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**ELEMENTARY CALCULATION OF THE COHOMOLOGY  
RINGS OF REAL GRASSMANN MANIFOLDS**

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## ELEMENTARY CALCULATION OF THE COHOMOLOGY RINGS OF REAL GRASSMANN MANIFOLDS

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**We give elementary proofs of the Takeuchi and He theorems on the real cohomology rings and real equivariant cohomology rings of real Grassmann manifolds.**

### 1. Introduction

In an influential paper, Borel [1953] developed a general technique of computing cohomology rings of compact symmetric spaces. However, there are some exceptional cases including those of real Grassmann manifolds of odd dimension that do not immediately fit the Borel theory. In these cases the cohomology rings with real coefficients were determined by Takeuchi [1962].

Let  $\tilde{G}(m, n)$  denote the Grassmann manifold of oriented planes of dimension  $m$  in  $\mathbb{R}^{m+n}$ . Its tautological  $m$ - and  $n$ -vector bundles support the total Pontrjagin classes

$$p = 1 + p_1 + \cdots + p_{\lfloor m/2 \rfloor}, \quad \bar{p} = 1 + \bar{p}_1 + \cdots + \bar{p}_{\lfloor n/2 \rfloor},$$

as well as the Euler classes  $e_m$  and  $\bar{e}_n$ . If  $mn$  is odd, there is also a cohomology class  $r$  in  $\tilde{G}(m, n)$  of degree  $m + n - 1$ . Let  $\mathbb{P} = \mathbb{P}(m, n)$  denote the symmetric algebra over  $\mathbb{R}$  on the Pontrjagin classes  $p_i, \bar{p}_j$  subject to the relation  $p \cdot \bar{p} = 1$ .

**Theorem 1** [Takeuchi 1962]. *For  $m, n > 1$ , the cohomology algebra  $H^*\tilde{G}(m, n)$  over  $\mathbb{R}$  is isomorphic to*

- $\mathbb{P} \otimes \Lambda(r)$  if  $mn$  is odd,
- $\mathbb{P}[e_m]$  subject to  $e_m^2 = p_{m/2}$  if  $m$  is even and  $n$  is odd,
- $\mathbb{P}[\bar{e}_n]$  subject to  $\bar{e}_n^2 = \bar{p}_{n/2}$  if  $m$  is odd and  $n$  is even,
- $\mathbb{P}[e_m, \bar{e}_n]$  subject to  $e_m \bar{e}_n = 0$ ,  $e_m^2 = p_{m/2}$  and  $\bar{e}_n^2 = \bar{p}_{n/2}$ , if  $m$  and  $n$  are even.

The original proof of Theorem 1 by Takeuchi [1962] relies on the Borel theory [Borel 1953] as well as the Borel–Hirzebruch theory [Borel and Hirzebruch 1958]. The algebra  $H^*\tilde{G}(m, n)$  as well as its equivariant version were also recently

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computed by means of the GKM theory by He [2016]. Furthermore, there is an elegant computation of these algebras by means of pure Sullivan models by Carlson [2016] whose work relies on a model constructed by Kapovitch [2002]. We give a short elementary proof of Theorem 1 based on an observation that the total spaces of the tautological  $(m - 1)$ -sphere bundle over  $\tilde{G}(m, n)$  and  $n$ -sphere bundle over  $\tilde{G}(n + 1, m - 1)$  are isomorphic. We also deduce from Theorem 1 its equivariant version, see Theorem 6.

### 2. Proof of Theorem 1

**2.1. The case of even  $mn$ .** The calculations in these three cases can be carried out directly as in the case where both  $m$  and  $n$  are odd, see Section 2.2. Alternatively, it suffices to observe that if  $mn$  is even, then the Lie groups  $SO_m \times SO_n$  and  $SO_{m+n}$  are of the same rank, and therefore, Theorem 1 follows from the Borel Theorem [Borel 1953, §26].

**2.2. The case of odd  $mn$ .** To simplify notation, we fix the dimension  $m + n$  of the ambient space and write  $\tilde{G}_m$  for  $\tilde{G}(m, n)$ . Let  $S\tilde{G}_m$  denote the total space of the (tautological) sphere bundle associated with the tautological  $m$ -vector bundle  $E\tilde{G}_m$  over  $\tilde{G}_m$ . There are isomorphisms

$$(1) \quad \tilde{G}_{m-1} = \tilde{G}_{n+1}, \quad S\tilde{G}_m = S\tilde{G}_{n+1}.$$

**Remark 2.** Since  $\tilde{G}_{n+m}$  consists of two points, it is reasonable to define  $\tilde{G}_0$  to be a two-point set.

