Pacific Journal of Mathematics

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LEE TIM WENG
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Volume 290 No. 1 September 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 1, 2017
https://doi.org/10.2140/pjm.2017.290.139

FUNDAMENTAL DOMAINS OF ARITHMETIC QUOTIENTS OF REDUCTIVE GROUPS OVER NUMBER FIELDS

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For a connected reductive algebraic group G over a number field $\Bbbk$, we investigate the Ryshkov domain R_Q associated to a maximal $\Bbbk$-parabolic subgroup Q of G. By considering the arithmetic quotients $G(\Bbbk)\backslash G(\mathbb{A})^1/K$ and $\Gamma_i\backslash G(\Bbbk)/K_\infty$, with K a maximal compact subgroup of the adele group $G(\mathbb{A})$ and the Γ_i arithmetic subgroups of $G(\Bbbk)$, we present a method of constructing fundamental domains for $Q(\Bbbk)\backslash R_Q$ and $\Gamma_i\backslash G(\Bbbk_\infty)^1$. We also study the particular case when $G = \mathrm{GL}_n$, and subsequently construct fundamental domains for P_n, the cone of positive definite Humbert forms over $\Bbbk$, with respect to the subgroups Γ_i.

1. Introduction

Let $\Bbbk$ be an arbitrary algebraic number field with ring of integers $\mathcal{O}$. This paper mainly focuses on the determination and construction of fundamental domains associated to certain arithmetic quotients of reductive algebraic groups over $\Bbbk$.

For the first part of the paper we consider a general connected reductive isotropic algebraic group G over $\Bbbk$ and investigate fundamental domains associated to the arithmetic quotients $G(\Bbbk)\backslash G(\mathbb{A})^1/K$ and $\Gamma_i\backslash G(\Bbbk_\infty)^1/K_\infty$, with K a maximal compact subgroup of $G(\mathbb{A})$ and subgroups Γ_i of $G(\Bbbk)$ to be described below.

The discussion and results here in the preliminary sections are an extension of Watanabe's results [2014]. A maximal $\Bbbk$-parabolic subgroup Q of G is taken and we consider its associated height function H_Q and Hermite function $\mathsf{m}_Q(g) = \min_{x\in Q(\Bbbk)\backslash G(\Bbbk)} H_Q(xg)$ on $G(\mathbb{A})^1$. Watanabe [2014] introduced the Ryshkov domain of m_Q, $R_Q = \{g \in G(\mathbb{A})^1 : \mathsf{m}_Q(g) = H_Q(g)\}$, for the purpose of constructing a fundamental domain for $G(\Bbbk)\backslash G(\mathbb{A})^1$ well matched with m_Q. Watanabe also considered the case when G is of class number 1, that is, when $|G(\Bbbk)\backslash G(\mathbb{A})^1/G^1_{\mathbb{A},\infty}| = 1$, and obtained a fundamental domain for $G(\Bbbk_\infty)$ with respect to $G_{\mathcal{O}} = G(\Bbbk)\cap G_{\mathbb{A},\infty}$.

MSC2010: primary 11H55; secondary 11F06.
Keywords: fundamental domain, arithmetic quotient, Ryshkov domain, Humbert form, Voronoi reduction.

Here however, we consider algebraic groups of any general class number n_G. Particularly for class numbers higher than 1, for each $i = 1, \dots, n_G$ we are required to consider different arithmetic subgroups Γ_i of $G(\Bbbk)$ in place of just $G_{\mathcal{O}}$.

Let R_Q^* denote the closure in $G(\mathbb{A})^1$ of the interior of R_Q. It was established in [Watanabe 2014] that by starting from a fundamental domain Ω of R_Q^* with respect to $Q(\Bbbk)$, a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\Bbbk)$ can be obtained by taking the interior of Ω in $G(\mathbb{A})^1$. In order to explicitly construct such an Ω, we define groups

$$G_{\mathbb{A},\infty} = G(\Bbbk_\infty) \times K_f \quad \text{and} \quad \Gamma_i = \eta_i G^1_{\mathbb{A},\infty} \eta_i^{-1} \cap G(\Bbbk),$$

where the $\eta_1, \dots, \eta_{n_G}$ are representatives of $G(\Bbbk)\backslash G(\mathbb{A})^1/G^1_{\mathbb{A},\infty}$. Also for each i take a complete set of representatives $\{\xi_{ij}\}_{j=1}^{h_i}$ for $Q(\Bbbk)\backslash G(\Bbbk)/\Gamma_i$, define sets

$$R_{i,j,\infty} = \{g \in G(\Bbbk_\infty)^1 : \mathrm{m}_Q(g\xi_{ij}\eta_i) = H_Q(g\xi_{ij}\eta_i)\}$$

and let $Q_{i,j} = Q(\Bbbk) \cap \xi_{ij}\Gamma_i\xi_{ij}^{-1}$. By considering the action of $Q_{i,j}$ on $R_{i,j,\infty}$, we find that starting with arbitrary open fundamental domains $\Omega_{i,j,\infty}$ for $Q_{i,j}\backslash R_{i,j,\infty}$ we can construct the required Ω. From this we obtain the following results.

Theorem. $\Omega = \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \Omega_{i,j,\infty}\xi_{ij}\eta_i K_f$ *is an open fundamental domain of* R_Q^* *with respect to* $Q(\Bbbk)$.

Theorem. *For each* $i = 1, \dots, n_G$, *the set* $\bigcup_{j=1}^{h_i} \xi_{ij}^{-1}\Omega_{i,j,\infty}\xi_{ij}$ *is an open fundamental domain of* $G(\Bbbk_\infty)^1$ *with respect to* Γ_i.

In particular we can take η_1 to be the identity element of G, in which case Γ_1 coincides with the group $G_{\mathcal{O}} = G(\Bbbk) \cap G_{\mathbb{A},\infty}$ used in [Watanabe 2014] when $n_G = 1$.

The second topic of interest in this paper is the special case when G is the general linear group GL_n defined over $\Bbbk$. This time we consider the maximal $\Bbbk$-parabolic subgroup

$$Q = Q^{n,m} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a \in \mathrm{GL}_m(\Bbbk),\ b \in M_{m,n-m}(\Bbbk),\ d \in \mathrm{GL}_{n-m}(\Bbbk) \right\}$$

for a fixed $1 \le m < n$. The class number of G in this case is equal to h, the class number of $\Bbbk$. Using $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$, a complete set of representatives for the ideal class group of $\Bbbk$, we can produce a corresponding set of matrices $\{\eta_1, \dots, \eta_h\}$ representing $\mathrm{GL}_n(\Bbbk)\backslash \mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty}$. The Γ_i in this case are the subgroups of $\mathrm{GL}_n(\Bbbk)$ stabilizing the respective $\mathcal{O}$-lattices $\sum_{k=1}^{n-1} \mathcal{O}\boldsymbol{e}_k + \mathfrak{a}_i\boldsymbol{e}_n$. The main result established in this part is:

Theorem. $|Q(\Bbbk)\backslash \mathrm{GL}_n(\Bbbk)/\Gamma_i| = h$ *for every* $i = 1, \dots, h$.

This can be proved by identifying $Q(\Bbbk)\backslash \mathrm{GL}_n(\Bbbk)$ with the set of all m-dimensional subspaces of $\Bbbk^n$ and establishing a bijection between this set modulo Γ_i and the ideal class group of $\Bbbk$. This bijection also allows us to obtain suitable matrix representatives $\{\xi_{ij}\}_{j=1}^{h}$ for $Q(\mathbb{k})\backslash \mathrm{GL}_n(\mathbb{k})/\Gamma_i$. Relations between the field class number and the number of double cosets in quotients of similar type involving other algebraic groups, e.g., SL_n, Sp_{2n} and Chevalley groups, modulo a minimal parabolic subgroup instead are noted by Borel [1962, Section 4.7].

In the final sections we consider P_n, the space of positive definite Humbert forms over $\mathbb{k}$, with the usual identification $P_n = \prod_\sigma P_n(\mathbb{k}_\sigma)$, where $P_n(\mathbb{k}_\sigma)$ denotes the set of $n \times n$ positive definite real symmetric/complex Hermitian matrices, depending on whether σ is real or imaginary, and the product is taken over all infinite places σ of $\mathbb{k}$.

If $\mathbb{k} = \mathbb{Q}$, then P_n is just the cone of positive definite real symmetric matrices, and fundamental domains for $P_n/\mathrm{GL}_n(\mathbb{Z})$ in this case have been historically constructed by Korkin and Zolotarev [1873], Minkowski [1905] and later on Grenier [1988]. For P_n over a general number field, Humbert [1939] previously provided a fundamental domain constructed with respect to the particular group $\mathrm{GL}_n(\mathcal{O})$. As $\mathrm{GL}_n(\mathcal{O})$ coincides with one of the Γ_i we study in this paper, the question can be raised about fundamental domains for P_n with respect to each of the groups Γ_i when $n_G > 1$.

As such, we proceed in the final sections to provide a general way of constructing fundamental domains for P_n/Γ_i given any number field. The method of construction given here follows and generalizes the example given by Watanabe [2014] for the specific case $\mathbb{k} = \mathbb{Q}$. As already noted in that paper, when $\mathbb{k} = \mathbb{Q}$ the fundamental domain for $P_n/\mathrm{GL}_n(\mathbb{Z})$ resulting from this method coincides with Grenier's [1988]. It was observed by Dutour Sikirić and Schürmann that Grenier's fundamental domain is in fact equivalent to the one previously developed by Korkin and Zolotarev. Regarding $P_n/\mathrm{GL}_n(\mathcal{O})$ for general number fields however, we note that the fundamental domain produced by the method here differs from Humbert's construction, which utilizes the matrix trace, whereas the domain here is defined using the adele norm of matrix determinants.

Using the matrix representatives $\{\eta_i\}_{i=1}^{h}$ and $\{\xi_{ij}\}_{j=1}^{h}$, we associate to each pair (η_i, ξ_{ij}) a maximal compact subgroup $K_{i,j,\infty}$ of $\mathrm{GL}_n(\mathbb{k}_\infty)$ and a map π_{ij} inducing an isomorphism between $\mathrm{GL}_n(\mathbb{k}_\infty)/K_{i,j,\infty}$ and P_n. Then the results of our discussions on GL_n can be transferred to P_n via the maps π_{ij}, which finally lead up to an iterative method of constructing fundamental domains for P_n with respect to the groups Γ_i for any general dimension n. Watanabe has also graciously provided an appendix to this paper on Voronoi reduction over general number fields that are not necessarily totally real, which settles the base case of dimension 1.

We also demonstrate that this fundamental domain construction for P_n/Γ_i is well matched with certain automorphisms of $\mathrm{GL}(\mathbb{k}_\infty)$. Namely we see that the fundamental domain for P_n/Γ_i constructed using a set of ideals $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ representing the ideal class group and the maximal $\mathbb{k}$-parabolic subgroup $Q^{n,m}$ can be directly mapped by an automorphism to the one constructed with the representative set $\{\mathfrak{a}_1^{-1}, \dots, \mathfrak{a}_h^{-1}\}$ and $Q^{n,n-m}$.

Notation

In this paper we use $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the fields of rational, real, and complex numbers respectively, and $\mathbb{Z}$ for the ring of integers. $\mathbb{R}_{>0}$ will denote the set of positive reals.

For positive integers r and s, we denote by $M_{r,s}(S)$ the set of all $r \times s$ matrices with entries in the set S, and we write $M_r(S)$ for $M_{r,r}(S)$. The identity matrix of size r will be denoted by I_r. The transpose of a matrix A will be written by ${}^tA$. If $A \in M_{r,s}(\mathbb{C})$, we write $\overline{A}$ for the matrix whose entries are the complex conjugates of the original entries of A.

We will fix and consider $\Bbbk$, an algebraic number field of finite degree over $\mathbb{Q}$, and denote its ring of integers by $\mathcal{O}$. We denote by $\boldsymbol{p}_\infty$ and $\boldsymbol{p}_f$ the sets of infinite and finite places of $\Bbbk$ respectively and we let $\boldsymbol{p} = \boldsymbol{p}_\infty \cup \boldsymbol{p}_f$. For $\sigma \in \boldsymbol{p}$, we write $\Bbbk_\sigma$ for the completion of $\Bbbk$ at σ, while for any subring $\mathbb{B}$ of $\Bbbk$, the closure of $\mathbb{B}$ in $\Bbbk_\sigma$ will be denoted by $\mathbb{B}_\sigma$. We denote by $\Bbbk_\infty$ the étale $\mathbb{R}$-algebra $\Bbbk \otimes_{\mathbb{Q}} \mathbb{R}$ which we identify with $\prod_{\sigma \in \boldsymbol{p}_\infty} \Bbbk_\sigma$. The ideal class group of $\Bbbk$ will be denoted by $\mathrm{Cl}(\Bbbk)$.

The adele ring and idele group of $\Bbbk$ are denoted by $\mathbb{A}$ and $\mathbb{A}^\times$ respectively. For an adele $a \in \mathbb{A}$ we write a_∞ and a_f for its infinite and finite components respectively. Similarly for any matrix $A = [a_{ij}]_{i,j}$ with elements in $\mathbb{A}$ we write A_∞ to denote the matrix $[(a_{ij})_\infty]_{i,j}$.

For each place σ, we write $|\ |_\sigma$ for the absolute value on $\Bbbk_\sigma$ taken as follows: at each infinite place we use the standard complex absolute value on $\Bbbk_\sigma$, while for $\sigma \in \boldsymbol{p}_f$ we use the normalized absolute value satisfying $|x|_\sigma = |\mathcal{O}_\sigma / \mathfrak{p}_\sigma|^{-1}$ for any arbitrary $x \in \mathfrak{p}_\sigma \setminus \mathfrak{p}_\sigma^2$, where $\mathfrak{p}_\sigma$ is the prime ideal of $\mathcal{O}_\sigma$. For an $a = (a_\sigma) \in \mathbb{A}^\times$ we write $|a|_{\mathbb{A}}$ to denote the idele norm of a, and $|a|_\infty$ for the idele norm of a restricted to $\Bbbk_\infty^\times$, $\prod_{\sigma \in \boldsymbol{p}_\infty} |a_\sigma|_\sigma^{[\Bbbk_\sigma:\mathbb{R}]}$.

Given a finite-dimensional $\Bbbk$-vector space V and $\sigma \in \boldsymbol{p}$, we will write V_σ for the $\Bbbk_\sigma$-vector space $V \otimes_{\Bbbk} \Bbbk_\sigma$. Also we will use the term *$\mathcal{O}$-lattice* in V to mean a finitely generated $\mathcal{O}$-submodule of V containing a $\Bbbk$-basis of V. If L is such an $\mathcal{O}$-lattice in V, we write L_σ to denote the $\mathcal{O}_\sigma$-linear span of L in V_σ when $\sigma \in \boldsymbol{p}_f$.

For an affine algebraic group G defined over $\Bbbk$ and any $\Bbbk$-algebra $\mathbb{B}$, we write $G(\mathbb{B})$ for the set of all $\mathbb{B}$-rational points of G. Also, the set of all $\Bbbk$-rational characters of G will be written as $\boldsymbol{X}^*(G)_{\Bbbk}$. We define $G(\mathbb{A})^1$ to be the set $\{g \in G(\mathbb{A}) : |\chi(g)|_{\mathbb{A}} = 1 \text{ for all } \chi \in \boldsymbol{X}^*(G)_{\Bbbk}\}$.

Lastly given a topological space X and a subset $Y \subset X$, we denote by Y_X° and Y_X^- (or just Y° and Y^- if the underlying space X is clear) the interior and closure of Y in X respectively.

