(p^2, pq-1)$. As the image of μ_1 is given as the 0-framed push-off of the attaching circle of the central 1-framed unknot at the bottom of Figure 21. We will trace where the curve, γ in Figure 3, goes as well — finding that it too goes to the 0-framed push-off of the central 1-framed unknot via an appropriately defined diffeomorphism. We want to define these diffeomorphisms similarly to those of Proposition 4.1.

Lemma 4.5. *$A_{m,n}$ is given by Figure 22.*

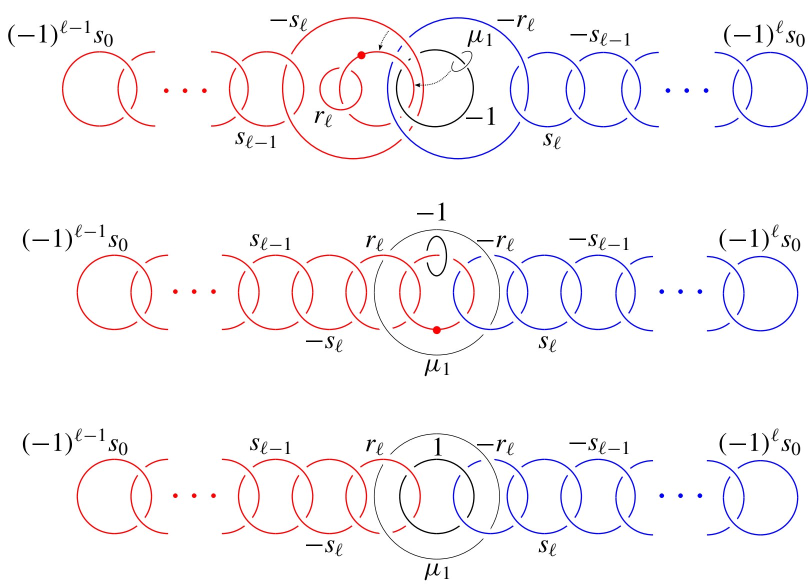

Figure 21. From top to bottom: The introduction of a canceling pair to $B_{p,q}^{\ell+1}$ after surgery; the result of the indicated slides; a linear plumbing associated to $\partial B_{p,q}$.

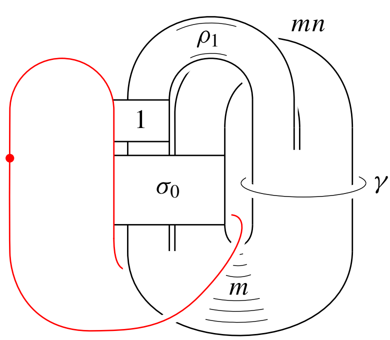

Figure 22. An alternative description of $A_{m,n}$.

Proof. The result follows from an isotopy of the 2-handle's attaching circle. First, view the $m+n$ strands of the attaching circle in Figure 2 as a band of n strands going over the 1-handle once with the remaining m strands going over twice (left side of Figure 23). Viewing the band of m strands going over the 1-handle completely σ_0 times with ρ_1 strands traversing an extra time (right side of Figure 23) gives the result. □

Using Lemma 4.5, we prove the analog of Proposition 4.1 in the $\partial A_{m,n}$ case.

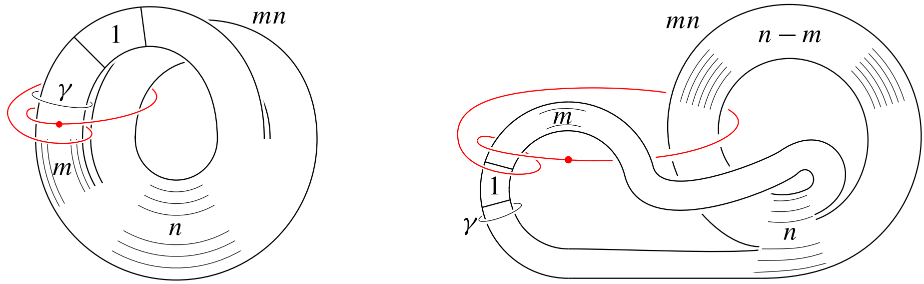

Figure 23. The isotopy of the 2-handle in $A_{m,n}$ used in the proof of Lemma 4.5.