The multiplication by  $\bar{e}_{n+1}$  defines an endomorphism of  $H^*\tilde{G}_{n+1}$ . By the result in Section 2.1, its cokernel  $I$  is the quotient of the algebra  $\mathbb{P}[e_{m-1}]$  by the ideal generated by  $e_{m-1}^2 - p_{\lfloor m/2 \rfloor}$ , while its kernel  $K$  is the ideal  $e_{m-1}I$ . Let  $r$  be a cohomology class in  $S\tilde{G}_m$  such that  $\delta(r) = e_{m-1}$ , where  $\delta$  is the coboundary homomorphism in the Gysin exact sequence

$$(*) \quad \dots \xrightarrow{\sim \bar{e}_{n+1}} H^*\tilde{G}_{n+1} \xrightarrow{i^*} H^*S\tilde{G}_m \xrightarrow{\delta} H^{*-n}\tilde{G}_{n+1} \xrightarrow{\sim \bar{e}_{n+1}} \dots$$

of the tautological sphere bundle over  $\tilde{G}_{n+1}$  with total space  $S\tilde{G}_{n+1} = S\tilde{G}_m$ .

**Proposition 3.** *The cohomology algebra of  $S\tilde{G}_m$  is isomorphic to  $I \otimes \Lambda(r)$ .*

*Proof.* Since the restriction of  $i^*$  to  $I$  is injective, we will identify its image with  $I$ . By the Leibniz formula [Dold 1972, VII.8.10], the restriction of  $\delta$  to the vector space  $rI$  is an isomorphism onto  $K$ . It follows now from (\*) that the vector space  $H^*S\tilde{G}_m$  is isomorphic to  $I \oplus rI$ . Finally, the class  $r \smile r$  is trivial since  $r$  is of odd degree.  $\square$

*Proof in the case of odd  $mn$ .* In the Gysin exact sequence of the tautological sphere bundle over  $\tilde{G}_m$ ,

$$(**) \quad \dots \xrightarrow{0} H^*\tilde{G}_m \xrightarrow{i^*} H^*S\tilde{G}_m \xrightarrow{\bar{\delta}} H^{*-m+1}\tilde{G}_m \xrightarrow{0} \dots$$

the (surjective) coboundary homomorphism  $\bar{\delta}$  restricted to  $I = \mathbb{P} \oplus e_{m-1}\mathbb{P}/\sim$  is trivial on  $\mathbb{P}$  and takes  $e_{m-1}p$  to  $p$  for all  $p \in \mathbb{P}$ ; compare  $\bar{\delta}$  with the coboundary homomorphism in the Gysin exact sequence of the tautological sphere bundle over  $BSO_m$ , see [Milnor and Stasheff 1974, p.180]. Since the Euler class  $e_m$  of the tautological bundle over  $\tilde{G}_m$  is trivial, the group  $H^n\tilde{G}_m$  is a subgroup of the trivial group  $H^n S\tilde{G}_m$ . Hence  $\bar{\delta}(r) \in H^n\tilde{G}_m$  is trivial, and therefore  $r$  extends to a class in  $H^*\tilde{G}_m$ . This completes the proof of Theorem 1 in the case of odd  $mn$ .  $\square$

**Remark 4.** There is a free involution  $\sigma$  on  $\tilde{G}(m, n)$  whose orbit space is the Grassmann manifold  $G(m, n)$  of nonoriented planes. Hence  $H^*G(m, n)$  is isomorphic to the subring of  $H^*\tilde{G}(m, n)$  of  $\sigma$ -invariant classes. Casian and Kodama [2013, Theorem 3.2] gave a description of the adjacencies of Schubert cells in  $G(m, n)$ , from which it follows that the class  $r$  corresponds to the Schubert cell with the Young diagram  $(n) \times 1^{m-1}$ ; alternatively, it also follows from the Ehresmann’s adjacency formulas.

**Remark 5.** From Giambelli’s formula, the mod 2 reduction of  $r$  is  $\bar{w}_n w_{m-1} = \bar{w}_{n-1} w_m$ .

### 3. Equivariant case

Recall that  $\tilde{G} = \tilde{G}(m, n)$  can be identified with the quotient of  $SO(m + n)$  by  $SO(m) \times SO(n)$ . Let  $T$  denote the maximal torus of the latter group. There is a left action of  $T < SO(m + n)$  on the Grassmann manifold  $\tilde{G}$ . Let  $k$  be the dimension of  $T$ ; it equals  $\lfloor (m + n)/2 \rfloor$  if  $mn$  is even, and  $\lfloor (m + n - 1)/2 \rfloor$  if  $mn$  is odd. In this section we give a short computation of the equivariant cohomology ring  $H_T^*\tilde{G}$  which was earlier computed by He [2016] and Carlson [2016].

