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DONG WANG

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CONCENTRATION FOR A BIHARMONIC SCHRÖDINGER EQUATION

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We consider the fourth-order problem

$$\begin{cases} \epsilon^4 \Delta^2 u + V(x)u = P(x) f(|u|)u, & x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where V and P are spatial distributions of external potentials. We study the concentration phenomena of the solutions as $\epsilon \to 0$ using variational methods.

1. Introduction

This work is devoted to the analysis of solutions that solve the following nonlinear stationary biharmonic Schrödinger equation:

$$\begin{cases} \epsilon^4 \Delta^2 u + V(x)u = P(x) f(|u|)u, & x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1-1}$$

where ϵ denotes Planck's constant, and V, P are spatial distributions of external potentials. To simplify the idea of this work, we are going to describe certain concentration phenomena of the solutions of (1-1) as $\epsilon \to 0$, for physical purposes.

Problem (1-1) is the biharmonic version of the usual Schrödinger equation which has been extensively studied in literature [Floer and Weinstein 1986; Coti Zelati and Rabinowitz 1992; Rabinowitz 1992; Wang 1993; Willem 1996; del Pino and Felmer 1996; Gui 1996; Ambrosetti et al. 1997; Bartsch et al. 2001; Sirakov 2002; Byeon and Wang 2003; Ding and Tanaka 2003; Ni and Wei 2006; Ambrosetti and Malchiodi 2006; Byeon and Jeanjean 2007; Ding and Szulkin 2007; Ding and Wei 2007; Ding and Liu 2013] and references therein. In general, if we omit the exponent 2 of the first term in (1-1), we have an equation

$$(-i\epsilon\nabla + A(x))^2 w + V(x)w = f(x, w), \qquad w \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1-2}$$

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which arises when one seeks the standing wave solutions of the Schrödinger equation

$$i\hbar\frac{\partial\varphi}{\partial t}=(-i\hbar\nabla+A(x))^2\varphi+W(x)\varphi-n(x,|\varphi|)\varphi. \tag{1-3}$$

Considerable effort has gone into the study of the nonlinear Schrödinger equations without the magnetic field (i.e., $A(x)=0$) for studying the existence, multiplicity and qualitative properties of standing wave solutions. When the magnetic vector $A(x)\not\equiv 0$, the first work seems to be involved in [Esteban and Lions 1989] in which the existence of solutions of (1-2) via a constrained minimization argument with $\epsilon=1$ is studied. Later, under certain assumptions, the existence and multiplicity of solutions of (1-2) ($\epsilon=1$) were obtained in [Arioli and Szulkin 2003; Pankov 2003; Wang 2008; Liang and Zhang 2011]. The existence and concentration phenomena of semiclassical solutions of (1-2) were studied in [Kurata 2000], where $f(x,w)=g(|w|^2)w$ is subcritical and $\inf_{x\in\mathbb{R}^N}V(x)>0$ such that the Palais–Smale condition holds for any energy level and for any $\epsilon>0$. For the case $A(x)=0$, Floer and Weinstein [1986], proved in the one dimensional case and for $f(w)=w^3$ that a single spike solution concentrates around any given nondegenerate critical point of the linear potential $V(x)$. Oh [1988; 1990] extended this result in higher dimensions and for $f(u)=|u|^{p-1}u$ $(1<p<N+2/N-2)$. Subsequently, variational methods were found suitable for such issues and the existence of spike layer solutions in the semiclassical limit were established under various conditions of $V(x)$. Particularly, initiated by Rabinowitz [1992], the existence of positive solutions of the Schrödinger equation for small $\epsilon>0$ is proved whenever

$$\liminf_{|x|\to\infty}V(x)>\inf_{x\in\mathbb{R}^N}V(x).$$

These solutions concentrate around the global minimum points of V when $\epsilon\to 0$, as was shown by Wang [1993]. It should be pointed out that M. del Pino and P. Felmer [1996] first succeeded in proving a localized version of the concentration behavior of semiclassical solutions.

By using a combination of stability analysis and numerical simulations, the role of small fourth-order dispersion has been considered in a series of papers by Karpman and Shagalov ([2000] and the references therein), who studied the equation

$$i\psi_t(t,x)+\Delta\psi+|\psi|^{2\sigma}\psi+\epsilon\Delta^2\psi=0,$$

in the case when $\epsilon<0$. Later, in [Ben-Artzi et al. 2000], Ben-Artzi, Koch, and Saut obtained sharp dispersive estimates for the biharmonic Schrödinger operator in

$$i\partial_t u+\Delta^2u+\epsilon\Delta u+f(|u|^2)u=0,$$

namely for the linear group associated to $i\partial_t + \Delta^2 \pm \Delta$. Parallel to this, some specific nonlinear fourth-order Schrödinger equations have received deep consideration. Fibich, Ilan, and Papanicolaou, in [Fibich et al. 2002], analyzed self-focusing and singularity formation in the nonlinear Schrödinger equation (NLS) with high-order dispersion $i\psi_t \pm \Delta^q \psi + |\psi|^{2\sigma}\psi = 0$, in the isotropic mixed-dispersion NLS $i\psi_t + \Delta\psi + \epsilon\Delta^2\psi + |\psi|^{2\sigma}\psi = 0$, and in nonisotropic mixed-dispersion NLS equations which model propagation in fiber arrays. Almost at the same time, Guo and Wang [2002] studied the existence and scattering theory for the nonlinear Schrödinger equations $iu_t + (-\Delta)^m u + f(u) = 0$, with $u(0, x) = \phi(x)$, where $u(t, x)$ defined on $\mathbb{R} \times \mathbb{R}^n$ is a complex valued function, $m \geq 1$ is an integer and f is a scalar nonlinear function. Not much later, Hao, Hsiao and Wang, in [Hao et al. 2006; 2007], discussed the Cauchy problem in a high regularity setting. Subsequently, Segata [2006] proved scattering in the case the space dimension is one and considered the three-dimensional motion of an isolated vortex filament by using the method of Fourier restriction norm.

Motivated by the previously mentioned works, we are mainly interested in (1-1) with the biharmonic operator

$$\Delta^2 u = \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} u + \sum_{i \neq j}^{N} \frac{\partial^4}{\partial x_i^2 x_j^2} u.$$

The biharmonic Schrödinger equation (1-1) appears when one considers the stationary solutions $w(t, x) = e^{i\lambda t} u(x)$ of the t-dependent equation of the form

$$i\epsilon\partial_t w - \epsilon^4 \Delta^2 w - M(x)w + P(x) f(|w|)w = 0, \tag{1-4}$$

where $\lambda \in \mathbb{R}$. Such stationary solutions, also called standing waves, are finite energy waveguide solutions of (1-1) after rearranging terms in (1-4). It is worth pointing out that although there are many works dealing with problems related to (1-2), so many problems appear when dealing with the fourth-order problem. The main reason for this difficulty is the lack of a general maximum principle to the biharmonic operator. This leads to a series of technical problems in trying to adapt some second-order classical arguments.