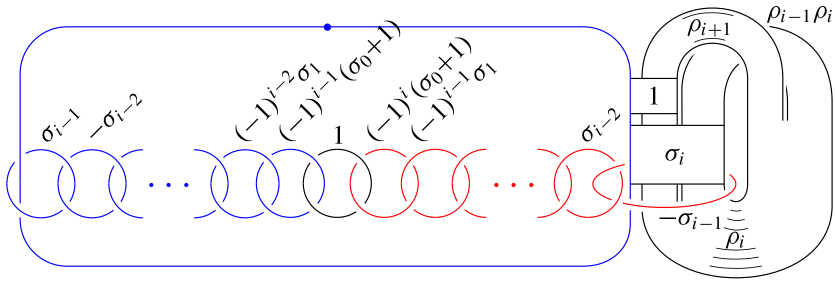

Figure 24. The 4-manifold $A^i_{m,n}$

Proposition 4.6. *Let* $\{\rho_i\}_{i=-1}^{\ell+2}$ *and* $\{\sigma_i\}_{i=0}^{\ell+1}$ *be as defined in Definition 2.3 (associated to* $n > m \geq 1$*). Then for each* $i \in \{0, \ldots, \ell+1\}$,

$$A_{m,n} \overset{\partial}{\approx} A^i_{m,n}$$

where $A^i_{m,n}$ *is the 4-manifold given by Figure 24.*

Proof. We induct on i, treating the base case and the induction step simultaneously. For the base case, start with the handle decomposition from Lemma 4.5. For the induction step, suppose that the result holds for some $i \leq \ell$. Let K^i_1 be the attaching circle of the $\rho_{i-1}\rho_i$-framed 2-handle in $A^i_{m,n}$. Surger the 1-handle and introduce a canceling 1- and 2-handle (for the base case see the left side of Figure 25, for the induction step see Figure 26). Notice, similar to Proposition 4.1 the framing of K^i_1 changes from $\rho_{i-1}\rho_i$ to $\rho_i\rho_{i+1}$.

Slide the now surgered 1-handle as indicated in the respective figures and, for the base case, blow-up once (right side of Figure 25). From here the base case follows similarly to the induction step; both of which are similar to Proposition 4.1. Indeed, isotope K^i_1 to view a band with ρ_{i+1} stands traversing the 1-handle σ_{i+1}-times along with ρ_{i+2} of those strands traversing an extra time as in Figure 27.

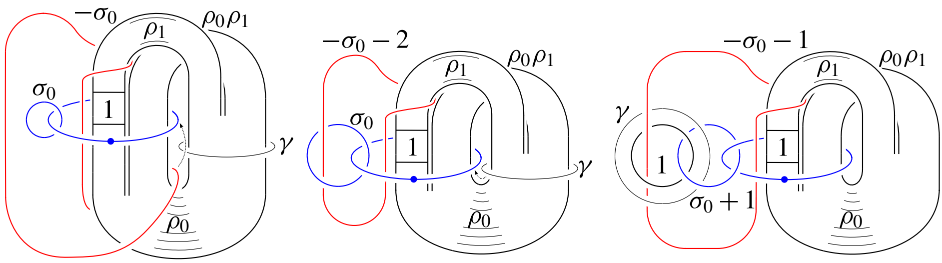

Figure 25. The base case of Proposition 4.6.

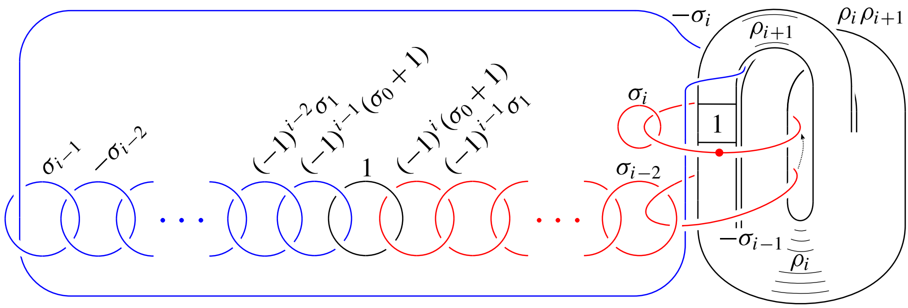

Figure 26. Introducing a canceling pair.

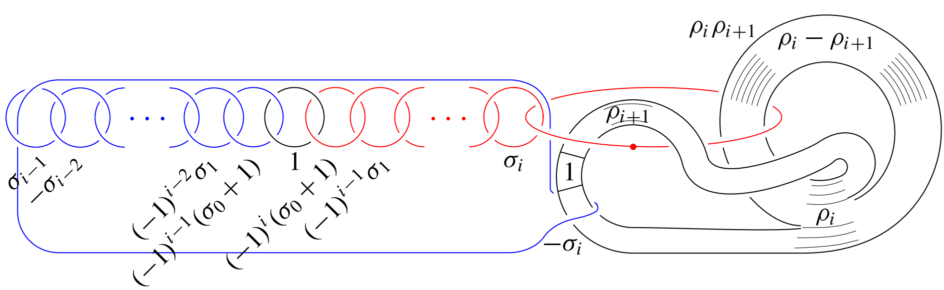

Figure 27. Isotoping K_1^i in $A_{m,n}^i$.

