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A PALEY–WIENER THEOREM FOR THE SPECTRAL PROJECTION OF SYMMETRIC GRAPHS

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We prove a Paley–Wiener theorem for the spectral projection of symmetric graphs and, as a corollary, derive a Paley–Wiener theorem for the Helgason–Fourier transform. The proof is based on contour integration arguments similar to those used to prove the Paley–Wiener theorem for Euclidean spaces and symmetric spaces.

1. Introduction

The theory of representations of free groups has been studied by many authors in analogy with the semisimple theory. This arises from the realization of a free group as a homogeneous tree and relies upon the use of the Poisson boundary and spherical function. Mantero and Zappa [1983] characterized the image of the Poisson transform of free groups and studied the uniform boundedness of the spherical representation. In [Cowling et al. 1998], Cowling, Meda and Setti studied the images of the Abel transform for various function spaces on homogeneous trees. Cowling and Setti [1999] gave the characterizations of the images of the spaces of compactly supported functions and rapidly decreasing functions.

The concept of tree has been extended in several aspects. For instance, Iozzi and Picardello [1983a; 1983b] extended the context of tree to symmetric graphs and gave an explicit expression of the spherical function. Later, the Plancherel measure on symmetric graphs was explicitly computed in [Kuhn and Soardi 1983; Faraut and Picardello 1984]. Recently Eddine [2013; 2015] investigated the characterization of the Abel transform for symmetric graphs and, as an application, solved the shifted wave equations on it.

In [Koizumi 2013], we studied the spectral projection on homogeneous trees and proved the Paley–Wiener theorem of the spectral projection, which is an analogue of that given by Bray [1996]. In this paper, we shall extend the works in [Koizumi 2013] to the case of symmetric graphs. Unlike the works of Cowling and Setti

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[1999], our proof is based on contour integration arguments, which are usually used to prove the Paley–Wiener theorem for the cases of the Euclidean spaces and the symmetric spaces [Johnson 1979; Campoli 1980].

A brief outline of this paper is as follows: Section 2 is devoted to the overview of the notation of symmetric graphs. In Section 3, we concretely write down the expressions of the Poisson transform on symmetric graphs. In Section 4, we construct the intertwining operators between the spherical representations and give the explicit expressions of the intertwining operators. In Section 5, we study the properties of the spectral projection for symmetric graphs. Finally in Section 6, we show the Paley–Wiener theorem of the spectral projection and prove the Paley–Wiener theorem of the Helgason–Fourier transform.

2. Notation and preliminaries

The standard symbols $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are used for the integers, the real numbers and the complex numbers, respectively. Let us set $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. Throughout this paper, the imaginary unit is denoted as i. If $x \in \mathbb{C}$, $\Re x$ and $\Im x$ denote its real part and its imaginary part, respectively.

A graph $\mathfrak{X}$ is symmetric of type $k \geq 2$ and order $r \geq 2$ if every vertex v belongs exactly to r polygons with k sides each with no sides and no vertex in common except v, and if every nontrivial loop in $\mathfrak{X}$ runs through all edges of at least one polygon. If $k = 2$, $\mathfrak{X}$ reduces to a homogeneous tree of degree r. In what follows, we write $q = (k-1)(r-1)$, $\tau = 2\pi/\log q$ and $\mathbb{T} = \mathbb{R}/\tau\mathbb{Z}$. Different notions of length on a symmetric graph were introduced in [Iozzi and Picardello 1983a]. Here we use the definition of the length $d(x, y)$ between two vertices $x, y \in \mathfrak{X}$ to denote the minimal number of polygons crossed by a path connecting x and y. We fix a reference point o in $\mathfrak{X}$ and write $|x| = d(x, o)$.

By the same arguments as in [Betori and Pagliacci 1984, Theorem 1], if $k > 2$, it is easy to see that every group acting simply transitively on $\mathfrak{X}$ and isometrically with respect to the metric induced by this length is isometric to the free group $G = \oplus_{i=1}^{r} \mathbb{Z}_k$, while, for $k = 2$, G is isometric to the free product of t copies of $\mathbb{Z}$ and s copies of $\mathbb{Z}_2$, where $2t + s = r$. Hence every vertex of $\mathfrak{X}$ is identified with an element of G and, under this identification, every polygon corresponds to an orbit under right translations by one of the factors $\mathbb{Z}_k$. For $x \in \mathfrak{X}$ and $n \leq |x|$, we write $x^{(n)}$ for the word of length n consisting of the first n blocks of x and simply write x' for $x^{(|x|-1)}$.

Let $\mathfrak{S}_n$ be the set of words of length n in $\mathfrak{X}$. We write Ω for the Poisson boundary of $\mathfrak{X}$. For $\omega \in \Omega$ and $n \in \mathbb{Z}_{\geq 0}$, we denote by ω_n the word of length n consisting the first n blocks. Let $E(x)$ denote the subset of Ω of words that begin with the reduced word $x \in \mathfrak{X}$. We write $\mathcal{M}$ and $\mathcal{M}_n$ for the σ-algebra generated by $\{E(x) : x \in \mathfrak{X}\}$ and σ-subalgebra generated by $\{E(x) : |x| \leqq n\}$ respectively. Then $\mathcal{M}$ makes

Ω into a compact topological space and there exists a natural G-quasi-invariant probability measure ν on $(\Omega, \mathcal{M})$. We write $\mathcal{F}(\Omega)$ for the space comprised of the $\mathcal{M}_n$-measurable functions on Ω. We denote by $\mathcal{F}(\Omega)_c$ the linear span of the characteristic functions of $E(x)$ for $x \in \mathfrak{X}$. The dual space $\mathcal{F}'(\Omega)$ is identified with the space of the martingales on Ω with respect to $\{\mathcal{M}_n\}$.

We write $s_0 = \big(\frac{1}{2} - \log_q(k-1)\big)i + \frac{1}{2}\tau$ and set

$$\Upsilon = \big\{s \in \mathbb{C} : s = \tfrac{1}{2}i + h\tau,\ s = s_0 + h\tau\ (h \in \mathbb{Z})\big\}.$$

We define the subsets b_x, c_x, d_x of $\mathfrak{X}$ by the following: for $x \in \mathfrak{X} \setminus \{o\}$

$$b_x = \{y \in \mathfrak{X} : d(y, x) = 1, |y| = |x|\},$$
$$c_x = \{y \in \mathfrak{X} : d(y, x) = 2, |y| = |x|\},$$
$$d_x = \{y \in \mathfrak{X} : d(y, x) \geq 3, |y| = |x|\},$$

and $b_o = c_o = d_o = \varnothing$. The subsets $B(x)$ and $C(x)$ of Ω are defined by

$$B(x) = \bigcup_{y \in b_x} E(y), \qquad C(x) = \bigcup_{y \in c_x} E(y).$$

For a function η on Ω and $n \in \mathbb{Z}_{\geq 0}$, we define the averages $E_n\eta$ and $B_n\eta$ as follows:

$$E_n\eta(\omega) = \frac{1}{\nu(E(\omega_n))}\int_{E(\omega_n)} \eta(\omega')\,d\nu(\omega'), \quad B_n\eta(\omega) = \frac{1}{\nu(B(\omega_n))}\int_{B(\omega_n)} \eta(\omega')\,d\nu(\omega').$$

Then, as shown in [Mantero and Zappa 1983, p. 375], the set $\{E_n\eta\}$ is a martingale associated to $\eta \in L^1(\Omega)$ and the n-th martingale difference of η is given by $D_n\eta = E_n\eta - E_{n-1}\eta$. Here we set $E_{-1} = 0$. For $x \in \mathfrak{X}$ and $\omega \in \Omega$, the Poisson kernel $p(x, \omega)$ is defined to be the Radon–Nikodym derivative $d\nu(x^{-1}\omega)/d\nu(\omega)$ and is computed as

$$p(x, \omega) = q^{\zeta(x,\omega)},$$

where $\zeta(x, \omega) = \lim_{m\to\infty}(m - d(x, \omega_m))$ is the Busemann function. As shown in [Iozzi and Picardello 1983b, Proposition 2], for $x \in \mathfrak{S}_n$, we have

$$p(x, \omega) = q^n \chi_{E(x)}(\omega) + \sum_{j=1}^{n} q^{2j-n-1}\chi_{B(x^{(j)})}(\omega) + \sum_{j=1}^{n} q^{2j-n-2}\chi_{C(x^{(j)})}(\omega). \tag{2-1}$$

For $\eta \in L^1(\Omega)$ and $s \in \mathbb{T}$, we define the Poisson transform $P^s\eta$ by

$$P^s\eta(x) = \int_{\Omega} p(x, \omega)^{1/2+is}\eta(\omega)\,d\nu(\omega). \tag{2-2}$$

By duality, the Poisson transform is naturally extended to $\mathcal{F}'(\Omega)$ and is denoted by the same symbol P^s.

