Pacific Journal of Mathematics

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Volume 289 No. 2 August 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 289, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.289.443

ELEMENTARY CALCULATION OF THE COHOMOLOGY RINGS OF REAL GRASSMANN MANIFOLDS

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We give elementary proofs of the Takeuchi and He theorems on the real cohomology rings and real equivariant cohomology rings of real Grassmann manifolds.

1. Introduction

In an influential paper, Borel [1953] developed a general technique of computing cohomology rings of compact symmetric spaces. However, there are some exceptional cases including those of real Grassmann manifolds of odd dimension that do not immediately fit the Borel theory. In these cases the cohomology rings with real coefficients were determined by Takeuchi [1962].

Let $\tilde{\mathrm{G}}(m,n)$ denote the Grassmann manifold of oriented planes of dimension m in $\mathbb{R}^{m+n}$. Its tautological m- and n-vector bundles support the total Pontrjagin classes

$$p = 1 + p_1 + \cdots + p_{\lfloor m/2 \rfloor}, \qquad \bar{p} = 1 + \bar{p}_1 + \cdots + \cdots \bar{p}_{\lfloor n/2 \rfloor},$$

as well as the Euler classes e_m and $\bar{e}_n$. If mn is odd, there is also a cohomology class r in $\tilde{\mathrm{G}}(m,n)$ of degree $m+n-1$. Let $\mathbb{P} = \mathbb{P}(m,n)$ denote the symmetric algebra over $\mathbb{R}$ on the Pontrjagin classes p_i, $\bar{p}_j$ subject to the relation $p \cdot \bar{p} = 1$.

Theorem 1 [Takeuchi 1962]. *For $m, n > 1$, the cohomology algebra $H^*\tilde{\mathrm{G}}(m,n)$ over $\mathbb{R}$ is isomorphic to*

- $\mathbb{P} \otimes \Lambda(r)$ *if mn is odd,*
- $\mathbb{P}[e_m]$ *subject to $e_m^2 = p_{m/2}$ if m is even and n is odd,*
- $\mathbb{P}[\bar{e}_n]$ *subject to $\bar{e}_n^2 = \bar{p}_{n/2}$ if m is odd and n is even,*
- $\mathbb{P}[e_m, \bar{e}_n]$ *subject to $e_m \bar{e}_n = 0$, $e_m^2 = p_{m/2}$ and $\bar{e}_n^2 = \bar{p}_{n/2}$, if m and n are even.*

The original proof of Theorem 1 by Takeuchi [1962] relies on the Borel theory [Borel 1953] as well as the Borel–Hirzebruch theory [Borel and Hirzebruch 1958]. The algebra $H^*\tilde{\mathrm{G}}(m,n)$ as well as its equivariant version were also recently

MSC2010: primary 14M15, 57T15; secondary 22C05.
Keywords: Grassmann manifolds, equivariant cohomology.

computed by means of the GKM theory by He [2016]. Furthermore, there is an elegant computation of these algebras by means of pure Sullivan models by Carlson [2016] whose work relies on a model constructed by Kapovitch [2002]. We give a short elementary proof of Theorem 1 based on an observation that the total spaces of the tautological $(m-1)$-sphere bundle over $\tilde{\mathrm{G}}(m,n)$ and n-sphere bundle over $\tilde{\mathrm{G}}(n+1,m-1)$ are isomorphic. We also deduce from Theorem 1 its equivariant version, see Theorem 6.

2. Proof of Theorem 1

2.1. ***The case of even mn.*** The calculations in these three cases can be carried out directly as in the case where both m and n are odd, see Section 2.2. Alternatively, it suffices to observe that if mn is even, then the Lie groups $\mathrm{SO}_m \times \mathrm{SO}_n$ and SO_{m+n} are of the same rank, and therefore, Theorem 1 follows from the Borel Theorem [Borel 1953, §26].

2.2. ***The case of odd mn.*** To simplify notation, we fix the dimension $m+n$ of the ambient space and write $\tilde{\mathrm{G}}_m$ for $\tilde{\mathrm{G}}(m,n)$. Let $\mathrm{S}\tilde{\mathrm{G}}_m$ denote the total space of the (*tautological*) sphere bundle associated with the tautological m-vector bundle $E\tilde{\mathrm{G}}_m$ over $\tilde{\mathrm{G}}_m$. There are isomorphisms

$$\tilde{\mathrm{G}}_{m-1} = \tilde{\mathrm{G}}_{n+1}, \qquad \mathrm{S}\tilde{\mathrm{G}}_m = \mathrm{S}\tilde{\mathrm{G}}_{n+1}. \tag{1}$$

Remark 2. Since $\tilde{\mathrm{G}}_{n+m}$ consists of two points, it is reasonable to define $\tilde{\mathrm{G}}_0$ to be a two-point set.

