Pacific Journal of Mathematics

DIFFERENTIAL HARNACK ESTIMATES FOR FISHER'S EQUATION

XIAODONG CAO, BOWEI LIU, IAN PENDLETON AND ABIGAIL WARD

Volume 290 No. 2 October 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.273

DIFFERENTIAL HARNACK ESTIMATES FOR FISHER'S EQUATION

XIAODONG CAO, BOWEI LIU, IAN PENDLETON AND ABIGAIL WARD

We derive several differential Harnack estimates (also known as Li–Yau–Hamilton-type estimates) for positive solutions of Fisher's equation. We use the estimates to obtain lower bounds on the speed of traveling wave solutions and to construct classical Harnack inequalities.

1. Introduction

Fisher's equation, or the Fisher–KPP partial differential equation, is given by

$$f_t = \Delta f + cf(1 - f), \tag{1}$$

where f is a real-valued function on an n-dimensional Riemannian manifold M^n, and c is a positive constant. The equation was proposed by R. A. Fisher [1937] to describe the propagation of an evolutionarily advantageous gene in a population, and was also independently described in a seminal paper by A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov [1937] in the same year; for this reason, it is often referred to in the literature as the Fisher–KPP equation. The density of the gene evolves according to diffusion (the term Δf) and reaction (the term $cf(1-f)$). Since the two papers in 1937, the equation has found many applications including in the description of the branching Brownian motion process [McKean 1975], in neuropsychology [Tuckwell 1988], and in describing certain chemical reactions [Ó Náraigh and Kamhawi 2013]. Because a solution f often describes a concentration or density, it is natural to study solutions to the equation for which $0 < f < 1$; our main theorems will simply assume positive solutions.

It is clear that $f = 0$ and $f = 1$ are stationary solutions to this equation on any manifold; it is also known that when $M^n = \mathbb{R}^n$ the equation admits traveling wave solutions, i.e., solutions $f(x, t)$ that we can express as a function of $z = x + \eta t$ for some vector $\eta \in \mathbb{R}^n$. Under a broad range of conditions, general solutions to the equation in $\mathbb{R}^1$ approach a traveling wave solution with a unique minimal speed (see for example, [Kolmogorov et al. 1937, Theorem 17] or [Fisher 1937; Sherratt 1998]).

MSC2010: 58J35, 35K59.
Keywords: differential Harnack, classical Harnack, Fisher's equation, Fisher–KPP, traveling wave.

A bound on the minimum speed of such a traveling wave solution on $\mathbb{R}^1$ was known to Kolmogorov, Petrovskii and Piskunov [1937]; our work results in bounds for the minimum speed of a solution on $\mathbb{R}^n$ for $n = 1, 2, 3$. While our bound in dimension 1 is weaker than the previously known bounds, the bounds in higher dimensions are new and suggest that the study of Harnack inequalities may be used to bound the minimal speed of traveling waves in higher dimensions as well.

Our work introduces and proves three Li–Yau–Hamilton-type Harnack inequalities which constrain positive functions satisfying the Fisher–KPP equation on an arbitrary Riemannian manifold M^n. Depending on the setting we obtain different inequalities. The study of differential Harnack inequalities was first initiated by P. Li and S.-T. Yau [1986] (also see [Aronson and Bénilan 1979]). Harnack inequalities have since played an important role in the study of geometric analysis and geometric flows (for example, see [Hamilton 1993; Perelman 2002]). Applications have also been found to the study of nonlinear parabolic equations, e.g., in [Hamilton 2011]. One of these is a recent reproof of the classical result of H. Fujita [1966], which states that any positive solution to the endangered species equation in dimension n,

$$f_t = \Delta f + f^p,$$

blows up in finite time provided $0 < n(p-1) < 2$; see [Cao et al. 2015].

When the dimension falls into a certain range we can integrate our differential Harnack inequality along any spacetime curve to obtain a classical Harnack inequality which allows us to compare the values of positive solutions at any two points in spacetime when time is large.

