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WEAKENING IDEMPOTENCY IN K-THEORY

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We show that the K-theory of C^*-algebras can be defined by pairs of matrices a, b satisfying less strict relations than idempotency, namely $p(a) = p(b)$ for any polynomial p with $p(0) = p(1) = 0$, which means that a and b have to be "projections" only where $a \neq b$.

1. Introduction

The K-theory of a C^*-algebra A is patently defined by pairs (formal differences) of idempotent matrices (projections) over A. Regretfully, projection is a very strict property, and it is usually very hard to find projections in a given C^*-algebra. Many famous conjectures (Kadison, Novikov, Baum–Connes, Bass, etc.) are related to projections and would become more tractable if one could provide enough projections for a given C^*-algebra. Our aim is to show that the K-theory can be defined using less-restrictive relations in the hope that it will be easier to find elements satisfying these relations than the genuine idempotency. We show that K-theory is generated by pairs a, b of matrices over A satisfying $p(a) = p(b)$ for any polynomial p with $p(0) = p(1) = 0$, which means that a and b have to be "projections" only where $a \neq b$.

2. Definitions and some properties

Let A be a C^*-algebra. For $a, b \in A$, consider the relations

(1) $\|a\| \leq 1, \quad \|b\| \leq 1, \quad a, b \geq 0, \quad (a-a^2)(a-b) = 0, \quad (b-b^2)(a-b) = 0.$

Definition 2.1. A pair (a, b) of elements in a C^*-algebra is called *balanced* if it satisfies the relations (1).

Two balanced pairs (a_0, b_0) and (a_1, b_1) of elements in A are *homotopy equivalent* if there are paths $a = (a_t)$, $b = (b_t) : [0, 1] \to A$, connecting a_0 with a_1 and b_0 with b_1 respectively, such that the pair (a_t, b_t) is balanced for each $t \in [0, 1]$.

A balanced pair (a, b) is *homotopy trivial* if it is homotopy equivalent to $(0, 0)$.

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Lemma 2.2. *A balanced pair (a, a) is homotopy trivial for any $a \in A$.*

Proof. The linear homotopy $a_t = t \cdot a$ would do. □

Lemma 2.3. *If a pair (a, b) is balanced then $f(a) = f(b)$ and $f(a)(a-b) = 0$ for any $f \in C_0(0, 1)$.*

Proof. It follows from $(a-a^2)(a-b) = 0$, or, equivalently, from $(a-a^2)a = (a-a^2)b$, that

$$(a-a^2)a^2 = a(a-a^2)a = a(a-a^2)b = (a-a^2)b^2;$$

hence

$$(a-a^2)(a-a^2) = (a-a^2)(b-b^2).$$

Similarly,

$$(b-b^2)(b-b^2) = (a-a^2)(b-b^2);$$

therefore

$$(a-a^2)^2 = (b-b^2)^2. \tag{2}$$

Then (2) and the positivity of $a-a^2$ and $b-b^2$ imply that

$$a-a^2 = b-b^2.$$

Also,

$$(a-a^2)a = (a-a^2)b = (b-b^2)b.$$

Since the two functions g and h given by $g(t) = t-t^2$ and $h(t) = tg(t)$ generate $C_0(0, 1)$, and since $g(a) = g(b)$ and $h(a) = h(b)$, we conclude that the same holds for any $f \in C_0(0, 1)$. Similarly, $g(a)(a-b) = 0$ and $h(a)(a-b) = 0$ imply $f(a)(a-b) = 0$ for any $f \in C_0(0, 1)$. □

Corollary 2.4. *If $\|a\| < 1$, $\|b\| < 1$ and the pair (a, b) is balanced then $a = b$; hence the pair (a, b) is homotopy trivial.*

Proof. Take $f \in C_0(0, 1)$ such that $f(t) = t \in \mathrm{Sp}(a) \cup \mathrm{Sp}(b)$ and $f(1) = 0$. Then $a = f(a)$, $b = f(b)$, and the claim follows from Lemma 2.3. □

Lemma 2.5. *Let $f : [0, 1] \to [0, 1]$ be a continuous map such that $f(0) = 0$ and $f(1) = 1$. If (a, b) is a balanced pair then the pair $(f(a), f(b))$ is also balanced and is homotopy equivalent to (a, b).*

Proof. As the set of all functions with the stated properties is convex, it suffices to show that for any such function f, the pair $(f(a), f(b))$ satisfies the relations (1).

Set $f_0(t) = f(t) - t$. Then $f_0 \in C_0(0, 1)$. As $f_0(a) = f_0(b)$ by Lemma 2.3,

$$f(a) - f(b) = a - b.$$

Set

$$g(t) = t - t^2 + f_0(t) - f_0^2(t) - 2tf_0(t).$$

Then $g \in C_0(0, 1)$ and

$$(f(a) - f^2(a))(f(a) - f(b)) = g(a)(a - b) = 0,$$
$$(f(b) - f^2(b))(f(a) - f(b)) = g(a)(a - b) = 0. \qquad \square$$

Corollary 2.6. *If a pair* (a, b) *is balanced then* $\mathrm{Sp}(a) \setminus \{0, 1\} = \mathrm{Sp}(b) \setminus \{0, 1\}$.

