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ON RELATIVE RATIONAL CHAIN CONNECTEDNESS OF THREEFOLDS WITH ANTI-BIG CANONICAL DIVISORS IN POSITIVE CHARACTERISTICS

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Let X be a projective klt threefold over an algebraically closed field of positive characteristic, and $f : X \to Y$ a morphism from X to a projective variety Y of dimension 1 or 2. We study how bigness and relative bigness of $-K_X$ influences the rational chain connectedness of X and fibers of f, respectively. We construct a canonical bundle formula and use it as well as the minimal model program to prove two results in this context.

1. Introduction

It is widely recognized that the geometry of a higher-dimensional variety is closely related to the geometry of rational curves on it. A classical result by Campana [1992] and Kollár, Miyaoka and Mori [Kollár et al. 1992] says that smooth Fano varieties are rationally connected in characteristic zero and are rationally chain connected in positive characteristics. This was generalized in characteristic zero in [Zhang 2006; Hacon and McKernan 2007]. More recently, using the minimal model program of [Hacon and Xu 2015; Birkar 2016], Gongyo, Li, Patakfalvi, Schwede, Tanaka and Zong [Gongyo et al. 2015a] proved that projective globally F-regular threefolds in characteristic ≥ 11 are rationally chain connected and this was later generalized to threefolds of log Fano type by Gongyo, Nakamura and Tanaka [Gongyo et al. 2015b].

The main result of Hacon and McKernan is as follows:

Theorem 1.1 [Hacon and McKernan 2007, Theorem 1.2]. *Let (X, Δ) be a log pair, and let $f : X \to S$ be a proper morphism such that $-K_X$ is relatively big and $-(K_X + \Delta)$ is relatively semiample. Let $g : Y \to X$ be any birational morphism. Then the connected components of every fiber of $f \circ g$ are rationally chain connected modulo the inverse image of the locus of log canonical singularities of (X, Δ).*

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In this paper we prove a theorem similar to Theorem 1.1 for morphisms from a klt threefold to a variety of dimension ≥ 1. More precisely, we have

Theorem 3.1. *Let X be a normal $\mathbb{Q}$-factorial threefold over an algebraically closed field k of characteristic ≥ 7 and (X, D) a klt pair. Let $f : X \to Z$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\dim(Z)$ is 1 or 2, Z is klt, $-K_X$ is relatively big, $-(K_X + D)$ is relatively semiample, and (X_z, D_z) is klt for general $z \in Z$. Let $g : Y \to X$ be any birational morphism. Then the connected components of every fiber of $f \circ g$ are rationally chain connected.*

Motivated by Theorem 3.1, we construct a global version of rational chain connectedness for threefolds.

Theorem 5.1. *Let X be a projective threefold over an algebraically closed field k of characteristic $p > 0$, $f : X \to Y$ a projective surjective morphism from X to a projective variety Y such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let D be an effective $\mathbb{Q}$-divisor, and $X_{\bar{\eta}}$ the geometric generic fiber of f. Assume that the following conditions hold:*

(1) *(X, D) is klt, $-K_X$ is big, and f-ample, $K_X + D \sim_{\mathbb{Q}} 0$, and the general fibers of f are smooth.*

(2) *$p > 2/\delta$, where δ is the minimum nonzero coefficient of D.*

(3) *$D = E + f^*L$ where E is an effective $\mathbb{Q}$-Cartier divisor such that $p \nmid \operatorname{ind}(E)$, $(X_{\bar{\eta}}, E|_{X_{\bar{\eta}}})$ is globally F-split, and L is a big $\mathbb{Q}$-divisor on Y.*

(4) $\dim(Y)$ *is* 1 *or* 2.

Then X is rationally chain connected.

Here $\operatorname{ind}(E)$ means the Cartier index of E.

The main ingredients of the proofs of Theorems 3.1 and 5.1 are the minimal model program constructed in [Hacon and Xu 2015; Birkar 2016; Gongyo et al. 2015a]; some facts, especially [Gongyo et al. 2015a, Theorem 2.1]; some positivity results [Patakfalvi 2014; Ejiri 2015]; a canonical bundle formula constructed in Section 4 in the spirit of [Prokhorov and Shokurov 2009]. Note that condition (3) in Theorem 5.1 is used in order to apply the result [Ejiri 2015, Theorem 1.1] to deduce that $-K_Y$ is big, and to apply Theorem 4.3 when $\dim Y = 2$. This creates enough rational curves on Y. Note that by [Ejiri 2015, Example 3.4], $(X_{\bar{\eta}}, E|_{X_{\bar{\eta}}})$ being globally F-split is equivalent to $S^0(X_{\bar{\eta}}, E|_{X_{\bar{\eta}}}, \mathcal{O}_{X_{\bar{\eta}}}) = H^0(X_{\bar{\eta}}, \mathcal{O}_{X_{\bar{\eta}}})$.

We note that although its proof is independent, Theorem 3.1 is implied by [Gongyo et al. 2015b, Theorem 4.1], which was put on arXiv before this paper. The proof of that result relies on the minimal model program in dimension 3 in positive characteristic, which is only established in characteristic ≥ 7 so far. On the other hand, Theorem 5.1 covers some cases in characteristic < 7. It does not rely on the minimal model program and is not implied by [Gongyo et al. 2015b].

2. Preliminaries

We work over an algebraically closed field k of characteristic $p > 0$.

Preliminaries on rational connected varieties and the minimal model program.

