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We prove an analogue of a Gouvêa–Mazur conjecture on local constancy of dimension of slope subspaces of modular forms on the upper half plane for automorphic forms on reductive algebraic groups $\tilde{G}/\mathbb{Q}$ having discrete series. The proof uses a comparison of Bewersdorff's elementary trace formula for pairs of congruent weights and does not make use of methods from p-adic Banach space theory, overconvergent forms or rigid analytic geometry.

We also compare two Goresky–MacPherson trace formulas computing Lefschetz numbers on weighted cohomology for pairs of congruent weights; this has an application to a more explicit version of the Gouvêa–Mazur conjecture for symplectic groups of rank 2.

Introduction

0.1. We fix a prime $p \in \mathbb{N}$ and an integer N not divisible by p. Generalizing Hida's theory [1993; 1988] of ordinary modular forms, Gouvêa and Mazur [1992] conjectured that the dimension $d(\beta, k)$ of the slope β subspace of the space $S_k(\Gamma_0(pN))$ of cuspidal modular forms of level pN and weight k is locally constant in the p-adic topology as a function of k. More precisely, they conjectured that there is a linear polynomial $\boldsymbol{m}(x)$ such that the conditions $k, k' \geq 2\beta + 2$ and $k \equiv k' \pmod{(p-1)p^{m-1}}$ with $m \geq \boldsymbol{m}(\beta)$ imply

$$d(\beta, k) = d(\beta, k'). \tag{1}$$

Using work of Coleman [1997] which is based on rigid analytic geometry and p-adic spectral theory, as well as Katz's theory of p-adic modular forms and results of Gouvêa and Mazur, Wan [1998] proved that there is a quadratic polynomial $\boldsymbol{m}(x)$ such that equation (1) holds. On the other hand, Buzzard and Calegari [2004] showed that in general there is no linear polynomial $\boldsymbol{m}(x)$ such that (1) holds, hence, Wan's result is best possible.

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0.2. In this article we prove a higher-rank analogue of the Gouvêa–Mazur conjecture. To describe this in more detail, we denote by $\mathbb{A}$ the adeles of $\mathbb{Q}$ and fix a prime $p \in \mathbb{N}$. We let $\tilde{\boldsymbol{G}}/\mathbb{Q}$ be a connected reductive algebraic group which contains a maximal torus $\tilde{\boldsymbol{T}}/\mathbb{Q}$ which splits over $\mathbb{Q}_p$. We select a basis Δ of the root system Φ of $\tilde{\boldsymbol{G}}/F$ where $F/\mathbb{Q}$ is a (minimal) splitting field for $\tilde{\boldsymbol{T}}/\mathbb{Q}$. We let $\tilde{K} \le \tilde{\boldsymbol{G}}(\mathbb{A}_f)$ be a compact open subgroup with p-component $\tilde{K}_p$ equal to the Iwahori subgroup $\tilde{\mathcal{I}}$ of $\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$. We denote by $\mathbb{T}$ the Hecke operator attached to the double coset $\tilde{K}h^{-1}r\tilde{K}$ where $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})^{++} \le \tilde{\boldsymbol{T}}(\mathbb{Q}_p)^{++}$ is a strictly dominant element and $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$, i.e., r has trivial p-component. For any dominant weight $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ we understand by $L_{\tilde{\lambda}}$ the irreducible representation of $\tilde{\boldsymbol{G}}/F$ of highest weight $\tilde{\lambda}$. The normalization $\mathbb{T}_{\tilde{\lambda}}$ of $\mathbb{T}$ acts on full cohomology $H^i(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))$ as well as on cuspidal cohomology $H^i_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))$ where $S_{\tilde{K}} = \tilde{\boldsymbol{G}}(\mathbb{Q})\backslash\tilde{\boldsymbol{G}}(\mathbb{A})/\tilde{K}\tilde{K}_\infty$ is the locally symmetric space. We write $H^i(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))^\beta$ (resp. $H^i(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))^{\le\beta}$) for the subspace of slope β (resp. slope $\le \beta$) w.r.t. $\mathbb{T}_{\tilde{\lambda}}$ and we use analogous notation for cuspidal cohomology. Our main results then are as follows.

Theorem A (see 3.10 Corollary, 4.11.4 Theorem). *Let $s = |\Phi^+|$ be the number of positive roots of $\tilde{\boldsymbol{G}}/\mathbb{Q}_p$, $\sigma = \max_{\alpha\in\Phi^+} \mathrm{ht}(\alpha)$ the maximal height of a positive root and $\boldsymbol{g}_i$ the number of i-cells in a finite cell complex $\mathcal{Z}$ which is homotopy equivalent to the Borel Serre compactification $\bar{S}_{\tilde{K}}$ of $S_{\tilde{K}}$. Then for all $\beta \in \mathbb{Q}_{\ge 0}$, $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ and $i \in \mathbb{N}_0$ we obtain*

$$\dim H^i(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))^{\le\beta} \le \mathtt{m}\beta^s + \mathtt{n};$$

here, $\mathtt{m} = 12\boldsymbol{g}_i\sigma^{s+1}/s$ and $\mathtt{n} \in \mathbb{N}$ is an integer which also only depends on $\tilde{K}$ (and, hence, on $\tilde{\boldsymbol{G}}$ and p) and on i.

Theorem B (see 5.2 Theorem). *We assume that $\tilde{\boldsymbol{G}}$ has discrete series and we denote by $d = d_{\tilde{\boldsymbol{G}}}$ the middle degree. There are polynomials $\boldsymbol{m}_1(x), \boldsymbol{m}_2(x) \in \mathbb{Q}[x]$ both of degree $s+1$ and leading term $12\boldsymbol{g}_d\sigma^{s+1}/s$ which only depend on $\tilde{K}$ (hence, on $\tilde{\boldsymbol{G}}$ and p) and $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ with the following property. Let $\beta \in \mathbb{Q}_{\ge 0}$. Suppose the dominant weights $\tilde{\lambda}, \tilde{\lambda}' \in X(\tilde{\boldsymbol{T}})$ satisfy*

- $\langle\tilde{\lambda}, \alpha^\vee\rangle > 2\boldsymbol{m}_1(\beta)$ *and* $\langle\tilde{\lambda}', \alpha^\vee\rangle > 2\boldsymbol{m}_1(\beta)$ *for all* $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$;
- $\tilde{\lambda} \equiv \tilde{\lambda}' \pmod{(p-1)p^{m-1}X(\tilde{\boldsymbol{T}})}$ *with* $m \ge \boldsymbol{m}_2(\beta)$ $(m \in \mathbb{N})$.

Then

$$\dim H^d_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))^\gamma = \dim H^d_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}'}(\mathbb{C}))^\gamma \quad \textit{for all } 0 \le \gamma \le \beta.$$

Remark. In the $\mathbf{GL}_2$-case $\boldsymbol{m}_2(x)$ is a quadratic polynomial; i.e., we obtain the same growth as that of $\boldsymbol{m}(x)$ in [Wan 1998], except that the weights have to satisfy a stronger lower bound (quadratic in β instead of linear as in that paper).

0.3. To prove Theorems A and B we will mostly work in a non-adelic setting; i.e., $\Gamma \leq \tilde{\boldsymbol{G}}(\mathbb{Q})$ denotes an arithmetic subgroup contained in $\tilde{\mathcal{I}}$. Theorem A then is an extension of the main result of [Mahnkopf 2014] (see Section 3.1) and the proof is based on an extension of the notion of truncation of an irreducible representation of $\tilde{\boldsymbol{G}}/\mathbb{Q}_p$ introduced in [Mahnkopf 2013; 2014].

Using the boundedness result of Theorem A the proof of Theorem B reduces to proving certain congruences between traces of powers of the Hecke operator $\mathbb{T}_{\tilde{\lambda}}$ on $H^i_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))$ and on $H^i_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}'}(\mathbb{C}))$ for p-adically close weights $\tilde{\lambda}, \tilde{\lambda}'$. We first verify these congruences on full cohomology and our principal tool for this is a comparison of a simple and elementary trace formula of Bewersdorff [1985] for cohomology with coefficients in $L_{\tilde{\lambda}}$ and in $L_{\tilde{\lambda}'}$. The equality of mod p^n reductions of geometric sides essentially follows from p-adic properties of the diagonalization of elements in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}} \subseteq \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$, $e \in \mathbb{N}$, which are proved using basic algebra (see 4.3 Lemma and 4.4 Proposition); we note that $\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}$ is the p-component of $\mathbb{T}^e$. To obtain congruences on cuspidal cohomology we directly prove congruences on the Eisenstein part of full cohomology and subtract from congruences on full cohomology.

Since the Bewersdorff trace formula is elementary we obtain an elementary proof of the congruences on full cohomology and the proofs of Theorems A and B do not make use of methods from p-adic Banach space theory, overconvergent cohomology or rigid analytic geometry (but use the spectral decomposition of full cohomology for regular weight).

0.4. *Weighted cohomology.* Goresky and MacPherson [Goresky and MacPherson 2003] proved a trace formula for Lefschetz numbers of Hecke operators on weighted cohomology. Unlike Bewersdorff's formula it contains contributions not only from $\tilde{\boldsymbol{G}}$ but from all $\mathbb{Q}$-parabolic subgroups of $\tilde{\boldsymbol{G}}$. Nevertheless, the same diagonalization of elements in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}} \subseteq \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ as in 0.3 allows to compare two Goresky–MacPherson trace formulas for pairs of congruent weights. This then yields certain congruences on weighted cohomology groups and has an application to a version of the Gouvêa–Mazur conjecture for symplectic groups of rank 2 which is more explicit since we avoid use of the spectral decomposition of full cohomology (see Section 5.8). We note that this depends on properties of the root system C_2 but using instead the Goresky–Kottwitz–MacPherson trace formula [Goresky et al. 1997] together with the calculations of Spallone [2009] it might be possible to extend this to arbitrary reductive groups $\tilde{\boldsymbol{G}}/\mathbb{Q}$.

0.5. Buzzard [Buzzard 2001] gave an elementary proof of boundedness of dimension of slope subspaces in the case $\mathbf{GL}_2/\mathbb{Q}$ also based on an analysis of representations of $\mathbf{GL}_2(\mathbb{Z}_p)$. In the case of quaternion algebras over $\mathbb{Q}$ he also proved in [Buzzard 1998] local constancy of dimension of slope subspaces and his results were generalized to $\mathbf{GL}_2$ over totally real fields by Pande [2009]. Following the method of Ash and Stevens [2008], who introduced overconvergent cohomology, Urban [2011] obtained p-adic families of systems of Hecke eigenvalues; he uses this to also derive a p-adic trace formula on overconvergent cohomology. Andreatta, Iovita and Pilloni also proved existence of p-adic families of eigenforms using rigid analytic geometry; see [Andreatta et al. 2015].

More closely related to our approach is work of Koike [1975; 1976] (and some unpublished work of Clozel); like Buzzard, Koike does not make use of methods from rigid analytic geometry or p-adic Banach space theory. In the case of cuspidal modular forms, i.e., in the case $\mathbf{GL}_2/\mathbb{Q}$ he uses a Selberg trace formula which yields an explicit expression for the trace of Hecke operators to deduce congruences between traces of Hecke operators. Since the Selberg trace formula becomes much more involved this seems difficult to generalize to higher rank. We therefore do not attempt to determine an explicit expression for the trace of Hecke operators but only equate mod p^n reductions of traces for p-adically close weights $\tilde{\lambda}$, $\tilde{\lambda}'$. This can be done even in higher rank by comparing the simple (non-explicit) trace formula of Bewersdorff for weights $\tilde{\lambda}$ and $\tilde{\lambda}'$.

1. Chevalley groups

We recall some basic facts from the theory of Chevalley groups and their representations and we give proofs for some (technical) results for which we do not know a reference.

1.1. ***Complex semisimple Lie algebras.*** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and by $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{g}$ w.r.t. to $\mathfrak{h}$. We choose a basis Δ of Φ and we denote by Φ^+ the set of positive roots. For each root $\alpha \in \Phi$ we write $\mathfrak{g}(\alpha)$ for the corresponding root subspace of $\mathfrak{g}$ and we select elements $h_\alpha \in \mathfrak{h}$, $\alpha \in \Delta$, and $x_\alpha \in \mathfrak{g}(\alpha)$, $\alpha \in \Phi$, such that $\{h_\alpha,\ \alpha \in \Delta,\ x_\beta,\ \beta \in \Phi\}$ is a *Chevalley basis* of $\mathfrak{g}$. In particular, h_α is the coroot corresponding to $\alpha \in \Delta$. The Chevalley basis yields $\mathbb{Z}$-forms $\mathfrak{g}(\mathbb{Z}) = \bigoplus_{\beta\in\Phi} \mathbb{Z}x_\beta \oplus \bigoplus_{\alpha\in\Delta} \mathbb{Z}h_\alpha$ (resp. $\mathfrak{h}(\mathbb{Z}) = \bigoplus_{\alpha\in\Delta} \mathbb{Z}h_\alpha$) of $\mathfrak{g}$ (resp. of $\mathfrak{h}$). We denote by $\mathcal{U}_\mathbb{Z}$ the $\mathbb{Z}$-form of the universal enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$ which as a ring is generated by the elements $x_\alpha^n/n!$, $\alpha \in \Phi$, $n \in \mathbb{N}_0$ (see [Humphreys 1972, Theorem 26.4, p. 156]). We set $\mathfrak{g}(R) = \mathfrak{g}(\mathbb{Z}) \otimes R$, $\mathfrak{h}(R) = \mathfrak{h}(\mathbb{Z}) \otimes R$ and $\mathcal{U}_R = \mathcal{U}_\mathbb{Z} \otimes R$, R a $\mathbb{Z}$-algebra. We set $s = |\Phi^+|$ and we fix an ordering $\Phi^+ = \{\alpha_1, \dots, \alpha_s\}$ of the set of positive roots and we set

$$X_\pm^{\boldsymbol{n}} = \frac{x_{\pm\alpha_1}^{n_1}}{n_1!} \cdots \frac{x_{\pm\alpha_s}^{n_s}}{n_s!} \in \mathcal{U}_\mathbb{Z},$$

where $\boldsymbol{n} = (n_1, \dots, n_s) \in \mathbb{N}_0^s$. The $\mathbb{Z}$-span of the elements $X_-^{\boldsymbol{n}}$, where $\boldsymbol{n} \in \mathbb{N}_0^s$, is a

$\mathbb{Z}$-form $\mathcal{U}^-_{\mathbb{Z}}$ of the universal envelopping algebra $\mathcal{U}^-$ of $\mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{g}(\alpha)$. Finally, $\mu \leq \lambda$, for $\lambda, \mu \in \mathfrak{h}^*$, means that $\lambda - \mu$ is a linear combination of positive roots with nonnegative coefficients. For any $\lambda \in \mathfrak{h}^*$ we define a relative height function $\mathrm{ht}_\lambda : \{\mu \in \mathfrak{h}^* : \mu \leq \lambda\} \to \mathbb{N}_0$ by $\mathrm{ht}_\lambda(\mu) = \mathrm{ht}(\lambda - \mu)$ (see [Mahnkopf 2013, 1.3]); here, $\mathrm{ht} = \mathrm{ht}_\Delta$ is the height function corresponding to Δ, i.e., $\mathrm{ht}(\lambda - \mu) = \sum_{\alpha\in\Delta} n_\alpha$ if $\lambda - \mu = \sum_{\alpha\in\Delta} n_\alpha \alpha$. By $\omega_\alpha \in \mathfrak{h}^*$, $\alpha \in \Delta$, we understand the fundamental (dominant) weights, i.e., $\omega_\beta(h_\alpha) = \delta_{\alpha,\beta}$. The fundamental weights span the weight lattice Γ_{sc} of $\mathfrak{g}$ which contains the root lattice Γ_{ad}.

For any integral and dominant weight $\lambda \in \mathfrak{h}^*$ we denote by $(\rho_\lambda, L_\lambda)$ the complex irreducible $\mathfrak{g}$-module of highest weight λ. We denote by Γ_λ the subgroup of the weight lattice Γ_{sc} of $\mathfrak{g}$ which is generated by the (finite) set of weights P_λ of L_λ. The representation ρ_λ is defined over $\mathbb{Z}$, i.e., $L_\lambda = L_\lambda(\mathbb{Z}) \otimes \mathbb{C}$ where $L_\lambda(\mathbb{Z})$ is $\mathcal{U}_{\mathbb{Z}}$-invariant. We select a highest weight vector $v_\lambda \in L_\lambda$; the lattice then is defined as $L_\lambda(\mathbb{Z}) = \mathcal{U}_{\mathbb{Z}} v_\lambda$ (see [Humphreys 1972, proof of Theorem 27.1, p. 158]). Moreover, we set $L_\lambda(\mathbb{Z}, \mu) = L_\lambda(\mathbb{Z}) \cap L_\lambda(\mu)$ where $L_\lambda(\mu) \subseteq L_\lambda$ is the weight μ subspace and obtain $L_\lambda(\mathbb{Z}) = \bigoplus_{\mu\leq\lambda} L_\lambda(\mathbb{Z}, \mu)$ by [Humphreys 1972, Theorem 27.1, p. 158]. More generally, for any $\mathbb{Z}$-algebra R we put $L_\lambda(R) = R \otimes L_\lambda(\mathbb{Z})$ and $L_\lambda(R, \mu) = R \otimes L_\lambda(\mathbb{Z}, \mu)$. The space $L_\lambda(R)$ is a $\mathcal{U}_R$-module and $L_\lambda(R, \mu)$ is a $\mathfrak{h}(R)$-module and the weight decomposition of $L_\lambda(R)$ w.r.t. $\mathfrak{h}(R)$ reads

$$L_\lambda(R) = \bigoplus_{\mu\leq\lambda} L_\lambda(R, \mu).$$

More generally, let (π, L_π) be a faithful complex finite dimensional representation of $\mathfrak{g}$. Since $\pi = \bigoplus_i \rho_{\lambda_i}$ is semisimple (by the theorem just cited) there is a $\mathcal{U}_{\mathbb{Z}}$-invariant lattice $L_\pi(\mathbb{Z})$ in L_π i.e., $L_\pi = L_\pi(\mathbb{Z}) \otimes \mathbb{C}$. Furthermore, for any weight $\mu \in \mathfrak{h}^*$ we set $L_\pi(\mathbb{Z}, \mu) = L_\pi(\mathbb{Z}) \cap L_\pi(\mu)$ and $L_\pi(R, \mu) = R \otimes L_\pi(\mathbb{Z}, \mu)$ (R a $\mathbb{Z}$-algebra) and obtain

$$L_\pi(R) = \bigoplus_{\mu\in P_\pi} L_\pi(R, \mu),$$

where $P_\pi \subseteq \mathfrak{h}^*$ is the set of weights of π. We note that $P_\pi = \bigcup_i P_{\lambda_i}$ and we set $\Gamma_\pi = \langle P_\pi \rangle$.

1.2. ***Chevalley groups /$\mathbb{Z}_p$.*** From now on we fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and we denote by $\mathcal{O}_{\bar{\mathbb{Q}}_p}$ the integer ring in $\bar{\mathbb{Q}}_p$. We recall some basic facts from the theory of Chevalley groups. Let (π, L_π) be a finite dimensional complex representation of $\mathfrak{g}$ and let R by a $\mathbb{Z}_p$-algebra. For any $t \in R$ and any root $\alpha \in \Phi$ we define the element $x_\alpha(t) = x^\pi_\alpha(t) = \exp(\pi(t x_\alpha)) \in \mathrm{Aut}(L_\pi(R))$. The subgroup

$$G_{\pi,R} = \langle x_\alpha(t_\alpha), \alpha \in \Phi, t_\alpha \in R \rangle \leq \mathrm{Aut}(L_\pi(R))$$

is called the *Chevalley group* attached to π and R.

The group $G_{\pi,\bar{\mathbb{Q}}_p}$ is a semisimple connected algebraic group, i.e., it is the set of $\bar{\mathbb{Q}}_p$-points of an algebraic group (group scheme) $\boldsymbol{G}_\pi$ which is defined over $\mathbb{Z}_p$. To make this more precise, we denote by $\mathbf{GL}_n$ the general linear group with canonical $\mathbb{Z}_p$-structure $\mathbb{Z}_p[\mathbf{GL}_n] = \mathbb{Z}_p[x_{ij}, \det^{-1}]$, i.e., for any $\mathbb{Z}_p$-algebra R we obtain

$$\mathbf{GL}_n(R) = \mathrm{Mor}_{\mathbb{Z}_p-alg}(\mathbb{Z}_p[\mathbf{GL}_n], R) = \{(x_{ij}) \in R^{n^2} : \det(x_{ij}) \in R^*\}.$$

We select a basis $\mathcal{B}$ of the free $\mathbb{Z}_p$-module $L_\pi(\mathbb{Z}_p)$ and obtain for any $\mathbb{Z}_p$-algebra R an identification

$$\mathrm{Aut}(L_\pi(R)) \overset{\mathcal{B}}{=} \mathbf{GL}_n(R) \quad (n = \dim L_\pi).$$

In particular, $G_{\pi,\bar{\mathbb{Q}}_p}$ is a subset of $\mathrm{Aut}(L_\pi(\bar{\mathbb{Q}}_p)) = \mathbf{GL}_n(\bar{\mathbb{Q}}_p)$ and it is the set of $\bar{\mathbb{Q}}_p$-points $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ of a closed algebraic subgroup $\boldsymbol{G}_\pi = \boldsymbol{G}_\pi/\mathbb{Q}_p$ of $\mathbf{GL}_n/\mathbb{Q}_p$ which is defined over $\mathbb{Q}_p$ (see [Borel 1970, 3.3(1), p. 14 and 3.4, p. 18]); in particular, $\mathbb{Q}_p[\boldsymbol{G}_\pi] = \mathbb{Q}_p[\mathbf{GL}_n]/J'$ for some ideal $J' \leq \mathbb{Q}_p[\mathbf{GL}_n]$.

We set $J = J' \cap \mathbb{Z}_p[\mathbf{GL}_n]$ and $\mathbb{Z}_p[\boldsymbol{G}_\pi] := \mathbb{Z}_p[\mathbf{GL}_n]/J$ then is a $\mathbb{Z}_p$-form on $\mathbb{Q}_p[\boldsymbol{G}_\pi]$ which yields a $\mathbb{Z}_p$-structure on $\boldsymbol{G}_\pi$, i.e., which yields a $\mathbb{Z}_p$-group scheme $\boldsymbol{G}_\pi/\mathbb{Z}_p$ whose extension to $\mathbb{Q}_p$ is $\boldsymbol{G}_\pi$ (ibid., 3.4, p. 18). Thus, for any $\mathbb{Z}_p$-algebra R contained in $\bar{\mathbb{Q}}_p$ the group of R-points $\boldsymbol{G}_\pi(R)$ is defined and

$$\boldsymbol{G}_\pi(R) = \boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p) \cap \mathbf{GL}_n(R) = \boldsymbol{G}_{\pi,\bar{\mathbb{Q}}_p} \cap \mathrm{Aut}(L_\pi(R)).$$

In particular, since $x_\alpha(t_\alpha) \in \mathrm{Aut}(L_\pi(R))$, $t_\alpha \in R$, we obtain $x_\alpha(t_\alpha) \in \boldsymbol{G}_\pi(R)$ if $t_\alpha \in R$ which yields

$$G_{\pi,R} \subseteq \boldsymbol{G}_\pi(R). \tag{2}$$

For each $\alpha \in \Phi$ there is a unique morphism $\mu_\alpha = \mu_\alpha^\pi : \mathbf{SL}_2(\bar{\mathbb{Q}}_p) \to \boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ such that $\left(\begin{smallmatrix}1 & t\\ & 1\end{smallmatrix}\right) \mapsto x_\alpha^\pi(t)$ and $\left(\begin{smallmatrix}1 & \\ t & 1\end{smallmatrix}\right) \mapsto x_{-\alpha}^\pi(t)$ ($t \in \bar{\mathbb{Q}}_p$) map (see [Borel 1970, 3.2(1), p. 13]). The μ_α is defined over $\mathbb{Z}_p$ (ibid., 3.3(2), p. 15 and 4.3, p. 22). We denote by $h_\alpha(t) = h_\alpha^\pi(t)$ the image of $\left(\begin{smallmatrix}t & \\ & t^{-1}\end{smallmatrix}\right)$ under μ_α (ibid., 3.2(1), p. 13). The algebraic group $\boldsymbol{G}_\pi/\mathbb{Q}_p$ contains a $\mathbb{Q}_p$-split maximal torus $\boldsymbol{T}/\mathbb{Q}_p = \boldsymbol{T}^\pi/\mathbb{Q}_p$ such that the group of $\bar{\mathbb{Q}}_p$-rational points of $\boldsymbol{T}$ is given as

$$\boldsymbol{T}(\bar{\mathbb{Q}}_p) = \langle h_\alpha(t_\alpha),\ \alpha \in \Delta,\ t_\alpha \in \bar{\mathbb{Q}}_p^*\rangle$$

(ibid., 3.2(1), p. 13 and 3.3(3), p. 15). To any $\lambda \in \Gamma_\pi$ we attach a rational character $\lambda^\circ \in X(\boldsymbol{T})$ by setting

$$\lambda^\circ\biggl(\prod_{\alpha\in\Delta} h_\alpha(t_\alpha)\biggr) = \prod_{\alpha\in\Delta} t_\alpha^{\lambda(h_\alpha)}$$

for all $t_\alpha \in \bar{\mathbb{Q}}_p^*$ (ibid., 3.3, p. 15). This defines an isomorphism $\Gamma_\pi \to X(\boldsymbol{T})$ (ibid., 3.3(3), p. 15). We note that $d\lambda^\circ = \lambda$ (ibid., 3.3 equation (2), p. 16). The characters α°, $\alpha \in \Phi$, are the roots of $\boldsymbol{G}_\pi/\mathbb{Q}_p$ with respect to $\boldsymbol{T}$ (i.e., the weights of the adjoint action of $\boldsymbol{T}$ on $\mathrm{Lie}(\boldsymbol{G}_\pi/\mathbb{Q}_p)$). To simplify notation we denote the exponential $\alpha^\circ : \boldsymbol{T}(\bar{\mathbb{Q}}_p) \to \bar{\mathbb{Q}}_p^*$ of a root α also by α.

There are closed subgroups $\boldsymbol{N} = \boldsymbol{N}^\pi$ and $\boldsymbol{N}^- = \boldsymbol{N}^{-,\pi}$ of $\boldsymbol{G}_\pi/\mathbb{Q}_p$ such that

$$\begin{aligned} \boldsymbol{N}(\bar{\mathbb{Q}}_p) &= \langle x_\alpha(t),\ t_\alpha \in \bar{\mathbb{Q}}_p, \alpha \in \Phi^+\rangle, \\ \boldsymbol{N}^-(\bar{\mathbb{Q}}_p) &= \langle x_\alpha(t),\ t_\alpha \in \bar{\mathbb{Q}}_p, \alpha \in \Phi^-\rangle. \end{aligned} \tag{3}$$

The subgroups $\boldsymbol{N}$ and $\boldsymbol{N}^-$ are defined over $\mathbb{Q}_p$ and they are maximal unipotent (see [Borel 1970, 3.3(3), p. 15]). In the same way as above the $\mathbb{Z}_p$-structure on $\boldsymbol{G}_\pi$ induces $\mathbb{Z}_p$-structures on closed subgroups H of $\boldsymbol{G}_\pi/\mathbb{Q}_p$ such that

$$H(\mathbb{Z}_p) = H(\bar{\mathbb{Q}}_p) \cap \boldsymbol{G}_\pi(\mathbb{Z}_p) = H(\bar{\mathbb{Q}}_p) \cap \mathrm{Aut}(L_\pi(\mathbb{Z}_p)).$$

For example, this applies to the groups $\boldsymbol{N}$, $\boldsymbol{N}^-$, $\boldsymbol{T}$, which thus have $\mathbb{Z}_p$-structures.

We set $\boldsymbol{B} = \boldsymbol{T}\boldsymbol{N}$, which is a subgroup of $\boldsymbol{G}_\pi$ defined over $\mathbb{Q}_p$. Thus, $\boldsymbol{B}(\bar{\mathbb{Q}}_p)$ is the subgroup of $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ which is generated by the root subgroups $\boldsymbol{G}_{\pi,\alpha}(\bar{\mathbb{Q}}_p)$ of $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ with $\alpha \in \Phi^+$ together with $\boldsymbol{T}(\bar{\mathbb{Q}}_p)$ and its existence as a closed subgroup defined over $\mathbb{Q}_p$ also follows from [Popov and Vinberg 1994, 5.3.4 Proposition, p. 70]. In particular, $\boldsymbol{B}$ is a minimal parabolic subgroup. We also define the subgroup $\boldsymbol{B}^- = \boldsymbol{T}\boldsymbol{N}^-$ of $\boldsymbol{G}_\pi/\mathbb{Q}_p$. Since $h_\alpha(t) \in \boldsymbol{G}_\pi(\mathbb{Z}_p)$, $t \in \mathbb{Z}_p^*$, because μ_α is defined over $\mathbb{Z}_p$ we obtain

$$h_\alpha(t) \in \boldsymbol{T}(\bar{\mathbb{Q}}_p) \cap \boldsymbol{G}_\pi(\mathbb{Z}_p) = \boldsymbol{T}(\mathbb{Z}_p) \quad (t \in \mathbb{Z}_p^*).$$

Analogously, we obtain for any $t \in \mathbb{Z}_p$ and any positive root α

$$x_\alpha(t) \in \boldsymbol{N}(\bar{\mathbb{Q}}_p) \cap \boldsymbol{G}_\pi(\mathbb{Z}_p) = \boldsymbol{N}(\mathbb{Z}_p)$$

while for a negative root α we get $x_\alpha(t) \in \boldsymbol{N}^-(\mathbb{Z}_p)$.

Notation. If $\pi = \rho_\lambda$ is an irreducible representation then we simplify notation and set $x_\alpha^\lambda(t) = x_\alpha^{\rho_\lambda}(t)$, $\boldsymbol{G}_\lambda = \boldsymbol{G}_{\rho_\lambda}$, $\mu_\alpha^\lambda = \mu_\alpha^{\rho_\lambda}$, $h_\alpha^\lambda(t) = h_\alpha^{\rho_\lambda}(t)$ and $\boldsymbol{T}^\lambda = \boldsymbol{T}^{\rho_\lambda}$; we note that in Section 1.1 we already used the notation L_λ for L_{ρ_λ} and Γ_λ for Γ_{ρ_λ}.

1.3. ***Mod p reduction.*** Let $p \in \mathbb{N}$ be a prime element. We denote by $\boldsymbol{G}_{\pi,(p)}$ or by $\boldsymbol{G}_\pi/\mathbb{F}_p$ the mod p reduction of $\boldsymbol{G}_\pi/\mathbb{Z}_p$; i.e., $\boldsymbol{G}_{\pi,(p)}(\bar{\mathbb{F}}_p)$ is the set of zeros of $\mathbb{F}_p \otimes \boldsymbol{J} \leq \mathbb{F}_p \otimes \mathbb{Z}_p[\mathbf{GL}_n]$ in $\mathbf{GL}_n(\bar{\mathbb{F}}_p)$. Hence, $\boldsymbol{G}_{\pi,(p)}$ is an affine variety defined over $\mathbb{F}_p$ (a closed subgroup of $\mathbf{GL}_n/\mathbb{F}_p$). Moreover, since $\boldsymbol{G}_\pi/\mathbb{Z}_p$ has good reduction (see [Borel 1970, 3.4, p. 18]) we know that $\mathbb{F}_p \otimes J$ equals the ideal consisting of all $f \in \mathbb{F}_p \otimes \mathbb{Z}_p[\mathbf{GL}_n]$ which vanish on $\boldsymbol{G}_{\pi,(p)}(\bar{\mathbb{F}}_p)$, hence, $\mathbb{F}_p[\boldsymbol{G}_{\pi,(p)}] = \mathbb{F}_p \otimes \mathbb{Z}_p[\boldsymbol{G}_\pi]$. The mod p reduction $\boldsymbol{G}_{\pi,(p)}$ is a semisimple group defined over $\mathbb{F}_p$ (ibid., 4.3, p. 21/22). Analogously, the mod p reductions $\boldsymbol{N}_{(p)}$, $\boldsymbol{B}_{(p)}$, $\boldsymbol{N}^-_{(p)}$, $\boldsymbol{B}^-_{(p)}, \ldots$ are defined.

We denote by

$$\wp : \mathrm{Aut}(L_\pi(\mathbb{Z}_p)) \overset{\mathcal{B}}{=} \mathbf{GL}_n(\mathbb{Z}_p) \to \mathbf{GL}_n(\mathbb{F}_p) \overset{\bar{\mathcal{B}}}{=} \mathrm{Aut}(L_\pi(\mathbb{F}_p))$$

the mod p reduction map which sends (x_{ij}) to $(x_{ij} \pmod p)$ ($\bar{\mathcal{B}}$ is the basis of $L_\pi(\mathbb{F}_p) = \mathbb{F}_p \otimes L_\pi(\mathbb{Z}_p)$ induced by the basis $\mathcal{B}$ of $L_\pi(\mathbb{Z}_p)$). If $(x_{ij}) \in \boldsymbol{G}_\pi(\mathbb{Z}_p) \le \mathbf{GL}_n(\mathbb{Z}_p)$ then $\wp(x)$ obviously is contained in $\boldsymbol{G}_\pi(\mathbb{F}_p)$, hence, $\wp$ induces a map $\boldsymbol{G}_\pi(\mathbb{Z}_p) \to \boldsymbol{G}_\pi(\mathbb{F}_p)$.

The Iwahori subgroup $\mathcal{I}$ of $\boldsymbol{G}_\pi(\mathbb{Z}_p)$ then is defined as the set of all $k \in \boldsymbol{G}_\pi(\mathbb{Z}_p)$ such that $\wp(k) \in \boldsymbol{B}^-(\mathbb{F}_p)$, i.e., $\mathcal{I} = \wp^{-1}(\boldsymbol{B}^-(\mathbb{F}_p))$.

1.4. ***Irreducible representations of $\boldsymbol{G}_\pi/\mathbb{Z}_p$.*** The group $\boldsymbol{G}_\pi$ is a semisimple connected $\mathbb{Q}_p$-split group with $\mathbb{Q}_p$-split maximal torus $\boldsymbol{T} = \boldsymbol{T}^\pi$. Let $\lambda^\circ \in X(\boldsymbol{T})$ be a dominant weight. We set $\lambda = \mathrm{d}\,\lambda^\circ \in \mathfrak{h}^*$. As in Section 1.2 the choice of a $\mathbb{Z}_p$-basis $\mathcal{B}$ of the $\mathcal{U}_{\mathbb{Z}_p}$-invariant lattice $L_\lambda(\mathbb{Z}_p)$ $(= L_{\rho_\lambda}(\mathbb{Z}_p))$ yields an identification $\mathrm{Aut}(L_\lambda(\bar{\mathbb{Q}}_p)) \overset{\mathcal{B}}{=} \mathbf{GL}_m(\bar{\mathbb{Q}}_p)$ (where $m = \dim(L_\lambda)$).

If $\Gamma_\pi \supseteq \Gamma_\lambda$ we define a representation of algebraic groups

$$\rho_{\lambda^\circ} : \boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p) \to \mathrm{Aut}(L_\lambda(\bar{\mathbb{Q}}_p)) = \mathbf{GL}_m(\bar{\mathbb{Q}}_p)$$

by mapping $x^\pi_\alpha(t)$ to $x^\lambda_\alpha(t)$, $\alpha \in \Phi$, $t \in \bar{\mathbb{Q}}_p$ (see [Borel 1970, 3.2(4), p. 14 and 3.3(2), p. 15], where ρ_{λ° is denoted by $\lambda_{\rho_\lambda,\pi}$). We note that ρ_{λ° has image $\boldsymbol{G}_\lambda(\bar{\mathbb{Q}}_p)$ (which equals $\boldsymbol{G}_{\rho_\lambda}(\bar{\mathbb{Q}}_p)$).

Lemma. *The map ρ_{λ° induces a map of tori*

$$\rho_{\lambda^\circ} : \boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p) \to \boldsymbol{T}^\lambda(\bar{\mathbb{Q}}_p)$$

which maps $h^\pi_\alpha(t) \mapsto h^\lambda_\alpha(t)$ for all $\alpha \in \Delta$ and $t \in \bar{\mathbb{Q}}^*_p$.