Proof. The inner points of $[0, 1]$ in the two spectra must coincide by Lemma 2.3. $\square$

Let $M_n(A)$ denote the $n \times n$ matrix algebra over A. Two balanced pairs (a_0, b_0) and (a_1, b_1), where $a_0, a_1, b_0, b_1 \in M_n(A)$, are equivalent if there is a homotopy trivial balanced pair (a, b), $a, b \in M_m(A)$ for some integer m, such that the balanced pairs $(a_0 \oplus a, b_0 \oplus b)$ and $(a_1 \oplus a, b_1 \oplus b)$ are homotopy equivalent in $M_{n+m}(A)$. Using the standard inclusion $M_n(A) \subset M_{n+k}(A)$ (as the upper-left corner) we may speak about the equivalence of balanced pairs of different matrix size.

Let $[(a, b)]$ denote the equivalence class of the balanced pair (a, b), $a, b \in M_n(A)$. For two balanced pairs (a, b), $a, b \in M_n(A)$, and (c, d), $c, d \in M_m(A)$, set

$$[(a, b)] + [(c, d)] = [(a \oplus c, b \oplus d)].$$

The result obviously doesn't depend on the choice of representatives. Also $[(a, b)] + [(c, d)] = [(a, b)]$ when (c, d) is homotopy trivial.

Lemma 2.7. *The addition is commutative and associative.*

Proof. If $(u_t)_{t\in[0,1]}$ is a path of unitaries in A with $u_1 = 1$ and $u_0 = u$, then $[(u^*au, u^*bu)] = [(a, b)]$ for any $a, b \in A$, as the relations (1) are not affected by unitary equivalence. The standard argument with a unitary path connecting $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ proves commutativity. A similar argument proves associativity. $\square$

Lemma 2.8. $\qquad [(a, b)] + [(b, a)] = [(0, 0)] \quad$ *for any* a, b.

Proof. Set

$$U_t = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}, \qquad B_t = U_t^* \begin{pmatrix} b & 0\\ 0 & a\end{pmatrix} U_t.$$

We claim that the pair $\left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right), B_t\right)$ is balanced for all t.

One has

$$B_t = \begin{pmatrix} b\cos^2 t + a\sin^2 t & (a-b)\cos t\sin t\\ (a-b)\cos t \sin t & b\sin^2 t + a\cos^2 t\end{pmatrix} = \begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} + (a-b)C_t, \tag{3}$$

where

$$C_t = \begin{pmatrix} -\cos^2 t & \cos t\sin t\\ \cos t\sin t & \cos^2 t\end{pmatrix}.$$

Then

$$\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2\right)\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t\right) = \begin{pmatrix} a-a^2 & 0 \\ 0 & b-b^2 \end{pmatrix}(a-b)C_t$$
$$= \begin{pmatrix} (a-a^2)(a-b) & 0 \\ 0 & (b-b^2)(a-b) \end{pmatrix}C_t = 0.$$

It remains to show that

$$A = (B_t - B_t^2)\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t\right) = 0.$$

Using (3) we have

$$\begin{aligned}
A &= \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + (a-b)C_t - \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + (a-b)C_t\right)^2\right)(a-b)C_t \\
&= \left(\begin{pmatrix} a-a^2 & 0 \\ 0 & b-b^2 \end{pmatrix} + (a-b)C_t - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(a-b)C_t \right. \\
&\qquad \left. - C_t(a-b)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - (a-b)^2C_t^2\right)(a-b)C_t \\
&= \left((a-b)C_t - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(a-b)C_t - C_t(a-b)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - (a-b)^2C_t^2\right)(a-b)C_t \\
&= \left(\begin{pmatrix} a-b-a^2+ab & 0 \\ 0 & a-b-ba+b^2 \end{pmatrix}C_t - C_t(a-b)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right. \\
&\qquad \left. - (a-b)^2\cos^2 t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)(a-b)C_t \\
&= \left(\begin{pmatrix} -b+ab & 0 \\ 0 & a-ba \end{pmatrix}C_t - C_t\begin{pmatrix} a-ba & 0 \\ 0 & ab-b \end{pmatrix} - (a-b)^2\cos^2 t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)(a-b)C_t \\
&= \left(\begin{pmatrix} (ab+ba-a-b)\cos^2 t & 0 \\ 0 & (ab+ba-a-b)\cos^2 t \end{pmatrix} \right. \\
&\qquad \left. - \begin{pmatrix} (a-b)^2\cos^2 t & 0 \\ 0 & (a-b)^2\cos^2 t \end{pmatrix}\right)(a-b)C_t \\
&= 0.
\end{aligned}$$

Thus, the balanced pair $(a \oplus b, b \oplus a)$ is homotopy equivalent to the balanced pair $(a \oplus b, a \oplus b)$, and the latter is homotopy trivial by Lemma 2.2. □

So we see that the equivalence classes of balanced pairs in matrix algebras over A form an abelian group for any C^*-algebra A. Let us denote this group by $L(A)$.

Note that pairs of projections are patently balanced. If A is a unital C^*-algebra then $K_0(A)$ consists of formal differences $[p]-[q]$ with p, q projections in matrices over A. Then

$$\iota([p]-[q]) = [(p, q)]$$

gives rise to a morphism $\iota : K_0(A) \to L(A)$.

