Pacific
Journal of
Mathematics

# GROWTH AND DISTORTION THEOREMS FOR SLICE MONOGENIC FUNCTIONS

GUANGBIN REN AND XIEPING WANG

Volume 290 No. 1 September 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 1, 2017
https://doi.org/10.2140/pjm.2017.290.169

# GROWTH AND DISTORTION THEOREMS FOR SLICE MONOGENIC FUNCTIONS

GUANGBIN REN AND XIEPING WANG

**We establish the sharp growth and distortion theorems for slice monogenic extensions of univalent functions on the unit disc $\mathbb{D} \subset \mathbb{C}$ in the setting of Clifford algebras, based on a new convex combination identity. The analogous results are also valid in the quaternionic setting for slice regular functions and we can even prove a Koebe type one-quarter theorem in this case. Our growth and distortion theorems for slice regular (slice monogenic) extensions to higher dimensions of univalent holomorphic functions hold without extra geometric assumptions, in contrast to the setting of several complex variables in which the growth and distortion theorems fail in general and hold only for some subclasses with the starlike or convex assumption.**

## 1. Introduction

In geometric function theory of holomorphic functions of one complex variable, the following well-known growth and distortion theorems (see, e.g., [Duren 1983; Graham and Kohr 2003]) mark the beginning of the systematic study of univalent functions.

**Theorem 1.1** (growth and distortion theorems). *Let $F$ be a univalent function on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = 0$ and $F'(0) = 1$. Then for each $z \in \mathbb{D}$, the following inequalities hold*:

$$\frac{|z|}{(1+|z|)^2} \le |F(z)| \le \frac{|z|}{(1-|z|)^2}; \tag{1-1}$$

$$\frac{1-|z|}{(1+|z|)^3} \le |F'(z)| \le \frac{1+|z|}{(1-|z|)^3}; \tag{1-2}$$

$$\frac{1-|z|}{1+|z|} \le \left|\frac{zF'(z)}{F(z)}\right| \le \frac{1+|z|}{1-|z|}. \tag{1-3}$$

This work was supported by the NNSF of China (11371337).
*MSC2010:* primary 30G35; secondary 30C45.
*Keywords:* quaternions, Clifford algebra, slice regular (slice monogenic) functions, growth and distortion theorems, Koebe one-quarter theorem.

*Moreover, equality holds for one of these six inequalities at some point* $z_0 \in \mathbb{D} \setminus \{0\}$ *if and only if* $F$ *is a rotation of the Koebe function, i.e.,*

$$F(z) = \frac{z}{(1 - e^{i\theta} z)^2}, \qquad \forall z \in \mathbb{D},$$

*for some* $\theta \in \mathbb{R}$.

The extension of geometric function theory to higher dimensions was suggested by H. Cartan [1933], but the first meaningful result was only made in 1991 by Barnard, Fitzgerald and Gong [Barnard et al. 1991]. Since then, the geometric function theory in several complex variables has been extensively studied, see, for example, [Gong 1998; Graham and Kohr 2003]. In particular, the growth theorem holds for starlike mappings on starlike circular domains [Liu and Ren 1998a], and for convex mappings on convex circular domains [Liu and Ren 1998b].

However, as far as we know, nearly nothing has been done about the corresponding theory for other classes of functions, such as the classical regular (monogenic) functions in the sense of Fueter and the recently introduced slice regular (slice monogenic) functions, mainly because both regularity (monogenicity) and slice regularity (slice monogenicity) of functions are seldom preserved under multiplication and composition, because of the noncommutativity of the underlying algebras on which these functions are defined.

In this paper, we shall focus on slice regular and slice monogenic functions and aim to generalize Theorem 1.1 to the noncommutative setting for slice regular and slice monogenic extensions of univalent functions on the unit disc $\mathbb{D} \subset \mathbb{C}$. The theory of slice regular functions of one quaternionic variable was initiated recently by Gentili and Struppa [2006; 2007], and was also extended by the same authors to octonions [2010] for octonionic slice regular functions. The related theory of slice monogenic functions on domains in the paravector space $\mathbb{R}^{n+1}$ with values in the Clifford algebra $\mathbb{R}_n$ was introduced in [Colombo et al. 2009; 2010]. For a more complete insight and further references, we refer the reader to the monographs [Gentili et al. 2013; Colombo et al. 2011a]. These function theories were also unified and generalized in [Ghiloni and Perotti 2011a] using the concept of slice functions on the so-called quadratic cone of a real alternative *-algebra, based on a slight modification of a well-known construction due to Fueter. The theory of slice regular functions on real alternative *-algebras is now well-developed through a series of papers mainly due to Ghiloni and Perotti after their seminal work [Ghiloni and Perotti 2011a]. It is also well worth mentioning that this recently introduced theory of slice regular (slice monogenic) functions is significantly different from the more classical theory of regular (monogenic) functions in the sense of Fueter (cf. [Brackx et al. 1982; Colombo et al. 2004; Gürlebeck et al. 2008]), and has elegant applications to the functional calculus for noncommutative operators [Colombo et al. 2011a], to Schur analysis [Alpay et al. 2016], and to the construction and classification of orthogonal complex structures on dense open subsets of $\mathbb{R}^4 \simeq \mathbb{H}$ [Gentili et al. 2014].

We are now in a position to state one of our main results in the case of the Clifford algebra $\mathbb{R}_n$ for slice monogenic extensions to the open unit ball

$$\mathbb{B} := \{x \in \mathbb{R}^{n+1} : |x| < 1\}$$

of univalent functions on the unit disc $\mathbb{D} \subset \mathbb{C}$.

**Theorem 1.2.** *Let $F : \mathbb{D} \to \mathbb{C}$ be a univalent function such that $F(0) = 0$ and $F'(0) = 1$, and let $f : \mathbb{B} \to \mathbb{R}_n$ be the slice monogenic extension of $F$. Then for each $x \in \mathbb{B}$, the following inequalities hold*:

$$\frac{|x|}{(1+|x|)^2} \leq |f(x)| \leq \frac{|x|}{(1-|x|)^2}; \tag{1-4}$$

$$\frac{1-|x|}{(1+|x|)^3} \leq |f'(x)| \leq \frac{1+|x|}{(1-|x|)^3}; \tag{1-5}$$

$$\frac{1-|x|}{1+|x|} \leq \left|xf'(x) * f^{-*}(x)\right| \leq \frac{1+|x|}{1-|x|}. \tag{1-6}$$

*Moreover, equality holds for one of these six inequalities at some point $x_0 \in \mathbb{B} \setminus \{0\}$ if and only if*

$$f(x) = x(1 - xe^{i\theta})^{-*2}, \qquad \forall x \in \mathbb{B},$$

*for some $\theta \in \mathbb{R}$.*

Although Theorem 1.2 coincides in form with Theorem 1.1, the classical approach to Theorem 1.1 cannot be directly applied in this new case of the Clifford algebra $\mathbb{R}_n$, since there lacks a fruitful theory of compositions for slice monogenic functions. We shall reduce Theorem 1.2 to Theorem 1.1 via a new convex combination identity; see (3-11). We remark that in contrast to the setting of several complex variables in which the growth and distortion theorems fail to hold in general [Cartan 1933] and can only be restricted to the starlike or convex subclasses, our result for slice monogenic extensions of univalent functions holds without extra geometric assumptions. This new phenomenon is in a certain sense related to the rigidity of the functions under consideration. There is a significant difference between slice monogenic functions and holomorphic functions of several complex variables, although they are both the generalizations in higher dimensions of holomorphic functions of one complex variable. The former are closer to holomorphic functions of one complex variable, and each of them can be completely determined by its values on a set that lies in a complex slice and has an accumulation point in its domain of definition. However, this is not the case for the latter, each of which is not always determined by its values on a complex submanifold of positive codimensions in its domain of definition. From this perspective, we realize that holomorphic functions of several complex variables are less rigid than slice monogenic functions so that certain extra geometric assumptions such as starlikeness and convexity are naturally present in the geometric function theory in several complex variables.

A result analogous to Theorem 1.2 also holds in the setting of quaternions (see Theorem 4.7). As an application, we can prove a covering theorem, i.e., the so-called Koebe type one-quarter theorem (see Theorem 4.10, a generalization of [Gal et al. 2015, Theorem 3.11 (1)]), with the help of the open mapping theorem, which is now known to hold only for slice regular functions defined on symmetric slice domains in $\mathbb{H}$ with values in $\mathbb{H}$ rather than slice monogenic functions defined on symmetric slice domains in paravector space $\mathbb{R}^{n+1}$ with values in the Clifford algebra $\mathbb{R}_n$.

We now describe in more detail the structure of the paper. In Section 2, we set up basic notation and give some preliminary results. In Section 3, we first prove in Proposition 3.1 a general formula to express the squared norm of a slice monogenic function defined on a symmetric slice domain in the paravector space $\mathbb{R}^{n+1}$, in terms of the values of the function at two conjugate points on some fixed slice of the domain. For slice monogenic functions that preserve one slice, we provide in Lemma 3.2 the aforementioned convex combination identity, which is the key ingredient in proving Theorem 1.2. Section 4 is devoted to the detailed proofs of the analogous results and the Koebe type one-quarter theorem (Theorem 4.10) for slice regular functions in the quaternionic setting. Thanks to the specialty of quaternions, we can also provide in Corollary 4.4 a sufficient and necessary condition under which the aforementioned convex combination identity holds identically. Finally, Section 5 provides a concluding remark and an open question.

## 2. Preliminaries

We recall in this section some necessary definitions and preliminary results on real Clifford algebras and slice monogenic functions. For a more complete insight, we refer the reader to the monograph [Colombo et al. 2011a].