Proof. We first claim that for each $\alpha \in \Phi$ the diagram

$$\begin{tikzcd}
\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p) \arrow[rr, "\rho_{\lambda^\circ}"] & & \boldsymbol{G}_\lambda(\bar{\mathbb{Q}}_p) \\
& \mathbf{SL}_2(\bar{\mathbb{Q}}_p) \arrow[ul, "\mu^\pi_\alpha"] \arrow[ur, "\mu^\lambda_\alpha"'] &
\end{tikzcd}$$

commutes. It is sufficient to show commutativity of the diagram for all $\left(\begin{smallmatrix}1 & t\\ & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & \\ t & 1\end{smallmatrix}\right)$ with $t \in \bar{\mathbb{Q}}_p$ because these elements generate $\mathbf{SL}_2(\bar{\mathbb{Q}}_p)$. But

$$\rho_{\lambda^\circ}\big(\mu^\pi_\alpha\big(\left(\begin{smallmatrix}1 & t\\ & 1\end{smallmatrix}\right)\big)\big) = \rho_{\lambda^\circ}(x^\pi_\alpha(t)) = x^\lambda_\alpha(t) = \mu^\lambda_\alpha\big(\left(\begin{smallmatrix}1 & t\\ & 1\end{smallmatrix}\right)\big),$$

and analogously for the lower unipotent matrices. Hence, the diagram commutes and we obtain

$$\rho_{\lambda^\circ}(h^\pi_\alpha(t)) = \rho_{\lambda^\circ}\big(\mu^\pi_\alpha\big(\left(\begin{smallmatrix}t & \\ & t^{-1}\end{smallmatrix}\right)\big)\big) = \mu^\lambda_\alpha\big(\left(\begin{smallmatrix}t & \\ & t^{-1}\end{smallmatrix}\right)\big) = h^\lambda_\alpha(t).$$

Since $\boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$ is generated by the $h^\pi_\alpha(t)$, where $\alpha \in \Delta$ and $t \in \bar{\mathbb{Q}}^*_p$, this implies that $\rho_{\lambda^\circ}(\boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)) \subseteq \boldsymbol{T}^\lambda(\bar{\mathbb{Q}}_p)$ and the lemma is proven. $\square$

The lemma implies that for all $t \in \boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$ and any weight vector $v_\mu \in L_\lambda(\bar{\mathbb{Q}}_p, \mu)$, $\mu \in P_\lambda$, (i.e., v_μ has weight μ w.r.t. $\mathfrak{h}$) we have

$$\rho_{\lambda^\circ}(t)(v_\mu) = \mu^\circ(t) v_\mu. \tag{4}$$

In fact since $\boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$ is generated by the $h^\pi_\alpha(s)$ we may assume that $t = h^\pi_\alpha(s)$ for some $\alpha \in \Delta$ and $s \in \bar{\mathbb{Q}}_p^*$. Using the lemma and the equation in [Borel 1970, 3.2(1), p. 13], we obtain

$$\rho_{\lambda^\circ}(t)(v_\mu) = \rho_{\lambda^\circ}(h^\pi_\alpha(s))(v_\mu) = h^\lambda_\alpha(s) v_\mu = s^{\mu(h_\alpha)} v_\mu = \mu^\circ(t) v_\mu.$$

The following result seems to be well known. Since we could not find a direct reference we add a proof.

Proposition. 1. *The $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$-module $L_\lambda(\bar{\mathbb{Q}}_p)$ contains a vector v_{λ° which is invariant under $\boldsymbol{N}^\pi(\bar{\mathbb{Q}}_p)$ and satisfies $t v_{\lambda^\circ} = \lambda^\circ(t) v_{\lambda^\circ}$ for all $t \in \boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$.*

2. *The representation ρ_{λ° is the irreducible representation of $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ of highest weight λ°.*

Proof. 1. We choose for v_{λ° the highest weight vector $v_\lambda \in L_\lambda(\mathbb{Z})$, which we selected in Section 1.1. Since $x_\alpha v_{\lambda^\circ}$ $(= \rho_\lambda(x_\alpha) v_{\lambda^\circ})$ vanishes for all $\alpha \in \Phi^+$ we obtain

$$\rho_{\lambda^\circ}(x^\pi_\alpha(t)) v_{\lambda^\circ} = x^\lambda_\alpha(t) v_{\lambda^\circ} = v_{\lambda^\circ} + t\rho_\lambda(x_\alpha)(v_{\lambda^\circ}) + \cdots = v_{\lambda^\circ}.$$

Since $\boldsymbol{N}^\pi(\bar{\mathbb{Q}}_p)$ is generated by the $x^\pi_\alpha(t)$ with $\alpha \in \Phi^+$, $t \in \bar{\mathbb{Q}}_p$, this yields the first claim about v_{λ°. The second claim is immediate by equation (4) since $v_{\lambda^\circ} = v_\lambda$ has $\mathfrak{h}$-weight λ.

2. For the moment we denote by $(\sigma_{\mu^\circ}, \Sigma_{\mu^\circ})$ the irreducible representation of $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ of highest weight $\mu^\circ \in X(\boldsymbol{T}^\pi)$. The derived representation of σ_{μ° is $(\rho_\mu, L_\mu(\bar{\mathbb{Q}}_p))$, where $\mu = d\,\mu^\circ$; hence, $\dim \Sigma_{\mu^\circ} = \dim L_\mu(\bar{\mathbb{Q}}_p)$. Since $\boldsymbol{G}_\pi$ is semisimple and since we are in characteristic 0 any representation of $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ is semisimple, hence, we can write

$$\rho_{\lambda^\circ} = \bigoplus_{i=1}^r \Sigma_{\mu_i^\circ}.$$

Any representation $\Sigma_{\mu_i^\circ}$ contains a unique (up to scalars) nontrivial vector $v_{\mu_i^\circ}$ invariant under $\boldsymbol{N}^\pi(\bar{\mathbb{Q}}_p)$. This vector $v_{\mu_i^\circ}$ then satisfies $t v_{\mu_i^\circ} = \mu_i^\circ(t) v_{\mu_i^\circ}$, $t \in \boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$ (i.e., $\bar{\mathbb{Q}}_p v_{\mu_i^\circ}$ is the unique line which is stable under $\boldsymbol{B}(\bar{\mathbb{Q}}_p)$). The vector v_{λ° decomposes as

$$v_{\lambda^\circ} = \sum_{i=1}^r v_i,$$

where $v_i \in \Sigma_{\mu_i^\circ}$ and at least one vector v_j does not vanish. Since v_{λ° is invariant under $\boldsymbol{N}(\bar{\mathbb{Q}}_p)$ by part 1, we obtain $\sum_i n v_i = \sum_i v_i$ for any $n \in \boldsymbol{N}(\bar{\mathbb{Q}}_p)$, hence, $n v_i = v_i$

for all i and all $n \in \boldsymbol{N}(\bar{\mathbb{Q}}_p)$. Thus, $v_i = c_i v_{\mu_i^\circ}$ for some $c_i \in \bar{\mathbb{Q}}_p$ by the uniqueness of $v_{\mu_i^\circ}$; in particular, v_i has weight μ_i° w.r.t. $\boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$. On the other hand, since $tv_{\lambda^\circ} = \lambda^\circ(t)v_{\lambda^\circ}$ by part 1, we obtain $\sum_i tv_i = \sum_i \lambda^\circ(t)v_i$; hence, $tv_i = \lambda^\circ(t)v_i$ for all i and $t \in \boldsymbol{T}^\pi(\bar{\mathbb{Q}}_p)$. Since $v_j \neq 0$ we deduce that $\mu_j^\circ = \lambda^\circ$. The representation ρ_{λ° therefore decomposes as a direct sum $\rho_{\lambda^\circ} = \Sigma_{\lambda^\circ} \oplus C$. Since ρ_{λ° is a representation on the space $L_\lambda(\bar{\mathbb{Q}}_p)$ we know that $\dim \rho_{\lambda^\circ} = \dim L_\lambda(\bar{\mathbb{Q}}_p) = \dim \Sigma_{\lambda^\circ}$. This implies that $C = 0$, hence, $\rho_{\lambda^\circ} = \Sigma_{\lambda^\circ}$ is the irreducible representation of $\boldsymbol{G}_\pi(\bar{\mathbb{Q}}_p)$ of highest weight λ°. Thus, the proof is complete. □

From [Borel 1970, 3.5, p. 19], we know that the morphism ρ_{λ° is defined over $\mathbb{Z}_p$, i.e., it is associated to a morphism of $\mathbb{Z}_p$-group schemes $\rho_{\lambda^\circ} : \boldsymbol{G}_\pi/\mathbb{Z}_p \to \boldsymbol{G}_\lambda/\mathbb{Z}_p$. Since $\boldsymbol{G}_\lambda$ is a closed subscheme of $\mathbf{Aut}(L_\lambda) = \mathbf{GL}_m/\mathbb{Z}_p$ we obtain that the representation ρ_{λ° is defined over $\mathbb{Z}_p$, i.e.,

$$\rho_{\lambda^\circ} : \boldsymbol{G}_\pi/\mathbb{Z}_p \to \mathbf{Aut}(L_\lambda) = \mathbf{GL}_m/\mathbb{Z}_p$$

In particular, $L_\lambda(R)$ is a $\boldsymbol{G}_\pi(R)$-module for all $\mathbb{Z}_p$-algebras R. Using equation (4) we deduce that $\boldsymbol{T}^\pi(\mathbb{Z}_p)$ leaves $L_\lambda(\mathbb{Z}_p, \mu) = L_\lambda(\mathbb{Z}_p) \cap L_\lambda(\bar{\mathbb{Q}}_p, \mu)$ invariant and acts via the character μ°.

1.5. ***The level subgroup $K_*(p, \sigma)$.*** From now on we fix a prime element $p \in \mathbb{N}$. For any $\sigma \in \mathbb{N}$ we define the level subgroup

$$K_*(\sigma) = K_*(p, \sigma) = K_*^\pi(p, \sigma) \leq \boldsymbol{G}_\pi(\mathbb{Z}_p)$$

as the subgroup generated by the following elements: all $x_\alpha(t_\alpha)$ with $\alpha \in \Phi^-$ and $t_\alpha \in \mathbb{Z}_p$, all $x_\alpha(t_\alpha)$ with $\alpha \in \Phi^+$ and $t_\alpha \in p^{\lceil \frac{1}{\sigma} \mathrm{ht}(\alpha) \rceil}\mathbb{Z}_p$ and all $h_\alpha^\pi(t_\alpha)$ with $\alpha \in \Delta$ and $t_\alpha \in \mathbb{Z}_p^*$. We note that the equations at the end of Section 1.2 imply that $K_*(\sigma) \leq \boldsymbol{G}_\pi(\mathbb{Z}_p)$ and even that $K_*(\sigma) \leq \mathcal{I}$, because $\wp(x_\alpha(t))$ is the identity in $\mathrm{Aut}(L_\pi(\mathbb{F}_p))$ if $t \in p\mathbb{Z}_p$. If $\sigma \geq \max_{\alpha \in \Phi^+} \mathrm{ht}(\alpha)$ we conclude that $K_*(\sigma)$ equals

$$\left\langle x_\alpha(t_\alpha),\ t_\alpha \in p\mathbb{Z}_p \text{ if } \alpha > 0 \text{ and } t_\alpha \in \mathbb{Z}_p \text{ if } \alpha < 0,\ h_\alpha(t_\alpha), \alpha \in \Delta, t_\alpha \in \mathbb{Z}_p^* \right\rangle = \mathcal{I};$$

the latter equality follows from [Iwahori and Matsumoto 1965, p. 259] (the Iwahori subgroup is denoted by B there).

We define the subgroups $N_{\mathbb{Z}_p} = N_{\mathbb{Z}_p}^\pi := \langle x_\alpha(t_\alpha),\ \alpha > 0,\ t_\alpha \in \mathbb{Z}_p \rangle \subseteq \boldsymbol{N}(\mathbb{Z}_p)$, $N_{p\mathbb{Z}_p} = N_{p\mathbb{Z}_p}^\pi := \langle x_\alpha(t_\alpha),\ \alpha > 0,\ t_\alpha \in p\mathbb{Z}_p \rangle$, $N_{\mathbb{Z}_p}^- = N_{\mathbb{Z}_p}^{-,\pi} := \langle x_\alpha(t_\alpha),\ \alpha < 0,\ t_\alpha \in \mathbb{Z}_p \rangle$ and $T_{\mathbb{Z}_p} = T_{\mathbb{Z}_p}^\pi := \langle h_\alpha^\pi(t_\alpha),\ \alpha \in \Delta,\ t_\alpha \in \mathbb{Z}_p^* \rangle$ of $\boldsymbol{G}_\pi(\mathbb{Z}_p)$. The Iwahori subgroup then satisfies the decomposition

$$\mathcal{I} = N_{p\mathbb{Z}_p} T_{\mathbb{Z}_p} N_{\mathbb{Z}_p}^-$$

(see [Iwahori and Matsumoto 1965, Theorem 2.5, p. 263]; note that $T_{\mathbb{Z}_p} = \mathfrak{h}_{\mathfrak{O}}$). We note the following consequences.

1. Assume $h \in \boldsymbol{T}(\mathbb{Q}_p)^{++}$; i.e., $v_p(\alpha(h)) > 0$ for all simple roots $\alpha \in \Delta$. Then, for all $e, f \in \mathbb{N}_0$ we have

$$\mathcal{I}h^e\mathcal{I}h^f\mathcal{I} = \mathcal{I}h^{e+f}\mathcal{I}. \tag{5}$$

2. Assume $t, t' \in \boldsymbol{T}(\mathbb{Q}_p)^{++}$. Then

$$\mathcal{I}t\mathcal{I} = \mathcal{I}t'\mathcal{I} \quad \Longleftrightarrow \quad T_{\mathbb{Z}_p}tT_{\mathbb{Z}_p} = T_{\mathbb{Z}_p}t'T_{\mathbb{Z}_p}. \tag{6}$$

Proof of equation (6). The leftward implication is trivial. To prove the reverse implication we note that $t \in \mathcal{I}t'\mathcal{I}$ implies that there are $k^+, m^+ \in N_{p\mathbb{Z}_p}$, $k^\circ, m^\circ \in T_{\mathbb{Z}_p}$ and $k^-, m^- \in N^-_{\mathbb{Z}_p}$ such that $tk^+k^\circ k^- = m^+m^\circ m^- t'$. Since $\mathrm{Ad}(t)(x_\alpha(t_\alpha)) = x_\alpha(\alpha(t)t_\alpha)$ the element t normalizes $N_{p\mathbb{Z}_p}$ and $(t')^{-1}$ normalizes $N^-_{\mathbb{Z}_p}$, hence, we obtain $\tilde{k}^+tk^\circ k^- = m^+m^\circ t'\tilde{m}^-$ with $\tilde{k}^+ \in N_{p\mathbb{Z}_p}$, $\tilde{m}^- \in N^-_{\mathbb{Z}_p}$. Equivalently,

$$(m^+)^{-1}\tilde{k}^+tk^\circ = m^\circ t'\tilde{m}^-(k^-)^{-1}. \tag{7}$$

The left-hand side is contained in $\boldsymbol{B}(\mathbb{Q}_p)$ and the right-hand side is contained in $\boldsymbol{B}^-(\mathbb{Q}_p)$, whose intersection is $\boldsymbol{T}(\mathbb{Q}_p)$. Hence, $(m^+)^{-1}\tilde{k}^+ \in \boldsymbol{T}(\mathbb{Q}_p) \cap N_{p\mathbb{Z}_p} = \{1\}$ (note that $N_{p\mathbb{Z}_p} \subseteq N_{\mathbb{Z}_p} \subseteq \boldsymbol{N}(\mathbb{Z}_p)$) and, similarly, $\tilde{m}^-(k^-)^{-1} \in \boldsymbol{T}(\mathbb{Q}_p) \cap N^-_{\mathbb{Z}_p} = \{1\}$. Equation (7) thus implies that $tk^\circ = m^\circ t'$, which proves the claim. □

2. Hecke algebra and cohomology

2.1. ***Reductive algebraic groups.*** From now on, $\tilde{\boldsymbol{G}}$ denotes a connected reductive algebraic group defined over $\mathbb{Q}$. Since $\tilde{\boldsymbol{G}}$ is defined over $\mathbb{Q}$ it contains a maximal torus which is defined over $\mathbb{Q}$ and we assume that $\tilde{\boldsymbol{G}}$ contains a maximal torus $\tilde{\boldsymbol{T}}$ which is defined over $\mathbb{Q}$ and split over $\mathbb{Q}_p$ (hence, $\tilde{\boldsymbol{G}}$ is $\mathbb{Q}_p$-split). This assumption is in particular satisfied if $\tilde{\boldsymbol{G}}$ is $\mathbb{Q}$-split. We denote by $\boldsymbol{G} = \tilde{\boldsymbol{G}}^{\mathrm{der}}$ the derived group and by $\tilde{\boldsymbol{Z}}$ the center of $\tilde{\boldsymbol{G}}$; hence, $\tilde{\boldsymbol{G}} = (\boldsymbol{G} \times \tilde{\boldsymbol{Z}})/\boldsymbol{Z}$ as algebraic groups over $\mathbb{Q}$, where $\boldsymbol{Z}$ is the center of $\boldsymbol{G}$ (embedded via $z \mapsto (z, z^{-1})$). We denote by $\mathrm{Lie}(\boldsymbol{G})$ the Lie algebra of $\boldsymbol{G}$. We use the notations introduced in Section 1.1 for the complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(\boldsymbol{G}) \otimes_{\mathbb{Q}} \mathbb{C}$; e.g., $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$, $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ the set of roots and Δ a choice of a basis of Φ. Since $\boldsymbol{G}$ is a $\mathbb{Q}_p$-split, semisimple algebraic group, there is a finite dimensional complex representation π of $\mathfrak{g}$ such that

$$\boldsymbol{G}/\mathbb{Q}_p \cong \boldsymbol{G}_\pi/\mathbb{Q}_p$$

as $\mathbb{Q}_p$-groups, where $\boldsymbol{G}_\pi/\mathbb{Z}_p$ is the Chevalley group attached to π; see Section 1.2. In the following we may assume that $\boldsymbol{G}/\mathbb{Q}_p = \boldsymbol{G}_\pi/\mathbb{Q}_p$.

The $\mathbb{Q}_p$-structure on $\tilde{\boldsymbol{G}}$. We denote by $\boldsymbol{T}$ the $\mathbb{Q}_p$-split maximal torus and by $\boldsymbol{N}$, $\boldsymbol{N}^-$ the maximal unipotent subgroups in $\boldsymbol{G}/\mathbb{Q}_p = \boldsymbol{G}_\pi/\mathbb{Q}_p$ defined in Section 1.2. The subgroups $\boldsymbol{N}$, $\boldsymbol{N}^-$ remain maximal unipotent in $\tilde{\boldsymbol{G}}$ and $\boldsymbol{T}$ is a $\mathbb{Q}_p$-defined and $\mathbb{Q}_p$-split torus in $\tilde{\boldsymbol{G}}$. We denote by $\tilde{\boldsymbol{B}}/\mathbb{Q}_p$ the Borel subgroup in $\tilde{\boldsymbol{G}}$ containing $\tilde{\boldsymbol{T}}$ which corresponds to Δ. The torus $\tilde{\boldsymbol{T}}$ decomposes $\tilde{\boldsymbol{T}} = (\boldsymbol{T}' \times \tilde{\boldsymbol{Z}})/\boldsymbol{Z}$ over $\mathbb{Q}_p$, where $\boldsymbol{T}'/\mathbb{Q}_p$ is a $\mathbb{Q}_p$-split maximal torus in $\boldsymbol{G}/\mathbb{Q}_p$. Since any two $\mathbb{Q}_p$-split maximal tori in $\boldsymbol{G}/\mathbb{Q}_p$ are conjugate by an element $x \in \boldsymbol{G}(\mathbb{Q}_p)$ (see [Springer 1981, 15.2.6 Theorem, p. 256]) we may assume after composing the isomorphism $\boldsymbol{G}/\mathbb{Q}_p \cong \boldsymbol{G}_\pi/\mathbb{Q}_p$ with conjugation by x that $\boldsymbol{T} = \boldsymbol{T}'$, hence, $\tilde{\boldsymbol{T}} = (\boldsymbol{T} \times \tilde{\boldsymbol{Z}})/\boldsymbol{Z}$ as algebraic groups over $\mathbb{Q}_p$. We denote by $X(\tilde{\boldsymbol{T}})$ (resp. $X_*(\tilde{\boldsymbol{T}})$) the (additively written) group of $\mathbb{Q}_p$-characters (resp. $\mathbb{Q}_p$-cocharacters) of $\tilde{\boldsymbol{T}}/\mathbb{Q}_p$ and by $\langle\,\cdot\,,\cdot\,\rangle : X(\tilde{\boldsymbol{T}}) \times X_*(\tilde{\boldsymbol{T}}) \to \mathbb{Z}$ the canonical pairing defined by $\chi \circ \eta(x) = x^{\langle \chi, \eta \rangle}$ for all $x \in \mathbb{G}_m(\bar{\mathbb{Q}}_p)$. We recall that by α we denote a root in $\Phi \subseteq \mathfrak{h}^*$ and also its exponential in $X(\boldsymbol{T})$ (i.e., we write α for α°; see Section 1.2). Any root $\alpha \in X(\boldsymbol{T})$ vanishes on the center $\boldsymbol{Z}$ of $\boldsymbol{G}$, hence, it extends to a character on $\tilde{\boldsymbol{T}} = (\boldsymbol{T} \times \tilde{\boldsymbol{Z}})/\boldsymbol{Z}$ by setting it equal to 1 on $\tilde{\boldsymbol{Z}}$; we denote this extension of the root again by α; hence, $\alpha(t) = \alpha(t^\circ)(= \alpha^\circ(t^\circ))$ if $t = t^\circ z \in \boldsymbol{T}(\bar{\mathbb{Q}}_p)\tilde{\boldsymbol{Z}}(\bar{\mathbb{Q}}_p)$. We denote by $\alpha^\vee \in X_*(\boldsymbol{T}) \subseteq X_*(\tilde{\boldsymbol{T}})$ the coroot corresponding to α; explicitly, $\alpha^\vee(t) = h_\alpha(t)$, $t \in \mathbb{G}_m(\bar{\mathbb{Q}}_p)$. Any character $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ is of the form $\tilde{\lambda} = \lambda^\circ \otimes \kappa$, where $\kappa = \tilde{\lambda}|_{\tilde{\boldsymbol{Z}}} \in X(\tilde{\boldsymbol{Z}})$ and $\lambda^\circ = \tilde{\lambda}|_{\boldsymbol{T}} \in X(\boldsymbol{T})$ satisfy $\lambda^\circ|_{\boldsymbol{Z}} = \kappa|_{\boldsymbol{Z}}$. We note that λ° corresponds to a weight $\lambda \in \Gamma_\pi$, i.e., $\lambda = \mathrm{d}\lambda^\circ$; see Section 1.2. We call $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ dominant if

$$\langle \tilde{\lambda}, \alpha^\vee \rangle = \langle \lambda^\circ, \alpha^\vee \rangle = \lambda(h_\alpha) \geq 0$$

for all $\alpha \in \Delta$. We denote by $\tilde{\boldsymbol{T}}(\mathbb{Q}_p)^+$ (resp. $\tilde{\boldsymbol{T}}(\mathbb{Q}_p)^{++}$, $\tilde{\boldsymbol{T}}(\mathbb{Q}_p)^{--}$) the set of all elements $t \in \tilde{\boldsymbol{T}}(\mathbb{Q}_p)$ such that $v_p(\alpha(t)) \geq 0$ (resp. $v_p(\alpha(t)) > 0$, $v_p(\alpha(t)) < 0$) for all $\alpha \in \Delta$ and by $X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ the set of dominant characters.

The $\mathbb{Z}_p$-structure on $\tilde{\boldsymbol{G}}$. We recall that the derived group $\boldsymbol{G}/\mathbb{Q}_p = \boldsymbol{G}_\pi/\mathbb{Q}_p$ has a $\mathbb{Z}_p$-structure; see Section 1.2. In Section 1.5 we defined the level subgroup $K_*(\sigma) = K_*^\pi(p, \sigma)$ which is a subgroup of $\boldsymbol{G}(\mathbb{Z}_p)$. We define a $\mathbb{Z}_p$-structure on $\tilde{\boldsymbol{Z}}$ by selecting as a $\mathbb{Z}_p$-form of $\mathbb{Q}_p[\tilde{\boldsymbol{Z}}]$ the algebra $\mathbb{Z}_p[\mathrm{res}_{\tilde{\boldsymbol{T}}/\tilde{\boldsymbol{Z}}}\ X(\tilde{\boldsymbol{T}})] = \mathbb{Z}_p[X(\tilde{\boldsymbol{T}})/X'(\tilde{\boldsymbol{T}})]$, where $X'(\tilde{\boldsymbol{T}}) = X(\tilde{\boldsymbol{T}}) \cap \sum_{\alpha\in\Phi} \mathbb{Q}\alpha$. It follows that for any $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$

$$\tilde{\lambda}|_{\tilde{\boldsymbol{Z}}} : \tilde{\boldsymbol{Z}} \to \mathbb{G}_m \tag{8}$$

is defined over $\mathbb{Z}_p$. The $\mathbb{Z}_p$-structures on $\boldsymbol{G}$ and $\tilde{\boldsymbol{Z}}$ yield a $\mathbb{Z}_p$-structure on $\tilde{\boldsymbol{G}}$.

2.2. From now on, we fix a prime $p \in \mathbb{N}$ and we define the subgroup

$$\tilde{\mathcal{I}} := \langle \mathcal{I}, \tilde{\boldsymbol{Z}}(\mathbb{Z}_p) \rangle \leq \tilde{\boldsymbol{G}}(\mathbb{Z}_p).$$

We select an arithmetic subgroup $\Gamma \leq \tilde{\boldsymbol{G}}(\mathbb{Q})$ satisfying $\Gamma \leq \tilde{\mathcal{I}}$.

2.3. ***The Hecke algebra.*** Also, from now on, we let h be an element in $\tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$; i.e., $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})$ and $v_p(\alpha(h)) > 0$ for all $\alpha \in \Delta$. We denote by $\mathcal{K} = \mathcal{K}_h = \langle h, \tilde{\mathcal{I}} \rangle_{\mathrm{semigrp}} \leq$

$\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ the (sub)semigroup of $\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ which is generated by h and $\tilde{\mathcal{I}}$ and we set

$$\Delta = \Delta_h = \{g \in \tilde{\boldsymbol{G}}(\mathbb{Q}) : g \in \mathcal{K}\}.$$

Thus, $\Delta \leq \tilde{\boldsymbol{G}}(\mathbb{Q})$ is a subsemigroup containing Γ and we denote by

$$\mathcal{H} = \mathcal{H}_h = \mathcal{H}(\Gamma\backslash\Delta/\Gamma)$$

the Hecke algebra attached to the pair (Δ, Γ). Thus, $\mathcal{H}$ is a $\mathbb{Z}$-algebra which is a free $\mathbb{Z}$-module with basis $\{\Gamma\zeta\Gamma,\ \zeta \in \Delta\}$. For any $\mathbb{Z}$-algebra R we set $\mathcal{H}_R = \mathcal{H} \otimes R$ and we put

$$T_\zeta = \Gamma\zeta\Gamma \in \mathcal{H} \quad (\zeta \in \Delta).$$

2.4. ***Irreducible representations of $\tilde{\boldsymbol{G}}$.*** Let $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ be dominant. We set $\lambda^\circ = \tilde{\lambda}|_{\boldsymbol{T}} \in X(\boldsymbol{T})$ and $\lambda = \mathrm{d}\,\lambda^\circ \in \mathfrak{h}^*$. Since $X(\boldsymbol{T}) \cong \Gamma_\pi$ (see Section 1.2) we know that $\lambda \in \Gamma_\pi$ and we let $\rho_{\lambda^\circ} : \boldsymbol{G}_\pi/\mathbb{Z}_p \to \mathbf{Aut}(L_\lambda)$ be the irreducible representation of $\boldsymbol{G}/\mathbb{Z}_p = \boldsymbol{G}_\pi/\mathbb{Z}_p$ of highest weight λ°; see Section 1.4. The morphism $\rho_{\lambda^\circ} \otimes \tilde{\lambda}|_{\tilde{\boldsymbol{Z}}} : \boldsymbol{G} \times \tilde{\boldsymbol{Z}} \to \mathbf{Aut}(L_\lambda)$, given by sending $(g, z) \in \boldsymbol{G}(R) \times \tilde{\boldsymbol{Z}}(R)$ to $\tilde{\lambda}(z)\rho_{\lambda^\circ}(g) \in \mathbf{Aut}(L_\lambda(R))$, R any $\mathbb{Z}_p$-algebra, is defined over $\mathbb{Z}_p$ (see equation (8)) and factorizes over $\boldsymbol{Z}$, hence, we obtain a representation

$$\rho_{\tilde{\lambda}} : \tilde{\boldsymbol{G}} = (\boldsymbol{G} \times \tilde{\boldsymbol{Z}})/\boldsymbol{Z} \to \mathbf{Aut}(L_\lambda)$$

of $\mathbb{Z}_p$-groups (group schemes). The representation $(\rho_{\tilde{\lambda}}, L_\lambda)$ is irreducible of highest weight $\tilde{\lambda}$ and $\rho_{\lambda^\circ} = \rho_{\tilde{\lambda}}|_{\boldsymbol{G}}$. In particular, for any $\mathbb{Z}_p$-algebra R the $\boldsymbol{G}(R)$-module $L_\lambda(R)$ also is a $\tilde{\boldsymbol{G}}(R)$-module and we write $L_{\tilde{\lambda}}(R)$ for $L_\lambda(R)$ if we view it as $\tilde{\boldsymbol{G}}(R)$-module. Thus, $L_{\tilde{\lambda}}(R)$ and $L_\lambda(R)$ are isomorphic as $\boldsymbol{G}(R)$-modules, but on $L_{\tilde{\lambda}}(R)$ we have an action of $\tilde{\boldsymbol{Z}}(R)$ via $\tilde{\lambda}|_{\tilde{\boldsymbol{Z}}}$ and, hence, an action of $\tilde{\boldsymbol{T}}(R)$. Similarly, we obtain a representation

$$\tilde{\boldsymbol{T}} = (\boldsymbol{T} \times \tilde{\boldsymbol{Z}})/\boldsymbol{Z} \to \mathbf{Aut}(L_\lambda(\mu)), \quad \mu \in P_\lambda,$$

by sending $(t, z) \in \mathbf{T}(R) \times \tilde{\boldsymbol{Z}}(R)$ to $\tilde{\lambda}(z)\rho_{\lambda^\circ}(t) \in \mathbf{Aut}(L_\lambda(\mu, R))$, R any $\mathbb{Z}_p$-algebra (note that $L_\lambda(\mathbb{Z}_p, \mu)$ is a $\mathbb{Z}_p$-module, hence, $\mathbf{Aut}(L_\lambda(\mu))$ is a $\mathbb{Z}_p$-group). If we view the weight space $L_\lambda(R, \mu)$ as $\tilde{\boldsymbol{T}}(R)$-module we write it as $L_{\tilde{\lambda}}(R, \mu)$. Thus, $L_{\tilde{\lambda}}(R, \mu) = L_\lambda(R, \mu)$ as abelian groups and also as $\boldsymbol{T}(R)$-modules but on $L_{\tilde{\lambda}}(R, \mu)$ the torus $\tilde{\boldsymbol{T}}(R)$ acts via the character $\tilde{\mu} := \mu^\circ \otimes \tilde{\lambda}|_{\tilde{\boldsymbol{Z}}}$ of $\tilde{\boldsymbol{T}}$ (see Section 1.4 and equation (4) in particular). The weight decomposition of $L_{\tilde{\lambda}}(R)$ w.r.t. $\tilde{\boldsymbol{T}}(R)$ then reads

$$L_{\tilde{\lambda}}(R) = \bigoplus_{\mu \leq \lambda} L_{\tilde{\lambda}}(R, \mu), \tag{9}$$

where $L_{\tilde{\lambda}}(R, \mu)$ is the weight $\tilde{\mu}$-subspace of $L_{\tilde{\lambda}}(R)$ w.r.t. $\tilde{\boldsymbol{T}}(R)$.

2.5. ***Splitting field.*** Since the maximal torus $\tilde{\boldsymbol{T}}/\mathbb{Q}$ is assumed to be split over $\mathbb{Q}_p$ there is a subfield $F \subseteq \mathbb{Q}_p$ which is a finite extension of $\mathbb{Q}$ such that $\tilde{\boldsymbol{T}}/F$ is split. In particular, $\tilde{\boldsymbol{G}}$ is F-split and $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ and the irreducible highest weight representation $(\rho_{\tilde{\lambda}}, L_{\tilde{\lambda}})$ are defined over F; hence, $L_{\tilde{\lambda}}(F)$ is defined and is a $\tilde{\boldsymbol{G}}(\mathbb{Q})$-module.

We fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ with valuation v_p normalized by $v_p(p) = 1$. Since $F \subseteq \mathbb{Q}_p \subseteq \bar{\mathbb{Q}}_p$ this induces a p-adic valuation v_p on F and we obtain $F_{v_p} = \mathbb{Q}_p$. We also fix an embedding $F \subseteq \mathbb{C}$. We extend the embeddings of F to embeddings of its algebraic closure $\bar{F} \subseteq \bar{\mathbb{Q}}_p$ and $\bar{F} \subseteq \mathbb{C}$; hence, we may view F as a subfield of $\mathbb{Q}_p$ and of $\mathbb{C}$ and $\bar{F}$ as a subfield of $\mathbb{C}$ and of $\bar{\mathbb{Q}}_p$.

2.6. ***Cohomology with coefficients $L_{\tilde{\lambda}}$.*** We denote by $\Delta^{-1} \leq \tilde{\boldsymbol{G}}(\mathbb{Q})$ the sub semigroup consisting of the inverses of elements in Δ. The representation space $L_{\tilde{\lambda}}(F)$ in particular is a Δ^{-1}-module, hence, the Hecke algebra $\mathcal{H}$ acts on cohomology $H^i(\Gamma, L_{\tilde{\lambda}}(F))$. For later use we recall the definition of this action. Let $T_\zeta = \Gamma\zeta\Gamma \in \mathcal{H}$ ($\zeta \in \Delta$). We select a system of representatives $\gamma_1, \ldots, \gamma_r$ for $(\zeta^{-1}\Gamma\zeta \cap \Gamma)\backslash\Gamma$, hence,

$$T_\zeta = \bigcup_{i=1,\ldots,r} \Gamma\zeta\gamma_i.$$

Thus, for any $\eta \in \Gamma$ and any index i satisfying $1 \leq i \leq r$ there is an index $\eta(i)$ such that

$$\Gamma\zeta\gamma_i\eta = \Gamma\zeta\gamma_{\eta(i)}.$$

In particular, there are $\rho_i(\eta) \in \Gamma$, $i = 1, \ldots, r$, such that $\zeta\gamma_i\eta = \rho_i(\eta)\zeta\gamma_{\eta(i)}$. Let now $c \in C^d(\Gamma, L_{\tilde{\lambda}}(F))$ be any cochain; we then define $T_\zeta(c)$ as the cochain $c' \in C^d(\Gamma, L_{\tilde{\lambda}}(F))$, which is given by

$$c'(\eta_0, \ldots, \eta_d) = \sum_{1 \leq i \leq r} (\zeta\gamma_i)^{-1} c(\rho_i(\eta_0), \ldots, \rho_i(\eta_d)). \tag{10}$$

Since T_ζ commutes with the coboundary operator, T_ζ acts on cohomology with coefficients in $L_{\tilde{\lambda}}(F)$, i.e., T_ζ defines an element in $\mathrm{End}(H^i(\Gamma, L_{\tilde{\lambda}}(F)))$ which does not depend on the choice of the representatives $\gamma_1, \ldots, \gamma_r$ (see [Kuga et al. 1981, p. 227]). We note that this also yields $\mathcal{H}$-module structures on $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p))$ and $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{C}))$. We denote by

$$H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\mathrm{int}}$$

the image of the canonical mapping $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p)) \to H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p))$; this defines a lattice in $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p))$.

Cuspidal cohomology. We select a maximal compact open subgroup $\tilde{K}_\infty \leq \tilde{\boldsymbol{G}}(\mathbb{R})$. We denote by $A_{\tilde{\boldsymbol{G}}}$ the connected component of the real points of a maximal $\mathbb{Q}$-split torus $A_{\tilde{G}}$ in the center of $\tilde{G}$ and we set $X = \tilde{G}(\mathbb{R})/\tilde{K}_\infty A_{\tilde{G}}$. The cuspidal cohomology $H^i_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))$ is a subspace of full cohomology $H^i(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))$. We note that if the highest weight $\tilde{\lambda}$ is regular and $\tilde{G}$ has discrete series then there are isomorphisms

$$H^i_{(2)}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C})) = H^i_!(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C})) = H^i_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))$$

and cuspidal cohomology vanishes in all degrees except for the middle degree $d = d_{\tilde{G}} = \frac{1}{2}\dim X$.

Weighted cohomology. For use in Section 4.14 involving the Goresky–MacPherson trace formula we briefly recall the relation between cuspidal cohomology and weighted cohomology. We denote by $W^\nu H^i(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(F))$ the weighted cohomology groups of Γ (see [Goresky et al. 1994]). If ν is the middle weight profile and $\tilde{G}$ has discrete series then [Nair 1999, Corollary B, p. 3] (see also Section 5.1 there) implies that there is an isomorphism

$$W^\nu H^i(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(\mathbb{C})) = H^i_{(2)}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C})).$$

(in the hermitian case this also follows from the Zucker conjecture, which was proven by Saper and Stern, and independently by Looijenga). Thus, if in addition the highest weight $\tilde{\lambda} \in X(\tilde{T})$ is regular then there is a canonical isomorphism of Hecke modules

$$W^\nu H^i(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(\mathbb{C})) = H^i_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))$$

where the cohomology groups are nonvanishing only if $i = d$; in particular, we obtain

$$(-1)^d \text{tr}(T \,|\, H^d_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))) = \text{Lef}(T \,|\, W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(\mathbb{C})))$$

where $T \in \mathcal{H}$ is a Hecke operator.

