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NONCONTRACTIBLE HAMILTONIAN LOOPS IN THE KERNEL OF SEIDEL'S REPRESENTATION

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The main purpose of this note is to exhibit a Hamiltonian diffeomorphism loop undetected by the Seidel morphism of a 1-parameter family of 2-point blow-ups of $S^2 \times S^2$, exactly one of which is monotone. As side remarks, we show that Seidel's morphism is injective on all Hirzebruch surfaces, and discuss how to adapt the monotone example to the Lagrangian setting.

1. Introduction

The motivation for this work is the search for homotopy classes of loops of Hamiltonian diffeomorphisms which are not detected by Seidel's morphism. Given a symplectic manifold (M, ω) and its Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$, recall that Seidel's morphism

$$\mathcal{S} : \pi_1(\mathrm{Ham}(M, \omega)) \to \mathrm{QH}_*(M, \omega)^\times$$

was defined on a covering of $\pi_1(\mathrm{Ham}(M, \omega))$ by Seidel [1997] for strongly semi-positive symplectic manifolds and then on the fundamental group itself and for any closed symplectic manifold by Lalonde, McDuff and Polterovich [1999].

The target space, $\mathrm{QH}_*(M, \omega)^\times$, is the group of invertible elements of the quantum homology of (M, ω). More precisely, the small quantum homology of (M, ω) is $\mathrm{QH}_*(M, \omega) = H_*(M; \mathbb{Z}) \otimes \Pi$, where Π is equal to $\Pi^{\mathrm{univ}}[q, q^{-1}]$, with q a variable of degree 2 and the ring Π^{univ} consisting of generalized Laurent series in a variable t of degree 0:

$$\Pi^{\mathrm{univ}} := \Big\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \;\Big|\; r_\kappa \in \mathbb{Q} \text{ and } \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \text{ for all } c \in \mathbb{R} \Big\}. \tag{1}$$

Since its construction, Seidel's morphism has been successfully used to detect many Hamiltonian loops (see, e.g., [McDuff 2010]), and was extended or generalized

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to various situations (see, e.g., [Hutchings 2008; Savelyev 2008; Hu and Lalonde 2010; Hu et al. 2011; Fukaya et al. 2017]). One particular extension consists of secondary-type invariants, whose construction is based on Seidel's construction after enriching Floer homology by considering Leray–Serre spectral sequences introduced by Barraud and Cornea [2007], and which should detect loops undetected by Seidel's morphism [Barraud and Cornea ≥ 2017]. However, there were no Hamiltonian loops with nontrivial homotopy class known to be undetected by Seidel's morphism (as far as we know). This short note intends to provide the first example of such a loop on a family of symplectic manifolds. Moreover, the example is explicit and thus can easily be used to test other constructions. Notice finally that this example can also be used to construct other examples (e.g., by products, see [Leclercq 2009]).

First try: symplectically aspherical manifolds. Looking for elements in the kernel of the Seidel morphism, one might first consider symplectically aspherical manifolds, by which we mean that both the symplectic form and the first Chern class vanish on the second homotopy group of the manifold. Indeed, such manifolds have trivial Seidel morphism.

The geometric reason for this is that, by construction, the Seidel morphism of (M, ω) counts pseudo-holomorphic section classes of a fibration over S^2 with fiber (M, ω). The difference between two such classes is thus given by elements of $\pi_2(M)$ admitting a pseudo-holomorphic representative, whose existence is prevented by symplectic asphericity.

Alternatively, this can be proved via purely algebraic methods, using the equivalent description of Seidel's morphism, as a representation of $\pi_1(\mathrm{Ham}(M, \omega))$ into the Floer homology of (M, ω). Given a loop of Hamiltonian diffeomorphisms, one gets an automorphism of $\mathrm{HF}_*(M, \omega)$ which can be shown to act trivially by using the following facts:

(i) Morse homology (the quantum homology of symplectically aspherical manifolds) is a ring over which Floer homology is a module.

(ii) All involved morphisms (PSS, Seidel, continuation) are module morphisms.

(iii) Any automorphism of Morse homology preserves the fundamental class, since it generates the top degree homology group.

(iv) The fundamental class is the unit of the Morse homology ring.

This line of ideas, which goes back to Seidel, has been used by McDuff and Salamon [2004] to simplify Schwarz's original proof of invariance of spectral invariants. It has been adapted by Leclercq [2008] to Lagrangian spectral invariants and used to prove the triviality of the relative (i.e., Lagrangian) Seidel morphism by Hu, Lalonde and Leclercq [2011] (see Lemma 5.5).

Now, even though aspherical manifolds seem to be ideal candidates, there are no homotopically nontrivial loops of Hamiltonian diffeomorphisms known to the authors in such manifolds.

Second try: symplectic toric manifolds. Symplectic toric geometry provides a large class of natural examples of symplectic manifolds which are complicated enough to be interesting while simple enough that many rather involved constructions can be explicitly performed. In [Anjos and Leclercq 2015], we computed the Seidel morphism on NEF toric 4-manifolds following work of McDuff and Tolman [2006]. Recall that by definition (M, J) is an NEF pair if there are no J-pseudo-holomorphic spheres in M with negative first Chern number. This gave, in the particular case of 4-dimensional toric manifolds, an elementary and somehow purely symplectic way to perform these computations previously obtained by Chan, Lau, Leung, and Tseng [2017] (and using works by Fukaya, Oh, Ohta, and Ono [2016], and González and Iritani [2012]). We also showed that one could then deduce the Seidel morphism of some non-NEF symplectic manifolds and, as an example, we made explicit computations for some Hirzebruch surface.

The easiest symplectic toric 4-manifolds for which we can exhibit a nontrivial element in the kernel of the Seidel morphism are 2-point blow-ups of $S^2 \times S^2$. More precisely, start with the monotone product $(S^2 \times S^2, \omega_1)$[1] on which we perform two blow-ups. Notice that the resulting symplectic manifold is monotone only when the respective sizes of the blow-ups coincide and are equal to $\frac{1}{2}$.