Following [Mantero and Zappa 1983], we define the operators ε and Δ on $\mathfrak{X}$, which are essentially the analogue of E_n and D_n. We set for $n \in \mathbb{Z}_{\geq 0}$

$$S(n,x) = \begin{cases} \{x\}, & |x| \leq n, \\ \{y \in \mathfrak{X} : |y| = |x|, \, y^{(n)} = x^{(n)}\}, & |x| > n. \end{cases}$$

For a function ϕ on $\mathfrak{X}$ and $n \in \mathbb{Z}_{\geq 0}$, we define its average $\varepsilon_n \phi$ by

$$\varepsilon_n \phi(x) = \frac{1}{\operatorname{Card} S(n,x)} \sum_{y \in S(n,x)} \phi(y). \tag{2-3}$$

We also define $\Delta_n \phi$ by

$$\Delta_n \phi(x) = \varepsilon_n \phi(x) - \varepsilon_{n-1} \phi(x).$$

Here we set $\varepsilon_{-1}\phi = 0$. We write μ_1 for the probability measure equidistributed on words of length 1. We also use the notation κ_1 to denote the following:

$$(\phi * \kappa_1)(x) = \frac{1}{k-2} \sum_{y \in b_x} \phi(y).$$

We write $C_c(\mathfrak{X})$ for the set of all compactly supported functions on $\mathfrak{X}$. For $N \in \mathbb{Z}_{\geq 0}$, we denote by $C_N(\mathfrak{X})$ the subset of $C_c(\mathfrak{X})$ consisting of all $f \in C_c(\mathfrak{X})$ such that $\operatorname{supp} f \subseteq \mathfrak{B}_N$. A function ϕ on $\mathfrak{X}$ is said to be radial if $\varepsilon_0 \phi = \phi$ and cylindrical if $\varepsilon_N \phi(x) = \phi(x)$ for some $N \in \mathbb{Z}_{\geq 0}$. For any function space $E(\mathfrak{X})$, we denote by $E(\mathfrak{X})^{\#}$ and $E(\mathfrak{X})_c$ the subspaces of $E(\mathfrak{X})$ consisting of all radial functions and cylindrical functions, respectively. A function f on $\mathbb{T}$ is said to be Weyl-invariant if $f(s+\tau) = f(s)$ and $f(-s) = f(s)$.

Finally we pointed out that it is meaningful to study harmonic analysis for symmetric graphs using methods similar to that in symmetric spaces. For example, the explicit expressions of the intertwining operators obtained in Section 4 can be used to construct the composition series of the spherical representations and determine which parts of the subquotients are unitarizable. In Section 6, using this information, we can concretely characterize the image of the compactly supported functions under the Helgason–Fourier transform.

3. The Poisson transform on symmetric graphs

Iozzi and Picardello [1983b] studied the Poisson transform for symmetric graphs. They showed in their paper that the Poisson transform P^s is injective on $\mathcal{F}(\Omega)_c$ if and only if $s \notin \Upsilon$. In this section, by carrying out similar arguments to that in [Mantero and Zappa 1983], we show that P^s is also surjective on $\mathcal{F}'(\Omega)$ when $s \notin \Upsilon$.

As shown in [Iozzi and Picardello 1983b, Theorem 1], we have

$$(P^s \eta * \mu_1)(x) = \gamma(s) P^s \eta(x),$$

where

$$\gamma(s) = \frac{1}{r(k-1)}(q^{1/2+is} + q^{1/2-is} + k - 2).$$

By using the equation

$$\chi_{E(x^{(j)})}(\omega) - \chi_{E(x^{(j+1)})}(\omega) = \chi_{C(x^{(j+1)})}(\omega) + \chi_{B(x^{(j+1)})}(\omega),$$

(2-1) is expressed as

$$p(x,\omega) = q^{-|x|}\chi_{E(x^{(0)})}(\omega) + (1-q^{-2})\sum_{j=1}^{|x|} q^{2j-|x|}\chi_{E(x^{(j)})}(\omega) + (1-q^{-1})\sum_{j=1}^{|x|} q^{2j-|x|-1}\chi_{B(x^{(j)})}(\omega). \tag{3-1}$$

Therefore for $\omega \in \Omega$, substituting (3-1) into (2-2), we have

$$P^s\eta(\omega_n) = \sum_{j=0}^{n} b_{j,n}(s)E_j\eta(\omega) + \frac{k-2}{q^{1/2+is}+1}\sum_{j=1}^{n} b_{j,n}(s)B_j\eta(\omega),$$

where $b_{0,n}(s) = q^{-n(1/2+is)}$ and

$$b_{j,n}(s) = \frac{q}{r(k-1)}(1-q^{-1-i2s})q^{-n(1/2+is)+i2js}.$$

By the definitions of B_n and E_n, it is easy to verify that

$$E_m B_n = \begin{cases} B_n, & m \ge n, \\ E_m, & m < n, \end{cases} \qquad B_m E_n = \begin{cases} E_n, & m > n, \\ B_m, & m \le n. \end{cases}$$

And hence we obtain that

$$B_m D_n\eta = \begin{cases} D_n\eta, & m > n, \\ B_n\eta - E_{n-1}\eta, & m = n, \\ 0, & m < n. \end{cases}$$

Hereafter we suppose that $D_M\eta = \eta$ for some $M \ge 0$. We first consider the case when $M > 0$. Since $E_j\eta = 0$ and $B_j\eta = 0$ for $j < M$, we have

$$P^s\eta(\omega_n) = 0 \quad \text{for } n < M,$$

and

$$P^s\eta(\omega_{M+\ell}) = \sum_{j=M}^{M+\ell} b_{j,M+\ell}(s)\eta(\omega) + \frac{k-2}{q^{1/2+is}+1}\sum_{j=M+1}^{M+\ell} b_{j,M+\ell}(s)\eta(\omega) + \frac{k-2}{q^{1/2+is}+1} b_{M,M+\ell}(s)B_M\eta(\omega). \tag{3-2}$$

We define the function $Q_M^0(s)$ on $\mathbb{T}$ by $Q_0^0(s)=1$ and

$$Q_M^0(s)=\frac{\sqrt{q}}{r(k-1)}q^{-M/2}q^{i(M-1)s}(q^{1/2+is}-q^{-1/2-is})$$

for $M>0$. By using this, (3-2) can be written

$$\begin{aligned}P^s\eta(\omega_{M+\ell})&\\ =q^{-\ell(1/2+is)}Q_M^0(s)\Biggl\{\sum_{j=M}^{M+\ell}q^{i2s(j-M)}\eta(\omega)&+\frac{k-2}{q^{1/2+is}+1}\sum_{j=M+1}^{M+\ell}q^{i2s(j-M)}\eta(\omega)\\ &+\frac{k-2}{q^{1/2+is}+1}B_M\eta(\omega)\Biggr\}.\end{aligned}$$