Precisely, we formulate the fundamental assumption on the potential functions V, P as

(VP) $V, P \in L^\infty(\mathbb{R}^N)$ are uniformly continuous such that $\inf_{x \in \mathbb{R}^N} V(x) > 0$ and $\inf_{x \in \mathbb{R}^N} P(x) > 0$.

To obtain the concentration results, let us introduce the following restrictions on the nonlinear function f:

(f_1) $f(0)=0$, $f\in C^1(0,\infty)$, $f'(s)>0$ for $s>0$, and there is a $p\in(2,2N/(N-4))$ such that $\lim_{s\to\infty} f(s)/s^{p-2}<\infty$,

(f_2) denoting $F(s)=\int_0^s f(t)t\,dt$, there is $\theta>2$ such that $0<F(s)\le(1/\theta)f(s)s^2$ for $s>0$.

Now we are ready to describe our concentration results. First let us set

$$\begin{aligned}
\tau &:= \inf_{x\in\mathbb{R}^N} V(x), & \tau_\infty &:= \lim_{|x|\to\infty}\inf V(x), & \tau_p &:= \inf_{x\in\mathcal{P}} V(x),\\
\gamma &:= \sup_{x\in\mathbb{R}^N} P(x), & \gamma_\infty &:= \lim_{|x|\to\infty}\sup P(x), & \gamma_v &:= \sup_{x\in\mathcal{V}} P(x),\\
\mathcal{V} &:= \{x\in\mathbb{R}^N : V(x)=\tau\}, & \mathcal{P} &:= \{x\in\mathbb{R}^N : P(x)=\gamma\},
\end{aligned}$$

and suppose that

(V) V leads the behavior, i.e., $\tau<\tau_\infty$ and there exists $R>0$ such that $\gamma_v\ge P(x)$ for any $|x|\ge R$,

(P) P leads the behavior, i.e., $\gamma>\gamma_\infty$ and there exists $R>0$ such that $\tau_p\le V(x)$ for any $|x|\ge R$.

Let us define for (V)

$$\mathscr{A}_v := \{x\in\mathcal{V} : P(x)=\gamma_v\}\cup\{x\notin\mathcal{V} : P(x)>\gamma_v\},$$

and for (P)

$$\mathscr{A}_p := \{x\in\mathcal{P} : V(x)=\tau_p\}\cup\{x\notin\mathcal{P} : V(x)<\tau_p\}.$$

Remark that, generally, $\mathcal{V}\cap\mathcal{P}=\varnothing$. And notice that, for example, $\mathcal{V}\cap\mathcal{P}\ne\varnothing$ implies $\tau=\tau_p$ and $\gamma=\gamma_v$, from which we deduce that

$$\mathscr{A}_v=\mathcal{V}\cap\mathcal{P}.$$

Under our assumptions (V) and (P), $\mathscr{A}_v$ and $\mathscr{A}_p$ are nonempty bounded sets in $\mathbb{R}^N$, and $\mathscr{A}_v=\mathscr{A}_p=\mathcal{V}\cap\mathcal{P}$ if and only if $\mathcal{V}\cap\mathcal{P}\ne\varnothing$; see also [Ding and Liu 2013].

To give a better description of our results, let us set

$$\mathscr{C}=\begin{cases}\mathscr{A}_v & \text{if } (V) \text{ holds},\\ \mathscr{A}_p & \text{if } (P) \text{ holds}.\end{cases}$$

We have the following results:

Theorem 1.1. *Let (VP), (f_1) and (f_2) hold. Assume additionally that either (V) or (P) holds. Then (1-1) has (at least) one ground state solution for all small ϵ.*

And for the concentration of the solutions of (1-1) as $\epsilon\to0$, we have:

Theorem 1.2. *Let (VP), (f_1) and (f_2) hold. Assume additionally that either (V) or (P) holds. Let u_ϵ be the solution of* (1-1)*, given by Theorem 1.1. For any $\epsilon_n > 0$ with $\lim_{n\to\infty} \epsilon_n = 0$, up to a subsequence, there exists $x_{\epsilon_n} \in \mathbb{R}^N$ which is a maximum point of $|u_{\epsilon_n}|$, such that*

$$\operatorname{dist}(\epsilon_n x_{\epsilon_n}, \mathscr{C}) \to 0 \quad \textit{as} \quad n \to \infty.$$

Let $w_{\epsilon_n}(x) := u_{\epsilon_n}(\epsilon_n(x + x_{\epsilon_n}))$. Then $w_{\epsilon_n} \to w_0$ in $H^2(\mathbb{R}^N)$, where w_0 is a ground state solution of

$$\Delta^2 w + V(x_0)w = P(x_0) f(|w|)w, \qquad x_0 \in \mathscr{C}.$$

Our arguments are variational with a mixture of the mountain pass technique and Nehari Manifolds. The paper is organized as follows. In the next section, we introduce some notations and the variational framework for such problem. We prove the existence and concentration results for (1-1) in the remaining two sections.

2. Variational framework

Hereafter we use the following notation:

- $E := H^2(\mathbb{R}^N)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + uv)\, dx, \qquad \|u\|^2 = \int_{\mathbb{R}^N} (|\Delta u|^2 + u^2)\, dx.$$

- $L^q(\Omega)$, $1 \le q \le +\infty$, denotes a Lebesgue space. The norm in $L^q(\Omega)$ is denoted by $|u|_{q,\Omega}$ when Ω is a proper subset of $\mathbb{R}^N$, by $|\cdot|_p$ when $\Omega = \mathbb{R}^N$.
- For any $\rho > 0$ and for any $z \in \mathbb{R}^N$, $B_\rho(z)$ denotes the ball of radius ρ centered at z, $|B_\rho(z)|$ denotes its Lebesgue measure and $\partial B_\rho(z)$ denotes its boundary.
- For ease of notation, let us set $2^* = 2N/(N-4)$. Without loss of generality, we assume that $0 \in \mathscr{C}$.

By assumption (VP), the following energy functional I of (1-1) defined in E is well defined,

$$I(u) = \frac{1}{2}\epsilon^4 \int_{\mathbb{R}^N} |\Delta u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2\, dx - \int_{\mathbb{R}^N} P(x)F(|u|)\, dx.$$

Moreover, the solutions of (1-1) are the critical points of I.

Equivalent problem. Making the change of variable $x \to \epsilon x$, (1-1) becomes

$$\begin{cases} \Delta^2 z + V_\epsilon(x)z = P_\epsilon(x) f(|z|)z, & x \in \mathbb{R}^N, \\ z \in E, \end{cases} \tag{2-1}$$

where $V_\epsilon(x) = V(\epsilon x)$, $P_\epsilon(x) = P(\epsilon x)$, and $z(x) = u(\epsilon x)$.