In Section 4, we exhibit a specific loop of Hamiltonian diffeomorphisms whose homotopy class is in the kernel of Seidel's morphism if and only if the size of the two blow-ups coincide. Since this loop, obtained from two circle actions, can easily be seen to be nontrivial (Anjos and Pinsonnault [2013] computed the rational homotopy of symplectomorphism groups of these manifolds), this obviously yields a family of symplectic manifolds, only one of which is monotone, with noninjective Seidel morphism.

Theorem 1.1. *The Seidel morphism of the* 2*-point blow-ups of* $(S^2 \times S^2, \omega_1)$ *with blow-ups of equal* (*arbitrary*) *sizes is not injective.*

In our search for undetected Hamiltonian loops, we realized the following:

Theorem 1.2. *Seidel's morphism is injective on all Hirzebruch surfaces.*

While this is not hard to prove and might be well known to experts, we did not find it in the literature and therefore include a proof in Section 3.

[1] Traditionally, ω_μ denotes the product symplectic form with total area $\mu \geq 1$ on the first factor and area 1 on the second one.

Discussion on the adaptation to the Lagrangian setting. As mentioned above, there is a relative (i.e., Lagrangian) version of the Seidel morphism defined by Hu and Lalonde [2010] and further studied by Hu, Lalonde and Leclercq [2011]. There are two ways to adapt the example of Theorem 1.1 to the Lagrangian setting which we discuss here. (However, in order to keep this note short, and to avoid too many technical details on the standard tools involved, we will not investigate these ideas further here.)

First, let us remark that to get the Lagrangian version of the Seidel morphism, we need to consider a monotone Lagrangian of minimal Maslov at least 2. So, in what follows, we have in mind the only monotone symplectic manifold of the family mentioned above, i.e., the monotone product $S^2 \times S^2$ with the area of each factor equal to 1 on which we perform two blow-ups of size $\frac{1}{2}$.

The first way to relate absolute and relative settings is to consider the diagonal of the symplectic product. More precisely, let (M, ω) be a monotone symplectic manifold. The diagonal $\Delta \simeq M$ is a monotone Lagrangian of the product $(M \times M, \omega \oplus (-\omega))$, which we denote $(\widehat{M}, \widehat{\omega})$ for short, with minimal Maslov number equal to twice the minimal first Chern number of (M, ω) and thus greater than or equal to 2. This allows us to consider the Lagrangian Seidel morphism:

$$\mathcal{S}_\Delta : \pi_1\big(\mathrm{Ham}(\widehat{M}, \widehat{\omega}), \mathrm{Ham}_\Delta(\widehat{M}, \widehat{\omega})\big) \to \mathrm{QH}_*(\Delta)^\times,$$

where Ham_Δ denotes the subgroup of Ham formed by Hamiltonian diffeomorphisms which preserve Δ, and $\mathrm{QH}_*(\Delta)$ denotes the Lagrangian quantum homology of Δ.

An element $\phi \in \pi_1(\mathrm{Ham}(M, \omega))$ generated by the Hamiltonian $H : M \times [0,1] \to \mathbb{R}$, induces $\widehat{\phi} \in \pi_1(\mathrm{Ham}(\widehat{M}, \widehat{\omega}), \mathrm{Ham}_\Delta(\widehat{M}, \widehat{\omega}))$, generated by $\widehat{F} = F \oplus 0 : \widehat{M} \times [0,1] \to \mathbb{R}$. To get an element in the kernel of the Lagrangian Seidel morphism, it only remains to prove that

(i) $\mathcal{S}(\phi) = \mathcal{S}_\Delta(\widehat{\phi})$ in $\mathrm{QH}_*(M, \omega) \simeq \mathrm{QH}_*(\Delta)$, and

(ii) $\widehat{\phi}$ is nonzero.

Note that in (i), not only are the quantum homologies canonically identified but the chain complexes themselves coincide and this identification agrees with the PSS morphisms in the following sense:

$$\begin{array}{ccc} \mathrm{QH}_*(M, \omega) & = & \mathrm{QH}_*(\Delta) \\ \big\downarrow{\scriptstyle \mathrm{PSS}} & & \big\downarrow{\scriptstyle \mathrm{PSS}} \\ \mathrm{HF}_*(H, J) & = & \mathrm{HF}_*(\widehat{H}, \widehat{J} : \Delta) \end{array}$$

as proved in the monotone setting by Leclercq and Zapolsky [2017] (J denotes an almost complex structure on M, compatible with and tamed by ω, while $\widehat{J}$

denotes an almost complex structure on $\widehat{M}$ adapted to J). This suggests that it is straightforward to show that (i) holds.

On the other hand, proving (ii) will require a different technique.

The second way to the Lagrangian setting is to use Albers's comparison map [2008] between Hamiltonian and Lagrangian Floer homologies, denoted below by $\mathcal{A}$, which relates the absolute and relative Seidel morphisms via the following commutative diagram (see [Hu and Lalonde 2010]):

$$\begin{tikzcd}
\pi_1(\mathrm{Ham}(M,\omega)) \arrow[r] \arrow[d, "\mathcal{S}"'] & \pi_1(\mathrm{Ham}(M,\omega), \mathrm{Ham}_L(M,\omega)) \arrow[r] \arrow[d, "\mathcal{S}_L"] & \pi_0(\mathrm{Ham}_L(M,\omega)) \\
\mathrm{HF}_*(M,\omega) \arrow[r, "\mathcal{A}"'] & \mathrm{HF}_*(M,\omega; L) &
\end{tikzcd}$$

where L is a closed monotone Lagrangian of (M,ω) with minimal Maslov number at least 2.