Definition 2.1. For a variety X and a $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of (X, Δ) and write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

where E_i is a prime divisor. We say that (X, Δ) is

- *sub-Kawamata log terminal* (*sub-klt* for short) if $a_i > -1$ for any i;
- *Kawamata log terminal* (*klt* for short) if $a_i > -1$ for any i and $\Delta \geq 0$;
- *log canonical* if $a_i \geq -1$ for any i and $\Delta \geq 0$.

Definition 2.2. [Kollár 1996, IV.3.2] Suppose that X is a variety over k.

(1) We say that X is *rationally chain connected* (*RCC*) if there is a family of proper and connected algebraic curves $g : U \to Y$ whose geometric fibers have only rational components and there is a cycle morphism $u : U \to X$ such that $u^{(2)} : U \times_Y U \to X \times_k X$ is dominant.

(2) We say that X is *rationally connected (RC)* if (1) holds and moreover the geometric fibers of g in (1) are irreducible.

Proposition 2.3. *Let X be a klt $\mathbb{Q}$-factorial threefold over an algebraically closed field k and* $\operatorname{char}(k) \geq 7$. *Let $g : W \to X$ be a log resolution and assume that $K_W + E = g^*K_X + B$, where E and B are exceptional divisors and the coefficients in E are all* 1. *Then relative minimal model for (W, E) over X exists. Denote this process by*

$$W = W_0 \overset{f_0}{\dashrightarrow} W_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{N-1}}{\dashrightarrow} W_N = W'.$$

Then we actually have $W' = X$. Moreover if we have a morphism $h : X \to Y$ such that every fiber of h is RCC, then every fiber of $h \circ g$ is RCC.

Proof. The existence of this minimal model program is by [Gongyo et al. 2015a, Theorem 3.2]. So we have a morphism $g' : W' \to X$ and we want to show that g' is the identity. Denote the strict transform of E by E', then $K_{W'} + E' = g'^*K_X + B'$ for some exceptional $\mathbb{Q}$-divisor B'. By construction of the minimal model program we know that $g'^*K_X + B'$ is nef over X which means that B' is g'-nef and since X is klt the support of B' is the whole exceptional locus of g'. So we can get that $B' = 0$ by the negativity lemma, and since X is $\mathbb{Q}$-factorial we will get $W' = X$.

The proof of the last statement follows the proof of Proposition 3.6 in the same reference. Without loss of generality we can do a base change and assume that the base field k is uncountable. Define F in the following way: if f_i is a divisorial contraction, then let $E_0 = E,\ E_{i+1} = f_{i,*}E_i$, and F be an arbitrary component of E_i; if f_i is a flip and C is any flipping curve then let F be a component of E_i that contains C. Let $K_F + \Delta_F := \big(K_{W_i} + E_i - \frac{1}{n}(E_i - F)\big)\big|_F$, where $n \gg 0$. By assumption $K_{W_i} + E_i - \frac{1}{n}(E_i - F)$ is plt, then by adjunction $K_F + \Delta_F$ is klt, hence by [Tanaka 2014, Theorem 14.4] F is $\mathbb{Q}$-factorial. We also know that $-(K_{W_i} + E_i)$ is f_i-ample by assumption, then $-(K_F + \Delta_F)$ is ample. Moreover by [Prokhorov 2001, Corollary 2.2.8] the coefficients of Δ_F are in the standard set $\big\{1 - \frac{1}{n} \,\big|\, n \in \mathbb{N}\big\}$. Let $\tilde{F}$ be the normalization of F. Then by [Hacon and Xu 2015, Theorem 3.1] we know that $(\tilde{F}, \Delta_{\tilde{F}})$ is strongly F-regular and by Theorem 4.1 from that reference F is a normal surface.

Next we consider three cases.

Case 1: If f_i is a divisorial contraction and the exceptional divisor is contracted to a point, then since $-(K_F + \Delta_F)$ is ample, by [Kawamata 1994, Lemma 2.2] F is a rational surface, in particular it is rationally connected.

Case 2: If f_i is a divisorial contraction and the exceptional divisor is contracted to a curve, then let $p : F \to B$ be the Stein factorization of $f_i|_F$. By assumption $-(K_F + \Delta_F)$ is f_i-ample, so it is p-ample. Then for a general fiber D of p,

$$(K_F + D)\cdot D = (K_F + \Delta_F + D - \Delta_F)\cdot D = (K_F + \Delta_F)\cdot D - \Delta_F \cdot D < 0.$$

Here D is reduced and irreducible by [Bădescu 2001, Theorem 7.1], hence by [Tanaka 2014, Theorem 5.3] $D \cong \mathbb{P}^1$. Therefore every component of every fiber of f_i is a rational curve.

Case 3: If f_i is a flip, then let C be an arbitrary flipping curve. By assumption we have $(K_F + \Delta_F)\cdot C < 0,\ C^2 < 0$, and $0 \le \operatorname{coeff}_C \Delta_F < 1$, so $(K_F + C)\cdot C < 0$. Again by [op. cit., Theorem 5.3] $C \cong \mathbb{P}^1$.