The organization for the paper is as follows: In Section 2 we present the precise formulations and the proofs of our two inequalities governing closed manifolds. In Section 3 we state and prove a similar Harnack inequality for complete noncompact manifolds. In Section 4, we end the paper with the aforementioned results on the minimum speed of traveling wave solutions and classical Harnack inequalities.

2. On closed manifolds

In this section, we will deal with the case when the Riemannian manifold M is closed, and we also assume that its Ricci curvature is nonnegative.

In what follows, the time derivative will always be taken to mean the derivative from the left if the two-sided derivative does not exist.

Theorem 1. *Let (M^n, g) be an n-dimensional closed Riemannian manifold with nonnegative Ricci curvature and let $f(x,t) : M \times [0, \infty) \to \mathbb{R}$ be a positive solution of the Fisher–KPP equation $f_t = \Delta f + cf(1-f)$, where f is C^2 in x and C^1 in t, and $c > 0$.*

(A) *Let* $u = \log f$ *and define*

$$\phi_0^<(t) = \frac{\left(\frac{\beta cn}{cn+8\beta(1-\alpha)}\right)e^{-ct} - \beta}{1-e^{-ct}}.$$

Then we have

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0^<(t) \geq 0 \tag{2}$$

for all x *and* t, *provided that*

(i) $0 < \alpha < 1$, (ii) $\beta \leq \dfrac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n} < 0$ and (iii) $\dfrac{8\beta(1-\alpha)}{n} + c < 0$.

(B) *Now set*

$$\phi_0^>(t) = \begin{cases} \dfrac{n}{2(1-\alpha)t} & \textit{if } t \leq T_2, \\ \dfrac{-\beta c(e^{c(t-T_2)}+1)}{c+\dfrac{8\beta(1-\alpha)}{n}+ce^{c(t-T_2)}} & \textit{otherwise}, \end{cases}$$

where

$$T_2 := \frac{n}{2(1-\alpha)(-\beta c)}\left(\frac{4\beta(1-\alpha)}{n}+c\right).$$

If instead of (iii) *we have*

(iv) $\dfrac{8\beta(1-\alpha)}{n} + c \geq 0$,

in addition to (i) *and* (ii), *then*

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0^>(t) \geq 0. \tag{3}$$

In summary, our theorem is that $\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0(t) \geq 0$, where

$$\phi_0(t) = \begin{cases} \dfrac{(\beta cn/(cn+8\beta(1-\alpha)))e^{-ct} - \beta}{1-e^{-ct}} & \text{if (iii) holds,} \\ \dfrac{n}{2(1-\alpha)t} & \text{if (iv) holds and } t \leq T_2, \\ \dfrac{-\beta c(e^{c(t-T_2)}+1)}{c+\dfrac{8\beta(1-\alpha)}{n}+ce^{c(t-T_2)}} & \text{if (iv) holds and } t > T_2. \end{cases}$$

We briefly describe the main idea of our proof here, which uses the parabolic maximum principle and an argument by contradiction. We first define a quantity

$$h(x,t) : M \times (0,\infty) \to \mathbb{R},$$

which will depend on a given solution to Fisher's equation. We start with $h(x,\varepsilon) > 0$ for any sufficiently small $\varepsilon > 0$, and our goal is to prove this quantity $h(x,t)$ remains positive for all points in $M \times \mathbb{R}^+$. As suggested in [Cao 2008; Cao and Hamilton 2009], we then compute what we call the time evolution of h, namely $\partial h/\partial t$, in the following form:

$$\frac{\partial h}{\partial t}(x,t) = \Delta h(x,t) + A_1(x,t)\cdot\nabla h(x,t) + A_2(x,t),$$

for some $A_1 : M \times (0,\infty) \to \mathbb{R}^n$, and $A_2 : M \times (0,\infty) \to \mathbb{R}$. We then assume for the sake of a contradiction that there exists a first (with respect to t) point (x_1,t_1) where $h(x,t) \le 0$; it follows that $(\partial h/\partial t)(x_1,t_1) \le 0$. Since $h(x_1,t_1)$ must be a local minimum in M of the function $h(x,t_1) : M \to \mathbb{R}$, it also follows that $\Delta h(x_1,t_1) \ge 0$, and $\nabla h(x_1,t_1) = (0,\dots,0)$. Thus our time evolution simplifies to