In the sequel, we will in fact focus on finding the critical points of the energy functional associated to (2-1) which is defined by

$$\phi_\epsilon(z)=\frac{1}{2}\int_{\mathbb{R}^N}|\Delta z|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}V_\epsilon(x)|z|^2\,dx-\int_{\mathbb{R}^N}P_\epsilon(x)F(|z|)\,dx.$$

Remark. Denote $\|z\|_\epsilon^2=\int_{\mathbb{R}^N}[|\Delta z|^2+V_\epsilon(x)|z|^2]\,dx$. Recall that $\tau=\inf_{x\in\mathbb{R}^N}V(x)$. Here norm $\|\cdot\|_\epsilon$ is equivalent to $\|\cdot\|$. Indeed, by assumption (VP),

$$\|z\|_\epsilon^2\geq\int_{\mathbb{R}^N}[|\Delta z|^2+\tau|z|^2]\,dx\geq\delta\int_{\mathbb{R}^N}(|\Delta z|^2+|z|^2)\,dx=\delta\|z\|^2,$$

where

$$\delta=\min\{1,\tau\}>0$$

and also we have

$$\begin{aligned}\|z\|_\epsilon^2&\leq\int_{\mathbb{R}^N}(|\Delta z|^2+|V|_\infty\cdot|z|^2)\,dx\\&\leq(1+|V|_\infty)\int_{\mathbb{R}^N}(|\Delta z|^2+|z|^2)\,dx=(1+|V|_\infty)\|z\|^2.\end{aligned}$$

For notational convenience, let us write $\phi_\epsilon(z)=\frac{1}{2}\|z\|_\epsilon^2-\int_{\mathbb{R}^N}P_\epsilon(x)F(|z|)\,dx$. We observe that ϕ_ϵ satisfies the so-called mountain pass structure. For details, recall that by (f_1), (f_2), we have

$$\hat{F}(s):=\frac{1}{2}f(s)s^2-F(s)\geq\frac{\theta-2}{2\theta}f(s)s^2\qquad\text{for any s>0.}$$

Moreover there exists δ_1 small enough and $c_1>0$ such that

$$f(s)\leq\delta_1+c_1s^{p-2}.$$

Hence

$$F(s)\leq\frac{\delta_1}{2}s^2+c_1's^p,$$

which implies

$$\phi_\epsilon(z)=\frac{1}{2}\|z\|_\epsilon^2-\int_{\mathbb{R}^N}P_\epsilon(x)F(|z|)\,dx\geq\frac{1}{4}\|z\|_\epsilon^2-c_1''\|z\|_\epsilon^p,$$

where we have used the Sobolev embedding theorems, $E\hookrightarrow L^p(\mathbb{R}^N)$, $|z|_p\leq C\cdot\|z\|_\epsilon$ for some positive constant C. Notice that when $p>2$, then by direct computation, we have $\phi_\epsilon(z)\geq\alpha>0$, where $z\in\partial B_\rho(0)=\{z\in E:\|z\|=\rho\}$ for some $\rho>0$.

By (f_2) we have $F(s)\geq cs^\theta-s^2$ for some $c>0$. Thus, for any $z\in E\setminus\{0\}$ and for any positive real number t, we have

$$\phi_\epsilon(tz)\leq c_2t^2\|z\|_\epsilon^2-c_3t^\theta\int_{\mathbb{R}^N}|z|^\theta\,dx.$$

Since $p > 2$, we know $\phi_\epsilon(tz) \to -\infty$ as $t \to \infty$. Also, $\phi_\epsilon(0) = 0$. Thus there exists a sufficiently large positive real number t_z, such that $\|t_z z\| > \rho$ and $\max\{\phi_\epsilon(0), \phi_\epsilon(t_z z)\} \le 0 < \alpha$.

Based on the above discussion, by the classical mountain pass theorem, there exists a sequence $\{z_{n,\epsilon} : n = 1, 2, \ldots\} \subseteq E \setminus \{0\}$ such that

$$\phi_\epsilon(z_{n,\epsilon}) \to c_\epsilon \quad \text{as } n \to \infty, \quad \text{and} \quad \phi'_\epsilon(z_{n,\epsilon}) \to 0 \quad \text{as } n \to \infty, \tag{2-2}$$

where

$$c_\epsilon = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \phi_\epsilon(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \phi_\epsilon(\gamma(1)) < \alpha\}.$$

Moreover, by (2-2), we have

$$o_n(1)\|z_{n,\epsilon}\| = \phi'_\epsilon(z_{n,\epsilon})z_{n,\epsilon} = \|z_{n,\epsilon}\|^2_\epsilon - \int_{\mathbb{R}^N} P_\epsilon(x) f(|z_{n,\epsilon}|)z^2_{n,\epsilon}\, dx,$$

$$c_\epsilon + o_n(1)\|z_{n,\epsilon}\| = \phi_\epsilon(z_{n,\epsilon}) - \frac{1}{2}\phi'_\epsilon(z_{n,\epsilon})z_{n,\epsilon} = \int_{\mathbb{R}^N} P_\epsilon(x)\hat{F}(|z_{n,\epsilon}|)\, dx.$$

It follows that

$$\|z_{n,\epsilon}\|^2_\epsilon \le M + o_n(1)\|z_{n,\epsilon}\|$$

for some $M > 0$. Thus $\{z_{n,\epsilon} : n = 1, 2, \ldots\}$ is a bounded sequence and $\|z_{n,\epsilon}\| \ge D$ for some positive constant D, $n = 1, 2, \ldots$.

The limiting equation. Now we consider the special form of the equation (2-1)

$$\begin{cases} \Delta^2 z + \mu z = \lambda f(|z|)z, & x \in \mathbb{R}^N, \\ z \in E, \end{cases} \tag{2-3}$$

where $\mu > 0$ and $\lambda > 0$ are both constants.

Then the energy functional of (2-3) is

$$\phi_{\mu,\lambda}(z) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta z|^2\, dx + \frac{\mu}{2}\int_{\mathbb{R}^N} |z|^2\, dx - \lambda\int_{\mathbb{R}^N} F(|z|)\, dx.$$

Denote $\|z\|^2_\mu = \int_{\mathbb{R}^N}(|\Delta z|^2 + \mu|z|^2)\, dx$. Then similarly, norm $\|\cdot\|_\mu$ is also equivalent to $\|\cdot\|$.

Rewrite $\phi_{\mu,\lambda}(z) = \frac{1}{2}\|z\|^2_\mu - \lambda\int_{\mathbb{R}^N} F(|z|)\, dx$. Similarly, $\phi_{\mu,\lambda}$ also satisfies the conditions of the mountain pass theorem, i.e.,

- there exists $\rho, \alpha > 0$ such that $\phi_{\mu,\lambda}|_{\partial B_\rho(0)} \ge \alpha$,
- there exists $e \in E$, $\|e\| > \rho$ such that $\max\{\phi_{\mu,\lambda}(0), \phi_{\mu,\lambda}(e)\} < \alpha$.

By the mountain pass lemma, there exists a sequence $\{z_{n,\mu,\lambda} : n=1,2,\dots\} \subseteq E\backslash\{0\}$ such that

(2-4) $$\phi_{\mu,\lambda}(z_{n,\mu,\lambda}) \to c_{\mu,\lambda} \quad \text{as } n \to \infty, \quad \text{and} \quad \phi'_{\mu,\lambda}(z_{n,\mu,\lambda}) \to 0 \quad \text{as } n \to \infty,$$

where

$$c_{\mu,\lambda} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \phi_{\mu,\lambda}(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Similarly, $\{z_{n,\mu,\lambda} : n = 1, 2, \dots\}$ is also a bounded sequence, and $\|z_{n,\mu,\lambda}\| \geq D_{\mu,\lambda}$ for some positive constant $D_{\mu,\lambda}$, $n = 1, 2, \dots$.