We denote a fiber of h over $y \in Y$ by $F_{X,y}$. There is a morphism from W_i to Y for every i, and we denote the fiber of this morphism over y as $F_{W_i,y}$. Then there is a rational map $F_{W_i,y} \dashrightarrow F_{W_{i+1},y}$. From the above Cases 1–3 we see that compared to $F_{W_i,y}$, there are only rational curves or a rational surface generated in $F_{W_{i+1},y}$. So the RCC-ness of $F_{W_{i+1},y}$ implies the RCC-ness of $F_{W_i,y}$. By assumption $F_{X,y}$ is RCC, so $F_{W,y}$ is RCC. □

Proposition 2.4. *Let X be a klt $\mathbb{Q}$-factorial threefold over an algebraically closed field k and* $\operatorname{char}(k) \ge 7$. *Let $f : X \to Y$ be a morphism from X to a normal surface Y. Suppose we run a K_X-minimal model program and it terminates at $g : X' \to Y$. If every fiber of g is RCC then every fiber of f is RCC.*

Proof. This can be easily deduced from Proposition 2.3 by taking a common resolution of X and X'. The proof of [Gongyo et al. 2015a, Proposition 3.6] works as well. □

Preliminaries on F-singularities. In this article, for a proper variety X, a $\mathbb{Q}$-divisor Δ, and the line bundle M, we will use the concepts of *strongly F-regular*, the *non-F-pure ideal* $\sigma(X, \Delta)$ and $S^0(X, \sigma(X, \Delta) \otimes M)$. The definitions of these can be found in many papers related to F-singularities, e.g., [Hacon and Xu 2015]. For a pair (X, Δ) where Δ is a $\mathbb{Q}$-Cartier divisor we also follow the definition of *globally F-split* in [Ejiri 2015].

Lemma 2.5. *Let X be a surface, D an effective $\mathbb{Q}$-divisor on X, $f : X \to C$ a morphism from X to a smooth curve C, and (X_c, D_c) is a strongly F-regular pair for general $c \in C$. Assume that $-K_X$ is big, $K_X + D \sim_{\mathbb{Q}} 0$, then $C \cong \mathbb{P}^1$.*

Proof. By Kodaira's lemma we can write $D \sim_{\mathbb{Q}} \epsilon f^*H + E$ where H is an ample $\mathbb{Q}$-divisor on C, $0 < \epsilon \in \mathbb{Q}$, E is an effective $\mathbb{Q}$-divisor on X and (X_c, E_c) is also strongly F-regular for general $c \in C$ (since X_c is a curve). Suppose that C is not isomorphic to $\mathbb{P}^1$. We know that $K_{X/C} + E \sim_{\mathbb{Q}} f^*(-K_C - \epsilon H)$ is f-nef and $K_{X_c} + E_c$ is semiample for general $c \in C$, so by [Patakfalvi 2014, Theorem 3.16], $K_{X/C} + E = K_X - f^*K_C + E$ is nef. Since we have assumed that $g(C) > 0$ we have that $K_X + E$ is nef. However this is impossible since $K_X + E \sim_{\mathbb{Q}} -\epsilon f^*H$ where H is ample and $\epsilon > 0$. □

Weak positivity. Let Y be a nonsingular projective variety, $\mathcal{F}$ a torsion-free coherent sheaf on Y. We take $i : \hat{Y} \to Y$ to be the biggest open subvariety such that $\mathcal{F}|_{\hat{Y}}$ is locally free. Let $\hat{S}^k(\mathcal{F}) := i_* S^k(i^*\mathcal{F})$.

Definition 2.6 [Viehweg 1983, Definition 1.2]. We call $\mathcal{F}$ *weakly positive*, if there is an open subset $U \subseteq Y$ such that for every ample line bundle $\mathcal{H}$ on Y and every positive number α there exists some positive number β such that $\hat{S}^{\alpha \cdot \beta}(\mathcal{F}) \otimes \mathcal{H}^{\beta}$ is generated by global sections over U.

Lemma 2.7. *Weakly positive line bundles are nef.*

Proof. This easily follows from Definition 2.6. □

3. Relative rational chain connectedness

In this section we prove the following

Theorem 3.1. *Let X be a normal $\mathbb{Q}$-factorial threefold over an algebraically closed field k of characteristic ≥ 7 and (X, D) a klt pair. Let $f : X \to Z$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\dim(Z)$ is 1 or 2, Z is klt, $-K_X$ is relatively big, $-(K_X + D)$ is relatively semiample, and (X_z, D_z) is klt for general $z \in Z$. Let $g : Y \to X$ be any birational morphism. Then the connected components of every fiber of $f \circ g$ are rationally chain connected.*

Remark 3.2. In Theorem 3.1, if $\dim Z = 2$, by adjunction and a theorem of Tate (see [Liedtke 2013, Theorem 5.1]) we have that the generic fiber of f is smooth. So in this case the condition that (X_z, D_z) is klt for general $z \in Z$ is not necessary.

Proof. First we observe that (X_z, D_z) being klt implies that X_z is normal (in particular reduced) and irreducible.

Next we prove that if every fiber of f is RCC, then every fiber of $f \circ g$ is RCC. We take a log resolution of Y and denote it by $p : Y' \to Y$ and let $q = g \circ p$. If $K_{Y'} = q^* K_X + \tilde{B}$ then $K_{Y'} - \tilde{B} = q^* K_X$ and the coefficients of $-\tilde{B}$ are < 1. Then we can add another effective divisor to make all the coefficients 1, and we denote this divisor by $\tilde{E}$. Now we run a relative $(K_{Y'} + \tilde{E})$-minimal model program of Y' over X. By Proposition 2.3 we see that if every fiber of f is RCC then every fiber of $f \circ g \circ p$ is RCC, hence every fiber of $f \circ g$ is RCC.

Therefore it suffices to show that every fiber of f is RCC. We consider the cases of $\dim(Z) = 2$ and $\dim(Z) = 1$, respectively.