Here we set $Q_0(s)=R_0(s)=1$ and

$$\begin{aligned}Q_M(s)&=\frac{\sqrt{q}}{r(k-1)}q^{-M/2}q^{i(M-1)s}(q^{1/2+is}-(k-1)q^{-1/2-is}+k-2),\\ R_M(s)&=\frac{\sqrt{q}}{r(k-1)}q^{-M/2}q^{i(M-1)s}(1-q^{-1/2-is})\end{aligned}$$

for $M>0$. Then a direct computation yields that

$$\begin{aligned}P^s\eta(\omega_{M+\ell})&=q^{-\ell/2}\psi(\ell+1,s)Q_M(s)\eta(\omega)\\ &\quad+(k-2)R_M(s)q^{-\ell(1/2+is)}(B_M\eta(\omega)-\eta(\omega)),\end{aligned}\tag{3-3}$$

where

$$\psi(n,s)=\frac{\sin(ns\log q)}{\sin(s\log q)}.$$

As pointed out in [Cowling and Setti 1999, p. 242], $D_M\eta=\eta$ if and only if η is constant on $E(x)$ for every $x\in\mathfrak{S}_M$ and the average of η with respect to $E(y)$ for $|y|<M$ is equal to 0. Therefore we can regard η as a function on $\mathfrak{X}$ by setting $\eta(x)=E_{|x|}\eta(\omega)$ for $\omega\in E(x)$. Under this identification, we have that $\eta(x)=0$ when $|x|<M$ and $\eta(x)=\eta(x^{(M)})$ when $|x|\geq M$. Moreover, for $x\in\mathfrak{X}$ such that $|x|>M$ and $y\in b_x$, it holds that $\eta(x)=\eta(y)$ because $|x'|=|y'|\geq M$. We also remark that $B_M\eta(\omega)$ corresponds to $\eta*\kappa_1(x^{(M)})$. For these reasons, (3-3) can be rewritten as follows:

$$\begin{aligned}P^s\eta(x)&=q^{-(|x|-M)/2}\psi(|x|-M+1,s)Q_M(s)\eta(x^{(M)})\\ &\quad+(k-2)R_M(s)q^{-(|x|-M)(1/2+is)}(\eta*\kappa_1(x^{(M)})-\eta(x^{(M)})).\end{aligned}\tag{3-4}$$

In the case $M=0$, η is a constant function on Ω and so $P^s\eta$ is expressed in terms of the spherical function ϕ_s given in [Iozzi and Picardello 1983b, Theorem 2] as:

$$P^s\eta(x)=\phi_s(x)\eta(o).$$

We summarize these results in the following proposition.

Proposition 3.1. *Let $\eta \in L^1(\Omega)$ be such that $D_M\eta = \eta$. Then the Poisson transform $P^s\eta$ has the following forms*:

(1) *If* $M > 0$,

$$P^s\eta(x) = 0, \qquad |x| < M,$$
$$P^s\eta(x) = q^{-(|x|-M)/2}\psi(|x|-M+1, s)Q_M(s)\eta(x^{(M)}) + (k-2)R_M(s)q^{-(|x|-M)(1/2+is)}(\eta * \kappa_1(x^{(M)}) - \eta(x^{(M)})), \quad |x| \geq M.$$

(2) *If* $M = 0$,

$$P^s\eta(x) = \phi_s(x)\eta(o).$$

Let $\mathscr{L}_{\gamma(s)}(\mathfrak{X})$ denote the space consisting of the functions ϕ on $\mathfrak{X}$ satisfying the condition $\phi * \mu_1 = \gamma(s)\phi$. For $M \in \mathbb{Z}_{\geq 0}$, we denote by $\mathscr{L}^M_{\gamma(s)}(\mathfrak{X})$ the subspace of $\mathscr{L}_{\gamma(s)}(\mathfrak{X})$ consisting of the functions ϕ which satisfy the following conditions:

(1) $\Delta_M\phi = \phi$,

(2) for $x \in \mathfrak{X}$ such that $|x| > M$ and $y \in b_x$, $\phi(x) = \phi(y)$.

Then $\mathscr{L}^0_{\gamma(s)}(\mathfrak{X})$ is just the space of radial harmonic functions on $\mathfrak{X}$. We prove the following lemma, which is an analogue of [Mantero and Zappa 1983, Lemma 3.2].

Lemma 3.2. *Let $\phi \in \mathscr{L}^M_{\gamma(s)}(\mathfrak{X})$ and $\omega \in \Omega$. Then we have the following*:

(1) *If* $M > 0$,

$$\phi(\omega_n) = 0 \qquad (n < M),$$
$$\phi(\omega_{M+\ell}) = q^{-\ell/2}\psi(\ell+1, s)\phi(\omega_M) + (k-2)q^{-(\ell+1)/2}\psi(\ell, s) \times (\phi(\omega_M) - \phi * \kappa_1(\omega_M)). \tag{3-5}$$

(2) *If* $M = 0$,

$$\phi(\omega_\ell) = \phi_s(\omega_\ell)\phi(o).$$

Proof. Since for $\omega \in \Omega$ and $n \geq M$,

$$\phi*\mu_1(\omega_n) = \frac{1}{r(k-1)}\biggl(\phi(\omega_{n-1}) + (k-2)\phi*\kappa_1(\omega_n) + \phi(\omega_{n+1}) + \sum_{y\in b_{\omega_{n+1}}}\phi(y) + \sum_{y\in c_{\omega_{n+1}}}\phi(y)\biggr),$$

and

$$\Delta_{n+1}\phi(\omega_{n+1}) = \phi(\omega_{n+1}) - \frac{1}{q}\biggl(\phi(\omega_{n+1}) + \sum_{y\in b_{\omega_{n+1}}}\phi(y) + \sum_{y\in c_{\omega_{n+1}}}\phi(y)\biggr) = 0,$$

we have

$$\phi * \mu_1(\omega_n) = \frac{1}{r(k-1)}\{\phi(\omega_{n-1}) + (k-2)\phi * \kappa_1(\omega_n) + q\phi(\omega_{n+1})\}.$$

Hence the condition $\phi * \mu_1 = \gamma(s)\phi$ implies that

$$(q\beta(s)+k-2)\phi(\omega_n) = \phi(\omega_{n-1}) + (k-2)\phi * \kappa_1(\omega_n) + q\phi(\omega_{n+1}),$$

where $\beta(s) = q^{-1/2+is} + q^{-1/2-is}$. When $n > M$, we have that $\phi * \kappa_1(\omega_n) = \phi(\omega_n)$ and therefore we have the recursion formulae:

$$\phi(\omega_n) = 0, \qquad n < M, \tag{3-6}$$

$$\phi(\omega_{M+1}) = \beta(s)\phi(\omega_M) + (k-2)q^{-1}(\phi(\omega_M) - \phi * \kappa_1(\omega_M)), \tag{3-7}$$

$$\phi(\omega_{M+\ell}) = \beta(s)\phi(\omega_{M+\ell-1}) - q^{-1}\phi(\omega_{M+\ell-2}), \qquad \ell \geq 2. \tag{3-8}$$

In the case $s \notin (\tau/2)\mathbb{Z}$, the difference equation (3-8) has the fundamental solutions $q^{-1/2+is}$ and $q^{-1/2-is}$. So using the initial condition (3-7) to determine the coefficients of the fundamental solutions, we obtain the following expression:

$$\phi(\omega_{M+\ell}) = C_1 q^{\ell(-1/2+is)} + C_2 q^{\ell(-1/2-is)}, \tag{3-9}$$

where

$$C_1 = \frac{q^{is}\phi(\omega_M) + (k-2)q^{-1/2}\{\phi(\omega_M) - \phi * \kappa_1(\omega_M)\}}{q^{is} - q^{-is}},$$

$$C_2 = \frac{-q^{-is}\phi(\omega_M) - (k-2)q^{-1/2}\{\phi(\omega_M) - \phi * \kappa_1(\omega_M)\}}{q^{is} - q^{-is}}.$$

Similarly, when $s = \frac{1}{2}m\tau$ $(m \in \mathbb{Z})$, we also have

$$\phi(\omega_{M+\ell}) = (C_1' + C_2'\ell)(-1)^{m\ell} q^{-\ell/2}, \tag{3-10}$$

where

$$C_1' = \phi(\omega_M), \quad C_2' = \phi(\omega_M) + (k-2)(-1)^m q^{-1/2}(\phi(\omega_M) - \phi * \kappa_1(\omega_M)).$$