The real Clifford algebra $\mathbb{R}_n = \mathrm{Cl}_{0,n}$ is an associative algebra over $\mathbb{R}$ generated by $n$ basis elements $e_1, e_2, \ldots, e_n$, subject to the relation

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, 2, \ldots, n.$$

As a real vector space, $\mathbb{R}_n$ has dimension $2^n$. Each element $b$ in $\mathbb{R}_n$ can be represented *uniquely* as

$$b = \sum_A b_A e_A,$$

where $b_A \in \mathbb{R}$, $e_0 = 1$, $e_A := e_{h_1} e_{h_2} \cdots e_{h_r}$, and $A = h_1 \cdots h_r$ is a multi-index such that $1 \le h_1 < \cdots < h_r \le n$. The real number $b_0$ is called the *scalar* part of

$b$ and is denoted by $\mathrm{Sc}(b)$ as usual. The *Clifford conjugate* of each generator $e_i$, $i = 1, 2, \ldots, n$, is defined to be $\bar{e}_i = -e_i$, and thus extends to each $e_A$ by setting

$$\bar{e}_A := \bar{e}_{h_r}\bar{e}_{h_{r-1}}\cdots\bar{e}_{h_1} = (-1)^r e_{h_r}e_{h_{r-1}}\cdots e_{h_1} = (-1)^{r(r+1)/2}e_A,$$

and further extends by linearity to each element $b = \sum_A b_A e_A \in \mathbb{R}_n$ so that

$$\bar{b} = \sum_A b_A \bar{e}_A.$$

Therefore, the Clifford conjugate is an antiautomorphism of $\mathbb{R}_n$, i.e., $\overline{ab} = \bar{b}\bar{a}$ for any $a, b \in \mathbb{R}_n$. Moreover, the Euclidean inner product on $\mathbb{R}_n \simeq \mathbb{R}^{2^n}$ is given by

$$\langle a, b\rangle := \mathrm{Sc}(a\bar{b}) = \sum_A a_A b_A \tag{2-1}$$

for any $a = \sum_A a_A e_A,\ b = \sum_A b_A e_A \in \mathbb{R}_n$, so it follows from the simple identity

$$\langle a, b\rangle = \tfrac{1}{2}\big(|a+b|^2 - |a|^2 - |b|^2\big)$$

that

$$\langle a, b\rangle = \langle b, a\rangle = \langle \bar{a}, \bar{b}\rangle = \langle \bar{b}, \bar{a}\rangle. \tag{2-2}$$

It is worth remarking here that for $\mathbb{R}_n (n \geq 3)$ the multiplicative property of the Euclidean norm fails in general, and holds only for some special cases; see [Colombo et al. 2011a, Proposition 2.1.17] or [Gürlebeck et al. 2008, Theorem 3.14 (ii)]. In particular, it holds that

$$|ab| = |ba| = |a||b| \tag{2-3}$$

whenever one of $a$ and $b$ is a paravector (see below for this definition). This simple fact will be useful for our argument in Section 3.

For convenience, some specific elements in $\mathbb{R}_n$ can be identified with vectors in the Euclidean space $\mathbb{R}^{n+1}$: an element $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ will be identified with a so-called 1-*vector* in the Clifford algebra $\mathbb{R}_n$ through the map

$$(x_1, x_2, \ldots, x_n) \mapsto \underline{x} = x_1e_1 + e_2x_2 + \cdots + x_ne_n;$$

and an element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with

$$x = x_0 + \underline{x} = x_0 + x_1e_1 + \cdots + x_ne_n,$$

which is called a *paravector*. Now for any two 1-vectors $x, y \in \mathbb{R}^n$, the Euclidean inner product becomes

$$\langle x, y\rangle = \mathrm{Sc}(x\bar{y}) = -\tfrac{1}{2}(xy + yx),$$

and consequently,

$$xy = -\langle x, y\rangle + x \wedge y,$$

where

$$x \wedge y := \tfrac{1}{2}(xy - yx)$$

is called the *outer product* (see [Brackx et al. 1982, p.4; Gürlebeck et al. 2008, p.58]) or *wedge product* (see [Colombo et al. 2004, p.218, Definition 4.1.9; Colombo et al. 2011a, p.21]) of $x$ and $y$. It is noteworthy here that *in general the operator* $\wedge$ *is a mapping from* $\mathbb{R}^n \times \mathbb{R}^n$ *to* $\mathbb{R}_n$, *not to* $\mathbb{R}^n$. Furthermore, under the identifications above, a vector $x$ in $\mathbb{R}^{n+1}$ can be taken as a Clifford number

$$x = x_0 + \sum_{i=1}^{n} x_i e_i$$

so that it has inverse

$$x^{-1} = \frac{\bar{x}}{|x|^2},$$

where $\bar{x}$ is the *conjugate* of $x$ given by $\bar{x} = x_0 - \sum_{i=1}^{n} x_i e_i$, and the norm of $x$ is induced by the inner product given above, that is, $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Every $x = x_0 + x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^{n+1}$ is composed by the *scalar* part $\mathrm{Sc}(x) = x_0 \in \mathbb{R}$ and the *vector* part $\underline{x} = x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^n$, and it can be expressed alternatively as $x = u + Iv$, where $u, v \in \mathbb{R}$ and

$$I = \frac{\underline{x}}{|\underline{x}|}$$

if $\underline{x} \neq 0$, otherwise we take $I$ arbitrarily in $\mathbb{R}^n$ such that $I^2 = -1$. Then $I$ is an element of the unit $(n-1)$-sphere of 1-vectors in $\mathbb{R}^n$,

$$\mathbb{S} = \left\{\underline{x} = x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\right\}.$$

For every $I \in \mathbb{S}$ we will denote by $\mathbb{C}_I$ the plane $\mathbb{R} \oplus I\mathbb{R}$, isomorphic to $\mathbb{C}$, and, if $U \subseteq \mathbb{R}^{n+1}$, by $U_I$ the intersection $U \cap \mathbb{C}_I$. Also, for $R > 0$, we will denote the open ball of $\mathbb{R}^{n+1}$ centered at the origin with radius $R$ by

$$B(0, R) = \{x \in \mathbb{R}^{n+1} : |x| < R\}.$$

We can now recall the definition of slice monogenicity.

**Definition 2.1.** Let $U$ be a domain in $\mathbb{R}^{n+1}$. A function $f : U \to \mathbb{R}_n$ is called *slice monogenic* if, for all $I \in \mathbb{S}$, its restriction $f_I$ to $U_I$ is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(u + vI) := \tfrac{1}{2}\Big(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\Big) f_I(u + vI) = 0,$$

for all $u + vI \in U_I$.

For slice monogenic functions, the natural domains of definition are symmetric slice domains.

**Definition 2.2.** Let $U$ be a domain in $\mathbb{R}^{n+1}$.

(i) $U$ is called a *slice domain* if it intersects the real axis and if for each $I \in \mathbb{S}$, $U_I$ is a domain in $\mathbb{C}_I$.

(ii) $U$ is called an *axially symmetric domain* if for every point $u + vI \in U$, with $u, v \in \mathbb{R}$ and $I \in \mathbb{S}$, the entire sphere $u + v\mathbb{S}$ is contained in $U$.

A domain in $\mathbb{R}^{n+1}$ is called a *symmetric slice domain* if it is not only a slice domain, but also an axially symmetric domain. By the very definition, an open ball $B(0, R)$ is a typical symmetric slice domain. From now on, we will focus mainly on slice monogenic functions on $B(0, R)$. In most cases, the following results hold, with appropriate changes, for symmetric slice domains more general than open balls of the type $B(0, R)$. For slice monogenic functions a natural definition of derivative is given by the following.

**Definition 2.3.** Let $f : B(0, R) \to \mathbb{R}_n$ be a slice monogenic function. The *slice derivative* of $f$ is defined to be

$$\partial_I f(u + vI) := \tfrac{1}{2}\Big(\frac{\partial}{\partial u} - I\frac{\partial}{\partial v}\Big) f_I(u + vI).$$

Notice that the operators $\partial_I$ and $\bar{\partial}_I$ commute, and

$$\partial_I f(u + vI) = \frac{\partial}{\partial u} f(u + vI)$$

holds for slice monogenic functions. Therefore, the slice derivative of a slice monogenic function is still slice monogenic so we can iterate the differentiation to obtain the $k$-th slice derivative,

$$\partial_I^k f(u + vI) = \Big(\frac{\partial}{\partial u}\Big)^k f(u + vI), \quad \forall k \in \mathbb{N}.$$

In what follows, for the sake of simplicity, we will directly denote the $k$-th slice derivative $\partial_I^k f$ by $f^{(k)}$ for every $k \in \mathbb{N}$.

As shown in [Colombo et al. 2009], a paravector power series $\sum_{k=0}^{\infty} x^k a_k$ with $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_n$ defines a slice monogenic function in its domain of convergence, which proves to be an open ball $B(0, R)$ with $R$ equal to the radius of convergence of the power series. The converse result is also true.

**Theorem 2.4.** *A function $f$ is slice monogenic on $B = B(0, R)$ if and only if $f$ has a power series expansion*

$$f(x) = \sum_{k=0}^{\infty} x^k a_k \quad \textit{with} \quad a_k = \frac{f^{(k)}(0)}{k!}.$$

A fundamental result in the theory of slice monogenic functions is described by the splitting lemma, which relates the notion of slice monogenicity to the classical notion of holomorphicity; see [Colombo et al. 2009].

**Lemma 2.5.** *Let $f$ be a slice monogenic function on $B = B(0, R)$. For each $I_1 = I \in \mathbb{S}$, let $I_2, \ldots, I_n$ be a completion to a basis of $\mathbb{R}^n$ satisfying the defining relations $I_i I_j + I_j I_i = -2\delta_{ij}$. Then there exist $2^{n-1}$ holomorphic functions $F_A : B_I \to \mathbb{C}_I$ such that for every $z = u + vI \in B_I$,*

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A,$$

*where $I_0 = 1$ when $r = 0$, or $I_A = I_{i_1} I_{i_2} \cdots I_{i_r}$, with $A = i_1 i_2 \cdots i_r$ a multi-index such that $2 \le i_1 < \cdots < i_r \le n$ when $r > 0$.*

The following version of the identity principle is one of the direct consequences of the preceding lemma; see [Colombo et al. 2009].

**Theorem 2.6.** *Let $f$ be a slice monogenic function on $B = B(0, R)$. Denote by $\mathcal{Z}_f$ the zero set of $f$,*

$$\mathcal{Z}_f = \{x \in B : f(x) = 0\}.$$

*If there exists an $I \in \mathbb{S}$ such that $B_I \cap \mathcal{Z}_f$ has an accumulation point in $B_I$, then $f$ vanishes identically on $B$.*

Another useful result is Theorem 2.7; see [Colombo and Sabadini 2009].

**Theorem 2.7.** *Let $f$ be a slice monogenic function on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$ and let $I \in \mathbb{S}$. Then for all $u + vJ \in U$ with $J \in \mathbb{S}$,*

$$f(u + vJ) = \tfrac{1}{2}\big(f(u + vI) + f(u - vI)\big) + \tfrac{1}{2} JI\big(f(u - vI) - f(u + vI)\big).$$

*In particular, for each sphere of the form $u + v\mathbb{S}$ contained in $U$, there exist $b, c \in \mathbb{R}_n$ such that $f(u + vI) = b + Ic$ for all $I \in \mathbb{S}$.*

Thanks to this result, it is possible to recover the values of a slice monogenic function on symmetric slice domains, which are more general than open balls centered at the origin, from its values on a single slice. This yields an extension theorem that, in the special case of functions that are slice monogenic on $B(0, R)$, can be obtained by means of their power series expansions.