We mention that this implies an F-structure on cuspidal cohomology: we denote by $H^d_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(F))$ the image of $W^\nu H^d(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(F))$ in $H^d_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))$ and obtain

$$H^d_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C})) = H^d_{\text{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(F)) \otimes \mathbb{C}.$$

2.7. ***Normalization of Hecke operators.*** We want to normalize the Hecke operators so that they act on cohomology with p-adically integral coefficients. We recall the following diagram of inclusions:

$$\begin{array}{ccc} \tilde{\mathcal{I}} & \subseteq & \mathcal{K} \\ \cup & & \cup \\ \Gamma & \subseteq & \Delta. \end{array}$$

Lemma. 1. $\mathcal{K} = \bigcup_{e \in \mathbb{N}_0} \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$, *i.e., any element* $g \in \mathcal{K}$ *can be written* $g = k_1 h^e k_2$ *with* $k_1, k_2 \in \tilde{\mathcal{I}}$ *and* $e \in \mathbb{N}_0$.

2. *If* $\tilde{\mathcal{I}} h^e \tilde{\mathcal{I}} = \tilde{\mathcal{I}} h^f \tilde{\mathcal{I}}$, $e, f \in \mathbb{N}_0$, *then* $e = f$.

3. *Let* $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ *be a dominant weight. The mapping*

$$\hat{\lambda} : \mathcal{K} \to F^*, \qquad k_1 h^e k_2 \mapsto \tilde{\lambda}(h^e) \quad (k_1, k_2 \in \tilde{\mathcal{I}}, e \in \mathbb{N}_0)$$

is a well defined morphism of semigroups.

Proof. Equations (5) and (6) in Section 1.5 remain valid with the same proof if $\mathcal{I}$ is replaced by $\tilde{\mathcal{I}}$, $T_{\mathbb{Z}_p}$ is replaced by $\tilde{T}_{\mathbb{Z}_p} = T_{\mathbb{Z}_p} \tilde{\boldsymbol{Z}}(\mathbb{Z}_p)$ and if $h, t, t' \in \tilde{\boldsymbol{T}}(\mathbb{Q}_p)$. Conclusion 1 is then immediate by equation (5). As for 2 we note that equation (6) implies $h^e = \delta h^f$, where $\delta \in \tilde{T}_{\mathbb{Z}_p} \subseteq \tilde{\boldsymbol{T}}(\mathbb{Z}_p)$. Applying an arbitrary simple root α and taking p-adic values yields $e v_p(\alpha(h)) = v_p(\alpha(\delta)) + f v_p(\alpha(h))$ and since $\alpha(\delta) \in \mathbb{Z}_p^*$ and $v_p(\alpha(h)) > 0$ we deduce that $e = f$. As for 3 we remark that parts 1 and 2 show that $\hat{\lambda}$ is well defined and equation (5) in Section 1.5 implies that $\hat{\lambda}$ is a morphism of semigroups. Thus, the lemma is proven. □

Let $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ be a dominant weight. By restriction, $\hat{\lambda}$ induces a mapping $\hat{\lambda} : \Delta \to F^*$. For any F-algebra R we define an R-linear mapping

$$\mathcal{H}_R \to \mathcal{H}_R$$

by sending $\Gamma \zeta \Gamma \mapsto \hat{\lambda}(\zeta) \Gamma \zeta \Gamma$, $\zeta \in \Delta$; note that $\{\Gamma \zeta \Gamma,\ \zeta \in \Delta\}$ is a basis for $\mathcal{H}_R$ and that the assignment is well defined since $\hat{\lambda}$ vanishes on $\tilde{\mathcal{I}}$ by definition and, hence, vanishes on $\Gamma \subseteq \tilde{\mathcal{I}}$. We denote the image of $T \in \mathcal{H}_R$ under the above mapping by $T_{\tilde{\lambda}} \in \mathcal{H}_R$ and we call $T_{\tilde{\lambda}}$ the $\tilde{\lambda}$-normalization of T. In particular, if $\zeta \in \Delta$ with $\zeta \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ then

$$(T_\zeta)_{\tilde{\lambda}} = \hat{\lambda}(\zeta) T_\zeta = \tilde{\lambda}(h^e) T_\zeta.$$

The normalization $T_{\tilde{\lambda}}$ of any $T \in \mathcal{H}_{\mathbb{Q}_p}$ leaves $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p))$ invariant and we want to show that $T_{\tilde{\lambda}}$ leaves cohomology with integral coefficients $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ invariant. To this end we first show that the "normalization $\hat{\lambda}(g) g^{-1}$" of any $g \in \mathcal{K}$ leaves the lattice $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ invariant.

2.8. ***Lemma.*** *For all* $g \in \mathcal{K}$ *and* $v \in L_{\tilde{\lambda}}(\mathbb{Z}_p)$ *we have* $\hat{\lambda}(g) g^{-1} v \in L_{\tilde{\lambda}}(\mathbb{Z}_p)$.

Proof. Any $g \in \mathcal{K}$ has the form $g = k_1 h^e k_2$ with $k_1, k_2 \in \tilde{\mathcal{I}}$. Since $\hat{\lambda}(g) = \tilde{\lambda}(h^e)$ and since $\tilde{\mathcal{I}} \subseteq \tilde{\boldsymbol{G}}(\mathbb{Z}_p)$ leaves $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ invariant it is sufficient to show that $\tilde{\lambda}(h^e) h^{-e}$ leaves $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ invariant. Since, as we saw in equation (9), we further have

$$L_{\tilde{\lambda}}(\mathbb{Z}_p) = \bigoplus_{\mu \le \lambda} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu),$$

it is sufficient to show that $\tilde{\lambda}(h^e) h^{-e} v_\mu \in L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)$ for any weight vector v_μ

in $L_{\tilde\lambda}(\mathbb{Z}_p,\mu)$. Equation (9) implies that $\tilde\lambda(h^e)h^{-e}v_\mu=\tilde\lambda(h^e)\tilde\mu(h^{-e})v_\mu$. We write $\mu=\lambda-\nu$ where $\nu=\sum_{\alpha\in\Delta}n_\alpha\alpha$ with $n_\alpha\in\mathbb{N}_0$ and $h=tz$ with $t\in \boldsymbol{T}(\bar{\mathbb{Q}}_p)$, $z\in\tilde{\boldsymbol{Z}}(\bar{\mathbb{Q}}_p)$ and obtain

$$\begin{aligned}\tilde\lambda(h^e)\tilde\mu(h^{-e})&=\lambda^\circ(t^e)\tilde\lambda(z^e)\mu^\circ(t^{-e})\tilde\lambda(z^{-e})=\lambda^\circ(t^e)\mu^\circ(t^{-e})\\&=\lambda^\circ(t^e)\lambda^\circ(t^{-e})\nu^\circ(t^e)=\prod_{\alpha\in\Delta}\alpha(t)^{en_\alpha}.\end{aligned}$$

Since $v_p(\alpha(t))=v_p(\alpha(h))\geq 1$ for all simple roots α we deduce that

$$v_p\Bigl(\prod_{\alpha\in\Delta}\alpha(t)^{en_\alpha}\Bigr)\geq e\sum_{\alpha\in\Delta}n_\alpha=e\,\mathrm{ht}(\nu)\geq 0.$$

Thus, taking into account that $\mathrm{ht}(\nu)=\mathrm{ht}_\lambda(\mu)$ we obtain

$$\tilde\lambda(h^e)h^{-e}v_\mu\in p^{e\,\mathrm{ht}_\lambda(\mu)}L_{\tilde\lambda}(\mathbb{Z}_p,\mu)\subseteq L_{\tilde\lambda}(\mathbb{Z}_p,\mu), \tag{11}$$

which implies that $\tilde\lambda(h^e)h^{-e}$ leaves $L_{\tilde\lambda}(\mathbb{Z}_p)$ invariant. The lemma is proven. □

2.9. Corollary. 1. *For any $T\in\mathcal{H}_{\mathbb{Z}_p}$ the normalized operator $T_{\tilde\lambda}$ leaves the group $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Z}_p))$ invariant. In particular, $T_{\tilde\lambda}$ acts on integral cohomology $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Z}_p))_{\mathrm{int}}$.*

2. *For any $T\in\mathcal{H}$ the eigenvalues of $T_{\tilde\lambda}\in\mathcal{H}_F$ on $H^i(\Gamma\backslash X,L_{\tilde\lambda}(\mathbb{C}))$ and on $H^i_{\mathrm{cusp}}(\Gamma\backslash X,L_{\tilde\lambda}(\mathbb{C}))$ are algebraic over F (note that $F\subseteq\mathbb{C}$) and are contained in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$ (note that $\bar F\subseteq\bar{\mathbb{Q}}_p$).*

Proof. 1. We may assume that $T=T_\zeta$ for some $\zeta\in\Delta$. The claim then follows directly from the definition of the action of T_ζ on cohomology given in equation (10) and the lemma just proved (note that $\zeta\in\Delta\subseteq\mathcal{K}$ and that $\gamma_i^{-1}\in\Gamma\subseteq\tilde{\boldsymbol{G}}(\mathbb{Z}_p)$ leaves $L_{\tilde\lambda}(\mathbb{Z}_p)$ invariant).

2. The cuspidal cohomology $H^i_{\mathrm{cusp}}(\Gamma\backslash X,L_{\tilde\lambda}(\mathbb{C}))\subseteq H^i(\Gamma\backslash X,L_{\tilde\lambda}(\mathbb{C}))$ is a Hecke submodule of full cohomology, hence, the eigenvectors and (complex) eigenvalues of $T_{\tilde\lambda}$ on $H^i_{\mathrm{cusp}}(\Gamma\backslash X,L_{\tilde\lambda}(\mathbb{C}))$ are contained in the set of eigenvectors and eigenvalues of $T_{\tilde\lambda}$ on $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{C}))$. Since $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{C}))=H^i(\Gamma,L_{\tilde\lambda}(F))\otimes\mathbb{C}$ all eigenvalues of $T_{\tilde\lambda}$ on $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{C}))$ are algebraic over F and, hence, they already appear as eigenvalues of $T_{\tilde\lambda}$ on $H^i(\Gamma,L_{\tilde\lambda}(\bar F))$. In particular, they also appear as eigenvalues of $T_{\tilde\lambda}$ on $H^i(\Gamma,L_{\tilde\lambda}(\bar{\mathbb{Q}}_p))$ where the latter contains the $T_{\tilde\lambda}$-invariant $\mathbb{Z}_p$-lattice $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Z}_p))_{\mathrm{int}}$. Thus, all eigenvalues of $T_{\tilde\lambda}$ on $H^i(\Gamma\backslash X,L_{\tilde\lambda}(\mathbb{C}))$ are algebraic over F and contained in the integer ring $\mathcal{O}_{\bar{\mathbb{Q}}_p}$ after embedding $\bar F$ in $\bar{\mathbb{Q}}_p$. This completes the proof. □

The diagram of inclusions on the next page recapitulates the objects appearing in the proof above and groups them together for easy lookup as they come up later in the discussion.

$$\begin{array}{ccccc}
H^i_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{C})) & \subseteq & H^i(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{C})) & & \\
 & & \cup & & \\
 & & H^i(\Gamma\backslash X, L_{\tilde\lambda}(\bar F)) & \subseteq & H^i(\Gamma\backslash X, L_{\tilde\lambda}(\bar{\mathbb{Q}}_p)) \\
 & & \cup & & \cup \\
 & & H^i(\Gamma\backslash X, L_{\tilde\lambda}(F)) & \subseteq & H^i(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{Q}_p)) \\
 & & & & \cup \\
 & & & & H^i(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{Z}_p))_{\mathrm{int}}
\end{array}$$

2.10. ***Mod p reduction of irreducible representations.*** We denote by $\boldsymbol{T}_m$ the maximal torus in $\mathbf{GL}_m$ consisting of diagonal matrices, by $\boldsymbol{B}_m^-$ the Borel subgroup in $\mathbf{GL}_m$ consisting of all lower triangular matrices and by $\wp : \mathbf{GL}_m(\mathbb{Z}_p) \to \mathbf{GL}_m(\mathbb{F}_p)$ the mod p reduction map for $\mathbf{GL}_m$. Since $\tilde{\boldsymbol{G}}/\mathbb{Z}_p$ is smooth the reduction map $\wp : \tilde{\boldsymbol{G}}(\mathbb{Z}_p) \to \tilde{\boldsymbol{G}}(\mathbb{F}_p)$ is surjective. We let $\tilde\lambda \in X(\tilde{\boldsymbol{T}})$ be a dominant weight and as before we set $\lambda = d\,(\tilde\lambda|_{\boldsymbol{T}}) \in \mathfrak{h}^*$. We define the mod p reduction

$$\bar\rho_{\tilde\lambda} : \tilde{\boldsymbol{G}}(\mathbb{F}_p) \to \mathbf{GL}_m(\mathbb{F}_p)$$

of the representation $\rho_{\tilde\lambda} : \tilde{\boldsymbol{G}}/\mathbb{Z}_p \to \mathbf{GL}_m/\mathbb{Z}_p$ by $\bar\rho_{\tilde\lambda}(\bar g) = \wp\,(\rho_{\tilde\lambda}(g))$, where $g \in \tilde{\boldsymbol{G}}(\mathbb{Z}_p)$ satisfies $\wp\,(g) = \bar g$. We denote by $\mathcal{J}_m \le \mathbf{GL}_m(\mathbb{Z}_p)$ the Iwahori subgroup consisting of all elements $g \in \mathbf{GL}_m(\mathbb{Z}_p)$ such that $\wp\,(g) \in \boldsymbol{B}_m^-(\mathbb{F}_p)$.

Lemma. $$\rho_{\tilde\lambda}(\tilde{\mathcal{I}}) \subseteq \mathcal{J}_m.$$

Proof. In Section 1.4 we selected a $\mathbb{Z}_p$-basis $\mathcal{B}$ of $L_\lambda(\mathbb{Z}_p)$ to identify $\mathrm{Aut}(L_\lambda(\mathbb{Z}_p)) = \mathbf{GL}_m(\mathbb{Z}_p)$. Since $L_\lambda(\mathbb{Z}_p) = \bigoplus_{\mu \in P_\lambda} L_\lambda(\mathbb{Z}_p, \mu)$ — see Section 1.1 — we may choose a basis $\mathcal{B}$ consisting of weight vectors w.r.t. $\mathfrak{h}$. We order $\mathcal{B}$ so that, if $v_\mu, v_{\mu'} \in \mathcal{B}$ are vectors of respective weights $\mu, \mu' \in \mathfrak{h}^*$, then $\mathrm{ht}_\lambda(\mu) < \mathrm{ht}_\lambda(\mu')$ implies that $v_\mu < v_{\mu'}$. We consider the image $\rho_{\tilde\lambda}(x_\alpha^\pi(t_\alpha)) = \rho_{\lambda^\circ}(x_\alpha^\pi(t_\alpha)) = x_\alpha^\lambda(t_\alpha) \in \mathrm{Aut}(L_\lambda(\bar{\mathbb{Q}}_p))$, where $\alpha \in \Phi^-$ ($t_\alpha \in \bar{\mathbb{Q}}_p$). Let $v_\mu \in \mathcal{B}$ be a basis vector of weight μ. Since $x_\alpha^\lambda(t_\alpha)v_\mu = v_\mu + t_\alpha x_\alpha v_\mu + \frac{1}{2}t_\alpha^2 x_\alpha^2 v_\mu + \cdots$, we see that $x_\alpha^\lambda(t_\alpha)v_\mu$ is a sum of vectors of weights μ, $\mu{+}\alpha$, $\mu{+}2\alpha, \ldots$, which are of strictly increasing relative height (since $\alpha < 0$). Hence, $\rho_{\tilde\lambda}(x_\alpha^\pi(t_\alpha))$ has lower triangular form w.r.t. $\mathcal{B}$, i.e., $\rho_{\tilde\lambda}(x_\alpha^\pi(t_\alpha)) \in \boldsymbol{B}_m^-(\bar{\mathbb{Q}}_p)$. This shows that $\rho_{\tilde\lambda}(\boldsymbol{N}^-(\bar{\mathbb{Q}}_p)) \subseteq \boldsymbol{B}_m^-(\bar{\mathbb{Q}}_p)$. Since $\boldsymbol{T}(\bar{\mathbb{Q}}_p)$ preserves weight spaces by equation (4) in Section 1.4 we find quite analogous that $\rho_{\tilde\lambda}(\boldsymbol{T}(\bar{\mathbb{Q}}_p)) \subseteq \boldsymbol{T}_m(\bar{\mathbb{Q}}_p)$, hence, $\rho_{\tilde\lambda}(\boldsymbol{B}^-(\bar{\mathbb{Q}}_p)) \subseteq \boldsymbol{B}_m^-(\bar{\mathbb{Q}}_p)$. Thus, we obtain $\rho_{\tilde\lambda}(\boldsymbol{B}^-(\mathbb{Z}_p)) \subseteq \boldsymbol{B}_m^-(\bar{\mathbb{Q}}_p) \cap \mathbf{GL}_m(\mathbb{Z}_p) = \boldsymbol{B}_m^-(\mathbb{Z}_p)$ and, hence,

$$\bar\rho_{\tilde\lambda}(\boldsymbol{B}^-(\mathbb{F}_p)) \subseteq \boldsymbol{B}_m^-(\mathbb{F}_p).$$

We obtain $\wp\,(\rho_{\tilde\lambda}(\mathcal{I})) = \bar\rho_{\tilde\lambda}(\wp\,(\mathcal{I})) \subseteq \bar\rho_{\tilde\lambda}(\boldsymbol{B}^-(\mathbb{F}_p)) \subseteq \boldsymbol{B}_m^-(\mathbb{F}_p)$ and since $\rho_{\tilde\lambda}(\mathcal{I}) \subseteq \rho_{\tilde\lambda}(\tilde{\boldsymbol{G}}(\mathbb{Z}_p)) \subseteq \mathbf{GL}_m(\mathbb{Z}_p)$ we deduce that $\rho_{\tilde\lambda}(\mathcal{I}) \subseteq \mathcal{J}_m$. Equation (8) implies that $\rho_{\tilde\lambda}(z) = \tilde\lambda(z)1_{\mathbf{GL}_m} \in \mathcal{J}_m$ for all $z \in \tilde{\boldsymbol{Z}}(\mathbb{Z}_p)$, hence, we finally obtain $\rho_{\tilde\lambda}(\tilde{\mathcal{I}}) \subseteq \mathcal{J}_m$ and the lemma is proven. $\square$

3. Boundedness of dimension of slope subspaces

3.1. We keep the assumptions from the previous sections. In particular, $\tilde{\boldsymbol{G}}/\mathbb{Q}$ is a connected reductive group containing a $\mathbb{Q}_p$-split maximal torus $\tilde{\boldsymbol{T}}/\mathbb{Q}$ and $\Gamma \leq \tilde{\boldsymbol{G}}(\mathbb{Q})$ is an arithmetic subgroup such that $\Gamma \subseteq \tilde{\mathcal{I}}$. We will obtain bounds for the dimension of the slope subspaces of $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p))$; see 3.10 Corollary. This extends the main result in [Mahnkopf 2014], since (i) we allow Γ to be an arbitrary subgroup in $\tilde{\mathcal{I}}$ (i.e., we do not assume that Γ is contained in the smaller group $K_*(p) < \tilde{\mathcal{I}}$ defined in [Mahnkopf 2013; 2014]); (ii) we do not assume that $\Gamma \leq \boldsymbol{G}(\mathbb{Q})$ where $\boldsymbol{G}$ is the derived group of $\tilde{\boldsymbol{G}}$; (iii) we obtain stronger bounds for the dimension of the slope subspaces than those in [Mahnkopf 2014]. The proof follows the one in that paper. To deal with arithmetic subgroups Γ which are only contained in $\tilde{\mathcal{I}}$ we have to generalize the notion of truncation of an irreducible representation introduced in [Mahnkopf 2013; 2014] (see Section 3.3).

3.2. We note the following corrections to the works just cited.

1. In [Mahnkopf 2014] we considered a connected reductive group $\tilde{\boldsymbol{G}}$ which is defined over a number field F with $F_{\mathfrak{p}}$-split maximal torus $\tilde{\boldsymbol{T}}$ (Section 1.4 there). As in the present article, we have to assume that $\tilde{\boldsymbol{T}}$ is defined over F (and split over $F_{\mathfrak{p}}$); thus, the F-points $\tilde{\boldsymbol{T}}(F)$ are defined and we may select $h \in \tilde{\boldsymbol{T}}(F)$ as done in Section 1.6 of [Mahnkopf 2014].

2. Let $\boldsymbol{G}$ denote the derived group of the connected reductive group $\tilde{\boldsymbol{G}}$ which we considered in [Mahnkopf 2013; 2014]. Hence, $\boldsymbol{G}$ is a semisimple group and in those two papers we assumed that it is isomorphic over a splitting field to a Chevalley group $\boldsymbol{G}_{\lambda_0}$ for an irreducible representation ρ_{λ_0} of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\boldsymbol{G}) \otimes \mathbb{C}$. In general, $\boldsymbol{G}$ over its splitting field only is isomorphic to a Chevalley group $\boldsymbol{G}_\pi$ for a semisimple representation π of $\mathfrak{g}$ (if one restricts to irreducible representations π one does not obtain all covering groups of the adjoint group with Lie algebra $\mathfrak{g}$). Since in the cited papers we did not make use of the irreducibility of the representation ρ_{λ_0} the results also hold if we consider a Chevalley group $\boldsymbol{G}_\pi$ which is attached to a semisimple representation π of $\mathfrak{g}$.

3. In the summation formula (7) in Section 2.5.3 of [Mahnkopf 2014], the Bernoulli number $B_s(0)$ has to be replaced by $B_s(1)$ (note that $B_s(1) = B_s(0)$ for all $s > 1$ but $B_1(1) = -B_1(0) = \frac{1}{2}$).

3.3. ***Truncations with slope parameter.*** For the moment we let $\sigma \in \mathbb{N}$ be any natural number and we consider the subgroup

$$\tilde{K}_*(\sigma) = \tilde{K}_*(p, \sigma) = \langle K_*(p, \sigma), \tilde{\boldsymbol{Z}}(\mathbb{Z}_p)\rangle \leq \tilde{\boldsymbol{G}}(\mathbb{Z}_p).$$

Thus, if $\sigma \geq \max_{\alpha\in\Phi^+} \mathrm{ht}(\alpha)$ then $\tilde{K}_*(\sigma) = \tilde{\mathcal{I}}$. For any $r \in \mathbb{N}_0$ we define the $\mathbb{Z}_p$-submodule

$$(12)\quad L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma) := \bigoplus_{\substack{\mu\leq\lambda \\ 0\leq \mathrm{ht}_\lambda(\mu)\leq r\sigma}} p^{r-\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \oplus \bigoplus_{\substack{\mu\leq\lambda \\ \mathrm{ht}_\lambda(\mu)>r\sigma}} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)$$

of $L_{\tilde{\lambda}}(\mathbb{Z}_p)$.

Lemma. *The $\mathbb{Z}_p$-module $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ of $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ is $\tilde{K}_*(\sigma)$-invariant.*

Proof. In view of the definition of $\tilde{K}_*(\sigma)$ we have to show that the three types of generators of $\tilde{K}_*(\sigma)$ map any of the weight subspaces of $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ (see equation (12)) to $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$. Since $t_\alpha^n x_\alpha^n/n!$, where $t_\alpha \in \mathbb{Z}_p$ and $\alpha \in \Phi$, maps $L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)$ to $L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha)$, it is immediate that any $t_\alpha^n x_\alpha^n/n!$ with $\alpha < 0$ and $t_\alpha \in \mathbb{Z}_p$ maps any weight subspace $p^m L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)$ contained in $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ to $p^m L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha)$, which is contained in $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ because $\mathrm{ht}_\lambda(\mu+n\alpha) \geq \mathrm{ht}_\lambda(\mu)$. Hence, any generator $x_\alpha(t_\alpha)$, $\alpha \in \Phi^-$, $t_\alpha \in \mathbb{Z}_p$, leaves $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ invariant. We look at generators $x_\alpha(t_\alpha)$ where $\alpha \in \Phi^+$—hence, $t_\alpha \in p^{\lceil \frac{1}{\sigma} \mathrm{ht}(\alpha)\rceil}\mathbb{Z}_p$. Using the inequalities $\lceil x\rceil - \lceil y\rceil \leq \lceil x-y\rceil$ and $n\lceil x\rceil \geq \lceil nx\rceil$, $x, y \in \mathbb{R}$, $n \in \mathbb{N}$, we find for all $n \in \mathbb{N}$ and all weights $\mu \leq \lambda$ with $\mathrm{ht}_\lambda(\mu) \leq r\sigma$:

$$\begin{aligned} t_\alpha^n \frac{x_\alpha^n}{n!} p^{r-\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) &\subseteq p^{n\lceil \frac{1}{\sigma} \mathrm{ht}(\alpha)\rceil} p^{r-\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha) \\ &\subseteq p^{r-\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu)\rceil + \lceil n\frac{1}{\sigma} \mathrm{ht}(\alpha)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha) \\ &\subseteq p^{r-(\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu) - n\frac{1}{\sigma} \mathrm{ht}(\alpha)\rceil)} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha) \\ &= p^{r-(\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu+n\alpha)\rceil)} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha) \\ &\subseteq L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma); \end{aligned}$$

for the last inclusion note that $\mathrm{ht}_\lambda(\mu+n\alpha) \leq \mathrm{ht}_\lambda(\mu) \leq r\sigma$ (we remark that if $\mu+n\alpha$ is not $\leq \lambda$ then $t_\alpha^n(x_\alpha^n/n!)p^{r-\lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) = 0$). For weights $\mu \leq \lambda$ with $\mathrm{ht}_\lambda(\mu) > r\sigma$ we find

$$t_\alpha^n \frac{x_\alpha^n}{n!} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \subseteq p^{n\lceil \frac{1}{\sigma} \mathrm{ht}(\alpha)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha) \subseteq p^{\lceil \frac{n}{\sigma} \mathrm{ht}(\alpha)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu+n\alpha).$$

Since $\lceil \frac{n}{\sigma} \mathrm{ht}(\alpha)\rceil \geq 0$ this shows that $t_\alpha^n(x_\alpha^n/n!)L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \subseteq L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ if $\mathrm{ht}_\lambda(\mu+n\alpha) > r\sigma$. If $\mathrm{ht}_\lambda(\mu+n\alpha) \leq r\sigma$ we note that

$$r - \left\lceil \tfrac{1}{\sigma} \mathrm{ht}_\lambda(\mu+n\alpha)\right\rceil \leq r - \tfrac{1}{\sigma}(\mathrm{ht}_\lambda(\mu) - n\,\mathrm{ht}(\alpha)) \leq \tfrac{n}{\sigma} \mathrm{ht}(\alpha),$$

which shows that again $t_\alpha^n(x_\alpha^n/n!)L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \subseteq L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$. Hence, the generators $x_\alpha(t_\alpha)$, $\alpha \in \Phi^+$, $t_\alpha \in p^{\lceil \frac{1}{\sigma} \mathrm{ht}(\alpha)\rceil}\mathbb{Z}_p$, also leave $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ invariant. Finally if $t \in \tilde{\boldsymbol{T}}(\mathbb{Z}_p)$ then t leaves all weight spaces $L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)$ invariant because the representation

$\rho_{\tilde{\lambda}}$ is defined over $\mathbb{Z}_p$. Hence, $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ is invariant under all generators of $\tilde{K}_*(\sigma)$ and the lemma is proven. □

Definition. The quotient

$$\boldsymbol{L}_{\tilde{\lambda}}^{[r]}(\mathbb{Z}_p, \sigma) = \frac{L_{\tilde{\lambda}}(\mathbb{Z}_p)}{L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)} = \bigoplus_{\substack{\mu \leq \lambda \\ \mathrm{ht}_\lambda(\mu) \leq r\sigma}} \frac{L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)}{p^{r - \lceil \frac{1}{\sigma} \mathrm{ht}_\lambda(\mu) \rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)}$$

is called the truncation of $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ of height $r \in \mathbb{N}_0$ and slope $\frac{1}{\sigma}$, $\sigma \in \mathbb{N}$.

Remark. The lemma just proved implies that $\boldsymbol{L}_{\tilde{\lambda}}^{[r]}(\mathbb{Z}_p, \sigma)$ is a $\tilde{K}_*(\sigma)$-module. Therefore, if $\sigma \geq \max_{\alpha \in \Phi^+} \mathrm{ht}(\alpha)$, i.e., $\tilde{K}_*(\sigma) = \tilde{\mathcal{I}}$, then $\boldsymbol{L}_{\tilde{\lambda}}^{[r]}(\mathbb{Z}_p, \sigma)$ is a Γ-module (recall that $\Gamma \leq \tilde{\mathcal{I}}$).

3.4. *Lemma.* *For any dominant and integral weight λ and any $r \in \mathbb{N}$, $\sigma \in \mathbb{N}$ there is an embedding (of $\mathbb{Z}_p$-modules)*

$$\boldsymbol{L}_{\tilde{\lambda}}^{[r]}(\mathbb{Z}_p, \sigma) \leq \bigoplus_{h=0}^{r} \left(\frac{\mathbb{Z}_p}{p^{r-h} \mathbb{Z}_p} \right)^{M_{\sigma,h}},$$

where $M_{\sigma,h} = \sigma(\sigma h + 1)^{s-1}$ ($s = |\Phi^+|$).

Proof. We write

$$\boldsymbol{L}_{\tilde{\lambda}}^{[r]}(\mathbb{Z}_p, \sigma) = \bigoplus_{\substack{h \in \mathbb{N}_0 \\ 0 \leq h \leq r\sigma}} \bigoplus_{\substack{\mu \leq \lambda \\ \mathrm{ht}_\lambda(\mu) = h}} \frac{\mathbb{Z}_p}{p^{r - \lceil \frac{1}{\sigma} h \rceil} \mathbb{Z}_p} \otimes L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu).$$

In the proof of 2.2 Lemma in [Mahnkopf 2014] we have seen that

$$\dim_{\mathbb{Z}_p} \bigoplus_{\substack{\mu \leq \lambda \\ \mathrm{ht}_\lambda(\mu) = h}} L_\lambda(\mathbb{Z}_p, \mu) \leq N_h,$$

where N_h is the number of tuples $\boldsymbol{n} = (n_1, \dots, n_s) \in \mathbb{N}_0^s$ such that

$$\sum_{i=1}^{s} n_i \, \mathrm{ht}(\alpha_i) = h$$

(recall that $\Phi^+ = \{\alpha_1, \dots, \alpha_s\}$). Hence,

$$\boldsymbol{L}_{\tilde{\lambda}}^{[r]}(\mathbb{Z}_p, \sigma) \leq \bigoplus_{\substack{h \in \mathbb{N}_0 \\ 0 \leq h \leq r\sigma}} \left(\frac{\mathbb{Z}_p}{p^{r - \lceil \frac{1}{\sigma} h \rceil} \mathbb{Z}_p} \right)^{N_h}. \tag{13}$$

We select an $a \in \mathbb{N}$. The terms of the form $r - \lceil \frac{1}{\sigma} h \rceil$, $h \in \mathbb{N}_0$, which equal $r - a$ are then precisely those with $h = \sigma a - \sigma + 1, \sigma a - \sigma + 2, \dots, \sigma a$. Thus, the term

$\dfrac{\mathbb{Z}_p}{p^{r-a}\mathbb{Z}_p}$ appears with multiplicity

$$N_{\sigma a-\sigma+1}+N_{\sigma a-\sigma+2}+\cdots+N_{\sigma a}$$

in the right-hand side of equation (13). It is easy to see that $N_h \leq (h+1)^{s-1}$, which implies that

$$N_{\sigma a-\sigma+1}+N_{\sigma a-\sigma+2}+\cdots+N_{\sigma a} \leq \sigma(\sigma a+1)^{s-1}.$$

Thus, we obtain, as desired,

$$\boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p,\sigma) \leq \bigoplus_{\substack{a\in\mathbb{N}_0\\ 0\leq a\leq r}} \left(\frac{\mathbb{Z}_p}{p^{r-a}\mathbb{Z}_p}\right)^{M_{\sigma,a}}. \qquad \square$$

3.5. From now on we set

$$\sigma = \max_{\alpha\in\Phi^+} \mathrm{ht}(\alpha).$$

Hence, σ only depends on $\tilde{\boldsymbol{G}}$ and $\tilde{K}_*(\sigma)=\tilde{\mathcal{I}}$. In particular, $L(\mathbb{Z}_p,r,\sigma)$ is a $\tilde{\mathcal{I}}$ and, hence, a Γ-module. The inclusion $i : L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma) \subseteq L_{\tilde{\lambda}}(\mathbb{Z}_p)$ induces a mapping

$$i^* : H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma)) \to H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p)).$$

We recall that h is an element in $\tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ (see Section 2.3).

Lemma. 1. *The mapping i^* induces an injection*

$$i^* : H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma))^{\mathrm{TF}} \hookrightarrow H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))^{\mathrm{TF}},$$

where superscript TF denotes the maximal torsion-free quotient. In particular, we may identify $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma))^{\mathrm{TF}}$ with its image in $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))^{\mathrm{TF}}$ under i^*.

2. *Let $\zeta\in\Delta$; hence, $\zeta\in\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}}$ for some $e\in\mathbb{N}_0$ and we assume that $e\in\mathbb{N}$. Then the Hecke operator $(T_\zeta)_{\tilde{\lambda}}$ induces an operator on $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))^{\mathrm{TF}}$ and we obtain*

$$(T_\zeta)_{\tilde{\lambda}}\big(H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma))^{\mathrm{TF}}\big) \subseteq p^r H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))^{\mathrm{TF}}.$$

Proof. 1. The exact sequence

$$0 \to L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma) \xrightarrow{i} L_{\tilde{\lambda}}(\mathbb{Z}_p) \xrightarrow{\pi} \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p,\sigma) \to 0$$

yields an exact sequence

$$H^{i-1}(\Gamma, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p,\sigma)) \to H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p,r,\sigma)) \xrightarrow{i^*} H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p)) \xrightarrow{\pi^*} H^i(\Gamma, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p,\sigma)).$$

Since $H^i(\Gamma, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p, \sigma))$ is a finite abelian group we further obtain an exact sequence

$$0 \to H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma))^{\mathrm{TF}} \xrightarrow{i^*} H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))^{\mathrm{TF}} \xrightarrow{\pi^*} Q \to 0, \tag{14}$$

where Q is a certain subquotient of $H^i(\Gamma, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p, \sigma))$. Thus, i^* is injective.

2. The first claim follows from 2.9 Corollary. In equation (11) in Section 2.8 we have seen that for any $v_\mu \in L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)$

$$\tilde{\lambda}(h^e)h^{-e}v_\mu \in p^{e\,\mathrm{ht}_\lambda(\mu)} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu).$$

Hence, for all weights $\mu \le \lambda$ satisfying $\mathrm{ht}_\lambda(\mu) \le r\sigma$ we obtain

$$\begin{aligned}\tilde{\lambda}(h^e)h^{-e}p^{r-\lceil \frac{1}{\sigma}\mathrm{ht}_\lambda(\mu)\rceil} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) &\subseteq p^{r-\lceil \frac{1}{\sigma}\mathrm{ht}_\lambda(\mu)\rceil + e\,\mathrm{ht}_\lambda(\mu)} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu)\\ &\subseteq p^{r+(e-1)\,\mathrm{ht}_\lambda(\mu)} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \subseteq p^r L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu),\end{aligned}$$

and for all weights $\mu \le \lambda$ satisfying $\mathrm{ht}_\lambda(\mu) > r\sigma (\ge r)$ we obtain

$$\tilde{\lambda}(h^e)h^{-e} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \subseteq p^{e\,\mathrm{ht}_\lambda(\mu)} L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu) \subseteq p^r L_{\tilde{\lambda}}(\mathbb{Z}_p, \mu).$$

Hence, we obtain $\tilde{\lambda}(h^e)h^{-e}L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma) \subseteq p^r L_{\tilde{\lambda}}(\mathbb{Z}_p)$. Since $\zeta \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ with $e \ge 1$ and $\tilde{\mathcal{I}}$ leaves $L_{\tilde{\lambda}}(\mathbb{Z}_p)$ and $L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma)$ invariant (see 3.3.1 Lemma) we obtain

$$\tilde{\lambda}(h^e)\zeta^{-1} L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma) \subseteq p^r L_{\tilde{\lambda}}(\mathbb{Z}_p)$$

which yields

$$(T_\zeta)_{\tilde{\lambda}}(C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma))) \subseteq p^r C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))\ (\subseteq C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p)))$$

(here, we view $C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p, r, \sigma))$ as embedded in $C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))$ via i^*). The last equation implies the claim. □

We note that $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))^{\mathrm{TF}} \cong H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\mathrm{int}}$.

3.6. We select a resolution of the trivial Γ-module $\mathbb{Z}$,

$$0 \to M_d \to \cdots \to M_1 \to M_0 \to \mathbb{Z} \to 0,$$

where M_i is a free $\mathbb{Z}\Gamma$-module of finite rank (see [Brown 1982, p. 199]; note that Γ is of type FL; see p. 218 in the same work). The groups $H^i(\Gamma, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p, \sigma))$ then may be computed as the cohomology of the complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}_p\Gamma}(M_{0,p}, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p, \sigma)) \to \cdots \to \mathrm{Hom}_{\mathbb{Z}_p\Gamma}(M_{d,p}, \boldsymbol{L}^{[r]}_{\tilde{\lambda}}(\mathbb{Z}_p, \sigma)) \to 0$$

where $M_{i,p} = \mathbb{Z}_p \otimes M_i$. We set

$$g_i = g_{i,\Gamma} = \mathrm{rk}_{\mathbb{Z}\Gamma} M_i.$$

Thus, g_i depends on i and the arithmetic group Γ.