A standard concentration compactness argument shows that there exists $R > 0$, $\alpha > 0$ and $\{x_n : n = 1, 2, \dots\} \subseteq \mathbb{R}^N$, such that

$$\int_{B_R(x_n)} |z_{n,\mu,\lambda}|^2 \, dx \geq \alpha.$$

Let be $\tilde{z}_{n,\mu,\lambda}(x) = z_{n,\mu,\lambda}(x + x_n)$. Then $\{\tilde{z}_{n,\mu,\lambda} : n = 1, 2, \dots\}$ is also a bounded sequence and $\|\tilde{z}_{n,\mu,\lambda}\| \geq D_{\mu,\lambda}$, $n = 1, 2, \dots$. Thus, up to a subsequence, we have $\tilde{z}_{n,\mu,\lambda} \rightharpoonup z_{\mu,\lambda} \in E$. Particularly, $z_{\mu,\lambda} \neq 0$. Indeed, by Sobolev embedding theorems, $\tilde{z}_{n,\mu,\lambda} \rightharpoonup z_{\mu,\lambda}$ in E implies $\tilde{z}_{n,\mu,\lambda} \to z_{\mu,\lambda}$ in $L^2_{loc}(\mathbb{R}^N)$. Together with

$$\int_{B_R(0)} |\tilde{z}_{n,\mu,\lambda}|^2 \, dx = \int_{B_R(x_n)} |z_{n,\mu,\lambda}|^2 \, dx \geq \alpha,$$

we have

$$\int_{B_R(0)} |z_{\mu,\lambda}|^2 \, dx \geq \alpha,$$

i.e., $z_{\mu,\lambda} \neq 0$.

By using the weak sequential continuity of $\phi'_{\mu,\lambda} : E \to E^*$, we know that $\phi'_{\mu,\lambda}(z_{\mu,\lambda})z_{\mu,\lambda} = 0$, i.e., $z_{\mu,\lambda}$ is a critical point of the functional $\phi_{\mu,\lambda}$ which is a weak solution of (2-3).

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Initially, we would like to collect some useful results concerning the limiting equation. A first observation is that if $\gamma_{\mu,\lambda}$ denotes the ground state energy of $\phi_{\mu,\lambda}$, we have

$$c_{\mu,\lambda} = \gamma_{\mu,\lambda} = \phi_{\mu,\lambda}(z_{\mu,\lambda}).$$

Indeed, since $\phi'_{\mu,\lambda}(z_{\mu,\lambda})z_{\mu,\lambda}=0$, we have

$$\begin{aligned}\phi_{\mu,\lambda}(z_{\mu,\lambda}) &= \phi_{\mu,\lambda}(z_{\mu,\lambda}) - \tfrac{1}{2}\phi'_{\mu,\lambda}(z_{\mu,\lambda})z_{\mu,\lambda}\\ &= \lambda\int_{\mathbb{R}^N}\hat{F}(|z_{\mu,\lambda}|)\,dx\\ &\le \liminf_{n\to\infty}\lambda\int_{\mathbb{R}^N}\hat{F}(|\tilde{z}_{n,\mu,\lambda}|)\,dx\\ &= \liminf_{n\to\infty}\big[\phi_{\mu,\lambda}(\tilde{z}_{n,\mu,\lambda}) - \tfrac{1}{2}\phi'_{\mu,\lambda}(\tilde{z}_{n,\mu,\lambda})\tilde{z}_{n,\mu,\lambda}\big]\\ &= c_{\mu,\lambda},\end{aligned}$$

the above inequality follows from the Fatou's lemma. Now we consider the Nehari manifold

$$\mathcal{N}_{\mu,\lambda} := \{z\in E\setminus\{0\} : \phi'_{\mu,\lambda}(z)z=0\}.$$

By a direct observation, for any $z\in E\setminus\{0\}$, there exists $t_z>0$, such that $t_z z\in\mathcal{N}_{\mu,\lambda}$. Notice that

$$\max_{t>0}\phi_{\mu,\lambda}(tz)=\phi_{\mu,\lambda}(t_z z).$$

Indeed, denote $f(t)=\phi_{\mu,\lambda}(tz)$. Then it is easy to check that $f(t)\to-\infty$ as $t\to\infty$, $f(0)=0$ and there is small t such that $f(t)>0$. Since $f'(t_z)=\phi'_{\mu,\lambda}(t_z z)z=0$ and $f''(t)<0$, we know that t_z is the unique critical point of $f(t)$. Therefore $\max_{t>0}\phi_{\mu,\lambda}(tz)=\phi_{\mu,\lambda}(t_z z)$. Let

$$\bar{c}_{\mu,\lambda}=\inf_{z\in E\setminus\{0\}}\max_{t>0}\phi_{\mu,\lambda}(tz).$$

Then a standard argument shows that

$$\phi_{\mu,\lambda}(z_{\mu,\lambda})\le c_{\mu,\lambda}\le\bar{c}_{\mu,\lambda}=\inf_{z\in\mathcal{N}_{\mu,\lambda}}\phi_{\mu,\lambda}(z)\le\phi_{\mu,\lambda}(z_{\mu,\lambda}).$$

Thus $\phi_{\mu,\lambda}(z_{\mu,\lambda})=c_{\mu,\lambda}$.

The following lemma is a direct application of the above observation:

Lemma 3.1. *If $\mu_2\ge\mu_1$ and $\lambda_2\le\lambda_1$, then $c_{\mu_2,\lambda_2}\ge c_{\mu_1,\lambda_1}$. Particularly, if*

$$\max\{\mu_2-\mu_1,\lambda_1-\lambda_2\}>0,$$

then $c_{\mu_2,\lambda_2}>c_{\mu_1,\lambda_1}$.

Proof. Let $z_{\mu_2,\lambda_2}\in E\setminus\{0\}$ be the critical point such that

$$\phi_{\mu_2,\lambda_2}(z_{\mu_2,\lambda_2})=c_{\mu_2,\lambda_2}.$$

A standard argument implies that

$$c_{\mu_2,\lambda_2}=\max_{t>0}\phi_{\mu_2,\lambda_2}(tz_{\mu_2,\lambda_2}).$$

Observe that if $\mu_2 \geq \mu_1$ and $\lambda_2 \leq \lambda_1$, then

$$\phi_{\mu_2,\lambda_2}(z) \geq \phi_{\mu_1,\lambda_1}(z), \quad \text{for all } z \in E \setminus \{0\}.$$

Thus

$$c_{\mu_2,\lambda_2} = \max_{t>0} \phi_{\mu_2,\lambda_2}(tz_{\mu_2,\lambda_2}) \geq \max_{t>0} \phi_{\mu_1,\lambda_1}(tz_{\mu_2,\lambda_2}) \geq \inf_{z\in E\setminus\{0\}} \max_{t>0} \phi_{\mu_1,\lambda_1}(tz) = c_{\mu_1,\lambda_1}.$$