2. The Ryshkov domain of G associated to Q

Let G denote a connected reductive isotropic affine algebraic group over $\Bbbk$, S a fixed maximal $\Bbbk$-split torus of G, and P_0 a minimal $\Bbbk$-parabolic subgroup of G

containing S. Let M_0 be the centralizer of S in G and U_0 the unipotent radical of P_0 so that P_0 has the Levi decomposition $P_0 = M_0U_0$. We consider a relative root system of G with respect to S and denote the set of simple roots with respect to P_0 in this system by $\Delta_{\Bbbk}$.

A $\Bbbk$-parabolic subgroup of G containing P_0 is called a standard $\Bbbk$-parabolic subgroup. A standard $\Bbbk$-parabolic subgroup R has a unique Levi subgroup M_R containing M_0, which gives the Levi decomposition $R = M_R U_R$, where U_R denotes the unipotent radical of R. We write Z_R for the largest central $\Bbbk$-split torus of M_R.

We fix a maximal compact subgroup $K = \prod_{\sigma\in \mathfrak{p}} K_\sigma$ of $G(\mathbb{A})$, where each K_σ is a maximal compact subgroup of $G(\Bbbk_\sigma)$, satisfying the property that for every standard $\Bbbk$-parabolic subgroup R of G,

- $K \cap M_R(\mathbb{A})$ is a maximal compact subgroup in $M_R(\mathbb{A})$,
- $M_R(\mathbb{A}) = (M_R(\mathbb{A}) \cap U_0(\mathbb{A}))\, M_0(\mathbb{A})\, (K \cap M_R(\mathbb{A}))$ (Iwasawa decomposition) holds.

Consider a standard proper maximal $\Bbbk$-parabolic subgroup Q of G, which we now fix. There exists a unique simple root in $\Delta_{\Bbbk}$ that restricts nontrivially on Z_Q, which we denote by χ_0. Let m_Q be the positive integer such that $m_Q^{-1}\chi_0|_{Z_Q}$ is a $\mathbb{Z}$-basis of the $X^*(Z_Q/Z_G)_{\Bbbk}$. We write χ_Q for the character

$$[X^*(Z_Q/Z_G)_{\Bbbk} : X^*(M_Q/Z_G)_{\Bbbk}]\, m_Q^{-1}(\chi_0|_{Z_Q}),$$

which is a $\mathbb{Z}$-basis for $X^*(M_Q/Z_G)_{\Bbbk}$.

Next we define the map

$$z_Q : G(\mathbb{A}) \ni umh \longmapsto Z_G(\mathbb{A})M_Q(\mathbb{A})^1 m \in Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A}),$$

where $u \in U_Q(\mathbb{A}),\ m \in M_Q(\mathbb{A}),\ h \in K$. This is a well-defined left $Q(\mathbb{A})^1$-invariant map, which gives rise to the following map, which we also denote by z_Q:

$$Q(\mathbb{A})^1\backslash G(\mathbb{A})^1 \ni Q(\mathbb{A})^1 g \longmapsto z_Q(g) \in M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1).$$

Here we have used $Z_G(\mathbb{A})^1 = Z_G(\mathbb{A}) \cap G(\mathbb{A})^1 \subset M_Q(\mathbb{A})^1$.

We can now define the *height function* $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ by

$$H_Q(g) = |\chi_Q(z_Q(g))|_{\mathbb{A}}^{-1}, \quad g \in G(\mathbb{A}),$$

as well as the *Hermite function* $\mathsf{m}_Q : G(\mathbb{A})^1 \to \mathbb{R}_{>0}$ by

$$\mathsf{m}_Q(g) = \min_{x\in Q(\Bbbk)\backslash G(\Bbbk)} H_Q(xg), \quad g \in G(\mathbb{A})^1.$$

Definition [Watanabe 2014, §4]. The set R_Q defined by

$$\{g \in G(\mathbb{A})^1 : \mathsf{m}_Q(g) = H_Q(g)\}$$

is called the *Ryshkov domain of* m_Q.

3. Fundamental domains of $G(\Bbbk)\backslash G(\mathbb{A})^1$ and $\Gamma_i\backslash G(\Bbbk_\infty)^1$

Definition. Let T be a locally compact Hausdorff space and Γ a discrete group with a properly discontinuous action on T. An open subset Ω of T satisfying

(i) $T = \Gamma\Omega^-$,

(ii) $\Omega \cap \gamma\Omega^- = \varnothing$ for all $\gamma \in \Gamma \setminus \{e\}$

is called an *open fundamental domain of T with respect to* Γ. (Here we have assumed that Γ acts on T from the left. In the case of a right action the same definition holds with the group action written on the right instead.)

We call a subset F of T a *fundamental domain of T with respect to* Γ, or simply a *fundamental domain of* $\Gamma\backslash T$ (T/Γ in the case of a right action) if there exists an open fundamental domain Ω of T with respect to Γ such that $\Omega \subset F \subset \Omega^-$.

Further Notation. Hereafter we will use the following notation:

- $K_\infty = \prod_{\sigma\in \boldsymbol{p}_\infty} K_\sigma,\ \ K_f = \prod_{\sigma\in\boldsymbol{p}_f} K_\sigma,$
- $G_{\mathbb{A},\infty} = G(\Bbbk_\infty)\times K_f,\ \ G^1_{\mathbb{A},\infty} = G_{\mathbb{A},\infty}\cap G(\mathbb{A})^1,$
- $G(\Bbbk_\infty)^1 = G(\Bbbk_\infty)\cap G(\mathbb{A})^1$, where we identify $G(\Bbbk_\infty)$ with the subgroup $\{g \in G(\mathbb{A}) : g_f = e\}$ of $G(\mathbb{A})$.

We will denote the *class number of* G, i.e., the finite number $|G(\Bbbk)\backslash G(\mathbb{A})/G_{\mathbb{A},\infty}|$, by n_G. We note here that $|G(\Bbbk)\backslash G(\mathbb{A})^1/G^1_{\mathbb{A},\infty}|$ is also equal to n_G.

The case when G is of class number 1 is discussed in [Watanabe 2014], where a fundamental domain for $G(\Bbbk_\infty)^1$ with respect to the group $G(\Bbbk)\cap G_{\mathbb{A},\infty}$ is determined. In the following we discuss and obtain a similar fundamental domain in the general case.

We take a complete set of representatives $\{\eta_1,\dots,\eta_{n_G}\}$ for $G(\Bbbk)\backslash G(\mathbb{A})^1/G^1_{\mathbb{A},\infty}$. Then, for $i = 1,\dots,n_G$, define the groups

$$G_i = \eta_i G^1_{\mathbb{A},\infty}\eta_i^{-1} \quad\text{and}\quad \Gamma_i = G_i \cap G(\Bbbk).$$

We note that since $(\eta_i)_\infty G(\Bbbk_\infty)^1(\eta_i)_\infty^{-1} = G(\Bbbk_\infty)^1$, we can also write G_i as $G(\Bbbk_\infty)^1 \times (\eta_i)_f K_f (\eta_i)_f^{-1}$ or $G(\Bbbk_\infty)^1\eta_i K_f \eta_i^{-1}$.

From $G(\mathbb{A})^1 = \bigsqcup_{i=1}^{n_G} G(\Bbbk)\eta_i G^1_{\mathbb{A},\infty} = \bigsqcup_{i=1}^{n_G} G(\Bbbk)G_i\eta_i$ we have

$$G(\Bbbk)\backslash G(\mathbb{A})^1 = \bigsqcup_{i=1}^{n_G} \Gamma_i\backslash G_i\eta_i = \bigsqcup_{i=1}^{n_G} \Gamma_i\backslash (G(\Bbbk_\infty)^1\eta_i K_f),$$

which gives us the isomorphism

$$G(\Bbbk)\backslash G(\mathbb{A})^1/K \simeq \bigsqcup_{i=1}^{n_G} \Gamma_i\backslash G(\Bbbk_\infty)^1/K_\infty.$$

Also for each $i = 1, \dots, n_G$ we take a complete set of representatives $\{\xi_{ij}\}_{j=1}^{h_i}$ for $Q(\Bbbk)\backslash G(\Bbbk)/\Gamma_i$ (where the number of double cosets h_i is finite; see [Borel 1963, §7]) and define groups

$$Q_{i,j} = Q \cap \xi_{ij}\Gamma_i\xi_{ij}^{-1} = Q(\Bbbk) \cap \xi_{ij}G_i\xi_{ij}^{-1}$$

and the sets

$$R_{i,j,\infty} = \{g \in G(\Bbbk_\infty)^1 : \mathsf{m}_Q(g\xi_{ij}\eta_i) = H_Q(g\xi_{ij}\eta_i)\}$$

for $j = 1, \dots, h_i$. Also since $G_i = G(\Bbbk_\infty)^1\eta_i K_f \eta_i^{-1}$ as previously noted,

$$\xi_{ij}G_i\xi_{ij}^{-1} = \xi_{ij}G(\Bbbk_\infty)^1\eta_i K_f \eta_i^{-1}\xi_{ij}^{-1} = G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f \eta_i^{-1}\xi_{ij}^{-1}.$$

Lemma 1.
$$G(\mathbb{A})^1 = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i} Q(\Bbbk)G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f.$$

Proof. We first show that for a fixed i the union $\bigcup_{j=1}^{h_i} Q(\Bbbk)\xi_{ij}G_i\eta_i$ is disjoint. Suppose for some $1 \le j, j' \le h_i$ that $Q(\Bbbk)\xi_{ij}G_i\eta_i \cap Q(\Bbbk)\xi_{ij'}G_i\eta_i$ is nonempty. Then there exist $q, q' \in Q(\Bbbk)$ and $g, g' \in G_i$ such that $q\xi_{ij}g = q'\xi_{ij'}g'$. Rearranging gives us $gg'^{-1} = \xi_{ij}^{-1}q^{-1}q'\xi_{ij'} \in G_i \cap G(\Bbbk) = \Gamma_i$. This shows that $Q(\Bbbk)\xi_{ij'}\Gamma_i = Q(\Bbbk)\xi_{ij}\Gamma_i$, implying $j = j'$. The result then follows from

$$\begin{aligned} G(\mathbb{A})^1 &= \bigsqcup_i G(\Bbbk)\eta_i G^1_{\mathbb{A},\infty} = \bigsqcup_i G(\Bbbk)G_i\eta_i \\ &= \bigsqcup_i \Bigl(\bigsqcup_j Q(\Bbbk)\xi_{ij}\Gamma_i\Bigr)G_i\eta_i \subset \bigsqcup_i\bigsqcup_j Q(\Bbbk)\xi_{ij}G_i\eta_i \end{aligned}$$

and $\xi_{ij}G_i\eta_i = G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f$. □

The lemma also gives us the disjointedness of the union in the following result.

Proposition 2.
$$R_Q = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i} Q(\Bbbk)R_{i,j,\infty}\xi_{ij}\eta_i K_f.$$

Proof. From the previous lemma, we see that any $g \in G(\mathbb{A})^1$ can be written as $qg'\xi_{ij}\eta_i h$ for some i, j and $q \in Q(\Bbbk)$, $g' \in G(\Bbbk_\infty)^1$, $h \in K_f$. Since both H_Q and m_Q are left $Q(\Bbbk)$-invariant and right K-invariant, we see that

$$H_Q(g) = H_Q(g'\xi_{ij}\eta_i), \quad \mathsf{m}_Q(g) = \mathsf{m}_Q(g'\xi_{ij}\eta_i).$$

Hence $g \in R_Q$ if and only if $g' \in R_{i,j,\infty}$. □

The following two lemmas hold for any fixed $1 \le i \le n_G$ and $1 \le j \le h_i$.

Lemma 3. *Let $q \in Q(\Bbbk)$. If the sets $q(G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f)$ and $G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f$ intersect, then $q \in Q_{i,j}$.*

Proof. Suppose that $g \in q(G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f) \cap (G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f)$. By rewriting $G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f$ as $\xi_{ij}G_i\eta_i$, we have $q^{-1}g,\ g \in \xi_{ij}G_i\eta_i$, from which we get $q^{-1} \in \xi_{ij}G_i\xi_{ij}^{-1}$. Hence $q \in Q(\Bbbk) \cap \xi_{ij}G_i\xi_{ij}^{-1} = Q_{i,j}$. □

Lemma 4. $$Q_{i,j}(R_{i,j,\infty}\xi_{ij}\eta_i K_f) = R_{i,j,\infty}\xi_{ij}\eta_i K_f.$$

Proof. Consider $q \in Q_{i,j}$ and $g \in R_{i,j,\infty}$. Since $q \in G(\Bbbk_\infty)^1\xi_{ij}\eta_i K_f \eta_i^{-1}\xi_{ij}^{-1}$, we have $q_f \in (\xi_{ij}\eta_i)K_f(\xi_{ij}\eta_i)^{-1}$. Let $q_f = (\xi_{ij}\eta_i)h(\xi_{ij}\eta_i)^{-1}$, with $h \in K_f$. Then

$$H_Q((q_\infty g)\xi_{ij}\eta_i) = H_Q(q_\infty g(\xi_{ij}\eta_i)h) = H_Q(q_\infty g q_f(\xi_{ij}\eta_i)) = H_Q(qg\xi_{ij}\eta_i),$$

which is equal to $H_Q(g\xi_{ij}\eta_i)$. Similarly

$$\mathsf{m}_Q((q_\infty g)\xi_{ij}\eta_i) = \mathsf{m}_Q(q_\infty g q_f \xi_{ij}\eta_i) = \mathsf{m}_Q(qg\xi_{ij}\eta_i) = \mathsf{m}_Q(g\xi_{ij}\eta_i);$$

thus $q_\infty g \in R_{i,j,\infty}$. Finally $q_f\xi_{ij}\eta_i K_f \subset \xi_{ij}\eta_i K_f$. Hence we get $q(g\xi_{ij}\eta_i K_f) \subset R_{i,j,\infty}\xi_{ij}\eta_i K_f$, as required. □

By taking a complete set of representatives $\{\theta_{ijk}\}_k$ for $Q(\Bbbk)/Q_{i,j}$ and using both Proposition 2 and Lemma 4, we obtain

$$R_Q = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i} Q(\Bbbk)R_{i,j,\infty}\xi_{ij}\eta_i K_f = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i}\Bigl(\bigsqcup_k \theta_{ijk}Q_{i,j}\Bigr)R_{i,j,\infty}\xi_{ij}\eta_i K_f = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i}\bigsqcup_k \theta_{ijk}R_{i,j,\infty}\xi_{ij}\eta_i K_f, \tag{1}$$

where the final unions are disjoint as a result of Lemma 3.