3.7. ***Borel–Serre compactification.*** Following [Borel and Serre 1973] (see also [Brown 1982, pp. 14 and 218]) we can construct a finite free resolution $(M_i)_{d\geq i\geq 0}$ of $\mathbb{Z}$ as follows. We denote by $Y=\Gamma\backslash\bar{X}$ the Borel–Serre compactification of the locally symmetric space $\Gamma\backslash X$ attached to $\tilde{\boldsymbol{G}}/\mathbb{Q}$. By the work of Borel and Serre Y is a compact $K(\Gamma,1)$ space; hence, there is a *finite* CW complex $\mathcal{Z}$ having the same homotopy type as Y. The universal cover $\tilde{Y}$ of Y inherits a structure of CW complex $\tilde{\mathcal{Z}}$ from $\mathcal{Z}$ and the cellular complex $\tilde{C}_\bullet=(\tilde{C}_i)_{d\geq i\geq 0}$ ($d=\dim X$) attached to $\tilde{\mathcal{Z}}$ is a complex consisting of free $\mathbb{Z}\Gamma$-modules $\tilde{C}_i$. The module $\tilde{C}_i$ has a natural $\mathbb{Z}\Gamma$-basis which is in bijection with the set of i-cells of $\mathcal{Z}$ (see [Brown 1982, p. 15]); hence, $\tilde{C}_\bullet$ is a finite complex consisting of free, finitely generated $\mathbb{Z}\Gamma$-modules. Since $\tilde{Y}$ is contractible, its cohomology vanishes except in degree 0; hence, the complex

$$0\to\tilde{C}_d\to\cdots\to\tilde{C}_0\to\mathbb{Z}\to 0$$

is exact and thus a resolution of $\mathbb{Z}$. In particular, we may select $M_i=\tilde{C}_i$ and since $\mathrm{rk}_{\mathbb{Z}\Gamma}\,\tilde{C}_i$ equals the number of i-cells of $\mathcal{Z}$ this shows that we can take for g_i the number of i-cells of a CW complex $\mathcal{Z}$ which has the same homotopy type as $\Gamma\backslash\bar{X}$.

3.8. ***Slope subspaces.*** We select a Hecke operator $T\in\mathcal{H}_{\mathbb{Z}_p}$. Let $\tilde{\lambda}\in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ and let $E/\mathbb{Q}_p$ be an extension which is contained in $\bar{\mathbb{Q}}_p$. For any $\beta\in\mathbb{Q}_{\geq 0}$ we denote by

$$H^i(\Gamma,L_{\tilde{\lambda}}(E))^\beta=p_\beta(T_{\tilde{\lambda}})H^i(\Gamma,L_{\tilde{\lambda}}(E))$$

the slope β subspace of $H^i(\Gamma,L_{\tilde{\lambda}}(E))$ w.r.t. to the (normalized) Hecke operator $T_{\tilde{\lambda}}$. Here, $p(X)\in\mathbb{Z}_p[X]$ is the characteristic polynomial of $T_{\tilde{\lambda}}$ acting on $H^i(\Gamma,L_{\tilde{\lambda}}(E))$ and $p_\beta(X)=\prod_{\mu,\,v_p(\mu)\neq\beta}(X-\mu)\in\mathbb{Z}_p[X]$, where μ runs over all roots of $p(X)$ whose p-adic value is different from β. Thus, we obtain $H^i(\Gamma,L_{\tilde{\lambda}}(\bar{\mathbb{Q}}_p))^\beta=\bigoplus_{\mu\in\mathcal{O}_{\bar{\mathbb{Q}}_p},\,v_p(\mu)=\beta}H^i(\Gamma,L_{\tilde{\lambda}}(\bar{\mathbb{Q}}_p))(\mu)$ where $H^i(\Gamma,L_{\tilde{\lambda}}(\bar{\mathbb{Q}}_p))(\mu)$ is the generalized eigenspace attached to the eigenvalue μ. We set

$$H^i(\Gamma,L_{\tilde{\lambda}}(\mathbb{Q}_p))^{\leq\beta}=\bigoplus_{0\leq\gamma\leq\beta}H^i(\Gamma,L_{\tilde{\lambda}}(\mathbb{Q}_p))^\gamma$$

and we denote by $H^i(\Gamma,L_{\tilde{\lambda}}(\mathbb{Q}_p))^{<\infty}=\bigoplus_{0\leq\gamma<\infty}H^i(\Gamma,L_{\tilde{\lambda}}(\mathbb{Q}_p))^\gamma$ the finite slope subspace.

3.9. ***An estimate for the Newton polygon.*** We denote by

$$\mathcal{H}_{\mathbb{Z}_p}^{\mathrm{reg}}\subseteq\mathcal{H}_{\mathbb{Z}_p}$$

the set of all Hecke operators $T=\sum_\zeta c_\zeta T_\zeta\in\mathcal{H}_{\mathbb{Z}_p}$ ($\zeta\in\Delta$, $c_\zeta\in\mathbb{Z}_p$) where $\zeta\in\tilde{\mathcal{I}}h^{e_\zeta}\tilde{\mathcal{I}}$ with $e_\zeta\geq 1$ for all ζ with $c_\zeta\neq 0$. We let $T\in\mathcal{H}_{\mathbb{Z}_p}^{\mathrm{reg}}$. We set $t'=t'(\tilde{\lambda},i)=$

$\dim H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{<\infty}$ and we denote by $p'(X) = \sum_{i=0}^{t'} a_i X^{t'-i} \in \mathbb{Z}_p[X]$ the characteristic polynomial and by

$$\mathcal{N}^{<\infty} = \mathcal{N}^{<\infty}_{\tilde\lambda,i} : [0, t'] \to \mathbb{R}_{\geq 0}$$

the Newton polygon of $T_{\tilde\lambda}$ acting on $H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{<\infty}$ which contains the $T_{\tilde\lambda}$-invariant lattice $H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{<\infty}_{\mathrm{int}} = H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{<\infty} \cap H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))_{\mathrm{int}}$. Thus, $\mathcal{N}^{<\infty}$ is the lower convex hull of the points $(i, v_p(a_i))$, $i = 0, \ldots, t'$, where we omit all points with $a_i = 0$ (note that $p'(0) \neq 0$, hence, $a_{t'} \neq 0$). We recall that $g_i = \mathrm{rk}_{\mathbb{Z}\Gamma} M_i$ (see Sections 3.6 and 3.7) and that $B_s \in \mathbb{Q}[X]$ denotes the s-th Bernoulli polynomial.

Theorem. *For all dominant weights $\tilde\lambda \in X(\tilde{\boldsymbol{T}}))$ and all $i \in \mathbb{N}_0$ the Newton polygon $\mathcal{N}^{<\infty} = \mathcal{N}^{<\infty}_{\tilde\lambda,i}$ lies above the restriction to $[0, t']$ of the piecewise linear function $f^*_\infty = f^*_{i,\infty} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which connects the points $(0, 0)$ and*

$$P_j = \left(g_i \sigma \frac{B_s(\sigma(j+1)+1) - B_s(1)}{s}, g_i \sigma^{s+1} \frac{B_{s+1}(j+1) - B_{s+1}(1)}{s+1} \right),$$

where $j = 0, 1, 2, \ldots.$

Proof. We proceed in steps.

3.9.1. We let $\tilde\lambda \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ and set $\lambda = \mathrm{d}\,\tilde\lambda|_{\boldsymbol{T}} \in \mathfrak{h}^*$. Moreover, we select a natural number $r \in \mathbb{N}$. Since $\sigma = \max_{\alpha \in \Phi^+} \mathrm{ht}(\alpha)$ the $\mathbb{Z}_p$-module $\boldsymbol{L}^{[r]}_{\tilde\lambda}(\mathbb{Z}_p, \sigma)$ is a Γ-module and by 3.4 Lemma we know that

$$\boldsymbol{L}^{[r]}_{\tilde\lambda}(\mathbb{Z}_p, \sigma) \leq \bigoplus_{h=0}^{r} (\mathbb{Z}_p / p^{r-h} \mathbb{Z}_p)^{M_{\sigma,h}},$$

which implies that

$$\mathrm{Hom}_{\mathbb{Z}_p\Gamma}(M_{i,p}, \boldsymbol{L}^{[r]}_{\tilde\lambda}(\mathbb{Z}_p, \sigma)) \leq \bigoplus_{h=0}^{r} (\mathbb{Z}_p / p^{r-h} \mathbb{Z}_p)^{g_i M_{\sigma,h}}. \tag{15}$$

We denote by $(p^{a_l})_l$, $a_1 \geq a_2 \geq \cdots \geq a_n > 0$, the sequence of elementary divisors of the right-hand side of equation (15), i.e.,

$$(p^{a_l})_{l=1,\ldots,n} = (p^r, \ldots, p^r, p^{r-1}, \ldots, p^{r-1}, \ldots, p, \ldots, p), \tag{16}$$

where p^{r-h} appears $g_i M_{\sigma,h}$-times. From 3.5 Lemma it follows that there is a natural embedding of $H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p, r, \sigma))^{\mathrm{TF}}$ in $H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{\mathrm{TF}}$ and the exact sequence in equation (14) shows that

$$\frac{H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{\mathrm{TF}}}{H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p, r, \sigma))^{\mathrm{TF}}} \quad \text{is a subquotient of } H^i(\Gamma, \boldsymbol{L}^{[r]}_{\tilde\lambda}(\mathbb{Z}_p, \sigma)). \tag{17}$$

We denote by t the rank of $H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{\mathrm{TF}}$ and by $(p^{b_l})_l$, $b_1 \geq b_2 \geq \cdots \geq b_m > 0$ ($m \leq t$) the sequence of elementary divisors of the quotient on the left in (17). Equation (17) implies that this quotient is a subquotient of the Hom space on the left-hand side of equation (15); hence, it is a subquotient of the right-hand side of (15) and equation (16) yields $m \leq n$ and

$$b_1 \leq a_1, b_2 \leq a_2, \ldots, b_m \leq a_m. \tag{18}$$

We set $b_l = 0$ for $m < l \leq t$ and $a_l = 0$ for $n < l \leq t$ (if $n < t$); hence, $b_i \leq a_i$ for $i = 1, \ldots, t$.

3.9.2. Using the results so far we can give a lower bound for $\mathcal{N}^{<\infty}$. Equations (16) and (18) imply that the b_l are all smaller than or equal to r. Moreover, since $T = \sum_\zeta c_\zeta T_\zeta$ where $c_\zeta \in \mathbb{Z}_p$ and $\zeta \in \tilde{\mathcal{I}} h^{e_\zeta} \tilde{\mathcal{I}}$ with $e_\zeta \geq 1$ if $c_\zeta \neq 0$ 3.5 Lemma implies that

$$T_{\tilde\lambda}(H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p, r, \sigma))^{\mathrm{TF}}) \subseteq p^r H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{\mathrm{TF}}.$$

Thus, we may apply Lemma 1 in [Buzzard 2001], as recalled in Section 1.7 of [Mahnkopf 2014], to the pair $L = H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{\mathrm{TF}}$ and $K = H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p, r, \sigma))^{\mathrm{TF}}$ and the operator $\xi = T_{\tilde\lambda}$. More precisely, we denote by $f_b = f_{b,r} : [0, t'] \to \mathbb{R}_{\geq 0}$ the piecewise linear function attached to the sequence $(b_1, \ldots, b_t)$; i.e., f_b is the piecewise linear function joining the points $(j, C(j))$, $j = 0, \ldots, t'$, where $C(j) = \sum_{l=1}^{j}(r - b_l)$. Then (1.7) Lemma in [Mahnkopf 2014] states that the Newton polygon $\mathcal{N}^{<\infty}$ of $T_{\tilde\lambda}$ acting on

$$\left(H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Z}_p))^{\mathrm{TF}} \otimes \mathbb{Q}_p\right)^{<\infty} = H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{<\infty}$$

is bounded from below by the graph of f_b.

3.9.3. We further estimate the function f_b. Equation (18) implies that f_b lies above the piecewise linear function $f_{a,r} : [0, t'] \to \mathbb{R}_{\geq 0}$ attached to the sequence $(a_1, \ldots, a_{t'})$, i.e., $f_{a,r}$ joins the points $(j, A(j))$, $j = 0, \ldots, t'$, where $A(j) = \sum_{l=1}^{j} r - a_l$ (the $A(j)$'s are equal to or smaller than the $C(j)$'s). Thus, we have

$$\mathcal{N}^{<\infty} \geq f_{b,r} \geq f_{a,r} \tag{19}$$

and this inequality holds for all $r \in \mathbb{N}$ since r was chosen arbitrarily. Using equation (16) it is not difficult to see that the function $f_{a,r}$ is the restriction to $[0, t']$ of the piecewise linear function on $\mathbb{R}_{\geq 0}$ which starts in $(0, 0)$ and has slope j for $g_i \sum_{h=0}^{j-1} M_{\sigma,h} \leq x \leq g_i \sum_{h=0}^{j} M_{\sigma,h}$, $j = 0, \ldots, r-1$, and slope r for $x \geq g_i \sum_{h=0}^{r-1} M_{\sigma,h}$. Since $M_{\sigma,h} = \sigma(\sigma h + 1)^{s-1}$, $h \geq 0$, we see that the function $f_{a,r}$ may be equivalently described as the piecewise linear function $f_{a,r} : \mathbb{R}_{\geq 0} \to \mathbb{R}$

which starts in (0, 0), has slope j for

$$g_i\sigma\sum_{h=0}^{j-1}(\sigma h+1)^{s-1}\le x\le g_i\sigma\sum_{h=0}^{j}(\sigma h+1)^{s-1},\quad j=0,\dots,r-1, \tag{20}$$

and slope r for $x\ge g_i\sigma\sum\limits_{h=0}^{r-1}(\sigma h+1)^{s-1}$. We set

$$x_s(j)=g_i\sigma\frac{B_s(\sigma(j+1)+1)-B_s(1)}{s},\quad j=0,1,2,\dots,$$

and we denote by $f_\infty:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ the piecewise linear function which starts in (0, 0) and has slope 0 in the interval

$$0\le x\le x_s(0)$$

and slope j in the interval

$$x_s(j-1)\le x\le x_s(j),\quad j=1,2,\dots$$

The function f_∞ like the function $f_{a,r}$ is monotonely increasing. Taking into account that for all $j\in\mathbb{N}_0$

$$\sum_{h=0}^{\sigma j-1}(h+1)^{s-1}=\sum_{h=1}^{\sigma j}h^{s-1}=\begin{cases}\dfrac{B_s(\sigma j+1)-B_s(1)}{s} & \text{if } j\ge 1,\\ 0 & \text{if } j=0,\end{cases} \tag{21}$$

we deduce that

$$x_s(0)=g_i\sigma\frac{B_s(\sigma+1)-B_s(1)}{s}\ge g_i\sigma 1^{s-1}=g_i\sigma$$

and for all $j\in\mathbb{N}$

$$\begin{aligned}x_s(j)-x_s(j-1)&=g_i\sigma\sum_{h=\sigma j}^{\sigma(j+1)-1}(h+1)^{s-1}\\&\ge g_i\sigma(\sigma j+1)^{s-1}=g_i\sigma\left(\sum_{h=0}^{j}(\sigma h+1)^{s-1}-\sum_{h=0}^{j-1}(\sigma h+1)^{s-1}\right).\end{aligned}$$

Thus, equation (20) implies that the segments of slope $0,\dots,r-1$ of f_∞ are longer than those of $f_{a,r}$, hence, $f_{a,r}(x)\ge f_\infty(x)$ for all $x\in[0,x_s(r-1)]$. This implies by equation (19) that

$$\mathcal{N}^{<\infty}(x)\ge f_\infty(x)$$

for all $x\in[0,x_s(r-1)]$. Since equation (19) holds for arbitrarily large $r\in\mathbb{N}$, and since $x_s(r-1)\to\infty$ for $r\to\infty$ by equation (21), we finally obtain

$$\mathcal{N}^{<\infty}(x)\ge f_\infty(x),\quad x\in[0,\infty). \tag{22}$$

3.9.4. We show that $f_\infty \geq f_\infty^*$. In view of equation (22) this completes the proof. By definition f_∞ is the piecewise linear function joining the points

$$(0,0), \quad Q_j = \left(x_s(j), \sum_{h=1}^{j} h(x_s(h) - x_s(h-1))\right), \quad j = 0, 1, 2, 3, \ldots$$

We obtain the following estimate for the second coordinate $\mathtt{y}_s(j)$ of Q_j, i.e., for the value $f_\infty(x_s(j))$:

$$\mathtt{y}_s(0) = 0 = f_\infty^*(x_s(0))$$

and if $j \geq 1$ then equation (21) implies that

$$\begin{aligned}
\mathtt{y}_s(j) &= \sum_{h=1}^{j} h(x_s(h) - x_s(h-1)) \\
&= \sum_{h=1}^{j} h g_i \sigma \left(\frac{B_s(\sigma(h+1)+1) - B_s(1)}{s} - \frac{B_s(\sigma h+1) - B_s(1)}{s} \right) \\
&\overset{(21)}{=} g_i \sigma \sum_{h=1}^{j} h \sum_{k=\sigma h}^{\sigma(h+1)-1} (k+1)^{s-1} \geq g_i \sigma \sum_{h=1}^{j} h \sum_{k=\sigma h}^{\sigma h+\sigma-1} k^{s-1} \\
&\geq g_i \sigma \sum_{h=1}^{j} h\, \sigma (\sigma h)^{s-1} = g_i \sigma^{s+1} \sum_{h=1}^{j} h^s \\
&\overset{(21)}{=} g_i \sigma^{s+1} \frac{B_{s+1}(j+1) - B_{s+1}(1)}{s+1} = f_\infty^*(x_s(j)).
\end{aligned}$$

Thus, $f_\infty \geq f_\infty^*$ and the theorem is proven. □

3.10. ***A bound for the dimension of slope subspaces.*** We recall that $s = |\Phi^+|$, $\sigma = \max_{\alpha\in\Phi^+} \mathrm{ht}(\alpha)$ and g_i is the number of i-cells in a cell complex $\tilde{\mathcal{Z}}$ which is homotopy equivalent to $\Gamma\backslash\bar{X}$.

Corollary. *For all $\beta \in \mathbb{Q}_{\geq 0}$, all dominant weights $\tilde\lambda \in X(\tilde{\boldsymbol{T}})$, all i and all Hecke operators $T \in \mathcal{H}_{\mathbb{Z}_p}^{\mathrm{reg}}$ we have*

$$\dim H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{\leq\beta} \leq \mathtt{m}\beta^s + \mathtt{n};$$

here, $\mathtt{m} = \mathtt{m}_\Gamma = 12(g_i/s)\sigma^{s+1} \in \mathbb{Q}_{\geq 0}$ *and* $\mathtt{n} = \mathtt{n}_\Gamma \in \mathbb{N}$ *is an integer which also only depends on* g_i, σ, s (see (26) below); *in particular,* $\mathtt{m}$ *and* $\mathtt{n}$ *only depend on* Γ (*and so on* $\tilde{\boldsymbol{G}}$ *and* p) *and* i, *but not on* $\tilde\lambda$, h *and* T.

Proof. Let $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be any function such that $f_\infty^*(x) \geq h(x)$ for all $x \geq 0$ and let $(d(\epsilon), y)$ with $d(\epsilon) > 0$ be an intersection point of h and the function

$w_\epsilon : x \mapsto (\beta+\epsilon)x$ ($\epsilon > 0$). Since

$$(\beta+\epsilon)x > \mathcal{N}^{<\infty}(x) \geq h(x)$$

for all $x \in [0, \dim H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{\leq\beta}]$ by 3.9 Theorem we deduce that $d(\epsilon) \geq \dim H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{\leq\beta}$; hence, we obtain an upper bound for the dimension of the slope $\leq\beta$-subspace. We explicitly define a lower bound h for f_∞^* as follows. Since B_s is a polynomial of degree s and leading coefficient 1 there is a natural number $M = M(\sigma, s) \in \mathbb{N}$ such that

$$x_s(j) = g_i\sigma\frac{B_s(\sigma(j+1)+1) - B_s(1)}{s} \leq 2^{\frac{1}{s+1}}\frac{g_i\sigma^{s+1}}{s}j^s \tag{23}$$

and

$$y_s(j) := g_i\sigma^{s+1}\frac{B_{s+1}(j+1) - B_{s+1}(1)}{s+1} \geq 2^{-\frac{1}{s}}\frac{g_i\sigma^{s+1}}{s+1}j^{s+1} \tag{24}$$

for all $j \geq M$. We define the function

$$h : [x_s(M), \infty) \to \mathbb{R}_{\geq 0}, \quad x \mapsto cx^{\frac{s+1}{s}},$$

where $c = 4^{-\frac{1}{s}}g_i^{-\frac{1}{s}}s^{\frac{s+1}{s}}\frac{1}{(s+1)}\sigma^{-\frac{s+1}{s}}$. We note that $x_s(M) \geq 0$ by equation (21). We then obtain for all $j \geq M$

$$\begin{aligned} h(x_s(j)) = h\left(g_i\sigma\frac{B_s(\sigma(j+1)+1) - B_s(1)}{s}\right) &\overset{(23)}{\leq} c\Big(2^{\frac{1}{s+1}}\frac{g_i\sigma^{s+1}}{s}j^s\Big)^{\frac{s+1}{s}} \\ &= c\,2^{\frac{1}{s}}\left(\frac{g_i}{s}\right)^{\frac{s+1}{s}}\sigma^{\frac{(s+1)^2}{s}}j^{s+1} \\ &= 2^{-\frac{1}{s}}\frac{g_i\sigma^{s+1}}{s+1}j^{s+1} \overset{(24)}{\leq} y_s(j). \end{aligned}$$

Since f_∞^* is the piecewise linear function connecting the points $P_j = (x_s(j), y_s(j))$, $j \in \mathbb{N}_0$ and $(0,0)$, and since h passes below the points P_j, $j \geq M$, and is convex this implies that $h(x) \leq f_\infty^*(x)$ for all $x \geq x_s(M)$. We extend h to a function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by setting $h(x) = f_\infty^*(x)$ for $x \in [0, x_s(M)]$ and $h(x) = cx^{\frac{s+1}{s}}$ if $x > x_s(M)$, hence, $f_\infty^*(x) \geq h(x)$ for all $x \in [0, \infty)$. As in the proof of 3.3 Corollary in [Mahnkopf 2014] we see that for all $\epsilon > 0$ the functions h and $x \mapsto (\beta+\epsilon)x$ always intersect in a point $(d(\epsilon), y)$ with $d(\epsilon) > 0$ and this point satisfies $d(\epsilon) \leq \max((\frac{\beta+\epsilon}{c})^s, x_s(M))$. Since $(\beta/c)^s$, $x_s(M) \geq 0$ are positive we obtain

$$\dim H^i(\Gamma, L_{\tilde\lambda}(\mathbb{Q}_p))^{\leq\beta} \leq \max((\beta/c)^s, x_s(M)) \leq (\beta/c)^s + x_s(M).$$

Since further

$$c^{-s} = 4g_is^{-(s+1)}(s+1)^s\sigma^{s+1} = 4g_is^{-1}\Big(1+\frac{1}{s}\Big)^s\sigma^{s+1} \leq 4\frac{g_i}{s}\sigma^{s+1}\exp(1) \tag{25}$$

and $\exp(1) \le 3$ the claim of the corollary holds with $\mathtt{m} = 12(g_i/s)\sigma^{s+1}$ and $\mathtt{n} = x_s(M)$. The claim still holds if we replace $\mathtt{n}$ with any larger number and since equation (23) implies that $x_s(M) \le 2^{\frac{1}{s+1}}(g_i\sigma^{s+1}/s)M^s$ the corollary in particular holds with

$$\mathtt{n} = \left\lceil 2^{\frac{1}{s+1}} \frac{g_i\sigma^{s+1}}{s} M^s \right\rceil + 1 \in \mathbb{N}. \tag{26}$$ □

4. Mod p^n reduction of traces of Hecke operators

4.1. In this section we will prove congruences between traces of powers of normalized Hecke operators on cuspidal cohomology for varying weight $\tilde{\lambda}$. Our main tool will be a comparison of Bewersdorff's elementary trace formula for pairs $\tilde{\lambda}$, $\tilde{\lambda}'$ of congruent weights. The equality of mod p^n reductions of geometric sides follows from p-adic properties of the diagonalization of elements in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}} \subseteq \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$; see 4.4 Proposition (note that the Hecke operator $\Gamma\zeta\Gamma$, $\zeta \in \Delta$, is contained in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}}$ for some $e \in \mathbb{N}_0$). In particular, the comparison is elementary and does not make use of advanced methods such as rigid analytic geometry or p-adic Banach space methods such as overconvergent cohomology. Using an adelic setting we prove analogous congruences on the Eisenstein part of cohomology and subtracting from full cohomology we obtain congrences congruences on cuspidal cohomology (Sections 4.11–4.13).

In Section 4.14 we compare two Goresky–MacPherson trace formulas for two congruent weights. Equality of mod p^n reductions of the geometric sides again follows from the same diagonalization of elements in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}} \subseteq \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ but now applied for all Levi subgroups $\tilde{\boldsymbol{M}}$ of $\mathbb{Q}$-parabolic subgroups of $\tilde{\boldsymbol{G}}$. This yields congruences on weighted cohomology groups and also has an application to a more explicit version of the Gouvêa–Mazur conjecture for symplectic groups of rank 2 (see Section 5.8).

As before, $\tilde{\boldsymbol{G}}/\mathbb{Q}$ is a connected reductive group containing a $\mathbb{Q}_p$-split maximal torus $\tilde{\boldsymbol{T}}/\mathbb{Q}$ and $\Gamma \subseteq \tilde{\boldsymbol{G}}(\mathbb{Q})$ is an arithmetic subgroup satisfying $\Gamma \subseteq \tilde{\mathcal{I}}$.

4.2. ***The fixed point principle of Bewersdorff.*** As in Section 2.6 we denote by $X = \tilde{\boldsymbol{G}}(\mathbb{R})/\tilde{K}_\infty A_{\tilde{\boldsymbol{G}}}$ the symmetric space attached to $\tilde{\boldsymbol{G}}$. The Γ-module $L_{\tilde{\lambda}}(\mathbb{Q}_p)$ defines a locally constant sheaf on the locally symmetric space $\Gamma\backslash X$ and its Borel–Serre compactification $\Gamma\backslash\tilde{X}$, which we will also denote by $L_{\tilde{\lambda}}(\mathbb{Q}_p)$, and the Hecke algebra $\mathcal{H}$ acts on the cohomology groups

$$H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p)) \cong H^i(\Gamma\backslash\bar{X}, L_{\tilde{\lambda}}(\mathbb{Q}_p)).$$

We recall that in Section 2.5 we selected a finite extension $F/\mathbb{Q}$ which splits $\tilde{\boldsymbol{G}}$ and which embeds in $\mathbb{C}$, $\mathbb{Q}_p$; in particular, $(\rho_{\tilde{\lambda}}, L_{\tilde{\lambda}})$ is defined over F. We write

$\Gamma\zeta\Gamma/\sim_\Gamma$ for the set of Γ-conjugacy classes contained in the double coset $\Gamma\zeta\Gamma$, $\zeta \in \Delta$, and $[\xi]_\Gamma$ denotes the Γ-conjugacy class of $\xi \in \tilde{\boldsymbol{G}}(\mathbb{Q})$. We now borrow from [Bewersdorff 1985, Satz 2.6] a simple and elementary formula for the Lefschetz number of Hecke correspondences on full cohomology:

Theorem (Bewersdorff). *Let $\Gamma\zeta\Gamma \in \mathcal{H}$ (i.e., $\zeta \in \Delta$). There are rational integers $c_{[\xi]_\Gamma} \in \mathbb{Z}$, $[\xi]_\Gamma \in \Gamma\zeta\Gamma/\sim_\Gamma$, such that for all dominant weights $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$*

$$\operatorname{Lef}(\Gamma\zeta\Gamma|H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(F))) = \sum_{[\xi]_\Gamma \in \Gamma\zeta\Gamma/\sim_\Gamma} c_{[\xi]_\Gamma} \operatorname{tr}(\xi^{-1}|L_{\tilde{\lambda}}(F)) \tag{27}$$

and $c_{[\xi]_\Gamma}$ vanishes if $\xi x \neq x$ for all $x \in \bar{X}$.

Remark. 1. The integers $c_{[\xi]_\Gamma}$ do not depend on the weight $\tilde{\lambda}$.

2. The trace formula (27) is of an elementary nature. Apart from the existence of a nice compactification of the locally symmetric space $\Gamma\backslash X$ (the Borel–Serre compactification) its proof is a direct application of the Lefschetz fixed point principle which is a general and basic principle of algebraic topology.

4.3. The following lemma will be applied in the proof of 4.4 Proposition, where representations $\tilde{\boldsymbol{G}}/\mathbb{Q}_p \hookrightarrow \mathbf{GL}_m/\mathbb{Q}_p$ of $\tilde{\boldsymbol{G}}$ as matrix group are used.

Lemma. *Let $\beta = (\beta_{ij}) \in \mathcal{J}_m$ and let $t = \operatorname{diag}(t_1, \dots, t_m) \in \boldsymbol{T}_m(\mathbb{Q}_p)$ with $v_p(t_1) > v_p(t_i)$ for all $i = 2, \dots, m$. Then the characteristic polynomial $\operatorname{ch}_{\beta t}$ of $\beta t \in \mathbf{GL}_m(\mathbb{Q}_p)$ has m roots $t'_1, t'_2, \dots, t'_m$ in $\bar{\mathbb{Q}}_p$ (roots appearing several times according their multiplicity) such that $v_p(t'_1) > v_p(t'_i)$ for all $i = 2, \dots, m$ (in particular, t'_1 has multiplicity 1) and*

$$v_p(t'_1) = v_p(t_1) \quad \textit{and} \quad t'_1 \equiv \beta_{11} t_1 \pmod{p^{v_p(t_1)+1}\mathcal{O}_{\bar{\mathbb{Q}}_p}}.$$

Proof. We put $[a, b] = \{a, a+1, a+2, \dots, b\}$ ($a, b \in \mathbb{N}$, $a \le b$) and we denote by S_M the symmetric group on the set M. We write the characteristic polynomial of βt as $\operatorname{ch}_{\beta t}(X) = (-1)^m X^m + (-1)^{m-1} c_1 X^{m-1} + \cdots - c_{m-1} X + c_m$, $c_i \in \mathbb{Q}_p$ (i.e., $c_0 = 1$). The Leibniz formula

$$\operatorname{ch}_{\beta t}(X) = \det(\beta t - X\mathbf{1}) = \sum_{\pi \in S_{[1,m]}} \operatorname{sgn}(\pi) \prod_{i=1}^{m} (\beta_{\pi(i),i} t_i - \delta_{\pi(i),i} X)$$

yields

$$c_i = \sum_{\substack{T \subseteq [1,m] \\ |T| = i}} \sum_{\pi \in S_T} c_{T,\pi}$$

for all $i = 1, \dots, m$, where

$$c_{T,\pi} = \operatorname{sgn}(\pi) \prod_{h \in T} \beta_{\pi(h),h} t_h \in \mathbb{Q}_p.$$

Since

$$v_p(\beta_{g,h}t_h)\begin{cases}\geq v_p(t_h)+1 & \text{if } g<h,\\ = v_p(t_h) & \text{if } g=h,\\ \geq v_p(t_h) & \text{if } g>h,\end{cases}\tag{28}$$

we obtain $v_p(c_{T,\pi})\geq v_p(c_{T,\mathrm{id}})+1$ for all $T\subseteq[1,m]$ and all $\pi\in S_T$, $\pi\neq\mathrm{id}$. Hence,

$$c_m=c_{[1,m],\mathrm{id}}+\text{terms with } p\text{-adic value equal to or greater than } v_p(c_{[1,m],\mathrm{id}})+1,$$

which implies that

$$c_m\equiv\prod_{h=1}^{m}\beta_{hh}t_h \pmod{p^{v_p(c_m)+1}\mathbb{Z}_p}\quad\text{and}\quad v_p(c_m)=v_p\Bigl(\prod_{h=1}^{m}t_h\Bigr).\tag{29}$$

Since, moreover, $v_p(t_1)\geq v_p(t_i)+1$ for all $i=2,\dots,m$ we obtain from equation (28) that

- for any subset $T\subset[1,m]$, $T\neq[2,m]$, of cardinality $m-1$ and any $\pi\in S_T$ we have $v_p(c_{T,\pi})\geq v_p(c_{T,\mathrm{id}})\geq v_p(c_{[2,m],\mathrm{id}})+1$
- for any $\pi\in S_{[2,m]}$, $\pi\neq\mathrm{id}$, we have $v_p(c_{[2,m],\pi})\geq v_p(c_{[2,m],\mathrm{id}})+1$.

Hence,

$$c_{m-1}=c_{[2,m],\mathrm{id}}+\text{terms with } p\text{-adic value equal to or greater than } v_p(c_{[2,m],\mathrm{id}})+1,$$

which implies that

$$c_{m-1}\equiv\prod_{h=2}^{m}\beta_{hh}t_h \pmod{p^{v_p(c_{m-1})+1}\mathbb{Z}_p}\quad\text{and}\quad v_p(c_{m-1})=v_p\Bigl(\prod_{h=2}^{m}t_h\Bigr).\tag{30}$$

In particular,

$$v_p(c_m)-v_p(c_{m-1})=v_p(t_1).\tag{31}$$

Finally, for $i=1,\dots,m-1$ we denote by $T_{i,\min}\subset[1,m]$ a subset of cardinality i such that $v_p\bigl(\prod_{h\in T_{i,\min}}t_h\bigr)\leq v_p\bigl(\prod_{h\in T}t_h\bigr)$ for all subsets $T\subseteq[1,m]$ of cardinality i (thus, $T_{m-1,\min}=[2,m]$). As above equation (28) implies

$$v_p(c_i)\geq v_p\Bigl(\prod_{h\in T_{i,\min}}t_h\Bigr)=\sum_{h\in T_{i,\min}}v_p(t_h)\tag{32}$$

for all $i=1,\dots,m-1$. Since $v_p(t_1)>v_p(t_h)$ for all $h=2,\dots,m$ we obtain $T_{i,\min}\subseteq[2,m]$ and equations (30) and (32) imply that

$$v_p(c_{m-1})-v_p(c_i)\leq\sum_{h\in[2,m]-T_{i,\min}}v_p(t_h)\leq(m-1-i)r$$

for all $i = 1, \ldots, m-1$, where $r = \max_{h=2}^{m} v_p(t_h)$. Equivalently,

$$v_p(c_i) \geq v_p(c_{m-1}) - (m-1-i)r \tag{33}$$

for all $i = 1, \ldots, m-1$. Since $(m-1)r \geq \sum_{h=2}^{m} v_p(t_h) = v_p(c_{m-1})$ in view of equation (30), we see that equation (33) also holds for $i = 0$ ($v_p(c_0) = 0$). Thus:

- The line connecting the points $(m-1, v_p(c_{m-1}))$ and $(m, v_p(c_m))$ has slope $v_p(t_1)$; see equation (31).
- All points $(i, v_p(c_i))$ with $0 \leq i \leq m-1$ lie on or above the line g, which has slope r and passes through $(m-1, v_p(c_{m-1}))$; see equation (33).

Since r is strictly smaller than $v_p(t_1)$ this shows that the Newton polygon $\mathcal{N}$ of $\mathrm{ch}_{\beta t}$ has the segment connecting $(m, v_p(c_m))$ and $(m-1, v_p(c_{m-1}))$ as one of its sides while all other segments have slope less than or equal to r. We deduce that there is precisely one root $t_1' \in \bar{\mathbb{Q}}_p$ of $\mathrm{ch}_{\beta t}$ (counted with multiplicity) such that

$$v_p(t_1') = v_p(t_1),$$

while all remaining roots $t_h' \in \bar{\mathbb{Q}}_p$ of $\mathrm{ch}_{\beta t}$, $h = 2, \ldots, m$, have p-adic value smaller than or equal to r (in particular t_1' appears with multiplicity 1). Since $r = \max_{h=2}^{m} v_p(t_h) < v_p(t_1)$ we obtain $v_p(t_1') \geq v_p(t_h') + 1$ for all $h = 2, \ldots, m$. This implies that

$$c_m = \prod_{h=1}^{m} t_h' \quad \text{and} \quad c_{m-1} \equiv \prod_{h=2}^{m} t_h' \pmod{p^{v_p(c_{m-1})+1} \mathcal{O}_{\bar{\mathbb{Q}}_p}}$$

(note that $\mathrm{ch}_{\beta t}(X) = \prod_{h=1}^{m} (t_h' - X)$ because both sides have leading coefficient $(-1)^m$). Together with equations (29) and (30) we obtain

$$\prod_{h=1}^{m} \beta_{hh} t_h \equiv \prod_{h=1}^{m} t_h' \pmod{p^{v_p(c_m)+1} \mathbb{Z}_p}$$

and

$$\prod_{h=2}^{m} \beta_{hh} t_h \equiv \prod_{h=2}^{m} t_h' \pmod{p^{v_p(c_{m-1})+1} \mathcal{O}_{\bar{\mathbb{Q}}_p}}.$$

Since $v_p(c_m) - v_p(c_{m-1}) = v_p(t_1)$ the above two equations imply that $\beta_{11} t_1 \equiv t_1' \pmod{p^{v_p(t_1)+1} \mathcal{O}_{\bar{\mathbb{Q}}_p}}$ and the proof of the lemma is complete. □

4.4. The following proposition is an extension of 4.3 Lemma to closed subgroups of $\mathbf{GL}_m$ and is used in the proof of 4.7 Proposition. We denote by $W_{\tilde{G}}$ the Weyl group of $\tilde{\boldsymbol{T}}/F \leq \tilde{\boldsymbol{G}}/F$.

