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We show that, in the setting of Galois pairs, the spherical characters of unitary supercuspidal representations satisfy an orthogonality relation.

1. Main result

Let F be a finite extension field of $\mathbb{Q}_p$ for an odd prime p, and E a quadratic field extension of F. Let G be a connected reductive group over F, and $\boldsymbol{G} = \mathrm{R}_{E/F} G$ the Weil restriction of G with respect to E/F. The nontrivial automorphism in $\mathrm{Gal}(E/F)$ induces an involution θ, defined over F, on $\boldsymbol{G}$. The pair $(\boldsymbol{G}, G)$ is called a Galois pair, which is a kind of symmetric pair.

Let π be an irreducible admissible unitary representation of $G(E) = \boldsymbol{G}(F)$. We say that π is G-distinguished if the space $\mathrm{Hom}_{G(F)}(\pi, \mathbb{C})$ is nonzero. We fix a Haar measure $\mathrm{d}g$ on $G(E)$. Given an element ℓ in $\mathrm{Hom}_{G(F)}(\pi, \mathbb{C})$, the *spherical character* $\Phi_{\pi,\ell}$ associated to ℓ is the distribution on $G(E)$ defined by

$$\Phi_{\pi,\ell}(f) := \sum_{v \in \mathrm{ob}(\pi)} \ell(\pi(f)v)\overline{\ell(v)}, \quad f \in C_c^\infty(G(E)),$$

where $\mathrm{ob}(\pi)$ is an orthonormal basis of the representation space V_π of π. In this note, our main goal is to show that spherical characters satisfy an orthogonality relation when π is unitary supercuspidal.

Before stating our result, we introduce some notation. Recall that an element $g \in G(E)$ is called θ-regular if $s(g) := g^{-1}\theta(g)$ is regular semisimple in $G(E)$ in the usual sense; a θ-regular element g is called θ-elliptic if the identity component of the centralizer of $s(g)$ in G is an elliptic F-torus. We denote by $G(E)_{\mathrm{reg}}$ (resp. $G(E)_{\mathrm{ell}}$) the subset of θ-regular (resp. θ-elliptic) elements of $G(E)$.

Theorem 1.1 [Hakim 1994, Theorem 1]. *The spherical character $\Phi_{\pi,\ell}$ is locally integrable on $G(E)$ and locally constant on the θ-regular locus $G(E)_{\mathrm{reg}}$.*

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We denote by $\phi_{\pi,\ell}$ the locally integrable function on $G(E)$ representing the distribution $\Phi_{\pi,\ell}$, that is,

$$\Phi_{\pi,\ell}(f) = \int_{G(E)} \phi_{\pi,\ell}(g) f(g)\,\mathrm{d}g, \quad f \in C_c^\infty(G(E)).$$

We will also call $\phi_{\pi,\ell}$ a *spherical character*. Note that $\phi_{\pi,\ell}$ is bi-$G(F)$-invariant and independent of the choice of Haar measures dg. Theorem 1.1 is analogous to the classical result of Harish-Chandra [1999, Theorem 16.3] on admissible invariant distributions on connected reductive p-adic groups.

When π is unitary supercuspidal, we will show that the spherical characters $\phi_{\pi,\ell}$ satisfy an orthogonality relation (see Theorem 1.2). Before stating this relation, we need to review the Weyl integration formula in the setting of symmetric pairs. We refer the reader to [Rader and Rallis 1996, §3] or [Hakim 2003, §6] for the notation below and more details on this integration formula.

Let $\mathscr{T}$ be a set of representatives for the equivalence classes of Cartan subsets of $G(E)$ with respect to the involution θ. For $T \in \mathscr{T}$, denote $T_{\mathrm{reg}} = T \cap G(E)_{\mathrm{reg}}$. For $T \in \mathscr{T}$, the map

$$\mu : G(F) \times T_{\mathrm{reg}} \times G(F) \to G(E)_{\mathrm{reg}}, \quad (h_1, t, h_2) \mapsto h_1 t h_2,$$

is submersive and

$$G(E)_{\mathrm{reg}} = \coprod_{T \in \mathscr{T}} G(F) T_{\mathrm{reg}} G(F).$$