Denote $(R_{i,j,\infty}^\circ)^-$ by $R_{i,j,\infty}^*$, where the interior and closure is taken in $G(\Bbbk_\infty)^1$. Similarly write R_Q^* for $(R_Q^\circ)^-$ in $G(\mathbb{A})^1$. From (1) we have

$$R_Q^* = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i}\bigsqcup_k \theta_{ijk}R_{i,j,\infty}^*\xi_{ij}\eta_i K_f. \tag{2}$$

Taking open fundamental domains $\Omega_{i,j,\infty}$ of $R_{i,j,\infty}^*$ with respect to $Q_{i,j}$ for each $i = 1, \dots, n_G$ and $j = 1, \dots, h_i$, we consider the set

$$\Omega = \bigsqcup_{i=1}^{n_G}\bigsqcup_{j=1}^{h_i}\Omega_{i,j,\infty}\xi_{ij}\eta_i K_f.$$

Theorem 5. *Ω is an open fundamental domain of R_Q^* with respect to $Q(\Bbbk)$.*

Corollary 6. *Ω° ($= \Omega^\circ_{G(\mathbb{A})^1}$) is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\Bbbk)$.*

Proof. From (2) we have

$$R_Q^* = \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \bigsqcup_{k} \theta_{ijk} R^*_{i,j,\infty} \xi_{ij} \eta_i K_f = \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \bigsqcup_{k} \theta_{ijk} (Q_{i,j} \Omega^-_{i,j,\infty}) \xi_{ij} \eta_i K_f$$
$$= \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} Q(\Bbbk) \Omega^-_{i,j,\infty} \eta_i K_f = Q(\Bbbk)\Omega^-.$$

Now suppose $\Omega \cap q\Omega^- \neq \varnothing$ for $q \in Q(\Bbbk)$. So for some i, i', j, j' we must have $q(\Omega_{i,j,\infty}\xi_{ij}\eta_i K_f) \cap (\Omega^-_{i',j',\infty}\xi_{i'j'}\eta_{i'}K_f) \neq \varnothing$. Writing $q = \theta_{ijk}q'$ with $q' \in Q_{i,j}$ and some k, we have

$$\theta_{ijk}(q')_\infty \Omega_{i,j,\infty}\xi_{ij}\eta_i K_f \cap \Omega^-_{i',j',\infty}\xi_{i'j'}\eta_{i'}K_f \neq \varnothing$$

since $(q')_f \xi_{ij}\eta_i K_f \subset \xi_{ij}\eta_i K_f$. Then (2) implies $i = i'$, $j = j'$, and $\theta_{ijk} = e$. Thus $\Omega_{i,j,\infty} \cap (q')_\infty \Omega^-_{i,j,\infty} = \Omega_{i,j,\infty} \cap q'\Omega^-_{i,j,\infty}$ must be nonempty, which means $q' = e$ and hence $q = e$. This proves the theorem, and the corollary follows from [Watanabe 2014, Theorem 15]. □

Finally, for any fixed $1 \leq i \leq n_G$, we have the following theorem.

Theorem 7. *The set* $\Omega_{i,\infty} = \bigcup_{j=1}^{h_i} \xi_{ij}^{-1}\Omega_{i,j,\infty}\xi_{ij}$ *is a fundamental domain of* $G(\Bbbk_\infty)^1$ *with respect to* Γ_i.

Proof. The following proof was suggested by Professor Watanabe. To show that $G(\Bbbk_\infty)^1 = \Gamma_i \Omega^-_{i,\infty}$, consider an arbitrary $g \in G(\Bbbk_\infty)^1$. From Corollary 6,

$$G(\mathbb{A})^1 = G(\Bbbk)\Omega^- = G(\Bbbk) \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \Omega^-_{i,j,\infty}\xi_{ij}\eta_i K_f$$
$$= G(\Bbbk) \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \xi_{ij}(\xi_{ij}^{-1}\Omega^-_{i,j,\infty}\xi_{ij})\eta_i K_f \subset G(\Bbbk) \bigcup_{i=1}^{n_G} \Omega^-_{i,\infty}\eta_i K_f,$$

so we may write $g\eta_i = g'\omega\eta_i h$ with $g' \in G(\Bbbk)$, $\omega \in \Omega^-_{i,\infty}$ and $h \in K_f$. Rearranging we get $g' = (g\omega^{-1})(\eta_i h^{-1}\eta_i^{-1})$, which belongs to $G(\Bbbk_\infty)^1\eta_i K_f \eta_i^{-1} = G_i$. Hence $g' \in \Gamma_i$. Since $g = (g'\omega)(\eta_i h \eta_i^{-1})$ and $g \in G(\Bbbk_\infty)^1$, we know $\eta_i h \eta_i^{-1}$ must necessarily be trivial. Thus $g \in \Gamma_i \Omega^-_{i,\infty}$.

Now suppose that $\Omega^\circ_{i,\infty} \cap g\Omega^-_{i,\infty}$ is nonempty for a $g \in \Gamma_i$. Then we must have $\xi_{ij}^{-1}\Omega^\circ_{i,j,\infty}\xi_{ij} \cap g\xi_{ij'}^{-1}\Omega^-_{i,,j',\infty}\xi_{ij'} \neq \varnothing$ for some j, j'. Since $g_f\eta_i K_f = \eta_i K_f$,

$$\xi_{ij}^{-1}\Omega^\circ_{i,j,\infty}\xi_{ij} \cap g\xi_{ij'}^{-1}\Omega^-_{i,,j',\infty}\xi_{ij'} \neq \varnothing$$
$$\Rightarrow \quad (\Omega_{i,j,\infty}\xi_{ij}\eta_i K_f)^\circ \cap \xi_{ij} g \xi_{ij'}^{-1}(\Omega_{i,j',\infty}\xi_{ij'}\eta_i K_f)^- \neq \varnothing$$
$$\Rightarrow \quad \Omega^\circ \cap (\xi_{ij} g \xi_{ij'}^{-1})\Omega^- \neq \varnothing,$$

and thus $\xi_{ij} g \xi_{ij'}^{-1} = e$ by Corollary 6. Hence $Q(\Bbbk)\xi_{ij}\Gamma_i = Q(\Bbbk)\xi_{ij'}\Gamma_i$, which implies $j = j'$ whereby $g = \xi_{ij}^{-1}\xi_{ij'} = e$. □

4. The case $G = \mathrm{GL}_n$

We will now consider the case where G is a general linear group GL_n defined over $\Bbbk$. We use the group of diagonal matrices as the maximal $\Bbbk$-split torus S, and the group of upper triangular matrices in G as the minimal $\Bbbk$-parabolic subgroup P_0. Also fixing an integer $1 \le m < n$, we will consider the maximal standard $\Bbbk$-parabolic subgroup Q defined by

$$Q(\Bbbk) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a \in \mathrm{GL}_m(\Bbbk), b \in M_{m,n-m}(\Bbbk), d \in \mathrm{GL}_{n-m}(\Bbbk) \right\}$$

and the Levi subgroup M_Q is given by

$$M_Q(\Bbbk) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a \in \mathrm{GL}_m(\Bbbk), d \in \mathrm{GL}_{n-m}(\Bbbk) \right\}.$$

For the maximal compact subgroup K of $G(\mathbb{A})$ let $K = K_\infty \times K_f$, where

$$K_\infty = \{g \in \mathrm{GL}_n(\Bbbk_\infty) : {}^t\bar{g}g = I_n\}, \quad K_f = \prod_{\sigma \in \boldsymbol{p}_f} \mathrm{GL}_n(\mathcal{O}_\sigma).$$

Here we identify $\mathrm{GL}_n(\Bbbk_\infty)$ with $\prod_{\sigma\in\boldsymbol{p}_\infty} \mathrm{GL}_n(\Bbbk_\sigma)$, and for $g = (g_\sigma)_{\sigma\in\boldsymbol{p}_\infty} \in \mathrm{GL}_n(\Bbbk_\infty)$ we write ${}^t\bar{g}$ for the element $({}^t\bar{g}_\sigma)_{\sigma\in\boldsymbol{p}_\infty}$ of $\mathrm{GL}_n(\Bbbk_\infty)$.

The character χ_Q described in the first section is then given by

$$\chi_Q\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = (\det a)^{(n-m)/l}(\det d)^{-m/l}$$

and the height function H_Q by

$$H_Q\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = |\det a|_{\mathbb{A}}^{-(n-m)/l}|\det d|_{\mathbb{A}}^{m/l},$$

where l is the greatest common divisor of $n - m$ and m.

We shall see that in this case the number of double cosets of $Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)/\Gamma_i$ for each i is invariant and equal to $|\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty}|$, the class number of GL_n.

Denote the set of all $\mathcal{O}$-lattices in $\Bbbk^r$ $(r \ge 1)$ by $\mathfrak{L}_r$, and the standard unit vectors of $\Bbbk^r$ by $\boldsymbol{e}_1^{(r)}, \dots, \boldsymbol{e}_r^{(r)}$. For this section we simply write $\mathfrak{L}$ for $\mathfrak{L}_n$ and $\boldsymbol{e}_k$ for $\boldsymbol{e}_k^{(n)}$ $(1 \le k \le n)$.

For $L \in \mathfrak{L}_r$ and $g = (g_\sigma)_{\sigma\in\boldsymbol{p}} \in \mathrm{GL}_r(\mathbb{A})$ put

$$gL = \left((\Bbbk_\infty)^r \times \prod_{\sigma\in\boldsymbol{p}_f} g_\sigma L_\sigma\right) \cap \Bbbk^r \in \mathfrak{L}_r. \tag{3}$$

This defines a transitive left action of $\mathrm{GL}_r(\mathbb{A})^1$ on $\mathfrak{L}_r$. Note that if $g \in \mathrm{GL}_r(\Bbbk)$ then gL as defined above coincides with the usual image of L under the linear transformation $v \mapsto gv$ of $\Bbbk^r$. The subset of $\mathfrak{L}$ consisting of all $\mathcal{O}$-lattices of the form gL with $g \in \mathrm{GL}_n(\Bbbk)$ will be referred to as the *$\mathcal{O}$-lattice class of L* or just the lattice class of L in $\mathfrak{L}$.

There is known to be a one-to-one correspondence between the $\mathcal{O}$-lattice classes in $\mathfrak{L}$ and the double cosets in $\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty}$, which we give explicitly later on in this section. For now we note that this means the number of distinct lattice classes in $\mathfrak{L}$ and the class number $|\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty}|$ are equal.

Lemma 8. *Let L be an $\mathcal{O}$-lattice in a $\Bbbk$-vector space V of dimension $s \geq 1$. Then there exists a $\Bbbk$-basis $\{x_j\}_{j=1}^s$ of V and s fractional ideals $A_1, \dots, A_s$ such that $L = A_1x_1 + \cdots + A_sx_s$. Moreover*:

(i) *If W is a $\Bbbk$-subspace of V of dimension $r \leq s$, the x_j can be chosen such that $x_1, \dots, x_r \in W$.*

(ii) *The ideal class of $A_1 \cdots A_s$ is uniquely determined by the isomorphism class of L as an $\mathcal{O}$-module. In particular, $L \simeq \left(\bigoplus_{j=1}^{s-1} \mathcal{O}\right) \oplus (A_1 \cdots A_s)$.*

(iii) *In the case $V \subseteq \Bbbk^n$ $(s \leq n)$, we can find $g \in \mathrm{GL}_n(\Bbbk)$ such that*

$$gL = \left(\sum_{j=1}^{s-1} \mathcal{O}\boldsymbol{e}_j\right) + (A_1 \cdots A_s)\boldsymbol{e}_s.$$

Proof. See [Shimura 2010, Theorem 10.19]. We prove (iii) here. Consider the case $s = 2$, where $L = A_1x_1 + A_2x_2$. We can find $k_1, k_2 \in \Bbbk^\times$ such that $A'_1 = k_1A_1$ and $A'_2 = k_2A_2$ are integral ideals and $A'_1 + A'_2 = \mathcal{O}$ [Shimura 2010, Lemma 10.15(i)]. Let g' be the matrix formed by substituting the first two columns of the $n \times n$ unit matrix with $k_1^{-1}x_1$ and $k_2^{-1}x_2$. Then $g'^{-1}L = A'_1e_1 + A'_2e_2$. Next let

$$g'' = \begin{bmatrix} 1 & 1 & \\ -a_2 & a_1 & \\ & & I_{n-2} \end{bmatrix},$$

where $a_1 \in A'_1$ and $a_2 \in A'_2$ are taken such that $a_1 + a_2 = 1$. It is easily verified that $g''(A'_1e_1 + A'_2e_2) = Oe_1 + A'_1A'_2e_2$. Hence $g = \mathrm{diag}(1, k_1^{-1}k_2^{-1}, 1, \dots, 1)g''g'^{-1}$ maps L to $\mathcal{O}e_1 + A_1A_2e_2$. The general case when $s > 2$ follows inductively from this result. □

The ideal class associated to the $\mathcal{O}$-lattice L mentioned above in (ii) is known as the *Steinitz class of L*, denoted by $\lambda(L)$. We may also speak of the Steinitz class of an entire lattice class in $\mathfrak{L}$ since every $\mathcal{O}$-lattice in a lattice class has the same Steinitz class.

It follows directly that mapping each lattice class to its Steinitz class gives a bijection between the set of lattice classes in $\mathfrak{L}$ and $\mathrm{Cl}(\Bbbk)$. As a result the class number of GL_n, which we have noted to be equivalent to the number of distinct lattice classes in $\mathfrak{L}$, is equal to the class number of $\Bbbk$, which we write as h.

We now proceed to prove that $h_i = |Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)/\Gamma_i|$ is also equal to h for every $i = 1, \dots, h$. As we did in the previous section, let $\{\eta_1, \dots, \eta_h\}$ be a complete set of representatives for $\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty}$. Then for each $i = 1, \dots, h$ put $L_i = \eta_i(\mathcal{O}\boldsymbol{e}_1 + \cdots + \mathcal{O}\boldsymbol{e}_n) \in \mathfrak{L}$.

Next we identify $Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)$ with the set of all m-dimensional linear subspaces of $\Bbbk^n$ denoted by Gr_m (the Grassmannian) via the bijection

$$Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk) \ni Q(\Bbbk)g \longmapsto g^{-1}\left(\sum_{k=1}^{m} \Bbbk\boldsymbol{e}_k\right) \in \mathrm{Gr}_m. \tag{4}$$

From here up to the end of Theorem 11 we fix $i \in \{1, \dots, h\}$. Considering the left action of $\Gamma_i \subset \mathrm{GL}_n(\Bbbk)$ on Gr_m, the map (4) gives rise to the bijection

$$Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)/\Gamma_i \ni Q(\Bbbk)g\Gamma_i \longmapsto \Gamma_i g^{-1}\left(\sum_{k=1}^{m} \Bbbk\boldsymbol{e}_k\right) \in \Gamma_i\backslash\mathrm{Gr}_m, \tag{5}$$

which lets us identify $Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)/\Gamma_i$ with $\Gamma_i\backslash\mathrm{Gr}_m$.

Lemma 9. *Γ_i is the stabilizer of L_i in $\mathrm{GL}_n(\Bbbk)$, under the action of $\mathrm{GL}_n(\mathbb{A})^1$ on $\mathfrak{L}$, i.e.,*

$$\Gamma_i = \{g \in \mathrm{GL}_n(\Bbbk) : gL_i = L_i\}.$$

Proof. Since $\Gamma_i = \left(\mathrm{GL}_n(\Bbbk_\infty) \times \eta_i \prod_{\sigma\in\boldsymbol{p}_f} \mathrm{GL}_n(\mathcal{O}_\sigma)\eta_i^{-1}\right) \cap \mathrm{GL}_n(\Bbbk)$, this is obvious from our choice of L_i. ☐

Proposition 10. *Let $V_1, V_2 \in \mathrm{Gr}_m$ and put $\tilde{L}_1 = L_i \cap V_1$, $\tilde{L}_2 = L_i \cap V_2$, which are $\mathcal{O}$-lattices in V_1 and V_2 respectively. Then $\lambda(\tilde{L}_1) = \lambda(\tilde{L}_2)$ if and only if there exists $g \in \Gamma_i$ such that $V_1 = gV_2$.*

Proof. Suppose that $V_1 = gV_2$ for some $g \in \Gamma_i$. From Lemma 8 we can find a $\Bbbk$-basis $\{y_j\}_{j=1}^n$ for $\Bbbk^n$ contained in L_i with $y_1, \dots, y_m \in V_2$. Put $x_j = gy_j$ for $j = 1, \dots, m$. Then $\{x_j\}_{j=1}^m$ and $\{y_j\}_{j=1}^m$ span V_1 and V_2 respectively and since g stabilizes L_i, they are also contained in $\tilde{L}_1$ and $\tilde{L}_2$ respectively.