Particularly, if $\max\{\mu_2 - \mu_1, \lambda_1 - \lambda_2\} > 0$, then $c_{\mu_2,\lambda_2} > c_{\mu_1,\lambda_1}$ follows from the following equality:

$$\phi_{\mu_2,\lambda_2}(z) = \phi_{\mu_1,\lambda_1}(z) + \frac{\mu_2 - \mu_1}{2} |z|_2^2 + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^N} F(|z|)\, dx. \qquad \square$$

By assumption (VP), we have

$$V_\epsilon(x) = V(\epsilon x) \to V(0) \quad \text{and} \quad P_\epsilon(x) = P(\epsilon x) \to P(0) \quad \text{in } L^2_{loc}(\mathbb{R}^N) \tag{3-1}$$

as $\epsilon \to 0$. Let be $\mu_0 = V(0)$ and $\lambda_0 = P(0)$. The following lemma is the key to the concentration behavior:

Lemma 3.2.
$$\limsup_{\epsilon\to 0} c_\epsilon \leq c_{\mu_0,\lambda_0}.$$

Proof. It is equivalent to prove that

$$c_\epsilon \leq c_{\mu_0,\lambda_0} + o_\epsilon(1).$$

Let z_0 be the critical point such that $\phi_{\mu_0,\lambda_0}(z_0) = c_{\mu_0,\lambda_0}$. A standard argument implies that there exists $t_0 > 0$, such that $\phi_{\mu_0,\lambda_0}(t_0 z_0) < -1$. For any $t \in [0, t_0]$, let

$$f_0(t) = \phi_{\mu_0,\lambda_0}(tz_0) \quad \text{and} \quad f_\epsilon(t) = \phi_\epsilon(tz_0).$$

Claim: For each $z \in E$ fixed,

$$\phi_\epsilon(z) \to \phi_{\mu_0,\lambda_0}(z). \tag{3-2}$$

First observe that, in view of the Sobolev embedding theorems, $z \in E$ implies that $z \in L^p(\mathbb{R}^N)$, $p \in [2, 2^*]$. Thus, for any $\eta > 0$, we have for large ρ

$$|z|^2_{2,\mathbb{R}^N\setminus B_\rho(0)} < \eta \quad \text{and} \quad |z|^p_{p,\mathbb{R}^N\setminus B_\rho(0)} < \eta. \tag{3-3}$$

Furthermore, considering (3-1), we can assert that for any $x \in B_\rho(0)$, the relations

$$|V_\epsilon(x) - \mu_0| < \eta \quad \text{and} \quad |P_\epsilon(x) - \lambda_0| < \eta \tag{3-4}$$

hold for small ϵ.

Hence, by using (3-3) and (3-4), we deduce, for small ϵ,

$$
\begin{aligned}
|\phi_\epsilon(z) - \phi_{\mu_0,\lambda_0}(z)| \\
&= \left| \frac{1}{2} \int_{\mathbb{R}^N} [V_\epsilon(x) - \mu_0]|z|^2\,dx - \int_{\mathbb{R}^N} [P_\epsilon(x) - \lambda_0] F(|z|)\,dx \right| \\
&\le \frac{1}{2} \int_{\mathbb{R}^N} |V_\epsilon(x) - \mu_0| \cdot |z|^2\,dx + \int_{\mathbb{R}^N} |P_\epsilon(x) - \lambda_0| F(|z|)\,dx \\
&= \frac{1}{2} \int_{B_\rho(0)} |V_\epsilon(x) - \mu_0| \cdot |z|^2\,dx + \frac{1}{2} \int_{\mathbb{R}^N \setminus B_\rho(0)} |V_\epsilon(x) - \mu_0| \cdot |z|^2\,dx \\
&\qquad + \int_{B_\rho(0)} |P_\epsilon(x) - \lambda_0| F(|z|)\,dx + \int_{\mathbb{R}^N \setminus B_\rho(0)} |P_\epsilon(x) - \lambda_0| F(|z|)\,dx \\
&\le c \cdot \eta,
\end{aligned}
$$

proving (3-2).

Now together with $\phi_{\mu_0,\lambda_0}(t_0 z_0) < -1$, we have for small ϵ, $\phi_\epsilon(t_0 z_0) < -\frac{1}{2}$. This implies that f_ϵ admits a maximum in $(0, t_0)$. Observe that $\{f_\epsilon\}_{\epsilon>0}$ and $\{f'_\epsilon\}_{\epsilon>0}$ are both uniformly bounded.

By a simple application of Arzela–Ascoli theorem, we have that $\{f_\epsilon\}_{\epsilon>0} \subseteq C([0, t_0])$ is compact. Our claim implies that f_ϵ converges pointwise to f_0. The compactness of $\{f_\epsilon\}_{\epsilon>0}$ tells us $|f_\epsilon - f_0|_\infty \to 0$ as $\epsilon \to 0$. Together with $t_0 > 1$ and $\phi_\epsilon(t_0 z_0) < -\frac{1}{2}$, we deduce that

$$
\begin{aligned}
c_\epsilon = \inf_{z \in E\setminus\{0\}} \max_{t>0} \phi_\epsilon(tz) &\le \max_{t\in[0,t_0]} \phi_\epsilon(t z_0) \\
&= \max_{t\in[0,t_0]} f_\epsilon(t) \\
&\le \max_{t\in[0,t_0]} f_0(t) + o_\epsilon(1) \\
&= \max_{t\in[0,t_0]} \phi_{\mu_0,\lambda_0}(t z_0) + o_\epsilon(1) = c_{\mu_0,\lambda_0} + o_\epsilon(1). \qquad \square
\end{aligned}
$$

The following lemma is our main result, which shows that (2-1) has (at least) one ground state solution for all small ϵ.

Lemma 3.3. *Under the assumptions of Theorem 1.1, if $\epsilon > 0$ is small enough, then c_ϵ is attained.*

Proof. Ekeland's variational principle implies that there exists

$$\{z_n : n = 1, 2, \ldots\} \subseteq \mathcal{N}_\epsilon,$$

such that

$$\phi_\epsilon(z_n) \to c_\epsilon \quad \text{as } n \to \infty, \quad \text{and} \quad \phi'_\epsilon(z_n) \to 0 \quad \text{as } n \to \infty. \tag{3-5}$$

We also know that $\{z_n : n = 1, 2, \ldots\}$ is bounded. Thus $z_n \rightharpoonup z_\epsilon$ for some $z_\epsilon \in E$.