Proposition. *Let $t \in \tilde{\boldsymbol{T}}(\mathbb{Q}_p)^{++}$ and let $x \in \tilde{\mathcal{I}} t \tilde{\mathcal{I}}$. Then the semisimple part x_s of $x \in \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to a uniquely determined element $t' = t'_x \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$. The element t' satisfies*

$$v_p(\tilde{\lambda}(t')) = v_p(\tilde{\lambda}(t)) \quad \textit{and} \quad \tilde{\lambda}(t') \equiv \epsilon \tilde{\lambda}(t) \pmod{p^{v_p(\tilde{\lambda}(t))+1} \mathcal{O}_{\bar{\mathbb{Q}}_p}}$$

for all $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$, where $\epsilon = \epsilon_{\tilde{\lambda},x} \in \mathbb{Z}_p^*$ is a p-adic unit in $\mathbb{Z}_p$.

Proof. Conjugating x by an element in $\tilde{\mathcal{I}} \subseteq \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ we may assume that $x = \beta t$ with $\beta \in \tilde{\mathcal{I}}$. Since $\mathbb{Q}_p$ is a perfect field we know that the semisimple part $(\beta t)_s$ of $\beta t \in \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ also is contained in $\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ (see [Sp 1], 12.1.7 (c), p. 211). By 6.4.5 Theorem (ii) in [Sp 1], p. 109, $(\beta t)_s$ is contained in $\tilde{\boldsymbol{S}}(\bar{\mathbb{Q}}_p)$, where $\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{S}}_{\beta t}$ is a maximal $\bar{\mathbb{Q}}_p$-torus in $\tilde{\boldsymbol{G}}/\bar{\mathbb{Q}}_p$. Since all maximal tori in $\tilde{\boldsymbol{G}}/\bar{\mathbb{Q}}_p$ are conjugate over $\bar{\mathbb{Q}}_p$, there is $g \in \tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$ such that ${}^g\tilde{\boldsymbol{T}} := g\tilde{\boldsymbol{T}}g^{-1} = \tilde{\boldsymbol{S}}$; in particular,

$$t' := g^{-1}(\beta t)_s g \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p).$$

Conjugating further by some $w \in W_{\tilde{\boldsymbol{G}}}$ we may assume that $t' \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^+$. Thus, t' is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to $x_s = (\beta t)_s$ and we will show that it satisfies the conditions of the Proposition. To this end we let $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ and we first assume that $\tilde{\lambda}$ is dominant. We denote by $(\rho_{\tilde{\lambda}}, L_{\tilde{\lambda}})$ the irreducible representation of $\tilde{\boldsymbol{G}}/\mathbb{Z}_p$ of highest weight $\tilde{\lambda}$ (see Section 1.4 and 2.4). We select a basis $(v_1, v_2, \ldots, v_m)$ of $L_{\tilde{\lambda}}(\mathbb{Q}_p)$ consisting of weight vectors w.r.t. $\mathfrak{h}$ as in the proof of 2.10 Lemma, i.e., if μ_i denotes the weight of v_i then $\mathrm{ht}_\lambda(\mu_i) > \mathrm{ht}_\lambda(\mu_j)$ implies $i > j$. We note that v_i has weight $\tilde{\mu}_i = \mu_i^\circ \otimes \tilde{\lambda}|_{\tilde{\boldsymbol{Z}}}$ w.r.t. $\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$ (see equation (9) in Section 2.4). In particular, v_1 is the highest weight vector, i.e., $\tilde{\mu}_1 = \tilde{\lambda}$. The above choice of a basis of $L_{\tilde{\lambda}}$ yields a matrix representation

$$\rho_{\tilde{\lambda}} : \tilde{\boldsymbol{G}}(\mathbb{Q}_p) \to \mathbf{GL}_m(\mathbb{Q}_p)$$

and 2.10 Lemma implies that

$$\rho_{\tilde{\lambda}}(\tilde{\mathcal{I}}) \subseteq \mathcal{J}_m.$$

Moreover, we obtain

$$\rho_{\tilde{\lambda}}(t) = \mathrm{diag}(\tilde{\lambda}(t), \tilde{\mu}_2(t), \ldots, \tilde{\mu}_m(t)) \in \boldsymbol{T}_m(\mathbb{Q}_p)$$

and

$$\rho_{\tilde{\lambda}}(t') = \mathrm{diag}(\tilde{\lambda}(t'), \tilde{\mu}_2(t'), \ldots, \tilde{\mu}_m(t')) \in \boldsymbol{T}_m(\bar{\mathbb{Q}}_p).$$

Any weight μ_i, $i \geq 2$, has the form $\mu_i = \lambda - \sum_{\alpha \in \Delta} n_\alpha \alpha$ where not all $n_\alpha \in \mathbb{N}_0$ are equal to zero and since $t \in \tilde{\boldsymbol{T}}(\mathbb{Q}_p)^{++}$ (i.e., $v_p(\alpha(t)) > 0$ for all $\alpha \in \Delta$) we obtain

$$v_p(\tilde{\mu}_i(t)) = v_p(\tilde{\lambda}(t)) - \sum_{\alpha \in \Delta} n_\alpha v_p(\alpha(t)) < v_p(\tilde{\lambda}(t)) \tag{34}$$

for all weights $\tilde{\mu}_i$, $i = 2, \ldots, m$. Analogously, since $t' \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^+$ we obtain

$$v_p(\tilde{\mu}_i(t')) = v_p(\tilde{\lambda}(t')) - \sum_{\alpha\in\Delta} n_\alpha v_p(\alpha(t')) \leq v_p(\tilde{\lambda}(t')) \tag{35}$$

for all weights $\tilde{\mu}_i$, $i = 2, \dots, m$. Since $\rho_{\tilde{\lambda}}(\beta) \in \rho_{\tilde{\lambda}}(\tilde{\mathcal{I}}) \subseteq \mathcal{J}_m$ and $\rho_{\tilde{\lambda}}(t) \in \boldsymbol{T}_m(\mathbb{Q}_p)$ where the first entry of $\rho_{\tilde{\lambda}}(t)$ has p-adic value strictly bigger than the remaining entries (see equation (34)) we may apply 4.3 Lemma to $\rho_{\tilde{\lambda}}(\beta t) = \rho_{\tilde{\lambda}}(\beta)\rho_{\tilde{\lambda}}(t) \in \mathbf{GL}_m(\mathbb{Q}_p)$; we obtain that $\rho_{\tilde{\lambda}}(\beta t)$ has an eigenvalue $t_1' \in \bar{\mathbb{Q}}_p$ of multiplicity 1 whose p-adic value is strictly larger than the p-adic values of the $m-1$ remaining eigenvalues of $\rho_{\tilde{\lambda}}(\beta t)$ and which satisfies

$$v_p(\tilde{\lambda}(t)) = v_p(t_1') \quad \text{and} \quad \tilde{\lambda}(t) \equiv \epsilon t_1' \pmod{p^{v_p(\tilde{\lambda}(t))+1}\mathcal{O}_{\bar{\mathbb{Q}}_p}} \tag{36}$$

for some $\epsilon = \epsilon_{\tilde{\lambda},\beta} \in \mathbb{Z}_p^*$. Since the matrix $\rho_{\tilde{\lambda}}(t')$ has the same eigenvalues as $\rho_{\tilde{\lambda}}((\beta t)_s) = (\rho_{\tilde{\lambda}}(\beta t))_s$ it has the same eigenvalues as $\rho_{\tilde{\lambda}}(\beta t)$. In particular, $\rho_{\tilde{\lambda}}(t')$ has the eigenvalue t_1' with multiplicity 1 whose p-adic value is strictly bigger than the p-adic values of the $m-1$ remaining eigenvalues of $\rho_{\tilde{\lambda}}(t')$. Thus, equation (35) shows that $t_1' = \tilde{\lambda}(t')$ and equation (36) yields

$$v_p(\tilde{\lambda}(t)) = v_p(\tilde{\lambda}(t')) \quad \text{and} \quad \tilde{\lambda}(t) \equiv \epsilon\tilde{\lambda}(t') \pmod{p^{v_p(\tilde{\lambda}(t))+1}\mathcal{O}_{\bar{\mathbb{Q}}_p}}. \tag{37}$$

We recall that we have proven (37) under the assumption that $\tilde{\lambda}$ is dominant. For the general case we will need the following consequence of (37). Let $\tilde{\lambda}$ be dominant and write $\tilde{\lambda} = \tilde{\lambda}|_{\tilde{\boldsymbol{Z}}}\lambda^\circ$ where $\lambda^\circ = \tilde{\lambda}|_{\boldsymbol{T}}$. We write $t = t^\circ z$ where $t^\circ \in \boldsymbol{T}(\bar{\mathbb{Q}}_p)$, $z \in \tilde{\boldsymbol{Z}}(\bar{\mathbb{Q}}_p)$, hence, $t' = t'^\circ z$ where $t'^\circ = g^{-1}(\beta t^\circ)_s g$. Equation (37) then implies that

$$1 \equiv \epsilon \frac{\lambda^\circ(t'^\circ)}{\lambda^\circ(t^\circ)} \pmod{p\mathcal{O}_{\bar{\mathbb{Q}}_p}}. \tag{38}$$

Since $\boldsymbol{T} \subseteq \tilde{\boldsymbol{T}}$ is a closed subset the restriction map $X(\tilde{\boldsymbol{T}}) \to X(\boldsymbol{T})$ is surjective, hence, any dominant character $\lambda^\circ \in X(\boldsymbol{T})$ is the restriction of a dominant character $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ and we deduce that equation (38) holds for all dominant $\lambda^\circ \in X(\boldsymbol{T})$.

We now let $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ be arbitrary and we show that equation (37) still holds. We write $\tilde{\lambda} = \tilde{\lambda}|_{\tilde{\boldsymbol{Z}}}\lambda^\circ$ where $\lambda^\circ = \tilde{\lambda}|_{\boldsymbol{T}}$. Applying 4.4.1 Lemma below to $\lambda = \mathrm{d}\,\lambda^\circ \in \Gamma_\pi$ we can write $\lambda^\circ = \prod_i (\mu_i^\circ)^{n_i}$, where $\mu_i^\circ \in X(\boldsymbol{T})$ is dominant and $n_i \in \mathbb{Z}$. Equation (38) implies that

$$1 \equiv \prod_i \epsilon_i^{n_i} \frac{\prod_i \mu_i^\circ(t'^\circ)^{n_i}}{\prod_i \mu_i^\circ(t^\circ)^{n_i}} \pmod{p\mathcal{O}_{\bar{\mathbb{Q}}_p}}$$

with certain $\epsilon_i \in \mathbb{Z}_p^*$. Multiplying this by $\tilde{\lambda}(t) = \tilde{\lambda}(z)\lambda^\circ(t^\circ)$ we finally obtain

$$\tilde{\lambda}(t) \equiv \prod_i \epsilon_i^{n_i}\,\tilde{\lambda}(t') \pmod{p^{v_p(\tilde{\lambda}(t))+1}\mathcal{O}_{\bar{\mathbb{Q}}_p}}.$$

The last equation also implies $v_p(\tilde{\lambda}(t)) = v_p(\tilde{\lambda}(t'))$. Since t was strictly dominant this shows in particular that $t' \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$.

It only remains to prove uniqueness of t'. Let $t'' \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$ be another element which is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to $x_s = (\beta t)_s$. Hence, $t'' = gt'g^{-1}$ for some $g \in \tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$. Since t', t'' are regular we know that $\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$ is the centralizer of t' and of t''. This implies $g\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)g^{-1} = \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$, hence, g yields an element in $W_{\tilde{\boldsymbol{G}}}$. Since t', t'' are both strictly dominant g is the unit element in $W_{\tilde{\boldsymbol{G}}}$, i.e., $g \in \mathcal{C}(\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p))$ (centralizer of $\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$ in $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$), hence, $t'' = t'$. Thus, the proposition is proven. □

4.4.1. *Lemma. The lattice $\Gamma_\pi \subseteq \mathfrak{h}^*$ is generated by dominant weights.*

Proof. We write the representation π of $\mathfrak{g}$ defining $\boldsymbol{G} = \boldsymbol{G}_\pi$ as $\pi = \bigoplus_{i=1}^n \rho_{\lambda_i}$ with dominant weights $\lambda_i \in \Gamma_\pi$. Since $P_\pi = \bigcup_i P_{\lambda_i}$ we obtain

$$\Gamma_\pi = \langle P_\pi \rangle = \left\langle \bigcup_i P_{\lambda_i} \right\rangle \subseteq \left\langle \bigcup_i \lambda_i + \Gamma_{\mathrm{ad}} \right\rangle \subseteq \Gamma_\pi;$$

hence, $\Gamma_\pi = \langle \bigcup_i \lambda_i + \Gamma_{\mathrm{ad}} \rangle$ (for the last inclusion note that π is faithful; hence, $\Gamma_{\mathrm{ad}} \subseteq \Gamma_\pi$). We select a weight $\gamma \in \Gamma_{\mathrm{ad}}$ which for the moment is arbitrary and we put $\mu_i = \lambda_i + \gamma$, $i = 1, \dots, n$. Since Γ_{ad} is generated by the simple roots we obtain

$$\Gamma_\pi = \left\langle \bigcup_i \mu_i + \Gamma_{\mathrm{ad}} \right\rangle = \langle \mu_i, \mu_i - \alpha,\ i = 1, \dots, n,\ \alpha \in \Delta \rangle.$$

If we choose the weight $\gamma \in \Gamma_{\mathrm{ad}}$ dominant and sufficiently regular, i.e., $\langle \gamma, h_\beta \rangle > 0$ is positive and sufficiently large for all $\beta \in \Delta$ then μ_i and $\mu_i - \alpha$ are dominant for all $i = 1, \dots, n$ and all $\alpha \in \Delta$. This implies that Γ_π is generated by dominant weights. □

4.5. We denote by ${}^w\chi$ or sometimes by $w\chi$, $\chi \in X(\tilde{\boldsymbol{T}})$, $w \in W_{\tilde{\boldsymbol{G}}}$, the character sending t to $\chi(w^{-1}tw)$. We write $\rho = \rho_{\tilde{\boldsymbol{G}}} \in \Gamma_{\mathrm{sc}}$ for the half sum of the positive roots and we put

$$w \cdot \tilde{\lambda} = w(\tilde{\lambda} + \rho^\circ) - \rho^\circ \in X(\tilde{\boldsymbol{T}}) \quad \text{for } \tilde{\lambda} \in X(\tilde{\boldsymbol{T}}),$$

with $\rho^\circ = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha \in X(\tilde{\boldsymbol{T}}) \otimes \mathbb{Q}$, where $\alpha = \alpha^\circ \in X(\tilde{\boldsymbol{T}})$ is the exponential of the root α; see Section 1.2.

Lemma. *Let $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ be a dominant weight. For any $w \in W_{\tilde{\boldsymbol{G}}}$, $w \neq 1$, and any $t \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$ we have $w\tilde{\lambda}(t) = \tilde{\lambda}(t)\left(\sum_{\alpha\in\Delta} -b_\alpha\alpha\right)(t)$, where $b_\alpha \in \mathbb{N}_0$ and*

$$b_{\alpha_0} \geq \frac{\langle \tilde{\lambda}, \alpha_0^\vee \rangle}{2}$$

for at least one root $\alpha_0 \in \Delta$. Also, if $w \neq 1$ we have $w \cdot \tilde{\lambda}(t) = \tilde{\lambda}(t)\left(\sum_{\alpha\in\Delta} -b_\alpha\alpha\right)(t)$, where $b_\alpha \in \mathbb{N}_0$ and

$$b_{\alpha_0} \geq \frac{\langle \tilde{\lambda}, \alpha_0^\vee \rangle}{2}$$

for at least one root $\alpha_0 \in \Delta$.

Proof. We prove the claim about $w\cdot\tilde\lambda$. We write $\tilde\lambda=\tilde\lambda|_{\tilde{\boldsymbol{Z}}}\lambda^\circ$, where $\lambda^\circ=\tilde\lambda|_{\boldsymbol{T}}$ and $t=zt^\circ\in\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$, where $z\in\tilde{\boldsymbol{Z}}(\bar{\mathbb{Q}}_p)$ and $t^\circ\in\boldsymbol{T}(\bar{\mathbb{Q}}_p)$. Since $w\rho^\circ-\rho^\circ\in\sum_{\alpha\in\Delta}\mathbb{Z}\alpha$ we obtain $w\cdot\tilde\lambda(z)=(w\tilde\lambda+(w\rho^\circ-\rho^\circ))(z)=\tilde\lambda(z)$. Since $\tilde\lambda=\lambda^\circ$ on $\boldsymbol{T}$ we obtain $w\cdot\tilde\lambda(t^\circ)=w\cdot\lambda^\circ(t^\circ)$. To determine $w\cdot\lambda^\circ(t^\circ)$ we set as before $\lambda=d\,\lambda^\circ\in\Gamma_\pi$, i.e., λ° corresponds to λ under the isomorphism $(\,\cdot\,)^\circ:\Gamma_\pi\to X(\boldsymbol{T})$ (see Section 1.2). The Weyl group $W_{\tilde{\boldsymbol{G}}}$ acts on $\Gamma_\pi\subseteq\mathfrak{h}^*$ via $\lambda\mapsto\lambda\circ\mathrm{Ad}(w^{-1})$, $w\in W_{\tilde{\boldsymbol{G}}}$, and since $(\,\cdot\,)^\circ$ is equivariant w.r.t. the action of $W_{\boldsymbol{G}}$ (see [B], Section 3.3, Remarks (1), p. 16) we obtain $w\cdot\lambda^\circ=(w\cdot\lambda)^\circ$, where $w\cdot\lambda:=w(\lambda+\rho)-\rho$. Since $w\lambda$ is a weight of the irreducible $\mathfrak{g}$-module of highest weight λ we know that

$$w\lambda=\lambda-\sum_{\alpha\in\Delta}c_\alpha\alpha$$

for certain $c_\alpha\in\mathbb{N}_0$. Since λ is a dominant element in the weight lattice Γ_{sc} we may write $\lambda=\sum_{\alpha\in\Delta}d_\alpha\omega_\alpha$, where ω_α, $\alpha\in\Delta$, are the fundamental weights and $d_\alpha\in\mathbb{N}_0$. On the other hand, $w\neq1$ implies that $w\lambda$ is not contained in the Weyl chamber corresponding to the basis Δ, hence, $\langle w\lambda,h_{\alpha_0}\rangle\le0$ for some root $\alpha_0\in\Delta$. We obtain

$$0\ge\langle w\lambda,h_{\alpha_0}\rangle=\Big\langle\sum_{\alpha\in\Delta}d_\alpha\omega_\alpha-\sum_{\alpha\in\Delta}c_\alpha\alpha,\,h_{\alpha_0}\Big\rangle=d_{\alpha_0}-\sum_{\alpha\in\Delta}c_\alpha\langle\alpha,h_{\alpha_0}\rangle.$$

Since $\langle\alpha,h_{\alpha_0}\rangle=2$ if $\alpha=\alpha_0$ and $\langle\alpha,h_{\alpha_0}\rangle\le0$ if $\alpha\neq\alpha_0$ this yields $0\ge d_{\alpha_0}-2c_{\alpha_0}$. Thus,

$$c_{\alpha_0}\ge\tfrac12 d_{\alpha_0}=\tfrac12\langle\lambda,h_{\alpha_0}\rangle=\tfrac12\langle\lambda^\circ,\alpha_0^\vee\rangle=\tfrac12\langle\tilde\lambda,\alpha_0^\vee\rangle.$$

Altogether we obtain

$$\begin{aligned}(w\cdot\tilde\lambda)(t)&=\tilde\lambda(z)\,(w\cdot\lambda^\circ)(t^\circ)=\tilde\lambda(z)\,(w\cdot\lambda)^\circ(t^\circ)=\tilde\lambda(z)(w\lambda+w\rho-\rho)^\circ(t^\circ)\\&=\tilde\lambda(z)\big(\lambda+\sum_{\alpha\in\Delta}-c_\alpha\alpha+w\rho-\rho\big)^\circ(t^\circ)=\tilde\lambda(t)\big(\sum_{\alpha\in\Delta}-c_\alpha\alpha+w\rho-\rho\big)^\circ(t^\circ).\end{aligned}$$

Since $w\rho-\rho\in\mathbb{Z}\Phi$ is a sum of negative roots, this shows that $w\cdot\tilde\lambda(t)$ has the claimed form (note that $\alpha(t)=\alpha(t^\circ)$ since $\alpha=\alpha^\circ$ vanishes on $\tilde{\boldsymbol{Z}}(\bar{\mathbb{Q}}_p)$). The claim about $w\tilde\lambda$ follows analogously. □

4.6. ***Notation.*** We recall that in Section 2.3 we selected an element $h\in\tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$. We set

$$\kappa_1=\kappa_{1,\tilde{\boldsymbol{G}}}(h)=\sum_{\alpha\in\Phi^+_{\tilde{\boldsymbol{G}}}}v_p(\alpha(h))\in\mathbb{N}.$$

Thus, κ_1 depends on $\tilde{\boldsymbol{G}}$ and h. Since $\rho-{}^w\rho$, $w\in W_{\tilde{\boldsymbol{G}}}$, is a sum of certain positive roots all of which occur with multiplicity 1 we obtain

$$v_p((\rho^\circ-{}^w\rho^\circ)(h^e))\le e\kappa_1.$$

We write $2\rho = 2\rho_{\tilde{\boldsymbol{G}}} = \sum_{\beta\in\Delta_{\tilde{\boldsymbol{G}}}} m_\beta\beta$, where $m_\beta \in \mathbb{N}_0$ for all $\beta \in \Delta_{\tilde{\boldsymbol{G}}}$, and we set

$$\kappa_2 = \kappa_{2,\tilde{\boldsymbol{G}}} = \max_{\beta\in\Delta_{\tilde{\boldsymbol{G}}}} m_\beta \in \mathbb{N}_0;$$

i.e., κ_2 is the maximum multiplicity with which a simple root can occur in $2\rho_{\tilde{\boldsymbol{G}}}$ and only depends on $\tilde{\boldsymbol{G}}$. Since $\rho - {}^w\rho$ is a sum of certain positive roots each of which occurs with multiplicity 1 we can write $\rho - {}^w\rho = \sum_{\beta\in\Delta_{\tilde{\boldsymbol{G}}}} n_\beta\beta$ where $n_\beta \in \mathbb{N}_0$ and $n_\beta \le m_\beta$ for all $\beta \in \Delta_{\tilde{\boldsymbol{G}}}$. Since $\langle\beta, \alpha^\vee\rangle$, $\alpha, \beta \in \Delta_{\tilde{\boldsymbol{G}}}$, equals 2 if $\alpha = \beta$ and is ≤ 0 otherwise we obtain for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$

$$\langle \rho - {}^w\rho, \alpha^\vee\rangle \le 2n_\alpha \le 2m_\alpha \le 2\kappa_2.$$

Let $\tilde{\boldsymbol{P}}/\mathbb{Q}_p \le \tilde{\boldsymbol{G}}/\mathbb{Q}_p$ be a standard parabolic subgroup (i.e., $\tilde{\boldsymbol{P}}/\mathbb{Q}_p \supseteq \tilde{\boldsymbol{B}}/\mathbb{Q}_p$) with Levi decomposition $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{N}}$. The Levi subgroup $\tilde{\boldsymbol{M}}$ contains the $\mathbb{Q}_p$-split maximal torus $\tilde{\boldsymbol{T}}$ and we denote by $W^{\tilde{P}}$ the set of Kostant representatives for the quotient of Weyl groups $W_{\tilde{\boldsymbol{M}}}\backslash W_{\tilde{\boldsymbol{G}}}$. The intersection $\tilde{\boldsymbol{B}} \cap \tilde{\boldsymbol{M}} \le \tilde{\boldsymbol{M}}/\mathbb{Q}_p$ is a Borel subgroup in $\tilde{\boldsymbol{M}}/\mathbb{Q}_p$ and for any dominant (w.r.t. to $\tilde{\boldsymbol{B}}\cap\tilde{\boldsymbol{M}}$) weight $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$ we then denote by $\rho^{\tilde{M}}_{\tilde{\lambda}} : \tilde{\boldsymbol{M}} \to \mathbf{Aut}(L^{\tilde{M}}_{\tilde{\lambda}})$ the irreducible representation of $\tilde{\boldsymbol{M}}/\mathbb{Q}_p$ of highest weight $\tilde{\lambda}$ (see Section 2.4). Any $p \in \tilde{\boldsymbol{P}}(\bar{\mathbb{Q}}_p)$ can be written $p = \bar{p}u$, $\bar{p} \in \tilde{\boldsymbol{M}}(\bar{\mathbb{Q}}_p)$, $u \in \tilde{\boldsymbol{N}}(\bar{\mathbb{Q}}_p)$ and we denote by $\nu_{\tilde{\boldsymbol{P}}} : \tilde{\boldsymbol{P}}/\mathbb{Q}_p \to \tilde{\boldsymbol{M}}/\mathbb{Q}_p$, $p \mapsto \bar{p}$, the morphism to the Levi subgroup.

In the next proposition we will use the following notation: for $c \in \mathbb{Q}$ and $x, y \in \bar{\mathbb{Q}}_p$ we write $x \equiv y \pmod{p^c\mathcal{O}_{\bar{\mathbb{Q}}_p}}$ to denote that $v_p(x-y) \ge c$. Thus, in case $c \in \mathbb{Z}$ the term "$x \equiv y \pmod{p^c\mathcal{O}_{\bar{\mathbb{Q}}_p}}$" has two meanings that coincide.

4.7. Proposition. *Let $C \in \mathbb{Q}_{>0}$ and assume that $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ satisfies*

$$\langle\tilde{\lambda}, \alpha^\vee\rangle > 2C$$

for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$. Select a standard parabolic subgroup $\tilde{\boldsymbol{P}}/\mathbb{Q}_p \le \tilde{\boldsymbol{G}}/\mathbb{Q}_p$ with Levi decomposition $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{N}}$ and let $\xi \in \tilde{\boldsymbol{M}}(\mathbb{Q}_p)$. Assume that there is $u \in \tilde{\boldsymbol{N}}(\mathbb{Q}_p)$ such that $\xi u \in \tilde{\boldsymbol{P}}(\mathbb{Q}_p)$ is $\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$-conjugate to an element in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}}$, $e \in \mathbb{N}$. We denote by $t = t_{\xi u}$ the unique element in $\tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$ which is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to $(\xi u)_s$ (see 4.4 Proposition; note that ξu is conjugate to an element $x \in \tilde{\mathcal{I}}h^e\tilde{\mathcal{I}}$). *Then there is an element $s = s_{\xi u} \in W^{\tilde{P}}$ such that for all $w \in W^{\tilde{P}}$ the following congruence holds*:

$$\tilde{\lambda}(h^e)\,\mathrm{tr}\,(\xi^{-1}|L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) \equiv \begin{cases} \varepsilon_{\tilde{\boldsymbol{P}}}\,\tilde{\lambda}(h^e t^{-1}) \pmod{p^{(C-\kappa_1-\kappa_2)e}\mathcal{O}_{\bar{\mathbb{Q}}_p}} & \textit{if } w = s, \\ 0 \pmod{p^{(C-\kappa_1-\kappa_2)e}\mathcal{O}_{\bar{\mathbb{Q}}_p}} & \textit{if } w \ne s. \end{cases}$$

Here, $\varepsilon_{\tilde{\boldsymbol{P}}} = \varepsilon_{\tilde{\boldsymbol{P}},\xi u}$ is an element in $\bar{\mathbb{Q}}_p$ which does not depend on $\tilde{\lambda}$ and satisfies

$$v_p(\varepsilon_{\tilde{\boldsymbol{P}},\xi u}) \ge -e\kappa_1.$$

Proof. By 12.1.7(c) (p. 211) of [Springer 1981], the semisimple part $(\xi u)_s$ is contained in $\tilde{\boldsymbol{P}}(\mathbb{Q}_p)$, and Theorem 6.4.5(ii) (p. 109) of the same reference shows that it is contained in a maximal $\bar{\mathbb{Q}}_p$-torus $\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{S}}_{\xi u}$ in $\tilde{\boldsymbol{P}}/\bar{\mathbb{Q}}_p$. Since all maximal tori in $\tilde{\boldsymbol{P}}/\bar{\mathbb{Q}}_p$ are conjugate there is $y \in \tilde{\boldsymbol{P}}(\bar{\mathbb{Q}}_p)$ such that $\tilde{\boldsymbol{S}} = {}^y\tilde{\boldsymbol{T}}$. In particular there is $t' \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$ such that

$$(\xi u)_s = {}^y t'.$$

Modifying y by an element in $W_{\tilde{\boldsymbol{M}}}$ we may even assume that t' is dominant w.r.t. $\Delta_{\tilde{\boldsymbol{M}}}$, i.e., $v_p(\alpha(t')) \geq 0$ for all $\alpha \in \Delta_{\tilde{\boldsymbol{M}}}$.

Since ξu is conjugate to an element $x \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ the semisimple part $(\xi u)_s$ is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to an element $t = t_{\xi u} \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$ satisfying

$$v_p(\chi(t)) = v_p(\chi(h^e)) \quad \text{and} \quad \chi(t) \equiv \epsilon \chi(h^e) \pmod{p^{v_p(\chi(t))+1}\mathcal{O}_{\bar{\mathbb{Q}}_p}} \tag{39}$$

for all $\chi \in X(\tilde{\boldsymbol{T}})$ where $\epsilon = \epsilon_\chi \in \mathbb{Z}_p^*$ (see 4.4 Proposition). Since t' also is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to $(\xi u)_s$ we find that $t, t' \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)$ are conjugate by an element $s = s_{\xi u} \in \tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$:

$$t' = {}^s t.$$

Since t is regular, t' also is regular, and it follows that $\tilde{\boldsymbol{T}} = C(t)^0 = C(t')^0$. This implies ${}^s\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{T}}$; hence, s is contained in the normalizer of $\tilde{\boldsymbol{T}}$ and we therefore can select $s \in W_{\tilde{\boldsymbol{G}}}$ (i.e., s is representative of an element in $W_{\tilde{\boldsymbol{G}}}$). Since t' is dominant w.r.t. $\Delta_{\tilde{\boldsymbol{M}}}$ and t is strictly dominant w.r.t. $\Delta_{\tilde{\boldsymbol{G}}}$ we see that s^{-1} maps $\Phi^+_{\tilde{\boldsymbol{M}}}$ to $\Phi^+_{\tilde{\boldsymbol{G}}}$, which implies $s \in W^{\tilde{P}}$. Denote by $L^{\tilde{P}}_{w\cdot\tilde{\lambda}}$ the extension of the representation $L^{\tilde{M}}_{w\cdot\tilde{\lambda}}$ to $\tilde{\boldsymbol{P}}/\mathbb{Q}_p$ via the morphism $\nu_{\tilde{\boldsymbol{P}}} : \tilde{\boldsymbol{P}} \to \tilde{\boldsymbol{M}}$. We have obtained

$$\begin{aligned}
\operatorname{tr}(\xi^{-1} | L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) &= \operatorname{tr}(\xi^{-1} | L^{\tilde{P}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) = \operatorname{tr}((\xi u)^{-1} | L^{\tilde{P}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) \\
&= \operatorname{tr}((\xi u)_s^{-1} | L^{\tilde{P}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) = \operatorname{tr}((t')^{-1}) | L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) \\
&= \operatorname{tr}({}^s t^{-1}) | L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)).
\end{aligned}$$

The Weyl character formula (see [Popov and Vinberg 1994, I.4.6.4 Theorem, p. 45] or [Jantzen 2003, II.5.10 Proposition, p. 223]) then yields

$$\tilde{\lambda}(h^e) \operatorname{tr}(\xi^{-1} | L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p)) = \tilde{\lambda}(h^e) \frac{\sum_{v \in W_{\tilde{M}}} (-1)^{\ell(v)}\, v \cdot_{\tilde{M}} (w \cdot \tilde{\lambda})({}^s t^{-1})}{\prod_{\alpha \in \Phi^+_{\tilde{M}}} (1 - \alpha^{-1}(\underbrace{{}^s t^{-1}}_{= t'^{-1}}))}. \tag{40}$$

Here, we use the notation $v \cdot_{\tilde{M}} \tilde{\mu} = v(\tilde{\mu} + \rho^\circ_{\tilde{\boldsymbol{M}}}) - \rho^\circ_{\tilde{\boldsymbol{M}}}$. We denote the denominator appearing on the right-hand side of (40) by

$$N(t') = N_{\tilde{\boldsymbol{M}}}(t').$$

For all $\alpha \in \Phi_{\tilde{G}}$ equation (39) implies that $v_p(\alpha(t)) \geq e$ or $v_p(\alpha(t)) \leq -e$. Since $\alpha({}^s t) = (s^{-1}\alpha)(t)$ the same then is true of $t' = {}^s t$, i.e., $v_p(\alpha(t')) \geq e$ or $v_p(\alpha(t')) \leq -e$ for all $\alpha \in \Phi_{\tilde{G}}$. Since $v_p(\alpha(t')) \geq 0$ for all $\alpha \in \Phi^+_{\tilde{M}}$ we obtain

$$v_p(\alpha(t')) \geq e \quad \text{for all } \alpha \in \Phi^+_{\tilde{M}}. \tag{41}$$

Hence,

$$v_p(N(t')) = v_p\Bigl(\prod_{\alpha \in \Phi^+_{\tilde{M}}} (1 - \alpha(t'))\Bigr) = 0,$$

i.e., $N(t')$ is a p-adic unit in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$; in particular, $N(t') \neq 0$.

We look at the individual summands indexed by $v \in W_{\tilde{M}}$ which appear in the numerator in equation (40) and distinguish cases.

4.7.1. We first assume that $v \neq 1$. Using 4.5 Lemma with $\tilde{M}$ in place of $\tilde{G}$ and the definition of $w \cdot \tilde{\lambda}$ we can write

$$\begin{aligned} &\tilde{\lambda}(h^e)\, v \cdot_{\tilde{M}} (w \cdot \tilde{\lambda})({}^s t^{-1}) \\ &\qquad = \tilde{\lambda}(h^e)\, (w \cdot \tilde{\lambda})({}^s t^{-1}) \Bigl(\sum_{\alpha \in \Delta_{\tilde{M}}} -b_{v,\alpha}\alpha\Bigr)({}^s t^{-1}) \\ &\qquad = \tilde{\lambda}(h^e)(s^{-1}w\tilde{\lambda})(t^{-1})(w\rho^\circ - \rho^\circ)({}^s t^{-1}) \Bigl(\sum_{\alpha \in \Delta_{\tilde{M}}} b_{v,\alpha}\alpha\Bigr)({}^s t) \end{aligned} \tag{42}$$

with $b_{v,\alpha} \in \mathbb{N}_0$ and $b_{v,\alpha_v} \geq \frac{1}{2}\langle w \cdot \tilde{\lambda}, \alpha_v^\vee \rangle$ for (at least) one root $\alpha_v \in \Delta_{\tilde{M}}$ (note that $w \cdot \tilde{\lambda}$ is dominant for $\Delta_{\tilde{M}}$ since $w \in W^{\tilde{P}}$). Since $w \in W^{\tilde{P}}$ we know that ${}^{w^{-1}}\alpha_v \in \Phi^+_{\tilde{G}}$, hence, we obtain

$$b_{v,\alpha_v} \geq \frac{\langle w \cdot \tilde{\lambda}, \alpha_v^\vee \rangle}{2} = \frac{\langle \tilde{\lambda}, {}^{w^{-1}}\alpha_v^\vee \rangle}{2} + \frac{\langle {}^w\rho^\circ - \rho^\circ, \alpha_v^\vee \rangle}{2} \geq C - \kappa_2.$$

Equation (41) implies that

$$v_p\Bigl(\Bigl(\sum_{\alpha \in \Delta_{\tilde{M}}} b_{v,\alpha}\alpha\Bigr)({}^s t)\Bigr) \geq (C - \kappa_2)e. \tag{43}$$

Since $w\rho - \rho$ is a sum of certain negative roots all appearing with multiplicity 1 and since t is strictly dominant we obtain using equation (39)

$$v_p((w\rho^\circ - \rho^\circ)({}^s t^{-1})) \geq -\sum_{\alpha \in \Phi^+_{\tilde{G}}} v_p(\alpha(t)) = -\sum_{\alpha \in \Phi^+_{\tilde{G}}} v_p(\alpha(h^e)) = -e\kappa_1. \tag{44}$$

If $s^{-1}w \neq 1$ then 4.5 Lemma yields

$$\tilde{\lambda}(h^e)(s^{-1}w\tilde{\lambda})(t^{-1}) = \tilde{\lambda}(h^e t^{-1}) \Bigl(\sum_{\alpha \in \Delta_{\tilde{G}}} -c_{w,\alpha}\alpha\Bigr)(t^{-1}),$$

where $c_{w,\alpha} \in \mathbb{N}_0$ and $c_{w,\alpha_w} \geq \frac{1}{2}\langle \tilde{\lambda}, \alpha_w^\vee \rangle \geq C$ for (at least) one root $\alpha_w \in \Delta_{\tilde{G}}$. It follows from (39) that $\tilde\lambda(h^e)\tilde\lambda(t^{-1})$ is a p-adic unit and that $v_p(\alpha(t)) = v_p(\alpha(h^e)) \geq e$ for all $\alpha \in \Delta_{\tilde{G}}$; hence,

$$v_p(\tilde\lambda(h^e)(s^{-1}w\tilde\lambda)(t^{-1})) = v_p\left(\left(\sum_{\alpha\in\Delta_{\tilde{G}}} c_{w,\alpha}\alpha\right)(t)\right) \geq Ce\,(>0). \tag{45}$$

If $s = w$ then $\tilde\lambda(h^e)(s^{-1}w\tilde\lambda)(t^{-1}) = \tilde\lambda(h^e t^{-1})$ is a p-adic integer. Thus, if $v \neq 1$ equations (42), (43), (44), (45) yield

$$v_p\big(\tilde\lambda(h^e)\, v \cdot_{\tilde{M}} (w\cdot\tilde\lambda)({}^s t^{-1})\big) \geq (C-\kappa_1-\kappa_2)e \tag{46}$$

for all $w \in W^{\tilde{P}}$. Hence, modulo $p^{(C-\kappa_1-\kappa_2)e}\mathcal{O}_{\bar{\mathbb{Q}}_p}$ we may neglect all summands with $v \neq 1$.