Let A be the split component of the center of G. For each θ-regular element g, we choose a Haar measure on $G_\gamma(F)$ where $\gamma = s(g)$ and G_γ is the split component of the centralizer of γ in G. Fix Haar measures on $A(F)$ and $G(F)$. For $\phi \in C_c^\infty(G(E)/A(F))$ and $g \in G(E)_{\mathrm{reg}}$, the orbital integral $O(g, \phi)$ of ϕ at g is defined to be

$$O(g, \phi) = \int_{A(F)\backslash G(F)} \int_{G_\gamma(F)\backslash G(F)} \phi(h_1 g h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2,$$

where $\gamma = s(g)$ and the measures inside the integral are quotient measures. From the definition we see that orbital integrals are bi-$G(F)$-invariant functions on $G(E)_{\mathrm{reg}}$. For $T \in \mathscr{T}$, the group $G_{s(t)}$ is the same for each $t \in T_{\mathrm{reg}}$. Let $D_{G(E)}$ be the usual Weyl discriminant function on $G(E)$. Then the Weyl integration formula reads as follows: with suitably normalized measures, for each $\phi \in C_c^\infty(G(E)/A(F))$, we have

$$(1) \qquad \int_{G(E)/A(F)} \phi(g)\,\mathrm{d}g = \sum_{T \in \mathscr{T}} \frac{1}{w_T} \int_T |D_{G(E)}(s(t))|_E \cdot O(t, \phi)\,\mathrm{d}t,$$

where w_T are some positive constants only depending on T (see [Rader and Rallis 1996, Theorem 3.4] and [Hakim 2003, Lemma 5]). Let $\mathscr{T}_{\text{ell}}$ be the subset of $\mathscr{T}$ consisting of elliptic Cartan subsets, that is, for $T \in \mathscr{T}$, T belongs to $\mathscr{T}_{\text{ell}}$ if and only if $T_{\text{reg}} \subset G(E)_{\text{ell}}$.

Theorem 1.2. (1) *Suppose that π is unitary supercuspidal and G-distinguished. Let ℓ be a nonzero element of* $\text{Hom}_{G(F)}(\pi, \mathbb{C})$. *Then*

$$\sum_{T \in \mathscr{T}_{\text{ell}}} \frac{1}{w_T} \int_T |D_{G(E)}(s(t))|_E \cdot |\phi_{\pi,\ell}(t)|^2 \, \mathrm{d}t$$

is nonzero.

(2) *Suppose that π and π' are two unitary supercuspidal representations of $G(E)$ and $\pi \not\cong \pi'$. Then for any $\ell \in \text{Hom}_{G(F)}(\pi, \mathbb{C})$ and $\ell' \in \text{Hom}_{G(F)}(\pi', \mathbb{C})$, we have*

$$\sum_{T \in \mathscr{T}_{\text{ell}}} \frac{1}{w_T} \int_T |D_{G(E)}(s(t))|_E \cdot \phi_{\pi,\ell}(t) \cdot \overline{\phi_{\pi',\ell'}(t)} \, \mathrm{d}t = 0.$$

Theorem 1.2 is an analog of the classical orthogonality relation for characters of discrete series (see [Clozel 1991] or [Kazhdan 1986] for this classical result). The following corollary, which has potential application in simple relative trace formula, is a direct consequence of Theorem 1.2.

Corollary 1.3. *Suppose that π is unitary supercuspidal and G-distinguished. Let ℓ be a nonzero element of* $\text{Hom}_{G(F)}(\pi, \mathbb{C})$. *Then the spherical character $\Phi_{\pi,\ell}$ does not vanish identically on $G(E)_{\text{ell}}$.*

2. Proof of Theorem 1.2

Lemma 2.1. *Suppose that $\gamma = s(g)$ with $g \in G(E)$ lies in an F-Levi subgroup M of G. Then there exists $m \in M(E)$ such that $\gamma = s(m)$.*

Proof. First we recall some basic facts about symmetric spaces. Denote $\boldsymbol{G} = \mathrm{R}_{E/F} G$ and $\boldsymbol{M} = \mathrm{R}_{E/F} M$. Let $\boldsymbol{X} = \boldsymbol{G}/G$ and $\boldsymbol{X}_M = \boldsymbol{M}/M$ be the quotient varieties. As F-varieties, $\boldsymbol{X}$ and $\boldsymbol{X}_M$ are isomorphic to the identity components of the varieties defined by the equations

$$\tilde{\boldsymbol{X}} = \{x \in \boldsymbol{G} : x\theta(x) = 1\} \quad \text{and} \quad \tilde{\boldsymbol{X}}_M = \{x \in \boldsymbol{M} : x\theta(x) = 1\}$$