Notice that if $z_\epsilon \neq 0$, we are done. Indeed, since $\phi'_\epsilon(z_n) \to 0$ as $n \to \infty$, for any $v \in E$, we have

$$\langle z_n, v\rangle_\epsilon - \int_{\mathbb{R}^N} P_\epsilon(x) f(|z_n|) z_n v \, dx = \phi'_\epsilon(z_n)v \to 0$$

as $n \to \infty$. By using of $z_n \rightharpoonup z_\epsilon$ and the Sobolev embedding theorem, we have that

$$\begin{aligned}\phi'_\epsilon(z_\epsilon)v &= \langle z_\epsilon, v\rangle_\epsilon - \int_{\mathbb{R}^N} P_\epsilon(x) f(|z_\epsilon|) z_\epsilon v \, dx \\ &= \lim_{n\to\infty}\left[\langle z_n, v\rangle_\epsilon - \int_{\mathbb{R}^N} P_\epsilon(x) f(|z_n|) z_n v \, dx\right] = 0\end{aligned}$$

holds for any $v \in E$, which implies that $z_\epsilon \in \mathcal{N}_\epsilon$. Furthermore, in view of Fatou's lemma, (3-5) implies

$$\begin{aligned}\phi_\epsilon(z_\epsilon) \geq \inf_{z\in\mathcal{N}_\epsilon} \phi_\epsilon(z) = c_\epsilon &= \phi_\epsilon(z_n) + o_n(1) \\ &= \phi_\epsilon(z_n) - \tfrac{1}{2}\phi'_\epsilon(z_n)z_n + o_n(1) \\ &= \int_{\mathbb{R}^N} P_\epsilon(x) \hat{F}(|z_n|) \, dx + o_n(1) \\ &\geq \int_{\mathbb{R}^N} P_\epsilon(x) \hat{F}(|z_\epsilon|) \, dx = \phi_\epsilon(z_\epsilon).\end{aligned}$$

Thus $c_\epsilon = \phi_\epsilon(z_\epsilon)$.

Now assume for contradiction that $z_\epsilon = 0$ for all small ϵ. Here we assume that V leads the behavior. Similarly, we can deal with the case where P leads the behavior.

Take $\kappa \in (\tau, \tau_\infty)$ and let be $\eta := \gamma_v$. Denote

$$\begin{aligned}V^\kappa(x) &= \max\{\kappa, V(x)\}, \quad V^\kappa_\epsilon(x) = V^\kappa(\epsilon x). \\ P^\eta(x) &= \min\{\eta, P(x)\}, \quad P^\eta_\epsilon(x) = P^\eta(\epsilon x).\end{aligned}$$

Let

$$\phi^{\kappa,\eta}_\epsilon(z) = \frac{1}{2}\int_{\mathbb{R}^N} [|\Delta z|^2 + V^\kappa_\epsilon(x)|z|^2] \, dx - \int_{\mathbb{R}^N} P^\eta_\epsilon(x) F(|z|) \, dx,$$

and $c^{\kappa,\eta}_\epsilon = \inf_{z\in\mathcal{N}^{\kappa,\eta}_\epsilon} \phi^{\kappa,\eta}_\epsilon(z)$, where $\mathcal{N}^{\kappa,\eta}_\epsilon$ is the Nehari manifold corresponding to $\phi^{\kappa,\eta}_\epsilon$. Then

$$\phi_\epsilon(z) = \phi^{\kappa,\eta}_\epsilon(z) + \frac{1}{2}\int_{\mathbb{R}^N} [V_\epsilon(x) - V^\kappa_\epsilon(x)]|z|^2 \, dx - \int_{\mathbb{R}^N} [P_\epsilon(x) - P^\eta_\epsilon(x)] F(|z|) \, dx.$$

Denote

$$\begin{aligned}\mathcal{O} &= \{x \in \mathbb{R}^N : V(x) \leq \kappa\}, \quad \mathcal{O}_\epsilon = \{x \in \mathbb{R}^N : \epsilon x \in \mathcal{O}\}. \\ \mathcal{O}' &= \{x \in \mathbb{R}^N : P(x) > \eta\}, \quad \mathcal{O}'_\epsilon = \{x \in \mathbb{R}^N : \epsilon x \in \mathcal{O}'\}.\end{aligned}$$

Notice that $\mathcal{O}_\epsilon$ and $\mathcal{O}'_\epsilon$ are both bounded for given ϵ, and for any z_n, there exists a positive real number t_n, such that $t_n z_n \in \mathcal{N}_\epsilon^{\kappa,\eta}$. Since $\{z_n : n = 1, 2, \ldots\}$ is bounded and the distance between 0 and $\mathcal{N}_\epsilon^{\kappa,\eta}$ is strictly positive, we know that the sequence of positive numbers $\{t_n : n = 1, 2, \ldots\}$ is also bounded. Without loss of generality, assume that $t_n \le D$ for some constant D, $n = 1, 2, \ldots$.

By the Sobolev embedding theorem, we have a compact embedding $E \hookrightarrow L^q_{loc}(\mathbb{R}^N)$, where $q \in [2, 2^*)$, thus $z_n \rightharpoonup z_\epsilon = 0$ implies that $z_n \to 0$ in $L^q_{loc}(\mathbb{R}^N)$. Since V and P are both L^∞-functions, we have by Hölder's inequalities

$$\left|\frac{1}{2}\int_{\mathcal{O}_\epsilon}[V_\epsilon(x) - V_\epsilon^\kappa(x)]|t_n z_n|^2\,dx\right| \le D^2|V|_\infty \int_{\mathcal{O}_\epsilon}|z_n|^2\,dx \to 0,$$

as $n \to \infty$. Thus

$$\frac{1}{2}\int_{\mathcal{O}_\epsilon}[V_\epsilon(x) - V_\epsilon^\kappa(x)]|t_n z_n|^2\,dx = o_n(1).$$

Similarly,

$$\left|\int_{\mathcal{O}'_\epsilon}[P_\epsilon(x) - P_\epsilon^\eta(x)]F(|t_n z_n|)\,dx\right| \to 0 \quad \text{as } n \to \infty.$$

Now taking all the above into one package, together with $\phi_\epsilon(t_n z_n) \le \phi_\epsilon(z_n)$ and (3-5), we have

$$\begin{aligned}
c_\epsilon^{\kappa,\eta} &= \inf_{z\in\mathcal{N}_\epsilon^{\kappa,\eta}} \phi_\epsilon^{\kappa,\eta}(z) \le \phi_\epsilon^{\kappa,\eta}(t_n z_n)\\
&= \phi_\epsilon(t_n z_n) - \frac{1}{2}\int_{\mathbb{R}^N}[V_\epsilon(x) - V_\epsilon^\kappa(x)]|t_n z_n|^2\,dx\\
&\qquad + \int_{\mathbb{R}^N}[P_\epsilon(x) - P_\epsilon^\eta(x)]F(|t_n z_n|)\,dx\\
&\le \phi_\epsilon(z_n) - \frac{1}{2}\int_{\mathcal{O}_\epsilon}[V_\epsilon(x) - V_\epsilon^\kappa(x)]|t_n z_n|^2\,dx\\
&\qquad + \int_{\mathcal{O}'_\epsilon}[P_\epsilon(x) - P_\epsilon^\eta(x)]F(|t_n z_n|)\,dx\\
&= c_\epsilon + o_n(1).
\end{aligned}$$

Thus $c_\epsilon^{\kappa,\eta} \le c_\epsilon$.