4.7.2. We assume $v = 1$. If $w \neq s$ equations (42), (44), (45) yield

$$\begin{aligned} v_p\big(\tilde\lambda(h^e)\, v \cdot_{\tilde{M}} (w\cdot\tilde\lambda)({}^s t^{-1})\big) &= v_p(\tilde\lambda(h^e)(s^{-1}w\tilde\lambda)(t^{-1})(w\rho^\circ-\rho^\circ)({}^s t^{-1})) \\ &\geq (C-\kappa_1)e. \end{aligned} \tag{47}$$

If $w = s$ we obtain as above

$$\tilde\lambda(h^e)\, v \cdot_{\tilde{M}} (w\cdot\tilde\lambda)({}^s t^{-1}) = \tilde\lambda(h^e)\tilde\lambda(t^{-1})(s\rho^\circ-\rho^\circ)({}^s t^{-1}). \tag{48}$$

4.7.3. Taking into account that $C-\kappa_1$ is bigger than or equal to $C-\kappa_1-\kappa_2$, equations (40) and (46), (47), (48) now yield (note that $N(t')$ is a p-adic unit)

$$\begin{aligned} &\tilde\lambda(h^e)\,\mathrm{tr}\,(\xi^{-1}|L^{\tilde{M}}_{w\cdot\tilde\lambda}(\mathbb{Q}_p)) \\ &\qquad\equiv \begin{cases} \dfrac{\tilde\lambda(h^e)\tilde\lambda(t^{-1})}{N(t')}(s\rho^\circ-\rho^\circ)({}^s t^{-1}) \pmod{p^{(C-\kappa_1-\kappa_2)e}\mathcal{O}_{\bar{\mathbb{Q}}_p}} & \text{if} w = s, \\ 0 \pmod{p^{(C-\kappa_1-\kappa_2)e}\mathcal{O}_{\bar{\mathbb{Q}}_p}} & \text{if} w \neq s. \end{cases} \end{aligned}$$

We put $\varepsilon_{\tilde{P}} = \varepsilon_{\tilde{P},\xi u} = \dfrac{(s\rho^\circ-\rho^\circ)({}^s t^{-1})}{N(t')} \in \bar{\mathbb{Q}}_p$. Since $N(t')$ is a p-adic unit, equation (44) shows that

$$v_p(\varepsilon_{\tilde{P},\xi u}) \geq -e\kappa_1.$$

This completes the proof of the proposition. □

4.8. We look at the special case $\tilde{P} = \tilde{G}$ in 4.7 Proposition which is sufficient for application to Bewersdorff's trace formula. (The general case will be needed in application to the Goresky–MacPherson trace formula which involves contributions from parabolic subgroups of $\tilde{G}$ as well; see Section 4.14). In this case, $\xi \in \tilde{G}(\mathbb{Q}_p)$, $u = 1$, $W^{\tilde{P}} = 1$ and $s = w = 1$; hence, $t = t_\xi = t'$. In particular, $\varepsilon_{\tilde{P}} = \varepsilon_{\tilde{G},1,\xi} = 1/N(t_\xi)$ is a p-adic integer. Moreover, we can choose $\kappa_1 = \kappa_2 = 0$ since then the equations involving ρ and κ_1, κ_2 in Section 4.6 still hold (note that $w = 1$; see also equation (44) and the equation following (42)). We thus obtain:

Corollary. *Let $C \in \mathbb{Q}_{>0}$ and assume that $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ satisfies*

$$\langle \tilde{\lambda}, \alpha^\vee \rangle > 2C$$

for all $\alpha \in \Delta$. Then for any $\xi \in \tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ which is $\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$-conjugate to an element in $\tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$, $e \in \mathbb{N}$, the following congruence holds:

$$\tilde{\lambda}(h^e)\,\mathrm{tr}\,(\xi^{-1} | L_{\tilde{\lambda}}(\mathbb{Q}_p)) \equiv \frac{\tilde{\lambda}(h^e t_\xi^{-1})}{N(t_\xi)} \pmod{p^{Ce} \mathcal{O}_{\bar{\mathbb{Q}}_p}}.$$

Here, $t_\xi \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$ denotes the unique element which is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to ξ_s (see 4.4 Proposition) *and*

$$N(t_\xi) = \prod_{\alpha \in \Phi^+_{\tilde{\boldsymbol{G}}}} (1 - \alpha(t_\xi))$$

is a p-adic unit in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$.

4.9. Proposition. *Let $C \in \mathbb{Q}_{>0}$ and assume that $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ satisfies*

$$\langle \tilde{\lambda}, \alpha^\vee \rangle > 2C$$

for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$. Let ζ be contained in the semigroup Δ, so $\zeta \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ for some $e \in \mathbb{N}_0$ (by 2.7 Lemma); *we assume that $e \in \mathbb{N}$. Then the Lefschetz number* $\mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{Q}_p)))$ *is contained in $\mathbb{Z}_p$ and the following congruence holds*:

$$\mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{Q}_p))) \equiv \sum_{[\xi]_\Gamma \in \Gamma\zeta\Gamma/\sim_\Gamma} c_{[\xi]_\Gamma} \frac{\tilde{\lambda}(h^e t_\xi^{-1})}{N(t_\xi)} \pmod{p^{Ce} \mathcal{O}_{\bar{\mathbb{Q}}_p}}.$$

Proof. By 2.9 Corollary, the Lefschetz numbers of normalized Hecke operators are contained in $\mathbb{Z}_p$. Since $\zeta \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ we know that $(\Gamma\zeta\Gamma)_{\tilde{\lambda}} = \tilde{\lambda}(h^e)\Gamma\zeta\Gamma$. On the other hand, a representative ξ of a Γ-conjugacy class contained in $\Gamma\zeta\Gamma$ is contained in $\tilde{\boldsymbol{G}}(\mathbb{Q})$ and in $\tilde{\mathcal{I}}\zeta\tilde{\mathcal{I}} = \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$. The second claim thus follows from Bewersdorff's trace formula (see 4.2 Theorem) and 4.8 Corollary (note that $F \subseteq \mathbb{Q}_p$). □

4.10. Proposition. *Let $C \in \mathbb{Q}_{>0}$ and let the dominant weights $\tilde{\lambda}, \tilde{\lambda}' \in X(\tilde{\boldsymbol{T}})$ satisfy*

- *$\langle \tilde{\lambda}, \alpha^\vee \rangle > 2C$ and $\langle \tilde{\lambda}', \alpha^\vee \rangle > 2C$ for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$;*
- *$\tilde{\lambda} \equiv \tilde{\lambda}' \pmod{(p-1)p^{m-1} X(\tilde{\boldsymbol{T}})}$ ($m \in \mathbb{N}$).*

Let ζ be contained in the semigroup Δ, so $\zeta \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ for some $e \in \mathbb{N}_0$ which we assume positive. Then the Lefschetz number $\mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(F)))$ *is contained in $\mathbb{Z}_p$ (note that $F \subseteq \mathbb{Q}_p$) and the following congruence holds*:

$$\begin{aligned}&\mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(F)))\\ &\qquad \equiv \mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}'} | H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}'}(F))) \pmod{p^{\lceil \min(m, Ce) \rceil} \mathbb{Z}_p}.\end{aligned}$$

Proof. Since $\mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}}|H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(F))) = \mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}}|H^\bullet(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{Q}_p)))$, in order to prove the proposition we may consider Lefschetz numbers over $\mathbb{Q}_p$. Integrality of the Lefschetz number then follows from 4.9 Proposition. We let $[\xi]_\Gamma$ be any Γ-conjugacy class contained in $\Gamma\zeta\Gamma$. Hence, $\xi \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ and we denote by $t_\xi \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$ the element which is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to ξ_s (see 4.4 Proposition). Since $\tilde{\lambda} \equiv \tilde{\lambda}' \pmod{(p-1)p^{m-1}X(\tilde{\boldsymbol{T}})}$ there is $\chi \in X(\tilde{\boldsymbol{T}})$ such that $\tilde{\lambda} - \tilde{\lambda}' = (p-1)p^{m-1}\chi$. Taking into account that $\chi(h^e t_\xi^{-1}) \equiv \epsilon \pmod{p\mathcal{O}_{\bar{\mathbb{Q}}_p}}$ by 4.4 Proposition, where $\epsilon = \epsilon_\chi \in \mathbb{Z}_p^*$, we therefore obtain

$$\frac{\tilde{\lambda}(h^e t_\xi^{-1})}{\tilde{\lambda}'(h^e t_\xi^{-1})} = \chi(h^e t_\xi^{-1})^{(p-1)p^{m-1}} \in 1 + p^m \mathcal{O}_{\bar{\mathbb{Q}}_p}.$$

Since also $\tilde{\lambda}'(h^e t_\xi^{-1})$ is a p-adic unit by the same proposition, this implies

$$\tilde{\lambda}(h^e t_\xi^{-1}) \equiv \tilde{\lambda}'(h^e t_\xi^{-1}) \pmod{p^m \mathcal{O}_{\bar{\mathbb{Q}}_p}}.$$

The claim now follows from 4.9 Proposition taking into account that the Lefschetz numbers are contained in $\mathbb{Q}_p$, hence, their p-adic valuations are integers and that $c_{[\xi]_\Gamma} \in \mathbb{Z}$ and $N(t_\xi)$ is a p-adic unit. Thus, the proof is complete. □

Remark. The proposition also holds trivially for $e = 0$ since both sides of the congruence are integers by 2.9 Corollary.

4.11. ***Adelic formulation.*** Using adelic formulation in Section 4.13 we will prove congruences between traces of Hecke operators on Eisenstein cohomology and, hence, on cuspidal cohomology. In this section we therefore reformulate 3.10 Corollary and 4.10 Proposition in adelic language.

We denote by $\mathbb{A}$ (resp. $\mathbb{A}_f$) the ring of adeles (resp. of finite adeles) of $\mathbb{Q}$. For any compact open subgroup $\tilde{K} \leq \tilde{\boldsymbol{G}}(\mathbb{A}_f)$ we set $S_{\tilde{K}} = \tilde{\boldsymbol{G}}(\mathbb{Q})\backslash\tilde{\boldsymbol{G}}(\mathbb{A})/\tilde{K}\tilde{K}_\infty A_{\tilde{\boldsymbol{G}}}$. We assume that $\boldsymbol{G}/\mathbb{Q}$ satisfies strong approximation; in particular, $\tilde{\boldsymbol{G}}(\mathbb{A})$ is a finite disjoint union $\tilde{\boldsymbol{G}}(\mathbb{A}) = \bigcup_{i=1}^t \tilde{\boldsymbol{G}}(\mathbb{Q}) g_i \tilde{\boldsymbol{G}}(\mathbb{R})\tilde{K}$, $g_i \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)$, and we obtain

$$S_{\tilde{K}} = \bigcup_{i=1}^t \Gamma_i\backslash X$$

where

$$\Gamma_i = \tilde{\boldsymbol{G}}(\mathbb{Q}) \cap g_i \tilde{K} g_i^{-1}.$$

We assume that we can choose a system of double coset representatives g_i as above which is contained in $\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$, where $\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)} \leq \tilde{\boldsymbol{G}}(\mathbb{A}_f)$ is the subgroup consisting of elements whose p-component equals 1 (e.g., $\tilde{\boldsymbol{G}}$ satisfies weak approximation at p; note that weak approximation holds for almost all primes p).

We fix a compact open subgroup $\tilde{K}=\prod_{\ell\neq\infty}\tilde{K}_\ell\leq\tilde{\boldsymbol{G}}(\mathbb{A}_f)$ such that $\tilde{K}_p=\tilde{\mathcal{I}}$ and we set $\tilde{K}^{(p)}=\prod_{\ell\neq p,\infty}\tilde{K}_\ell$. Since the p-component of g_i is equal to 1 any of the arithmetic subgroups Γ_i is contained in $\tilde{\mathcal{I}}$.

4.11.1. *Hecke algebra.* We fix a Haar measure $dg=\bigotimes_{\ell\neq\infty}dg_\ell$ on $\tilde{\boldsymbol{G}}(\mathbb{A}_f)$ and we denote by $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f))$ the Hecke algebra consisting of compactly supported smooth $\mathbb{Q}$-valued functions on $\tilde{\boldsymbol{G}}(\mathbb{A}_f)$. We have the decomposition $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f))=\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{Q}_p))\otimes\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)})$, where the two factors are the Hecke algebras consisting respectively of compactly supported smooth $\mathbb{Q}$-valued functions on $\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ and on $\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$. Let the subalgebra consisting of $\tilde{K}$- (resp. $\tilde{\mathcal{I}}$- or $\tilde{K}^{(p)}$-) bi-invariant functions be denoted by $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})\leq\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f))$ (resp. $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})\leq\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{Q}_p))$ or $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}//\tilde{K}^{(p)})\leq\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)})$). Let

$$[\tilde{K}x\tilde{K}]=\frac{1}{\mathrm{vol}(\tilde{K})}\mathbf{1}_{\tilde{K}x\tilde{K}},\quad x\in\tilde{\boldsymbol{G}}(\mathbb{A}_f),$$

where $\mathbf{1}_X$ is the characteristic function of the set X, and define likewise $[\tilde{\mathcal{I}}x\tilde{\mathcal{I}}]$ for $x\in\tilde{\boldsymbol{G}}(\mathbb{Q}_p)$ and $[\tilde{K}^{(p)}x\tilde{K}^{(p)}]$ for $x\in\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$, by replacing $\tilde{K}$ with $\tilde{\mathcal{I}}$ and with $\tilde{K}^{(p)}$. The elements $[\tilde{K}x\tilde{K}]$ (resp. $[\tilde{\mathcal{I}}x\tilde{\mathcal{I}}]$ or $[\tilde{K}^{(p)}x\tilde{K}^{(p)}]$) form a $\mathbb{Q}$-basis of $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})$ (resp. $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})$ or $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}//\tilde{K}^{(p)})$); their $\mathbb{Z}$-spans define $\mathbb{Z}$-structures $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})$ (resp. $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})$ or $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}//\tilde{K}^{(p)})$) in the respective Hecke algebras. Thus, the $\mathbb{Z}$-structure on $\mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})$ is given as the subspace of $\mathrm{vol}(\tilde{K})^{-1}\cdot\mathbb{Z}$-valued functions and analogously for the other two Hecke algebras.

In Section 2.3 we selected an element $h\in\tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ and we now denote by

$$\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})_h$$

the $\mathbb{Z}$-subalgebra of $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})$ generated by $[\tilde{\mathcal{I}}h^{-1}\tilde{\mathcal{I}}]$ and we set

$$\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h=\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})_h\otimes\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}//\tilde{K}^{(p)}).$$

Since $[\tilde{\mathcal{I}}h^{-1}\tilde{\mathcal{I}}]^e=[\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}]$ the algebra $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{Q}_p)//\tilde{\mathcal{I}})_h$ is the $\mathbb{Z}$-span of $[\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}]$, $e\in\mathbb{N}_0$, hence, $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$ is the $\mathbb{Z}$-span of

$$[\tilde{K}[h]_p^{-e}r\tilde{K}]=[\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}]\otimes[\tilde{K}^{(p)}r\tilde{K}^{(p)}],\ e\in\mathbb{N}_0,\ r\in\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)};$$

here, $[h]_p\in\tilde{\boldsymbol{G}}(\mathbb{A}_f)$ is the element with h in the p-component and all remaining components equal to 1.

For any $\mathbb{Z}$-algebra R we put $\mathcal{C}_{0,R}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_?=\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_?\otimes R$ where $?=blank,h$. We define the $\tilde{\lambda}$-normalization of $[\tilde{K}[h]_p^{-e}r\tilde{K}]$ as

$$[\tilde{K}[h]_p^{-e}r\tilde{K}]_{\tilde{\lambda}}=\tilde{\lambda}(h)^e[\tilde{K}[h]_p^{-e}r\tilde{K}]\in\mathcal{C}_{0,F}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$$

(note that $\tilde{\lambda}(h)^e \in F$; see Section 2.5) and we extend linearly to $\mathcal{C}_{0,F}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$. Since $[\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}][\tilde{\mathcal{I}}h^{-f}\tilde{\mathcal{I}}] = [\tilde{\mathcal{I}}h^{-(e+f)}\tilde{\mathcal{I}}]$ we obtain that the assignment

$$\mathcal{C}_{0,F}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h \to \mathcal{C}_{0,F}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h, \quad \mathbb{T} \mapsto \mathbb{T}_{\tilde{\lambda}},$$

is a morphism of F-algebras.

4.11.2. *Cohomology.* The algebra $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})$, and hence also $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$, acts on $H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(F)) = \bigoplus_i H^n(\Gamma_i\backslash X, L_{\tilde{\lambda}}(F))$. We define the integral cohomology $H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\text{int}}$ as the image of $H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Z}_p))$ in $H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))$; hence, $H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\text{int}} = \bigoplus_i H^n(\Gamma_i\backslash X, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\text{int}}$. Consider $e \in \mathbb{N}_0$ and $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$; we write $g_i[h]_p^{-e}r = \zeta_i g_{j(i)}k$ with $\zeta_i \in \tilde{\boldsymbol{G}}(\mathbb{Q})$, $k = (k_\ell)_\ell \in \tilde{K}$ and obtain

$$[\tilde{K}[h]_p^{-e}r\tilde{K}] = \bigoplus_i \Gamma_i\zeta_i^{-1}\Gamma_{j(i)}.$$

Looking at the p-component and recalling that $g_i \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)$ has trivial p-component we find $h^{-e} = \zeta_i k_p$. Hence, $\zeta_i^{-1} = k_p h^e$ is contained in $\tilde{\mathcal{I}}h^e\tilde{\mathcal{I}}$ and, thus, in $\Delta = \Delta_h$ and we deduce that the normalization $\tilde{\lambda}(h^e)\Gamma_i\zeta_i^{-1}\Gamma_{j(i)}$ maps $H^n(\Gamma_i\backslash X, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\text{int}}$ to $H^n(\Gamma_{j(i)}\backslash X, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\text{int}}$. Hence, $[\tilde{K}[h]_p^{-e}r\tilde{K}]_{\tilde{\lambda}}$ and, thus, any $\mathbb{T}_{\tilde{\lambda}}$ with $\mathbb{T}$ in $\mathcal{C}_{0,\mathbb{Z}_p}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$ leaves $H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Z}_p))_{\text{int}}$ invariant. In particular, the Lefschetz number $\text{Lef}(\mathbb{T}_{\tilde{\lambda}}|H^\bullet(S_{\tilde{K}}, L_{\tilde{\lambda}}(F)))$, $\mathbb{T} \in \mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$, is contained in F and in $\mathbb{Z}_p$ (note that $F \subseteq \mathbb{Q}_p$). Moreover, as in the proof of 2.9 Corollary we see that the eigenvalues of $\mathbb{T}_{\tilde{\lambda}}$, $\mathbb{T} \in \mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$, on $H^n_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C})) \subseteq H^n(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))$ are algebraic over F (note that $F \subseteq \mathbb{C}$) and integral over $\mathbb{Z}_p$, hence, they are contained in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$ (note that $\bar{F} \subseteq \bar{\mathbb{Q}}_p$).

4.11.3. We denote by

$$\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h^{\text{reg}} \subseteq \mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$$

the $\mathbb{Z}$-submodule generated by all Hecke operators $[\tilde{K}[h]_p^{-e}r\tilde{K}]$ with $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$ and $e \in \mathbb{N}$ (i.e., $e \geq 1$). Keeping in mind that $[\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}][\tilde{\mathcal{I}}h^{-f}\tilde{\mathcal{I}}] = [\tilde{\mathcal{I}}h^{-(e+f)}\tilde{\mathcal{I}}]$ we find that $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h^{\text{reg}}$ is an ideal in $\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h$.

Proposition. *Let $C \in \mathbb{Q}_{>0}$. If the dominant weights $\tilde{\lambda}, \tilde{\lambda}' \in X(\tilde{\boldsymbol{T}})$ satisfy*

- $\langle\tilde{\lambda}, \alpha^\vee\rangle > 2C$ *and* $\langle\tilde{\lambda}', \alpha^\vee\rangle > 2C$ *for all* $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$,
- $\tilde{\lambda} \equiv \tilde{\lambda}' \pmod{(p-1)p^{m-1}X(\tilde{\boldsymbol{T}})}$ $(m \in \mathbb{N})$,

then for all $e \in \mathbb{N}$ and $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$ the Lefschetz number of $[\tilde{K}[h]_p^{-e}r\tilde{K}]_{\tilde{\lambda}}$ on $H^\bullet(S_{\tilde{K}}, L_{\tilde{\lambda}}(F))$ is contained in $\mathbb{Z}_p$ and the following congruence holds:

$$\text{Lef}([\tilde{K}[h]_p^{-e}r\tilde{K}]_{\tilde{\lambda}}|H^\bullet(S_{\tilde{K}}, L_{\tilde{\lambda}}(F))) \equiv \text{Lef}([\tilde{K}[h]_p^{-e}r\tilde{K}]_{\tilde{\lambda}'}|H^\bullet(S_{\tilde{K}}, L_{\tilde{\lambda}'}(F))) \pmod{p^{\lceil \min(m, Ce)\rceil}\mathbb{Z}_p}.$$

Proof. We have

$$\mathrm{Lef}([\tilde K[h]_p^{-e} r\tilde K]_{\tilde\lambda}|H^\bullet(S_{\tilde K}, L_{\tilde\lambda}(F))) = \sum_{\substack{i\\ i=j(i)}} \tilde\lambda(h^e)\ \mathrm{Lef}(\Gamma_i\zeta_i^{-1}\Gamma_i|H^\bullet(\Gamma_i\backslash X, L_{\tilde\lambda}(F)))$$

where $\zeta_i^{-1}\in\Delta_h$ and $\zeta_i^{-1}\in\tilde{\mathcal I}h^e\tilde{\mathcal I}$. Hence, $\tilde\lambda(h^e)=\hat\lambda(\zeta_i^{-1})$ and the claim follows from 4.10 Proposition. $\square$

4.11.4. *Slope subspaces.* We select a Hecke operator $\mathbb T\in\mathcal C_{0,\mathbb Z_p}(\tilde{\boldsymbol G}(\mathbb A_f)/\tilde K)_h^{\mathrm{reg}}$ and we denote by $H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Q_p))^\beta$ the slope β subspace of $H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Q_p))$ w.r.t. $\mathbb T_{\tilde\lambda}$. We also denote by $\boldsymbol g_i=\boldsymbol g_{i,\tilde K}$ the number of i-cells in a cell complex which is homotopy equivalent to the Borel–Serre compactification $\bar S_{\tilde K}$ of $S_{\tilde K}$.

Theorem. *For all $\beta\in\mathbb Q_{\ge0}$, all dominant weights $\tilde\lambda\in X(\tilde{\boldsymbol T})$, all i and all $\mathbb T\in\mathcal C_{0,\mathbb Z_p}(\tilde{\boldsymbol G}(\mathbb A_f)/\tilde K)_h^{\mathrm{reg}}$ we have*

$$\dim H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Q_p))^{\le\beta}\le \mathtt m\beta^s+\mathtt n;$$

here, $\mathtt m=\mathtt m_{\tilde K}=12\frac{\boldsymbol g_i}{s}\sigma^{s+1}\in\mathbb Q_{\ge0}$ *and* $\mathtt n=\mathtt n_{\tilde K}\in\mathbb N$ *is an integer which also only depends on* $\boldsymbol g_i,\sigma,s$ (see (50) below); *in particular*, $\mathtt m$ *and* $\mathtt n$ *only depend on* $\tilde K$ (*and*, *hence*, *on* $\tilde{\boldsymbol G}$ *and* p) *and* i, *i.e., they do not depend on* $\tilde\lambda$, h *and* $\mathbb T$.

The proof follows those of 3.9 Theorem and 3.10 Corollary. More precisely, for any $r\in\mathbb N_0$ we define the $\mathbb Z_p$-submodule

$$H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p,r,\sigma))=\bigoplus_{j=1}^t H^i(\Gamma_j, L_{\tilde\lambda}(\mathbb Z_p,r,\sigma))$$

of $H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p))=\bigoplus_{j=1}^t H^i(\Gamma_j, L_{\tilde\lambda}(\mathbb Z_p))$ (note that $\Gamma_j\subseteq\tilde{\mathcal I}$). Using the decomposition $[\tilde K[h]_p^{-e}r\tilde K]=\bigoplus_i\Gamma_i\zeta_i^{-1}\Gamma_{j(i)}$ where $\zeta_i^{-1}\in\tilde{\mathcal I}h^e\tilde{\mathcal I}$ and following the proof in 3.5 Lemma we see that the submodule $H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p,r,\sigma))$ satisfies the following properties.

- $H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p))^{\mathrm{TF}}$ is $\mathbb T_{\tilde\lambda}$-invariant.
- $\mathbb T_{\tilde\lambda}H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p,r,\sigma))^{\mathrm{TF}}\subseteq p^rH^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p))^{\mathrm{TF}}$.

We denote by $(p^{b_l})_l$ the elementary divisors of the quotient

$$\frac{H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p))^{\mathrm{TF}}}{H^i(S_{\tilde K}, L_{\tilde\lambda}(\mathbb Z_p,r,\sigma))^{\mathrm{TF}}}\tag{49}$$

in decreasing order, i.e., $b_1\ge b_2\ge b_3\ge\cdots$. As in Section 3.9.1 we see that (49) is a subquotient of $\bigoplus_j H^i(\Gamma_j, \boldsymbol L_{\tilde\lambda}^{[r]}(\mathbb Z_p,\sigma))$, which is a subquotient of

$$\bigoplus_j\bigoplus_{h=0}^r\Big(\frac{\mathbb Z_p}{p^{r-h}\mathbb Z_p}\Big)^{g_{i,\Gamma_j}M_{\sigma,h}}=\bigoplus_{h=0}^r\Big(\frac{\mathbb Z_p}{p^{r-h}\mathbb Z_p}\Big)^{\boldsymbol g_iM_{\sigma,h}}.$$

We denote by $(p^{a_l})_l$ the elementary divisors of the latter sum in decreasing order, i.e., $a_1 \geq a_2 \geq \cdots$ (the elementary divisor p^{r-h} appears $\boldsymbol{g}_i M_{\sigma,h}$-times). We obtain $b_i \leq a_i \leq r$ for all i and following the arguments in Section 3.9.2 - 3.9.4 with $\boldsymbol{g}_i$ in place of g_i we obtain that the piecewise linear function defined in 3.9 Theorem but with g_i replaced by $\boldsymbol{g}_i$ is a lower bound for the Newton polygon of $\mathbb{T}_{\tilde{\lambda}}$ acting on $H^i(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))^{<\infty}$. Following the proof of 3.10 Corollary we find the bound for the dimension of the slope β subspace as in the statement of the theorem where we can choose

$$\mathtt{n} = \left\lceil 2^{\frac{1}{s+1}} \frac{\boldsymbol{g}_i \sigma^{s+1}}{s} M^s \right\rceil + 1 \in \mathbb{N}; \tag{50}$$

here, M is defined in equations (23) and (24) (note that the definition of M does not depend on g_i). This finishes the proof of the theorem.

4.12. ***Induced representations.*** We look at traces of Hecke operators on induced representations. This will be applied in the next section where we consider Hecke operators on Eisenstein cohomology.

4.12.1. We select a $\mathbb{Q}$-parabolic subgroup $\tilde{\boldsymbol{Q}} \leq \tilde{\boldsymbol{G}}$ with Levi decomposition $\tilde{\boldsymbol{Q}} = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{N}}$ and a representation π of $\tilde{\boldsymbol{M}}(\mathbb{A}_f)$. By $\mathrm{Ind}_{\tilde{\boldsymbol{Q}}(\mathbb{A}_f)}^{\tilde{\boldsymbol{G}}(\mathbb{A}_f)}\pi$ we understand the non-unitarily induced representation. We select a maximal compact open subgroup $\tilde{K}_0 = \prod_{\ell\neq\infty} \tilde{K}_{0,\ell} \leq \tilde{\boldsymbol{G}}(\mathbb{A}_f)$ and we may assume that $\tilde{K} \leq \tilde{K}_0$. We set

$$\tilde{K}^{\tilde{M}} = \prod_{\ell\neq\infty} \tilde{K}_\ell^{\tilde{M}} \quad \text{and} \quad \tilde{K}^{(p),\tilde{M}} = \prod_{\ell\neq p,\infty} \tilde{K}_\ell^{\tilde{M}},$$

where $\tilde{K}_\ell^{\tilde{M}} = \tilde{K}_\ell \cap \tilde{\boldsymbol{M}}(\mathbb{Q}_\ell)$, and we use the same definition with $\tilde{K}$ replaced by $\tilde{K}_0$. We also set $\tilde{K}^{\tilde{N}} = \tilde{K} \cap \tilde{\boldsymbol{N}}(\mathbb{A}_f)$ and $\tilde{K}^{(p),\tilde{N}} = \tilde{K}^{(p)} \cap \tilde{\boldsymbol{N}}(\mathbb{A}_f)^{(p)}$.

Let $f \in \mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f))$; we define the constant term $f_{\tilde{\boldsymbol{M}}} \in \mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{M}}(\mathbb{A}_f))$ by

$$f_{\tilde{\boldsymbol{M}}}(x) = \int_{\tilde{K}_0} \int_{\tilde{\boldsymbol{N}}(\mathbb{A}_f)} f(k^{-1}xnk)\, dn\, dk.$$

Here and below we normalize Haar measures on $\tilde{\boldsymbol{G}}(\mathbb{A}_f)$ (and, hence, on $\tilde{K}_0$), $\tilde{\boldsymbol{M}}(\mathbb{A}_f)$ and $\tilde{\boldsymbol{N}}(\mathbb{A}_f)$ so that $\mathrm{vol}(\tilde{K}) = 1$, $\mathrm{vol}(\tilde{K}^{\tilde{M}}) = 1$, $\mathrm{vol}(\tilde{K}^{\tilde{N}}) = 1$. With these conventions we have

$$\mathrm{tr}(f|\mathrm{Ind}_{\tilde{\boldsymbol{Q}}(\mathbb{A}_f)}^{\tilde{\boldsymbol{G}}(\mathbb{A}_f)}\pi) = \mathrm{tr}(f_{\tilde{\boldsymbol{M}}}|\pi).$$

If $f^{(p)} \in \mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)})$ we define $f_{\tilde{\boldsymbol{M}}}^{(p)} \in \mathcal{C}_{0,\mathbb{Q}}(\tilde{\boldsymbol{M}}(\mathbb{A}_f)^{(p)})$ by replacing f with $f^{(p)}$, $\tilde{K}_0$ with $\tilde{K}_0^{(p)} = \prod_{\ell\neq p,\infty} \tilde{K}_{0,\ell}$ and $\tilde{\boldsymbol{N}}(\mathbb{A}_f)$ with $\tilde{\boldsymbol{N}}(\mathbb{A}_f)^{(p)}$; an analogous identity for the trace of $f^{(p)}$ on representations induced from $\tilde{\boldsymbol{M}}(\mathbb{A}_f)^{(p)}$ then holds. We will also need the following classical identity which holds for a representation π_p of $\tilde{\boldsymbol{M}}(\mathbb{Q}_p)$. Assume that $\tilde{\boldsymbol{Q}}/F$ contains $\tilde{\boldsymbol{B}}^-/F$, i.e., the unipotent radical of $\tilde{\boldsymbol{Q}}/F$

is generated by root subgroups attached to negative roots (compare the definition of $\mathcal{I}$ in Section 1.3); then

$$\mathrm{tr}([\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}]|(\mathrm{Ind}_{\tilde{\boldsymbol{Q}}(\mathbb{Q}_p)}^{\tilde{G}(\mathbb{Q}_p)}\pi_p)^{\tilde{\mathcal{I}}}) = \sum_{v\in W^{\tilde{Q}}} c_{v,h^{-e}}\mathrm{tr}([\tilde{\mathcal{I}}^{\tilde{M}}vh^{-e}v^{-1}\tilde{\mathcal{I}}^{\tilde{M}}]|\pi_p^{\tilde{\mathcal{I}}^{\tilde{M}}}), \tag{51}$$

where $c_{v,h^{-e}}\in\mathbb{N}$ and $\tilde{\mathcal{I}}^{\tilde{M}}=\tilde{\mathcal{I}}\cap\tilde{\boldsymbol{M}}(\mathbb{Q}_p)$ (see [Urban 2011, p. 1751], for example). We obtain an analogous formula when $\tilde{\boldsymbol{Q}}$ is standard parabolic, i.e., $\tilde{\boldsymbol{B}}/F\leq\tilde{\boldsymbol{Q}}/F$. To this end we denote by $a\in W_{\tilde{\boldsymbol{G}}}$ and $b\in W_{\tilde{\boldsymbol{M}}}=W(\tilde{\boldsymbol{T}}/\mathbb{Z}_p,\tilde{\boldsymbol{M}}/\mathbb{Z}_p)$ the longest elements; i.e., a maps $\Phi^+_{\tilde{\boldsymbol{G}}}$ to $\Phi^-_{\tilde{\boldsymbol{G}}}$ and b maps $\Phi^+_{\tilde{\boldsymbol{M}}}$ to $\Phi^-_{\tilde{\boldsymbol{M}}}$. Using (51) but with positivity defined by $-\Delta_{\tilde{\boldsymbol{G}}}$ — that is, $\tilde{\boldsymbol{B}}/F\leq\tilde{\boldsymbol{Q}}/F$, $\tilde{\mathcal{I}}\equiv\tilde{\boldsymbol{B}}(\mathbb{F}_p)\pmod p$ and $h\in\tilde{\boldsymbol{T}}(\mathbb{Q})^{--}$ — we obtain

$$\begin{aligned}\mathrm{tr}([\tilde{\mathcal{I}}h^{-e}\tilde{\mathcal{I}}]|(\mathrm{Ind}_{\tilde{\boldsymbol{Q}}(\mathbb{Q}_p)}^{\tilde{G}(\mathbb{Q}_p)}\pi_p)^{\tilde{\mathcal{I}}}) &= \mathrm{tr}([{}^a\tilde{\mathcal{I}}({}^ah^{-e})\,{}^a\tilde{\mathcal{I}}]|(\mathrm{Ind}_{\tilde{\boldsymbol{Q}}(\mathbb{Q}_p)}^{\tilde{G}(\mathbb{Q}_p)}\pi_p)^{{}^a\tilde{\mathcal{I}}})\\ &= \sum_{v\in W^{\tilde{Q}}} c_{v,h^{-e}}\mathrm{tr}([({}^a\tilde{\mathcal{I}})^{\tilde{M}}v({}^ah^{-e})v^{-1}({}^a\tilde{\mathcal{I}})^{\tilde{M}}]|\pi_p^{({}^a\tilde{\mathcal{I}})^{\tilde{M}}})\\ &= \sum_{v\in W^{\tilde{Q}}} c_{v,h^{-e}}\mathrm{tr}([{}^b({}^a\tilde{\mathcal{I}}^{\tilde{M}})bv({}^ah^{-e})v^{-1}b^{-1}\,{}^b({}^a\tilde{\mathcal{I}}^{\tilde{M}})]|\pi_p^{{}^b({}^a\tilde{\mathcal{I}}^{\tilde{M}})})\\ &= \sum_{v\in W^{\tilde{Q}}} c_{v,h^{-e}}\mathrm{tr}([\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}]|\pi_p^{\tilde{\mathcal{I}}^{\tilde{M}}}),\end{aligned}$$

where $c_v=c_{v,h^{-e}}\in\mathbb{Z}$ are certain integers.