Obviously both the expressions (3-9) and (3-10) agree with the equation (3-5). The case $M = 0$ is analogous. This concludes the proof. □

Let $x \in \mathfrak{S}_M$ and $s \notin \Upsilon$. Then we have from (3-4) that

$$P^s\eta(x) = Q_M(s)\eta(x) + (k-2)R_M(s)(\eta * \kappa_1(x) - \eta(x)). \tag{3-11}$$

We put $\phi(x) = P^s\eta(x)$ and write down η in terms of ϕ and $\phi * \kappa_1$. Since

$$\phi * \kappa_1 * \kappa_1(x) = \frac{1}{k-2}\phi(x) + \frac{k-3}{k-2}\phi * \kappa_1(x), \tag{3-12}$$

we have from (3-11) that

$$\phi * \kappa_1(x) = Q_M(s)(\eta * \kappa_1)(x) + R_M(s)(\eta(x) - \eta * \kappa_1(x)). \tag{3-13}$$

Thus, solving the simultaneous equations (3-11) and (3-13), we get the expressions

$$\eta(x)=\frac{(q^{1/2+is}+k-2)\phi(x)-(k-2)\phi*\kappa_1(x)}{q^{1/2+is}Q_M(s)},$$

$$\eta*\kappa_1(x)=\frac{(q^{1/2+is}+1)\phi*\kappa_1(x)-\phi(x)}{q^{1/2+is}Q_M(s)}.$$

The above expressions suggest the following proposition.

Proposition 3.3 (cf., [Mantero and Zappa 1983, Proposition 3.4]). *Let $s\notin\Upsilon$. For $\phi\in\mathscr{L}^M_{\gamma(s)}(\mathfrak{X})$, there exists a function η on Ω such that $D_M\eta=\eta$ and $P^s\eta=\phi$.*

Proof. Suppose that $M>0$. Indeed, define $\eta(\omega)$ by

$$\eta(\omega)=\frac{(q^{1/2+is}+k-2)\phi(\omega_M)-(k-2)\phi*\kappa_1(\omega_M)}{q^{1/2+is}Q_M(s)}.$$

Then $\phi(\omega_M)=P^s\eta(\omega_M)$ is trivial. Applying Lemma 3.2 to our case together with (3-11) and (3-13), we see that

$$\begin{aligned}\phi(\omega_{M+\ell})&=q^{-\ell/2}\psi(\ell+1,s)\phi(\omega_M)+(k-2)q^{-(\ell+1)/2}\psi(\ell,s)\\&\qquad\times(\phi(\omega_M)-\phi*\kappa_1(\omega_M))\\&=q^{-\ell/2}\psi(\ell+1,s)\{Q_M(s)\eta(\omega_M)+(k-2)R_M(s)(\eta*\kappa_1(\omega_M)-\eta(\omega_M))\}\\&\qquad+(k-2)q^{-(\ell+1)/2}\psi(\ell,s)q^{1/2+is}R_M(s)(\eta(\omega_M)-\eta*\kappa_1(\omega_M))\\&=q^{-\ell/2}\psi(\ell+1,s)Q_M(s)\eta(\omega_M)+(k-2)q^{-\ell/2}R_M(s)\\&\qquad\times\{\psi(\ell+1,s)-q^{is}\psi(\ell,s)\}(\eta*\kappa_1(\omega_M)-\eta(\omega_M))\\&=P^s\eta(\omega_{M+\ell}).\end{aligned}$$

The case $M=0$ is analogous. This concludes the proof. □

The following proposition is proved in the same way as in [Mantero and Zappa 1983, Corollary 3.5] and hence we omit its proof.

Proposition 3.4. *Suppose that $s\notin\Upsilon$. Then the Poisson transform P^s is a bijective operator from $\mathcal{F}'(\Omega)$ onto $\mathscr{L}_{\gamma(s)}(\mathfrak{X})$.*

4. The construction of the intertwining operator

Mantero and Zappa [1983] defined the intertwining operator between the spherical representations for free groups and gave an explicit expression of the intertwining operator. In this section, we extend their results to the case of symmetric graphs.

Let $s\in\mathbb{C}$ and define the action π_s of G on $L^2(\Omega)$ by

$$(\pi_s(g)\eta)(\omega)=p(g\cdot o,\omega)^{1/2+is}\eta(g^{-1}\omega).$$

The representation $(\pi_s, L^2(\Omega))$ is called the spherical representation. We denote by λ the left regular representation of G on $\mathscr{L}_{\gamma(s)}(\mathfrak{X})$. Then as indicated in [Iozzi and Picardello 1983b, p. 372], the Poisson transform P^{-s} intertwines π_s and λ. Therefore, in the case $\pm s \notin \Upsilon$, we see from Proposition 3.4 that the operator on $\mathcal{F}'(\Omega)$ defined by $I_s = (P^s)^{-1} P^{-s}$ is bijective and satisfies the following relation:

$$I_s \pi_s(g) = \pi_{-s}(g) I_s. \tag{4-1}$$

Let $\eta \in L^1(\Omega)$ be such that $D_M \eta = \eta$ for some $M > 0$. Under this assumption, $I_s(B_M \eta) = B_M(I_s \eta)$ because

$$P^s(B_M I_s \eta)(x) = P^s I_s \eta * \kappa_1(x) = P^{-s} \eta * \kappa_1(x) = P^{-s}(B_M \eta)(x).$$

Since

$$P^s I_s \eta(\omega_{M+\ell}) = P^{-s} \eta(\omega_{M+\ell}),$$

we have from (3-3) that

$$\begin{aligned} &q^{-\ell/2} \psi(\ell+1, s) Q_M(s) I_s \eta(\omega) + (k-2) R_M(s) q^{-\ell(1/2+is)} (I_s B_M \eta(\omega) - I_s \eta(\omega)) \\ &= q^{-\ell/2} \psi(\ell+1, s) Q_M(-s) \eta(\omega) + (k-2) R_M(-s) q^{-\ell(1/2-is)} (B_M \eta(\omega) - \eta(\omega)). \end{aligned}$$

Taking $\ell = 0$ and $\ell = 1$ respectively, we obtain from the above equation that

$$\begin{aligned} &Q_M^0(s) I_s \eta(\omega) + (k-2) R_M(s) I_s B_M \eta(\omega) \\ &\qquad = Q_M^0(-s) \eta(\omega) + (k-2) R_M(-s) B_M \eta(\omega) \end{aligned} \tag{4-2}$$

and

$$\begin{aligned} &\beta(s) Q_M(s) I_s \eta(\omega) + (k-2) q^{-(1/2+is)} R_M(s) (I_s B_M \eta(\omega) - I_s \eta(\omega)) \\ &\qquad = \beta(s) Q_M(-s) \eta(\omega) + (k-2) q^{-(1/2-is)} R_M(-s) (B_M \eta(\omega) - \eta(\omega)). \end{aligned} \tag{4-3}$$

Solving the simultaneous equations (4-2) and (4-3), we have

$$\begin{aligned} I_s \eta(\omega) = {} & \frac{q^{-is} Q_M(-s) \eta(\omega) + (q^{is} - q^{-is}) Q_M^0(-s) \eta(\omega)}{q^{is} Q_M(s)} \\ & + \frac{(k-2)(q^{is} - q^{-is}) R_M(-s) B_M \eta(\omega)}{q^{is} Q_M(s)}, \end{aligned} \tag{4-4}$$

$$\begin{aligned} I_s B_M \eta(\omega) = {} & \frac{(q^{is} - q^{-is}) R_M(-s) \{\eta(\omega) - B_M \eta(\omega)\}}{q^{is} Q_M(s)} \\ & + \frac{q^{is} Q_M(-s) B_M \eta(\omega)}{q^{is} Q_M(s)}. \end{aligned} \tag{4-5}$$

For $\eta \in L^1(\Omega)$ satisfying $D_M\eta = \eta$, we define η^+ and η^- as

$$\eta^+(\omega) = \eta(\omega) + (k-2)B_M\eta(\omega), \tag{4-6}$$

$$\eta^-(\omega) = \eta(\omega) - B_M\eta(\omega). \tag{4-7}$$

Then we have from (4-4) and (4-5) that

$$I_s\eta^+(\omega) = \frac{Q_M(-s)}{Q_M(s)}\eta^+(\omega), \tag{4-8}$$

$$I_s\eta^-(\omega) = \frac{q^{-is}R_M(-s)}{q^{is}R_M(s)}\eta^-(\omega). \tag{4-9}$$

For $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{H}_n$ the subspace of $\mathcal{F}(\Omega)_c$ consisting of η such that $D_n\eta = \eta$. We write $\mathcal{H}_n^+$ and $\mathcal{H}_n^-$ for the subspaces of $\mathcal{H}_n$ generated by $\{\eta^+ : \eta \in \mathcal{H}_n\}$ and $\{\eta^- : \eta \in \mathcal{H}_n\}$, respectively. Then it holds that $\mathcal{H}_0 = \mathcal{H}_0^+$ and $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ for $n > 0$. The expressions (4-8) and (4-9) give the explicit forms of the intertwining operator I_s when restricted to $\mathcal{H}_n^+$ and $\mathcal{H}_n^-$, respectively.