To get an interesting example via this method, one must choose L such that $\mathrm{HF}_*(M,\omega; L) \neq 0$ and prove (again) that the image of $\phi \in \pi_1(\mathrm{Ham}(M,\omega))$ in $\pi_1(\mathrm{Ham}(M,\omega), \mathrm{Ham}_L(M,\omega))$ is nontrivial.

2. Background and user manual for Sections 3 and 4

In order to prove Theorems 1.1 and 1.2 in the following sections, we need to describe the setting and give some information whose nature we now explain. We also give some details about previous works on which it relies.

Step A: Geometric setting. We will first introduce the symplectic toric 4-manifold (M,ω) in which we are interested and describe the associated circle actions, moment map, and polytope. Then we will give topological information which will be useful:

- the fundamental group of $\mathrm{Ham}(M,\omega)$, on which the Seidel morphism is defined, and
- the second homology group of M, which consists of generators of the quantum homology of (M,ω) (as a module over the Novikov ring).

Background for Step A. (See [Cannas da Silva 2001] for more details.) First, consider a Hamiltonian circle action on (M,ω). It is generated by a function $\phi : M \to \mathbb{R}$, called the *moment map*, which is assumed to be normalized, that is, satisfying

$$\int_M \phi\, \omega^n = 0.$$

Now (M,ω) is called *toric* if it admits an effective action by a Hamiltonian torus $\mathbb{T}^2 \subset \mathrm{Ham}(M,\omega)$. We will denote by Φ the corresponding moment map and by $P = \Phi(M)$ the moment polytope. If η is an outward primitive normal to the facet

D_η of P, we consider the associated Hamiltonian circle action, Γ_η, whose moment map is $\phi := \langle \eta, \Phi(\cdot) \rangle$.[2]

Note that $\phi^{-1}(D_\eta)$ is a *semifree* maximum component for Γ_η, as the action is semifree (i.e., the stabilizer of every point is trivial or the whole circle) on some neighborhood of $\phi^{-1}(D_\eta)$.

Step B: The Seidel morphism. In this step, we will give the expression of the image of the aforementioned circle actions Γ_η via the Seidel morphism, $\mathcal{S}$.

Background for Step B. (See [McDuff and Tolman 2006; Anjos and Leclercq 2015].)

We consider a toric 4-manifold (M, ω, Φ) as above. To compute the image of a Hamiltonian circle action via the Seidel morphism, we pick a ω-compatible, S^1-invariant almost complex structure, J. The main case we are concerned with here is the Fano case. Recall that (M, J) is said to be *Fano* if any J-pseudo holomorphic sphere in M has positive first Chern number.

When this is the case, [McDuff and Tolman 2006, Theorem 1.10] or [Anjos and Leclercq 2015, Theorem 4.5] tells us that the associated Seidel element consists of only one term (the one of highest order). More precisely:

Theorem 2.1 [McDuff and Tolman 2006, Theorem 1.10]. *Let (M, ω, J, Φ) be a compact Fano toric symplectic 4-manifold. Let η be an outward primitive normal to the facet D_η of the moment polytope P and let Γ_η be the associated Hamiltonian circle action. Then*

$$\mathcal{S}(\Gamma_\eta) = [F_{\max}] \otimes q t^{\phi_{\max}},$$

where ϕ is the moment map associated to Γ_η, and $F_{\max} = \phi^{-1}(D_\eta)$ is the maximal fixed point component of ϕ and $\phi_{\max} = \phi(F_{\max})$.

Step C: The quantum homology of (M, ω). The computation of the Seidel elements $\mathcal{S}(\Gamma_\eta)$ in Step B also gives us explicit relations involving the quantum product. This allows us to complete the description of the quantum homology as an algebra. Since the generators of $\pi_1(\mathrm{Ham}(M, \omega))$ can be expressed in terms of the Γ_η, this also gives us the image of the Seidel morphism so that, by understanding $\mathrm{im}(\mathcal{S}) \subset \mathrm{QH}_*(M, \omega)^\times$, we can prove Theorems 1.1 and 1.2.

Background for Step C. (See [McDuff and Tolman 2006, Section 5.1] for the general setting.) Let us recall how to obtain the quantum homology algebra in our specific setting. Let $D_1, \ldots, D_n$ be the facets of P and $\eta_1, \ldots, \eta_n \in \mathbb{R}^2$ the respective outward primitive integral normal vectors. Let C be the set of *primitive sets*, i.e., subsets $I = \{i_1, i_2\} \subset \{1, \ldots, n\}$ such that $D_{i_1} \cap D_{i_2} = \varnothing$. Let $u_i = [D_i] \otimes q$.

[2] To lighten the notation, we will actually denote by D_i and Γ_i, respectively, the facet and the circle action associated to the normal η_i (instead of D_{η_i} and Γ_{η_i}).

There are two linear relations,

$$\sum_{i=1}^{n}\langle(1,0),\eta_i\rangle u_i = 0 \quad \text{and} \quad \sum_{i=1}^{n}\langle(0,1),\eta_i\rangle u_i = 0,$$

which generate the ideal of linear relations $\mathrm{Lin}(P)$ in $\mathbb{Q}[u_1,\dots,u_n]$. Moreover, relations between the normal vectors η_i yield equations satisfied by the corresponding Seidel elements $\mathcal{S}(\Gamma_i)$. Using these, it is then possible to exhibit the quantum product $u_{i_1} * u_{i_2}$, for every primitive set $\{i_1,i_2\}$, as a linear combination of the classes p (the class of a point), $\mathbb{1}$ (the fundamental class), and u_i: $f_{i_1 i_2} = (\alpha p\otimes q^2 + \beta\mathbb{1} + \sum\alpha_i u_i)t^{\gamma}$ for some $\alpha,\beta,\alpha_i \in \mathbb{Z}$ and $\gamma\in\mathbb{R}$. Then, the Stanley–Reisner ideal is defined by

$$\mathrm{SR}_Y(P) = \langle u_{i_1}\cdot u_{i_2} - f_{i_1 i_2} \mid \{i_1,i_2\}\in C\rangle.$$

Finally, there is an isomorphism of Π^{univ}-algebras

$$\mathrm{QH}_*(M,\omega) \simeq \mathbb{Q}\,[u_1,\dots,u_n]\otimes\Pi^{\mathrm{univ}}/(\mathrm{Lin}(P)+\mathrm{SR}_Y(P)). \tag{2}$$

3. Hirzebruch surfaces

We proceed in two steps as the "even" and "odd" Hirzebruch surfaces have to be dealt with separately. Throughout the section, we follow the notation and conventions used in [Anjos and Leclercq 2015] (in particular in Section 5.3), most of them having been recalled in Section 2 above.