**Remark 2.8.** Fix an element $I \in \mathbb{S}$ and denote by $B_I$ the intersection $B(0, R) \cap \mathbb{C}_I$ of the open ball $B(0, R)$ with the complex plane $\mathbb{C}_I$. Given a holomorphic function $f_I : B_I \to \mathbb{C}_I$ with the power series expansion taking the form

$$f_I(z) = \sum_{k=0}^{\infty} z^k a_k,$$

where $\{a_k\}_{k \in \mathbb{N}} \subset \mathbb{C}_I$, the unique slice monogenic extension of $f_I$ to the whole ball $B(0, R)$ is the function given by

$$f(x) := \mathrm{ext}(f_I)(x) = \sum_{k=0}^{\infty} x^k a_k,$$

which takes values in $\mathbb{R}_n$. The uniqueness is guaranteed by the identity principle; that is, Theorem 2.6. In Section 3, we will establish the growth and distortion theorems for such a class of slice monogenic functions that are injective on $B_I$.

Since slice monogenicity is not preserved under the usual pointwise product of two slice monogenic functions, a new multiplication operation, called the slice monogenic product (or $*$-product), appears via a suitable modification of the usual operation, subject to the noncommutative setting, and plays a key role in the theory of slice monogenic functions. On open balls centered at the origin, the slice monogenic product of two slice monogenic functions is defined by means of their power series expansions; see [Colombo et al. 2010; 2011a].

**Definition 2.9.** Let $f, g : B = B(0, R) \to \mathbb{R}_n$ be two slice monogenic functions and let

$$f(x) = \sum_{k=0}^{\infty} x^k a_k, \qquad g(x) = \sum_{k=0}^{\infty} x^k b_k$$

be their power series expansions. The *slice monogenic product* ($*$-*product*) of $f$ and $g$ is the function defined by

$$f * g(x) = \sum_{k=0}^{\infty} x^k \Bigl(\sum_{j=0}^{k} a_j b_{k-j}\Bigr),$$

which is slice monogenic on $B$.

We now recall more definitions (see, e.g., [Colombo et al. 2010; 2011a; Ghiloni and Perotti 2011a; 2011b]).

**Definition 2.10.** Letting $f(x) = \sum_{k=0}^{\infty} x^k a_k$ be a slice monogenic function on $B = B(0, R)$, we define the *slice monogenic conjugate* of $f$ as

$$f^c(x) = \sum_{k=0}^{\infty} x^k \bar{a}_k,$$

and the *symmetrization* of $f$ as

$$f^s(x) := \sum_{k=0}^{\infty} x^k \mathrm{Sc}\Bigl(\sum_{j=0}^{k} a_j \bar{a}_{k-j}\Bigr). \tag{2-4}$$

Moreover, we define the *normal function* of $f$ as

$$N(f)(x) := f * f^c(x) = \sum_{k=0}^{\infty} x^k \Bigl(\sum_{j=0}^{k} a_j \bar{a}_{k-j}\Bigr). \tag{2-5}$$

These three functions are slice monogenic on $B$.

**Remark 2.11.** Here are several useful remarks concerning Definitions 2.9 and 2.10:

(i) The slice monogenic product ($*$-product), the slice monogenic conjugate, and symmetrization can also be defined for slice monogenic functions $f$ on symmetric slice domains $U$ in $\mathbb{R}^{n+1}$ (we refer the interested reader to [Colombo et al. 2010] or [Colombo et al. 2011a, Section 2.6] for details). Moreover, for any two slice monogenic functions $f, g : U \to \mathbb{R}_n$ and each point $x_0 \in \mathbb{R}$, we can define two slice monogenic functions $f_{x_0}$ and $g_{x_0}$ on the symmetric slice domain $U_{x_0} := U - x_0$ by setting

$$f_{x_0}(x) = f(x + x_0), \qquad g_{x_0}(x) = g(x + x_0)$$

for each $x \in U_{x_0}$. Then we have the following identity

$$(f * g)_{x_0} = f_{x_0} * g_{x_0}.$$

This follows from the identity principle together with the fact that when restricted to the real axis, the slice monogenic product is just the usual pointwise one.

(ii) For slice monogenic functions on open balls of type $B := B(0, R)$, the notion of slice monogenic conjugate coincides with the one introduced in Definition 5.4 of [Colombo et al. 2010] (see also Proposition 5.5 therein). Further, the notion of symmetrization given here is equivalent to the one introduced in Definition 5.6. of that paper. To see this, we proceed as follows: For a slice monogenic function $f : B \to \mathbb{R}_n$, we denote by $f^{\mathrm{s}}$ the symmetrization of $f$ according to [Colombo et al. 2010, Definition 5.6]. By considering the power series expansion of $f^{\mathrm{s}}$, we may assume that

$$f^{\mathrm{s}}(x) = \sum_{k=0}^{\infty} x^k \alpha_k. \tag{2-6}$$

We also fix an element $I \in \mathbb{S}$. Then according to [Colombo et al. 2010, p.386] or [Colombo et al. 2011a, p.50], for each $x \in B_I$, we have

$$f^{\mathrm{s}}(x) = \mathrm{Sc}\big(f * f^c(x)\big) + \big\langle f * f^c(x), I\big\rangle I.$$

Now substituting (2-5) and (2-6) into the preceding equality, we see that for each $x \in B \cap \mathbb{R}$,

$$\sum_{k=0}^{\infty} x^k \alpha_k = \sum_{k=0}^{\infty} x^k \mathrm{Sc}\Big(\sum_{j=0}^{k} a_j \bar{a}_{k-j}\Big) + \sum_{k=0}^{\infty} x^k \Big\langle \sum_{j=0}^{k} a_j \bar{a}_{k-j}, I\Big\rangle I.$$

For each $k \in \mathbb{N}$, since $\sum_{j=0}^{k} a_j \bar{a}_{k-j}$ is invariant under the Clifford conjugate (see (2-9) below), the second summation on the right-hand side of the preceding equality must vanish identically. Indeed, in view of (2-2),

$$\left\langle \sum_{j=0}^{k} a_j \bar{a}_{k-j}, I \right\rangle = \left\langle \sum_{j=0}^{k} \overline{a_j \bar{a}_{k-j}}, \bar{I} \right\rangle = -\left\langle \sum_{j=0}^{k} a_j \bar{a}_{k-j}, I \right\rangle,$$

which must be zero. Consequently, we deduce that the equality

$$\sum_{k=0}^{\infty} x^k \alpha_k = \sum_{k=0}^{\infty} x^k \operatorname{Sc}\left( \sum_{j=0}^{k} a_j \bar{a}_{k-j} \right)$$

holds for all $x \in B \cap \mathbb{R}$. By uniqueness,

$$\alpha_k = \operatorname{Sc}\left( \sum_{j=0}^{k} a_j \bar{a}_{k-j} \right), \qquad \forall k \in \mathbb{N}.$$

This shows that $f^{\mathrm{s}}$ is the same as $f^s$ defined in (2-4).

(iii) In view of (i), the definition $N(f) := f * f^c$ is also valid for slice monogenic functions $f$ on symmetric slice domains in $\mathbb{R}^{n+1}$.

(iv) The notation $N(f)$ in the definition of normal functions is chosen in accordance with [Ghiloni and Perotti 2011a, Definition 11], which treated the case of slice functions on symmetric open subsets of the so-called *quadratic cone* of a finite-dimensional real alternative $^*$-algebra.

(v) For each slice monogenic function $f$ on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$ and each element $I \in \mathbb{S}$, the restriction $N(f)_I$ of $N(f)$ to $U_I := U \cap \mathbb{C}_I$ coincides with the function $f_I * f_I^c : U_I \to \mathbb{R}_n$ considered in [Colombo et al. 2010] or [Colombo et al. 2011a, Section 2.6].

With parts (i) and (iii) of Remark 2.11 in mind, the inverse element of a non-identically vanishing slice monogenic functions with respect to the $*$-product can be defined under a suitable condition.

**Definition 2.12.** Let $f$ be a slice monogenic function on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$ such that

$$N(f)(U_I) \subseteq \mathbb{C}_I$$

for *some* $I \in \mathbb{S}$. If $f$ does not vanish identically, its *slice monogenic inverse* is the function defined by

$$f^{-*}(x) := f^s(x)^{-1} f^c(x),$$

which is slice monogenic on $U \setminus \mathcal{Z}_{f^s}$. Here $\mathcal{Z}_{f^s}$ denotes the zero set of the symmetrization $f^s$ of $f$.

**Remark 2.13.** Two useful remarks concerning Definition 2.12 are in order:

(i) For each function $f$ as described in Definition 2.12, the requirement that

$$N(f)(U_I) \subseteq \mathbb{C}_I$$

for some $I \in \mathbb{S}$ guarantees that $f_I^s$ coincides with $N(f)_I = f_I * f_I^c$, see [Colombo et al. 2011a, Definition 2.6.10], although this fact is not explicitly proven in that paper.

(ii) Also we will see, in the proof of Proposition 2.14, that for each function $f$ as described in Definition 2.12, the coefficients which appeared in (2-5) are *real numbers*. This implies that for each such function $f$, its normal function $N(f)$ is the same as its symmetrization $f^s$, which is a slice preserving function so that its slice monogenic inverse

$$f^{-*}(x) = f^s(x)^{-1} f^c(x) = \big(N(f)(x)\big)^{-1} f^c(x)$$

is indeed slice monogenic on $U \setminus \mathcal{Z}_{f^s}$. Furthermore, it is well worth noting that in view of [Colombo et al. 2011a, Remark 2.6.8 and Lemma 2.5.12], the zero set $\mathcal{Z}_{f^s}$ of $f^s$ is precisely the union of isolated spheres of the form $u + v\mathbb{S}$ with $u, v \in \mathbb{R}$. This implies that $U \setminus \mathcal{Z}_{f^s}$ is a symmetric slice domain in $\mathbb{R}^{n+1}$.

The function $f^{-*}$ defined in Definition 2.12 deserves the name of slice monogenic inverse of $f$ due to the following:

**Proposition 2.14.** *Let $f$ be as described in Definition 2.12. Then we have*

$$f|_{U\setminus\mathcal{Z}_{f^s}} * f^{-*} = f^{-*} * f|_{U\setminus\mathcal{Z}_{f^s}} = 1, \tag{2-7}$$

*and*

$$(f^{-*})^{-*} = f|_{U\setminus\mathcal{Z}_{f^s}}. \tag{2-8}$$

This proposition is quite important in the theory of slice monogenic functions. The equalities in (2-7) first appeared in [Colombo et al. 2010, Proposition 5.9], but the proofs given there and in [Colombo et al. 2011a, Proposition 2.6.11] seem incomplete — the equality $f_I * f_I^c = f_I^c * f_I$ (which is equivalent to $N(f) = N(f^c)$, in view of Remark 2.11 (v) and the identity principle) is used without being proven. A different approach has been used in [Colombo et al. 2011b, Proposition 3.2]. A complete treatment has been given in [Ghiloni et al. 2016, Section 2] in the case of slice functions, which subsumes the case of slice monogenic functions. To keep our presentation self-contained, we provide here a detailed proof of Proposition 2.14.