The multiplication by $\bar{e}_{n+1}$ defines an endomorphism of $H^*\tilde{\mathrm{G}}_{n+1}$. By the result in Section 2.1, its cokernel I is the quotient of the algebra $\mathbb{P}[e_{m-1}]$ by the ideal generated by $e_{m-1}^2 - p_{\lfloor m/2 \rfloor}$, while its kernel K is the ideal $e_{m-1}I$. Let r be a cohomology class in $\mathrm{S}\tilde{\mathrm{G}}_m$ such that $\delta(r) = e_{m-1}$, where δ is the coboundary homomorphism in the Gysin exact sequence

$$\cdots \xrightarrow{\smile \bar{e}_{n+1}} H^*\tilde{\mathrm{G}}_{n+1} \xrightarrow{i^*} H^*\,\mathrm{S}\tilde{\mathrm{G}}_m \xrightarrow{\delta} H^{*-n}\tilde{\mathrm{G}}_{n+1} \xrightarrow{\smile \bar{e}_{n+1}} \cdots \tag{*}$$

of the tautological sphere bundle over $\tilde{\mathrm{G}}_{n+1}$ with total space $\mathrm{S}\tilde{\mathrm{G}}_{n+1} = \mathrm{S}\tilde{\mathrm{G}}_m$.

Proposition 3. *The cohomology algebra of* $\mathrm{S}\tilde{\mathrm{G}}_m$ *is isomorphic to* $I \otimes \Lambda(r)$.

Proof. Since the restriction of i^* to I is injective, we will identify its image with I. By the Leibniz formula [Dold 1972, VII.8.10], the restriction of δ to the vector space rI is an isomorphism onto K. It follows now from (*) that the vector space $H^*\mathrm{S}\tilde{\mathrm{G}}_m$ is isomorphic to $I \oplus rI$. Finally, the class $r \smile r$ is trivial since r is of odd degree. □

Proof in the case of odd mn. In the Gysin exact sequence of the tautological sphere bundle over $\tilde{\mathrm{G}}_m$,

$$\cdots \xrightarrow{0} H^*\tilde{\mathrm{G}}_m \xrightarrow{i^*} H^*\,\mathrm{S}\tilde{\mathrm{G}}_m \xrightarrow{\bar{\delta}} H^{*-m+1}\tilde{\mathrm{G}}_m \xrightarrow{0} \cdots \tag{**}$$

the (surjective) coboundary homomorphism $\bar{\delta}$ restricted to $I = \mathbb{P} \oplus e_{m-1}\mathbb{P}/_{\sim}$ is trivial on $\mathbb{P}$ and takes $e_{m-1}p$ to p for all $p \in \mathbb{P}$; compare $\bar{\delta}$ with the coboundary homomorphism in the Gysin exact sequence of the tautological sphere bundle over BSO_m, see [Milnor and Stasheff 1974, p.180]. Since the Euler class e_m of the tautological bundle over $\tilde{\mathrm{G}}_m$ is trivial, the group $H^n\tilde{\mathrm{G}}_m$ is a subgroup of the trivial group $H^n\,\mathrm{S}\tilde{\mathrm{G}}_m$. Hence $\bar{\delta}(r) \in H^n\tilde{\mathrm{G}}_m$ is trivial, and therefore r extends to a class in $H^*\tilde{\mathrm{G}}_m$. This completes the proof of Theorem 1 in the case of odd mn. $\square$

Remark 4. There is a free involution σ on $\tilde{\mathrm{G}}(m,n)$ whose orbit space is the Grassmann manifold $\mathrm{G}(m,n)$ of nonoriented planes. Hence $H^*\,\mathrm{G}(m,n)$ is isomorphic to the subring of $H^*\tilde{\mathrm{G}}(m,n)$ of σ-invariant classes. Casian and Kodama [2013, Theorem 3.2] gave a description of the adjacencies of Schubert cells in $\mathrm{G}(m,n)$, from which it follows that the class r corresponds to the Schubert cell with the Young diagram $(n) \times 1^{m-1}$; alternatively, it also follows from the Ehresmann's adjacency formulas.

Remark 5. From Giambelli's formula, the mod 2 reduction of r is $\overline{w}_n w_{m-1} = \overline{w}_{n-1} w_m$.

3. Equivariant case

Recall that $\tilde{\mathrm{G}} = \tilde{\mathrm{G}}(m,n)$ can be identified with the quotient of $\mathrm{SO}(m+n)$ by $\mathrm{SO}(m) \times \mathrm{SO}(n)$. Let T denote the maximal torus of the latter group. There is a left action of $T < \mathrm{SO}(m+n)$ on the Grassmann manifold $\tilde{\mathrm{G}}$. Let k be the dimension of T; it equals $\lfloor (m+n)/2 \rfloor$ if mn is even, and $\lfloor (m+n-1)/2 \rfloor$ if mn is odd. In this section we give a short computation of the equivariant cohomology ring $H^*_T\tilde{\mathrm{G}}$ which was earlier computed by He [2016] and Carlson [2016].