3.1. *Even Hirzebruch surfaces.* Recall that the toric "even" Hirzebruch surfaces $(\mathbb{F}_{2k},\omega_\mu)$, $0\le k\le \ell$ with $\ell\in\mathbb{N}$ and $\ell<\mu\le\ell+1$, can be identified with the symplectic manifolds $M_\mu = (S^2\times S^2,\omega_\mu)$ where ω_μ is the split symplectic form with area $\mu\ge 1$ for the first S^2-factor, and with area 1 for the second factor. The moment polytope of $\mathbb{F}_{2k}$ is

$$P_{2k} = \big\{(x_1,x_2)\in\mathbb{R}^2 \mid 0\le x_1\le 1,\ x_2+kx_1\ge 0,\ x_2-kx_1\le\mu-k\big\}.$$

Let $\Lambda^{2k}_{e_1}$ and $\Lambda^{2k}_{e_2}$ represent the circle actions whose moment maps are, respectively, the first and second components of the moment map associated to the torus action T_{2k} acting on $\mathbb{F}_{2k}$. We will also denote by $\Lambda^{2k}_{e_1}$ and $\Lambda^{2k}_{e_2}$ the corresponding generators in $\pi_1(T_{2k})$.

It is well known (see, e.g., [Abreu and McDuff 2000, Theorem 1.1 or Corollary 2.7]) that for $k=0$, $\pi_1(\mathrm{Ham}(\mathbb{F}_0,\omega_\mu)) = \mathbb{Z}/2\oplus\mathbb{Z}/2$ and that for $k\ge 1$, $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k},\omega_\mu)) = \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}$. Moreover, the authors explain in [Abreu and McDuff 2000] (see Section 2.5 and in particular Lemma 2.10) that the $\mathbb{Z}/2$ terms of the fundamental groups are respectively generated by $\Lambda^0_{e_1}$ and $\Lambda^0_{e_2}$, while the generator of the additional $\mathbb{Z}$ term is $\Lambda^2_{e_1}$.

Let $B=[S^2\times\{p\}]$ and $F=[\{p\}\times S^2]\in H_2(S^2\times S^2;\mathbb{Z})$ and denote $u=B\otimes q$ and $v=F\otimes q$ where q is the degree 2 variable entering into play in the definition of $\Pi=\Pi^{\mathrm{univ}}[q,q^{-1}]$ and Π^{univ} is the ring of generalized Laurent series defined by (1).

We now gather from [Anjos and Leclercq 2015] the results we will need for the proof of Theorem 1.2 in this case. First, in Section 5.3 of that paper, we computed the image of the generators $\Lambda^0_{e_1}$, $\Lambda^0_{e_2}$, and $\Lambda^2_{e_1}$ by the Seidel morphism, $\mathcal{S}$. Namely, we obtained:

$$\begin{aligned}&\mathcal{S}(\Lambda^0_{e_1})=B\otimes qt^{\frac12}=ut^{\frac12},\quad \mathcal{S}(\Lambda^0_{e_2})=F\otimes qt^{\frac{\mu}{2}}=vt^{\frac{\mu}{2}},\ \text{and}\\ &\mathcal{S}(\Lambda^2_{e_1})=(B+F)\otimes qt^{\frac12-\epsilon}=(u+v)t^{\frac12-\epsilon},\ \text{with}\ \epsilon=\frac{1}{6\mu}.\end{aligned}\tag{3}$$

Note that the circle action $\Lambda^2_{e_1}$ acts on the second Hirzebruch surface $\mathbb{F}_2$ and the almost complex structure in this case is not Fano, because the class $B-F$ is represented by a pseudo-holomorphic sphere and its first Chern number vanishes. Nevertheless, by Theorem 4.4 in [Anjos and Leclercq 2015], the Seidel element of this action still does not contain any lower order terms.

The computation of the Seidel elements associated to each one of the facets of the polytope yields the quantum product identities

$$F*F=\mathbb{1}\otimes q^{-2}t^{-\mu},\quad B*B=\mathbb{1}\otimes q^{-2}t^{-1},\quad\text{and}\quad F*B=p,\tag{4}$$

so $\mathcal{S}(\Lambda^0_{e_1})^2=\mathcal{S}(\Lambda^0_{e_2})^2=\mathbb{1}$. Finally recall that, thanks to [Anjos and Leclercq 2015, Proposition 5.1] (see (2) in our setting), we were able to express the (small) quantum homology algebra as

$$\mathrm{QH}_*(\mathbb{F}_{2k},\omega_\mu)\simeq\Pi^{\mathrm{univ}}[u,v]\,/\langle u^2=t^{-1},v^2=t^{-\mu}\rangle.$$

From (3) and (4), it is now easy to check that the inverse of $\mathcal{S}(\Lambda^2_{e_1})$ is given by

$$\mathcal{S}(\Lambda^2_{e_1})^{-1}=(B-F)\otimes q\frac{t^{\frac12+\epsilon}}{1-t^{1-\mu}}=(u-v)\frac{t^{\frac12+\epsilon}}{1-t^{1-\mu}}.\tag{5}$$

Let us now prove the theorem.