Observe that

$$\phi_\epsilon^{\kappa,\eta}(z) \ge \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta z|^2 + \kappa|z|^2)\,dx - \eta\int_{\mathbb{R}^N}F(|z|)\,dx = \phi_{\kappa,\eta}(z).$$

From this, we deduce that

$$c_{\kappa,\eta} = \inf_{z\in E\setminus\{0\}}\max_{t>0}\phi_{\kappa,\eta}(tz) \le \inf_{z\in E\setminus\{0\}}\max_{t>0}\phi_\epsilon^{\kappa,\eta}(tz) = c_\epsilon^{\kappa,\eta}.$$

This holds for any small ϵ, which implies that

$$c_{\kappa,\eta} \le \liminf_{\epsilon\to 0} c_\epsilon^{\kappa,\eta} \le \limsup_{\epsilon\to 0} c_\epsilon^{\kappa,\eta} \le c_{\kappa,\eta},$$

the last inequality followed by our key lemma (Lemma 3.2) since

$$V^\kappa(0) = \max\{\kappa, V(0)\} = \max\{\kappa, \tau\} = \kappa$$

and

$$P^\eta(0) = \min\{\eta, P(0)\} = \min\{\eta, \gamma_v\} = \eta.$$

Thus $c_\epsilon^{\kappa,\eta} \to c_{\kappa,\eta}$ as $\epsilon \to 0$. We have already seen that $c_\epsilon^{\kappa,\eta} \le c_\epsilon$. Letting $\epsilon \to 0$, we have

$$c_{\kappa,\eta} \le \liminf_{\epsilon\to 0} c_\epsilon \le \limsup_{\epsilon\to 0} c_\epsilon \le c_{\tau,\eta},$$

the last inequality is also followed by Lemma 3.2. The above inequality contradicts our Lemma 3.1 since $\kappa > \tau$. Thus $z_\epsilon \ne 0$, which is a ground state solution. □

Proof of Theorem 1.1. This theorem is another description of Lemma 3.3; that is, (2-1) has (at least) one ground state solution for small ϵ. □

4. Proof of Theorem 1.2

Now, using the same notation as in the previous section, we are now ready to show the concentration of the ground state solution given in Theorem 1.1.

Here we recall the description of Theorem 1.2.

Proposition 4.1. *Let* (VP), (f_1) *and* (f_2) *hold. Assume additionally that either* (V) *or* (P) *holds. Let* z_ϵ *be the solution of* (2-1), *given by Lemma 3.3. For any* $\epsilon_{\epsilon_n} > 0$ *with* $\lim_{n\to\infty} \epsilon_n = 0$, *up to a subsequence, there exists* $x_{\epsilon_n} \in \mathbb{R}^N$ *which is a maximum point of* $|z_{\epsilon_n}|$, *such that*

$$\operatorname{dist}(\epsilon_n x_{\epsilon_n}, \mathscr{C}) \to 0 \quad \text{as} \quad n \to \infty.$$

Let $w_{\epsilon_n}(x) := z_{\epsilon_n}(x + x_{\epsilon_n})$. *Then* $w_{\epsilon_n} \to w_0$ *in* E, *where* w_0 *is a ground state solution of*

$$\Delta^2 z + V(x_0)z = P(x_0) f(|z|)z, \qquad x_0 \in \mathscr{C}.$$

Proof. Without loss of generality, we assume (V) holds. Clearly, $\{z_\epsilon\} \subseteq \mathcal{N}_\epsilon$ is bounded.

Claim 1: $\{z_\epsilon\}$ is nonvanishing.

Suppose for contradiction $\{z_\epsilon\}$ is vanishing. This means $z_\epsilon \to 0$ as $\epsilon \to 0$ in $L^p(\mathbb{R}^N)$. Then

$$\|z_\epsilon\|_\epsilon^2 = \int_{\mathbb{R}^N} P_\epsilon(x) f(|z_\epsilon|) z_\epsilon^2 \, dx \to 0, \quad \text{as } \epsilon \to 0,$$

which contradicts the fact that 0 is bounded away from $\mathcal{N}_\epsilon$. Hence, the sequence $\{z_\epsilon\}$ is nonvanishing. Thus we know there exist $R > 0,\ \alpha > 0$ and $\{x_\epsilon\} \subseteq \mathbb{R}^N$ such that

$$\int_{B_R(x_\epsilon)} |z_\epsilon|^2\, dx \geq \alpha.$$

Claim 2: Let $\{x_\epsilon\}$ be the nonvanishing points found in Claim 1. Then $\{\epsilon x_\epsilon\}$ is bounded.

Suppose for contradiction that $|\epsilon x_\epsilon| \to \infty$. Up to a subsequence, we have

$$V_\epsilon(x_\epsilon) \to V_\infty \quad \text{and} \quad P_\epsilon(x_\epsilon) \to P_\infty,$$

for some positive numbers V_∞ and P_∞. Let

$$w_\epsilon(x) := z_\epsilon(x + x_\epsilon), \quad \tilde{V}_\epsilon(x) := V_\epsilon(x + x_\epsilon) \quad \text{and} \quad \tilde{P}_\epsilon(x) := P_\epsilon(x + x_\epsilon).$$

Notice that the boundedness of z_ϵ implies that of $\{w_\epsilon\}$, thus, up to a subsequence, we have $w_\epsilon \rightharpoonup w_0 \in E$. Particularly, $w_0 \neq 0$. Indeed, by Sobolev embedding theorems, $w_\epsilon \rightharpoonup w_0$ in E implies $w_\epsilon \to w_0$ in $L^2_{loc}(\mathbb{R}^N)$. Together with

$$\int_{B_R(0)} |w_\epsilon|^2\, dx = \int_{B_R(x_\epsilon)} |z_\epsilon|^2\, dx \geq \alpha,$$

we have $\int_{B_R(0)} |w_0|^2\, dx \geq \alpha$, i.e., $w_0 \neq 0$.

Observe that w_ϵ is a ground state solution to the following equation:

$$\Delta^2 w + \tilde{V}_\epsilon(x) w = \tilde{P}_\epsilon(x) f(|w|) w.$$

Now for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, by using the dominated convergence theorem, we have

$$\begin{aligned} 0 &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} [\Delta^2 w_\epsilon + \tilde{V}_\epsilon(x) w_\epsilon - \tilde{P}_\epsilon(x) f(|w_\epsilon|) w_\epsilon] \varphi\, dx \\ &= \int_{\mathbb{R}^N} (\Delta^2 w_0 + V_\infty w_0 - P_\infty f(|w_0|) w_0) \varphi\, dx \\ &= \phi'_{V_\infty, P_\infty}(w_0)\varphi. \end{aligned}$$

It follows from this that

$$\begin{aligned} c_\epsilon = \phi_\epsilon(z_\epsilon) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\Delta z_\epsilon|^2 + V_\epsilon(x)|z_\epsilon|^2]\, dx - \int_{\mathbb{R}^N} P_\epsilon(x) F(|z_\epsilon|)\, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} [|\Delta w_\epsilon|^2 + \tilde{V}_\epsilon(x)|w_\epsilon|^2]\, dx - \int_{\mathbb{R}^N} \tilde{P}_\epsilon(x) F(|w_\epsilon|)\, dx \\ &= \int_{\mathbb{R}^N} \tilde{P}_\epsilon(x) \hat{F}(|w_\epsilon|)\, dx \\ &\geq \int_{\mathbb{R}^N} P_\infty \hat{F}(|w_0|)\, dx = \phi_{V_\infty, P_\infty}(w_0) \geq c_{V_\infty, P_\infty}. \end{aligned}$$

Together with $c_\epsilon \le c_{\tau,\eta} + o_\epsilon(1)$ and $\max\{V_\infty - \tau, \eta - P_\infty\} > 0$, we have a contradiction with Lemma 3.1. Thus $\{\epsilon x_\epsilon\}$ is bounded.