For $v \in V_1$ and $w \in V_2$, we write $(\alpha_v)_j$ and $(\beta_w)_j$ for the $\Bbbk$-coefficients of x_j and y_j in v and w respectively $\left(\text{so } v = \sum_{j=1}^m (\alpha_v)_j x_j \text{ and } w = \sum_{j=1}^m (\beta_w)_j y_j\right)$. Let J_1 be the fractional ideal generated by $\{\det[(\alpha_{v_j})_l]_{j,l=1}^m \mid v_1, \dots, v_m \in \tilde{L}_1\}$. We can show that the ideal class of J_1 in $\mathrm{Cl}(\Bbbk)$ is $\lambda(\tilde{L}_1)$ as follows: From the lemma above we have $\tilde{L}_1 = A_1x'_1 + \cdots + A_mx'_m$, with fractional ideals $A_1, \dots, A_m$ and $\{x'_j\}_{j=i}^m$ a basis of V_1. Comparing $\bigwedge_{j=1}^m \tilde{L}_1 = A_1 \cdots A_m(x'_1 \wedge \cdots \wedge x'_m)$ with

$$\textstyle\bigwedge_{j=1}^m \tilde{L}_1 = \Bbbk\text{-span of } \{v_1 \wedge \cdots \wedge v_m \mid v_1, \dots, v_m \in \tilde{L}_1\} = J_1(x_1 \wedge \cdots \wedge x_m),$$

we see that $A_1 \cdots A_m$ is a $\Bbbk^\times$-multiple of J_1; hence their ideal classes are equivalent.

Similarly $\lambda(\tilde{L}_2)$ is the ideal class of the fractional ideal J_2 generated by the $\det[(\beta_{w_j})_l]_{j,l=1}^m$ for all $w_1, \dots, w_m \in \tilde{L}_2$. However, since any arbitrary $v \in \tilde{L}_1$ can be written as gw with some $w \in \tilde{L}_2$ and

$$v = gw \iff \sum_{j=1}^m (\alpha_v)_j x_j = g\left(\sum_{j=1}^m (\beta_w)_j y_j\right) = \sum_{j=1}^m (\beta_w)_j g y_j = \sum_{j=1}^m (\beta_w)_j x_j$$
$$\iff (\alpha_v)_j = (\beta_w)_j, \quad j = 1, \dots, m,$$

this shows that $J_1 = J_2$ and thus $\lambda(\tilde{L}_1) = \lambda(\tilde{L}_2)$.

Now suppose conversely that $\lambda(\tilde{L}_1) = \lambda(\tilde{L}_2)$. Using Lemma 8, we obtain $\Bbbk$-bases $\{x_j\}_{j=1}^n$, $\{y_j\}_{j=1}^n$ for $\Bbbk^n$ and fractional ideals $A_1, \dots, A_n, B_1, \dots, B_n$ such that $L_i = A_1 x_1 + \cdots + A_n x_n = B_1 y_1 + \cdots + B_n y_n$ and $x_1, \dots, x_m \in V_1$, $y_1, \dots, y_m \in V_2$. Since $\tilde{L}_1 = A_1 x_1 + \cdots + A_m x_m$ and $\tilde{L}_2 = B_1 y_1 + \cdots + B_m y_m$, the ideal classes of $A_1 \cdots A_m$ and $B_1 \cdots B_m$ are equivalent, and hence so are those of $A_{m+1} \cdots A_n$ and $B_{m+1} \cdots B_n$. By substituting the basis vectors and fractional ideals with suitable $\Bbbk^\times$-multiples, we may assume that $A_1 \cdots A_m = B_1 \cdots B_m$ and $A_{m+1} \cdots A_n = B_{m+1} \cdots B_n$.

Finally using Lemma 8(iii) we can find $g_1, g_2 \in \mathrm{GL}_n(\Bbbk)$ satisfying

$$g_1 L_i = \sum_{j=1}^{m-1} \mathcal{O}\boldsymbol{e}_j + (A_1 \cdots A_m)\boldsymbol{e}_m + \sum_{j=m+1}^{n-1} \mathcal{O}\boldsymbol{e}_j + (A_{m+1} \cdots A_n)\boldsymbol{e}_n,$$
$$g_2 L_i = \sum_{j=1}^{m-1} \mathcal{O}\boldsymbol{e}_j + (B_1 \cdots B_m)\boldsymbol{e}_m + \sum_{j=m+1}^{n-1} \mathcal{O}\boldsymbol{e}_j + (B_{m+1} \cdots B_n)\boldsymbol{e}_n,$$

chosen such that

$$g_1 \tilde{L}_1 = \sum_{j=1}^{m-1} \mathcal{O}\boldsymbol{e}_j + (A_1 \cdots A_m)\boldsymbol{e}_m, \quad g_2 \tilde{L}_2 = \sum_{j=1}^{m-1} \mathcal{O}\boldsymbol{e}_j + (B_1 \cdots B_m)\boldsymbol{e}_m.$$

Put $g = g_1^{-1} g_2$. Since $g_1 L_i = g_2 L_i$, the previous lemma gives us $g \in \Gamma_i$, while $gV_2 = V_1$ follows from $gy_j \in g\tilde{L}_2 = \tilde{L}_1 \subset V_1$ $(j = 1, \dots, m)$. □

Finally we consider the map

(6) $$\lambda_i : \Gamma_i \backslash \mathrm{Gr}_m \to \mathrm{Cl}(\Bbbk), \qquad \lambda_i(\Gamma_i V) = \lambda(L_i \cap V) \quad (V \in \mathrm{Gr}_m),$$

which is well-defined and injective as a result of the previous proposition.

Theorem 11. $$h_i = h.$$

Proof. Since $h_i = |Q(\Bbbk) \backslash \mathrm{GL}_n(\Bbbk) / \Gamma_i| = |\Gamma_i \backslash \mathrm{Gr}_m|$ we only need to prove that λ_i is surjective.

Take any ideal class in $\mathrm{Cl}(\Bbbk)$ and let A be a fractional ideal representing this class. Also let B be a fractional ideal representing $\lambda(L_i)$. Lemma 8(iii) allows us to find $g \in \mathrm{GL}_n(\Bbbk)$ such that

$$gL_i = \sum_{1\le k<n-1} \mathcal{O}\boldsymbol{e}_k + A\boldsymbol{e}_{n-1} + A^{-1}B\boldsymbol{e}_n.$$

Let V be the subspace of $\Bbbk^n$ spanned by $\boldsymbol{e}, \dots, \boldsymbol{e}_{m-1}, \boldsymbol{e}_{n-1}$ and put $V' = g^{-1}V \in \mathrm{Gr}_m$. Then $L_i \cap V' \simeq \big(\bigoplus_{j=1}^{m-1} \mathcal{O}\big) \oplus A$ so $\lambda_i(\Gamma_i V') = \lambda(L_i \cap V')$ is the class of A in $\mathrm{Cl}(\Bbbk)$, as required. □

The one-to-one correspondence between $\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty}$ and the set of $\mathcal{O}$-lattices classes in $\mathfrak{L}$ mentioned earlier in the section is given by mapping each η_i to the lattice class of L_i. That this is a bijection follows from $G^1_{\mathbb{A},\infty}$ being the stabilizer group of the $\mathcal{O}$-lattice $\mathcal{O}\boldsymbol{e}_1 + \cdots + \mathcal{O}\boldsymbol{e}_n$ under the action of $\mathrm{GL}_n(\mathbb{A})^1$ on $\mathfrak{L}$. Continuing this map to the Steinitz class of the lattice gives us the bijection

$$\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1/G^1_{\mathbb{A},\infty} \ni \eta_i \mapsto \lambda(L_i) \in \mathrm{Cl}(\Bbbk).$$

This gives us an explicit way to find candidates for $\{\eta_1, \dots, \eta_h\}$ as follows. Let $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ be a complete set of fractional ideals representing the ideal class of $\Bbbk$. For each $i = 1, \dots, h$, we shall require an element $\eta_i \in \mathrm{GL}_n(\mathbb{A})^1$ such that the Steinitz class of the resulting lattice $L_i = \eta_i\big(\sum_{k=1}^n \mathcal{O}\boldsymbol{e}_k\big)$ is the ideal class represented by $\mathfrak{a}_i$.

Let $D_n(x)$ $(x \in \mathbb{A})$ denote the unit matrix of size n with bottom-most diagonal entry replaced by x. For each $1 \le i \le h$ we can choose $\alpha_i \in \mathbb{A}^\times$ such that $\alpha_{i\sigma}$ generates the principal ideal $\mathfrak{a}_i\mathcal{O}_\sigma$ for every finite σ and $|\alpha_i|_\infty = N(\mathfrak{a}_i)$, the ideal norm of $\mathfrak{a}_i$. Then $D_n(\alpha_i) \in \mathrm{GL}_n(\mathbb{A})^1$ since $|\det D_n(\alpha_i)|_{\mathbb{A}} = |\alpha_i|_{\mathbb{A}} = 1$, and

$$D_n(\alpha_i)\bigg(\sum_{k=1}^n \mathcal{O}\boldsymbol{e}_k\bigg) = \sum_{1\le k<n} \mathcal{O}\boldsymbol{e}_k + \mathfrak{a}_i\boldsymbol{e}_n.$$

Hence putting $\eta_i = D_n(\alpha_i)$ $(1 \le i \le h)$ gives us our required set of representatives. The corresponding $\mathcal{O}$-lattice L_i and its stabilizer group Γ_i will be denoted by $L_n(\mathfrak{a}_i)$ and $\Gamma_n(\mathfrak{a}_i)$ respectively whenever we want to call to attention the fractional ideal $\mathfrak{a}_i$ or the dimension n.

We can also proceed similarly to find, for a fixed i, a suitable set of representatives for $Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)/\Gamma_i$. We do this using the bijection

$$Q(\Bbbk)\backslash\mathrm{GL}_n(\Bbbk)/\Gamma_i \ni Q(\Bbbk)g\Gamma_i \longmapsto \lambda(L_i \cap g^{-1}V_m) \in \mathrm{Cl}(\Bbbk)$$

formed by composing λ_i with the bijection (5), where $V_m = \sum_{k=1}^m \Bbbk\boldsymbol{e}_k$.

For each $j \in \{1, \dots, h\}$ the ideal $\mathfrak{a}_i\mathfrak{a}_j^{-1}$ shares the same ideal class as a unique $\mathfrak{a}_{j'}$ $(j' \in \{1, \dots, h\})$; that is $[\mathfrak{a}_j][\mathfrak{a}_{j'}] = [\mathfrak{a}_i]$. Putting $\tau_i(j) := j'$ defines a permutation τ_i on $\{1, \dots, h\}$.

Call a set of matrices $\{\xi_1, \dots, \xi_h\} \subset \mathrm{GL}_n(\Bbbk)$ an *(n, m)-splitting set for $L_n(\mathfrak{a}_i)$* if for each $j = 1, \dots, h$

$$\begin{aligned}(7)\qquad \xi_j L_n(\mathfrak{a}_i) &= \Bigl(\sum_{1\le k<m} \mathcal{O}\boldsymbol{e}_k + \mathfrak{a}_j \boldsymbol{e}_m\Bigr) + \Bigl(\sum_{m<k<n} \mathcal{O}\boldsymbol{e}_k + \mathfrak{a}_{\tau_i(j)}\boldsymbol{e}_n\Bigr)\\ &\simeq L_m(\mathfrak{a}_j) \oplus L_{n-m}(\mathfrak{a}_{\tau_i(j)}).\end{aligned}$$

Since $\lambda(L_i \cap \xi_j^{-1} V_m) = \lambda(\xi_j L_i \cap V_m) = [\mathfrak{a}_j]$ $(i \le j \le h)$, such a set of matrices completely represents $Q(\Bbbk)\backslash \mathrm{GL}_n(\Bbbk)/\Gamma_i$.

One such set is given as follows. For each $j = 1, \dots, h$, first take $\kappa_{ij} \in \Bbbk$ such that $\mathfrak{a}_j \mathfrak{a}_{\tau_i(j)} = \kappa_{ij}\mathfrak{a}_i$. Then choose elements $\alpha_{ij} \in \mathfrak{a}_j$, $\alpha'_{ij} \in \mathfrak{a}_{\tau_i(j)}$, $\beta_{ij} \in \mathfrak{a}_j^{-1}$ and $\beta'_{ij} \in \mathfrak{a}_{\tau_i(j)}^{-1}$ satisfying

$$\alpha_{ij}\beta_{ij} - \alpha'_{ij}\beta'_{ij} = 1$$

(see [Cohen 2000, §1, Proposition 1.3.12 or Algorithm 1.3.16]) and define the matrix

$$\xi_{ij} := \begin{bmatrix} I_{m-1} & & & \\ & \alpha_{ij} & & \kappa_{ij}\beta'_{ij} \\ & & I_{n-m+1} & \\ & \alpha'_{ij} & & \kappa_{ij}\beta_{ij} \end{bmatrix} \in \mathrm{GL}_n(\Bbbk).$$

By direct calculation it is easily verified that $\{\xi_{ij}\}_{j=1}^{h}$ is indeed an (n, m)-splitting set for $L_n(\mathfrak{a}_i)$ and thus fully represents $Q^{n,m}(\Bbbk)\backslash \mathrm{GL}_n(\Bbbk)/\Gamma_n(\mathfrak{a}_i)$.

5. Fundamental domains of $\mathrm{GL}_n(\Bbbk)\backslash \mathrm{GL}_n(\mathbb{A})^1$ and P_n/Γ_i

We use the results of Section 3 to determine suitable fundamental domains in our continued discussion of the general linear group.

5.1. *Local height functions.*

Definition. For each $\sigma \in \boldsymbol{p}$ define $H_\sigma : \bigwedge^m \Bbbk_\sigma^n \to \mathbb{R}_{>0}$ by

$$H_\sigma\Bigl(\sum_I a_I(\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_m})\Bigr) = \begin{cases} \Bigl(\sum_I |a_I|_\sigma^2\Bigr)^{[\Bbbk_\sigma:\mathbb{R}]/2}, & \sigma \in \boldsymbol{p}_\infty, \\ \sup_I |a_I|_\sigma, & \sigma \in \boldsymbol{p}_f, \end{cases}$$

where the sum and the supremum are taken over all $I = \{i_1 < \cdots < i_m\} \subset \{1, \dots, n\}$. We call this the *local height function* at σ.

In the following we extend each H_σ to a function of $\mathrm{GL}_n(\Bbbk_\sigma)$ by putting

$$H_\sigma(\gamma) = H_\sigma(\gamma\boldsymbol{e}_1 \wedge \cdots \wedge \gamma\boldsymbol{e}_m), \quad \gamma \in \mathrm{GL}_n(\Bbbk_\sigma).$$

The following lemma allows us to express the height function H_Q (restricted to $G(\mathbb{A})^1$) in terms of these local heights.

Lemma 12. *For $g = (g_\sigma)_{\sigma\in\boldsymbol{p}} \in \mathrm{GL}_n(\mathbb{A})^1$,*

$$H_Q(g) = \prod_{\sigma\in\boldsymbol{p}} H_\sigma({}^t g_\sigma^{-1})^{n/l}.$$

Proof. By noting that every local height H_σ as a function of $\mathrm{GL}_n(\Bbbk_\sigma)$ is left K_σ-invariant and writing

$$g = \begin{bmatrix} a & * \\ 0 & d \end{bmatrix} h \quad (a \in \mathrm{GL}_m(\mathbb{A}),\ d \in \mathrm{GL}_{n-m}(\mathbb{A}),\ h \in K),$$

we see that $H_\sigma({}^t g_\sigma^{-1}) = |\det(a_\sigma^{-1})|_\sigma^{r_\sigma}$ at every σ, where $r_\sigma = 2$ when σ is an imaginary infinite place and 1 otherwise. Hence the right-hand side of our equation becomes $|\det a|_{\mathbb{A}}^{-n/l}$, while $H_Q(g) = |\det a|_{\mathbb{A}}^{-(n-m)/l} |\det d|_{\mathbb{A}}^{m/l}$ by definition. Then since $g \in \mathrm{GL}_n(\mathbb{A})^1$, we have $1 = |\det g|_{\mathbb{A}} = |\det a|_{\mathbb{A}} |\det d|_{\mathbb{A}}$, which gives us our equality. □

We proceed to describe the sets $R_{i,j,\infty}$ using the matrices η_i and ξ_{ij} chosen at the end of the previous section. For the rest of this paper, for a square matrix A with entries in $\mathbb{A}$ or $\Bbbk_\infty$, we will write $|A|_{\mathbb{A}}$ and $|A|_\infty$ to denote $|\det A|_{\mathbb{A}}$ and $|\det A|_\infty$ respectively. When the size of A is at least m, we write $A^{[m]}$ for the top-left $m \times m$ submatrix of A, and use $|A|_\infty^{[m]}$ to denote $|A^{[m]}|_\infty$.