4.12.2. We look more closely at the constant term $f_{\tilde{\boldsymbol{M}}}^{(p)}$ of $f^{(p)}\in\mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}//\tilde{K}^{(p)})$, i.e., $f^{(p)}$ is $\mathbb{Z}$-valued and $\tilde{K}^{(p)}$ bi-invariant (note that $\mathrm{vol}(\tilde{K}^{(p)})=1$ by our normalizations). For simplicity we assume that $\tilde{K}^{(p)}\leq\tilde{K}_0^{(p)}$ is a normal subgroup. The definition yields

$$\begin{aligned}f_{\tilde{\boldsymbol{M}}}^{(p)}(x) &= \sum_{k\in\tilde{K}_0^{(p)}/\tilde{K}^{(p)}}\int_{\tilde{\boldsymbol{N}}(\mathbb{A}_f)^{(p)}} f^{(p)}(k^{-1}xnk)\,dn\\ &= \mathrm{vol}(\tilde{K}^{(p),\tilde{N}})\sum_{k\in\tilde{K}_0^{(p)}/\tilde{K}^{(p)}}\int_{\tilde{\boldsymbol{N}}(\mathbb{A}_f)^{(p)}/\tilde{K}^{(p),\tilde{N}}} f^{(p)}(k^{-1}xnk)\,dn,\end{aligned}$$

The first of these equalities shows that $f_{\tilde{\boldsymbol{M}}}^{(p)}$ is $\tilde{K}^{(p),\tilde{M}}$ bi-invariant (the modulus $\delta_{\tilde{\boldsymbol{Q}}(\mathbb{A}_f)^{(p)}}(m)$ vanishes for $m\in\tilde{K}^{(p),\tilde{M}}$ because $\tilde{K}^{(p),\tilde{M}}$ is contained in the compact group $\tilde{K}^{(p)}\cap\tilde{\boldsymbol{M}}(\mathbb{A}_f)^{(p)}$). Since $\mathrm{vol}(\tilde{K}^{(p),\tilde{N}})=\mathrm{vol}(\tilde{K}^{(p),\tilde{M}})=1$ the second equality implies

$$f_{\tilde{\boldsymbol{M}}}^{(p)}\in\mathcal{C}_0(\tilde{\boldsymbol{M}}(\mathbb{A}_f)^{(p)}//\tilde{K}^{(p),\tilde{M}}). \tag{52}$$

4.13. ***Eisenstein cohomology.***

4.13.1. We assume that the highest weight $\tilde{\lambda}\in X(\tilde{\boldsymbol{T}})$ is dominant and regular. The full cohomology then decomposes as

$$H^\bullet(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C})) = H^\bullet_{\mathrm{cusp}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C})) \oplus H^\bullet_{\mathrm{Eis}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C})),$$

where

$$H^\bullet_{\mathrm{Eis}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C})) = \bigoplus_{\tilde{\boldsymbol{M}}\neq\tilde{\boldsymbol{G}}} \bigoplus_{w} \Big(\mathrm{Ind}_{\tilde{\boldsymbol{Q}}(\mathbb{A}_f)}^{\tilde{\boldsymbol{G}}(\mathbb{A}_f)} H^{\bullet-\ell(w)}_{\mathrm{cusp}}(S^{\tilde{\boldsymbol{M}}}, L_{w\cdot\tilde\lambda}(\mathbb{C}))\Big)^{\tilde K}; \tag{53}$$

here, $\tilde{\boldsymbol{M}}$ runs over a system of representatives of $\tilde{\boldsymbol{G}}(\mathbb{Q})$-conjugacy classes of proper $\mathbb{Q}$-Levi subgroups of $\tilde{\boldsymbol{G}}$, i.e., there is a (standard) $\mathbb{Q}$-parabolic subgroup $\tilde{\boldsymbol{Q}} \leq \tilde{\boldsymbol{G}}$ with Levi decomposition

$$\tilde{\boldsymbol{Q}} = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{N}}, \quad S^{\tilde{\boldsymbol{M}}} = \tilde{\boldsymbol{M}}(\mathbb{Q})\backslash\tilde{\boldsymbol{M}}(\mathbb{A})/A_{\tilde{\boldsymbol{Q}}}\tilde K^{\tilde{\boldsymbol{M}}}_\infty,$$

where $\tilde K^{\tilde M}_\infty = \tilde K_\infty \cap \tilde{\boldsymbol{M}}(\mathbb{R})$, is the locally symmetric space attached to $\tilde{\boldsymbol{M}}$, $A_{\tilde{\boldsymbol{Q}}}$ is the connected component of the real points of a maximal $\mathbb{Q}$-split torus $\boldsymbol{A}_{\tilde{\boldsymbol{Q}}}$ in the center of $\tilde{\boldsymbol{M}}$ and w runs over those elements in $W^{\tilde Q}$ which satisfy the condition that $-w(\tilde\lambda+\rho^\circ)|_{A_{\tilde{\boldsymbol{Q}}}}$ is nonnegative, i.e., $\langle -\mathrm{Re}(w(\tilde\lambda+\rho^\circ))|_{A_{\tilde{\boldsymbol{Q}}}}, \alpha^\vee\rangle \geq 0$ for all roots α of $A_{\tilde{\boldsymbol{Q}}}$ acting on $\mathrm{Lie}(\tilde{\boldsymbol{N}}) \otimes \mathbb{R}$ (see [Franke 1998, Theorem 19 II, p. 257] and [Schwermer 1994, Proof of 6.3 Theorem, p. 505]).

4.13.2. Theorem. *Let $C \in \mathbb{Q}_{>0}$ and suppose the dominant weights $\tilde\lambda, \tilde\lambda' \in X(\tilde{\boldsymbol{T}})$ satisfy*

- $\langle\tilde\lambda, \alpha^\vee\rangle > 2C$ *and* $\langle\tilde\lambda', \alpha^\vee\rangle > 2C$ *for all* $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$,
- $\tilde\lambda \equiv \tilde\lambda' \pmod{(p-1)p^{m-1}X(\tilde{\boldsymbol{T}})}$ $(m \in \mathbb{N})$.

Then, for all $e \in \mathbb{N}$ and $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$, the Lefschetz number of $[\tilde K[h]_p^{-e} r \tilde K]_{\tilde\lambda}$ on $H^\bullet_{\mathrm{cusp}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C}))$ is contained in F and the following congruence holds (*note that $F \subseteq \mathbb{Q}_p$*):

$$\mathrm{Lef}([\tilde K[h]_p^{-e} r\tilde K]_{\tilde\lambda} | H^\bullet_{\mathrm{cusp}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C}))) \equiv \mathrm{Lef}([\tilde K[h]_p^{-e} r\tilde K]_{\tilde\lambda'} | H^\bullet_{\mathrm{cusp}}(S_{\tilde K}, L_{\tilde\lambda'}(\mathbb{C}))) \pmod{p^\dagger\mathbb{Z}_p}.$$

Here, $\dagger = \big\lceil \min\big(m, e(C-\kappa_2\mathrm{rk}(\tilde{\boldsymbol{G}}))\big)\big\rceil - e\kappa_1\mathrm{rk}(\tilde{\boldsymbol{G}})$ with $\kappa_i = \kappa_{i,\tilde{\boldsymbol{G}}}$ and $\mathrm{rk}(\tilde{\boldsymbol{G}}) = \mathrm{rk}_{\mathbb{Q}}(\tilde{\boldsymbol{G}})$ is the $\mathbb{Q}$-rank of $\tilde{\boldsymbol{G}}$.

Proof. We use induction on the $\mathbb{Q}$-rank of $\tilde{\boldsymbol{G}}$. If $\mathrm{rk}(\tilde{\boldsymbol{G}}) = 0$ then

$$H^\bullet_{\mathrm{cusp}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C})) = H^\bullet(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C}));$$

hence, 4.11.3 Proposition implies the claim. We assume $\mathrm{rk}(\tilde{\boldsymbol{G}}) > 0$. Equation (53) yields

$$\begin{aligned}(54)\quad &\mathrm{Lef}([\tilde K[h]_p^{-e} r\tilde K]|H^\bullet_{\mathrm{Eis}}(S_{\tilde K}, L_{\tilde\lambda}(\mathbb{C})))\\ &= \sum_{\tilde{\boldsymbol M}\neq\tilde{\boldsymbol G}}\sum_w\sum_i(-1)^{i+\ell(w)}\mathrm{tr}\big([\tilde K[h]_p^{-e} r\tilde K]|\big(\mathrm{Ind}_{\tilde{\boldsymbol Q}(\mathbb{A}_f)}^{\tilde G(\mathbb{A}_f)} H^i_{\mathrm{cusp}}(S^{\tilde{\boldsymbol M}}, L_{w\cdot\tilde\lambda}(\mathbb{C}))\big)^{\tilde K}\big)\\ &= \sum_{\tilde{\boldsymbol M}\neq\tilde{\boldsymbol G}}\sum_w(-1)^{\ell(w)}\,\mathrm{Lef}\big([\tilde K[h]_p^{-e} r\tilde K]|\bigoplus_i(\mathrm{Ind}_{\tilde{\boldsymbol Q}(\mathbb{A}_f)}^{\tilde G(\mathbb{A}_f)}\pi^i_{w\cdot\tilde\lambda})^{\tilde K}\big).\end{aligned}$$

Here, we have set

$$\pi^i_{w\cdot\tilde\lambda} = H^i_{\mathrm{cusp}}(S^{\tilde{\boldsymbol M}}, L_{w\cdot\tilde\lambda}(\mathbb{C}));$$

this is a module under the Hecke algebra attached to $\tilde{\boldsymbol M}(\mathbb{A}_f)$ and we have

$$H^i_{\mathrm{cusp}}(S^{\tilde{\boldsymbol M}}, L_{w\cdot\tilde\lambda}(\mathbb{C}))^{\tilde H} = H^i_{\mathrm{cusp}}(S^{\tilde{\boldsymbol M}}/\tilde H, L_{w\cdot\tilde\lambda}(\mathbb{C})),$$

with $\tilde H \leq \tilde{\boldsymbol M}(\mathbb{A}_f)$ compact open. We select a proper $\mathbb{Q}$-parabolic subgroup $\tilde{\boldsymbol Q} = \tilde{\boldsymbol M}\tilde{\boldsymbol N}$ of $\tilde{\boldsymbol G}$ and an element $w \in W^{\tilde Q}$ as in equation (53). We denote by $\Phi_{\tilde{\boldsymbol M}}$ the set of roots of $\tilde{\boldsymbol T}/F$ acting on $\mathrm{Lie}(\tilde{\boldsymbol M}/F)$ and we set $\Delta_{\tilde{\boldsymbol M}} = \Phi_{\tilde{\boldsymbol M}} \cap \Delta_{\tilde{\boldsymbol G}}$. The set $\Delta_{\tilde{\boldsymbol M}}$ is the basis for the root system $\Phi_{\tilde{\boldsymbol M}}$ corresponding to the Borel subgroup $\tilde{\boldsymbol B}^M/F = \tilde{\boldsymbol B}/F \cap \tilde{\boldsymbol M}/F$ of $\tilde{\boldsymbol M}/F$; in particular, this determines the set of positive roots $\Phi^+_{\tilde{\boldsymbol M}}$. The subgroup $\tilde{\mathcal I}^{\tilde M} = \tilde{\mathcal I} \cap \tilde{\boldsymbol M}(\mathbb{Q}_p)$ is a Iwahori subgroup in $\tilde{\boldsymbol M}(\mathbb{Z}_p)$, i.e., $\tilde{\mathcal I}^{\tilde M}$ (mod p) is contained in the Borel subgroup $(\tilde{\boldsymbol B}^- \cap \tilde{\boldsymbol M})(\mathbb{F}_p) \leq \tilde{\boldsymbol M}(\mathbb{F}_p)$. The identities in Section 4.12.1 for the traces of induced representations and equation (52) yield

$$\begin{aligned}(55)\quad &\mathrm{Lef}\big([\tilde K[h]_p^{-e} r\tilde K]|\bigoplus_i\big(\mathrm{Ind}_{\tilde{\boldsymbol Q}(\mathbb{A}_f)}^{\tilde G(\mathbb{A}_f)}\pi^i_{w\cdot\tilde\lambda}\big)^{\tilde K}\big)\\ &= \mathrm{Lef}\big([\tilde{\mathcal I}h^{-e}\tilde{\mathcal I}]\otimes[\tilde K^{(p)} r\tilde K^{(p)}]|\bigoplus_i\mathrm{Ind}_{\tilde{\boldsymbol Q}(\mathbb{A}_f)}^{\tilde G(\mathbb{A}_f)}\pi^i_{w\cdot\tilde\lambda}\big)\\ &= \sum_{v\in W^{\tilde Q}} c_v\,\mathrm{Lef}\big([\tilde{\mathcal I}^{\tilde M}({}^{bva}h^{-e})\tilde{\mathcal I}^{\tilde M}]\otimes[\tilde K^{(p)} r\tilde K^{(p)}]_{\tilde{\boldsymbol M}}|\bigoplus_i\pi^i_{w\cdot\tilde\lambda}\big)\\ &= \sum_{v\in W^{\tilde Q}} c_v\,\mathrm{Lef}\big([\tilde{\mathcal I}^{\tilde M}({}^{bva}h^{-e})\tilde{\mathcal I}^{\tilde M}]\otimes[\tilde K^{(p)} r\tilde K^{(p)}]_{\tilde{\boldsymbol M}}|\bigoplus_i(\pi^i_{w\cdot\tilde\lambda})^{\tilde K_{\tilde M}}\big),\end{aligned}$$

where $c_v = c_{v,h^{-e}} \in \mathbb{Z}$, a (resp. b) is the longest element in the Weyl group $W_{\tilde{\boldsymbol G}}$ (resp. $W_{\tilde{\boldsymbol M}}$) and $\tilde K^{\tilde M} = \tilde{\mathcal I}^{\tilde M} \times \tilde K^{(p),\tilde M}$. We select an element $v \in W^{\tilde Q}$. Since $v^{-1}\Phi^+_{\tilde{\boldsymbol M}} \subseteq \Phi^+_{\tilde{\boldsymbol G}}$ and ${}^a h \in \tilde{\boldsymbol T}(\mathbb{Q}_p)^{--}$ we obtain for all $\alpha \in \Phi^+_{\tilde{\boldsymbol M}}$ that $v_p(\alpha({}^{bva}h)) = v_p(({}^{v^{-1}b^{-1}}\alpha)({}^a h)) > 0$; hence, ${}^{bva}h$ is regular dominant w.r.t. $\Phi^+_{\tilde{\boldsymbol M}}$. Thus, using equation (52) we obtain that

$$(56)\qquad [\tilde{\mathcal I}^{\tilde M}({}^{bva}h^{-e})\tilde{\mathcal I}^{\tilde M}]\otimes[\tilde K^{(p)} r\tilde K^{(p)}]_{\tilde{\boldsymbol M}} \in \mathcal{C}_0(\tilde{\boldsymbol M}(\mathbb{A}_f)//\tilde K^{\tilde M})_{{}^{bva}h}$$

is contained in the integral Hecke algebra attached to $\tilde{\boldsymbol M}$ and the dominant regular element ${}^{bva}h \in \tilde{\boldsymbol T}(F)$. Now, in Section 4.6 we have seen that $\langle {}^w\rho - \rho, \alpha^\vee\rangle \geq -2\kappa_2$

for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$ ($\rho = \rho_{\tilde{\boldsymbol{G}}}$); hence, we obtain for all $\alpha \in \Delta_{\tilde{\boldsymbol{M}}}$

$$\langle w \cdot \tilde{\lambda}, \alpha^\vee \rangle = \langle \tilde{\lambda}, {}^{w^{-1}}\alpha^\vee \rangle + \langle {}^{w}\rho^\circ - \rho^\circ, \alpha^\vee \rangle \geq 2C - 2\kappa_2.$$

Since also $w \cdot \tilde{\lambda} \equiv w \cdot \tilde{\lambda}'$ (mod $(p-1)p^m X(\tilde{\boldsymbol{T}})$) the induction hypotheses for the group $\tilde{\boldsymbol{M}}$ which has strictly smaller rank than $\tilde{\boldsymbol{G}}$ (since $\tilde{\boldsymbol{M}} \neq \tilde{\boldsymbol{G}}$) is satisfied and we obtain that the following congruence holds:

$$\begin{aligned} (57)\quad & (w \cdot \tilde{\lambda})({}^{bva}h^e) \operatorname{Lef}\big([\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}] \otimes [\tilde{K}^{(p)} r \tilde{K}^{(p)}]_{\tilde{\boldsymbol{M}}} | \bigoplus_i (\pi^i_{w\cdot\tilde{\lambda}})^{\tilde{K}^{\tilde{M}}}\big) \\ & \equiv (w \cdot \tilde{\lambda}')({}^{bva}h^e) \operatorname{Lef}\big([\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}] \otimes [\tilde{K}^{(p)} r \tilde{K}^{(p)}]_{\tilde{\boldsymbol{M}}} | \bigoplus_i (\pi^i_{w\cdot\tilde{\lambda}'})^{\tilde{K}^{\tilde{M}}}\big) \\ & \hspace{20em} (\text{mod } p^\star \mathbb{Z}_p), \end{aligned}$$

where $\star = \lceil \min(m, e(C - \kappa_2 - \kappa_{2,\tilde{\boldsymbol{M}}} \operatorname{rk}(\tilde{\boldsymbol{M}}))) \rceil - e\kappa_{1,\tilde{\boldsymbol{M}}} \operatorname{rk}(\tilde{\boldsymbol{M}})$; moreover, the above Lefschetz numbers are contained in F. Equations (54) and (55) thus imply that

$$\tilde{\lambda}(h^e) \operatorname{Lef}([\tilde{K}[h]_p^{-e} r \tilde{K}] | H^\bullet_{\mathrm{Eis}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))) \in F$$

(note that $\tilde{\lambda}$ and all $\alpha \in \Phi_{\tilde{\boldsymbol{G}}}$ are defined over F, $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})$ and a, b, v, w normalize $\tilde{\boldsymbol{T}}(F)$) and since full cohomology is defined over F we obtain that

$$\operatorname{Lef}([\tilde{K}[h]_p^{-e} r \tilde{K}]_{\tilde{\lambda}} | H^\bullet_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))) \in F.$$

It remains to prove the congruences. Since $\tilde{\lambda} - (bva)^{-1}(w \cdot \tilde{\lambda}) = \tilde{\lambda} - ((bva)^{-1}w) \cdot \tilde{\lambda} + (bva)^{-1}\rho^\circ - \rho^\circ$ we obtain using 4.5 Lemma (note that $h \in \tilde{\boldsymbol{T}}(\mathbb{Q}_p)^{++}$) and $v_p(((bva)^{-1}\rho^\circ - \rho^\circ)(h^e)) \geq -e\kappa_1$ (see Section 4.6) that

$$(58)\qquad \frac{\tilde{\lambda}(h^e)}{(bva)^{-1}(w \cdot \tilde{\lambda})(h^e)} \begin{cases} \text{has } p\text{-adic value} \geq Ce - e\kappa_1 & \text{if } bva \neq w, \\ \text{equals } ((bva)^{-1}\rho^\circ - \rho^\circ)(h^e) & \text{if } bva = w, \end{cases}$$

and the same holds for $\tilde{\lambda}'$. Thus, if $bva = w$, we obtain from equation (57), after multiplying by $((bva)^{-1}\rho^\circ - \rho^\circ)(h^e)$,

$$\begin{aligned} & \tilde{\lambda}(h^e) \operatorname{Lef}\big([\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}] \otimes [\tilde{K}^{(p)} r \tilde{K}^{(p)}]_{\tilde{\boldsymbol{M}}} | \bigoplus_i (\pi^i_{w\cdot\tilde{\lambda}})^{\tilde{K}^{\tilde{M}}}\big) \\ & \equiv \tilde{\lambda}'(h^e) \operatorname{Lef}\big([\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}] \otimes [\tilde{K}^{(p)} r \tilde{K}^{(p)}]_{\tilde{\boldsymbol{M}}} | \bigoplus_i (\pi^i_{w\cdot\tilde{\lambda}'})^{\tilde{K}^{\tilde{M}}}\big) \ (\text{mod } p^{\star - e\kappa_1}\mathbb{Z}_p). \end{aligned}$$

We look at the case $bva \neq w$. Since

$$[\tilde{K}^{(p)} r \tilde{K}^{(p)}]_{\tilde{\boldsymbol{M}}} = \sum_{s \in \tilde{\boldsymbol{M}}(\mathbb{A}_f)^{(p)}} z_s [\tilde{K}^{(p),\tilde{M}} s \tilde{K}^{(p),\tilde{M}}]$$

with $z_s \in \mathbb{Z}$, by equation (52), and since the trace of the $w \cdot \tilde{\lambda}$-normalization of

$$[\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}] \otimes [\tilde{K}^{(p),\tilde{M}} s \tilde{K}^{(p),\tilde{M}}] \in C_0(\tilde{\boldsymbol{M}}(\mathbb{A}_f)//\tilde{K}^{\tilde{M}})_{{}^{bva}h}$$

on $H^i_{\mathrm{cusp}}(S^{\tilde{M}}, L_{w\cdot\tilde{\lambda}}(\mathbb{C}))$ is contained in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$ (see Section 4.11.2) we obtain that the Lefschetz number of $(w \cdot \tilde{\lambda})({}^{bva}h^e)\,[\tilde{\mathcal{I}}^{\tilde{M}}({}^{bva}h^{-e})\tilde{\mathcal{I}}^{\tilde{M}}] \otimes [\tilde{K}^{(p)} r \tilde{K}^{(p)}]_{\tilde{\boldsymbol{M}}}$ on cuspidal cohomology $\bigoplus_i (\pi^i_{w\cdot\tilde\lambda})^{\tilde K_{\tilde M}}$ is p-adically integral; thus using equation (58) we obtain

$$\begin{aligned}&\tilde\lambda(h^e)\,\mathrm{Lef}\big([\tilde{\mathcal I}^{\tilde M}({}^{bva}h^{-e})\tilde{\mathcal I}^{\tilde M}]\otimes[\tilde K^{(p)}r\tilde K^{(p)}]_{\tilde{\boldsymbol M}}\,|\,\bigoplus_i(\pi^i_{w\cdot\tilde\lambda})^{\tilde K_{\tilde M}}\big)\\ &\equiv\tilde\lambda'(h^e)\,\mathrm{Lef}\big([\tilde{\mathcal I}^{\tilde M}({}^{bva}h^{-e})\tilde{\mathcal I}^{\tilde M}]\otimes[\tilde K^{(p)}r\tilde K^{(p)}]_{\tilde{\boldsymbol M}}\,|\,\bigoplus_i(\pi^i_{w\cdot\tilde\lambda'})^{\tilde K_{\tilde M}}\big)\pmod{p^{Ce-e\kappa_1}\mathbb Z_p}.\end{aligned}$$

Since $Ce-e\kappa_1\ge\star-e\kappa_1$, in both cases the congruence holds modulo $p^{\star-e\kappa_1}\mathbb Z_p$. Using equations (54) and (55) we obtain

$$\begin{aligned}&\mathrm{Lef}([\tilde K[h]_p^{-e}r\tilde K]_{\tilde\lambda}\,|\,H^\bullet_{\mathrm{Eis}}(S_{\tilde K},L_{\tilde\lambda}(\mathbb C)))\\ &\qquad\equiv\mathrm{Lef}([\tilde K[h]_p^{-e}r\tilde K]_{\tilde\lambda'}\,|\,H^\bullet_{\mathrm{Eis}}(S_{\tilde K},L_{\tilde\lambda'}(\mathbb C)))\pmod{p^{\star-e\kappa_1}\mathbb Z_p}.\end{aligned}$$

The rank of $\tilde{\boldsymbol M}$ is strictly smaller than the rank of $\tilde{\boldsymbol G}$ and $\kappa_{i,\tilde{\boldsymbol M}}\le\kappa_i$, as follows from Section 4.6; hence, $\kappa_i+\kappa_{i,\tilde{\boldsymbol M}}\mathrm{rk}(\tilde{\boldsymbol M})\le\kappa_i\mathrm{rk}(\tilde{\boldsymbol G})$ which yields

$$\star-e\kappa_1\ge\lceil\min(m,e(C-\kappa_2\mathrm{rk}(\tilde{\boldsymbol G})))\rceil-\mathrm{rk}(\tilde{\boldsymbol G})e\kappa_1.$$

Together with 4.11.3 Proposition this implies the claim about congruences for the Lefschetz numbers of $[\tilde K[h]_p^{-e}r\tilde K]_{\tilde\lambda}$ on cuspidal cohomology for $S_{\tilde K}$. Thus, the theorem is proven. □

4.13.3. Remark. Section 4.11.2 implies that the Lefschetz number

$$\mathrm{Lef}(\mathbb T_{\tilde\lambda}\,|\,H^\bullet_{\mathrm{cusp}}(S_{\tilde K},L_{\tilde\lambda}(\mathbb C))),\qquad\mathbb T\in\mathcal C_0(\tilde{\boldsymbol G}(\mathbb A_f)//\tilde K)^{\mathrm{reg}}_h,$$

is contained in $\mathcal O_{\bar{\mathbb Q}_p}$. Thus, 4.13.2 Theorem implies that $\mathrm{Lef}(\mathbb T_{\tilde\lambda}\,|\,H^\bullet_{\mathrm{cusp}}(S_{\tilde K},L_{\tilde\lambda}(\mathbb C)))$ is contained in $F\cap\mathcal O_{\bar{\mathbb Q}_p}\subseteq\mathbb Z_p$.

4.14. ***Weighted cohomology.*** In this section we compare two Goresky–MacPherson trace formulas for two different weights $\tilde\lambda$ and $\tilde\lambda'$. This is analogous to the comparison of Bewersdorff's trace formula in previous sections and relies on the same diagonalization of elements in $\tilde{\mathcal I}h^e\tilde{\mathcal I}$ (see 4.4 Proposition) but now applied to all Levi subgroups $\tilde{\boldsymbol M}$ of parabolic subgroups in $\tilde{\boldsymbol G}/\mathbb Q$ (see 4.7 Proposition). As a result we obtain congruences for Hecke operators on weighted cohomology for varying weight $\tilde\lambda$.

Here, we will work again in a classical, non-adelic setting (see Section 2.2 and 2.3); e.g., $\Gamma\le\tilde{\boldsymbol G}(\mathbb Q)$ is an arithmetic subgroup contained in $\tilde{\mathcal I}$, $h\in\tilde{\boldsymbol T}(\mathbb Q)^{++}$ and we will consider Hecke operators $\Gamma\zeta\Gamma$ where $\zeta\in\Delta=\Delta_h$.

4.14.1. *The trace formula of Goresky–MacPherson.* We select a minimal $\mathbb Q$-parabolic subgroup $\tilde{\boldsymbol P}_0$ in $\tilde{\boldsymbol G}/\mathbb Q$ with Levi decomposition $\tilde{\boldsymbol P}_0=\tilde{\boldsymbol M}_0\tilde{\boldsymbol N}_0$ and we denote by $\boldsymbol A_0$ a maximal $\mathbb Q$-split torus in the center of the Levi subgroup $\tilde{\boldsymbol M}_0$. We may assume that $\tilde{\boldsymbol B}\subseteq\tilde{\boldsymbol P}_0/F$ and $\tilde{\boldsymbol T}\supseteq\boldsymbol A_0/F$. Let $\tilde{\boldsymbol P}=\tilde{\boldsymbol M}\tilde{\boldsymbol N}$ be a $\mathbb Q$-parabolic subgroup in $\tilde{\boldsymbol G}$. We denote by $\boldsymbol A_{\tilde P}$ a maximal $\mathbb Q$-split torus in the center of $\tilde{\boldsymbol M}$ and we write

$\Delta_{\tilde{P}} = \{\alpha_1, \dots, \alpha_m\} \subset X(A_{\tilde{P}}/A_{\tilde{G}})$ for the set of simple roots of $A_{\tilde{P}}$ occuring in $\mathrm{Lie}(\tilde{N})$ and

$$\{t_{\alpha_1}, \dots, t_{\alpha_m}\} \subset X_*(A_{\tilde{P}}/A_{\tilde{G}}) \otimes \mathbb{Q}$$

for the basis dual to $\Delta_{\tilde{P}}$. We select $h \in \tilde{T}(\mathbb{Q})^{++}$ and we let $\zeta \in \Delta = \Delta_h$. The double coset $\Gamma\zeta\Gamma$ induces an operator on the weighted cohomology groups $W^\nu H^i(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(\mathbb{C}))$ with weight profile $\nu \in X(A_0)\otimes\mathbb{Q}$. Goresky and MacPherson computed the Lefschetz number of $\Gamma\zeta\Gamma$ acting on weighted cohomology:

Theorem [Goresky and MacPherson 2003, 1.4 Theorem].

$$\mathrm{Lef}\big(\Gamma\zeta\Gamma | W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(\mathbb{C}))\big) = \sum_{\{\tilde{P}\}} \sum_i \sum_{\{\xi\}} \Xi_{\tilde{P},\xi} \sum_{\substack{w\in W^{\tilde{P}} \\ I_\nu(w,\tilde{\lambda}) = \Delta^+_{\tilde{P}}(\xi)}} (-1)^{\ell(w)} \operatorname{tr}\big(\xi^{-1} | L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(F)\big).$$

In this formula, $\tilde{P}$ runs over a choice of representatives for the Γ-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of $\tilde{G}$. For each such $\tilde{P}$ we write $\Gamma\zeta\Gamma \cap \tilde{P}(\mathbb{Q}) = \coprod_i \Gamma_{\tilde{P}}\zeta_i\Gamma_{\tilde{P}}$ where $\Gamma_{\tilde{P}} = \Gamma \cap \tilde{P}(\mathbb{Q})$ and $\zeta_i \in \tilde{P}(\mathbb{Q})$. The second sum runs over these finitely many double cosets. We set $\Gamma_{\tilde{M}} = \nu_{\tilde{P}}(\Gamma_{\tilde{P}}) \subseteq \tilde{M}(\mathbb{Q})$ and $\bar{\zeta}_i = \nu_{\tilde{P}}(\zeta_i) \in \tilde{M}(\mathbb{Q})$; here $\tilde{P} = \tilde{M}\tilde{N}$ is the Levi decomposition and $\nu_{\tilde{P}} : \tilde{P} \to \tilde{M}$ is the canonical mapping. The third sum is over a set of representatives ξ of the $\Gamma_{\tilde{M}}$-conjugacy classes of elliptic (modulo $A_{\tilde{P}}(\mathbb{R})$) elements in $\Gamma_{\tilde{M}}\bar{\zeta}_i\Gamma_{\tilde{M}} \subseteq \tilde{M}(\mathbb{Q})$. (The numbers $\Xi_{\tilde{P},\xi}$ are explained in [Goresky and MacPherson 2003, 1.4]; we only need to know that they are contained in $\mathbb{Z}$ and do not depend on the weight $\tilde{\lambda}$.) Moreover,

$$\Delta^+_{\tilde{P}}(\xi) = \{\alpha \in \Delta_{\tilde{P}} : \alpha(a_\xi) < 1\},$$

where a_ξ is the projection of ξ to the identity component $A_{\tilde{P}}$ of $A_{\tilde{P}}(\mathbb{R})$ and $I_\nu(w,\tilde{\lambda})$ is given as

$$I_\nu(w,\tilde{\lambda}) = \{\alpha_i \in \Delta_{\tilde{P}} : \langle w(\tilde{\lambda}+\rho) - \rho - \nu, t_{\alpha_i}\rangle < 0\},$$

where $\rho = \rho_{\tilde{G}}$. Finally, since $\xi \in \tilde{M}(\mathbb{Q})$ and $L^{\tilde{M}}_{w\cdot\tilde{\lambda}}$ is defined over F, the trace may be computed on F-points of $L^{\tilde{M}}_{w\cdot\tilde{\lambda}}$. We note that to make sense of the trace of $\xi^{-1} \in \tilde{M}(\mathbb{Q})$ on $L^{\tilde{M}}_{\tilde{\lambda}}(F)$ ($L^{\tilde{M}}_{\tilde{\lambda}}$ was defined for standard parabolic subgroups) as well as of the definition of $I_\nu(w,\tilde{\lambda})$ ($\tilde{\lambda}$, ρ are characters of $\tilde{T}$) we have to conjugate $\tilde{P}$, i.e., we select $x \in \tilde{G}(F)$ such that ${}^x\tilde{P}/F$ is standard parabolic.

4.14.2. We select a $\mathbb{Q}$-parabolic subgroup $\tilde{P}$ in $\tilde{G}$ with Levi decomposition $\tilde{P} = \tilde{M}\tilde{N}$. For any $w \in W^{\tilde{P}}$ and any $\alpha_i \in \Delta_{\tilde{P}}$ the assignment $L_{w,i} : X(\tilde{T})\otimes\mathbb{Q} \to \mathbb{Q}$ taking $\tilde{\lambda}$ to $\langle w\tilde{\lambda}, t_{\alpha_i}\rangle$ is linear and we denote by $H^+_{w,i}$ (resp. $H^-_{w,i}$) the half space consisting of all $\tilde{\lambda} \in X(\tilde{T})$ such that $\langle w(\tilde{\lambda}+\rho) - \rho - \nu, t_{\alpha_i}\rangle$ is positive (resp. negative). For any $\underline{\epsilon} = (\epsilon_{w,i}) \in \{\pm\}^{W^{\tilde{P}} \times [1,\dots,m]}$ we set

$$C(\underline{\epsilon}) = X(\tilde{\boldsymbol{T}})^{\mathrm{dom}} \cap \bigcap_{w,i} H_{w,i}^{\epsilon(w,i)}.$$

Thus, $C(\underline{\epsilon})$ is an intersection of half spaces which may be empty. For all $w \in W^{\tilde{P}}$ and $i = 1, \dots, m$ the values $\langle w(\tilde{\lambda} + \rho) - \rho - \nu, t_{\alpha_i} \rangle$ and $\langle w(\tilde{\lambda}' + \rho) - \rho - \nu, t_{\alpha_i} \rangle$, where $\tilde{\lambda}, \tilde{\lambda}' \in C(\underline{\epsilon})$ have the same sign, hence, for all $w \in W^{\tilde{P}}$ we obtain $I_\nu(w, \tilde{\lambda}) = I_\nu(w, \tilde{\lambda}')$, i.e., $I_\nu(w, \tilde{\lambda})$ does not depend on $\tilde{\lambda} \in C(\underline{\epsilon})$.