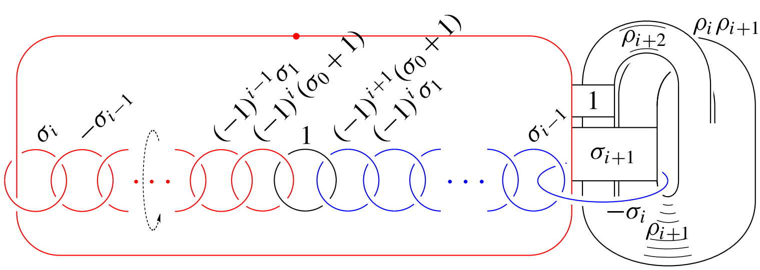

Figure 28. Further isotopy of K_1^i to K_1^{i+1} in $A_{m,n}^{i+1}$.

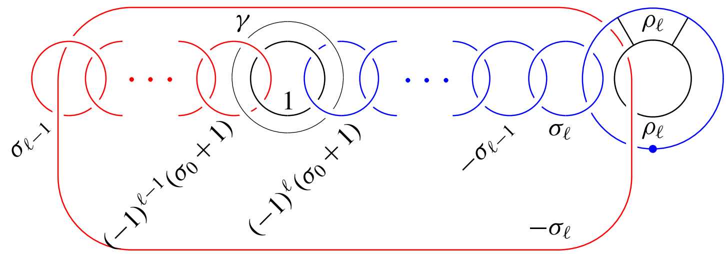

Figure 29. The space $A^{\ell+1}_{m,n}$.

A further isotopy of K^i_1 gives a closed braid on ρ_{i+1} strands geometrically linking the carving disk of the new 1-handle ρ_i-times. Finally, notice that to get the appropriate linking on the chain of unknots, we have to wind the chain (as indicated in Figure 28) to add a total of i positive half-twists to the left of the disk bundle of Euler class 1 along with i negative half-twists to the right. The result follows by induction. □

Corollary 4.7 [Yamada 2007, Theorem 1.1]. $\partial A_{m,n} \approx L(p^2, pq-1)$ *for* $(p-q, q) = A(m, n)$.

Proof. By Proposition 4.6, $\partial A_{m,n} \approx \partial A^{\ell+1}_{m,n}$; see Figure 29. We proceed as in Corollary 4.3. The boundary diffeomorphism from $\partial A^{\ell+1}_{m,n}$ to a linear plumbing of S^1-bundles over S^2 is contained in Figure 30. □

Remark 4.8. The fact that $\partial A_{m,n}$ is $L(p^2, pq-1)$ for $A(m,n) = (p-q, q)$ follows by noting that given p and q, or equivalently m and n, we can define the other pair by an appropriate identification of the linear plumbings in Corollaries 4.3 and 4.7, provided that $s_0 > 1$ (that is, provided that $p-q > q$, which we have assumed all along). In fact, as we have chosen to do in Remark 2.4, this can be taken as the definition of the function A defined by Yamada [2007]. Notice also that γ bounds a disk in each $\partial A^i_{m,n}$ as well as in the linear plumbing of Figure 30. Furthermore, each boundary diffeomorphism defined in Proposition 4.6 and those of Corollary 4.7 preserve the 0-framing of γ specified by those disks. Therefore, we can employ the carving method provided that carving along γ gives $S^1 \times B^3$, which it does:

Proposition 4.9 Proof of Corollary 1.4. *Carving* $A_{m,n}$ *along* γ *gives* $S^1 \times B^3$.

Proof. Carving $A_{m,n}$ along the curve γ means removing a neighborhood of the disk γ bounds inside $A_{m,n}$. The resulting handlebody decomposition is given by that of $A_{m,n}$ along with an extra 1-handle whose carving disk is γ. If we let γ_i be the analogous curve in A_{ρ_{i-1},ρ_i}, then the result of carving A_{ρ_{i-1},ρ_i} along γ_i is given in Figure 31.