Lemma 13. *Let X_{ij} be the $n \times m$ matrix formed by the first m columns of ξ_{ij}^{-1}. Then*

$$H_Q(\xi_{ij}\gamma g \eta_i) = N(\mathfrak{a}_j)^{n/l} \left| {}^t\bar{X}_{ij}\, {}^t\bar{\gamma}^{-1}\, {}^t\bar{g}^{-1} (\eta_i)_\infty^{-2} g^{-1} \gamma^{-1} X_{ij} \right|_\infty^{n/2l} \tag{8}$$

for any $1 \le i, j \le h$, $\gamma \in \Gamma_i$ and $g \in \mathrm{GL}_n(\Bbbk_\infty)^1$.

Proof. Let $x = \eta_i^{-1} g^{-1} \gamma^{-1} X_{ij}$ so that $H_\sigma({}^t(\xi_{ij}\gamma g\eta_i)_\sigma^{-1}) = H_\sigma(x_\sigma \boldsymbol{e}_1 \wedge \cdots \wedge x_\sigma \boldsymbol{e}_m)$. For $\sigma \in \boldsymbol{p}_\infty$, this computes to

$$\Bigg(\sum_{\substack{I\subset\{1,\dots,n\}\\|I|=m}} |\det[x_\sigma]_I|_\sigma^2\Bigg)^{\frac12[\Bbbk_\sigma:\mathbb{R}]} = \Bigg(\sum_I \det{}^t\overline{[x_\sigma]}_I \det[x_\sigma]_I\Bigg)^{\frac12[\Bbbk_\sigma:\mathbb{R}]} = \det({}^t\bar{x}_\sigma x_\sigma)^{\frac12[\Bbbk_\sigma:\mathbb{R}]},$$

where for each $I = \{i_1 < \cdots < i_m\}$ that the sums run through $[x_\sigma]_I$ denotes the $m \times n$ matrix formed by the i_1-th, …, i_m-th rows of x_σ arranged from top to bottom in that order. The final equality is due to the Cauchy–Binet formula; see [Bombieri and Gubler 2006, Proposition 2.8.8].

For $\sigma \in \boldsymbol{p}_f$, since g_σ is trivial and $\gamma_\sigma \in \eta_{i\,\sigma} \mathrm{GL}_n(\mathcal{O}_\sigma) \eta_{i\,\sigma}^{-1}$, we have $(\xi_{ij}\gamma g \eta_i)_\sigma = \xi_{ij\,\sigma}\eta_{i\,\sigma} h_\sigma$ for some $h_\sigma \in \mathrm{GL}_n(\mathcal{O}_\sigma)$. Hence $H_\sigma({}^t(\xi_{ij}\gamma g\eta_i)_\sigma^{-1})$ simplifies to

$$H_\sigma({}^t\eta_{i\,\sigma}^{-1}\, {}^t\xi_{ij\,\sigma}^{-1}) = H_\sigma\big(\beta_{ij}(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m) + \alpha_{i\,\sigma}^{-1} \kappa_{ij} \alpha'_{ij} (\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_{m-1} \wedge \boldsymbol{e}_n)\big)$$

or

$$\max\{|\beta_{ij}|_\sigma, |\alpha_{i\,\sigma}^{-1} \kappa_{ij}^{-1} \alpha'_{ij}|_\sigma\} = |\beta'_{ij}\kappa_{ij}\alpha_{i\,\sigma}|_\sigma^{-1} \max\{|\beta_{ij}\beta'_{ij}\kappa_{ij}\alpha_{i\,\sigma}|_\sigma, |\alpha'_{ij}\beta'_{ij}|_\sigma\}.$$

By the previous lemma, $H_Q(\xi_{ij}\gamma g\eta_i)$ is obtained by taking the n/l-th power of the product of all the $H_\sigma((\xi_{ij}\gamma g\eta_i)_\sigma^{-1})$. Thus it remains to verify that

$$\prod_{\sigma\in\boldsymbol{p}_f} |\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma^{-1}\max\{|\beta_{ij}\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma, |\alpha'_{ij}\beta'_{ij}|_\sigma\} = N(\mathfrak{a}_j).$$

First we see that $\prod_{\sigma\in\boldsymbol{p}_f} |\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma^{-1} = N(\beta'_{ij}\kappa_{ij}\mathfrak{a}_i) = N(\beta'_{ij}\mathfrak{a}_j\mathfrak{a}_{\tau_i(j)})$. It is then sufficient to show that the product of the remaining factors is $N(\beta'_{ij}\mathfrak{a}_{\tau_i(j)})^{-1}$.

Let $\mathfrak{p}_\sigma$ denote the prime ideal associated to a finite place $\sigma \in \boldsymbol{p}_f$. Write the prime ideal decompositions of $\beta_{ij}\mathfrak{a}_j$ and $\beta'_{ij}\mathfrak{a}_{\tau_i(j)}$ as $\prod_{\sigma\in\boldsymbol{p}_f}(\mathfrak{p}_\sigma\cap\mathcal{O})^{d_\sigma}$ and $\prod_{\sigma\in\boldsymbol{p}_f}(\mathfrak{p}_\sigma\cap\mathcal{O})^{e_\sigma}$ respectively, the exponents d_σ and e_σ being nonnegative.

Then $\beta_{ij}\beta'_{ij}\kappa_{ij}\mathfrak{a}_i = (\beta_{ij}\mathfrak{a}_j)(\beta'_{ij}\mathfrak{a}_{\tau_i(j)}) = \prod_{\sigma\in\boldsymbol{p}_f}(\mathfrak{p}_\sigma\cap\mathcal{O})^{d_\sigma+e_\sigma}$ and since each $\mathfrak{a}_{i\sigma}$ is generated by $\alpha_{i\sigma}$, this yields

$$|\beta_{ij}\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma = |\mathcal{O}_\sigma/\mathfrak{p}_\sigma|^{-d_\sigma-e_\sigma}, \quad \sigma\in\boldsymbol{p}_f.$$

Now $\alpha'_{ij}\beta'_{ij}\in\beta'_{ij}\mathfrak{a}_{\tau_i(j)}$ and hence $|\alpha'_{ij}\beta'_{ij}|_\sigma \le |\mathcal{O}_\sigma/\mathfrak{p}_\sigma|^{-e_\sigma}$. We have two cases.

Case 1: $d_\sigma = 0$. Then $|\alpha'_{ij}\beta'_{ij}|_\sigma \le |\beta_{ij}\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma = |O_\sigma/\mathfrak{p}_\sigma|^{-e_\sigma}$.

Case 2: $d_\mathfrak{p} > 0$. In this case

$$\alpha'_{ij}\beta'_{ij} = -1+\alpha_{ij}\beta_{ij}\in -1+\beta_{ij}\mathfrak{a}_j\subset -1+(\mathfrak{p}_\sigma\cap\mathcal{O})^{d_\mathfrak{p}}$$

shows us that $\alpha'_{ij}\beta'_{ij}\in\mathcal{O}_\sigma^\times$ and so $|\alpha'_{ij}\beta'_{ij}|_\sigma = 1 \ge |\beta_{ij}\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma$. We also note that since β_{ij} and β'_{ij} were chosen in such a way that $\beta_{ij}\mathfrak{a}_{ij}+\beta'_{ij}\mathfrak{a}_{ij} = \mathcal{O}$, the ideal $\beta_{ij}\mathfrak{a}_j$ is prime to $\beta'_{ij}\mathfrak{a}_{\tau_i(j)}$, which means $e_\sigma = 0$.

So in either case,

$$\max\{|\beta_{ij}\beta'_{ij}\kappa_{ij}\alpha_{i\sigma}|_\sigma, |\alpha'_{ij}\beta'_{ij}|_\sigma\} = |\mathcal{O}_\sigma/\mathfrak{p}_\sigma|^{-e_\sigma}$$

and thus the product over all finite places is $N(\beta'_{ij}\mathfrak{a}_{\tau_i(j)})^{-1}$, as required. □

Now fix $1\le i, j\le h$ and first consider the set $\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij}$. It is easy to directly verify that

$$\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij} = \{g\in G(\Bbbk_\infty)^1 : H_Q(\xi_{ij}g\eta_i) = \mathsf{m}_Q(g\eta_i)\}.$$

Hence for $g\in\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij}$ we have

$$H_Q(\xi_{ij}g\eta_i) = \mathsf{m}_Q(g\eta_i) = \min_{x\in Q(\Bbbk)\backslash \mathrm{GL}_n(\Bbbk)} H_Q(xg\eta_i) = \min_{\substack{1\le k\le h\\ \gamma\in\Gamma_i}} H_Q(\xi_{ik}\gamma g\eta_i),$$

which in this case can be written using (8) as

$$|{}^t\overline{X}_{ij}\,{}^t\overline{g}^{-1}(\eta_i)_\infty^{-2}g^{-1}X_{ij}|_\infty \le \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 |{}^t\overline{X}_{ik}\,{}^t\overline{\gamma}\,{}^t\overline{g}^{-1}(\eta_i)_\infty^{-2}g^{-1}\gamma X_{ik}|_\infty$$

for all $k = 1,\dots,h$ and $\gamma\in\Gamma_i$.

Now ${}^t\overline{X}_{ik}\,{}^t\bar{\gamma}\,{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2}g^{-1}\gamma X_{ik} = ({}^t\bar{\xi}_{ik}^{-1}\,{}^t\bar{\gamma}\,{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2}g^{-1}\gamma\xi_{ik}^{-1})^{[m]}$, which by letting $g_{[ij]} = \xi_{ij}g\xi_{ij}^{-1}$ can be rewritten as

$$\big({}^t\overline{(\xi_{ij}\gamma\xi_{ik}^{-1})}\,{}^t\bar{g}_{[ij]}^{-1}({}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1})g_{[ij]}^{-1}(\xi_{ij}\gamma\xi_{ik}^{-1})\big)^{[m]}.$$

This lets us express the set $R_{i,j,\infty}$ as follows. For $g \in \mathrm{GL}_n(\Bbbk_\infty)$ let $\pi_{ij}(g)$ denote ${}^t\bar{g}^{-1}({}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1})g^{-1}$. Then $g \in R_{i,j,\infty}$ if and only if

$$|\pi_{ij}(g)|_\infty^{[m]} \le \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 \left|{}^t\overline{(\xi_{ij}\gamma\xi_{ik}^{-1})}\pi_{ij}(g)(\xi_{ij}\gamma\xi_{ik}^{-1})\right|_\infty^{[m]} \tag{9}$$

for all $k = 1, \dots, h$ and $\gamma \in \Gamma_i$.

5.2. *Fundamental domains of P_n/Γ_i.* For each infinite place σ of $\Bbbk$ let $P_n(\Bbbk_\sigma)$ denote the subset of $\mathrm{GL}_n(\Bbbk_\sigma)$ consisting of all positive definite real symmetric matrices when σ is real and positive definite Hermitian matrices when σ is imaginary. We consider the subset of $\mathrm{GL}_n(\Bbbk_\infty)$ defined by $P_n = \prod_{\sigma\in\boldsymbol{p}_\infty} P_n(\Bbbk_\sigma)$. This is the space of positive definite Humbert forms in $\mathrm{GL}_n(\Bbbk)$.

We have the following right action of $\mathrm{GL}_n(\Bbbk_\infty)$ on P_n:

$$A \cdot g = {}^t\bar{g}Ag \quad (g \in \mathrm{GL}_n(\Bbbk_\infty),\ A \in P_n). \tag{10}$$

To determine fundamental domains in P_n with respect to subgroups of $\mathrm{GL}_n(\Bbbk)$, we consider instead the induced action $A \cdot gZ = {}^t\bar{g}Ag$ of $\mathrm{GL}_n(\Bbbk)/Z$ on P_n, where $Z = \{z \in \Bbbk : \bar{z}z = 1\}$, the set of roots of unity in $\Bbbk$. Here $\{zI_n : z \in Z\}$ is naturally seen to be the intersection of K_∞ and the center of $\mathrm{GL}_n(\Bbbk)$.

Hence given a discrete subgroup Γ of $\mathrm{GL}_n(\Bbbk)$ acting on a subset T of P_n, a fundamental domain Ω of a T/Γ is an open subset of T satisfying

(i) $T = \Omega^- \cdot \Gamma$,

(ii) for $\gamma \in \Gamma$, if $\Omega^\circ \cap (\Omega^- \cdot \gamma) \ne \varnothing$ then $\gamma \in Z$.

Now for each $1 \le i, j \le h$, put

$$K_{i,j,\infty} = (\xi_{ij}\eta_i)_\infty K_\infty(\xi_{ij}\eta_i)_\infty^{-1}, \quad P_n^{ij} = \{A \in P_n : |A|_\infty = N(\kappa_{ij}\mathfrak{a}_i)^{-2}\},$$

and define the map $\pi_{ij} : G(\Bbbk_\infty) \ni g \mapsto {}^t\bar{g}^{-1}({}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1})g^{-1} \in P_n$. Note that $K_{i,j,\infty}$ is the stabilizer of ${}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1} \in P_n$ under the action of $\mathrm{GL}_n(\Bbbk_\infty)$ on P_n and that π_{ij} preserves this action. Thus the surjective map π_{ij} gives us the isomorphisms

$$\mathrm{GL}_n(\Bbbk_\infty)/K_{i,j,\infty} \simeq P_n \quad \text{and} \quad \mathrm{GL}_n(\Bbbk_\infty)^1/K_{i,\infty} \simeq \pi_{ij}(\mathrm{GL}_n(\Bbbk_\infty)^1) = P_n^{ij}$$

since $|{}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1}|_\infty = N(\kappa_{ij}\mathfrak{a}_i)^{-2}$.

Lastly let $F_{i,j}^{n,m}$ denote the following closed subset of P_n:

$$\left\{A \in P_n : |A|_\infty^{[m]} \leq \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 \left|{}^t\overline{(\xi_{ij}\gamma\xi_{ik}^{-1})}A(\xi_{ij}\gamma\xi_{ik}^{-1})\right|_\infty^{[m]},\ 1 \leq k \leq h,\ \gamma \in \Gamma_i\right\}.$$

From (9), π_{ij} maps $R_{i,j,\infty}$ onto $F_{i,j}^{n,m} \cap P_n^{ij}$. We also note that following statement holds true, the proof of which will be given later in the section.

Proposition 14. *$F_{i,j}^{n,m}$ is right $Q_{i,j}$-invariant under the action* (10).

Thus the subgroup $Q_{i,j}$ of $\mathrm{GL}_n(\mathbb{k}_\infty)$ acts on $R_{i,j,\infty}$ from the left and on $F_{i,j}^{n,m}$ from the right, and π_{ij} preserves this. Hence by constructing a fundamental domain for $F_{i,j}^{n,m}/Q_{i,j}$, we can find one for $Q_{i,j}\backslash R_{i,j,\infty}$ by taking the inverse image under π_{ij}.