In the nonunital case, ι can be defined after unitalization. But, as we shall see, unlike K_0, there is no need to unitalize for L. The following example shows the reason for that in the commutative case.

Example 2.9. Let X be a compact Hausdorff space, $x \in X$, $Y = X \setminus \{x\}$, $A = C_0(Y)$, $A^+ = C(X)$. Let $[p]-[q] \in K_0(A)$, where $p, q \in M_n(A^+)$ are projections. Then $p = p_0 + \alpha$ and $q = p_0 + \beta$, where p_0 is constant on X and $\alpha, \beta \in M_n(A)$. Without loss of generality we may assume that $\alpha, \beta = 0$ not only at the point x, but also in a small neighborhood U of x. Let $h \in C(X)$ satisfy $0 \le h \le 1$, $h(x) = 0$ and $h(z) = 1$ for any $z \in X \setminus U$. Set $a = hp_0 + \alpha$, $b = hp_0 + \beta$. Then $a, b \in M_n(A)$ and $[(a, b)] \in L(A)$.

Lemma 2.10. $L(\mathbb{C}) \cong \mathbb{Z}$.

Proof. Let $a, b \in M_n$, and let the pair (a, b) be balanced. Let $e_1, \dots, e_n$ (resp. $e'_1, \dots, e'_n$) be an orthonormal basis of eigenvectors for a (resp. for b) with eigenvalues $\lambda_1, \dots, \lambda_n$ (resp. $\lambda'_1, \dots, \lambda'_n$). Let $0 < \lambda_i < 1$. Then e_i is an eigenvector for $a - a^2$ with a nonzero eigenvalue $\lambda_i - \lambda_i^2$. As $(a-a^2)(a-b) = 0$, we have $(a-b)(a-a^2) = 0$; hence

$$(a-b)(a-a^2)(e_i) = (\lambda_i - \lambda_i^2)(a-b)(e_i) = 0.$$

Thus $(a-b)(e_i) = 0$, or, equivalently, $a(e_i) = b(e_i)$. As e_i is an eigenvector for a, it is an eigenvector for b as well: $b(e_i) = \lambda_i e_i$. So the eigenvectors, corresponding to the eigenvalues $\neq 0, 1$, are the same for a and b.

Reorder, if necessary, the eigenvalues so that

$$\lambda_1, \dots, \lambda_k \in (0, 1), \quad \lambda_{k+1}, \dots, \lambda_n \in \{0, 1\},$$

and denote the linear span of $e_1, \dots, e_k$ by L. Similarly, assume that

$$\lambda'_1, \dots, \lambda'_{k'} \in (0, 1), \quad \lambda'_{k'+1}, \dots, \lambda'_n \in \{0, 1\},$$

and denote the linear span of $e'_1, \dots, e'_{k'}$ by L'. As $e_1, \dots, e_k \in L'$ and, symmetrically, $e'_1, \dots, e'_{k'} \in L$, we have $\dim L = \dim L'$, $k = k'$, and $\lambda_i = \lambda'_i$ for $i = 1, \dots, k$.

Then $L^\perp$ is an invariant subspace for both a and b, and the restrictions $a|_{L^\perp}$ and $b|_{L^\perp}$ are projections (as their eigenvalues equal 0 or 1). We may write a and b as matrices with respect to the decomposition $L \oplus L^\perp$:

$$a = \begin{pmatrix} c & 0 \\ 0 & p \end{pmatrix}, \quad b = \begin{pmatrix} c & 0 \\ 0 & q \end{pmatrix}, \tag{4}$$

where p, q are projections. The linear homotopy

$$a_t = \begin{pmatrix} tc & 0 \\ 0 & p \end{pmatrix}, \quad b_t = \begin{pmatrix} tc & 0 \\ 0 & q \end{pmatrix}, \qquad t \in [0, 1],$$

connects the pair (a, b) with the pair $(p, q) + (0, 0)$. Therefore, $L(\mathbb{C})$ is a quotient of $\mathbb{Z}$ (which is the set of homotopy classes of pairs of projections modulo stable equivalence). To see that $L(\mathbb{C})$ is exactly $\mathbb{Z}$, note that (4) implies that $\operatorname{tr}(a - b) \in \mathbb{Z}$ for any balanced pair (a, b), so this integer is homotopy invariant. □

Remark 2.11. One may think that the relations (1) imply that balanced pairs (a, b) are something like projections plus a common part and can be reduced to just a pair of projections by cutting out the common part. The following example shows that this is not that simple.

Example 2.12. Let $A = C(X)$, and let Y, Z be closed subsets in X with $Y \cap Z = K$. Let $p, q \in M_n(C(Y))$ be projection-valued functions on Y such that $p|_K = q|_K = r$, where r cannot be extended to a projection-valued function on Z due to a K-theory obstruction, but can be extended to a matrix-valued function $s \in M_n(C(Z))$ on Z (with $0 \le s \le 1$). Then set

$$a = \begin{cases} p & \text{on } Y, \\ s & \text{on } Z, \end{cases} \qquad \text{and} \qquad b = \begin{cases} q & \text{on } Y, \\ s & \text{on } Z. \end{cases}$$

3. Universal C^*-algebra for relations (1)

Let (a, b) be a balanced pair in a C^*-algebra A. Denote the C^*-subalgebra generated by a and b by $C^*(a, b)$. The universal C^*-algebra for the relations (1) is a C^*-algebra D generated by elements $\boldsymbol{a}, \boldsymbol{b} \in D$ satisfying the relations (1) such that for any balanced pair (a, b) there is a surjective $*$-homomorphism $\varphi : D \to C^*(a, b)$ with $\varphi(\boldsymbol{a}) = a$ and $\varphi(\boldsymbol{b}) = b$; see [Loring 1997].