Proof of Theorem 1.2 for even Hirzebruch surfaces. Since $\Lambda^0_{e_1}$ and $\Lambda^0_{e_2}$ are of order 2, any element in $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k},\omega_\mu))$ is of the form $\varepsilon_1\Lambda^0_{e_1}+\varepsilon_2\Lambda^0_{e_2}+\ell\Lambda^2_{e_1}$, with ε_1 and ε_2 in $\{0,1\}$ and $\ell\in\mathbb{Z}$. Moreover, it is in the kernel of $\mathcal{S}$ if and only if $\mathcal{S}(\Lambda^2_{e_1})^{-\ell}=\mathcal{S}(\Lambda^0_{e_1})^{\varepsilon_1}\mathcal{S}(\Lambda^0_{e_2})^{\varepsilon_2}$, which is equivalent to the fact that $\mathcal{S}(\Lambda^2_{e_1})^{-\ell}$ is either u, v, or uv, up to a power of t.

Let $\ell'\in\mathbb{N}\setminus\{0\}$, and expand the ℓ'-th power of $\mathcal{S}(\Lambda^2_{e_1})$ (whose expression is recalled in (3) above) using the binomial theorem to get

$$\mathcal{S}(\Lambda^2_{e_1})^{\ell'}=\sum_{k=0}^{\ell'}\binom{\ell'}{k}u^kv^{\ell'-k}t^{(\frac12-\epsilon)\ell'}.$$

The identities $u^2 = t^{-1}$ and $v^2 = t^{-\mu}$ ensure $\mathcal{S}(\Lambda^2_{e_1})^{\ell'}$ is of the form $C_1 \cdot u + C_2 \cdot v$ if ℓ' is odd, or $C_1 + C_2 \cdot uv$ otherwise, where (in both cases) C_1 and C_2 are linear combinations of powers of t with positive rational coefficients (hence nonzero), so

$$\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2} + \ell \Lambda^2_{e_1} \notin \ker(\mathcal{S})$$

for any $\varepsilon_1$and ε_2 in $\{0, 1\}$ and $\ell < 0$.

We proceed along the same lines for a positive ℓ: $\mathcal{S}(\Lambda^2_{e_1})^{-\ell}$ is, by the binomial theorem together with (5), of the form

$$\frac{C'_1 \cdot u - C'_2 \cdot v}{(1-t^{1-\mu})^{\ell}} \quad \text{or} \quad \frac{C'_1 - C'_2 \cdot uv}{(1-t^{1-\mu})^{\ell}},$$

which shows that $\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2} + \ell \Lambda^2_{e_1}$ is not in $\ker(\mathcal{S})$ for any $\ell > 0$ either.

This implies that the only elements of $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k}, \omega_\mu))$ which could be in $\ker(\mathcal{S})$ are of the form $\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2}$ so that in the end $\ker(\mathcal{S}) = \{0\}$. □

3.2. *Odd Hirzebruch surfaces.* Similarly, "odd" Hirzebruch surfaces $(\mathbb{F}_{2k-1}, \omega'_\mu)$, $1 \le k \le \ell$ with $\ell \in \mathbb{N}$ and $\ell < \mu \le \ell + 1$, can be identified with the symplectic manifolds $(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2, \omega'_\mu)$ where the symplectic area of the exceptional divisor is $\mu > 0$ and the area of the projective line is $\mu + 1$. Its moment polytope is

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le x_1 + x_2 \le 1,\ x_2(k-1) + kx_1 \ge 0, \\ kx_2 + (k-1)x_1 \ge k - \mu - 1 \end{array} \right\}.$$

Let $\Lambda^{2k-1}_{e_1}$ and $\Lambda^{2k-1}_{e_2}$ represent the circle actions whose moment maps are, respectively, the first and the second component of the moment map associated to the torus action T_{2k-1} acting on $\mathbb{F}_{2k-1}$. As before, we will also denote by $\Lambda^{2k-1}_{e_1}$ and $\Lambda^{2k-1}_{e_2}$ the generators of $\pi_1(T_{2k-1})$.

Similarly to the even case the fundamental group of $(\mathbb{F}_{2k-1}, \omega'_\mu)$ is computed in [Abreu and McDuff 2000, Theorem 1.4 or Corollary 2.7]. More precisely, $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k-1}, \omega'_\mu)) = \mathbb{Z}\langle \Lambda^1_{e_1} \rangle$ for all $k \ge 1$, that is, $\Lambda^1_{e_1}$ is the generator of the fundamental group as explained in [Abreu and McDuff 2000, Section 2.5 (in particular Lemma 2.11)]. So, in order to prove that the Seidel morphism is injective, we only need to show that the order of $\mathcal{S}(\Lambda^1_{e_1})$ in $\mathrm{QH}_*(\mathbb{F}_{2k+1}, \omega'_\mu)$ is infinite.

We now need to expand Remark 5.6 of [Anjos and Leclercq 2015] (which quickly dealt with the odd case), along the lines of [Anjos and Leclercq 2015, Section 5.3] (where we focused in more detail on the even case). Let $B \in H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2; \mathbb{Z})$ denote the homology class of the exceptional divisor with self intersection -1 and F the class of the fiber of the fibration $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \to S^2$. If we set $u_1 = (B + F) \otimes q$, $u_2 = u_4 = F \otimes q$, and $u_3 = B \otimes q$, clearly the additive relations are given by

$$u_2 = u_4 \quad \text{and} \quad u_1 = u_2 + u_3. \tag{6}$$

The normal vectors to the moment polytope of $\mathbb{F}_1$ are given by $\eta_1 = (1,1)$, $\eta_2 = (0,-1)$, $\eta_3 = (-1,-1)$, and $\eta_4 = (-1,0)$. We denote by Γ_i the actions associated to η_i.