We start by observing that $\xi_{ij}\Gamma_i\xi_{ij}^{-1}$ is the stabilizer in $\mathrm{GL}_n(\mathbb{k})$ of the $\mathcal{O}$-lattice $\xi_{ij}L_i$ described in (7). This gives us an expression for $Q_{i,j} = Q(\mathbb{k}) \cap \xi_{ij}\Gamma_i\xi_{ij}^{-1}$:

$$\left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a \in \Gamma_m(\mathfrak{a}_j),\ d \in \Gamma_{n-m}(\mathfrak{a}_{\tau_i(j)}),\ bL_{n-m}(\mathfrak{a}_{\tau_i(j)}) \subset L_m(\mathfrak{a}_j)\right\}.$$

Any $A \in P_n$ can be written uniquely in the form

$$A = \begin{bmatrix} I_m & 0 \\ {}^t\overline{u_{A,m}} & I_{n-m} \end{bmatrix} \begin{bmatrix} A^{[m]} & 0 \\ 0 & A_{[n-m]} \end{bmatrix} \begin{bmatrix} I_m & u_{A,m} \\ 0 & I_{n-m} \end{bmatrix} \tag{11}$$

with $A^{[m]} \in P_m,\ A_{[n-m]} \in P_{n-m}$ and $u_{A.m} \in M_{m,n-m}(\mathbb{k}_\infty)$. (The symbol $A^{[m]}$ here coincides with its prior use to denote the top left $m \times m$ submatrix of A). It is easy to verify that the action of $q = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in Q_{i,j}$ on A results in

$$({}^t\bar{q}Aq)^{[m]} = {}^t\bar{a}A^{[m]}a, \quad ({}^t\bar{q}Aq)_{[n-m]} = {}^t\bar{d}A_{[n-m]}d,$$
$$u_{{}^t\bar{q}Aq,m} = a^{-1}(u_{A,m}d + b).$$

These equations will determine the necessary form of our fundamental domain, as well as allow us to prove our previous proposition. Given $A \in F_{i,j}^{n,m}$ and q as above, we first see that

$$|{}^t\bar{q}Aq|_\infty^{[m]} = |{}^t\bar{a}|_\infty |A|_\infty^{[m]} |a|_\infty = |A|_\infty^{[m]}.$$

Next put $q = \xi_{ij}\gamma_q\xi_{ij}^{-1},\ \gamma_q \in \Gamma_i$, to get

$${}^t\overline{(\xi_{ij}\gamma\xi_{ik}^{-1})}\,{}^t\bar{q}Aq(\xi_{ij}\gamma\xi_{ik}^{-1}) = {}^t\overline{(\xi_{ij}\gamma_q\gamma\xi_{ik}^{-1})}A(\xi_{ij}\gamma_q\gamma\xi_{ik}^{-1})$$

for all $\gamma \in \Gamma_i$ and every k. Together, this shows that ${}^t\bar{q}Aq \in F_{i,j}^{n,m}$ as proposed.

Now for each $k = 1, \ldots, h$ choose sets $\mathfrak{d}_k$, $\mathfrak{d}'_k$ and $\mathfrak{d}_{ik}$ that are fundamental domains for $\mathbb{k}_\infty$ with respect to addition by $\mathfrak{a}_k$, $\mathfrak{a}_k^{-1}$ and $\mathfrak{a}_k\mathfrak{a}_{\tau_i(k)}^{-1}$ respectively. We require each of these sets to be closed under multiplication by Z. Then choose also a subset $\tilde{\mathfrak{d}}_{ik}$ of $\mathfrak{d}_{ik}$ that is a fundamental domain for $\mathfrak{d}_{ik}$ with respect to multiplication by Z. Also if necessary (which will be the case when $m > 1$ and $n - m > 1$) take a fundamental domain $\mathfrak{d}_{\mathcal{O}}$ of $\Bbbk_\infty$ with respect to addition by $\mathcal{O}$.

Using these, we define for $1 < i,\ j < h$ the sets

$$\mathfrak{D}^{n,m}_{i,j} = \left\{ \begin{bmatrix} d_{11} & \cdots & d_{1,n-m} \\ \vdots & \ddots & \vdots \\ d_{m1} & \cdots & d_{m,n-m} \end{bmatrix} : d_{m,n-m} \in \tilde{\mathfrak{d}}_{ij},\ d_{rs} \in \begin{cases} \mathfrak{d}_{\mathcal{O}}, & r < m,\ s < n-m, \\ \mathfrak{d}'_{\tau_i(j)}, & r < m,\ s = n-m, \\ \mathfrak{d}_j, & r = m,\ s < n-m \end{cases} \right\}$$

and

$$F^{n,m}_{i,j}(S, S') = \{A \in F^{n,m}_{i,j} : A^{[m]} \in S,\ A_{[n-m]} \in S',\ u_{A,m} \in \mathfrak{D}^{n,m}_{i,j}\}$$

with arbitrary subsets $S \subset P_m$ and $S' \subset P_{n-m}$.

In particular we will want to consider $F^{n,m}_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\sigma_i(j)})$ when $\mathfrak{B}_j$ and $\mathfrak{C}_{\tau_i(j)}$ are fundamental domains for $P_m / \Gamma_m(\mathfrak{a}_j)$ and $P_{n-m}/\Gamma_{n-m}(\mathfrak{a}_{\tau_i(j)})$ respectively. In this case, based on our observations on the action of $Q_{i,j}$ on $F^{n,m}_{i,j}$, we establish the following result.

Lemma 15. *$F^{n,m}_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)})$ is a fundamental domain of $F^{n,m}_{i,j}/Q_{i,j}$.*

Proof. We write $F = F^{n,m}_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)})$ for short. First consider an $A \in F^{n,m}_{i,j}$. We can find $b \in \mathfrak{B}^-_j$, $c \in \mathfrak{C}^-_{\tau_i(j)}$ and $a \in \Gamma_m(\mathfrak{a}_j)$, $d \in \Gamma_{n-m}(\mathfrak{a}_{\tau_i(j)})$ such that $A^{[m]} = {}^t\bar{a}ba$ and $A_{[n-m]} = {}^t\bar{d}cd$. Also, by substituting a with a suitable Z-multiple if necessary, we can find $f \in (\mathfrak{D}^{n,m}_{i,j})^-$ and a $g \in M_{m,n-m}(\Bbbk)$ mapping $L_{n-m}(\mathfrak{a}_{\tau_i(j)})$ to $L_m(\mathfrak{a}_j)$ such that $au_{A,m}d^{-1} = f + g$. Let

$$q = \begin{bmatrix} a & gd \\ 0 & d \end{bmatrix}, \quad A' = \begin{bmatrix} I_m & 0 \\ {}^t\bar{f} & I_{n-m} \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} I_m & f \\ 0 & I_{n-m} \end{bmatrix}.$$

Then $q \in Q_{i,j}$ and $A = {}^t\bar{q}A'q$. We have from the $Q_{i,j}$-invariance of $F^{n,m}_{i,j}$ that $A' \in F^{n,m}_{i,j}$ and so $A' \in F^-$. This shows that $F^{n,m}_{i,j} = F^- \cdot Q_{i,j}$.

Next suppose $F^\circ \cap (F^- \cdot q)$ is nonempty for a $q = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in Q_{i,j}$, so there exist $A \in F^\circ$ and $A' \in F^-$ such that $A = {}^t\bar{q}A'q$. We must show that $q \in Z$. From $A^{[m]} = {}^t\bar{a}A'^{[m]}a \in \mathfrak{B}_{ij}$ and $A_{[n-m]} = {}^t\bar{d}A'_{[n-m]}d \in \mathfrak{C}_{ij}$, we must have $a = a_1 I_m$ and $d = d_1 I_{n-m}$ with some $a_1, d_1 \in Z$. Since the entries of $u_{A,m}$ and $u_{A',m}$ are respectively in the interior and closure of either $\mathfrak{d}_{\mathcal{O}}$, $\mathfrak{d}_j$, $\mathfrak{d}'_{\tau_i(j)}$ or $\mathfrak{d}_{ij}$, which are all invariant under Z, we see that $b = au_{A,m} - u_{A',m}d$ must necessarily be 0. From this we get $a_1 u_{A,m} = d_1 u_{A',m}$, whose $(m, n{-}m)$-th entry belongs to $\tilde{\mathfrak{d}}_{ij}$, implying that $a_1 d_1^{-1} \in Z$. Hence $q \in Z$. □

As a result, the inverse image of $F^{n,m}_{i,j}(\mathfrak{B}_{ij}, \mathfrak{C}_{ij}) \cap P^{ij}_n$ under π_{ij} is a fundamental domain of $Q_{i,j}\backslash R_{i,j,\infty}$.

If we have fundamental domains $\mathfrak{B}_1, \dots, \mathfrak{B}_h$ for P_m with respect to the groups $\Gamma_m(\mathfrak{a}_1), \dots, \Gamma_m(\mathfrak{a}_h)$, as well as fundamental domains $\mathfrak{C}_1, \dots, \mathfrak{C}_h$ of P_{n-m} with respect to $\Gamma_{n-m}(\mathfrak{a}_1), \dots, \Gamma_{n-m}(\mathfrak{a}_h)$, we are able to construct the sets $F^{n,m}_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\sigma_i(j)})$

for each i and j. Then by Corollary 6 a fundamental domain for $\mathrm{GL}_n(\Bbbk)\backslash\mathrm{GL}_n(\mathbb{A})^1$ is given by the set

$$\bigsqcup_{1\le i,j\le h} \pi_{ij}^{-1}(F_{i,j}^{n,m}(\mathfrak{B}_j,\mathfrak{C}_{\tau_i(j)})\cap P_n^{ij})\xi_{ij}\eta_i K_f.$$

Also Theorem 7 shows us that $\bigcup_{j=1}^h \xi_{ij}^{-1}\pi_{ij}^{-1}(F_{i,j}^{n,m}(\mathfrak{B}_j,\mathfrak{C}_{\tau_i(j)})\cap P_n^{ij})\xi_{ij}$ is a fundamental domain for $\mathrm{GL}_n(\Bbbk_\infty)^1$ with respect to Γ_i. Now let

$$\Omega_i^{n,m}(\mathfrak{B}_1,\dots,\mathfrak{B}_h,\mathfrak{C}_1,\dots,\mathfrak{C}_h)=\bigcup_{j=1}^h {}^t\bar{\xi}_{ij}F_{i,j}^{n,m}(\mathfrak{B}_j,\mathfrak{C}_{\tau_i(j)})\xi_{ij}.$$

We have the following result.

Theorem 16. $\Omega_i^{n,m}(\mathfrak{B}_1,\dots,\mathfrak{B}_h,\mathfrak{C}_1,\dots,\mathfrak{C}_h)\cap P_n^{ij}$ *is a fundamental domain of* P_n^{ij} *with respect to* Γ_i. *In addition, by viewing* $\mathbb{R}_{>0}$ *as a subset of* $\Bbbk_\infty$ *via the usual diagonal embedding, if we assume for* $k=1,\dots,h$ *that*

$$\mathbb{R}_{>0}\mathfrak{B}_k=\mathfrak{B}_k,\quad \mathbb{R}_{>0}\mathfrak{C}_k=\mathfrak{C}_k,$$

then $\Omega_i^{n,m}(\mathfrak{B}_1,\dots,\mathfrak{B}_h,\mathfrak{C}_1,\dots,\mathfrak{C}_h)$ *is a fundamental domain of* P_n/Γ_i.

Proof. We write Ω for $\Omega_i^{n,m}(\mathfrak{B}_1,\dots,\mathfrak{B}_h,\mathfrak{C}_1,\dots,\mathfrak{C}_h)$ and Γ for Γ_i for short. If we define the map $G(\Bbbk_\infty)\ni g\mapsto {}^t\bar{g}^{-1}(\eta_i)_\infty^{-2}g^{-1}\in P_n$ we can directly verify that the image of $\bigcup_{j=1}^h \xi_{ij}^{-1}\pi_{ij}^{-1}(F_{i,j}^{n,m}(\mathfrak{B}_j,\mathfrak{C}_{\tau_i(j)})\cap P_n^{ij})\xi_{ij}$ under this map is Ω, which gives us the first result. For the second part, note that $\mathbb{R}_{>0}F_{i,j}^{n,m}=F_{i,j}^{n,m}$ and

$$(xA)^{[m]}=x(A^{[m]}),\quad (xA)_{[n-m]}=x(A_{[n-m]}),\quad u_{xA,m}=u_{A,m}$$

for any $x\in\mathbb{R}_{>0}$ and $A\in P_n$. Thus the conditions on the $\mathfrak{B}_k$ and $\mathfrak{C}_k$ imply that $\mathbb{R}_{>0}F_{i,j}^{n,m}(\mathfrak{B}_j,\mathfrak{C}_{\tau_i(j)})=F_{i,j}^{n,m}(\mathfrak{B}_j,\mathfrak{C}_{\tau_i(j)})$ for each j; hence $\mathbb{R}_{>0}\Omega=\Omega$. Since $P_n=\mathbb{R}_{>0}P_n^{ij}$, we see from $P_n^{ij}=(\Omega\cap P_n^{ij})^-\cdot\Gamma$ that $P_n=\Omega^-\cdot\Gamma$. Finally suppose that $\Omega^\circ\cap({}^t\bar{\gamma}\Omega^-\gamma)$ $(\gamma\in\Gamma)$ contains an element $g={}^t\bar{\gamma}g'\gamma$ $(g'\in\Omega^-)$. Put $x=(N(\kappa_{ij}\mathfrak{a}_i)^2|g|_\infty)^{-1/n[\Bbbk_\infty:\mathbb{R}]}$. Then $|xg|_\infty=|xg'|_\infty=N(\kappa_{ij}\mathfrak{a}_i)^{-2}$ and hence $xg={}^t\bar{\gamma}xg'\gamma\in(\Omega^\circ\cap P_n^{ij})\cap{}^t\bar{\gamma}(\Omega^-\cap P_n^{ij})\gamma$, which gives us $\gamma=I_n$, as required. □

Using the theorem, we can construct fundamental domains for P_n with respect to Γ_i for each i and $n\ge 1$. Since $\Gamma_i=\mathcal{O}^\times$ for any i when $n=1$, we can start by choosing a fixed fundamental domain, Ω^1, for P_1 with respect to $\mathcal{O}^\times/Z$ that is closed under multiplication by $\mathbb{R}_{>0}$. (The existence of such a set can be shown using Voronoi reduction; see the Appendix.) Then for each $i=1,\dots,h$, let $\Omega_i^1=\Omega^1$ and define

$$\Omega_i^n=\Omega_i^{n,n-1}(\Omega_1^{n-1}\dots,\Omega_h^{n-1},\Omega^1,\cdots,\Omega^1)$$

inductively for $n\ge 2$. By construction, $\mathbb{R}_{>0}\Omega_i^n=\Omega_i^n$ so for each $1\le i\le h$ and $n\ge 1$, Ω_i^n gives us a fundamental domain for P_n/Γ_i.

An example implementation of this construction for P_2 over the imaginary quadratic field $\mathbb{Q}(\sqrt{-5})$ of class number 2 is given in the following subsection. Similar work on fundamental domains in spaces over real quadratic fields of class number 1 can be found in [Cohn 1965].

5.3. ***An example*** **($\Bbbk = \mathbb{Q}(\sqrt{-5})$).** When $\Bbbk$ is an imaginary quadratic field, we have $\Bbbk_\infty = \mathbb{C}$. For $n = 1$ we have $P_1 = \mathbb{R}_{>0}(\subset \mathbb{C})$ and $\Gamma_i = \mathcal{O}^\times = Z$ acts trivially on P_1; hence P_1 itself is a fundamental domain for $P_1/\,\Gamma_1(\mathfrak{a}_i)$.