<u>Case 1: $\dim(Z) = 2$</u>. If $\dim(Z) = 2$ then a general fiber of f being normal and $-K_X$ being relatively big implies that a general fiber of f is a smooth rational curve. Next we run a relative minimal model program over Z and denote this process as

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{N-1}}{\dashrightarrow} X_n = X'.$$

Since $-K_X$ is relatively big we end up with a Mori fiber space $X' \xrightarrow{h} Z' \xrightarrow{p} Z$ where Z' is also a surface. Then the general fibers of h are rational curves. Moreover since $p_* \mathcal{O}_{Z'} = \mathcal{O}_Z$ we know that p is birational.

Now we prove that h is equidimensional. Suppose that this is not the case, then there is a fiber $\tilde{F}$ of h over a point $\tilde{z} \in Z'$ which contains a 2-dimensional irreducible component. If $\tilde{F}$ is reducible then let $\tilde{F}_1$ be a 2-dimensional component of $\tilde{F}$ and $\tilde{F}_2$ another component which intersects $\tilde{F}_1$. We can choose a curve $\tilde{C}_2 \subseteq \tilde{F}_2$ such that $\tilde{F}_1 \cdot \tilde{C}_2 > 0$. On the other hand if we take a general point $z' \in Z'$ then $h^{-1}(z')$ is an irreducible curve and $h^{-1}(z') \cdot \tilde{F}_2 = 0$. This contradicts the fact that $\rho(X'/Z') = 1$. If $\tilde{F}$ is irreducible, by Bertini's theorem we have a very ample divisor $H \subset X'$ such that $H \cap \tilde{F}$ is an irreducible curve which we denote by $\tilde{C}$. We do the Stein factorization of $h|_H$ and denote the process as

$$H \xrightarrow{h_1} Z'' \xrightarrow{h_2} Z',$$

then h_1 is birational and $\tilde{C}$ is an exceptional curve of h_1. After possibly replacing Z'' by its normalization we can assume that Z'' is normal. Now $\tilde{F} \cdot \tilde{C}$ is equal to $\tilde{C}^2$, viewed as the self-intersection of $\tilde{C}$ in H, so by the negativity lemma it is negative. On the other hand we can still take a general point $z' \in Z'$ as above such that $h^{-1}(z') \cdot \tilde{F} = 0$. This also contradicts the fact that $\rho(X'/Z') = 1$.

Since h is equidimensional, by [Debarre 2001, Lemma 3.7] the components of every fiber of h are rational curves. Then by Proposition 2.4 every fiber of f is RCC.

Case 2: $\dim(Z) = 1$. Without loss of generality we can do a base change and assume that the base field k is uncountable. By passing to the normalization of Z we can assume that Z is smooth. Then since every closed point of Z is a Cartier divisor, every fiber of f is also Cartier, hence f is equidimensional.

We first show that the general fibers of f are rationally chain connected. Let F be a general fiber of f. Since we assume that $(F, D|_F)$ is klt, by adjunction

$$K_X|_F \equiv_{\text{num}} (K_X + F)|_F = K_F + \mathrm{Diff}_F(0),$$

where $\mathrm{Diff}_F(0) \geq 0$; see [Kollár 1992, Proposition-Definition 16.5]. Therefore, $-(K_F + \mathrm{Diff}_F(0))$ is big, hence $-K_F$ as well. As a result, $\kappa(F) = -\infty$ and F is birationally ruled by classification of surfaces. To prove that the general fibers of f are RCC it suffices to prove that F is rational. By assumption $-(K_F + D|_F) = -(K_X + D)|_F$ is semiample, so there exists an effective $\mathbb{Q}$-divisor H such that $H \sim_{\mathbb{Q}} -(K_F + D|_F)$ and $(F, D|_F + H)$ is klt. We define $\Delta := D|_F + H$. Let $\pi : F' \to F$ be a minimal resolution of $(F, \mathrm{Diff}_F(0))$, then F' maps to a ruled surface F'' over a smooth curve B via a sequence of blowdowns and we denote the morphism by ψ. The situation is as follows:

$$\begin{array}{ccc} & F' & \\ {}^{\pi}\swarrow & & \searrow^{\psi} \\ F & & F'' \\ & & \downarrow q \\ & & B \end{array}$$

Since (F, Δ) is klt, by [Kollár and Mori 1998, Theorem 4.7] π and ψ only contract copies of $\mathbb{P}^1$. So F is RCC if and only if F'' is RCC. Define Δ'' on F'' via

$$K_{F''} + \Delta'' = \psi_* \pi^* (K_F + \Delta).$$

Then (F, Δ) being klt implies that (F'', Δ'') is klt.

We denote a general fiber of q by R. By construction $R \cong \mathbb{P}^1$, so we know that $(R, \Delta''|_R)$ is klt and hence strongly F-regular. Then by applying Lemma 2.5 on F'' we know that $B = \mathbb{P}^1$. So F is rational. Therefore we have proven that the general fibers of f are RCC.

Since we have assumed that the base field k is uncountable, by [Kollár 1996, Chapter IV, Corollary 3.5.2] we know that every fiber of f is RCC. □

4. A canonical bundle formula for threefolds in positive characteristics

In this section following the idea of the proof of [Prokhorov and Shokurov 2009] we construct a canonical bundle formula in characteristic p for a morphism from a threefold to a surface, whose general fibers are $\mathbb{P}^1$. There are similar constructions in [Cascini et al. 2015, 6.7; Das and Hacon 2016, Theorem 4.8].

Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of n-pointed stable curves of genus 0, let $f_{0,n} : \overline{\mathcal{U}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ be the universal family, and let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ be the sections of $f_{0,n}$ which correspond to the marked points. Let $d_j\,(j = 1, 2, \dots, n)$ be the rational numbers such that $0 < d_j \leq 1$ for all j, $\sum_j d_j = 2$, and $\mathcal{D} = \sum_j d_j \mathcal{P}_j$.

Lemma 4.1 [Das and Hacon 2016, Lemma 4.6; Kawamata 1997, Theorem 2].

(1) *There exists a smooth projective variety $\mathcal{U}^*_{0,n}$, a $\mathbb{P}^1$-bundle $g_{0,n} : \mathcal{U}^*_{0,n} \to \overline{\mathcal{M}}_{0,n}$, and a sequence of blowups with smooth centers*

$$\overline{\mathcal{U}}_{0,n} = \mathcal{U}^{(1)} \xrightarrow{\sigma_2} \mathcal{U}^{(2)} \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n-2}} \mathcal{U}^{(n-2)} = \mathcal{U}^*_{0,n}$$

(2) *Let $\sigma : \overline{\mathcal{U}_{0,n}} \to \mathcal{U}^*_{0,n}$ be the induced morphism, and let $\mathcal{D}^* = \sigma_* \mathcal{D}$. Then $K_{\overline{\mathcal{U}}_{0,n}} + \mathcal{D} - \sigma^*(K_{\mathcal{U}^*_{0,n}} + \mathcal{D}^*)$ is effective.*

(3) *There exists a semiample $\mathbb{Q}$-divisor $\mathcal{L}$ on $\overline{\mathcal{M}}_{0,n}$ such that*

$$K_{\mathcal{U}^*_{0,n}} + \mathcal{D}^* \sim_{\mathbb{Q}} g^*_{0,n}(K_{\overline{\mathcal{M}}_{0,n}} + \mathcal{L}).$$

Definition 4.2. Let $f : X \to Y$ be a surjective proper morphism between two normal varieties and $K_X + D \sim_{\mathbb{Q}} f^*L$, where D is a boundary divisor on X and L is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Let (X, D) be log canonical near the generic fiber of f, i.e., $(f^{-1}U, D|_{f^{-1}U})$ is log canonical for some Zariski dense open subset $U \subseteq Y$. We define

$$D_{\text{div}} := \sum (1 - c_Q) Q,$$

where $Q \subset Z$ are prime Weil divisors on Z and

$c_Q = \sup\{c \in \mathbb{R} : (X, D + cf^*Q)$is log canonical over the generic point η_Q of $Q\}$.

Next we define

$$D_{\text{mod}} := L - K_Y - D_{\text{div}},$$

so $K_X + D = f^*(K_Y + D_{\text{div}} + D_{\text{mod}})$.

Theorem 4.3. *Let $f : X \to Y$ be a proper surjective morphism, where X is a normal threefold and Y is a normal surface over an algebraically closed field k of characteristic $p > 0$. Assume that $Q = \sum_i Q_i$ is a divisor on Y such that f is smooth over $(Y - \operatorname{Supp}(Q))$ with fibers isomorphic to $\mathbb{P}^1$. Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on X where $d_i = 0$ is allowed, which satisfies the following conditions*:

(1) $(X, D \geq 0)$ *is klt on a general fiber of* f.

(2) *Suppose* $D = D^h + D^v$ *where* D^h *is the horizontal part and* D^v *is the vertical part of* D. *Then* $p = \mathrm{char}(k) > 2/\delta$, *where* δ *is the minimum nonzero coefficient of* D^h.

(3) $K_X + D \sim_{\mathbb{Q}} f^*(K_Y + M)$ *for some* $\mathbb{Q}$*-Cartier divisor* M *on* Y.

Then we have that D_{mod} *is* $\mathbb{Q}$*-linearly equivalent to an effective* $\mathbb{Q}$*-divisor. Here* D_{mod} *is defined as in Definition 4.2. Moreover if* (X, D) *is klt then there exists an effective* $\mathbb{Q}$*-divisor* $\bar{\mathcal{D}}_{\mathrm{mod}}$ *on* Y *such that* $\bar{\mathcal{D}}_{\mathrm{mod}} \sim_{\mathbb{Q}} D_{\mathrm{mod}}$ *and* $(Y, D_{\mathrm{div}} + \bar{\mathcal{D}}_{\mathrm{mod}})$ *is klt.*

Proof. First we reduce the problem to the case where all components of D^h are sections. Let D_{i_0} be a horizontal component of D and $D_{i_0} \to D_{i_0}^{\flat} \to Y$ be the Stein factorization of $f|_{D_{i_0}}$. Let $Y' \to D_{i_0}^{\flat}$ be the normalization of $D_{i_0}^{\flat}$, then $Y' \to Y$ is a finite surjective morphism of normal surfaces. Let X' be the normalization of the component of $X \times_Y Y'$ dominating Y.

$$\begin{array}{ccc} X & \xleftarrow{\nu'} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{\nu} & Y' \end{array}$$

Let $m = \deg(\mu : Y' \to Y)$ and l be a general fiber of f. Then

$$m = D_i \cdot l \leq \frac{1}{d_i}(D \cdot l) = \frac{1}{d_i}(-K_X \cdot l) = \frac{2}{d_i} \leq \frac{2}{\delta} < \mathrm{char}(k). \tag{4-1}$$

Therefore ν is a separable and tamely ramified morphism.