As explained in Section 2, since $\mathbb{F}_1$ is Fano, it follows from [McDuff and Tolman 2006, Theorem 1.10] that the Seidel elements associated to the Γ_i are given by

$$\begin{aligned}\mathcal{S}(\Gamma_1) &= (B+F)\otimes qt^{1+\mu-2\varepsilon} = u_1 t^{1+\mu-2\varepsilon},\\ \mathcal{S}(\Gamma_2) &= \mathcal{S}(\Gamma_4) = F\otimes qt^{\varepsilon} = u_2 t^{\varepsilon},\\ \mathcal{S}(\Gamma_3) &= B\otimes qt^{2\varepsilon-\mu} = u_3 t^{2\varepsilon-\mu},\end{aligned}$$

with $\varepsilon = (3\mu^2+3\mu+1)/(3(1+2\mu))$.

The relation $\eta_1+\eta_3=0$ yields $\mathcal{S}(\Gamma_1)*\mathcal{S}(\Gamma_3)=\mathbb{1}$, that is, $B*(B+F)\otimes q^2 t=\mathbb{1}$. Similarly, since $\eta_2+\eta_4=\eta_3$ it follows that $\mathcal{S}(\Gamma_2)*\mathcal{S}(\Gamma_4)=\mathcal{S}(\Gamma_3)$, which is equivalent to $F*F = B\otimes q^{-1}t^{-\mu}$. Therefore the primitive relations are given by

$$u_1u_3 = t^{-1} \quad\text{and}\quad u_2u_4 = u_3 t^{-\mu}. \tag{7}$$

Now, following Step C of Section 2 above, we set $u = F\otimes q$ and deduce from the relations (6) and (7) that

$$\mathrm{QH}_*(\mathbb{F}_{2k+1},\omega'_\mu) = \Pi^{\mathrm{univ}}[u]/(u^4t^{2\mu}+u^3t^{\mu}-t^{-1}). \tag{8}$$

Note that $\Lambda^1_{e_1}$, the generator of $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k-1},\omega'_\mu))$, is the action associated to the vector $(1,0)$. We thus get that $\mathcal{S}(\Lambda^1_{e_1}) = \mathcal{S}(\Gamma_4)^{-1}$.

Now we can proceed with the proof of the theorem.

Proof of Theorem 1.2 for odd Hirzebruch surfaces. From the discussion above, we see that $\mathcal{S}(\Lambda^1_{e_1})^{-1} = \mathcal{S}(\Gamma_4) = ut^{\varepsilon}$. So, in order to show that Seidel's morphism is injective we only need to show that

$$\mathcal{S}(\ell\Lambda^1_{e_1})^{-1} = u^{\ell}t^{\ell\varepsilon} \neq 1$$

for any $\ell\in\mathbb{N}\setminus\{0\}$.

First, note the polynomial $M(u)=u^4t^{2\mu}+u^3t^{\mu}-t^{-1}\in\Pi^{\mathrm{univ}}[u]$ in (8) above has invertible main coefficient, so that for any positive integer ℓ, there exist uniquely determined polynomials Q_ℓ and R_ℓ such that $u^{\ell}t^{\ell\varepsilon}-1 = M(u)Q_\ell(u)+R_\ell(u)$ and the degree of R_ℓ is less than the degree of M.

Assume Seidel's morphism is not injective: then there exists $\ell_0\in\mathbb{N}\setminus\{0\}$ such that $R_{\ell_0}=0$. To find the polynomial Q_{ℓ_0}, we proceed to the long division of $u^{\ell_0}t^{\ell_0\varepsilon}-1$ by M which consists of a finite number (at most ℓ_0-3) of steps. This ensures that the coefficients of Q_{ℓ_0} are *finite* linear combinations of powers of t (with rational coefficients). Therefore Q_{ℓ_0} induces a polynomial $Q^1_{\ell_0}$ in $\mathbb{Q}[u]$ when t is set to 1, satisfying $u^{\ell_0}-1 = (u^4+u^3-1)Q^1_{\ell_0}(u)$ in $\mathbb{Q}[u]$. Since the roots of u^4+u^3-1 are not roots of unity, we get a contradiction. So, there is no positive integer ℓ_0 such that $u^{\ell_0}t^{\ell_0\varepsilon}=1$, which concludes the proof. □

4. 2-point blow-ups of $S^2 \times S^2$

We now consider the manifold obtained from

$$(M_\mu, \omega_\mu) = (S^2 \times S^2, \omega_\mu)$$

(see Section 3.1) by performing two successive symplectic blow-ups of capacities c_1 and c_2 with $0 < c_2 \le c_1 < c_1 + c_2 \le 1 \le \mu$, which we denote by $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$. Let B, $F \in H_2(M_{\mu,c_1,c_2}; \mathbb{Z})$ be the homology classes defined by $B = [S^2 \times \{p\}]$ and $F = [\{p\} \times S^2]$ and let $E_i \in H_2(M_{\mu,c_1,c_2}; \mathbb{Z})$ be the exceptional class corresponding to the blow-up of capacity c_i.

Remark 4.1. There is an alternative description of this manifold as the 3-point blow-up of $\mathbb{CP}^2$. Indeed, consider $\mathbb{X}_3 = \mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ equipped with the symplectic form $\omega_{\nu;\delta_1,\delta_2,\delta_3}$ obtained from the symplectic blow-up of $(\mathbb{CP}^2, \omega_\nu)$ at three disjoint balls of capacities δ_1, δ_2 and δ_3, where ω_ν is the standard Fubini–Study form on $\mathbb{CP}^2$ rescaled so that $\omega_\nu(\mathbb{CP}^1) = \nu$. Let $\{L, V_1, V_2, V_3\}$ be the standard basis of $H_2(\mathbb{X}_3; \mathbb{Z})$ consisting of the class L of a line together with the classes V_i of the exceptional divisors. It is well known that $\mathbb{X}_3$ is diffeomorphic to M_{μ,c_1,c_2}. The diffeomorphism $\mathbb{X}_3 \to M_{\mu,c_1,c_2}$ can be chosen to map the ordered basis $\{L, V_1, V_2, V_3\}$ to $\{B + F - E_1, B - E_1, F - E_1, E_2\}$. When one considers this birational equivalence in the symplectic category, uniqueness of symplectic blow-ups implies that $(\mathbb{X}_3, \omega_{\nu;\delta_1,\delta_2,\delta_3})$ is symplectomorphic, after rescaling, to M_μ blown-up with capacities c_1 and c_2, where $\mu = (\nu - \delta_2)/(\nu - \delta_1)$, $c_1 = (\nu - \delta_1 - \delta_2)/(\nu - \delta_1)$, and $c_2 = \delta_3/(\nu - \delta_1)$. In Section 2.1 of [Anjos and Pinsonnault 2013], it is explained why it is sufficient to consider values of c_1 and c_2 in the range above: $0 < c_2 \le c_1 < c_1 + c_2 \le 1 \le \mu$. ◀