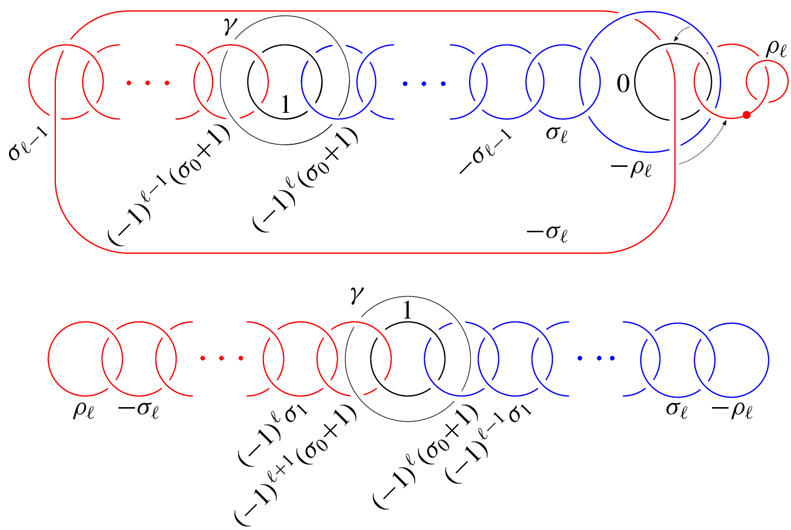

Figure 30. The result of surgering $A^{\ell+1}_{m,n}$ and introducing a canceling pair; the result of sliding and canceling as indicated gives a linear plumbing associated to $\partial A_{m,n}$.

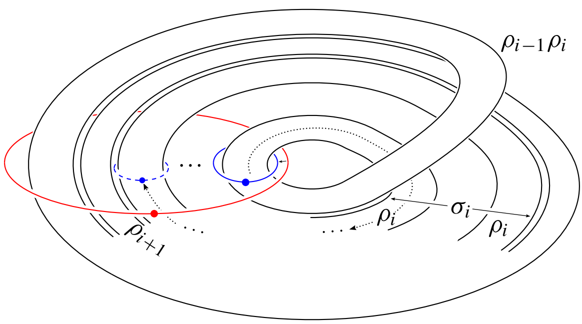

Figure 31. A_{ρ_{i-1},ρ_i} carved along γ_i.

Notice that $A_{m,n} = A_{\rho_0,\rho_{-1}}$ and $\gamma = \gamma_0$. By sliding the original 1-handle across the newly carved 1-handle σ_i times, twisting the 1-handle σ_i-times (negatively) and finally sliding as indicated in the left side of Figure 32 we arrive at $A_{\rho_i,\rho_{i+1}}$ carved along γ_{i+1} (right side of Figure 32). Therefore, the result of carving along γ_i in A_{ρ_{i-1},ρ_i} is diffeomorphic to carving along γ_{i+1} in $A_{\rho_i,\rho_{i+1}}$. As carving A_{1,ρ_ℓ} along γ_ℓ gives $S^1 \times B^3$ we have the result. $\square$

Proof of Theorem 1.3. As $A(p-q, q) = (m, n)$, we can identify the plumbings of Figures 21 and 30. By first, applying the diffeomorphisms of Proposition 4.1 we get a diffeomorphism from $\partial B_{p,q}$ to the boundary of the linear plumbing of the

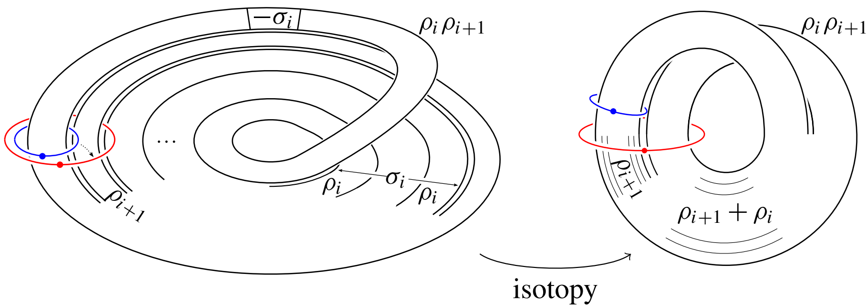

Figure 32. A_{ρ_{i-1},ρ_i} carved along γ_i after sliding and twisting σ_i-times.

bottom of Figure 21 carrying μ_1 as indicated. Applying the diffeomorphisms of Proposition 4.6 in reverse from the boundary of the linear plumbing of Figure 30 to $A_{m,n}$ gives the required diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$. $\square$

Acknowledgments

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LUKE WILLIAMS
DEPARTMENT OF MATHEMATICS
KANSAS STATE UNIVERSITY
MANHATTAN, KS 66502
UNITED STATES
lukewilliams@math.ksu.edu

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