Finally in this section, we list some properties of the Poisson transform. Analogously to (4-6) and (4-7), for a function ϕ on $\mathfrak{X}$, we define ϕ^+ and ϕ^- by

$$\phi^+(x) = \phi(x) + (k-2)\phi * \kappa_1(x), \quad \phi^-(x) = \phi(x) - \phi * \kappa_1(x).$$

Let $\eta \in L^1(\Omega)$ be such that $D_M\eta = \eta$. Then taking into account (3-12), we see from Proposition 3.1 that

$$(P^s\eta)^+(x) = q^{-(|x|-M)/2}\psi(|x|-M+1, s)Q_M(s)\eta^+(\omega) = (P^s\eta^+)(x) \tag{4-10}$$

and

$$\begin{aligned}(P^s\eta)^-(x) &= q^{-(|x|-M)/2}\psi(|x|-M+1, s)Q_M(s)\eta^-(\omega) \\ &\qquad -(k-1)q^{-(|x|-M)(1/2+is)}R_M(s)\eta^-(\omega) \\ &= (P^s\eta^-)(x).\end{aligned} \tag{4-11}$$

We here set

$$\psi^-(\ell+1, s) = \sum_{j=0}^{\ell} q^{i(\ell-2j)s} + q^{-1/2}(k-1)\sum_{j=1}^{\ell} q^{i(\ell-2j+1)s}.$$

Then the expression (4-11) can be written

$$(P^s\eta)^-(x) = q^{-(|x|-M)/2}\psi^-(|x|-M+1, s)q^{1/2+is}R_M(s)\eta^-(\omega). \tag{4-12}$$

We summarize these in the following corollary.

Corollary 4.1. *Let $\eta \in L^1(\Omega)$ be such that $D_M\eta = \eta$. Then*

$$(P^s\eta)^+(x) = (P^s\eta^+)(x), \qquad (P^s\eta)^-(x) = (P^s\eta^-)(x).$$

In addition, for any $x \in \mathfrak{X}$ satisfying $|x| > M$, we have

(1) $Q_M(s)^{-1}(P^s\eta)^+(x)$ *is an even entire holomorphic function on* $\mathbb{C}$ *with respect to the variable* s,

(2) $(q^{1/2+is}R_M(s))^{-1}(P^s\eta)^-(x)$ *is an even entire holomorphic function on* $\mathbb{C}$ *with respect to the variable* s.

5. The spectral projection on symmetric graphs

We first review the Helgason–Fourier transform for symmetric graphs and its inversion formula, which were introduced by Eddine [2013; 2015].

Let $(\pi_s, L^2(\Omega))$ be a spherical representation and let I_s be the intertwining operator defined in the previous section. The Helgason–Fourier transform $\tilde{f}(s,\omega)$ of $f \in C_c(\mathfrak{X})$ is defined by

$$\tilde{f}(s,\omega) = (\pi_s(f)1)(\omega) = \sum_{x\in\mathfrak{X}} f(x)p(x,\omega)^{1/2+is}. \tag{5-1}$$

Here 1 denotes the function identically one on Ω. In [Jamal Eddine 2015, Lemma 3.10], Eddine proved the following inversion formula:

$$f(x) = \frac{(k-r)_+}{k}\int_\Omega \tilde{f}(s_0,\omega)p(x,\omega)^{1/2-is_0}\,d\nu(\omega) \\ + c_G\int_\Omega\int_{\mathbb{T}} \tilde{f}(s,\omega)p(x,\omega)^{1/2-is}|c(s)|^{-2}\,ds\,d\nu(\omega),$$

where $c_G = q/\{2\tau r(k-1)\}$ and

$$c(s) = \frac{\sqrt{q}}{q+1}\cdot\frac{q^{1/2+is}-(k-1)q^{-1/2-is}+k-2}{q^{is}-q^{-is}}$$

is a c-function. Here $(k-r)_+$ stands for $k-r$ when $k > r$ and to 0 when $k \leqq r$. As described in [Cowling and Setti 1999, p. 240], we see that $I_s\tilde{f}(s,\omega) = \tilde{f}(-s,\omega)$ for almost all $s \in \mathbb{T}$ and thus we obtain the following symmetry condition:

$$\int_\Omega \tilde{f}(s,\omega)p(x,\omega)^{1/2-is}\,d\nu(\omega) = \int_\Omega \tilde{f}(-s,\omega)p(x,\omega)^{1/2+is}\,d\nu(\omega). \tag{5-2}$$

Following Bray [1996], we define the spectral projection $P_s f$ of $f \in C_c(\mathfrak{X})$ by

$$P_s f(x) = (f * \phi_s)(x) = \int_G f(g_1)\phi_s(g_1^{-1}g)\,dg_1, \tag{5-3}$$

where $x = g \cdot o$. Applying the functional equation of the spherical function [Jamal Eddine 2015, Lemma 3.9] and using Fubini's theorem, we obtain

$$\begin{aligned} P_s f(x) &= \int_G f(g_1 \cdot o) \int_\Omega p(g_1 \cdot o, \omega)^{1/2+is} p(g \cdot o, \omega)^{1/2-is} \, d\nu(\omega) \, dg_1 \\ &= \int_\Omega \tilde{f}(s, \omega) p(x, \omega)^{1/2-is} \, d\nu(\omega). \end{aligned}$$

Thus the spectral projection $P_s f(x)$ is Weyl-invariant with respect to the variable s and has the following inversion formula:

$$f(x) = \frac{(k-r)_+}{k} P_{s_0} f(x) + c_G \int_{\mathbb{T}} P_s f(x) |c(s)|^{-2} \, ds. \tag{5-4}$$

Let $a \in \mathfrak{X}$ and define the function ξ_a on Ω by $\xi_o(\omega) = 1$ and for $a \neq o$

$$\xi_a(\omega) = \nu(E(a))^{-1} \chi_{E(a)}(\omega) - \nu(E(a'))^{-1} \chi_{E(a')}(\omega).$$

Then it is easy to see that $D_{|a|} \xi_a = \xi_a$ and $B_{|a|} \xi_a = (\xi_* * \kappa_1)(a)$. For $a \in \mathfrak{X}$ and $s \in \mathbb{C}$, we define the generalized spherical function $\Phi_{a,s}$ on $\mathfrak{X}$ by

$$\Phi_{a,s}(x) = P^s \xi_a(x) = \int_\Omega p(x, \omega)^{1/2+is} \xi_a(\omega) \, d\nu(\omega).$$

By Proposition 3 in [Koizumi 2013] combined with Corollary 4.1, we have that

$$(\Delta_n P_s f)^{\pm}(x) = \int_\Omega \Phi^{\pm}_{\omega_n, -s}(x) \tilde{f}(s, \omega) \, d\nu(\omega).$$

The explicit expressions of $\Phi^{\pm}_{\omega_n,-s}$ are given by (4-10) and (4-12). We see from these that

$$\begin{aligned} (\Delta_n P_s f)^{+}(x) = 0 \quad &\text{when} \quad \pm s \in \Upsilon, \\ (\Delta_n P_s f)^{-}(x) = 0 \quad &\text{when} \quad \pm s \in \tfrac{1}{2} i + \tau \mathbb{Z}. \end{aligned}$$

Furthermore, we have the following proposition.