Consider in particular $\Bbbk = \mathbb{Q}(\sqrt{-5})$ of class number $h = 2$. We can choose representatives $\mathfrak{a}_1, \mathfrak{a}_2$ for $\mathrm{Cl}(\Bbbk)$ by putting $\mathfrak{a}_1 = \mathcal{O}$ and $\mathfrak{a}_2 = \langle 2, 1+\sqrt{-5}\rangle$. Then following the procedure at the end of Section 4, we see that

$$\mathfrak{a}_1^2 = \mathfrak{a}_1, \qquad \mathfrak{a}_2^2 = 2\mathfrak{a}_1 \qquad \Bigl(\tau_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},\ \kappa_{11} = 1,\ \kappa_{12} = 2\Bigr),$$

$$\mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{a}_2, \quad \mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_2 \qquad \Bigl(\tau_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ \kappa_{21} = \kappa_{22} = 1\Bigr),$$

and (2, 1)-splitting sets for $L_2(\mathfrak{a}_i)$ are given by

$$\left\{\xi_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \xi_{12} = \begin{bmatrix} 2 & 2+\sqrt{-5} \\ 2 & 3+\sqrt{-5} \end{bmatrix}\right\} \quad (i = 1),$$

$$\left\{\xi_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \xi_{22} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} \qquad (i = 2).$$

For $1 \le i, j, k \le 2$ denote by $\Xi_{i,j,k}$ the set of the first columns of the matrices $\xi_{ij}\gamma\xi_{ik}^{-1}$ as γ ranges over $\Gamma(\mathfrak{a}_i)$. Then for $A \in P_2$

$$\begin{aligned} \min_{\gamma\in\Gamma_i} \bigl|{}^t\overline{(\xi_{ij}\gamma\xi_{ik}^{-1})}A(\xi_{ij}\gamma\xi_{ik}^{-1})\bigr|_\infty^{[1]} &= \min_{\boldsymbol{x}\in\Xi_{i,j,k}} |{}^t\bar{\boldsymbol{x}}A\boldsymbol{x}| \\ &= \min_{\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]\in\Xi_{i,j,k}} A^{[1]}|e + u_{A,1}f|^2 + A_{[1]}|f|^2, \end{aligned}$$

and so $F_{i,j}^{2,1}$ can be expressed as

$$F_{i,j}^{2,1} = \left\{ \begin{bmatrix} 1 & 0 \\ \bar{d} & 1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : \begin{array}{l} b, c \in \mathbb{R}_{>0},\ d \in \mathbb{C}, \\ |e + df|^2 + \frac{c}{b}|f|^2 \ge 1, \\ \begin{bmatrix} e \\ f \end{bmatrix} \in \frac{1}{N(\mathfrak{a}_j)}\Xi_{i,j,1} \cup \frac{2}{N(\mathfrak{a}_j)}\Xi_{i,j,2} \end{array} \right\}.$$

Now for $\alpha, \beta \in \Bbbk$ let

$$\mathfrak{d}(\alpha, \beta) = \bigl\{x\alpha + y\beta : -\tfrac{1}{2} < x, y \le \tfrac{1}{2}\bigr\}.$$

When α and β generate a fractional ideal $\mathfrak{a}$, we have $\mathfrak{d}(\alpha, \beta)$ is a fundamental domain for $\mathbb{C}$ with respect to addition by $\mathfrak{a}$. Also if we let $\tilde{\mathfrak{d}}(\alpha, \beta)$ denote the subset of $\mathfrak{d}(\alpha, \beta)$ where the range of y is restricted to $0 \le y \le \frac{1}{2}$, this gives us a fundamental domain for $\mathfrak{d}(\alpha, \beta)$ with respect to multiplication by $Z = \{\pm 1\}$.

In particular $\mathfrak{d}(1, \sqrt{-5})$, $\mathfrak{d}(2, 1+\sqrt{-5})$, $\mathfrak{d}\big(1, \frac{1}{2}(1-\sqrt{-5})\big)$ are fundamental domains for $\mathbb{C}$ with respect to addition by $\mathcal{O}$, $\mathfrak{a}_2$ and $\mathfrak{a}_2^{-1}$ respectively, and we can put $\tilde{\mathfrak{d}}_{11} = \tilde{\mathfrak{d}}_{12} = \tilde{\mathfrak{d}}(1, \sqrt{-5})$, $\tilde{\mathfrak{d}}_{21} = \tilde{\mathfrak{d}}\big(1, \frac{1}{2}(1-\sqrt{-5})\big)$ and $\tilde{\mathfrak{d}}_{22} = \tilde{\mathfrak{d}}(2, \sqrt{-5})$. Then

$$F_{i,j}^{2,1}(P_1, P_1) = \left\{ \begin{bmatrix} 1 & 0 \\ \bar{d} & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : \begin{array}{l} b, c \in \mathbb{R}_{>0},\ d \in \tilde{\mathfrak{d}}_{ij}, \\ |e+df|^2 + \dfrac{c}{b}|f|^2 \ge 1, \\ \begin{bmatrix} e \\ f \end{bmatrix} \in \dfrac{1}{N(\mathfrak{a}_j)} \Xi_{i,j,1} \cup \dfrac{2}{N(\mathfrak{a}_j)} \Xi_{i,j,2} \end{array} \right\}.$$

Writing $F_{i,j}^{2,1}(P_1, P_1)$ as $F_{i,j}$, we obtain the fundamental domains $\Omega_1^2 = F_{1,1} \cup {}^t\bar{\xi}_{12} F_{1,2} \xi_{12}$ for $P_2/\Gamma_2(\mathfrak{a}_1)$ and $\Omega_2^2 = F_{1,1} \cup {}^t\bar{\xi}_{22} F_{2,2} \xi_{22}$ for $P_2/\Gamma_2(\mathfrak{a}_2)$.

5.4. ***Relations between the fundamental domains.*** So far we have used a representative set $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ for $\mathrm{Cl}(\Bbbk)$ and a standard parabolic subgroup $Q^{n,m}$ of GL_n in constructing our fundamental domains. This construction is of course possible with m varied and using any other representative set of fractional ideals. We will demonstrate in this section that the fundamental domain for $P_n/\Gamma_n(\mathfrak{a}_i)$ constructed using $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ and $Q^{n,m}$ can be mapped by an automorphism to a fundamental domain for $P_n/\Gamma_n(\mathfrak{a}_i^{-1})$ constructed with the representative set $\{\mathfrak{a}_1^{-1}, \dots, \mathfrak{a}_h^{-1}\}$ and $Q^{n,n-m}$.

For integers n and m where $1 \le m < n$, define the outer automorphism $\phi_{n,m}$ of $\mathrm{GL}_n(\Bbbk_\infty)$ by

$$\phi_{n,m}(g) := {}^tJ_{n,m}({}^tg^{-1})J_{n,m}, \quad g \in \mathrm{GL}_n(\Bbbk_\infty), \tag{12}$$

where

$$J_{n,m} = \begin{bmatrix} 0 & I_m \\ I_{n-m} & 0 \end{bmatrix}.$$

Note that ${}^tJ_{n,m} = (J_{n,m})^{-1} = J_{n,n-m}$ so that in particular we have $\phi_{n,m}^{-1} = \phi_{n,n-m}$. Also $\phi_{n,m}$ gives a one-to-one map between these two standard parabolic subgroups of GL_n since $\phi_{n,m}(Q^{n,m}(\Bbbk)) = Q^{n,n-m}(\Bbbk)$.

Let the ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_h$, the corresponding adeles $\alpha_1, \dots, \alpha_h$, and the matrices ξ_{ij} $(1 \le i, j \le h)$ be as they were chosen in the last section. Clearly $\{\mathfrak{a}_1^{-1}, \dots, \mathfrak{a}_h^{-1}\}$ is also a set of representative ideals for ideal class group. A corresponding set of matrices representing $\mathrm{GL}_n(\Bbbk)\backslash \mathrm{GL}_n(\mathbb{A})^1/(\mathrm{GL}_n)^1_{\mathbb{A},\infty}$ is given by

$$\{D_n(\alpha_1^{-1}), \dots, D_n(\alpha_h^{-1})\} = \{\eta_i^{-1}, \dots, \eta_h^{-1}\},$$

which gives us the subgroups

$$D_n(\alpha_i^{-1})(\mathrm{GL}_n(\Bbbk_\infty)^1 \times K_f)D_n(\alpha_i^{-1})^{-1} \cap \mathrm{GL}_n(\Bbbk) = \Gamma_n(\mathfrak{a}_i^{-1}),$$

which are the respective stabilizer subgroups in $\mathrm{GL}_n(\Bbbk)$ of the lattices $L_n(\mathfrak{a}_i^{-1})$ ($i = 1, \dots, h$).

Next for each $i, j = 1, \dots, h$ set

$$\tilde{\xi}_{ij} := {}^t J_{n,m}\, {}^t\xi_{i\tau_i(j)}^{-1} = \begin{bmatrix} & I_{n-m+1} & \\ & -\beta'_{i\tau_i(j)} & \kappa_{ij}^{-1}\alpha_{i\tau_i(j)} \\ I_{m-1} & & \\ & \beta_{i\tau_i(j)} & -\kappa_{ij}^{-1}\alpha'_{i\tau_i(j)} \end{bmatrix},$$

which is easily verified to satisfy

$$\begin{aligned}(13)\quad \tilde{\xi}_{ij} L_n(\mathfrak{a}_i^{-1}) &= \Big(\sum_{1\le k<n-m} \mathcal{O}\boldsymbol{e}_k^{(n)} + \mathfrak{a}_j^{-1}\boldsymbol{e}_m^{(n)}\Big) + \Big(\sum_{n-m<k<n} \mathcal{O}\boldsymbol{e}_k^{(n)} + \mathfrak{a}_{\tau_i(j)}^{-1}\boldsymbol{e}_n^{(n)}\Big)\\ &\simeq L_{n-m}(\mathfrak{a}_j^{-1}) \oplus L_m(\mathfrak{a}_{\tau_i(j)}^{-1}).\end{aligned}$$

Thus $\{\tilde{\xi}_{ij}\}_{j=1}^h$ is an $(n, n-m)$-splitting set for $L_n(\mathfrak{a}_i^{-1})$, and hence a complete set of representatives for $Q^{n,n-m}(\Bbbk)\backslash \mathrm{GL}_n(\Bbbk)/\Gamma_n(\mathfrak{a}_i^{-1})$.

We can also define

$$\tilde{Q}_{i.j}^{n,n-m} := Q^{n,n-m}(\Bbbk) \cap \tilde{\xi}_{ij}^{n,n-m}\Gamma_n(\mathfrak{a}_i^{-1})(\tilde{\xi}_{ij}^{n,n-m})^{-1},$$

$$\tilde{F}_{i,j}^{n,n-m} = \Big\{A \in P_n : |A|_\infty^{[n-m]} \le \Big(\frac{N(\mathfrak{a}_k^{-1})}{N(\mathfrak{a}_j^{-1})}\Big)^2 \big|{}^t(\overline{\tilde{\xi}_{ij}\gamma\tilde{\xi}_{ik}^{-1}})A(\tilde{\xi}_{ij}\gamma\tilde{\xi}_{ik}^{-1})\big|_\infty^{[n-m]},\ 1 \le k \le h,\ \gamma \in \Gamma_n(\mathfrak{a}^{-1})\Big\},$$

$$\tilde{\mathfrak{D}}_{i,j}^{n,n-m} = \left\{\begin{bmatrix} d_{11} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots \\ d_{n-m,1} & \cdots & d_{n-m,m}\end{bmatrix} : d_{n-m,m} \in \tilde{\mathfrak{d}}_{ij},\ d_{rs} \in \begin{cases} \mathfrak{d}_{\mathcal{O}}, & r<n-m,\ s<m, \\ \mathfrak{d}_{\tau_i(j)}, & r<n-m,\ s=m, \\ \mathfrak{d}'_j, & r=n-m,\ s<m \end{cases}\right\},$$

where the fundamental domains $\mathfrak{d}_k$, $\mathfrak{d}'_k$, $\tilde{\mathfrak{d}}_{ik}$, $\mathfrak{d}_{\mathcal{O}}$ are taken as in the previous section, and

$$\tilde{F}_{i,j}^{n,n-m}(S, S') = \big\{A \in \tilde{F}_{i,j}^{n,n-m} : A^{[n-m]} \in S,\ A_{[m]} \in S',\ u_{A,n-m} \in \tilde{\mathfrak{D}}_{i,j}^{n,n-m}\big\}$$

for arbitrary subsets $S \subset P_{n-m}$, $S' \subset P_m$. These are precisely the groups $Q_{i,j}^{n,m}$ and sets $F_{i,j}^{n,m}$, $\mathfrak{D}_{i,j}^{n,m}$ and $F_{i,j}^{n,m}(S, S')$ from the previous section with $\mathfrak{a}_i^{-1}$ and $\tilde{\xi}_{ik}$ in place of the $\mathfrak{a}_i$ and ξ_{ik} respectively, when $m = n-m$. It is easily verified that $\phi_{n,m}(Q_{i,j}^{n,m}) = \tilde{Q}_{i,\tau_i(j)}^{n,n-m}$.

Lemma 17. *For $A \in P_n$,*

$$\phi_{n,m}(A)^{[n-m]} = {}^tA_{[n-m]}^{-1}, \quad \phi_{n,m}(A)_{[m]} = {}^t(A^{[m]})^{-1},$$

$$u_{\phi_{n,m}(A),n-m} = -{}^tu_{A,m}.$$

Proof. Apply the automorphism $\phi_{n,m}$ to both sides of (11). □

Given a set S consisting of invertible matrices, denote the set $\{{}^t s^{-1} : s \in S\}$ by ${}^t S^{-1}$.