Recall that the equivariant cohomology ring  $H_T^*\tilde{G}$  is defined to be the cohomology ring of  $\tilde{G}_T = ET \times_T \tilde{G}$ , where  $ET$  is the total space of the principle  $T$ -bundle  $ET \rightarrow BT$ . In the cohomology ring of  $\tilde{G}_T$  there are total Pontrjagin classes  $p^T$  and  $\bar{p}^T$  and Euler classes  $e_m^T$  and  $\bar{e}_n^T$  of the tautological vector bundles over  $\tilde{G}_T$ , as well as the first Chern classes  $t_1, \dots, t_k$  of the  $k$  complex line bundles  $L_1, \dots, L_k$  that are pulled back from the tautological complex line bundles over  $BT = \mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty$ . Since the sum of the two tautological vector bundles over  $\tilde{G}_T$  is stably equivalent to  $L_1 \oplus \dots \oplus L_k$ , we have a relation  $p^T \bar{p}^T = \prod (1 + t_i^2)$ . Similarly,  $e_m^T \bar{e}_n^T = \prod t_i$  if  $m$  and  $n$  are even and  $m + n = 2k$ , and  $e_{m-1}^T \bar{e}_{n+1}^T = 0$  if  $m$  and  $n$  are odd and  $m + n = 2k + 2$ . Let  $\mathbb{P}^T$  denote the symmetric algebra over  $\mathbb{R}$  generated by the Pontrjagin classes  $p_i^T, \bar{p}_i^T$  as well as the Chern classes  $t_i$  subject to the relation  $p^T \bar{p}^T = \prod (1 + t_i^2)$ . When  $mn$  is odd, the equivariant version of the Gysin exact sequence (\*) defines a cohomology class  $\tilde{r}$  in  $S\tilde{G}_m$ , while from the equivariant version of the Gysin exact sequence (\*\*) it follows that  $\tilde{r}$  extends to a class in  $\tilde{G}_m$ .

**Theorem 6** [He 2016; Carlson 2016]. *For  $m, n > 1$ , the algebra  $H_T^* \tilde{G}(m, n)$  is isomorphic to*

- $\mathbb{P}^T \otimes \Lambda(\tilde{r})$  if  $mn$  is odd,
- $\mathbb{P}^T [e_m^T]$  subject to  $(e_m^T)^2 = p_{m/2}^T$  if  $m$  is even and  $n$  is odd,
- $\mathbb{P}^T [\bar{e}_n^T]$  subject to  $(\bar{e}_n^T)^2 = \bar{p}_{n/2}$  if  $m$  is odd and  $n$  is even,
- $\mathbb{P}^T [e_m, \bar{e}_n^T]$  subject to  $e_m^T \bar{e}_n^T = \prod t_i$ ,  $(e_m^T)^2 = p_{m/2}$  and  $(\bar{e}_n^T)^2 = \bar{p}_{n/2}$  if  $m$  and  $n$  are even.

*Proof.* We have seen that all cohomology classes of the fiber  $\tilde{G}$  of the fiber bundle  $\tilde{G}_T \rightarrow BT$  extend over the total space. Thus,  $H_T^* \tilde{G}$  is a free  $H^*BT$ -module on the set of generators given by a basis of the vector space  $H^* \tilde{G}$ . In particular, in  $H_T^* \tilde{G}$  there are no relations besides those listed in Theorem 6. Indeed, assume to the contrary that there is a trivial algebraic combination  $y$  of classes  $\tilde{r}$ ,  $e_m^T$ ,  $\bar{e}_n^T$ ,  $p_i$ ,  $\bar{p}_i$  and  $t_i$  not in the ideal  $\mathcal{I}$  generated by the relations in Theorem 6. Using relations in Theorem 6 we can reduce  $y$  to an  $H^*BT$ -linear combination of basis vectors of  $H^* \tilde{G}$ . Since  $y$  is trivial, all coefficients in the reduced linear combination are zero. Hence  $y \in \mathcal{I}$  contrary to the assumption.  $\square$

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### References

- [Borel 1953] A. Borel, “Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts”, *Ann. of Math. (2)* **57** (1953), 115–207. MR Zbl
- [Borel and Hirzebruch 1958] A. Borel and F. Hirzebruch, “Characteristic classes and homogeneous spaces, I”, *Amer. J. Math.* **80** (1958), 458–538. MR Zbl
- [Carlson 2016] J. D. Carlson, “The Borel equivariant cohomology of real Grassmannians”, preprint, 2016. arXiv
- [Casian and Kodama 2013] L. Casian and Y. Kodama, “On the cohomology of real Grassmann manifolds”, preprint, 2013. arXiv

- [Dold 1972] A. Dold, *Lectures on algebraic topology*, Grundlehren der math. Wissenschaften **200**, Springer, 1972. MR Zbl
- [Fehér 2013] L. Fehér, “Real Schubert calculus”, web page, Math Overflow, 2013, available at <http://mathoverflow.net/questions/119273/real-schubert-calculus>.
- [He 2016] C. He, “GKM theory, characteristic classes and the equivariant cohomology ring of real Grassmannian”, preprint, 2016. arXiv
- [Kapovitch 2002] V. Kapovitch, “A note on rational homotopy of biquotients”, preprint, 2002, available at <http://www.math.toronto.edu/vtk/biquotient.pdf>.
- [Milnor and Stasheff 1974] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies **76**, Princeton Univ. Press, 1974. MR Zbl
- [Takeuchi 1962] M. Takeuchi, “On Pontrjagin classes of compact symmetric spaces”, *J. Fac. Sci. Univ. Tokyo Sect. I* **9**:1962 (1962), 313–328. MR Zbl

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