Proposition 5.1. *Let $f \in C_N(\mathfrak{X})$. Then $(\Delta_n P_{s_0} f)^{-}(x) = 0$ when $n > N$.*

Proof. By the definition of the Poisson transform, we have

$$\begin{aligned} (\Delta_n P_{s_0} f)^{-}(x) &= \sum_{y \in \mathfrak{X}} f(y) \left\{ \int_\Omega p(x, \omega)^{1/2-is_0} \Phi^{-}_{\omega_n, s_0}(y) \, d\nu(\omega) \right\} \\ &= \sum_{y \in \mathfrak{X}} f(y) \left\{ \int_\Omega (1-r)^{\zeta(x,\omega)} \left(\frac{1}{1-k} \right)^{|y|} \frac{k(k-1)^{n-1}}{r(r-1)^{n-1}} \xi^{-}_{\omega_n}(\omega') \, d\nu(\omega) \right\}, \end{aligned}$$

where we choose $\omega' \in E(y)$. Taking into account

$$\xi^-_{\omega_n}(\omega') = \begin{cases} r(k-1)q^{n-1}, & \omega \in E(y^{(n)}), \\ -r(k-1)q^{n-1}/(k-2), & \omega \in B(y^{(n)}), \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}(\Delta_n P_{s_0} f)^-(x) = k(k-1)^{2n-1} &\sum_{\substack{y\in\mathfrak{X}\\ |y|\geq n}} f(y)\Big(\frac{1}{1-k}\Big)^{|y|}\int_{E(y^{(n)})}(1-r)^{\zeta(x,\omega)}\,d\nu(\omega)\\ &- \frac{k(k-1)^{2n-1}}{k-2}\sum_{\substack{y\in\mathfrak{X}\\ |y|\geq n}} f(y)\Big(\frac{1}{1-k}\Big)^{|y|}\int_{B(y^{(n)})}(1-r)^{\zeta(x,\omega)}\,d\nu(\omega).\end{aligned}$$

Since $f(y)=0$ for $|y|\geq n>N$, we have $(\Delta_n P_{s_0} f)^-(x)=0$. □

6. Paley–Wiener theorem for spectral projection

In this section, we shall characterize the image of $C_c(\mathfrak{X})$ under the spectral projection on $\mathfrak{X}$. As an application, we shall prove the Paley–Wiener theorem of the Helgason–Fourier transform for symmetric graphs.

Throughout this section, for a function ϕ on $\mathfrak{X}$, we denote $\Delta_n\phi(x)$ by $\phi_n(x)$. Let $N\in\mathbb{Z}_{\geq 0}$. We denote by $\mathscr{T}_N(\mathbb{T}\times\mathfrak{X})$ the set comprised of all functions F on $\mathbb{T}\times\mathfrak{X}$ satisfying the following conditions:

(N1) $F(s,x)$ is a Weyl-invariant smooth function on $\mathbb{R}$ with respect to the variable s.

(N2) For each $n\in\mathbb{Z}_{\geq 0}$ and $s\in\mathbb{R}$, $F_n(s,x)\in\mathscr{L}^n_{\gamma(s)}(\mathfrak{X})$,

(N3) For each $x\in\mathfrak{X}$, $F(s,x)$ extends to a Weyl-invariant holomorphic function on $\mathbb{C}$.

(N4) For each $n\in\mathbb{Z}_{\geq 0}$, $Q_n(-s)^{-1}F_n^+(s,x)$ is holomorphic on $\mathbb{C}$ and there exists a constant $C_N>0$ which does not depend on the choice of n such that

$$|Q_n(-s)^{-1}F_n^+(s,x)|\leq C_N q^{(|x|-n+N)|\Im s|}.$$

(N5) For each $n\in\mathbb{Z}_{>0}$, $(q^{1/2-is}R_n(-s))^{-1}F_n^-(s,x)$ is holomorphic on $\mathbb{C}$ and there exists a constant $C_N>0$ which does not depend on the choice of n such that

$$|(q^{1/2-is}R_n(-s))^{-1}F_n^-(s,x)|\leq C_N q^{(|x|-n+N)|\Im s|}.$$

(N6) $F_n^-(s_0,x)=0$ when $n>N$.

We set

$$\mathscr{T}(\mathbb{T}\times\mathfrak{X})=\bigcup_{N=0}^{\infty}\mathscr{T}_N(\mathbb{T}\times\mathfrak{X}).$$

The following proposition is obtained by the same arguments as in [Koizumi 2013, Proposition 4].

Proposition 6.1. *Let $f \in C_N(\mathfrak{X})$. Then $F(s,x) = P_s f(x)$ belongs to $\mathscr{T}_N(\mathbb{T} \times \mathfrak{X})$.*

To prove the sufficient condition in the Paley–Wiener theorem, we need the following lemma.

Lemma 6.2 (cf., [Koizumi 2013, Lemma 1]). *Let $N \in \mathbb{Z}_{>0}$, $F \in \mathscr{T}_N(\mathbb{T} \times \mathfrak{X})$ and $a \in \mathfrak{S}_n$. If $n > N$ then $F_n^+(s,a) = 0$ and $F_n^-(s,a) = 0$ for all $s \in \mathbb{T}$.*

Proof. We shall first show that $F_n^-(s,a) = 0$. Let us set

$$\phi^-(s) = (q^{1/2-is} R_n(-s))^{-1} F_n^-(s,a).$$

Then we see

$$\phi^-(-s) = (q^{1/2+is} R_n(s))^{-1} F_n^-(s,a) = \frac{q^{1/2-is} R_n(-s)}{q^{1/2+is} R_n(s)} \phi^-(s). \tag{6-1}$$

We put

$$c^-(n,s) = \frac{q^{1/2-is} R_n(-s)}{q^{1/2+is} R_n(s)}.$$

Obviously we have

$$c^-(n,s) = -q^{-1/2-is(2n-1)} + (1-q^{-1}) \sum_{\ell=0}^{\infty} q^{-\ell/2-is(2n+\ell)}. \tag{6-2}$$

The condition (N5) yields that $\phi^-(s)$ is an entire function of exponential type N. We use the Paley–Wiener theorem on $\mathbb{Z}$ to write

$$\phi^-(s) = \sum_{m \in \mathbb{Z}} \phi^-(m) q^{ims},$$

where $\phi^-(m) = 0$ unless $-N \leq m \leq N$. Substituting (6-2) to (6-1), we have

$$\sum_{m \in \mathbb{Z}} \phi^-(m) q^{-ims} = \sum_{m \in \mathbb{Z}} \left[-q^{-\frac{1}{2}-is(2n-1)} + (1-q^{-1}) \sum_{\ell=0}^{\infty} q^{-\frac{\ell}{2}-is(2n+\ell)} \right] \times \phi^-(m) q^{ims},$$

and thus we have the following recursion formula:

$$\phi^-(m) = -q^{-1/2} \phi^-(2n-1-m) + (1-q^{-1}) \sum_{\ell=0}^{\infty} q^{-\ell/2} \phi^-(2n+\ell-m). \tag{6-3}$$

From (6-3), when $n > N+1$, it is easily verified that $\phi^-(m) = 0$ for all $m \in \mathbb{Z}$. When $n = N+1$, (6-3) implies

$$\phi^-(m) = -q^{-1/2} \phi^-(2N+1-m),$$

and so $\phi^-(N) = 0$. We consequently have that $\phi^-(m) = 0$ for all $m \in \mathbb{Z}$.