*Proof.* We first prove (2-7). To this end, we need the following well known facts:

**Fact 1:** *For any $a, b \in \mathbb{R}_n$, $ab = 1$ if and only if $ba = 1$.*

**Fact 2:** *For each $a \in \mathbb{R}_n$, $a\bar{a} = 0$ if and only if $a = 0$.*

Indeed, Fact 1 holds for all finite-dimensional associative algebras (see, e.g., [Drozd and Kirichenko 1994, Theorem 1.2.1]), and Fact 2, which immediately follows from (2-1), is called *nonsingularity* of $\mathbb{R}_n$.

Note that $f$ does not vanish identically on $U$, and neither does the restriction $f|_{U\cap\mathbb{R}}$ of $f$ to $U\cap\mathbb{R}$, in view of the identity principle. Thus we can find one point $x_0\in U\cap\mathbb{R}$ and a positive number $R>0$ such that the open ball $B(x_0,R)$ is contained in $U$ and $f$ is nowhere vanishing on $B(x_0,R)$. Thanks to Remark 2.11(i), we may further assume that $x_0=0$ without loss of generality. Now we expand $f$ on $B:=B(0,R)$ as

$$f(x)=\sum_{k=0}^{\infty}x^k a_k.$$

Since there exists an element $I\in\mathbb{S}$ such that $N(f)=f*f^c$ maps $U_I$ into $\mathbb{C}_I$ (and also maps $B_I$ into $\mathbb{C}_I$), and

$$\sum_{j=0}^{k}\overline{a_j\bar{a}_{k-j}}=\sum_{j=0}^{k}a_{k-j}\,\bar{a}_j\ \overset{j\to k-j}{=}\ \sum_{j=0}^{k}a_j\bar{a}_{k-j},\tag{2-9}$$

we see that $\sum_{j=0}^{k}a_j\bar{a}_{k-j}$ must be a real number for each $k\in\mathbb{N}$. Therefore, $f*f^c$ is slice preserving and maps $B\cap\mathbb{R}$ into $\mathbb{R}$. We next show that

$$f^c*f=f*f^c.\tag{2-10}$$

We proceed as follows. In view of Definition 2.10,

$$f*f^c|_{B\cap\mathbb{R}}=(f\bar{f})|_{B\cap\mathbb{R}}.$$

Since $f*f^c(B\cap\mathbb{R})\subseteq\mathbb{R}$, we deduce that the restriction $(f\bar{f})|_{B\cap\mathbb{R}}$ takes values in $\mathbb{R}$ as well. This together with Facts 1 and 2 implies that

$$(f\bar{f})|_{B\cap\mathbb{R}}=(\bar{f}f)|_{B\cap\mathbb{R}}.$$

The right-hand side is no other than the restriction $f^c*f|_{\mathbb{B}\cap\mathbb{R}}$, according to Definitions 2.9 and 2.10. Now we obtain that $f*f^c$ coincides with $f^c*f$ on $B\cap\mathbb{R}\subset U$, and hence on $U$ by the identity principle. Now by using [Colombo et al. 2011a, Proposition 2.6.9], Remark 2.13 (ii) and equality (2-10), we can conclude the proof of equality (2-7) as follows:

$$f^{-*}*f=\frac{1}{f^s}(f^c*f)=\frac{1}{f^s}(f*f^c)=\frac{1}{f^s}N(f)=1,$$

and

$$f*f^{-*}=f*(\frac{1}{f^s}f^c)=\frac{1}{f^s}(f*f^c)=1.$$

Now it remains to prove (2-8). In view of the very definition, we first need to show that $f^{-*}$ satisfies the condition given in Definition 2.12. To see this, let $I$

be an element of $\mathbb{S}$ such that $f$ satisfies the assumption therein. From the above argument, we know that $f^s = N(f)$ is slice preserving. This, together with (2-10) and [Colombo et al. 2011a, Proposition 2.6.9], implies that

$$f^{-*} * (f^{-*})^c = \frac{1}{N(f)}$$

so that $f^{-*}$ satisfies the assumption in Definition 2.12 and hence $(f^{-*})^{-*}$ is well defined on $U \setminus \mathcal{Z}_{f^s}$. Now (2-8) follows from (2-7) and uniqueness of $(f^{-*})^{-*}$. □

## 3. Growth and distortion theorems for slice monogenic functions

In this section, in the setting of the Clifford algebra $\mathbb{R}_n$, we establish the growth and distortion theorems for slice monogenic extensions to the open unit ball $\mathbb{B} := \{x \in \mathbb{R}^{n+1} : |x| < 1\}$ of univalent functions on the unit disc $\mathbb{D} \subset \mathbb{C}$. We begin with a technical proposition. To present it more generally, we will digress for a moment to slice monogenic functions on general symmetric slice domains.

**Proposition 3.1.** *Let $U \subseteq \mathbb{R}^{n+1}$ be a symmetric slice domain and $f : U \to \mathbb{R}_n$ a slice monogenic function. Then for every $x = u + vJ \in U$ and every $I \in \mathbb{S}$, there holds the identity*

$$|f(x)|^2 = \frac{1+\langle I, J\rangle}{2}|f(y)|^2 + \frac{1-\langle I, J\rangle}{2}|f(\bar{y})|^2 - \big\langle f(y)\overline{f(\bar{y})}, I \wedge J \big\rangle, \tag{3-1}$$

*where $y = u + vI$ and $\bar{y} = u - vI$.*

*Proof.* Fix an arbitrary point $x = u + vJ \in U$ and an element $I \in \mathbb{S}$. Set $y := u + vI$ and $\bar{y} := u - vI$. It follows from Theorem 2.7 that

$$f(x) = \tfrac{1}{2}\big(f(y) + f(\bar{y})\big) - \tfrac{1}{2}JI\big(f(y) - f(\bar{y})\big). \tag{3-2}$$

Notice that, in vector notation,

$$\langle I, J\rangle = \mathrm{Sc}(I\bar{J}) = -\tfrac{1}{2}(IJ + JI), \tag{3-3}$$

and

$$I \wedge J = \tfrac{1}{2}(IJ - JI). \tag{3-4}$$

We shall use the simple identity that

$$|a + b|^2 = |a|^2 + |b|^2 + 2\langle a, b\rangle \tag{3-5}$$

for any $a, b \in \mathbb{R}_n \simeq \mathbb{R}^{2^n}$.

Observe that $I$ and $J$ are 1-vectors and hence are paravectors. In view of (2-3), it holds that

$$|JI(f(y) - f(\bar{y}))| = |f(y) - f(\bar{y})|.$$

Take the modulus on both sides of (3-2) and apply (3-5) to obtain

$$
\begin{aligned}
|f(x)|^2 &= \tfrac{1}{4}\big(|f(y)+f(\bar{y})|^2+|f(y)-f(\bar{y})|^2\big)\\
&\qquad -\tfrac{1}{2}\big\langle f(y)+f(\bar{y}), JI(f(y)-f(\bar{y}))\big\rangle\\
&=: A-\tfrac{1}{2}B.
\end{aligned}
\tag{3-6}
$$

Again applying (3-5), it is evident that

$$A=\tfrac{1}{2}(|f(y)|^2+|f(\bar{y})|^2). \tag{3-7}$$

To calculate the term $B$, it first follows from the very definition of inner product (see (2-1)) that

$$B=\big\langle\big(f(y)+f(\bar{y})\big)\big(\,\overline{f(y)}-\overline{f(\bar{y})}\,\big), JI\big\rangle=: B_1+B_2, \tag{3-8}$$

where $B_1=\big\langle f(y)\overline{f(y)}-f(\bar{y})\overline{f(\bar{y})}, JI\,\big\rangle$, and $B_2=\big\langle f(\bar{y})\overline{f(y)}-f(y)\overline{f(\bar{y})}, JI\,\big\rangle$.

We next claim that

$$B_1=-\langle I,J\rangle\big(|f(y)|^2-|f(\bar{y})|^2\big), \tag{3-9}$$

and

$$B_2=2\big\langle f(y)\overline{f(\bar{y})}, I\wedge J\,\big\rangle. \tag{3-10}$$

Indeed, applying the fact that $\langle a,b\rangle=\langle\bar{a},\bar{b}\rangle$ from (2-2) to $B_1$ yields that

$$B_1=\big\langle f(y)\overline{f(y)}-f(\bar{y})\overline{f(\bar{y})}, IJ\,\big\rangle.$$

Combining this, (3-3) and the initial notion of $B_1$, we thus obtain

$$
\begin{aligned}
B_1 &=\tfrac{1}{2}\big\langle f(y)\overline{f(y)}-f(\bar{y})\overline{f(\bar{y})}, IJ+JI\,\big\rangle\\
&=-\big\langle f(y)\overline{f(y)}-f(\bar{y})\overline{f(\bar{y})}, \langle I,J\,\rangle\big\rangle\\
&=-\langle I,J\rangle\big\langle f(y)\overline{f(y)}-f(\bar{y})\overline{f(\bar{y})}, 1\big\rangle\\
&=-\langle I,J\rangle\big(|f(y)|^2-|f(\bar{y})|^2\big).
\end{aligned}
$$

Similarly,

$$B_2=\Big\langle\,\overline{f(\bar{y})\overline{f(y)}}, \overline{JI}\,\Big\rangle-\big\langle f(y)\overline{f(\bar{y})}, JI\,\big\rangle=2\big\langle\, f(y)\overline{f(\bar{y})}, I\wedge J\,\big\rangle$$

as desired. In the second equality we have used (3-4). Now substituting (3-7)–(3-10) into (3-6) yields that

$$|f(x)|^2=\frac{1+\langle I,J\rangle}{2}|f(y)|^2+\frac{1-\langle I,J\rangle}{2}|f(\bar{y})|^2-\big\langle f(y)\overline{f(\bar{y})}, I\wedge J\big\rangle,$$

which completes the proof. □

Proposition 3.1 shows that when $f$ preserves at least one slice, the squared norm of $f$ can thus be expressed as a convex combination of those in the preserved slice.

**Lemma 3.2.** *Letting $f$ be a slice monogenic function on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$ such that $f(U_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$, the convex combination identity*

$$|f(u+vJ)|^2 = \frac{1+\langle I, J\rangle}{2}|f(u+vI)|^2 + \frac{1-\langle I, J\rangle}{2}|f(u-vI)|^2 \tag{3-11}$$

*holds for every $u+vJ \in U$.*

*Proof.* As mentioned before, this lemma is a direct consequence of the preceding proposition. But here, we would like to provide an alternative easier approach to it, making no use of Proposition 3.1.