Let D' be the log pullback of D under ν', i.e.,

$$K_{X'} + D' = \nu'^*(K_X + D).$$

More precisely by [Kollár 1992, 20.2],

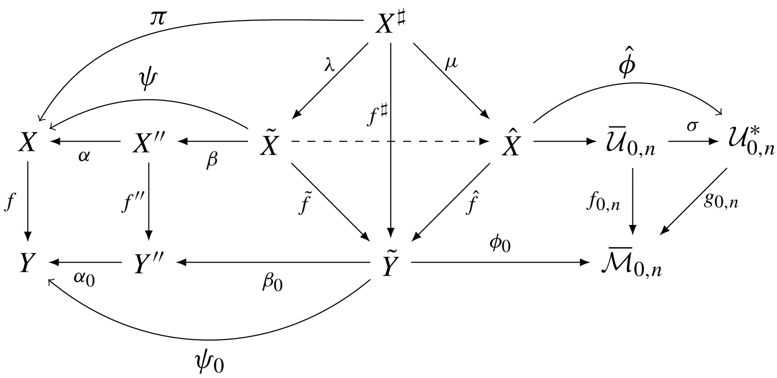
$$D' = \sum_{i,j} d'_{ij} D'_{ij}, \qquad \nu'(D'_{ij}) = D_i, \qquad d'_{ij} = 1 - (1 - d_i)e_{ij},$$

where e_{ij} is the ramification indices along D'_{ij}.

By construction X dominates Y. Also, since ν is étale over a dense open subset of Y, say $\nu^{-1}U \to U$, and étale morphisms are stable under base change, the map $(f' \circ \nu)^{-1}U \to f^{-1}U$ is étale. Thus the ramification locus Λ of ν' does not contain any horizontal divisor f', i.e., $f'(\Lambda) \neq Y'$. Therefore D' is a boundary near the generic fiber of f', i.e., D'^h is effective. We observe that the coefficients of D'^h can be computed by intersecting with a general fiber of $f' : X' \to Y'$, hence they are equal to the coefficient of $D^h \subseteq X$. Thus the condition $p > 2/\delta$ remains true for D' on X'.

After finitely many such base changes we get a family $f'' : X'' \to Y''$, such that all of the horizontal components of D'' are rational sections of f''. Here D'' is the log pullback of D via the induced finite morphism $\alpha : X'' \to X$, i.e., $K_{X''} + D'' = \alpha^*(K_X + D)$.

By construction of $\overline{\mathcal{M}}_{0,n}$ there is a generically finite rational map $Y'' \dashrightarrow \overline{\mathcal{M}}_{0,n}$. Let $\beta_0 : \tilde{Y} \to Y''$ be a morphism that resolves the indeterminacies of $Y'' \to \overline{\mathcal{M}}_{0,n}$ and $\tilde{X}$ the normalization of $X'' \times_{Y''} \tilde{Y}$. We have a morphism $\tilde{Y} \to \overline{\mathcal{M}}_{0,n}$ and let $\hat{X} = \tilde{Y} \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n}$. Let $X^\sharp$ be a common resolution of $\tilde{X}$ and $\hat{X}$. We have the following diagram:



Let $D^\sharp$ and $\hat{D}$ be $\mathbb{Q}$-divisors on $X^\sharp$ and $\hat{X}$ respectively, defined by

$$K_{X^\sharp} + D^\sharp = \pi^*(K_X + D) \quad \text{and} \quad K_{\hat{X}} + \hat{D} = \mu_*(K_{X^\sharp} + D^\sharp).$$

We also define D''_{mod} and D''_{div} on Y'' for (X'', D'') as in Definition 4.2, such that

$$K_{X''} + D'' = f''^*(K_{Y''} + D''_{\mathrm{mod}} + D''_{\mathrm{div}}),$$

and we define $\tilde{D}_{\mathrm{mod}}$ and $\tilde{D}_{\mathrm{div}}$ on $\tilde{Y}$ in a similar way. Since $K_{X^\sharp} + D^\sharp$ is the pullback of some $\mathbb{Q}$-divisor from the base $\tilde{Y}$ we get

$$K_{X^\sharp} + D^\sharp = \mu^*(K_{\hat{X}} + \hat{D}).$$

Since D_{div} does not depend on the birational modification of the family [Prokhorov and Shokurov 2009, Remark 7.3], we will define it with respect to $\hat{f} : \hat{X} \to \tilde{Y}$.

Since $\hat{\phi}$ is generically finite and $\mathcal{D}^*$ is horizontal it follows that $\hat{\phi}^*\mathcal{D}^*$ is horizontal too. Since $\hat{D}^h$ is also horizontal,

$$\hat{D}^h = \hat{\phi}^*\mathcal{D}^*. \tag{4-2}$$

From the construction of the map $\sigma : \overline{\mathcal{U}}_{0,n} \to \mathcal{U}^*_{0,n}$ we see that $(F, \mathcal{D}^*|_F)$ is log canonical for any fiber F of $g_{0,n} : \mathcal{U}^*_{0,n} \to \overline{\mathcal{M}}_{0,n}$. Since the fibers of $\hat{f} : \hat{X} \to \tilde{Y}$ are isomorphic to the fiber of $g_{0,n}$, we see that $(\hat{F}, \hat{D}^h|_{\hat{F}})$ is also log canonical, where $\hat{F}$ is any fiber of $\hat{f}$. Let $\hat{D}^v_i$ be a component of $\hat{D}^v$ and η the generic point of $\hat{f}(\hat{D}^v_i)$.