Let $I \subset C^*(a, b)$ denote the ideal generated by $a - a^2$, and let $C^*(a, b)/I$ be the quotient C^*-algebra. Then $C^*(a, b)/I$ is generated by $\dot{a} = q(a)$ and $\dot{b} = q(b)$, where q is the quotient map. But since $q(a - a^2) = q(b - b^2) = 0$, $\dot{a}$ and $\dot{b}$ are projections, and $C^*(a, b)/I$ is generated by two projections.

Then the C^*-algebra $C^*(a, b)$ is completely determined by the ideal I, by the quotient $C^*(a, b)/I$, and by the Busby invariant $\tau : C^*(a, b)/I \to Q(I)$ (we denote by $M(I)$ the multiplier algebra of I and by $Q(I) = M(I)/I$ the outer multiplier algebra). The latter is defined by the two projections $\tau(\dot{a})$, $\tau(\dot{b}) \in Q(C_0(Y))$, where $X = \operatorname{Sp}(a)$, $Y = X \setminus \{0, 1\}$. Let $C_b(Y)$ denote the C^*-algebra of bounded continuous functions on Y and let

$$\pi : C_b(Y) \to C_b(Y)/C_0(Y) = Q(C_0(Y))$$

be the quotient map. Using Gelfand duality, we identify a with the function id on $\mathrm{Sp}(a)$. Let $f \in C_0(Y)$. Then

$$\tau(\dot{a})\pi(f(a)) = \tau(\dot{b})\pi(f(a)) = \pi(af(a)),$$

so we can easily calculate these two projections.

If $1 \notin X$ then $\tau(\dot{a}) = \tau(\dot{b}) = 0$; if $X = \{1\}$ then $I = 0$; if $1 \in X$ and X has at least one more point x then $\tau(\dot{a}) = \tau(\dot{b})$ is the class of functions f on X such that $f(1) = 1$ and $f(t) = 0$ for all $t \leq x$.

Let $M_1 \subset M_2$ denote the upper-left corner in the 2-by-2 matrix algebra. Set

$$D = \{ f \in C([-1, 1]; M_2) : f(-1) = 0,\ f(1) \text{ is diagonal},\ f(t) \in M_1 \text{ for } t \in (-1, 0] \},$$

and let $\boldsymbol{a}, \boldsymbol{b}$ be functions in $C([-1, 1]; M_2)$ defined by

$$\boldsymbol{a}(t) = \begin{cases} \begin{pmatrix} \cos^2 \frac{\pi}{2}t & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [-1, 0], \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [0, 1], \end{cases} \tag{5}$$

$$\boldsymbol{b}(t) = \begin{cases} \begin{pmatrix} \cos^2 \frac{\pi}{2}t & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [-1, 0], \\ \begin{pmatrix} \cos^2 \frac{\pi}{2}t & \cos \frac{\pi}{2}t \sin \frac{\pi}{2}t \\ \cos \frac{\pi}{2}t \sin \frac{\pi}{2}t & \sin^2 \frac{\pi}{2}t \end{pmatrix} & \text{for } t \in [0, 1]. \end{cases} \tag{6}$$

Then $\boldsymbol{a}, \boldsymbol{b} \in D$, the pair $(\boldsymbol{a}, \boldsymbol{b})$ is balanced, and $D = C^*(\boldsymbol{a}, \boldsymbol{b})$ is generated by these $\boldsymbol{a}$ and $\boldsymbol{b}$.

Like all C^*-algebras of the form $C^*(a, b)$ defined by balanced pairs (a, b), the C^*-algebra D is an extension. It contains the ideal

$$J = \{ f \in D : f(t) = 0 \text{ for } t \in [0, 1] \} \cong C_0(-1, 0),$$

which is generated by $\boldsymbol{a} - \boldsymbol{a}^2$. Note that multiplication by $\boldsymbol{a}$ or by $\boldsymbol{b}$ determines the same multiplier $m_{\boldsymbol{a}} = m_{\boldsymbol{b}} \in M(J)$, and that the C^*-algebra $\bar{J}$ generated by J and by $m_{\boldsymbol{a}}$ is isomorphic to $C_0(-1, 0]$. It is the universal C^*-algebra for the relation $0 \leq a \leq 1$, so there exists a surjective $*$-homomorphism $\bar{\alpha}$ from $\bar{J}$ to the nonunital C^*-algebra generated by a such that $\alpha'(m_{\boldsymbol{a}}) = m_a$, where $m_a \in M(I)$ is the multiplier defined by multiplication by a on A. The restriction $\alpha = \bar{\alpha}|_J$ maps J onto I, and $\alpha(f(\boldsymbol{a})) = f(a)$ for any $f \in C_0(0, 1)$.