We shall next show that $F_n^+(s,a)=0$. We set $\phi^+(s)=Q_n(-s)^{-1}F_n^+(s,a)$. We use the Paley–Wiener theorem on $\mathbb{Z}$ to write

$$\phi^+(s)=\sum_{m\in\mathbb{Z}}\phi^+(m)q^{ims},$$

where $\phi^+(m)=0$ unless $-N\le m\le N$. Putting

$$c^+(n,s)=\frac{Q_n(-s)}{Q_n(s)},$$

we have

$$\phi^+(-s)=c^+(n,s)\phi^+(s).$$

Because

$$c^+(n,s)=\frac{1+(k-1)q^{-1/2+is}}{1+(k-1)q^{-1/2-is}}c^-(n,s),$$

we have

$$\begin{aligned}
\text{(6-4)}\quad \phi^+(m)& \\
={}&-(k-1)q^{-1}\phi^+(2n-2-m)\\
&+(k-1)(1-q^{-1})\sum_{\ell=0}^{\infty}q^{-(\ell+1)/2}\phi^+(2n+\ell-m-1)\\
&-(1-(1-k)^2q^{-1})\sum_{\ell=0}^{\infty}(1-k)^{\ell}q^{-(\ell+1)/2}\phi^+(2n+\ell-1-m)\\
&+\frac{(1-q^{-1})(1-(1-k)^2q^{-1})}{k}\sum_{\ell=0}^{\infty}(1-(1-k)^{\ell+1})q^{-\ell/2}\phi^+(2n+\ell-m).
\end{aligned}$$

From (6-4), when $n>N+1$, it is easily verified that $\phi^+(m)=0$ for all $m\in\mathbb{Z}$. In the case $n=N+1$, (6-4) implies

$$\phi^+(m)=-(k-1)q^{-1}\phi^+(2N-m),$$

and so $\phi^+(N)=0$. Therefore, in this case, $\phi^+(m)=0$ for all $m\in\mathbb{Z}$. □

The proof of the sufficient condition in the Paley–Wiener theorem is like the proof for the case of semisimple Lie groups given by Campoli [1980] and Johnson [1979]. We remark that $\Im s_0\ge 0$ when $k\le r$ and $\Im s_0<0$ when $k>r$. We also remark that the residue of $1/c(s)$ at $s=s_0$ is equal to $r(k-r)/\{ik(r-1)\log q\}$ and $1/c(-s_0)=1$. We first show the following proposition.

Proposition 6.3. *Let* $N\in\mathbb{Z}_{>0}$, $F\in\mathscr{T}_N(\mathbb{T}\times\mathfrak{X})$ *and set*

$$f_0(x)=c_G\int_{\mathbb{T}}F_0(s,x)|c(s)|^{-2}\,ds.$$

Then there exists a function $J_0 \in C_N(\mathfrak{X})^{\#}$ *such that*

$$f_0(x) - J_0(x) = \frac{(k-r)_+}{k} F_0(s_0, x).$$

Proof. We put $F(s) = F_0(s, o)$. Then it follows from Lemma 3.2 and the condition (N4) that $F(s)$ is an even entire function of exponential order N and

$$F_0(s, x) = \phi_s(x) F(s).$$

We thus have

$$\begin{aligned} f(x) &= c_G \int_{\mathbb{T}} \phi_s(x) F(s) |c(s)|^{-2}\, ds \\ &= c_G \int_{\mathbb{T}} F(s) \frac{1}{c(-s)} q^{(is-1/2)|x|}\, ds + c_G \int_{\mathbb{T}} F(s) \frac{1}{c(s)} q^{(-is-1/2)|x|}\, ds. \end{aligned} \tag{6-5}$$

We write $f_1(x)$ and $f_2(x)$ for the first term and the second term of the last expression of (6-5), respectively. For a sufficiently large $\eta > 0$, let $f_{1,\eta}$ denote the formula shifting the path of integral of f_1 from $\mathbb{T}$ to $\mathbb{T} + i\eta$ and let $f_{2,-\eta}$ denote the formula shifting the path of integral of f_2 from $\mathbb{T}$ to $\mathbb{T} - i\eta$.

Suppose that $k \le r$. Because f_1 is analytic inside the rectangle with corners $\pm\tau/2$ and $\pm\tau/2 + i\eta$, we have by Cauchy's theorem that $f_1 = f_{1,\eta}$. Similarly we can also obtain that $f_2 = f_{2,-\eta}$. In case $k > r$, we have

$$\begin{aligned} f_1(x) - f_{1,\eta}(x) &= 2\pi i c_G \operatorname{Res}_{s=-s_0} \Big\{ F(s) \frac{1}{c(-s)} q^{(is-1/2)|x|} \Big\} \\ &= \frac{k-r}{2k} F(-s_0) \Big(\frac{1}{1-k}\Big)^{|x|}, \\ f_2(x) - f_{2,-\eta}(x) &= 2\pi i c_G \operatorname{Res}_{s=s_0} \Big\{ F(s) \frac{1}{c(s)} q^{(-is-1/2)|x|} \Big\} \\ &= \frac{k-r}{2k} F(s_0) \Big(\frac{1}{1-k}\Big)^{|x|}. \end{aligned}$$

Then the assertion follows immediately. □

In the nonradial case, we need slightly complicated calculations.

Proposition 6.4. *Let* $N \in \mathbb{Z}_{>0}$, $F \in \mathscr{T}_N(\mathbb{T} \times \mathfrak{X})$ *and set*

$$f_n(x) = c_G \int_{\mathbb{T}} F_n(s, x) |c(s)|^{-2}\, ds$$

for $n > 0$. *Then there exists a function* $J_n \in C_N(\mathfrak{X})$ *such that*

$$f_n(x) - J_n(x) = \frac{(k-r)_+}{k} F_n(s_0, x).$$

Proof. We put $a = x^{(n)}$ and choose $\omega \in E(x)$. Because $F_n^{\pm}$ satisfies the condition (N2), we have from (4-10) and (4-12) that

$$F_n^{+}(s,x) = q^{-(|x|-|a|)/2}\psi(|x|-|a|+1,s)Q_n(s)F_n^{+}(s,a), \tag{6-6}$$

$$F_n^{-}(s,x) = q^{-(|x|-|a|)/2}\psi^{-}(|x|-|a|+1,s)q^{1/2+is}R_n(s)F_n^{-}(s,a). \tag{6-7}$$

In the case when $n > N$, Lemma 6.2 yields that $F_n^{+}(s,a) = 0$ and $F_n^{-}(s,a) = 0$. On the other hand, since $F_n^{-}(s_0,x) = 0$, it suffices to set $J_n(x) = 0$.

In the following, we suppose that $n \le N$. Substituting (6-6) and (6-7), we obtain

$$f_n^{\pm}(x) = \xi_a^{\pm}(\omega)^{-1} c_G \int_{\mathbb{T}} h_a^{\pm}(s)\Phi_{a,s}^{\pm}(x)|c(s)|^{-2}\,ds,$$

where $h_a^{+}(s) = Q_{|a|}(s)^{-1}F_n^{+}(s,a)$, $h_a^{-}(s) = (q^{1/2+is}R_{|a|}(s))^{-1}F_n^{-}(s,a)$ and $\omega \in E(x)$. We know that $\Phi_{a,s}^{+}(x)$ has the following expansion:

$$\Phi_{a,s}^{+}(x) = \{c(s)q^{(is-1/2)|x|} - q^{i2(|a|-1)s}c(s)q^{(-is-1/2)|x|}\}\xi_a^{+}(\omega). \tag{6-8}$$

Hence we can show $f_n^{+} \in C_N(\mathfrak{X})$ for all $k,\ r$ by the same arguments as in the proof of Proposition 6.3.