Then by inversion of adjunction we know that $(\hat{X}_\eta, (\hat{D}_i^v + \hat{D}^h)|_\eta)$ is log canonical. Since the fibers of $\hat{f}$ are reduced, the log canonical threshold of $(\hat{X}, \hat{D}; \hat{D}_i^v)$ over the generic point of $\hat{D}_i^v$ is $(1 - \operatorname{coeff}_{\hat{D}_i^v} \hat{D})$. Hence we get $\hat{D}^v = \hat{f}^* \tilde{D}_{\mathrm{div}}$. Note that the coefficients of $\hat{D}^v$ can be > 1. By definition of $\tilde{D}_{\mathrm{mod}}$ we have

$$K_{\hat{X}} + \hat{D}^h \sim_{\mathbb{Q}} \hat{f}^*(K_{\tilde{Y}} + \tilde{D}_{\mathrm{mod}}). \tag{4-3}$$

Then

$$K_{\hat{X}} + \hat{D}^h - f^*(K_{\tilde{Y}} + \phi_0^* \mathcal{L}) = K_{\hat{X}/\tilde{Y}} + \hat{D}^h - \hat{\phi}^* K_{\mathcal{U}_{0,n}^*/\overline{\mathcal{M}}_{0,n}} - \hat{\phi}^* \mathcal{D}^* \sim_{\mathbb{Q}} 0, \tag{4-4}$$

where the first equality follows from (4-3) and Lemma 4.1(3), and the second relation from (4-2) and [Liu 2002, Chapter 6, Theorem 4.9(b) and Example 3.18].

Since $\hat{f}$ has connected fibers, by (4-3) and (4-4) and projection formula we get

$$\tilde{D}_{\mathrm{mod}} \sim_{\mathbb{Q}} \phi_0^* \mathcal{L}, \tag{4-5}$$

i.e., $\tilde{D}_{\mathrm{mod}}$ is semiample.

Now since $\alpha_0 : Y'' \to Y$ is a composition of finite morphisms of degree strictly less than $\operatorname{char}(k)$ and β_0 is a birational morphism, by [Ambro 1999, Theorem 3.2 and Example 3.1],

$$K_{Y''} + D''_{\mathrm{div}} \sim_{\mathbb{Q}} \alpha_0^*(K_Y + D_{\mathrm{div}})$$

and

$$K_{\tilde{Y}} + \tilde{D}_{\mathrm{div}} \sim_{\mathbb{Q}} \beta_0^*(K_{Y''} + D''_{\mathrm{div}}).$$

So $\alpha_0^* D_{\mathrm{mod}} \sim_{\mathbb{Q}} D''_{\mathrm{mod}}$, and $\beta_0^* D''_{\mathrm{mod}} \sim_{\mathbb{Q}} \tilde{D}_{\mathrm{mod}}$. By the projection formula we have

$$D''_{\mathrm{mod}} \sim_{\mathbb{Q}} \beta_{0,*} \tilde{D}_{\mathrm{mod}}.$$

Then since α_0 is finite,

$$\psi_{0,*} \tilde{D}_{\mathrm{mod}} \sim_{\mathbb{Q}} \alpha_{0,*} \beta_{0,*} \tilde{D}_{\mathrm{mod}} \sim_{\mathbb{Q}} \alpha_{0,*} D''_{\mathrm{mod}} \sim_{\mathbb{Q}} \alpha_{0,*} \alpha_0^* D_{\mathrm{mod}} \sim_{\mathbb{Q}} D_{\mathrm{mod}}.$$

Here we view the pushforward through α_0 as pushforward of cycles. Therefore D_{mod} is $\mathbb{Q}$-linearly equivalent to an effective divisor.

Next we prove the second statement. Since α is finite, by [Kollár 2013, Corollary 2.42] we know that (X'', D'') is klt, and as β, λ, and μ are birational we know that $(\hat{X}, \hat{D})$ is sub-klt, in particular $\hat{D}^v$ has coefficients < 1. Since $\hat{f}$ is a $\mathbb{P}^1$ fibration and $(\tilde{Y}, \tilde{D}_{\mathrm{div}})$ is log smooth we have that $(\tilde{Y}, \tilde{D}_{\mathrm{div}})$ is sub-klt. By construction $\tilde{D}_{\mathrm{mod}}$ is semiample, so by [Tanaka 2015, Theorem 1] we know that $(\tilde{Y}, \tilde{D}_{\mathrm{div}} + \tilde{D}_{\mathrm{mod}})$ is sub-klt up to $\mathbb{Q}$-linear equivalence. Then $K_{Y''} + D''_{\mathrm{mod}} + D''_{\mathrm{div}} \sim_{\mathbb{Q}} \beta_{0,*}(K_{\tilde{Y}} + \tilde{D}_{\mathrm{div}} + \tilde{D}_{\mathrm{mod}})$ is also sub-klt. Finally using [Kollár 2013, Corollary 2.42] again and the fact that $D_{\mathrm{mod}} + D_{\mathrm{div}} \geq 0$ we get that $(Y, D_{\mathrm{mod}} + D_{\mathrm{div}})$ is klt. □

5. Global rational chain connectedness

In this section we prove the following theorem.