The quotient D/J is the universal (nonunital) C^*-algebra

$$D/J = \mathbb{C} * \mathbb{C} = \{ m \in C([0, 1], M_2) : m(1) \text{ is diagonal},\ m(0) \in M_1 \} \tag{7}$$

generated by two projections $\dot{\boldsymbol{a}}$ and $\dot{\boldsymbol{b}}$ [Raeburn and Sinclair 1989]. Therefore, D/J surjects onto any C^*-algebra generated by two projections in a canonical way. Note that D/J is an extension of $\mathbb{C}$ by the C^*-algebra

$$q\mathbb{C} = \{m \in C_0((0,1], M_2) : m(1) \text{ is diagonal}\}$$

used in the Cuntz picture of K-theory.

Lemma 3.1. *The C^*-algebra D is universal for the relations* (1).

Proof. For any balanced pair (a, b), the universality of $\bar{J}$ and of D/J implies the existence of surjective $*$-homomorphisms $\alpha : J \to I$ and $\gamma : D/J \to C^*(a, b)/I$ such that $\bar{\alpha}(\boldsymbol{a}) = a$ and $\gamma(\dot{\boldsymbol{a}}) = \dot{a}$, $\gamma(\dot{\boldsymbol{b}}) = \dot{b}$. Since α is surjective, it induces $*$-homomorphisms $M(\alpha) : M(J) \to M(I)$ and $Q(\alpha) : Q(J) \to Q(I)$ in a canonical way, and $M(\alpha)|_{\bar{J}} = \bar{\alpha}$. One has

$$D \cong \{(m, f) : m \in M(J),\ f \in D/J,\ q_J(m) = \tau(f)\}, \tag{8}$$

$$C^*(a, b) \cong \{(n, g) : n \in M(I),\ g \in C^*(a, b)/I,\ q_I(n) = \sigma(g)\}, \tag{9}$$

where $q_\bullet : M(\bullet) \to Q(\bullet)$ is the quotient map; hence the map $\varphi : D \to C^*(a, b)$ can be defined by $\varphi(m, f) = (M(\alpha)(m), \gamma(f))$. This map is well defined if the diagram

$$\begin{array}{ccc} D/J & \xrightarrow{\ \tau\ } & Q(J) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Q(\alpha)} \\ C^*(a, b)/I & \xrightarrow{\ \sigma\ } & Q(I) \end{array}$$

commutes. It does commute. The case $X = \mathrm{Sp}(a) = \{1\}$ is trivial. For the other cases, notice that the image of τ lies in $C_0(0, 1]/C_0(0, 1) \subset Q(J)$, and the image of σ lies in $C(X)/C_0(X \setminus \{0\})$, which is either $\mathbb{C}$ or 0 (when $1 \in X$ or $1 \notin X$, respectively), and the restriction of $Q(\alpha)$ from the image of τ to the image of σ is induced by the inclusion $X \subset [0, 1]$. So, there is a surjective $*$-homomorphism φ from D to $C^*(a, b)$.

Under the identification (8), $\boldsymbol{a} \in D$ corresponds to the pair $(m_{\boldsymbol{a}}, \dot{\boldsymbol{a}})$; hence $\varphi(\boldsymbol{a}) = (M(\alpha)(m_{\boldsymbol{a}}), \gamma(\dot{\boldsymbol{a}})) = (\alpha'(m_{\boldsymbol{a}}), \dot{a}) = (m_a, \dot{a})$, and the latter corresponds to a under the identification (9). Similarly, one can check that $\varphi(\boldsymbol{b}) = b$. □

The C^*-algebra D allows one more description. Set $A_0 = \mathbb{C}^2$ and $F = \mathbb{C} \oplus M_2$, and define a $*$-homomorphism $\gamma : A_0 \to F \oplus F$ by $\gamma = \gamma_0 \oplus \gamma_1$, where $\gamma_0, \gamma_1 : \mathbb{C}^2 \to \mathbb{C} \oplus M_2$ are given by

$$\gamma_0(\lambda, \mu) = \lambda \oplus \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma_1(\lambda, \mu) = 0 \oplus \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{C}.$$

Let $\partial : C([0,1]; F) \to F \oplus F$ be the boundary map given by $\partial(f) = f(0) \oplus f(1)$, $f \in C([0,1]; F)$. Then D can be identified with the pullback

$$\begin{array}{ccc} D = A_1 & \longrightarrow & A_0 \\ \downarrow & & \downarrow \gamma \\ C([0,1]; F) & \xrightarrow{\partial} & F \oplus F, \end{array}$$

$$D = \{(f, a) : f \in C([0,1]; F),\ a \in A_0,\ \partial(f) = \gamma(a)\}.$$

Such a pullback is called a 1-dimensional noncommutative CW complex (NCCW complex) in [Eilers et al. 1998]; in this terminology, A_0 is a 0-dimensional NCCW complex.

Recall [Blackadar 1985] that a C^*-algebra B is *semiprojective* if for any C^*-algebra A and increasing chain of ideals $I_n \subset A$, $n \in \mathbb{N}$, with $I = \overline{\bigcup_n I_n}$ and for any $*$-homomorphism $\varphi : B \to A/I$ there exist n and $\hat{\varphi} : B \to A/I_n$ such that $\varphi = q \circ \hat{\varphi}$, where $q : A/I_n \to A/I$ is the quotient map.