Lemma 18. *For $S \subset P_m$ and $S' \subset P_{n-m}$,*

$$\phi_{n,m}(F_{i,j}^{n,m}(S, S')) = \tilde{F}_{i,\tau_i(j)}^{n,n-m}({}^t S'^{-1}, {}^t S^{-1}).$$

Proof. We first show that $\phi_{n,m}(F_{i,j}^{n,m}) = \tilde{F}_{i,\tau_i(j)}^{n,n-m}$. First consider $A \in F_{i,j}^{n,m}$. Put

$$A(k,\gamma) = {}^t\overline{(\xi_{ij}\gamma\xi_{ik}^{-1})}A(\xi_{ij}\gamma\xi_{ik}^{-1})$$

for $1 \le k \le h$ and $\gamma \in \Gamma_i$. We have

$$|A(k,\gamma)| = \left(\frac{\kappa_{ij}}{\kappa_{ik}}\right)^2 |A| = \left(\frac{\kappa_{ij}}{\kappa_{ik}}\right)^2 |A^{[m]}||A_{[n-m]}|.$$

Substitute this and $|A(k,\gamma)^{[m]}| = |A(k,\gamma)||A(k,\gamma)_{[n-m]}|^{-1}$ into the inequality

$$|A^{[m]}|_\infty \le \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 |A(k,\gamma)^{[m]}|_\infty.$$

Rearranging, we get

$$|A_{[n-m]}|_\infty^{-1} \le \left(\frac{|\kappa_{ik}^{-1}|_\infty N(\mathfrak{a}_k)}{|\kappa_{ij}^{-1}|_\infty N(\mathfrak{a}_j)}\right)^2 |A(k,\gamma)_{[n-m]}|_\infty^{-1},$$

which, using the previous lemma, becomes

$$|\phi_{n,m}(A)|_\infty^{[n-m]} \le \left(\frac{N(\mathfrak{a}_{\tau_i(k)}^{-1})}{N(\mathfrak{a}_{\tau_i(j)}^{-1})}\right)^2 |\phi_{n,m}(A(k,\gamma))|_\infty^{[n-m]},$$

and since

$$\phi_{n,m}(A(k,\gamma)) = {}^t\overline{(\tilde{\xi}_{i\tau_i(j)}{}^t\gamma^{-1}\tilde{\xi}_{i\tau_i(k)}^{-1})}\phi_{n,m}(A)(\tilde{\xi}_{i\tau_i(j)}{}^t\gamma^{-1}\tilde{\xi}_{i\tau_i(k)}^{-1}),$$

this shows that $\phi_{n,m}(A) \in \tilde{F}_{i,\tau_i(j)}^{n,n-m}$. Thus $\phi_{n,m}(F_{i,j}^{n,m}) \subset \tilde{F}_{i,\tau_i(j)}^{n,n-m}$ and similarly $\phi_{n,n-m}(\tilde{F}_{i,\tau_i(j)}^{n,n-m}) \subset F_{i,j}^{n,m}$. The rest of our result follows from the previous lemma. □

Lemma 19. *Let Γ be a subgroup of $\mathrm{GL}_n(\Bbbk_\infty)$ acting on a subset X of P_n, the action being the one defined in* (10). *If F is a given fundamental domain for X/Γ and ϕ a group automorphism of $\mathrm{GL}_n(\Bbbk_\infty)$ that is also a topological isomorphism, then $\phi(F)$ is a fundamental domain for $\phi(X)/\phi(\Gamma)$.*

Proof. Since ϕ is both a group homomorphism and a topological isomorphism, $X = F^- \cdot \Gamma$ implies $\phi(X) = \phi(F)^- \cdot \phi(\Gamma)$. Also, for $g \in \Gamma$, if the intersection of $\phi(F)^\circ$ and $\phi(F)^- \cdot \phi(g)$ is nonempty, then so is $F^\circ \cap F^- \cdot g$, implying $g \in Z$. Since Z consists of all roots of unity in $\Bbbk$, we have $\phi(g) \in Z$. □

In particular, if for $k = 1, \ldots, h$ we let $\mathfrak{B}_k$ and $\mathfrak{C}_k$ be fundamental domains for $P_m/\Gamma_m(\mathfrak{a}_k)$ and $P_{n-m}/\Gamma_{n-m}(\mathfrak{a}_k)$ respectively as in the end of the previous section, then ${}^t\mathfrak{B}_k^{-1}$ and ${}^t\mathfrak{C}_k^{-1}$ are respectively fundamental domains for $P_{n-m}/\Gamma_{n-m}(\mathfrak{a}_k^{-1})$ and $P_m/\Gamma_m(\mathfrak{a}_k^{-1})$. Also we have:

Corollary 20. *The set*

$$\phi_{n,m}(F_{i,j}^{n,m}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)}))$$

is a fundamental domain for $\tilde{F}_{i,\tau_i(j)}^{n,n-m}/\tilde{Q}_{i,\tau_i(j)}^{n,n-m}$.

Corollary 21. *The set*

$${}^t\big(\Omega_i^{n,m}(\mathfrak{B}_1, \dots \mathfrak{B}_h, \mathfrak{C}_1, \dots, \mathfrak{C}_h)\big)^{-1}$$

is a fundamental domain for $P_n/\,\Gamma_n(\mathfrak{a}_i^{-1})$.

Since

$$\tilde{F}_{i.j}^{n,n-m}({}^t\mathfrak{C}_j^{-1}, {}^t\mathfrak{B}_{\tau_i(j)}^{-1}) = \phi_{n,m}(F_{i,\tau_i(j)}^{n,m}(\mathfrak{B}_{\tau_i(j)}, \mathfrak{C}_j)),$$

the first corollary is consistent with Lemma 15 in the previous section.

Similarly if we put

$$\tilde{\Omega}_i^{n,n-m}(\mathfrak{C}_1, \dots \mathfrak{C}_h, \mathfrak{B}_1, \dots, \mathfrak{B}_h) = \bigcup_{j=1}^{h} {}^t\bar{\tilde{\xi}}_{ij}\,\tilde{F}_{i,j}^{n,m}({}^t\mathfrak{C}_j^{-1}, {}^t\mathfrak{B}_{\tau_i(j)}^{-1})\tilde{\xi}_{ij}$$

then $\tilde{\Omega}_i^{n,n-m}(\mathfrak{C}_1, \dots \mathfrak{C}_h, \mathfrak{B}_1, \dots, \mathfrak{B}_h) = {}^t\big(\Omega_i^{n,m}(\mathfrak{B}_1, \dots \mathfrak{B}_h, \mathfrak{C}_1, \dots, \mathfrak{C}_h)\big)^{-1}$ and according to Theorem 16, this set is indeed a fundamental domain for $P_n/\,\Gamma_n(\mathfrak{a}_i^{-1})$.

Appendix: Voronoi reduction

by Takao Watanabe

We present here generalizations of results from [Watanabe et al. 2013, §4], without the assumption that the underlying number field is totally real.

Let $\Bbbk$, $\mathcal{O}$ and P_n be as previously defined in this paper. We consider the space of self-adjoint matrices in $M_n(\Bbbk_\infty)$ (with respect to the inner product $\langle\ ,\ \rangle$ as defined in [Watanabe et al. 2013, §1]), which we denote here by H_n. Identifying H_n with $\prod_{\sigma\in \boldsymbol{p}_\infty} H_n(\Bbbk_\sigma)$, where $H_n(\Bbbk_\sigma)$ denotes the set of $n\times n$ real symmetric (complex Hermitian) matrices when σ is real (imaginary respectively), we see that P_n is the set of positive definite matrices in H_n.

Also as per [Watanabe et al. 2013, §1], we use the inner product $(\ ,\ )$ on H_n defined by

$$(A, B) = \sum_{\sigma\in \boldsymbol{p}_\infty} \mathrm{Tr}_{\Bbbk_\sigma/\mathbb{R}}(\mathrm{Tr}(A_\sigma B_\sigma))$$

for $A = (A_\sigma)_{\sigma\in \boldsymbol{p}_\infty}$, $B = (B_\sigma)_{\sigma\in \boldsymbol{p}_\infty} \in H_n$.

Following [Watanabe et al. 2013, §2], we fix a projective $\mathcal{O}$-module $\Lambda \subset \Bbbk^n$ of rank n and consider the arithmetical minimum function

$$\mathrm{m}_\Lambda(A) = \inf_{x\in\Lambda\setminus\{0\}} \langle Ax, x\rangle$$

on P_n^-. The set

$$K_1(\mathsf{m}_\Lambda) = \{A \in P_n^- : \mathsf{m}_\Lambda(A) \geq 1\},$$

known as the Ryshkov polyhedron of m_Λ, is a locally finite polyhedron contained in P_n [Watanabe et al. 2013, Lemma 2.1 and Proposition 2.2]. The set of 0-dimensional faces of $K_1(\mathsf{m}_\Lambda)$, denoted by $\partial^0 K_1(\mathsf{m}_\Lambda)$, is characterized in [Watanabe et al. 2013, Theorem 2.5].

Now for a given $A \in P_n$ and a positive constant θ, define the sets

$$H_{A,\theta} = \{B \in H_n : (A, B) \leq \theta\},$$
$$[A]_\theta = \partial^0 K_1(\mathsf{m}_{\Lambda_0}) \cap H_{A,\theta}.$$

Lemma A1. *$[A]_\theta$ is a finite set.*

Proof. Since $H_{A,\theta} \cap P_n^-$ is compact [Faraut and Korányi 1994, Corollary I.1.6] and $K_1(\mathsf{m}_\Lambda)$ is a locally finite polyhedron, it follows that their intersection $K_1(\mathsf{m}_\Lambda) \cap H_{A,\theta}$ is a polytope. Hence $[A]_\theta$ must be finite. □

Lemma A2. *For an $A \in P_n$, there exists $B_0 \in \partial^0 K_1(\mathsf{m}_\Lambda)$ such that*

$$\inf_{B \in K_1(\mathsf{m}_\Lambda)} (A, B) = (A, B_0)$$

and hence A is in D_{B_0}, the perfect domain of B_0 [Watanabe et al. 2013, §3]. *Here*

$$D_{B_0} = \left\{ \sum_{x \in S_\Lambda(B_0)} \lambda_x x\, {}^t\bar{x} : \lambda_x \geq 0 \right\},$$

where

$$S_\Lambda(B_0) = \{x \in \Lambda : \mathsf{m}_\Lambda(B_0) = \langle B_0 x, x\rangle\}.$$

Proof. Take a sufficiently large $\theta > 0$ whereby $[A]_\theta$ is nonempty. Since $K_1(\mathsf{m}_\Lambda)$ is the convex hull of $\partial K_1(\mathsf{m}_\Lambda)$ [Watanabe et al. 2013, Theorem 2.6], we have

$$\inf_{B \in K_1(\mathsf{m}_\Lambda)} (A, B) = \inf_{B \in \partial K_1(\mathsf{m}_\Lambda)} (A, B) = \inf_{B \in [A]_\theta} (A, B),$$

which together with the previous lemma proves the existence of B_0. The proof that $A \in D_{B_0}$ is the same as in [Watanabe et al. 2013, Lemma 4.8]. □

Next consider the set

$$\Bbbk_\infty^+ = \{(\alpha_\sigma)_{\sigma \in \boldsymbol{p}_\infty} : \alpha_\sigma > 0 \text{ for all } \sigma \in \boldsymbol{p}_\infty\}.$$

Lemma A3. *The subset $\{\beta\bar{\beta} : \beta \in \Bbbk^\times\}$ of $\Bbbk_\infty$ is dense in $\Bbbk_\infty^+$.*

Proof. Define the norm $\|\cdot\|$ on $\Bbbk_\infty$ by

$$\|\alpha\| = \max_{\sigma \in \boldsymbol{p}_\infty} \sqrt{\alpha_\sigma \bar{\alpha}_\sigma}, \quad \alpha = (\alpha_\sigma) \in \Bbbk_\infty.$$

Now given a $\alpha \in \Bbbk_\infty^+$ there is an element $\sqrt{\alpha} \in \Bbbk_\infty^+$ such that $(\sqrt{\alpha})^2 = \alpha$. Since $\Bbbk$ is dense in $\Bbbk_\infty$, for a sufficiently small $\epsilon > 0$ we can find $\beta \in \Bbbk^\times$ such that

$$\|\sqrt{\alpha} - \beta\| < \frac{\epsilon}{2\|\sqrt{\alpha}\| + 1} < 1.$$

From $\|\beta\| < \|\sqrt{\alpha}\| + 1$, we have $\|\sqrt{\alpha} + \beta\| < 2\|\sqrt{\alpha}\| + 1$, and thus

$$\begin{aligned}\|\alpha - \beta\bar{\beta}\| &= \tfrac{1}{2}\left\|(\sqrt{\alpha} - \beta)(\sqrt{\alpha} + \bar{\beta}) + (\sqrt{\alpha} + \beta)(\sqrt{\alpha} - \bar{\beta})\right\| \\ &\le \tfrac{1}{2}\big(\|\sqrt{\alpha} - \beta\|\,\|\sqrt{\alpha} + \beta\| + \|\sqrt{\alpha} + \beta\|\,\|\sqrt{\alpha} - \beta\|\big) < \epsilon. \qquad \square\end{aligned}$$

Lemma A4. $$\Bbbk_\infty^+ \cup \{0\} = \left\{ \sum_{k=1}^{l} \lambda_k \beta_k {}^t\bar{\beta}_k : 1 \le l \in \mathbb{Z},\ \lambda_k \in \mathbb{R}_{\ge 0},\ \beta_k \in \Bbbk^\times \right\}.$$

Proof. See the proof of [Watanabe et al. 2013, Lemma 4.2]. □

As a result of the previous lemma, if we define the subsets

$$\Omega_1 = \left\{ \sum_{k=1}^{l} \alpha_k x_k {}^t\bar{x}_k : 1 \le l \in \mathbb{Z},\ \alpha_k \in \Bbbk_\infty^+ \cup \{0\},\ x_i \in \Bbbk^n \right\},$$

$$\Omega_2 = \left\{ \sum_{k=1}^{l} \lambda_k x_k {}^t\bar{x}_k : 1 \le l \in \mathbb{Z},\ \lambda_k \in \mathbb{R}_{\ge 0},\ x_i \in \Bbbk^n \right\}$$

of P_n^-, we have $\Omega_1 = \Omega_2$. Also by Lemma A2, $P_n \subset \Omega_2 = \Omega_1$.

Lemma A5. $$\Omega_2 = \bigcup_{B \in \partial^0 K_1(\mathsf{m}_\Lambda)} D_B.$$

Proof. For any $A \in \Omega_2 \backslash \{0\}$, following the same arguments as in the proofs of [Watanabe et al. 2013, Lemmas 4.7 and 4.8], we can find an element $B_0 \in \partial^0 K_1(\mathsf{m}_\Lambda)$ such that $\inf_{B \in K_1(\mathsf{m}_\Lambda)}(A, B) = (A, B_0)$ and hence $A \in D_{B_0}$. □

Finally take a complete set of representatives $B_1, \dots, B_t$ for $\partial^0 K_1(\mathsf{m}_\Lambda)/\mathrm{GL}(\Lambda)$, where the right action is the same one as (10), and for each $k = 1, \dots, t$ define the subgroups $\Gamma_{B_k} = \{\gamma \in \mathrm{GL}(\Lambda) : B_k \cdot {}^t\bar{\gamma} = B_k\}$. Since for any $A \in \partial^0 K_1(\mathsf{m})$ and $\gamma \in \mathrm{GL}(\Lambda)$ we have $S_\Lambda(A \cdot \gamma) = \gamma^{-1} S_\Lambda(A)$ and hence $D_{A \cdot {}^t\bar{\gamma}} = (D_A) \cdot \gamma^{-1}$, we see that Γ_{B_k} stabilizes D_{B_k} for each k. Thus we conclude from the previous lemma the following result.

Theorem A6. $$\Omega_2/\mathrm{GL}(\Lambda) = \bigcup_{k=1}^{t} D_{B_k}/\Gamma_{B_k}.$$

This is analogous to [Watanabe et al. 2013, Theorem 4.9]. In particular when $n = 1$, if we take $\Lambda = O$, we have $\mathrm{GL}(\Lambda) = O^\times$ and $P_1 = \Bbbk_\infty^+ = \Omega_1 \backslash \{0\} = \Omega_2 \backslash \{0\}$.

Since the action of $\mathcal{O}^\times$ on $\Bbbk_\infty^+$ is simply $x \cdot \epsilon = \bar{\epsilon}\epsilon x$ ($x \in \Bbbk_\infty^+$, $\epsilon \in \mathcal{O}^\times$), we have $\Gamma_{B_k} = Z$ acts trivially on D_{B_k} for each k. Thus we obtain the decomposition

$$P_1/\mathcal{O}^\times = \Bbbk_\infty^+/\mathcal{O}^\times = \bigcup_{k=1}^{t} (D_{B_k} \backslash \{0\}).$$

By definition each $D_{B_k} \backslash \{0\}$ is invariant under multiplication by $\mathbb{R}_{>0}$, so this establishes the existence of the fundamental domain Ω^1 in the conclusion of Section 5.2.

Acknowledgements

I would like to express my utmost gratitude to Professor Takao Watanabe for his patience and invaluable guidance in my research, as well as his work in [Watanabe 2014], which has served as a basis of the results in this paper. I also would like to thank Professor Achill Schürmann and Dr. Mathieu Dutour Sikirić for confirming via private communication their observation on the fundamental domains of Korkin and Zolotarev and Grenier.

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Received May 31, 2015. Revised January 23, 2017.

LEE TIM WENG
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
TOYONAKA
JAPAN
lt-weng@cr.math.sci.osaka-u.ac.jp
now_then_here_there@hotmail.com

TAKAO WATANABE
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
TOYONAKA
JAPAN
twatanabe@math.sci.osaka-u.ac.jp

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PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 290 No. 1 September 2017