The quantum algebra of $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ was computed by Entov and Polterovich [2008] (as $(\mathbb{X}_3, \omega_{\nu;\delta_1,\delta_2,\delta_3})$, see their proof of Proposition 4.3). More precisely, setting $u = (F - E_2) \otimes q$ and $v = (B - E_2) \otimes q$, they proved that:

Lemma 4.2. *As a Π^{univ}-algebra we have*

$$\mathrm{QH}_*(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2}) \cong \Pi^{\mathrm{univ}}[u, v]/I_{\mu,c_1,c_2}$$

where I_{μ,c_1,c_2} is the ideal generated by

$$u^2v^2 + u^2vt^{-c_2} = vt^{-\mu-c_2} + t^{c_1-\mu-1-c_2} \text{ and}$$
$$u^2v^2 + uv^2t^{-c_2} = ut^{-1-c_2} + t^{c_1-\mu-1-c_2}.$$

We recall here parts of this computation, using the formalism of [Anjos and Leclercq 2015], as they will be needed below to understand the proof of the noninjectivity result stated as Theorem 1.1. These parts correspond to Steps B and C of Section 2.

Sketch of proof. Consider $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ endowed with the standard action of the torus $T = S^1 \times S^1$ for which the moment polytope is given by

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_2 \leq \mu,\ -1 \leq x_1 \leq 0,\ c_1 \leq x_2 - x_1 \leq \mu + 1 - c_2\} \tag{9}$$

so the primitive outward normals to P are as follows:

$$\begin{aligned} \eta_1 &= (0,1), & \eta_2 &= (1,0), & \eta_3 &= (1,-1),\\ \eta_4 &= (0,-1), & \eta_5 &= (-1,0), & \eta_6 &= (-1,1). \end{aligned}$$

The Delzant construction gives a method to obtain, from the polytope P, the symplectic manifold $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ with the toric action T: first consider the standard action of the torus $\mathbb{T}^6$ on $\mathbb{C}^6$ and then perform a symplectic reduction at a regular level of that action (for more details, see, for example, [Cannas da Silva 2001, Section 29]). Then the normalized moment map $\Phi : M_{\mu,c_1,c_2} \to \mathbb{R}^2$ of the remaining T action, obtained through the Delzant construction, is given by

$$\Phi(z_1, \dots, z_6) = \left(-\tfrac{1}{2}|z_2|^2 + \epsilon_1, -\tfrac{1}{2}|z_1|^2 + \mu - \epsilon_2\right), \quad z_i \in \mathbb{C},$$

where ϵ_1 and ϵ_2 are given by the symplectic parameters μ, c_1, and c_2 as

$$\epsilon_1 = \frac{c_1^3 + 3c_2^2 - c_2^3 - 3\mu}{3(c_1^2 + c_2^2 - 2\mu)} \quad \text{and} \quad \epsilon_2 = \frac{c_1^3 - c_2^3 + 3c_2^2\mu - 3\mu^2}{3(c_1^2 + c_2^2 - 2\mu)}. \tag{10}$$

Moreover, the homology classes $A_i = [\Phi^{-1}(D_i)]$ of the pre-images of the corresponding facets D_i are: $A_1 = F - E_2$, $A_2 = B - E_1$, $A_3 = E_1$, $A_4 = F - E_1$, $A_5 = B - E_2$, and $A_6 = E_2$.

For $1 \leq i \leq 6$, let Γ_i be the circle action associated to the primitive outward normal η_i. Since the toric complex structure on M_{μ,c_1,c_2} is Fano and T-invariant, it follows from [McDuff and Tolman 2006, Theorem 1.10] or [Anjos and Leclercq 2015, Theorem 4.5] (recalled as Theorem 2.1 in Section 2) that the Seidel elements associated to the Γ_i are given by the expressions

$$\begin{aligned} \mathcal{S}(\Gamma_1) &= (F - E_2) \otimes qt^{\mu-\epsilon_2}, & \mathcal{S}(\Gamma_2) &= (B - E_1) \otimes qt^{\epsilon_1},\\ \mathcal{S}(\Gamma_3) &= E_1 \otimes qt^{\epsilon_1+\epsilon_2-c_1}, & \mathcal{S}(\Gamma_4) &= (F - E_1) \otimes qt^{\epsilon_2},\\ \mathcal{S}(\Gamma_5) &= (B - E_2) \otimes qt^{1-\epsilon_1}, & \mathcal{S}(\Gamma_6) &= E_2 \otimes qt^{\mu+1-c_2-\epsilon_1-\epsilon_2}. \end{aligned} \tag{11}$$

There are nine primitive sets: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$, and $\{4,6\}$ which yield nine multiplicative relations (which form the Stanley–Reisner ideal) that, combined with the two linear relations ($A_5 = A_1 + A_2 - A_4$ and $A_6 = A_3 + A_4 - A_1$), give the desired result as explained in Step C of Section 2 above. □

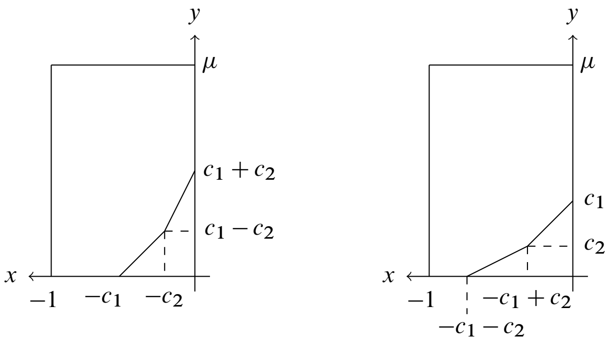

Figure 1. $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ with toric actions T_1 and T_2.