Corollary 3.2. *The C^*-algebra D is semiprojective.*

Proof. Essentially, this is Theorem 6.2.2 of [Eilers et al. 1998], where it is proved that all unital 1-dimensional NCCW complexes are semiprojective. The nonunital case is dealt with in Theorem 3.15 of [Thiel 2009], where is it noted that if A_1 is a 1-dimensional NCCW complex then A_1^+ is a 1-dimensional NCCW complex as well, and semiprojectivity of A_1 is equivalent to semiprojectivity of A_1^+. □

One more picture of D can be given in terms of an amalgamated free product: $D = C(0,1] *_{C_0(0,1)} C(0,1]$.

4. Identifying L with K_0

Our definition of $L(A)$ can be reformulated in terms of the universal C^*-algebra D as

$$L(A) = \varinjlim [D, M_n(A)],$$

where $[-,-]$ denotes the set of homotopy classes of $*$-homomorphisms. Recall that semiprojectivity is equivalent to stability of relations that determine D [Loring 1997, Theorem 14.1.4]. The latter means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $c, d \in A$ satisfy

$$\|c\| \le 1, \quad \|d\| \le 1, \quad c, d \ge 0, \quad \|(c - c^2)(c - d)\| < \delta, \quad \|(d - d^2)(c - d)\| < \delta,$$

there exist $a, b \in A$ such that $\|a - c\| < \varepsilon$, $\|b - d\| < \varepsilon$, and a, b satisfy the relations (1). Stability of the relations (1) implies that

$$L(A) = [D, A \otimes \mathbb{K}] = [\![D, A \otimes \mathbb{K}]\!],$$

where $\mathbb{K}$ denotes the C^*-algebra of compact operators and $[\![\,\cdot\,,\,\cdot\,]\!]$ is the set of homotopy classes of asymptotic homomorphisms.

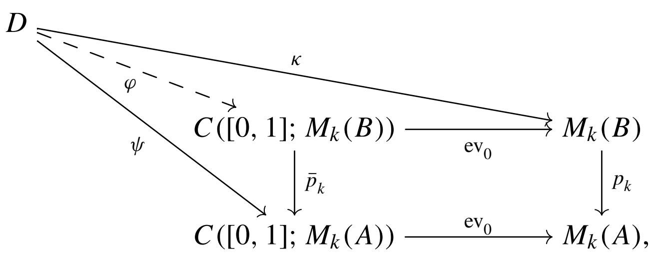
Lemma 4.1. *The functor L is half-exact.*

Proof. Let

$$0 \longrightarrow I \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be a short exact sequence of C^*-algebras. It is obvious that $p_* \circ i_* = 0$, so it remains to check that $\operatorname{Ker} p_* \subset \operatorname{Im} i_*$. Suppose that $a, b \in M_n(B)$, the pair (a, b) is balanced, and $(p(a), p(b)) = 0$ in $L(A)$. This means that there is a homotopy connecting $(p(a), p(b))$ to $(0, 0)$ in $M_k(A)$ for some $k \geq n$ such that the whole path satisfies (1). This homotopy is given by a $*$-homomorphism $\psi : D \to C([0, 1], M_k(A))$ such that $\mathrm{ev}_1 \circ \psi = 0$, where ev_t denotes the evaluation map at $t \in [0, 1]$.

When D is a semiprojective C^*-algebra, the homotopy lifting theorem [Blackadar 2016, Theorem 5.1] asserts that given a commuting diagram



where $\bar{p}_k$ and p_k are the $*$-homomorphisms induced by a surjection p, there exists a $*$-homomorphism φ completing the diagram. Replacing A and B by matrices over these C^*-algebras, we get a lifting φ for the given homotopy. It follows from $\mathrm{ev}_1 \circ \psi = 0$ that $\mathrm{ev}_1 \circ \varphi$ maps D to $M_k(I)$. Thus (a, b) lies in the image of i_*. $\square$

In the standard way, set $L_n(A) = L(S^n A)$, where SA denotes the suspension over A. Then, by Theorem 21.4.3 of [Blackadar 1986], $L_n(A)$, being homotopy invariant and half-exact, is a homology theory. Also, by Theorem 22.3.6 of that paper and by Lemma 2.10, it coincides with the K-theory on the bootstrap category of C^*-algebras. We shall show now that it coincides with the K-theory for any C^*-algebra.

Set

$$P = \begin{pmatrix} 1 - \boldsymbol{b} & f(\boldsymbol{a}) \\ f(\boldsymbol{a}) & \boldsymbol{a} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 - \boldsymbol{b} & f(\boldsymbol{a}) \\ f(\boldsymbol{a}) & \boldsymbol{b} \end{pmatrix},$$

where $\boldsymbol{a}, \boldsymbol{b}$ are generators for D ((5)–(6)), and $f \in C_0(0, 1)$ is given by $f(t) = (t - t^2)^{1/2}$. Then $P, Q \in M_2(D^+)$, where D^+ denotes the unitalization of D.