respectively [Richardson 1982, 2.1–2.4]. The exact sequences

$$1 \to G \to \boldsymbol{G} \to \boldsymbol{X} \to 1 \quad \text{and} \quad 1 \to M \to \boldsymbol{M} \to \boldsymbol{X}_M \to 1$$

induce the following exact cohomology sequences:

$$1 \to s(\boldsymbol{G}(F)) \to \boldsymbol{X}(F) \to H^1(F, G)$$

and

$$1 \to s(\boldsymbol{M}(F)) \to \boldsymbol{X}_M(F) \to H^1(F, M),$$

where we use the standard notation $H^1(F, \bullet)$ to denote the Galois cohomology of algebraic groups [Serre 1997, Chapter III. §2]. However, the above exact sequences have little to do with our assertion. What we need are the following exact sequences [Carmeli 2015, Lemma 4.1.1]:

$$\begin{array}{ccccccccc} 1 & \to & s(\boldsymbol{M}(F)) & \to & \tilde{\boldsymbol{X}}_M(F) & \to & H^1(\theta, \boldsymbol{M}(F)) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & s(\boldsymbol{G}(F)) & \to & \tilde{\boldsymbol{X}}(F) & \to & H^1(\theta, \boldsymbol{G}(F)) & \to & 1, \end{array}$$

where

$$H^1(\theta, \boldsymbol{M}(F)) := H^1(\mathrm{Gal}(E/F), M(E))$$

and

$$H^1(\theta, \boldsymbol{G}(F)) := H^1(\mathrm{Gal}(E/F), G(E)).$$

Note that $\gamma \in \tilde{\boldsymbol{X}}_M(F)$, and Lemma 2.1 asserts that $\gamma \in s(\boldsymbol{M}(F))$. Thus it suffices to show that the image $[\gamma]_M$ of γ in $H^1(\theta, \boldsymbol{M}(F))$ is trivial. On the other hand, we know that the image $[\gamma]_G$ of γ in $H^1(\theta, \boldsymbol{G}(F))$ is trivial, and $[\gamma]_G$ is also the image of $[\gamma]_M$ under the natural map

$$\iota : H^1(\theta, \boldsymbol{M}(F)) \to H^1(\theta, \boldsymbol{G}(F)).$$

We claim that ι is injective, which implies that $[\gamma]_M$ is trivial. Consider the exact sequences [Serre 1997, Chapter I. §5.8(a)]:

$$\begin{array}{ccccccc} 1 & \to & H^1(\theta, \boldsymbol{M}(F)) & \to & H^1(F, M) & \to & H^1(E, M)^{\mathrm{Gal}(E/F)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & H^1(\theta, \boldsymbol{G}(F)) & \to & H^1(F, G) & \to & H^1(E, G)^{\mathrm{Gal}(E/F)}. \end{array}$$

Let P an F-parabolic subgroup of G such that $P = M \ltimes U$ where U is the unipotent radical of P. We have natural isomorphisms (see [Gille 2007, Lemma 16.2])

$$H^1(F, P) \xrightarrow{\simeq} H^1(F, M), \quad \text{and} \quad H^1(E, P) \xrightarrow{\simeq} H^1(E, M),$$

and natural injections [Serre 1997, Chapter III. §2.1]

$$H^1(F, P) \hookrightarrow H^1(F, G) \quad \text{and} \quad H^1(E, P) \hookrightarrow H^1(E, G).$$

In summary we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 1 & \to & H^1(\theta, \boldsymbol{G}(F)) & \to & H^1(F, G) & \to & H^1(E, G) \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \to & H^1(\theta, \boldsymbol{P}(F)) & \to & H^1(F, P) & \to & H^1(E, P) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & H^1(\theta, \boldsymbol{M}(F)) & \to & H^1(F, M) & \to & H^1(E, M), \end{array}$$

which implies that ι is injective. □

Lemma 2.2. *Suppose that ϕ is a matrix coefficient of a unitary supercuspidal G-distinguished representation. Then, for any $g \in G(E)_{\mathrm{reg}}$, the orbital integral $O(g, \phi)$ vanishes unless g is θ-elliptic.*