First, we have the following simple fact, which can be easily verified:

**Fact:** *For any $I, J \in \mathbb{S}$, the set*

$$\{1, I, I \wedge J, I(I \wedge J)\}$$

*is an orthogonal set of $\mathbb{R}_n \simeq \mathbb{R}^{2^n}$.*

As in the preceding proposition, it follows from Theorem 2.7 that

$$f(x) = \tfrac{1}{2}\big(f(y)+f(\bar{y})\big) - \tfrac{1}{2}JI\big(f(y)-f(\bar{y})\big) \tag{3-12}$$

for every $x = u+vJ \in U$ with $y = u+vI$ and $\bar{y} = u-vI$. We can rewrite (3-12), in terms of the relation that

$$JI = -\langle I, J\rangle + J \wedge I,$$

as

$$\begin{aligned} f(x) =& \tfrac{1}{2}\big((1+\langle I, J\rangle)f(y) + (1-\langle I, J\rangle)f(\bar{y})\big) + \tfrac{1}{2}(J\wedge I)(f(\bar{y})-f(y)) \\ =&: \tfrac{1}{2}A + \tfrac{1}{2}(J\wedge I)B. \end{aligned}$$

By assumption $f(U_I) \subseteq \mathbb{C}_I$, we thus have

$$A \in \mathbb{C}_I, \qquad B \in \mathbb{C}_I.$$

From the fact above and equality (2-3), taking the modulus on both sides yields

$$|f(x)|^2 = \tfrac{1}{4}|A|^2 + \tfrac{1}{4}|J\wedge I|^2|B|^2. \tag{3-13}$$

A simple calculation shows that

$$\begin{aligned} |A|^2 =& (1+\langle I, J\rangle)^2|f(y)|^2 + (1-\langle I, J\rangle)^2|f(\bar{y})|^2 \\ &+ 2\big(1-\langle I, J\rangle^2\big)\langle f(y), f(\bar{y})\rangle \end{aligned} \tag{3-14}$$

and

$$|B|^2 = |f(y)|^2 + |f(\bar{y})|^2 - 2\langle f(y), f(\bar{y})\rangle. \tag{3-15}$$

Notice that

$$|J\wedge I|^2 = 1-\langle I, J\rangle^2. \tag{3-16}$$

Now inserting (3-14), (3-15) and (3-16) into (3-13) yields

$$|f(x)|^2 = \frac{1+\langle I, J\rangle}{2}|f(y)|^2 + \frac{1-\langle I, J\rangle}{2}|f(\bar{y})|^2,$$

which completes the proof. □

**Remark 3.3.** The counterpart of the convex combination identity (3-11) from Lemma 3.2 also holds for slice regular functions defined on octonions or more general real alternative algebras under the extra assumption that $f$ preserves at least one slice. This can be verified much as in the proof of Proposition 3.1; see [Wang 2015; Ren et al. 2016] for details.

As a direct consequence of Lemma 3.2, we conclude that the maximum and minimum moduli of $f$ are actually attained on the preserved slice.

**Corollary 3.4.** *Let $f$ be a slice monogenic function on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$ such that $f(U_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then for each sphere $u+v\mathbb{S} \subset U$, we have the equalities*:

$$\max_{J\in\mathbb{S}}|f(u+vJ)| = \max\big(|f(u+vI)|, |f(u-vI)|\big),$$
$$\min_{J\in\mathbb{S}}|f(u+vJ)| = \min\big(|f(u+vI)|, |f(u-vI)|\big).$$

We can now state the growth and distortion theorems for slice monogenic functions.

**Theorem 3.5** (growth and distortion theorems for paravectors). *Let $f$ be a slice monogenic function on $\mathbb{B}$ such that its restriction $f_I$ to $\mathbb{B}_I$ is injective and such that $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. If $f(0) = 0$ and $f'(0) = 1$, then for all $x \in \mathbb{B}$, the following inequalities hold*:

$$\frac{|x|}{(1+|x|)^2} \le |f(x)| \le \frac{|x|}{(1-|x|)^2}; \tag{3-17}$$

$$\frac{1-|x|}{(1+|x|)^3} \le |f'(x)| \le \frac{1+|x|}{(1-|x|)^3}; \tag{3-18}$$

$$\frac{1-|x|}{1+|x|} \le \big|xf'(x) * f^{-*}(x)\big| \le \frac{1+|x|}{1-|x|}. \tag{3-19}$$

*Moreover, equality holds for one of these six inequalities at some point $x_0 \in \mathbb{B}\setminus\{0\}$ if and only if $f$ is of the form*

$$f(x) = x\big(1-xe^{I\theta}\big)^{-*2}, \quad \forall x \in \mathbb{B},$$

*for some $\theta \in \mathbb{R}$.*

*Proof.* Notice that $f_I : \mathbb{B}_I \to \mathbb{C}_I$ is a univalent function by our assumption. Theorem 1.1 with $F$ replaced by $f_I$ implies that the inequalities

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \tag{3-20}$$

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \tag{3-21}$$

$$\frac{1-|z|}{1+|z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|z|} \tag{3-22}$$

hold for every $z = u + vI \in \mathbb{B}_I$. On the other hand, it follows from Lemma 3.2 that

$$|f(x)|^2 = \frac{1+\langle I, J\rangle}{2}|f(z)|^2 + \frac{1-\langle I, J\rangle}{2}|f(\bar{z})|^2$$

holds for every $x = u+vJ \in \mathbb{B}$. Since (3-20) holds for all $z = u+vI$, $\bar{z} = u-vI \in \mathbb{B}_I$, it immediately follows that the inequalities in (3-17) hold for all $x = u + vJ \in \mathbb{B}$, by virtue of the convex combination identity above. Since the condition $f'(\mathbb{B}_I) \subseteq \mathbb{C}_I$ holds trivially, Lemma 3.2 can also be used so that the inequalities in (3-18) can be proved in the same manner.

Now it remains to prove the inequalities in (3-19). To this end, we first need to show that the slice monogenic function $xf'(x) * f^{-*}(x)$ is well-defined on the whole ball $\mathbb{B}$. We proceed as follows. First of all, since $f(0) = 0$, by considering the Taylor expansion of $f$ at the origin 0 (see Theorem 2.4) and using the Cauchy–Hadamard formula for the radius of convergence of power series (which is valid in the situation here by following the classical proof and making use of (2-3)), or by Remark 2.8, we can write

$$f(x) = xg(x), \tag{3-23}$$

where $g$ is a slice monogenic function on $\mathbb{B}$. This together with the injectivity of $f_I$ and $f'(0) = 1$ implies that $g$ has no zeros on $\mathbb{B}_I$. Moreover, $g$ maps $\mathbb{B}_I$ into $\mathbb{C}_I$, since $f$ does by our assumption. Secondly, again from the assumption that $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$, i.e., all the coefficients of the Taylor expansion of $f$ at the origin belong to the complex plane $\mathbb{C}_I$, it follows that

$$f_I^c(z) = \overline{f_I(\bar{z})},$$

and hence

$$N(f)_I(z) = f_I(z)\overline{f_I(\bar{z})} = z^2 g_I(z)\overline{g_I(\bar{z})} = z^2 N(g)_I(z). \tag{3-24}$$

This implies that

$$N(f)(\mathbb{B}_I) \subseteq \mathbb{C}_I.$$

Furthermore, since $g$ maps $\mathbb{B}_I$ into $\mathbb{C}_I$ and has no zeros on $\mathbb{B}_I$, we obtain that $g_I^s$ is exactly $g_I\overline{g_I(\bar{\cdot})}$ and is zero free on $\mathbb{B}_I$. Thus it follows from Remark 2.13 (ii) and [Colombo et al. 2011a, Remark 2.6.8 and Lemma 2.5.12] that $g^s$ is zero free on $\mathbb{B}$ as well. This, together with the fact that

$$f^s(x) = x^2 g^s(x), \qquad \forall x \in \mathbb{B}, \tag{3-25}$$

(as obtained easily from (3-23)), implies that 0 is the only zero of $f^s$. Therefore, according to Definition 2.12, $f^{-*}$ and $g^{-*}$ can be defined on $\mathbb{B} \setminus \{0\}$ and $\mathbb{B}$, respectively. Finally, in view of (3-23),

$$f^c(x) = x g^c(x), \qquad \forall x \in \mathbb{B}, \tag{3-26}$$

from which and (3-25) it follows that the relation

$$x f'(x) * f^{-*}(x) = (f' * g^{-*})(x)$$

holds for all $x \in \mathbb{B} \setminus \{0\}$. Since the right-hand side is well-defined on the whole ball $\mathbb{B}$, the left-hand side can extend regularly to the whole ball $\mathbb{B}$, as desired.

Notice also that $x f'(x) * f^{-*}(x)$ is just the slice monogenic extension to $\mathbb{B}$ of the holomorphic function $z f_I'(z)/f_I(z)$, which also maps the unit disc $\mathbb{B}_I$ into $\mathbb{C}_I$. Now inequalities in (3-19) immediately follow from (3-22) and

$$\left| x f' * f^{-*}(x) \right|^2 = \frac{1+\langle I, J\rangle}{2} \left| \frac{z f'(z)}{f(z)} \right|^2 + \frac{1-\langle I, J\rangle}{2} \left| \frac{\bar{z} f'(\bar{z})}{f(\bar{z})} \right|^2,$$

in view of Lemma 3.2.

Furthermore, if equality holds for one of six inequalities in (3-17), (3-18) and (3-19) at some point $x_0 = u_0 + v_0 J \neq 0$ with $J \in \mathbb{S}$, then the corresponding equality also holds at $z_0 = u_0 + v_0 I$ or $\bar{z}_0 = u_0 - v_0 I$. Then from Theorem 1.1, we obtain

$$f_I(z) = \frac{z}{\left(1 - e^{I\theta} z\right)^2}, \qquad \forall z \in \mathbb{B}_I,$$

for some $\theta \in \mathbb{R}$, which implies

$$f(x) = x\left(1 - x e^{I\theta}\right)^{-*2}, \qquad \forall x \in \mathbb{B}.$$

The converse part is obvious. Now the proof is complete. $\square$

**Remark 3.6.** The right-hand inequalities in (3-17) and (3-18) can follow alternatively from the well-known but highly nontrivial Bieberbach–de Branges theorem for univalent functions on the open unit disc $\mathbb{D} \subset \mathbb{C}$.

Let $F : \mathbb{D} \to \mathbb{C}$ be a univalent function on the unit disc $\mathbb{D}$ of the complex plane with Taylor expansion

$$F(z) = z + \sum_{m=2}^{\infty} z^m a_m, \qquad a_m \in \mathbb{C}.$$

We consider the canonical imbedding $\mathbb{C} \subset \mathbb{R}^{n+1}$ by expanding the basis $\{1, i\}$ of $\mathbb{C}$ to the basis $\{1, e_1, \dots, e_n\}$ of $\mathbb{R}^{n+1}$ with $e_1 = i$. Therefore we can construct a natural extension of $F$ to $\mathbb{B}$ by setting

$$f(x) = x + \sum_{m=2}^{\infty} x^m a_m, \qquad x \in \mathbb{B}.$$

It is evident that $f$ is a slice monogenic function on the open unit ball $\mathbb{B} = B(0, 1)$ such that its restriction $f|_{\mathbb{D}} = F$ is injective and satisfies that $F(\mathbb{D}) \subseteq \mathbb{C}$. Clearly, $f(0) = 0$ and $f'(0) = 1$. Thus $f$ satisfies all the assumptions of Theorem 3.5 and thus Theorem 1.2 immediately follows.