Theorem. *Let $C \in \mathbb{Q}_{>0}$ and $\underline{\epsilon} \in \{\pm\}^{W^{\tilde{P}} \times [1,\dots,m]}$. Let $\tilde{\lambda}, \tilde{\lambda}' \in C(\underline{\epsilon})$ be (dominant) weights satisfying*

- *$\langle \tilde{\lambda}, \alpha^\vee \rangle > 2C$ and $\langle \tilde{\lambda}', \alpha^\vee \rangle > 2C$ for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$,*
- *$\tilde{\lambda} \equiv \tilde{\lambda}' \pmod{(p-1)p^{m-1} X(\tilde{\boldsymbol{T}})}$ ($m \in \mathbb{N}$).*

Let ζ be contained in the semigroup Δ_h, hence, $\zeta \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ for some $e \in \mathbb{N}_0$ and we assume that $e \in \mathbb{N}$. Then the Lefschetz number $\mathrm{Lef}\big((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(F))\big)$ is contained in F and the following congruence holds:

$$\mathrm{Lef}\big((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(F))\big) \equiv \mathrm{Lef}\big((\Gamma\zeta\Gamma)_{\tilde{\lambda}'} | W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}'}(F))\big) \pmod{p^{\lceil \min((C-\kappa_1-\kappa_2)e, m-e\kappa_1) \rceil} \mathbb{Z}_p}.$$

Proof. We look at the Goresky–MacPherson trace formula. Since $\Gamma\zeta\Gamma \cap \tilde{\boldsymbol{P}}(\mathbb{Q}) \supseteq \Gamma_{\tilde{\boldsymbol{P}}} \zeta_i \Gamma_{\tilde{\boldsymbol{P}}}$ we obtain $\zeta_i \in \Gamma\zeta\Gamma \cap \tilde{\boldsymbol{P}}(\mathbb{Q})$, hence, we can write $\zeta_i = \bar{\zeta}_i u$ where $\bar{\zeta}_i \in \tilde{\boldsymbol{M}}(\mathbb{Q})$ and $u \in \tilde{\boldsymbol{N}}(\mathbb{Q})$. Let $\xi \in \Gamma_{\tilde{\boldsymbol{M}}} \bar{\zeta}_i \Gamma_{\tilde{\boldsymbol{M}}}$ be a representative of a $\Gamma_{\tilde{\boldsymbol{M}}}$-conjugacy class. We may assume that $\xi = \gamma_M \bar{\zeta}_i$ for some $\gamma_M \in \Gamma_{\tilde{\boldsymbol{M}}}$, i.e., $\gamma_M = \nu_{\tilde{\boldsymbol{P}}}(\gamma_P)$ for some $\gamma_P \in \Gamma_{\tilde{\boldsymbol{P}}}$. Hence, we can write $\gamma_P = \gamma_N \gamma_M$, where $\gamma_N \in \tilde{\boldsymbol{N}}(\mathbb{Q})$, and obtain

$$\gamma_N \xi u = \gamma_N \gamma_M \bar{\zeta}_i u = \gamma_P \zeta_i \in \Gamma_{\tilde{\boldsymbol{P}}} \zeta_i \subseteq \Gamma\zeta\Gamma \subseteq \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}.$$

Since $\tilde{\boldsymbol{M}}(\mathbb{Q})$ normalizes $\tilde{\boldsymbol{N}}(\mathbb{Q})$ we can write $\gamma_N \xi u = \xi u'$ with $u' \in \tilde{\boldsymbol{N}}(\mathbb{Q})$. Thus, $\xi u' \in \tilde{\mathcal{I}} h^e \tilde{\mathcal{I}}$ and we may apply 4.7 Proposition to compute $\mathrm{tr}(\xi^{-1} | L^{\tilde{M}}_{w\cdot\tilde{\lambda}}(\mathbb{Q}_p))$. If we put the result in the trace formula of Goresky–MacPherson we obtain

$$\begin{aligned}(59)\quad &\mathrm{Lef}((\Gamma\zeta\Gamma)_{\tilde{\lambda}} | W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde{\lambda}}(F))) \\ &\equiv \sum_{\{\tilde{\boldsymbol{P}}\}} \sum_{i\xi} \sum_{\{\xi\}} \Xi_{\tilde{\boldsymbol{P}},\xi} \delta_s (-1)^{\ell(s)} \epsilon_{\tilde{\boldsymbol{P}},\xi u'} \tilde{\lambda}(h^e t^{-1}) \pmod{p^{(C-\kappa_1-\kappa_2)e} \mathcal{O}_{\bar{\mathbb{Q}}_p}},\end{aligned}$$

where $s = s_{\xi u'} \in W^{\tilde{P}}$, $t = t_{\xi u'} \in \tilde{\boldsymbol{T}}(\bar{\mathbb{Q}}_p)^{++}$ is the unique element which is $\tilde{\boldsymbol{G}}(\bar{\mathbb{Q}}_p)$-conjugate to $(\xi u')_s$, $\epsilon_{\tilde{\boldsymbol{P}},\xi u'}$ has p-adic value $\geq -e\kappa_1$ and $\delta_s = 1$ if $I_\nu(s, \tilde{\lambda}) = \Delta^+_{\tilde{P}}(\xi)$ and vanishes otherwise. We note that δ_s does not depend on $\tilde{\lambda}$ since we assume that $\tilde{\lambda}$ is contained in $C(\underline{\epsilon})$. By our assumption there is $\chi \in X(\tilde{\boldsymbol{T}})$ such that $\tilde{\lambda} - \tilde{\lambda}' = (p-1)p^{m-1}\chi$. Since $\chi(h^e t^{-1}) \equiv \epsilon \pmod{p\mathcal{O}_{\bar{\mathbb{Q}}_p}}$ where $\epsilon \in \mathbb{Z}_p^*$ (see 4.4 Proposition) this implies $\tilde{\lambda}(h^e t^{-1}) \equiv \tilde{\lambda}'(h^e t^{-1}) \pmod{p^m \mathcal{O}_{\bar{\mathbb{Q}}_p}}$ (see the proof of

4.10 Proposition) and we find that equation (59) (which also holds for $\tilde\lambda'$) implies that the Lefschetz numbers in the Theorem are congruent to each other modulo $p^{\lceil \min((C-\kappa_1-\kappa_2)e, m-e\kappa_1)\rceil}\mathbb{Z}_p$ (note that the Lefschetz numbers are contained in F). Thus the proof is complete. □

4.14.3. *The case C_2.* We assume that $\tilde{\boldsymbol{G}}/\mathbb{Q}$ is connected and $\mathbb{Q}$-split with root system of type C_2, hence, there are two simple roots α_1, α_2. We also assume that $\nu = -\rho$, i.e., ν is the middle weight profile. Thus, if $\tilde\lambda$ is regular then $W^\nu H^d(\overline{\Gamma\backslash X}, L_{\tilde\lambda}(\mathbb{C})) = H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{C}))$ and both these cohomology groups vanish for all degrees $i \neq d$. Let $\tilde\lambda \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$. Hence, $\tilde\lambda + \rho \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ is strictly dominant and for any $w \in W_{\tilde{\boldsymbol{G}}}$ we write $w(\tilde\lambda+\rho) = a_1\alpha_1 + a_2\alpha_2$ with $a_i = a_{i,w,\tilde\lambda} \in \mathbb{Q}$. For the root system of type C_2 it happens that the sign of a_1 as well as the sign of a_2 does not depend on $\tilde\lambda \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$, i.e., $\mathrm{sign}(a_{i,w,\tilde\lambda}) = \mathrm{sign}(a_{i,w,\tilde\lambda'})$ for any $\tilde\lambda, \tilde\lambda' \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ and all $w \in W_{\tilde{\boldsymbol{G}}}$ and all $i = 1, 2$. Thus, if $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{B}}$, hence, $\boldsymbol{A}_{\tilde{\boldsymbol{P}}} = \tilde{\boldsymbol{T}}/\mathbb{Q}$ and if $\{t_{\alpha_1}, t_{\alpha_2}\}$ denotes the basis of $X_*(\tilde{\boldsymbol{T}}) \otimes \mathbb{Q}$ dual to $\{\alpha_1, \alpha_2\}$ then we obtain that the sign of $\langle w(\tilde\lambda+\rho), t_{\alpha_i}\rangle = a_i$ does not depend on $\tilde\lambda \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$. Similarly if $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{P}}_{\alpha_i}$, hence, $\boldsymbol{A}_{\tilde{\boldsymbol{P}}} = \ker\alpha_j$, $j \neq i$, and if $\{t_{\alpha_i}\}$ denotes the basis of $X_*(\boldsymbol{A}_{\tilde{\boldsymbol{P}}}) \otimes \mathbb{Q}$ dual to $\{\alpha_i\}$ then we obtain that the sign of $\langle w(\tilde\lambda+\rho)|_{\boldsymbol{A}_{\tilde{\boldsymbol{P}}}}, t_{\alpha_i}\rangle = a_i$ does not depend on $\tilde\lambda \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$. This shows that $I_\nu(w, \tilde\lambda)$ does not depend on $\tilde\lambda \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ (if $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{G}}$ then $I_\nu(w, \tilde\lambda)$ is empty). 4.14.2 Theorem therefore holds with $X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ in place of $C(\underline{\epsilon})$:

Corollary. *Assume $\tilde{\boldsymbol{G}}/\mathbb{Q}$ is connected and $\mathbb{Q}$-split with root system C_2. Let $C \in \mathbb{Q}_{>0}$ and let $\tilde\lambda, \tilde\lambda' \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ be weights satisfying*

- *$\langle\tilde\lambda, \alpha^\vee\rangle > 2C$ and $\langle\tilde\lambda', \alpha^\vee\rangle > 2C$ for all $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$, and*
- *$\tilde\lambda \equiv \tilde\lambda' \pmod{(p-1)p^{m-1}X(\tilde{\boldsymbol{T}})}$ ($m \in \mathbb{N}$).*

Then $\mathrm{tr}((\Gamma\zeta\Gamma)_{\tilde\lambda}|H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{C})))$ is contained in F and

$$\mathrm{tr}\big((\Gamma\zeta\Gamma)_{\tilde\lambda}|H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{C}))\big) \equiv \mathrm{tr}\big((\Gamma\zeta\Gamma)_{\tilde\lambda'}|H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde\lambda'}(\mathbb{C}))\big) \pmod{p^{\lceil \min((C-\kappa_1-\kappa_2)e, m-e\kappa_1)\rceil}\mathbb{Z}_p}.$$

Proof. Use the equality

$$\mathrm{tr}\big((\Gamma\zeta\Gamma)_{\tilde\lambda}|H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde\lambda}(\mathbb{C}))\big) = (-1)^d\,\mathrm{Lef}\big((\Gamma\zeta\Gamma)_{\tilde\lambda}|W^\nu H^\bullet(\overline{\Gamma\backslash X}, L_{\tilde\lambda}(\mathbb{C}))\big). \quad \square$$

Remark. The $\mathbb{Q}$-rank of $\tilde{\boldsymbol{G}}$ does not appear in the modulus of these congruences.

5. Local constancy of dimension of slope subspaces

5.1. ***Slope subspaces of cuspidal cohomology.*** As before we let $\tilde{\boldsymbol{G}}/\mathbb{Q}$ be a connected reductive group with $\mathbb{Q}_p$-split maximal torus $\tilde{\boldsymbol{T}}/\mathbb{Q}$ and from now on we assume in addition that $\tilde{\boldsymbol{G}}/\mathbb{Q}$ has discrete series. We denote by ℓ the $\mathbb{Q}$-rank of

$\tilde{\boldsymbol{G}}$ and we keep the notations from Section 4.11. In particular, $\tilde{K} \leq \tilde{\boldsymbol{G}}(\mathbb{A}_f)$ is a compact open subgroup with p-component $\tilde{K}_p = \tilde{\mathcal{I}}$ and $S_{\tilde{K}}$ is the adelic locally symmetric space of level $\tilde{K}$. We select elements $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ and $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$ and we set

$$\mathbb{T} = [\tilde{K}[h]_p^{-1} r \tilde{K}] \in \mathcal{C}_0(\tilde{\boldsymbol{G}}(\mathbb{A}_f)//\tilde{K})_h^{\mathrm{reg}}.$$

As before we denote by $\mathbb{T}_{\tilde{\lambda}}$ the $\tilde{\lambda}$-normalization of $\mathbb{T}$. The (normalized) operator $\mathbb{T}_{\tilde{\lambda}}$ acts on

$$H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C}) = H^d_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))$$

where $d = d_{\tilde{\boldsymbol{G}}}$ is the middle degree. For $\beta \in \mathbb{Q}_{\geq 0}$ we denote by

$$H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})^\beta = \bigoplus_{\substack{\mu \in \bar{F} \hookrightarrow \bar{\mathbb{Q}}_p \\ v_p(\mu)=\beta}} H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})(\mu),$$

the slope β subspace of $H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})$ w.r.t. $\mathbb{T}_{\tilde{\lambda}}$; here, $H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})(\mu) = H^d_{\tilde{\lambda},\mathrm{cusp},\mathbb{T}}(\mathbb{C})(\mu)$ is the generalized eigenspace attached to $\mathbb{T}_{\tilde{\lambda}}$ and the eigenvalue μ and we remark that the eigenvalues μ of $\mathbb{T}_{\tilde{\lambda}}$ on $H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})$ are contained in $\bar{F}(\subseteq \mathbb{C})$ and are p-adically integral, i.e., contained in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$ (note that $\bar{F} \subseteq \bar{\mathbb{Q}}_p$; see Section 4.11.2). We set $H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})^{\leq\beta} = \bigoplus_{0\leq\gamma\leq\beta} H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})^\gamma$. Since

$$\begin{aligned}
\dim H^d_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))^{\leq\beta} &= \dim \bigoplus_{\substack{\mu \in \bar{F} \hookrightarrow \bar{\mathbb{Q}}_p \\ 0\leq v_p(\mu)\leq\beta}} H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})(\mu) \\
&\leq \dim \bigoplus_{\substack{\mu \in \bar{F} \\ 0\leq v_p(\mu)\leq\beta}} H^d(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))(\mu) \\
&= \dim \bigoplus_{\substack{\mu \in \bar{F} \\ 0\leq v_p(\mu)\leq\beta}} H^d(S_{\tilde{K}}, L_{\tilde{\lambda}}(\bar{F}))(\mu) \\
&= \dim \bigoplus_{\substack{\mu \in \bar{F} \hookrightarrow \bar{\mathbb{Q}}_p \\ 0\leq v_p(\mu)\leq\beta}} H^d(S_{\tilde{K}}, L_{\tilde{\lambda}}(\bar{\mathbb{Q}}_p))(\mu) \\
&= \dim H^d(S_{\tilde{K}}, L_{\tilde{\lambda}}(\bar{\mathbb{Q}}_p))^{\leq\beta} \\
&= \dim H^d(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{Q}_p))^{\leq\beta},
\end{aligned}$$

we obtain the following bound from 4.11.4 Theorem. Recall that $s = |\Phi^+_{\tilde{\boldsymbol{G}}}|$, $\sigma = \max_{\alpha\in\Phi^+_{\tilde{G}}} \mathrm{ht}(\alpha)$ and we denote by $\boldsymbol{g} = \boldsymbol{g}_{\tilde{K}}$ the number of d cells in a cell complex which is homotopy equivalent to the Borel–Serre compactification $\bar{S}_{\tilde{K}}$. We set

$$\mathtt{m} = \mathtt{m}_{\tilde{K}} = 12\frac{\boldsymbol{g}}{s}\sigma^{s+1} \in \mathbb{Q}_{\geq 0} \quad \text{and} \quad \mathtt{n} = \mathtt{n}_{\tilde{K}} = \left\lceil \frac{1}{2^{s+1}} \frac{\boldsymbol{g}\sigma^{s+1}}{s} M^s \right\rceil + 1 \in \mathbb{N},$$

where $M = M(\sigma, s) \in \mathbb{N}$ is defined in equations (23) and (24); then

$$\dim H^d_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))^{\leq \beta} \leq B(\beta) := \mathtt{m}\beta^s + \mathtt{n} \tag{60}$$

for all dominant $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$, all $\beta \in \mathbb{Q}_{\geq 0}$ and all $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ and $r \in \tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$.

5.2. We want to consider the function

$$d(\,\cdot\,,\cdot\,) : \mathbb{Q}_{\geq 0} \times X(\tilde{\boldsymbol{T}})^{\text{dom}} \to \mathbb{N}_0, \quad (\beta, \tilde{\lambda}) \mapsto \dim H^d_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))^{\beta};$$

i.e., we want to understand how the dimension of the slope subspace varies as a function of the weight. To this end, for any $\beta \in \mathbb{Q}_{\geq 0}$ we set

$$\begin{aligned} \boldsymbol{m}_1(\beta) &= \left(\beta + \tfrac{1}{p-1} + (\kappa_1 + \kappa_2)\ell\right) B(\beta) + 1 \\ &= \mathtt{m}\beta^{s+1} + \mathtt{m}\left(\tfrac{1}{p-1} + (\kappa_1 + \kappa_2)\ell\right)\beta^s + \mathtt{n}\beta + \mathtt{n}\left(\tfrac{1}{p-1} + (\kappa_1 + \kappa_2)\ell\right) + 1 \\ &\qquad \in \mathbb{Q}_{>0}. \end{aligned} \tag{61}$$

and

$$\begin{aligned} \boldsymbol{m}_2(\beta) &= \left(\beta + \tfrac{1}{p-1} + \kappa_1\ell\right) B(\beta) + 1 \\ &= \mathtt{m}\beta^{s+1} + \mathtt{m}\left(\tfrac{1}{p-1} + \kappa_1\ell\right)\beta^s + \mathtt{n}\beta + \mathtt{n}\left(\tfrac{1}{p-1} + \kappa_1\ell\right) + 1 \in \mathbb{Q}_{>0}. \end{aligned} \tag{62}$$

Thus, $\boldsymbol{m}_1(\beta), \boldsymbol{m}_2(\beta) \in \mathbb{Q}[\beta]$ are polynomials in β with positive coefficients, degree $s+1$ and leading term $\mathtt{m} = 12\frac{\boldsymbol{g}}{s}\sigma^{s+1}$ which depend on $\tilde{K}$ (and, hence, on $\tilde{\boldsymbol{G}}$ and p) and on h (since $\kappa_1 = \kappa_1(h)$) but do not depend on $\tilde{\lambda} \in X(\tilde{\boldsymbol{T}})$.

Theorem. *Assume that $\tilde{\boldsymbol{G}}$ has discrete series. Let $\beta \in \mathbb{Q}_{\geq 0}$ and assume the dominant weights $\tilde{\lambda}, \tilde{\lambda}' \in X(\tilde{\boldsymbol{T}})$ satisfy*

- $\langle \tilde{\lambda}, \alpha^\vee \rangle > 2\boldsymbol{m}_1(\beta)$ *and* $\langle \tilde{\lambda}', \alpha^\vee \rangle > 2\boldsymbol{m}_1(\beta)$ *for all* $\alpha \in \Delta_{\tilde{\boldsymbol{G}}}$,
- $\tilde{\lambda} \equiv \tilde{\lambda}' \pmod{(p-1)p^{m-1}X(\tilde{\boldsymbol{T}})}$ *with* $m \geq \boldsymbol{m}_2(\beta)$ $(m \in \mathbb{N})$.

Then

$$\dim H^d_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))^{\gamma} = \dim H^d_{\text{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}'}(\mathbb{C}))^{\gamma} \quad \textit{for all } 0 \leq \gamma \leq \beta.$$

5.3. ***Characteristic polynomial.*** The proof of the preceding theorem will be given in Section 5.6. To prepare it we consider the characteristic polynomial. We denote by $\text{ch}_{\tilde{\lambda}}(X) = \det(X\mathbf{1} - \mathbb{T}_{\tilde{\lambda}}) = \sum_{i=0}^m (-1)^i a_{i,\tilde{\lambda}} X^{m-i} \in \mathbb{C}[X]$, where $m = m_{\tilde{\lambda}} = \dim H^d_{\tilde{\lambda},\text{cusp}}(\mathbb{C})$, the characteristic polynomial of $\mathbb{T}_{\tilde{\lambda}}$ acting on $H^d_{\tilde{\lambda},\text{cusp}}(\mathbb{C})$. We set $a_{i,\tilde{\lambda}} = 0$ if $i > m_{\tilde{\lambda}}$. This definition of the characteristic polynomial differs from the one used in the proof of 4.3 Lemma by a factor $(-1)^m$, but we can refer directly to

[Koecher 1983, 3.4.6 Satz, p. 117], where this definition is used and which yields the following inductive formula for the coefficients of $\mathrm{ch}_{\tilde{\lambda}}(X)$: $a_{0,\tilde{\lambda}} = 1$ and

$$ia_{i,\tilde{\lambda}} = \sum_{e=1}^{i} (-1)^{e+1} \mathrm{tr}\, \mathbb{T}^e_{\tilde{\lambda}}\, a_{i-e,\tilde{\lambda}}, \quad i = 1, 2, \ldots, \tag{63}$$

where we have set

$$\mathrm{tr}\, \mathbb{T}^e_{\tilde{\lambda}} = \mathrm{tr}\, ((\mathbb{T}_{\tilde{\lambda}})^e \,|\, H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})).$$

Since $[\tilde{\mathcal{I}} h^{-e} \tilde{\mathcal{I}}][\tilde{\mathcal{I}} h^{-f} \tilde{\mathcal{I}}] = [\tilde{\mathcal{I}} h^{-(e+f)} \tilde{\mathcal{I}}]$ we obtain that $\mathbb{T}^e = \sum_s c_s [\tilde{K}[h]_p^{-e} s \tilde{K}]$ for all $e \geq 1$, where s runs over $\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$ and $c_s \in \mathbb{Z}$. Hence $(\mathbb{T}_{\tilde{\lambda}})^e = \sum_s c_s [\tilde{K}[h]_p^{-e} s \tilde{K}]_{\tilde{\lambda}}$. Since $\tilde{\boldsymbol{G}}$ has discrete series we therefore obtain if the highest weight $\tilde{\lambda}$ is regular

$$\begin{aligned} \mathrm{tr}\, \mathbb{T}^e_{\tilde{\lambda}} &= \sum_s c_s \mathrm{tr}\big([\tilde{K}[h]_p^{-e} s \tilde{K}]_{\tilde{\lambda}} \,|\, H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})\big) \\ &= \sum_s c_s (-1)^d\, \mathrm{Lef}\big([\tilde{K}[h]_p^{-e} s \tilde{K}]_{\tilde{\lambda}} \,|\, H^\bullet_{\mathrm{cusp}}(S_{\tilde{K}}, L_{\tilde{\lambda}}(\mathbb{C}))\big). \end{aligned} \tag{64}$$

Hence, equation (63) and 4.13.2 Theorem yield $a_{i,\tilde{\lambda}} \in F$ for all i, i.e.,

$$\mathrm{ch}_{\tilde{\lambda}}(X) \in F[X].$$

In particular, the roots of $\mathrm{ch}_{\tilde{\lambda}}$, which are the eigenvalues of $\mathbb{T}_{\tilde{\lambda}}$ on $H^d_{\tilde{\lambda},\mathrm{cusp}}(\mathbb{C})$, are algebraic over F and in Section 4.11.2 we have seen that after embedding $\bar{F} \subseteq \bar{\mathbb{Q}}_p$ they are contained in $\mathcal{O}_{\bar{\mathbb{Q}}_p}$. Hence, $a_{i,\tilde{\lambda}} \in \mathcal{O}_{\bar{\mathbb{Q}}_p}$, i.e., the coefficients are p-adically integral which implies that $a_{i,\tilde{\lambda}} \in \mathbb{Z}_p$, $i = 0, \ldots, m$; in particular,

$$\mathrm{ch}_{\tilde{\lambda}}(X) \in \mathbb{Z}_p[X].$$

5.4. Proposition. *Let $\beta \in \mathbb{Q}_{\geq 0}$. Assume that the weights $\tilde{\lambda}, \tilde{\lambda}' \in X(\tilde{\boldsymbol{T}})^{\mathrm{dom}}$ satisfy the two assumptions of 5.2 Theorem. Then for all $i = 0, 1, 2, 3, \ldots, B(\beta)$ the following congruence holds*:

$$a_{i,\tilde{\lambda}} \equiv a_{i,\tilde{\lambda}'} \pmod{p^{\lceil \beta B(\beta) + 1 \rceil} \mathbb{Z}_p}.$$

Proof. We will prove that for all $i = 0, 1, \ldots, B(\beta)$ the congruence

$$a_{i,\tilde{\lambda}} \equiv a_{i,\tilde{\lambda}'} \pmod{p^{\lceil \beta B(\beta) + \frac{B(\beta)}{p-1} + 1 \rceil - v_p(i!)} \mathbb{Z}_p}$$

holds. Since $v_p(i!) \leq i/(p-1) \leq B(\beta)/(p-1)$ these congruences imply that the congruences of the Proposition hold. The congruences hold trivially for $i = 0$ ($a_{0,\tilde{\lambda}} = a_{0,\tilde{\lambda}'} = 1$). To prove them for $i = 1, 2, 3, \ldots, B(\beta)$ we use equation (63). First, for all $e \in \mathbb{N}$ with $1 \leq e \leq B(\beta)$ we have

$$m - e\kappa_1 \ell \geq \boldsymbol{m}_2(\beta) - e\kappa_1 \ell \geq \beta B(\beta) + \frac{B(\beta)}{p-1} + 1$$

and

$$\begin{aligned}\left\lceil e(\boldsymbol{m}_1(\beta)-\kappa_2\ell)\right\rceil - e\kappa_1\ell &\geq \boldsymbol{m}_1(\beta)-e\kappa_2\ell-e\kappa_1\ell\\ &= \left(\beta+\tfrac{1}{p-1}\right)B(\beta)+(\kappa_1+\kappa_2)\ell B(\beta)+1-e\kappa_2\ell-e\kappa_1\ell\\ &\geq \beta B(\beta)+\frac{B(\beta)}{p-1}+1.\end{aligned}$$

Hence, 4.13.2 Theorem yields for all $1\leq e\leq B(\beta)$ and all $s\in\tilde{\boldsymbol{G}}(\mathbb{A}_f)^{(p)}$ that

$$\operatorname{Lef}\bigl([\tilde K[h]_p^{-e}s\tilde K]_{\tilde\lambda}\,|\,H^\bullet_{\mathrm{cusp}}(S_{\tilde K},L_{\tilde\lambda}(\mathbb{C}))\bigr)\equiv\operatorname{Lef}\bigl([\tilde K[h]_p^{-e}s\tilde K]_{\tilde\lambda'}\,|\,H^\bullet_{\mathrm{cusp}}(S_{\tilde K},L_{\tilde\lambda'}(\mathbb{C}))\bigr)$$
$$(\operatorname{mod} p^{\lceil\beta B(\beta)+\frac{B(\beta)}{p-1}+1\rceil}\mathbb{Z}_p).$$

Together with equation (64) this implies

$$\operatorname{tr}\mathbb{T}^e_{\tilde\lambda}\equiv\operatorname{tr}\mathbb{T}^e_{\tilde\lambda'}\ (\operatorname{mod} p^{\lceil\beta B(\beta)+\frac{B(\beta)}{p-1}+1\rceil}\mathbb{Z}_p)\tag{65}$$

for all $1\leq e\leq B(\beta)$. Since $a_{1,?}=\operatorname{tr}\mathbb{T}_?$, equation (65) implies

$$a_{1,\tilde\lambda}\equiv a_{1,\tilde\lambda'}\ \ (\operatorname{mod} p^{\lceil\beta B(\beta))+\frac{B(\beta)}{p-1}+1\rceil}\mathbb{Z}_p),$$

which is the claim for $i=1$. We now let $i\leq B(\beta)$ be arbitrary and assume that the claim holds for $0,1,2,\dots,i-1$. The recursive relation in equation (63)

$$ia_{i,?}=\sum_{e=1}^{i}(-1)^{e+1}\operatorname{tr}\mathbb{T}^e_?\,a_{i-e,?}$$

together with the induction assumption and equation (65) yields

$$ia_{i,\tilde\lambda}\equiv ia_{i,\tilde\lambda'}\ (\operatorname{mod} p^{\lceil\beta B(\beta)+\frac{B(\beta)}{p-1}+1\rceil-v_p((i-1)!)}\mathbb{Z}_p)$$

from which the claim for i is immediate. □

5.5. ***Newton polygon.*** The *Newton polygon* $\mathcal{N}_{\tilde\lambda}$ of $\mathrm{ch}_{\tilde\lambda}\in F[X]\subseteq\mathbb{Q}_p[X]$ is the lower convex hull of the points $(0,v_p(a_{0,\tilde\lambda})),\dots,(m,v_p(a_{m,\tilde\lambda}))$, where we omit from this list all points $(i,v_p(a_{i,\tilde\lambda}))$ with $v_p(a_{i,\tilde\lambda})=\infty$ (i.e., $a_{i,\tilde\lambda}=0$). Thus, if $a_{m-k+1,\tilde\lambda}=\cdots=a_{m,\tilde\lambda}=0$ and $a_{m-k,\tilde\lambda}\neq 0$ (i.e., if 0 occurs in $\mathrm{ch}_{\tilde\lambda}(X)$ with multiplicity $k=\operatorname{ord}_X(\mathrm{ch}_{\tilde\lambda})$), we omit the points $(m-k+1,v_p(a_{m-k+1,\tilde\lambda})),\dots,(m,v_p(a_{m,\tilde\lambda}))$. In particular, $\mathcal{N}_{\tilde\lambda}$ represents a piecewise linear function on the interval $[0,n_{\tilde\lambda}]$ which starts at the point $(0,0)$ corresponding to the leading term $a_{0,\tilde\lambda}=1$ of $\mathrm{ch}_{\tilde\lambda}$; here,

$$n_{\tilde\lambda}=m-k=\dim H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})-\operatorname{ord}_X(\mathrm{ch}_{\tilde\lambda})=\dim\bigoplus_{\gamma\in\mathbb{Q}_{\geq 0}}H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})^\gamma$$

is the dimension of the finite slope subspace $H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})^{<\infty}$ of $H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})$. We have $a_{i,\tilde\lambda}=0$ for all $i>n_{\tilde\lambda}$ (note that $a_{i,\tilde\lambda}=0$ for all $i>m_{\tilde\lambda}$). Since $\mathrm{ch}_{\tilde\lambda}/X^k$, for

$k = \mathrm{ord}_X(\mathrm{ch}_{\tilde\lambda})$, also has $\mathcal{N}_{\tilde\lambda}$ as its Newton polygon but has nonvanishing constant term we deduce that if the segment S_γ of (necessarily finite) slope $\gamma \in \mathbb{Q}_{\geq 0}$ of $\mathcal{N}_{\tilde\lambda}$ has length s_γ (if projected to the x-axis) then $\mathrm{ch}_{\tilde\lambda}/X^k$ and, hence, $\mathrm{ch}_{\tilde\lambda}$ has precisely s_γ many roots (counted with their multiplicities) in $\bar F(\subseteq \bar{\mathbb{Q}}_p)$ of p-adic value equal to γ.

5.6. ***Proof of 5.2 Theorem.*** We denote by $\mathcal{S}$ the finite set consisting of all $\gamma \in \mathbb{Q}_{\geq 0}$ such that $\gamma \leq \beta$ and the segment S_γ of slope γ of the Newton polygon $\mathcal{N}_{\tilde\lambda}$ or the segment S'_γ of slope γ of $\mathcal{N}_{\tilde\lambda'}$ has strictly positive length (i.e., $H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})^\gamma \neq 0$ or $H^d_{\tilde\lambda',\mathrm{cusp}}(\mathbb{C})^\gamma \neq 0$). We have to show that

$$\dim H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})^\gamma = \dim H^d_{\tilde\lambda',\mathrm{cusp}}(\mathbb{C})^\gamma \quad \text{for all } \gamma \in \mathcal{S}.$$

Since $\dim H^d_{\tilde\lambda,\mathrm{cusp}}(\mathbb{C})^\gamma$ equals the number of roots $\mu \in \bar F$ of $\mathrm{ch}_{\tilde\lambda}$ having p-adic value γ this is equivalent to showing that for all $\gamma \in \mathcal{S}$ the corresponding segments S_γ and S'_γ have the same length (length 0 if the slope γ subspace is trivial). We assume this is not the case and we denote by $\gamma \in \mathbb{Q}_{\geq 0}$ the smallest number in $\mathcal{S}$ such that S_γ and S'_γ have different length; without loss of generality we may assume that S'_γ is (strictly) longer than S_γ. For all $\gamma^* \in \mathcal{S}$ with $0 \leq \gamma* < \gamma$ the segments $S_{\gamma*}$ and $S'_{\gamma*}$ have the same length, hence, S_γ and S'_γ have the same initial point (note that $\mathcal{N}_{\tilde\lambda}, \mathcal{N}_{\tilde\lambda'}$ both start in $(0,0)$). We denote by $P' = (x', y')$ the endpoint of S'_γ. Hence, (x', y') is a vertex of $\mathcal{N}_{\tilde\lambda'}$ which implies that $x' \in \mathbb{N}$ and $y' = v_p(a_{x',\tilde\lambda'})$. We also know that $x' \leq B(\beta)$ because $x' = \dim H^d_{\tilde\lambda',\mathrm{cusp}}(\mathbb{C})^{\leq\gamma} \leq \dim H^d_{\tilde\lambda',\mathrm{cusp}}(\mathbb{C})^{\leq\beta} \leq B(\beta)$. Since $x' \leq B(\beta)$ and since all segments of $\mathcal{N}_{\tilde\lambda'}$ which are left to x' have slopes less than or equal to γ we deduce that

$$v_p(a_{x',\tilde\lambda'}) = y' \leq \gamma B(\beta) \leq \beta B(\beta). \tag{66}$$

Together with 5.4 Proposition this implies that

$$v_p(a_{x',\tilde\lambda}) = v_p(a_{x',\tilde\lambda'}). \tag{67}$$

We distinguish cases.

Case 1: $\mathcal{N}_{\tilde\lambda}$ is defined at x' (i.e., $x' \leq n_{\tilde\lambda}$). Since $\mathcal{N}_{\tilde\lambda}$ is convex and S_γ is strictly shorter than S'_γ in this case we know that $\mathcal{N}_{\tilde\lambda}(x')$ lies strictly above $\mathcal{N}_{\tilde\lambda'}(x') = y'$, hence, we obtain

$$v_p(a_{x',\tilde\lambda}) \geq \mathcal{N}_{\tilde\lambda}(x') > \mathcal{N}_{\tilde\lambda'}(x') = y' = v_p(a_{x',\tilde\lambda'}),$$

i.e., we obtain a contradiction to equation (67).

Case 2: $\mathcal{N}_{\tilde\lambda}$ is not defined at x' (i.e., $x' > n_{\tilde\lambda}$). In this case we know that $a_{x',\tilde\lambda} = 0$. Since $v_p(a_{x',\tilde\lambda'})$ is finite by equation (66), again, this contradicts equation (67).

Thus, the assumption does not hold and we have shown that all segments of slope $\le \beta$ in $\mathcal{N}_{\tilde{\lambda}}$ and $\mathcal{N}_{\tilde{\lambda}'}$ have the same length which finishes the proof.

5.7. Remark. The theorem in particular implies that for any $\beta \in \mathbb{Q}_{\ge 0}$ the function $\tilde{\lambda} \mapsto d(\beta, \tilde{\lambda})$ is constant on cosets modulo $(p-1)p^{\lceil \boldsymbol{m}_2(\beta) \rceil - 1} X(\tilde{\boldsymbol{T}})$.

5.8. Example: $C_2/\mathbb{Q}$. We look at a special case and assume that $\tilde{\boldsymbol{G}}/\mathbb{Q}$ is connected and $\mathbb{Q}$-split with root system of type C_2. As in Section 4.14 we use a non-adelic setting, i.e., $\Gamma \le \tilde{\boldsymbol{G}}(\mathbb{Q})$ is an arithmetic subgroup contained in $\tilde{\mathcal{I}}$, $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ and we set $T = \Gamma \zeta \Gamma \in \mathcal{H}$ where $\zeta \in \Delta_h$. We assume $\zeta \in \tilde{\mathcal{I}} h \tilde{\mathcal{I}}$. All eigenvalues of $T_{\tilde{\lambda}}$ on $H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))$ are algebraic over F and p-adically integral (see 2.9 Corollary) and we define the slope β subspace $H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))^{\beta}$ as the sum of the generalized $T_{\tilde{\lambda}}$-eigenspaces attached to eigenvalues of p-adic value β. As in Section 5.1 we obtain that 3.10 Corollary implies

$$\dim H^d_{\mathrm{cusp}}(\Gamma\backslash X, L_{\tilde{\lambda}}(\mathbb{C}))^{\le \beta} \le B_\Gamma(\beta),$$

where $B_\Gamma(\beta) = \mathtt{m}_\Gamma \beta^s + \mathtt{n}_\Gamma$. We define the polynomials

$$\begin{aligned}\boldsymbol{m}_1(\beta) &= \beta B_\Gamma(\beta) + \frac{B_\Gamma(\beta)}{p-1} + 1 + B_\Gamma(\beta)(\kappa_1 + \kappa_2) \\ &= \mathtt{m}_\Gamma \beta^{s+1} + \Big(\frac{\mathtt{m}_\Gamma}{p-1} + \mathtt{m}_\Gamma(\kappa_1+\kappa_2)\Big)\beta^s + \mathtt{n}_\Gamma \beta + \frac{\mathtt{n}_\Gamma}{p-1} + \mathtt{n}_\Gamma(\kappa_1+\kappa_2) + 1\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{m}_2(\beta) &= \beta B_\Gamma(\beta) + \frac{B_\Gamma(\beta)}{p-1} + 1 + B_\Gamma(\beta)\kappa_1 \\ &= \mathtt{m}_\Gamma \beta^{s+1} + \Big(\frac{\mathtt{m}_\Gamma}{p-1} + \mathtt{m}_\Gamma \kappa_1\Big)\beta^s + \mathtt{n}_\Gamma \beta + \frac{\mathtt{n}_\Gamma}{p-1} + \mathtt{n}_\Gamma \kappa_1 + 1.\end{aligned}$$

Following the arguments in the previous subsections but using the congruences for the traces of Hecke operators in 4.14.3 Corollary instead of those in 4.13.2 Theorem we obtain this:

Theorem. *If $\tilde{\boldsymbol{G}}$ is connected and $\mathbb{Q}$-split with root system of type C_2 then 5.2 Theorem holds with $\boldsymbol{m}_1, \boldsymbol{m}_2$ as defined above.*

We want to determine the polynomial $\boldsymbol{m}_2$ more explicitly. Since $\tilde{\boldsymbol{G}}/\mathbb{Q}$ is of type C_2 there are two simple roots $\Delta = \{\alpha_1, \alpha_2\}$, the positive roots are $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$, hence, $s = 4$ and $2\rho_{\tilde{\boldsymbol{G}}} = 4\alpha_1 + 3\alpha_2$. We denote by $g = g_d$ the number of d cells in a cell complex which is homotopy equivalent to the Borel–Serre compactification of $\Gamma\backslash X$. We assume that $h \in \tilde{\boldsymbol{T}}(\mathbb{Q})^{++}$ satisfies $v_p(\alpha_1(h)) = v_p(\alpha_2(h)) = 1$. We then obtain

$$\kappa_1 = \sum_{\alpha>0} v_p(\alpha(h)) = 7, \quad \kappa_2 = \max(m_{\alpha_1}, m_{\alpha_2}) = 4, \quad \sigma = \max_{\alpha>0} \mathrm{ht}(\alpha) = 3,$$

and hence

$$\mathtt{m}_\Gamma = \frac{12g\sigma^{s+1}}{s} = \frac{12g3^5}{4} = 729g.$$

To determine $\mathtt{n}_\Gamma$ we have to find $M \in \mathbb{N}$ such that equations (23) and (24) hold. A numerical evaluation shows that we may choose $M = 34$; hence, equation (26) yields

$$\mathtt{n}_\Gamma = \left\lceil 2^{1/(s+1)} \frac{g\sigma^{s+1}M^s}{s} \right\rceil + 1 = \left\lceil \frac{2^{1/5}g3^5 1336336}{4} \right\rceil + 1 \le 93254104g + 1.$$

This yields the bound

$$\begin{aligned} \boldsymbol{m}_2(X) &\le 729gX^5 + 729g\left(\tfrac{1}{p-1} + 7\right)X^4 + (93254104g + 1)X \\ &\qquad + (93254104g + 1)\left(\tfrac{1}{p-1} + 7\right) + 1 \\ &\le 729gX^5 + 5832gX^4 + (93254104g + 1)X + 746032832g + 9. \end{aligned}$$

We note that since p is in the level of Γ the number $g = g_\Gamma$ depends on the prime p.

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