Proof. Since ϕ is a matrix coefficient of a unitary supercuspidal G-distinguished representation, it belongs to $C_c^\infty(G(E)/A(F))$ and is a supercusp form [Harish-Chandra 1970, Part I. §3]. In particular, for any unipotent radical N of a proper parabolic subgroup P of G, we have

$$\int_{N(E)} \phi(gn)\,\mathrm{d}n = 0$$

for any $g \in G(E)$. Write $\gamma = s(g)$. Suppose that g is not θ-elliptic, which means that γ is not elliptic by definition. Therefore there exists a Levi subgroup M of a proper parabolic subgroup P of G such that $G_\gamma \subset M$. According to Lemma 2.1 there exists $m \in M(E)$ such that $\gamma = s(m)$. Since

$$O(g, \phi) = O(m, \phi),$$

we assume that g is in $M(E)$ from now on. Let N be the unipotent radical of P, and K a maximal open compact subgroup of $G(F)$ such that $G(F) = M(F)N(F)K$. Fix Haar measures $\mathrm{d}m$, $\mathrm{d}n$ and $\mathrm{d}k$ on $M(F)/A(F)$, $N(F)$ and $K/K \cap A(F)$ respectively so that $\mathrm{d}h = \mathrm{d}k\,\mathrm{d}n\,\mathrm{d}m$ on $G(F)/A(F)$. Denote $\bar{K} = K/K \cap A(F)$. Then the orbital integral $O(g, \phi)$ can be written as follows:

$$\begin{aligned} O(g, \phi) &= \int_{A(F)\backslash G(F)} \int_{G_\gamma(F)\backslash G(F)} \phi(h_1^{-1} g h_2)\,\mathrm{d}h_2\,\mathrm{d}h_1 \\ &= \int_{(A(F)\backslash M(F))\times N(F)\times \bar{K}} \int_{(G_\gamma(F)\backslash M(F))\times N(F)\times \bar{K}} \phi(k_1^{-1} n_1^{-1} m_1^{-1} g m_2 n_2 k_2) \\ &\qquad\qquad \cdot \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}m_1\,\mathrm{d}m_2 \\ &= \int_{(A(F)\backslash M(F))\times N(F)} \int_{(G_\gamma(F)\backslash M(F))\times N(F)} \phi'(n_1^{-1} m_1^{-1} g m_2 n_2) \\ &\qquad\qquad \cdot \mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}m_1\,\mathrm{d}m_2, \end{aligned}$$

where

$$\phi'(x) := \int_{\bar{K}\times\bar{K}} \phi(k_1xk_2)\,\mathrm{d}k_1\,\mathrm{d}k_2, \quad x \in G(E)/A(F).$$

Note that ϕ' is still a supercusp form on $G(E)$. From now on, for convenience, we write ϕ instead of ϕ' and g instead of $m_1^{-1}gm_2$. Let $\gamma = s(g)$ for this "new" g. We claim that:

$$\int_{N(F)\times N(F)} \phi(n_1^{-1}gn_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 = 0. \tag{2}$$

It is clear that this claim implies the lemma directly.

Now we begin to prove claim (2). Note that

$$\int_{N(F)\times N(F)} \phi(n_1^{-1}gn_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 = \int_{N(F)\times N(F)} \phi(g\cdot g^{-1}n_1gn_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

Denote $\boldsymbol{N} = \mathrm{R}_{E/F}N$. Consider the morphism of the algebraic varieties:

$$\eta_g : N\times N \to \boldsymbol{N}, \quad (n_1, n_2) \mapsto g^{-1}n_1gn_2.$$

We will show that η_g is an isomorphism. If $g^{-1}n_1gn_2 = g^{-1}n_1'gn_2'$, we have the relation

$$n_2^{-1}\gamma n_2 = s(n_1gn_2) = s(n_1'gn_2') = n_2'^{-1}\gamma n_2'. \tag{3}$$

Since γ is regular, according to [Harish-Chandra 1970, Lemma 22], the equality (3) implies $n_2 = n_2'$, and thus $n_1 = n_1'$. Hence η_g is injective. To show η_g is surjective, consider the Lie algebras $\mathfrak{n}' = \mathrm{Lie}(N')$, $\mathfrak{n}'' = \mathrm{Lie}(N)$ and $\mathfrak{n} = \mathrm{Lie}(\boldsymbol{N})$, where N' is the unipotent subgroup $g^{-1}Ng$. Since

$$2\dim_F \mathfrak{n}' = 2\dim_F \mathfrak{n}'' = \dim_F \mathfrak{n}$$

and $\mathfrak{n}'\cap\mathfrak{n}'' = \{0\}$ by the injectivity of η_g, we have $\mathfrak{n} = \mathfrak{n}'\oplus\mathfrak{n}''$. Therefore η_g is submersive and thus $N'\cdot N$ is open in $\boldsymbol{N}$. On the other hand, since N' and N are unipotent groups, the orbit $N'\cdot N$ of 1 under the left and right translations of N' and N is closed in $\boldsymbol{N}$. Hence $\boldsymbol{N} = N'\cdot N$, that is, η_g is surjective. It turns out that