**Remark 3.7.** The slice monogenic extension of holomorphic functions on the unit disc $\mathbb{D}$ of the complex plane can result in the theory of slice monogenic elementary functions. We refer to [Colombo et al. 2011a] for the corresponding functional calculus and applications.

The following proposition is of independent interest.

**Proposition 3.8.** *Let $f$ be a slice monogenic function on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$ such that its restriction $f_I$ to $U_I$ is injective and $f(U_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then the restriction $f_J : U_J \to \mathbb{R}_n$ is also injective for every $J \in \mathbb{S}$.*

*Proof.* Suppose that there are two points $x = \alpha + \beta J$ and $y = \gamma + \delta J$ such that $f(x) = f(y)$, then it suffices to prove that $x = y$. If $J = \pm I$, the result follows from the assumption. Otherwise, from Theorem 2.7 one can deduce that

$$f(x) = \tfrac{1}{2}(f(z) + f(\bar{z})) - \tfrac{1}{2} JI(f(z) - f(\bar{z}))$$

and

$$f(y) = \tfrac{1}{2}(f(w) + f(\bar{w})) - \tfrac{1}{2} JI(f(w) - f(\bar{w})).$$

Here $z = \alpha + \beta I$ and $w = \gamma + \delta I$ for the given $I \in \mathbb{S}$. Therefore,

$$\big((f(z) + f(\bar{z})) - (f(w) + f(\bar{w}))\big) - JI\big((f(z) - f(\bar{z})) - (f(w) - f(\bar{w}))\big) = 0.$$

Since $f(U_I) \subseteq \mathbb{C}_I$, 1 and $J$ are linearly independent on $\mathbb{C}_I$ we obtain that

$$f(z) + f(\bar{z}) = f(w) + f(\bar{w})$$

and

$$f(z) - f(\bar{z}) = f(w) - f(\bar{w}),$$

which imply that $f(z) = f(w)$. Thus it follows from the injectivity of $f_I$ that $z = w$ and consequently, $x = y$. $\square$

**Remark 3.9.** Let $f$ be as described in Theorem 3.5. Then $f_J : \mathbb{B}_J \to \mathbb{R}_n$ is injective for any $J \in \mathbb{S}$ by the preceding proposition. Unfortunately, the authors do not know whether $f : U \to \mathbb{R}_n$ is injective.

## 4. Growth, distortion and covering theorems for slice regular functions

Let $\mathbb{H}$ denote the noncommutative, associative, real algebra of quaternions with standard basis $\{1, i, j, k\}$, subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Let $\langle\ ,\ \rangle$ denote the standard inner product on $\mathbb{H} \cong \mathbb{R}^4$, i.e.,

$$\langle p, q\rangle = \mathrm{Re}(p\bar{q}) = \sum_{n=0}^{3} x_n y_n$$

for any

$$p = x_0 + x_1 i + x_2 j + x_3 k, \qquad q = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}.$$

In this section, we shall consider slice regular functions defined on domains in quaternions $\mathbb{H}$ with values also in $\mathbb{H}$. These functions are not slice monogenic functions obtained by setting $n = 2$ in the Clifford algebra $\mathbb{R}_n$. Such a class of functions enjoys many nice properties similar to those of classical holomorphic functions of one complex variable. For example, the open mapping theorem holds for slice regular functions on symmetric slice domains in $\mathbb{H}$, but fails for slice monogenic functions even in the quaternionic setting. A simple counterexample is the imbedding map $\imath : \mathbb{R}^3 \hookrightarrow \mathbb{R}_2 \simeq \mathbb{H}$. The open mapping theorem allows us to prove a Koebe type one-quarter theorem (see Theorem 4.10 below). Furthermore, in the quaternionic setting only, we have an explicit formula to express the regular product and regular quotient in terms of the usual pointwise product and quotient. It is exactly this explicit formula which plays a crucial role in many arguments; see the monograph [Gentili et al. 2013] and the recent papers [Ren and Wang 2017; Wang 2015] for more details. In higher dimensions, the formulas to express slice products and slice quotients in terms of the usual pointwise products hold true only under some special cases; see [Ghiloni et al. 2016, Corollary 3.5 and Theorem 3.7] for details. In a certain sense, this phenomenon distinguishes quaternions from other real alternative algebras.

To introduce the theory of slice regular functions, we will denote by $\mathbb{S}$ the unit 2-sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

For every $I \in \mathbb{S}$ we will denote by $\mathbb{C}_I$ the plane $\mathbb{R} \oplus I\mathbb{R}$, isomorphic to $\mathbb{C}$, and, if $\Omega \subseteq \mathbb{H}$, by $\Omega_I$ the intersection $\Omega \cap \mathbb{C}_I$. Also, we will denote by $\mathbb{B}$ the open unit ball centered at the origin in $\mathbb{H}$, i.e.,

$$\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}.$$

We can now recall the definition of slice regularity.

**Definition 4.1.** Let $\Omega$ be a domain in $\mathbb{H}$. A function $f:\Omega\to\mathbb{H}$ is called *slice regular* if, for all $I\in\mathbb{S}$, its restriction $f_I$ to $\Omega_I$ is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x+yI):=\tfrac{1}{2}\Big(\frac{\partial}{\partial x}+I\frac{\partial}{\partial y}\Big)f_I(x+yI)=0,$$

for all $x+yI\in\Omega_I$.

The notions of slice domain, of symmetric slice domain and of slice derivative are similar to those already given in Section 2. Moreover, the corresponding results still hold for the slice regular functions in the setting of quaternions, such as the splitting lemma, the representation formula, the power series expansion and so on.

Now we can establish the following result by some obvious modifications of the proof of Proposition 3.1.

**Proposition 4.2.** *Let $f$ be a slice regular function on a symmetric slice domain $\Omega\subseteq\mathbb{H}$. Then for every $q=x+yJ\in\Omega$ and every $I\in\mathbb{S}$, there holds the identity*

$$|f(q)|^2=\frac{1+\langle I,J\rangle}{2}|f(z)|^2+\frac{1-\langle I,J\rangle}{2}|f(\bar z)|^2-\big\langle \mathrm{Im}\big(f(z)\overline{f(\bar z)}\big),I\wedge J\big\rangle, \tag{4-1}$$

*where $z=x+yI$ and $\bar z=x-yI$.*

Before presenting the key ingredient in establishing the growth and distortion theorems, we first make an equivalent characterization of the vanishing of the third term on the right-hand side of (4-1), thanks to the specialty of quaternions.

**Theorem 4.3.** *Let $f$ be a slice regular function on a symmetric slice domain $\Omega\subseteq\mathbb{H}$ and let $I\in\mathbb{S}$. Then,*

$$\big\langle \mathrm{Im}\big(f(z)\overline{f(\bar z)}\big),I\wedge J\big\rangle=0,$$

*for all $J\in\mathbb{S}$ and all $z\in\Omega_I$ if and only if there exist $u\in\partial\mathbb{B}$ and a slice regular function $g$ on $\Omega$ that preserves the slice $\Omega_I$ such that*

$$f(q)=g(q)u$$

*on $\Omega$.*

*Proof.* We only prove the necessity, since the sufficiency is obvious. Let

$$f_I=F+GK$$

be the splitting of $f_I$, where $K\in\mathbb{S}$ is perpendicular to $I$, and $F,G:\Omega_I\to\mathbb{C}_I$ are holomorphic functions. Take $L\in\mathbb{S}$ such that $\{1,I,K,L\}$ is an orthonormal basis of quaternions $\mathbb{H}$ and let $V$ denote the real vector space generated by the set $\{I\wedge J:J\in\mathbb{S}\}$. Then it is clear that

$$V=K\mathbb{R}\oplus L\mathbb{R}. \tag{4-2}$$

Moreover, a simple calculation gives

$$f(z)\overline{f(\bar z)}=\big(F(z)\overline{F(\bar z)}+G(z)\overline{G(\bar z)}\big)+\big(F(\bar z)G(z)-F(z)G(\bar z)\big)K,$$

and from this combined with (4-2) it follows that

$$\left\langle \mathrm{Im}\big(f(z)\overline{f(\bar{z})}\big), I \wedge J \right\rangle = 0, \qquad \forall J \in \mathbb{S},$$

if and only if

$$F(z)G(\bar{z}) = F(\bar{z})G(z), \qquad \forall z \in \Omega_I. \tag{4-3}$$

If $G \equiv 0$ on $\Omega_I$, there is nothing to prove and the desired result follows. Otherwise, $G \not\equiv 0$. Then by the identity principle, its zero set $\mathcal{Z}_G$ has no accumulation points in $\Omega_I$, and neither does

$$\overline{\mathcal{Z}}_G := \{\bar{z} \in \Omega_I : z \in \mathcal{Z}_G\}.$$

Thus, by (4-3),

$$\frac{F(z)}{G(z)} = \frac{F(\bar{z})}{G(\bar{z})}$$

is both holomorphic and antiholomorphic on $\Omega_I \setminus \big(\mathcal{Z}_G \cup \overline{\mathcal{Z}}_G\big)$, which is still a domain of $\mathbb{C}_I$, therefore there exists a constant $\lambda \in \mathbb{C}_I$ such that

$$\frac{F(z)}{G(z)} = \frac{F(\bar{z})}{G(\bar{z})} = \lambda,$$

which implies that $F = \lambda G$ on $\Omega_I \setminus \big(\mathcal{Z}_G \cup \overline{\mathcal{Z}}_G\big)$ and hence on $\Omega_I$ by the identity principle.

Now let

$$g := \big(1 + |\lambda|^2\big)^{\frac{1}{2}} \mathrm{ext}(G),$$

and set

$$u := \big(1 + |\lambda|^2\big)^{-\frac{1}{2}} (\lambda + K) \in \partial\mathbb{B}.$$

Then $g$ is a slice regular function on $\Omega$ such that $g(\Omega_I) \subseteq \mathbb{C}_I$ and $f = gu$, which completes the proof. □

As a direct consequence, we obtain Corollary 4.4.

**Corollary 4.4.** *Let $I$ be an element of $\mathbb{S}$ and $f$ a slice regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$. Then the convex combination identity*

$$|f(x+yJ)|^2 = \frac{1+\langle I, J\rangle}{2}|f(x+yI)|^2 + \frac{1-\langle I, J\rangle}{2}|f(x-yI)|^2 \tag{4-4}$$

*holds for every $x + yJ \in \Omega$ if and only if there exists some $u \in \partial\mathbb{B}$ such that $f(\Omega_I) \subseteq \mathbb{C}_I u$.*

In particular, each element $f$ from the slice regular automorphism group of the open unit ball $\mathbb{B}$ of $\mathbb{H}$

$$\mathrm{Aut}(\mathbb{B}) = \big\{f(q) = (1 - q\bar{a})^{-*} * (q - a)u : a \in \mathbb{B}, u \in \partial\mathbb{B}\big\}$$

satisfies the condition that there exists some $u \in \partial\mathbb{B}$ such that $f(\Omega_I) \subseteq \mathbb{C}_I u$ so that equality (4-4) holds for such an $f$.