Recall that the equivariant cohomology ring $H^*_T\tilde{\mathrm{G}}$ is defined to be the cohomology ring of $\tilde{\mathrm{G}}_T = ET \times_T \tilde{\mathrm{G}}$, where ET is the total space of the principle T-bundle $ET \to BT$. In the cohomology ring of $\tilde{\mathrm{G}}_T$ there are total Pontrjagin classes p^T and $\bar{p}^T$ and Euler classes e^T_m and $\bar{e}^T_n$ of the tautological vector bundles over $\tilde{\mathrm{G}}_T$, as well as the first Chern classes $t_1, \dots, t_k$ of the k complex line bundles $L_1, \dots, L_k$ that are pulled back from the tautological complex line bundles over $BT = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$. Since the sum of the two tautological vector bundles over $\tilde{\mathrm{G}}_T$ is stably equivalent to $L_1 \oplus \cdots \oplus L_k$, we have a relation $p^T\bar{p}^T = \prod(1+t_i^2)$. Similarly, $e^T_m\bar{e}^T_n = \prod t_i$ if m and n are even and $m+n = 2k$, and $e^T_{m-1}\bar{e}^T_{n+1} = 0$ if m and n are odd and $m+n = 2k+2$. Let $\mathbb{P}^T$ denote the symmetric algebra over $\mathbb{R}$ generated by the Pontrjagin classes p^T_i, $\bar{p}^T_i$ as well as the Chern classes t_i subject to the relation $p^T\bar{p}^T = \prod(1+t_i^2)$. When mn is odd, the equivariant version of the Gysin exact sequence (*) defines a cohomology class $\tilde{r}$ in $\mathrm{S}\tilde{\mathrm{G}}_m$, while from the equivariant version of the Gysin exact sequence (**) it follows that $\tilde{r}$ extends to a class in $\tilde{\mathrm{G}}_m$.

Theorem 6 [He 2016; Carlson 2016]. *For $m, n > 1$, the algebra $H^*_T\tilde{\mathrm{G}}(m,n)$ is isomorphic to*

- $\mathbb{P}^T \otimes \Lambda(\tilde{r})$ *if mn is odd,*
- $\mathbb{P}^T[e^T_m]$ *subject to $(e^T_m)^2 = p^T_{m/2}$ if m is even and n is odd,*
- $\mathbb{P}^T[\bar{e}^T_n]$ *subject to $(\bar{e}^T_n)^2 = \bar{p}_{n/2}$ if m is odd and n is even,*
- $\mathbb{P}^T[e_m, \bar{e}^T_n]$ *subject to $e^T_m\bar{e}^T_n = \prod t_i$, $(e^T_m)^2 = p_{m/2}$ and $(\bar{e}^T_n)^2 = \bar{p}_{n/2}$ if m and n are even.*

Proof. We have seen that all cohomology classes of the fiber $\tilde{\mathrm{G}}$ of the fiber bundle $\tilde{\mathrm{G}}_T \to BT$ extend over the total space. Thus, $H^*_T\tilde{\mathrm{G}}$ is a free H^*BT-module on the set of generators given by a basis of the vector space $H^*\tilde{\mathrm{G}}$. In particular, in $H^*_T\tilde{\mathrm{G}}$ there are no relations besides those listed in Theorem 6. Indeed, assume to the contrary that there is a trivial algebraic combination y of classes $\tilde{r}$, e^T_m, $\bar{e}^T_n$, p_i, $\bar{p}_i$ and t_i not in the ideal $\mathcal{I}$ generated by the relations in Theorem 6. Using relations in Theorem 6 we can reduce y to an H^*BT-linear combination of basis vectors of $H^*\tilde{\mathrm{G}}$. Since y is trivial, all coefficients in the reduced linear combination are zero. Hence $y \in \mathcal{I}$ contrary to the assumption. $\square$

Acknowledgment

I am grateful to Toru Ohmoto; this project accidentally came out of our conversations at a Sapporo onsen during my visiting Hokkaido University. Ohmoto could not be convinced to coauthor the note.

I am thankful to the referee who informed me about the papers [Takeuchi 1962] and [He 2016], made many helpful comments and shared with me a copy of [Takeuchi 1962]. I am grateful to Osamu Saeki for many references, and to László Fehér for letting me know about his unpublished work. Namely, Fehér [2013] observed that the Schubert calculus of the Grassmann manifold $\mathrm{G}(2m, 2n)$ is analogous to the Schubert calculus of the Grassmann manifold of complex m-planes in $\mathbb{C}^n$.

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Received August 30, 2015. Revised February 5, 2017.

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PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 289 No. 2 August 2017

0030-8730(201708)289:2;1-O