$$\int_{N(E)} \phi(gn)\,\mathrm{d}n = \int_{N(F)\times N(F)} j_g(n_1, n_2)\cdot\phi(g\cdot g^{-1}n_1gn_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,$$

where $j_g(n_1, n_2)$ is the Jacobian of η_g at (n_1, n_2). Note that

$$j_g(n_1, n_2) = |\mathrm{ad}(g)|_{\mathfrak{n}(F)}|_E,$$

which is independent of (n_1, n_2). At last, the claim (2) follows from the condition that ϕ is a supercusp form. □

Proof of Theorem 1.2. Let π be a unitary supercuspidal representation of $G(E)$ and $\ell \in \mathrm{Hom}_{G(F)}(\pi, \mathbb{C})$. By [Zhang 2016, Theorem 1.5], there exists a vector u_0 in the space V_π such that $\ell = \mathscr{L}_{u_0}$, where the $G(F)$-invariant linear form $\mathscr{L}_{u_0}$ is defined by

$$\mathscr{L}_{u_0}(v) := \int_{G(F)/A(F)} \langle \pi(h)v, u_0\rangle \,\mathrm{d}h, \quad v \in V_\pi.$$

Set

$$\phi(g) = \langle \pi(g)u_0, u_0\rangle,$$

which is a matrix coefficient of π. Then, according to [Zhang 2016, Corollary 1.11], the spherical character $\Phi_{\pi,\ell}$ has the following expression:

$$\Phi_{\pi,\ell}(f) = \int_{G(F)/A(F)} \int_{G(F)/A(F)} \int_{G(E)} \phi(h_1 g h_2) f(g) \,\mathrm{d}g \,\mathrm{d}h_1 \,\mathrm{d}h_2. \tag{4}$$

Note that $G_{s(g)} = A$ for $g \in G(E)_{\mathrm{ell}}$. Therefore, when $f \in C_c^\infty(G(E)_{\mathrm{ell}})$, we get

$$\Phi_{\pi,\ell}(f) = \int_{G(E)} O(g,\phi) f(g)\,\mathrm{d}g.$$

On the other hand, by Theorem 1.1, we have

$$\Phi_{\pi,\ell}(f) = \int_{G(E)} \phi_{\pi,\ell}(g) f(g)\,\mathrm{d}g.$$

Therefore, for $g \in G(E)_{\mathrm{ell}}$, we obtain

$$\phi_{\pi,\ell}(g) = O(g,\phi). \tag{5}$$

Now let π' be another unitary supercuspidal representation of $G(E)$ and $\ell' \in \mathrm{Hom}_{G(F)}(\pi', \mathbb{C})$. Let ϕ' be a matrix coefficient of π' such that the distribution $\Phi_{\pi',\ell'}$ can be expressed as

$$\Phi_{\pi',\ell'}(f) = \int_{G(F)/A(F)} \int_{G(F)/A(F)} \int_{G(E)} \phi'(h_1 g h_2) f(g) \,\mathrm{d}g \,\mathrm{d}h_1 \,\mathrm{d}h_2$$

for any $f \in C_c^\infty(G(E))$. Thus

$$\phi_{\pi',\ell'}(g) = O(g,\phi') \tag{6}$$

for any $g \in G(E)_{\mathrm{ell}}$. We choose $f_1 \in C_c^\infty(G(E))$ so that $\phi_{f_1} = \bar{\phi}'$, where

$$\phi_{f_1}(g) := \int_{A(F)} f_1(ag)\,\mathrm{d}a.$$

Then, by the Weyl integration formula (1), we have

$$\Phi_{\pi,\ell}(f_1) = \int_{G(E)/A(F)} \phi_{\pi,\ell}(g) \cdot \phi_{f_1}(g)\,\mathrm{d}g$$
$$= \sum_{T\in\mathscr{T}} \frac{1}{w_T} \int_T |D_{G(E)}(s(t))|_E \cdot \phi_{\pi,\ell}(t) \cdot O(t,\bar{\phi}')\,\mathrm{d}t.$$