Theorem 5.1. *Let X be a projective threefold over an algebraically closed field k of characteristic $p > 0$, and $f : X \to Y$ a projective surjective morphism from X to a projective variety Y such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let D be an effective $\mathbb{Q}$-divisor, and $X_{\bar{\eta}}$ the geometric generic fiber of f. Assume that the following conditions hold*:

(1) *(X, D) is klt, $-K_X$ is big and f-ample, $K_X + D \sim_{\mathbb{Q}} 0$ and the general fibers of f are smooth.*

(2) *$p > 2/\delta$, where δ is the minimum nonzero coefficient of D.*

(3) *$D = E + f^*L$ where E is an effective $\mathbb{Q}$-Cartier divisor such that $p \nmid \mathrm{ind}(E)$, $(X_{\bar{\eta}}, E|_{X_{\bar{\eta}}})$ is globally F-split, and L is a big $\mathbb{Q}$-divisor on Y.*

(4) $\dim(Y)$ *is* 1 *or* 2.

Then X is rationally chain connected.

Remark 5.2. Under the assumptions of Theorem 5.1, the smoothness of the general fibers of f holds in characteristic $p \geq 11$ when $\dim Y = 1$ by [Hirokado 2004, Theorem 5.1(2)], and in characteristic $p \geq 5$ when $\dim Y = 2$, as is explained in Remark 3.2.

Proposition 5.3. *Let $f : X \to Y$ be a projective surjective morphism between normal varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume that the following conditions hold*:

(1) *The general fibers of f are isomorphic to $\mathbb{P}^1$.*

(2) *Y is rationally chain connected.*

Then X is rationally chain connected.

Proof. The proof is essentially the same as [Gongyo et al. 2015a, Lemma 3.12 and Proposition 3.13]. We take two general points $x_1, x_2 \in X$ and let $y_1 = f(x_1)$, $y_2 = f(x_2)$, so by construction $f^{-1}(y_1) \cong f^{-1}(y_2) \cong \mathbb{P}^1$. By assumption y_1 and y_2 can be connected by a chain of rational curves, say $C_1, C_2, \ldots, C_n$. Let $\overline{C_i} \to C_i$ be the normalization for each C_i, $S_i := f^{-1}(C_i)$, $\overline{S_i} := S_i \times_{\overline{C_i}} C_i$, and $g_i : \overline{S_i} \to S_i$ the induced morphisms. Now the morphism $\overline{S_i} \to \overline{C_i}$ is a flat projective morphism whose general fibers are $\mathbb{P}^1$, by [de Jong and Starr 2003, Theorem] it has a section which we denote by $\tilde{C_i}$. Then x_1 and x_2 is connected by $f^{-1}(y_1)$, $f^{-1}(y_2)$, $g_i(\tilde{C_i})$ and the fibers of f over the intersection points of $\{C_i\}$, which is a union of rational curves by [Debarre 2001, Lemma 3.7]. □

Proof of Theorem 5.1. We first prove the following lemma.

Lemma 5.4. *Under the condition of Theorem 5.1, $-K_Y$ is big.*

Proof. By assumption $m(K_{X_{\bar\eta}}+E|_{X_{\bar\eta}})\sim_{\mathbb{Q}} 0$ for sufficiently large and divisible m; in particular, the $k(\bar\eta)$-algebra

$$\bigoplus_{m\geq 0} H^0\big(m(K_{X_{\bar\eta}}+E|_{X_{\bar\eta}})\big)$$

is finitely generated. On the other hand since $(X_{\bar\eta}, E|_{X_{\bar\eta}})$ is globally F-split we have that

$$S^0\big(X_{\bar\eta}, \sigma(X_{\bar\eta}, E|_{X_{\bar\eta}})\otimes \mathcal{O}_{X_{\bar\eta}}(m(K_{X_{\bar\eta}}+E|_{X_{\bar\eta}}))\big)=H^0\big(X_{\bar\eta}, \mathcal{O}_{X_{\bar\eta}}(m(K_{X_{\bar\eta}}+E|_{X_{\bar\eta}}))\big).$$

Here we would like to mention that for a line bundle M and a $\mathbb{Q}$-Cartier divisor Δ, the notation $S^0(X, \Delta, M)$ is the same as the standard notation $S^0(X, \sigma(X, \Delta)\otimes M)$; see [Hacon and Xu 2015, between Lemma 2.2 and Proposition 2.3]. Therefore by [Ejiri 2015, Theorem 1.1],

$$f_*\mathcal{O}_X(m(K_{X/Y}+E))\cong f_*\mathcal{O}_X(f^*(-m(K_Y+L)))=\mathcal{O}_Y(-m(K_Y-L))$$

is weakly positive for m sufficiently large and divisible. By Lemma 2.7, $-K_Y-L$ is nef, so $-K_Y$ is big. □

Next we consider the following two cases.

Case 1: Y is 1-dimensional. After possibly taking the normalization of Y we can assume that Y is smooth. Then Lemma 5.4 implies that $g(Y)=0$, i.e., $Y\cong\mathbb{P}^1$. Let F be a general fiber of f. By assumption F is smooth and K_F is anti-ample, hence F is separably rationally connected. By [de Jong and Starr 2003, Theorem] we know that f has a section which we denote by s. Then $s(Y)$ is a rational curve in X which dominates Y. Therefore we get that X is rationally chain connected.

Case 2: Y is 2-dimensional. By assumption, a general fiber of f is isomorphic to $\mathbb{P}^1$. Now by Lemma 5.4 we know that $-K_Y$ is big. On the other hand since (X, D) is klt, by Theorem 4.3 there is a nonzero effective $\mathbb{Q}$-Cartier divisor M on Y such that $K_Y+M\sim_{\mathbb{Q}} 0$ and (Y, M) is klt. Then by the proof of Case 2 of Theorem 3.1 we know that Y is rational. Finally by Proposition 5.3 we get that X is rationally chain connected. □

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