Assume from now on that $\mu = 1$. Recall from [Anjos and Pinsonnault 2013, Theorem 1.1] that if $c_2 < c_1$ then

$$\pi_1(\mathrm{Ham}(M_{1,c_1,c_2}, \omega_{1,c_1,c_2})) \simeq \mathbb{Z}\langle x_0, x_1, y_0, y_1, z\rangle \simeq \mathbb{Z}^5,$$

where the generators x_0, x_1, y_0, y_1, z correspond to circle actions contained in maximal tori of the Hamiltonian group. In particular, the generators in which we will be most interested are $x_0 = \Gamma_2$ and $y_0 = \Gamma_1$ where the Γ_i are the circle actions associated to the primitive outward normals η_i to the polytope P defined in (9).

Remark 4.3. In order to understand the remaining generators, consider the two toric manifolds given by the polytopes in Figure 1. We denote by $\{x_{0,i}, y_{0,i}\}$ the generators in $\pi_1(T_i)$, where T_i, $i = 1, 2$, represent the two torus actions in this figure and the generators $\{x_{0,i}, y_{0,i}\}$ correspond to the circle actions whose moment maps are, respectively, the first and second components of the moment map associated to each one of the toric actions. It was shown in [Anjos and Pinsonnault 2013, Lemma 4.5] that $x_1 = x_{0,1}$, $z = y_{0,2}$, and $y_1 = y_{0,1} - x_1 = z - x_{0,2}$.

Note that the case $c_1 = c_2$ is an interesting limit case in terms of the topology of the Hamiltonian group since y_1 disappears. For more details see [Anjos and Pinsonnault 2013, Section 5.1]. ◀

To prove Theorem 1.1, we will prove Proposition 4.4.

Proposition 4.4. *The class of* $2(x_0 + y_0)$ *belongs to* $\ker(\mathcal{S})$ *if and only if* $\mu = 1$ *and* $c_1 = c_2$.

Proof. From the computation of the Seidel elements in (11) one gets that in the general case (by which we mean for all $\mu \geq 1$), $\mathcal{S}(\Gamma_1) = ut^{\mu-\epsilon_2}$ and $\mathcal{S}(\Gamma_5) = vt^{1-\epsilon_1}$. As the Seidel elements are *invertible* quantum classes, this yields invertibility of u and v. Note that

$$\mathcal{S}(x_0) = \mathcal{S}(\Gamma_2) = \mathcal{S}(\Gamma_5)^{-1} = v^{-1}t^{\varepsilon_1 - 1} \quad \text{and} \quad \mathcal{S}(y_0) = \mathcal{S}(\Gamma_1) = ut^{\mu-\varepsilon_2}.$$

Since $\mu \geq 1 > c_2^2$, it is straightforward to deduce from (10) that $\varepsilon_1 = \varepsilon_2$ if and only if $\mu = 1$: we now restrict our attention to this case and denote by ε the common value of $\varepsilon_1 = \varepsilon_2$. By invertibility of u and v, the fact that $2(x_0 + y_0)$ belongs to $\ker(\mathcal{S})$ is equivalent to $u^2 = v^2$, since

$$\mathcal{S}(2(x_0 + y_0)) = \mathcal{S}(x_0)^2 * \mathcal{S}(y_0)^2 = v^{-2} t^{\epsilon - 1} u^2 t^{1-\epsilon} = v^{-2} u^2.$$

On the other hand, note that multiplying the first and second relations in I_{1,c_1,c_2} by $v^{-1}t^{c_2}$ and $u^{-1}t^{c_2}$, respectively, these become equivalent to

$$u^2 = t^{-1} + v^{-1} t^{c_1 - 2} - u^2 v t^{c_2} \quad \text{and} \quad v^2 = t^{-1} + u^{-1} t^{c_1 - 2} - u v^2 t^{c_2}$$

so that $u^2 = v^2$ is equivalent to $v^{-1} t^{c_1 - 2} - u^2 v t^{c_2} = u^{-1} t^{c_1 - 2} - u v^2 t^{c_2}$.

Multiplying both relations in I_{1,c_1,c_2} by t^{2c_2}, we see that

$$\begin{aligned} -u^2 v t^{c_2} &= (u^2 v^2 t^{2c_2} - t^{c_1 + c_2 - 2}) - v t^{c_2 - 1}, \text{ and} \\ -u v^2 t^{c_2} &= (u^2 v^2 t^{2c_2} - t^{c_1 + c_2 - 2}) - u t^{c_2 - 1} \end{aligned} \tag{12}$$

so we can replace $u^2 v t^{c_2}$ and $u v^2 t^{c_2}$ in the previous equation to obtain

$$u^2 = v^2 \iff v^{-1} t^{c_1 - 1} + u t^{c_2} = u^{-1} t^{c_1 - 1} + v t^{c_2}. \tag{13}$$

Finally, (12) also induces, by subtracting one from the other, the equation

$$(u^2 v - u v^2) t^{-c_2} = (v - u) t^{-1 - c_2},$$

which is equivalent to $(v^{-1} - u^{-1}) t^{-1} = v - u$. Using these together with (13) we conclude that $u^2 = v^2$ if and only if $(u - v)(t^{c_1} - t^{c_2}) = 0$ which is equivalent to $c_1 = c_2$ since otherwise $t^{c_1} - t^{c_2}$ would be invertible. □

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