Combining Lemma 2.2 and (6), we get

$$\Phi_{\pi,\ell}(f_1) = \sum_{T\in\mathscr{T}_{\mathrm{ell}}} \frac{1}{w_T} \int_T |D_{G(E)}(s(t))|_E \cdot \phi_{\pi,\ell}(t) \cdot \overline{\phi_{\pi',\ell'}(t)}\,\mathrm{d}t. \tag{7}$$

For the first assertion, we take $\pi' = \pi$, $\ell' = \ell$ and $\phi' = \phi$. Then (7) implies

$$\Phi_{\pi,\ell}(f_1) = \sum_{T\in\mathscr{T}_{\mathrm{ell}}} \frac{1}{w_T} \int_T |D_{G(E)}(s(t))|_E \cdot |\phi_{\pi,\ell}(t)|^2\,\mathrm{d}t.$$

On the other hand, we set

$$v_0 = \frac{1}{\sqrt{\langle u_0, u_0\rangle}} u_0$$

and choose $\{v_i\}_{i\in\mathbb{N}}$ such that $\{v_i\}_{i\geq 0}$ is an orthonormal basis of V_π. Then

$$\pi(\bar{\phi})v_0 = \lambda v_0 \quad \text{for some nonzero } \lambda, \quad \text{and} \quad \pi(\bar{\phi})v_i = 0 \quad \text{for } i \geq 1.$$

Therefore

$$\Phi_{\pi,\ell}(f_1) = \lambda|\ell(v_0)|^2.$$

From the proof of [Zhang 2016, Theorem 1.4] (page 1542), we see that

$$\overline{\ell(u_0)} = c\int_{A(E)G(F)\backslash G(E)} |\ell(\pi(g)u_0)|^2\,\mathrm{d}g = c'\langle u_0, u_0\rangle,$$

where c and c' are some nonzero numbers. Hence $\Phi_{\pi,\ell}(f_1)$ is nonzero. This completes the proof of the first assertion.

As for the second assertion, note that

$$\Phi_{\pi,\ell}(f_1) = \int_{G(F)/A(F)} \int_{G(F)/A(F)} \int_{G(E)/A(F)} \phi(h_1 g h_2)\bar{\phi}'(g)\,\mathrm{d}g\,\mathrm{d}h_1\,\mathrm{d}h_2 = 0,$$

since the inner integral over $G(E)/A(F)$ vanishes by the Schur orthogonality relation. Hence the assertion is deduced from (7) directly. □

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References

[Carmeli 2015] S. Carmeli, "On the stability and Gelfand property of symmetric pairs", preprint, 2015. arXiv

[Clozel 1991] L. Clozel, "Invariant harmonic analysis on the Schwartz space of a reductive p-adic group", pp. 101–121 in *Harmonic analysis on reductive groups* (Brunswick, ME, 1989), edited by W. Barker and P. Sally, Progr. Math. **101**, Birkhäuser, Boston, 1991. MR Zbl

[Gille 2007] P. Gille, "Rationality properties of linear algebraic groups and Galois cohomology", course notes, 2007, available at http://math.univ-lyon1.fr/homes-www/gille/prenotes/mcm.pdf.

[Hakim 1994] J. Hakim, "Admissible distributions on p-adic symmetric spaces", *J. Reine Angew. Math.* **455** (1994), 1–19. MR Zbl

[Hakim 2003] J. Hakim, "Supercuspidal Gelfand pairs", *J. Number Theory* **100**:2 (2003), 251–269. MR Zbl

[Harish-Chandra 1970] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics **162**, Springer, Berlin, 1970. MR Zbl

[Harish-Chandra 1999] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, University Lecture Series **16**, American Mathematical Society, Providence, RI, 1999. MR Zbl

[Kazhdan 1986] D. Kazhdan, "Cuspidal geometry of p-adic groups", *J. Analyse Math.* **47** (1986), 1–36. MR Zbl

[Rader and Rallis 1996] C. Rader and S. Rallis, "Spherical characters on p-adic symmetric spaces", *Amer. J. Math.* **118**:1 (1996), 91–178. MR Zbl

[Richardson 1982] R. W. Richardson, "Orbits, invariants, and representations associated to involutions of reductive groups", *Invent. Math.* **66**:2 (1982), 287–312. MR Zbl

[Serre 1997] J.-P. Serre, *Galois cohomology*, Springer, Berlin, 1997. MR Zbl

[Zhang 2016] C. Zhang, "Local periods for discrete series representations", *J. Funct. Anal.* **271**:6 (2016), 1525–1543. MR Zbl

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