By Lemma 2.3, $f(\boldsymbol{a}) = f(\boldsymbol{b})$ and $\boldsymbol{a} f(\boldsymbol{a}) = \boldsymbol{b} f(\boldsymbol{a})$, so P and Q are projections. One also has $P - Q \in M_2(D)$; hence

$$x = [P] - [Q] \in K_0(D).$$

Lemma 4.2. *$K_0(D) \cong \mathbb{Z}$ with x as a generator.*

Proof. Consider the short exact sequence

$$0 \longrightarrow J \longrightarrow D \xrightarrow{\pi} \mathbb{C} * \mathbb{C} \longrightarrow 0,$$

where $\mathbb{C}*\mathbb{C}$ is the universal (nonunital) C^*-algebra (7) generated by two projections, p and q [Raeburn and Sinclair 1989], and π is given by restriction to $[0, 1]$, $\pi(\boldsymbol{a}) = p$, $\pi(\boldsymbol{b}) = q$. We have $\pi(P) = (1-q) \oplus p$ and $\pi(Q) = (1-q) \oplus q$, so $\pi_*(x) = [p] - [q] \in K_0(\mathbb{C} * \mathbb{C})$. For $t \in [-1, 0]$, one has $P(t) = Q(t)$; hence, for the boundary (exponential) map $\delta : K_0(\mathbb{C}*\mathbb{C}) \to K_1(J)$, we have $\delta(P) = \delta(Q)$. Recall that $J \cong C_0(-1, 0)$. Direct calculation shows that $\delta(P) = \delta(Q) \neq 0$. The claim follows now from the K-theory exact sequence

$$0 = K_0(J) \longrightarrow K_0(D) \xrightarrow{\pi_*} K_0(\mathbb{C} * \mathbb{C}) \xrightarrow{\delta} K_1(J) \cong \mathbb{Z}. \qquad \square$$

Let us define a map $\kappa : L(A) \to K_0(A)$. If $l = [(a, b)] \in L(A)$ then the balanced pair (a, b) determines a $*$-homomorphism $\varphi : D \to M_n(A)$ by $\varphi(\boldsymbol{a}) = a$ and $\varphi(\boldsymbol{b}) = b$. So, $l \in L(A)$ determines a $*$-homomorphism φ up to homotopy (for some n). Put

$$\kappa(l) = \varphi_*(x) \in K_0(A).$$

It is easy to see that the map κ is a well-defined group homomorphism.

Recall that there is also a map $\iota : K_0(A) \to L(A)$ given by $\iota([p] - [q]) = [(p, q)]$, where $[p] - [q] \in K_0(A)$.

Lemma 4.3. *For any unital C^*-algebra A, one has $\kappa \circ \iota = \mathrm{id}_{K_0(A)}$ and $\iota \circ \kappa = \mathrm{id}_{L(A)}$; hence $L(A) = K_0(A)$.*

Proof. To show the first identity, let $z \in K_0(A)$ and let $p, q \in M_n(A)$ be projections such that $z = [p] - [q]$. Let $\varphi : D \to M_n(A)$ be a $*$-homomorphism determined by the pair (p, q). Then, due to the universality of $\mathbb{C} * \mathbb{C}$, φ factorizes through $\mathbb{C} * \mathbb{C}$, $\varphi = \psi \circ \pi$, where $\pi : D \to \mathbb{C} * \mathbb{C}$ is the quotient map and $\psi : \mathbb{C} * \mathbb{C} \to M_n(A)$ is determined by $\psi(i_1(1)) = p$ and $\psi(i_2(1)) = q$, where $i_1, i_2 : \mathbb{C} \to \mathbb{C} * \mathbb{C}$ are inclusions onto the first and the second copy of $\mathbb{C}$. Then

$$\varphi(x) = \psi_*\big([i_1(1)] - [i_2(1)]\big) = [p] - [q];$$

hence $\kappa(\iota(z)) = z$.

Let us show the second identity. For $[(a, b)] \in L(A)$, let $\varphi : D \to M_n(A)$ be a $*$-homomorphism defined by the balanced pair (a, b) (i.e., by $\varphi(\boldsymbol{a}) = a$

and $\varphi(\boldsymbol{b}) = b$), and let $\varphi^+ : D^+ \to M_n(A)$ be its extension, $\varphi^+(1) = 1$. Then $\iota(\kappa([(a, b)])) = [(\varphi_2^+(P), \varphi_2^+(Q))]$, where $\varphi_2^+ = \varphi^+ \otimes \mathrm{id}_{M_2}$.

For $s \in [0, 1]$, set

$$P_s = C_s P C_s, \quad Q_s = C_s Q C_s, \qquad \text{where } C_s = \begin{pmatrix} s \cdot 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$P_s, Q_s \in M_2(D^+), \quad P_s - Q_s \in M_2(D), \quad 0 \le P_s, Q_s \le 1,$$
$$(P_s - P_s^2)(P_s - Q_s) = 0, \quad (Q_s - Q_s^2)(P_s - Q_s) = 0$$

for all $s \in [0, 1]$, $P_0, Q_0 \in M_2(D)$, and

$$P_1 = P, \quad Q_1 = Q, \qquad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{a} \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{b} \end{pmatrix}.$$