From Corollary 4.4, we also conclude that the maximum and minimum moduli of every slice regular function on a symmetric slice domain in $\mathbb{H}$ that preserves one slice are actually attained on its preserved slice.

**Corollary 4.5.** *Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$ such that $f(\Omega_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then for each sphere $x + y\mathbb{S} \subset \Omega$, the following equalities hold*:

$$\max_{J\in\mathbb{S}} |f(x+yJ)| = \max\big(|f(x+yI)|, |f(x-yI)|\big), \tag{4-5}$$

$$\min_{J\in\mathbb{S}} |f(x+yJ)| = \min\big(|f(x+yI)|, |f(x-yI)|\big). \tag{4-6}$$

*Consequently*,

$$\sup_{q\in\Omega} |f(q)| = \sup_{z\in\Omega_I} |f(z)| \tag{4-7}$$

*and*

$$\inf_{q\in\Omega} |f(q)| = \inf_{z\in\Omega_I} |f(z)|. \tag{4-8}$$

**Remark 4.6.** Equalities (4-5) and (4-6) were first proved in [Sarfatti 2013, Proposition 1.13] and [de Fabritiis et al. 2015, Proposition 2.6]. Together with the classical growth and distortion theorems, Corollary 4.5 is sufficient to prove Theorem 4.7 even without Corollary 4.4. Despite this trivial fact, Corollary 4.4 is of independent interest and has its own intrinsic value. It presents, additionally, a new convex combination identity (4-4) and provides a sufficient and necessary condition under which (4-4) holds identically. This convex combination identity is also quite useful for other purposes. For instance, it provides an effective approach to a quaternionic version of a well-known Forelli–Rudin estimate, which will play a fundamental role in the theory of various spaces of slice regular functions [Ren and Xu 2016].

Now we state the growth and distortion theorems for slice regular functions.

**Theorem 4.7** (growth and distortion theorems for quaternions). *Let $f$ be a slice regular function on $\mathbb{B}$ such that its restriction $f_I$ to $\mathbb{B}_I$ is injective and $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. If $f(0) = 0$ and $f'(0) = 1$, then for all $q \in \mathbb{B}$, the following inequalities hold*:

$$\frac{|q|}{(1+|q|)^2} \le |f(q)| \le \frac{|q|}{(1-|q|)^2}; \tag{4-9}$$

$$\frac{1-|q|}{(1+|q|)^3} \le |f'(q)| \le \frac{1+|q|}{(1-|q|)^3}; \tag{4-10}$$

$$\frac{1-|q|}{1+|q|} \le \big|qf'(q) * f^{-*}(q)\big| \le \frac{1+|q|}{1-|q|}. \tag{4-11}$$

*Moreover, equality holds for one of these six inequalities at some point* $q_0 \in \mathbb{B} \setminus \{0\}$ *if and only if* $f$ *is of the form*

$$f(q) = q\big(1 - qe^{I\theta}\big)^{-*2}, \quad \forall q \in \mathbb{B},$$

*for some* $\theta \in \mathbb{R}$.

Let $F : \mathbb{D} \to \mathbb{C}$ be a univalent function on the unit disc $\mathbb{D}$ of the complex plane with Taylor expansion

$$F(z) = z + \sum_{n=2}^{\infty} z^n a_n, \qquad a_n \in \mathbb{C}.$$

As in Section 3, with a canonical imbedding $\mathbb{C} \subset \mathbb{H}$, we can construct a natural slice regular extension of $F$ to $\mathbb{B}$ via

$$f(q) = q + \sum_{n=2}^{\infty} q^n a_n, \qquad q \in \mathbb{B}.$$

It is evident that $f$ is a slice regular function on the open unit ball $\mathbb{B} = B(0, 1)$ such that its restriction $f|_{\mathbb{D}} = F$ is injective and satisfies $F(\mathbb{D}) \subseteq \mathbb{C}$. Clearly, $f(0) = 0$ and $f'(0) = 1$. Thus $f$ satisfies all the assumptions of Theorem 4.7 and this results in Theorem 4.8.

**Theorem 4.8.** *Let* $F : \mathbb{D} \to \mathbb{C}$ *be a univalent function on* $\mathbb{D}$ *such that* $F(0) = 0$ *and* $F'(0) = 1$, *and let* $f : \mathbb{B} \to \mathbb{H}$ *be the slice regular extension of F. Then for all* $q \in \mathbb{B}$, *the following inequalities hold*:

$$\frac{|q|}{(1+|q|)^2} \leq |f(q)| \leq \frac{|q|}{(1-|q|)^2}; \tag{4-12}$$

$$\frac{1-|q|}{(1+|q|)^3} \leq |f'(q)| \leq \frac{1+|q|}{(1-|q|)^3}; \tag{4-13}$$

$$\frac{1-|q|}{1+|q|} \leq \big|qf'(q) * f^{-*}(q)\big| \leq \frac{1+|q|}{1-|q|}. \tag{4-14}$$

*Moreover, equality holds for one of these six inequalities at some point* $q_0 \in \mathbb{B} \setminus \{0\}$ *if and only if*

$$f(q) = q\big(1 - qe^{i\theta}\big)^{-*2}, \qquad \forall q \in \mathbb{B}.$$

Next we digress to the Koebe one-quarter theorem for slice regular functions on the open unit ball $\mathbb{B} \subset \mathbb{H}$. We recall the following definition (see [Gentili et al. 2013, Definition 7.5]):

**Definition 4.9.** Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subset \mathbb{H}$. The *degenerate set* of $f$ is defined to be the union $D_f$ of the 2-dimensional spheres $S = x + y\mathbb{S}$ (with $y \neq 0$) such that $f|_S$ is constant.

Now as a direct consequence of the open mapping theorem and the first inequality in (4-9), we have the following result, which is a generalization of [Gal et al. 2015, Theorem 3.11 (1)].

**Theorem 4.10** (Koebe one-quarter theorem). *Let $f$ be a slice regular function on $\mathbb{B}$ such that its restriction $f_I$ to $\mathbb{B}_I$ is injective and $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. If $f(0) = 0$ and $f'(0) = 1$, then $B\big(0, \frac{1}{4}\big) \subset f(\mathbb{B})$.*

*Proof.* By assumption, the degenerate set $D_f$ of $f$ is empty. Then $f$ is open by the open mapping theorem (see [Gentili et al. 2013, Theorem 7.7]). This together with the first inequality in (4-9) shows that the image set $f(\mathbb{B})$, containing the origin 0, is an open subset of $\mathbb{H}$, whose boundary $\partial f(\mathbb{B})$ lies outside of the ball $B(0, 1/4)$. Indeed, for each point $w \in \partial f(\mathbb{B})$, there exists a sequence $\{q_n\}_{n=1}^{\infty}$ in $\mathbb{B}$ such that $\lim_{n\to\infty} f(q_n) = w$. By passing to a subsequence, we may assume that the sequence $\{q_n\}_{n=1}^{\infty}$ itself converges to one point, say $q_\infty \in \overline{\mathbb{B}}$. By the openness of $f$, $q_\infty$ must lie on the boundary $\partial\mathbb{B}$. Thus in view of the first inequality in (4-9),

$$|w| = \lim_{n\to\infty} |f(q_n)| \geq \lim_{n\to\infty} \frac{|q_n|}{(1+|q_n|)^2} = \frac{1}{4}.$$

Consequently, $f(\mathbb{B})$ must contain the ball $B(0, 1/4)$. This completes the proof. □

Let $\mathcal{SR}(\mathbb{B})$ denote the set of slice regular functions on the open unit ball $\mathbb{B} \subset \mathbb{H}$. We define

$$\mathcal{S} := \big\{ f \in \mathcal{SR}(\mathbb{B}) : \exists\, I \in \mathbb{S} \text{ such that } f_I \text{ is injective and } f_I(\mathbb{B}_I) \subseteq \mathbb{C}_I \big\}$$

and

$$\mathcal{S}_0 := \big\{ f \in \mathcal{S} : f(0) = 0,\ f'(0) = 1 \big\}.$$

For each $f \in \mathcal{S}_0$, we use $r_0(f)$ to denote the radius of the smallest ball $B(0, r)$ contained in $f(\mathbb{B})$. Also for every $\theta \in \mathbb{R}$ and every $I \in \mathbb{S}$, denote by $k_{I,\theta}$ the slice regular function given by

$$k_{I,\theta}(q) = q\big(1 - qe^{I\theta}\big)^{-*2}, \quad \forall q \in \mathbb{B}, \tag{4-15}$$

which obviously belongs to the class $\mathcal{S}_0$. The image set of the unit disc $\mathbb{B}_I$ under $k_{I,\theta}$ is exactly the complex plane except for a radial slit from $\infty$ to $-e^{I\theta}/4$. This fact together with Theorem 4.10 gives the following result:

**Theorem 4.11.** *Let the notation be as above.*

(i) *For each $f \in \mathcal{S}_0$,*

$$r_0(f) \geq \tfrac{1}{4},$$

*with equality if and only if $f = k_{I,\theta}$ for some $I \in \mathbb{S}$ and some $\theta \in \mathbb{R}$.*

(ii)
$$\bigcap_{f \in \mathcal{S}_0} f(\mathbb{B}) = B\big(0, \tfrac{1}{4}\big).$$

*Proof.* We only prove (i). It suffices to consider the extremal case, since the remainder is clear. If $r_0(f) = 1/4$, from the proof of Theorem 4.10 and inequality (4-8), we conclude that there exists some $I_0 \in \mathbb{S}$ such that $1/4$ is exactly the radius of the smallest disc $\mathbb{B}_{I_0}(0, r)$ contained in the image set $f_{I_0}(\mathbb{B}_{I_0})$ of the unit disc $\mathbb{B}_{I_0}$ under the classical univalent function $f_{I_0} : \mathbb{B}_{I_0} \to \mathbb{C}_{I_0}$. This is possible only if $f = k_{I_0,\theta}$ for some $\theta \in \mathbb{R}$ (see the proof of [Graham and Kohr 2003, Theorem 1.1.5] or [Duren 1983, Theorem 2.3]). Now the proof is complete. □

**Remark 4.12.** Two remarks are in order:

(i) It is noteworthy here that Gal et al. [2015] dealt with the growth, distortion and covering theorems for *slice preserving* and *injective* slice regular functions on the open unit ball $\mathbb{B} \subset \mathbb{H}$ with certain normalized conditions. More precisely, they focused on injective slice functions $f$ on $\mathbb{B}$ of the form

$$f(q) = q + \sum_{n=2}^{\infty} q^n a_n,$$

with $\{a_n\}_{n\geq 2}$ being a sequence of real numbers; see [Gal et al. 2015, Theorem 3.11] for details, while, in the present paper we consider slice regular functions $f(q) = q + \sum_{n=2}^{\infty} q^n a_n$ on $\mathbb{B}$ for which there exists some $I \in \mathbb{S}$ such that the restriction $f_I$ is injective and $\{a_n\}_{n\geq 2}$ is a sequence of numbers in the complex plane $\mathbb{C}_I$ determined by $I$. Thus our result properly includes the former case. Moreover, our approach to the Koebe type one-quarter theorem (Theorem 4.10), which can be specialized to the complex case, depends only on the open mapping theorem and the first inequality in (4-9), and does not involve compositions of functions. We refer the interested reader to [Graham and Kohr 2003, p.14; Duren 1983, p.31] for a standard proof of the classical Koebe one-quarter theorem for univalent functions.