Claim 3: $\operatorname{dist}(\epsilon x_\epsilon, \mathscr{A}_v) \to 0$.

Suppose for contradiction, up to a subsequence, $\epsilon x_\epsilon \to x_0 \notin \mathscr{A}_v$ as $\epsilon \to 0$ since $\{\epsilon x_\epsilon\}$ is bounded.

Denote

$$V_\epsilon(x_\epsilon) \to V(x_0) \quad \text{and} \quad P_\epsilon(x_\epsilon) \to P(x_0),$$

as $\epsilon \to 0$. Then similarly to the proof of Claim 2, we have $c_\epsilon \ge c_{V(x_0),P(x_0)}$. Recall that

$$\mathscr{A}_v := \{x \in \mathcal{V} : P(x) = \gamma_v\} \cup \{x \notin \mathcal{V} : P(x) > \gamma_v\}.$$

It is easy to check that $x_0 \notin \mathscr{A}_v$ implies that $\max\{V(x_0) - \tau, \eta - P(x_0)\} > 0$. Together with $c_\epsilon \le c_{\tau,\eta} + o_\epsilon(1)$, we have a contradiction with Lemma 3.1. Thus $\operatorname{dist}(\epsilon x_\epsilon, \mathscr{A}_v) \to 0$. At this point, as was argued in Claim 2, the transformed solution $w_\epsilon(x) := z_\epsilon(x + x_\epsilon)$ will converge weakly (up to a subsequence) to w_0 which is a ground state solution of

$$\Delta^2 z + V(x_0) z = P(x_0) f(|z|) z, \qquad x_0 \in \mathscr{A}_v.$$

Claim 4: For any $\epsilon_n > 0$ with $\lim_{n\to\infty} \epsilon_n = 0$, up to a subsequence, we have $w_{\epsilon_n} \to w_0$ as $n \to \infty$ in $E \setminus \{0\}$.

Let $y_n := w_{\epsilon_n} - w_0$, then $y_n \rightharpoonup 0$ as $n \to \infty$. Recall that we have

$$\begin{gathered}\Delta^2 w_{\epsilon_n} + \tilde{V}_{\epsilon_n}(x) w_{\epsilon_n} = \tilde{P}_{\epsilon_n}(x) f(|w_{\epsilon_n}|) w_{\epsilon_n},\\ \Delta^2 w_0 + V(x_0) w_0 = P(x_0) f(|w_0|) w_0\end{gathered}$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^N} [\Delta w_{\epsilon_n} \Delta y_n + \tilde{V}_{\epsilon_n}(x) w_{\epsilon_n} y_n]\, dx &= \int_{\mathbb{R}^N} \tilde{P}_{\epsilon_n}(x) f(|w_{\epsilon_n}|) w_{\epsilon_n} y_n \, dx,\\ \int_{\mathbb{R}^N} [\Delta w_0 \Delta y_n + V(x_0) w_0 y_n]\, dx &= \int_{\mathbb{R}^N} P(x_0) f(|w_0|) w_0 y_n \, dx\end{aligned}$$

Notice that

$$\begin{aligned}\langle V(x_0) w_0, y_n\rangle_2 &= \langle w_0, V(x_0) y_n\rangle_2\\ &= \langle w_0, \tilde{V}_{\epsilon_n}(x) y_n + [V(x_0) - \tilde{V}_{\epsilon_n}(x)] y_n\rangle_2\\ &= \langle w_0, \tilde{V}_{\epsilon_n}(x) y_n\rangle_2 + \langle w_0, [V(x_0) - \tilde{V}_{\epsilon_n}(x)] y_n\rangle_2\\ &= \langle \tilde{V}_{\epsilon_n}(x) w_0, y_n\rangle_2 + o_{\epsilon_n}(1).\end{aligned}$$

Now we have

$$\int_{\mathbb{R}^N} [\Delta w_{\epsilon_n} \Delta y_n + \tilde{V}_{\epsilon_n}(x) w_{\epsilon_n} y_n]\, dx = \int_{\mathbb{R}^N} \tilde{P}_{\epsilon_n}(x) f(|w_{\epsilon_n}|) w_{\epsilon_n} y_n \, dx,$$
$$\int_{\mathbb{R}^N} [\Delta w_0 \Delta y_n + \tilde{V}_{\epsilon_n}(x) w_0 y_n]\, dx = \int_{\mathbb{R}^N} P(x_0) f(|w_0|) w_0 y_n \, dx + o_{\epsilon_n}(1)$$

Then, by (f_1), (f_2), it is easy to check

$$\int_{\mathbb{R}^N} [\Delta y_n \Delta y_n + \tilde{V}_{\epsilon_n}(x) y_n^2]\, dx = o_{\epsilon_n}(1),$$

which implies $\|y_n\|_{\epsilon_n} = o_{\epsilon_n}(1)$, ending the proof. □

Final remark

Our approach can be described in a more abstract way to deal with some general variational problems. Indeed, if we rewrite (2-1) as

$$\begin{cases} (\Delta^2 + \alpha) z + (V_\epsilon(x) - \alpha) z = P_\epsilon(x) f(|z|) z, & x \in \mathbb{R}^N, \\ z(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $0 < \alpha < \inf_{x \in \mathbb{R}^N} V(x)$, then we are led to the abstract equation of the form

$$Lz + M_\epsilon(x) z = P_\epsilon(x) \nabla G(z), \tag{4-1}$$

in which L is a positive defined differential operator on $L^2(\mathbb{R}^N)$ and $M(x)$, $P(x)$ satisfy the condition (VP). Let $E := \mathcal{D}(L^{1/2})$ be equipped with the scalar product

$$\langle u, v \rangle = \langle L^{1/2} u, L^{1/2} v \rangle_2$$

and the induced norm $\|u\| = \langle u, u \rangle^{1/2}$. Then the associated energy functional of (4-1) is of the form

$$\phi_\epsilon(z) = \frac{1}{2} \|z\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} M_\epsilon(x) |z|^2 \, dx - \int_{\mathbb{R}^N} P_\epsilon(x) G(z) \, dx,$$

and our arguments are in general feasible to such problems under some suitable assumptions on the nonlinear function G.

In fact, (4-1) is related to several equations appearing in quantum physics, including the Schrödinger equations and the fractional Schrödinger equations, etc. Therefore, our approach covers the semiclassical behavior of different equations under a general class of subcritical nonlinearities.

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DONG WANG
SCHOOL OF MATHEMATICS AND PHYSICS
CHANGZHOU UNIVERSITY
213164 JIANGSU
CHINA
wdflight@126.com

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