Therefore, $(\varphi_2^+(P_s), \varphi_2^+(Q_s))$ provides a homotopy connecting $(\varphi_2^+(P), \varphi_2^+(Q))$ with $(0 \oplus a, 0 \oplus b)$; hence, the balanced pair $(\varphi_2^+(P), \varphi_2^+(Q))$ is equivalent to the balanced pair (a, b). □

Theorem 4.4. *The functors L and K_0 coincide for any C^*-algebra A.*

Proof. Both functors are half-exact and coincide for unital C^*-algebras, so the claim follows. □

Remark 4.5. Similarly to D, one can define a C^*-algebra D_B for any C^*-algebra B as an appropriate extension of $B * B$ by CB, where CB is the cone over B (or by $D_B = CB *_{SB} CB$). Then one gets the group $[D_B, A \otimes \mathbb{K}]$. Regretfully, D_B has no nice presentation (unlike $D = D_{\mathbb{C}}$), so we don't pursue here the bivariant version.

5. Yet another picture for K-theory

Consider the relations

$$a^* = a, \quad b^* = b, \quad a - a^2 = b - b^2, \quad a(a - a^2) = b(b - b^2). \tag{10}$$

This is equivalent to

$$a^* = a, \quad b^* = b, \quad f(a) = f(b)$$

for any polynomial (or, equivalently, for any continuous function) f such that

$$f(0) = f(1) = 0. \tag{11}$$

As before, for a C^*-algebra A we can define a group $L'(A)$ of homotopy classes of pairs (a, b), where a, b are matrices over A satisfying the relations (10) instead of (1). Note that the relations (10) do not impose any bound for norms of a, b;

hence they do not determine a universal C^*-algebra. Nevertheless, the relations (10) give the same functor.

Proposition 5.1. *The group $L'(A)$ is canonically isomorphic to $K_0(A)$.*

Proof. Let us construct maps $i : L(A) \to L'(A)$ and $j : L'(A) \to L(A)$. In the proof of Lemma 2.3 it was shown that if (a, b) is balanced then they satisfy (10) too, so we can define $i([(a, b)]) = [(a, b)]$. For $r \geq 0$, set

$$c_r(t) = \begin{cases} -r & \text{for } t < -r, \\ t & \text{for } -r \leq t \leq r+1, \\ r+1 & \text{for } t > r+1. \end{cases}$$

It is obvious that the pair $(c_r(a), c_r(b))$ satisfies (10) for any $r \geq 0$.

We claim that the pair $(c_0(a), c_0(b))$ is balanced. Indeed, first we obviously have $c_0(a), c_0(b) \geq 0$ and $\|c_0(a)\|, \|c_0(b)\| \leq 1$. Then, $c_0(a) - c_0(a)^2 = f(a)$, where the function

$$f(t) = \begin{cases} t - t^2 & \text{for } t \in [0, 1], \\ 0 & \text{for } t \notin [0, 1] \end{cases}$$

satisfies (11); so $c_0(a) - c_0(a)^2 = c_0(b) - c_0(b)^2$. Similarly, $c_0(a)(c_0(a) - c_0(a)^2) = c_0(b)(c_0(b) - c_0(b)^2)$. Then

$$\begin{aligned}(c_0(a) - c_0(a)^2)(c_0(a) - c_0(b)) &= c_0(a)^2 - c_0(a)^3 - (c_0(a) - c_0(a)^2)c_0(b) \\ &= c_0(b)^2 - c_0(b)^3 - (c_0(b) - c_0(b)^2)c_0(b) = 0.\end{aligned}$$

Therefore, we can set $j([(a, b)]) = [(c_0(a), c_0(b))]$. Obviously, $j \circ i$ is the identity map, so it remains to check that $i \circ j$ is the identity map as well. Set

$$a_s = \begin{cases} a & \text{for } s = 1, \\ c_{\tan \frac{\pi}{2} s}(a) & \text{for } s \in [0, 1). \end{cases}$$

Then (a_s, b_s), $s \in [0, 1]$, is a required continuous homotopy that connects the balanced pairs (a, b) and $(c_0(a), c_0(b))$. □

References

[Blackadar 1985] B. Blackadar, "Shape theory for C^*-algebras", *Math. Scand.* **56**:2 (1985), 249–275. MR Zbl

[Blackadar 1986] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications **5**, Springer, New York, 1986. MR Zbl

[Blackadar 2016] B. Blackadar, "The homotopy lifting theorem for semiprojective C^*-algebras", *Math. Scand.* **118**:2 (2016), 291–302. MR Zbl

[Eilers et al. 1998] S. Eilers, T. A. Loring, and G. K. Pedersen, "Stability of anticommutation relations: an application of noncommutative CW complexes", *J. Reine Angew. Math.* **499** (1998), 101–143. MR Zbl

[Loring 1997] T. A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs **8**, American Mathematical Society, Providence, RI, 1997. MR Zbl

[Raeburn and Sinclair 1989] I. Raeburn and A. M. Sinclair, "The C^*-algebra generated by two projections", *Math. Scand.* **65**:2 (1989), 278–290. MR Zbl

[Thiel 2009] H. Thiel, *One-dimensional C^*-algebras and their K-theory*, Diplomarbeit, Westfälische Wilhelms-Universität Münster, 2009, available at http://tinyurl.com/hannesthiel.

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