(ii) Functions $k_{I,\theta}$ of the form in (4-15) are specific examples in $\mathcal{S}_0$. In view of Theorem 4.10, the image of $\mathbb{B}$ under the function $k_{I,\pi/2}$ contains the open ball $B(0, 1/4)$. However, it does not seem so easy to directly deduce this fact from the classical complex result, without using the open mapping theorem and the first inequality in (4-9).

The following proposition is the quaternionic version of Proposition 3.8 for slice regular functions.

**Proposition 4.13.** *Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$ such that its restriction $f_I$ to $\Omega_I$ is injective and $f(\Omega_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then its restriction $f_J : \Omega_J \to \mathbb{H}$ is also injective for every $J \in \mathbb{S}$.*

**Remark 4.14.** Let $f$ be as described in Theorem 4.7. Then according to the preceding proposition, $f_J : \mathbb{B}_J \to \mathbb{H}$ is injective for every $J \in \mathbb{S}$. It is well worth knowing whether $f : \mathbb{B} \to \mathbb{H}$ is injective. If it is indeed the case, together with the first inequality in (4-9) and invariance of domain theorem, it would provide an alternative approach to Theorem 4.10.

## 5. Concluding remarks

As pointed out in Remark 3.3, the counterpart of the convex combination identity (3-11) in Lemma 3.2 also holds for slice regular functions defined on octonions or more general real alternative algebras under the extra assumption that $f$ preserves at least one slice. Therefore some of the results given in the preceding sections can be easily generalized by slight modification to these new settings. Finally, we conclude with an open question connected with the subject of this paper.

Recall that $\mathcal{SR}(\mathbb{B})$ is the set of slice regular functions on the open unit ball $\mathbb{B} \subset \mathbb{H}$. We denote

$$\mathcal{SR}_0(\mathbb{B}) := \{f \in \mathcal{SR}(\mathbb{B}) : f(0) = 0,\, f'(0) = 1\}$$

and

$$\mathcal{S}_0 := \{f \in \mathcal{SR}_0(\mathbb{B}) : \exists I \in \mathbb{S} \text{ such that } f_I \text{ is injective and } f_I(\mathbb{B}_I) \subseteq \mathbb{C}_I\}.$$

**Open question:**[1] Is the class $\mathcal{S}_0$ the largest subclass of $\mathcal{SR}_0(\mathbb{B})$ in which the corresponding growth, distortion and covering theorems hold?

## Acknowledgment

The authors sincerely thank the referee for a careful reading of the paper and several valuable suggestions and comments, which have significantly improved the presentation of this paper.

## References

[Alpay et al. 2016] D. Alpay, F. Colombo, and I. Sabadini, *Slice hyperholomorphic Schur analysis*, Operator Theory, Advances and Applications **256**, Birkhäuser, Basel, 2016. Zbl

[Barnard et al. 1991] R. W. Barnard, C. H. FitzGerald, and S. Gong, "The growth and $1/4$-theorems for starlike mappings in $\mathbb{C}^n$", *Pacific J. Math.* **150**:1 (1991), 13–22. MR Zbl

[Brackx et al. 1982] F. Brackx, R. Delanghe, and F. Sommen, *Clifford analysis*, Research Notes in Mathematics **76**, Pitman, Boston, 1982. MR Zbl

[Cartan 1933] H. Cartan, "Sur la possibilité d'étendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalents", pp. 129–155 in *Leçons sur les fonctions univalents ou mutivalents*, edited by P. Montel, Gauthier-Villars, Paris, 1933. JFM

[Colombo and Sabadini 2009] F. Colombo and I. Sabadini, "A structure formula for slice monogenic functions and some of its consequences", pp. 101–114 in *Hypercomplex analysis*, edited by I. Sabadini et al., Birkhäuser, Basel, 2009. MR Zbl

[1] Very recently, Xu has found a negative answer to this question; see [Xu 2016, Example 3.1, Theorems 5.1 and 5.6] for details.

[Colombo et al. 2004] F. Colombo, I. Sabadini, F. Sommen, and D. C. Struppa, *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics **39**, Birkhäuser, Boston, 2004. MR Zbl

[Colombo et al. 2009] F. Colombo, I. Sabadini, and D. C. Struppa, "Slice monogenic functions", *Israel J. Math.* **171** (2009), 385–403. MR Zbl

[Colombo et al. 2010] F. Colombo, I. Sabadini, and D. C. Struppa, "An extension theorem for slice monogenic functions and some of its consequences", *Israel J. Math.* **177** (2010), 369–389. MR Zbl

[Colombo et al. 2011a] F. Colombo, I. Sabadini, and D. C. Struppa, *Noncommutative functional calculus: theory and applications of slice hyperholomorphic functions*, Progress in Mathematics **289**, Birkhäuser, Basel, 2011. MR Zbl

[Colombo et al. 2011b] F. Colombo, I. Sabadini, and D. C. Struppa, "The Runge theorem for slice hyperholomorphic functions", *Proc. Amer. Math. Soc.* **139**:5 (2011), 1787–1803. MR Zbl

[Drozd and Kirichenko 1994] Y. A. Drozd and V. V. Kirichenko, *Finite-dimensional algebras*, Springer, Berlin, 1994. MR Zbl

[Duren 1983] P. L. Duren, *Univalent functions*, Grundlehren Math. Wissenschaften **259**, Springer, New York, 1983. MR Zbl

[de Fabritiis et al. 2015] C. de Fabritiis, G. Gentili, and G. Sarfatti, "Quaternionic Hardy spaces", preprint, 2015. arXiv

[Gal et al. 2015] S. G. Gal, J. O. González-Cervantes, and I. Sabadini, "Univalence results for slice regular functions of a quaternion variable", *Complex Var. Elliptic Equ.* **60**:10 (2015), 1346–1365. MR Zbl

[Gentili and Struppa 2006] G. Gentili and D. C. Struppa, "A new approach to Cullen-regular functions of a quaternionic variable", *C. R. Math. Acad. Sci. Paris* **342**:10 (2006), 741–744. MR Zbl

[Gentili and Struppa 2007] G. Gentili and D. C. Struppa, "A new theory of regular functions of a quaternionic variable", *Adv. Math.* **216**:1 (2007), 279–301. MR Zbl

[Gentili and Struppa 2010] G. Gentili and D. C. Struppa, "Regular functions on the space of Cayley numbers", *Rocky Mountain J. Math.* **40**:1 (2010), 225–241. MR Zbl

[Gentili et al. 2013] G. Gentili, C. Stoppato, and D. C. Struppa, *Regular functions of a quaternionic variable*, Springer, Heidelberg, 2013. MR Zbl

[Gentili et al. 2014] G. Gentili, S. Salamon, and C. Stoppato, "Twistor transforms of quaternionic functions and orthogonal complex structures", *J. Eur. Math. Soc.* **16**:11 (2014), 2323–2353. MR Zbl

[Ghiloni and Perotti 2011a] R. Ghiloni and A. Perotti, "Slice regular functions on real alternative algebras", *Adv. Math.* **226**:2 (2011), 1662–1691. MR Zbl

[Ghiloni and Perotti 2011b] R. Ghiloni and A. Perotti, "Zeros of regular functions of quaternionic and octonionic variable: a division lemma and the camshaft effect", *Ann. Mat. Pura Appl.* (4) **190**:3 (2011), 539–551. MR Zbl

[Ghiloni et al. 2016] R. Ghiloni, A. Perotti, and C. Stoppato, "The algebra of slice functions", 2016. To appear in *Trans. Amer. Math. Soc.*

[Gong 1998] S. Gong, *Convex and starlike mappings in several complex variables*, Mathematics and its Applications (China) **435**, Kluwer, Dordrecht, 1998. MR Zbl

[Graham and Kohr 2003] I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Monographs and Textbooks in Pure and Applied Mathematics **255**, Marcel Dekker, New York, 2003. MR Zbl

[Gürlebeck et al. 2008] K. Gürlebeck, K. Habetha, and W. Sprößig, *Holomorphic functions in the plane and $n$-dimensional space*, Birkhäuser, Basel, 2008. MR Zbl

[Liu and Ren 1998a] T. Liu and G. Ren, "The growth theorem for starlike mappings on bounded starlike circular domains", *Chinese Ann. Math. Ser. B* **19**:4 (1998), 401–408. MR Zbl

[Liu and Ren 1998b] T. Liu and G. Ren, "Growth theorem of convex mappings on bounded convex circular domains", *Sci. China Ser. A* **41**:2 (1998), 123–130. MR Zbl

[Ren and Wang 2017] G. Ren and X. Wang, "Julia theory for slice regular functions", *Trans. Amer. Math. Soc.* **369**:2 (2017), 861–885. MR Zbl

[Ren and Xu 2016] G. Ren and Z. Xu, "Slice Lebesgue measure of quaternions", *Adv. Appl. Clifford Algebr.* **26**:1 (2016), 399–416. MR Zbl

[Ren et al. 2016] G. Ren, X. Wang, and Z. Xu, "Slice regular functions on regular quadratic cones of real alternative algebras", pp. 227–245 in *Modern trends in hypercomplex analysis*, edited by S. Bernstein et al., Birkhäuser, Basel, 2016.

[Sarfatti 2013] G. Sarfatti, *Elements of function theory in the unit ball of quaternions*, Ph.D. thesis, University of Florence, 2013, https://webusers.imj-prg.fr/∼giulia.sarfatti/PhDGiulia.pdf.

[Wang 2015] X. Wang, "On geometric aspects of quaternionic and octonionic slice regular functions", 2015. To appear in *J. Geom. Anal.* arXiv

[Xu 2016] Z. Xu, "Geometric function theory for certain slice regular functions", preprint, 2016. arXiv

Received July 21, 2014. Revised January 13, 2017.

GUANGBIN REN
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI, 230026
CHINA
rengb@ustc.edu.cn

XIEPING WANG
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI, 230026
CHINA
pwx@mail.ustc.edu.cn

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm



See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

# PACIFIC JOURNAL OF MATHEMATICS

Volume 290 No. 1 September 2017