Hereafter we shall compute $f_n^{-}(x)$. It follows from (6-8) that

$$\Phi_{a,s}^{-}(x) = \{c(s)q^{(is-1/2)|x|} - q^{i2(|a|-1)s}c(s)q^{(-is-1/2)|x|} - (k-1)R_{|a|}(s)q^{-(|x|-|a|)(1/2+is)}\}\xi_a^{-}(\omega).$$

Let us set

$$\begin{aligned} f_{n,1}^{-}(x) &= c_G\int_{\mathbb{T}} h_a^{-}(s)\frac{1}{c(-s)}q^{(is-1/2)|x|}\,ds,\\ f_{n,2}^{-}(x) &= c_G\int_{\mathbb{T}} h_a^{-}(s)\frac{1}{c(-s)}q^{i2(|a|-1)s}q^{(-is-1/2)|x|}\,ds,\\ f_{n,3}^{-}(x) &= c_G\int_{\mathbb{T}} h_a^{-}(s)R_{|a|}(s)q^{-(|x|-|a|)(1/2+is)|x|}|c(s)|^{-2}\,ds. \end{aligned}$$

Suppose first that $k \le r$. Then as shown in Proposition 6.3, we see that $f_{n,1}^{-} \in C_N(\mathfrak{X})$. Keeping the notation in the proof of Proposition 6.3, we have

$$\begin{aligned} f_{n,2}^{-}(x) - f_{n,2,-\eta}^{-}(x) &= 2\pi i c_G \operatorname{Res}_{s=-s_0}\Big\{h_a^{-}(s)\frac{1}{c(-s)}q^{i2(|a|-1)s}q^{(-is-1/2)|x|}\Big\}\\ &= \frac{r(k-r)}{2k}h_a^{-}(-s_0)(1-k)^{-|a|+1}(1-r)^{-|x|+|a|-1},\\ f_{n,3}^{-}(x) - f_{n,3,-\eta}^{-}(x) &= 2\pi i c_G \operatorname{Res}_{s=-s_0}\{h_a^{-}(s)R_{|a|}(s)q^{-(is+1/2)(|x|-|a|)}|c(s)|^{-2}\}\\ &= \frac{r(k-r)}{2k}h_a^{-}(-s_0)(1-k)^{-|a|}(1-r)^{-|x|+|a|-1}. \end{aligned}$$

Setting
$$f^-_{n,\eta}(x) = f^-_{n,1}(x) - f^-_{n,2,-\eta}(x) - (k-1) f^-_{n,3,-\eta}(x),$$
we obtain that
$$f^-_n(x) - f^-_{n,\eta}(x) = 0.$$
Thus we have by contour integration arguments that $f^-_n \in C_N(\mathfrak{X})$.

Suppose next that $k > r$. In this case, by the same discussion as in the proof of Proposition 6.3, we see that $f^-_{n,2} \in C_N(\mathfrak{X})$. Moreover we have

$$\begin{aligned} f^-_{n,1}(x) - f^-_{n,1,\eta}(x) &= 2\pi i c_G \operatorname{Res}_{s=-s_0}\left\{h^-_a(s)\frac{1}{c(-s)}q^{(is-1/2)|x|}\right\} \\ &= \frac{k-r}{2k}\left(\frac{1}{1-k}\right)^{|x|} h^-_a(-s_0), \\ f^-_{n,3}(x) - f^-_{n,3,-\eta}(x) &= 2\pi i c_G \operatorname{Res}_{s=s_0}\{h^-_a(s)R_{|a|}(s)q^{-(|x|-|a|)(1/2+is)}|c(s)|^{-2}\} \\ &= \frac{k-r}{2r}h^-_a(s_0)(1-k)^{-|x|+|a|-2}(1-r)^{-|a|+1}. \end{aligned}$$

Let us set
$$f^-_{n,\eta}(x) = f^-_{n,1,\eta}(x) - f^-_{n,2}(x) - (k-1) f^-_{n,3,-\eta}(x).$$
Then we have
$$f^-_n(x) - f^-_{n,\eta}(x) = \left(\frac{1}{1-k}\right)^{|x|}\left\{\frac{k-r}{2k}h_a(-s_0) + \frac{k-r}{2r}h^-_a(s_0)(1-k)^{|a|-1}(1-r)^{-|a|+1}\right\}.$$
We see from the definition of $h^-_a(s)$ that
$$h^-_a(-s_0) = (1-k)^{|a|}F^-_n(s_0,a), \quad h^-_a(s_0) = \frac{r(1-k)(1-r)^{|a|-1}}{k}F^-_n(s_0,a).$$
We therefore obtain
$$f^-_n(x) - f^-_{n,\eta}(x) = \left(\frac{1}{1-k}\right)^{|x|}\frac{k-r}{k}(1-k)^{|a|}F^-_n(s_0,a) = \frac{k-r}{k}F^-_n(s_0,x).$$
By contour integration arguments, we see that there exists $J^-_n \in C_N(\mathfrak{X})$ such that
$$f^-_n(x) - J^-_n(x) = \frac{(k-r)_+}{k}F^-_n(s_0,x),$$
concluding the proof. □

Summarizing the arguments in this section, we arrive at the following theorem.

Theorem 6.5. *Let $N \in \mathbb{Z}_{>0}$ and $F \in \mathscr{T}_N(\mathbb{T}\times\mathfrak{X})$. We set*
$$f(x) = c_G\int_{\mathbb{T}} F(s,x)|c(s)|^{-2}\,ds.$$
Then there exists a function $J \in C_N(\mathfrak{X})$ such that
$$f(x) - J(x) = \frac{(k-r)_+}{k}F(s_0,x).$$

Proof. Let $x \in \mathfrak{X}$ be such that $|x| > N$. We choose a positive integer M so that $|x| \leq M$. Then $f(x)$ can be written as the following finite sum:

$$f(x) = \varepsilon_M f(x) = f_0(x) + f_1(x) + \cdots + f_M(x).$$

Applying Propositions 6.3 and 6.4 to each F_n, we have

$$\sum_{n=0}^{M} f_n(x) - \sum_{n=0}^{\min(M,N)} J_n(x) = \frac{(k-r)_+}{k} \sum_{n=0}^{\min(M,N)} F_n(s_0, x).$$

We thus have the required result. □

In the remainder of this section, as a corollary of Theorem 6.5, we shall prove the Paley–Wiener theorem of the Helgason–Fourier transform.

Let $N \in \mathbb{Z}_{\geq 0}$. Denote by $\mathscr{Z}_N(\mathbb{T} \times \Omega)$ the set of all functions F on $\mathbb{T} \times \Omega$ satisfying the following conditions:

(H1) $F(s, \omega)$ is a smooth function on $\mathbb{T}$ with respect to the variable s,

(H2) $F(s + \tau, \omega) = F(s, \omega)$,

(H3) $F(s, \omega)$ extends to a τ-periodic entire function of exponential type N,

(H4) F satisfies the symmetry condition (5-2),

(H5) $(D_n F)^-(s_0, \omega) = 0$ when $n > N$.

With the notation above, we show the following theorem.

Theorem 6.6. *Let* $N \in \mathbb{Z}_{\geq 0}$, $F \in \mathscr{Z}_N(\mathbb{T} \times \Omega)$ *and set*

$$f(x) = c_G \int_{\mathbb{T}} \int_{\Omega} F(s, \omega) p(x, \omega)^{1/2 - is} \, ds \, d\nu(\omega).$$

Then there exists a function $J \in C_N(\mathfrak{X})$ *such that*

$$f(x) - J(x) = \frac{(k-r)_+}{k} \int_{\Omega} F(s_0, \omega) p(x, \omega)^{1/2 - is_0} \, d\nu(\omega).$$

Proof. Let $F \in \mathscr{Z}_N(\mathbb{T} \times \Omega)$. It suffices to show $P^{-s} F \in \mathscr{T}_N(\mathbb{T} \times \mathfrak{X})$. The conditions (H1), (H2) and (H6) are immediate from the conditions (H1), (H2) and (H5). Noting

$$|q^{-n/2} \psi(n+1, s)| \leq \frac{q+1}{q-1} q^{n|\Im s|}, \quad |q^{-n/2} \psi^-(n+1, s)| \leq \frac{q+1}{q-1} q^{n|\Im s|},$$

we have from (4-10) that

$$\begin{aligned} |Q_n(-s)^{-1} (P^{-s} F)_n^+(s, x)| &= |q^{-(|x|-n)/2} \psi(|x| - n + 1, s) (D_n F)^+(s, \omega)| \\ &\leq C'_N q^{(|x|-n+N)|\Im s|} \end{aligned}$$

for some constant C'_N which does not depend on the choice of n. The condition (N5) is obtained in the same fashion as above. This concludes the proof. □

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