$$\frac{\partial h}{\partial t}(x_1,t_1) \ge A_2(x_1,t_1).$$

By our construction of $h(x,t)$ we will force $A_2(x_1,t_1) > 0$, and so we will have

$$0 \ge \frac{\partial h}{\partial t}(x_1,t_1) \ge A_2(x_1,t_1) > 0,$$

which is a contradiction. Thereby we conclude that $h(x,t) > 0$ for all $(x,t) \in M \times (0,\infty)$.

Technical lemmas. In this section we prove the technical lemmas needed in the case that M is a closed manifold.

Lemma 2 gives us the time evolution of h in terms of 4 quantities P_1, P_2, P_3, P_4 (which sum to A_2 above). Lemma 3 gives a lower bound for P_2 which also applies in the noncompact case. Lemma 4 introduces quantities P_5, $P_{5.1}$, $P_{5.2}$ which depend only on ϕ and which give a lower bound for P_3. Lemma 5 puts a lower bound on P_5. Lemma 6, used for our second Harnack inequality, bounds P_3 when Lemma 5 is inapplicable. Finally, P_1 and P_4 are bounded in the proof of the main theorem.

Lemma 2. *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature bounded from below by* $\mathrm{Ric} \ge -K$. *Let $f(x,t) : M^n \to \mathbb{R}$ be a positive solution to $f_t = \Delta f + cf(1-f)$ which is C^2 in x and C^1 in t. Let $u(x,t) = \log f(x,t)$, and let α, β, c be any constants. Define $h(x,t)$ as follows*:

$$h(x,t) := \Delta u + \alpha|\nabla u|^2 + \beta e^u + \varphi,$$
$$\varphi = \varphi(x,t) = \phi(t) + \psi(x),$$

where $\phi(t)$ is any C^1 function and $\psi(x)$ is any C^2 function. Then the following inequality holds:

$$h_t - \Delta h - 2\nabla u\cdot\nabla h \ge P_1 h + P_2 + P_3 + P_4,$$

where

$$P_1 = \frac{2(1-\alpha)}{n}h - \frac{4(1-\alpha)}{n}(\alpha|\nabla u|^2 + \beta e^u + \phi + \psi) - ce^u,$$

$$P_2 = \frac{2(1-\alpha)}{n}(\alpha^2|\nabla u|^4 + 2\phi\psi) - 2K(1-\alpha)|\nabla u|^2 + \frac{4\alpha(1-\alpha)}{n}\phi|\nabla u|^2 + |\nabla u|^2 e^u\Big(\frac{4\alpha\beta(1-\alpha)}{n} - 2\beta - \alpha c - c\Big),$$

$$P_3 = e^{2u}\frac{2\beta^2(1-\alpha)}{n} + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t,$$

$$P_4 = \frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi + e^u\psi\Big(c + \frac{4\beta(1-\alpha)}{n}\Big) + \frac{2(1-\alpha)}{n}\psi^2 - \Delta\psi.$$

Lemma 2 will be used in the proofs of both Theorem 1 and Theorem 7, with different choices of α, β, c, ϕ and ψ. The statement of Lemma 2 is independent of these choices.

Proof. The proof is based on a straightforward but fairly long calculation. Let $f : M\times[0,\infty)\to\mathbb{R}$ satisfy (1); hence u must satisfy

$$u_t = \Delta u + |\nabla u|^2 + c(1-e^u).$$

We then compute

$$\begin{aligned}
(\partial_t - \Delta)u &= c - ce^u + |\nabla u|^2,\\
(\partial_t - \Delta)(\Delta u) &= \Delta|\nabla u|^2 - c(\Delta u)e^u - c|\nabla u|^2 e^u,\\
(\partial_t - \Delta)(\alpha|\nabla u|^2) &= 2\alpha\nabla u\cdot\nabla(\Delta u) + 2\alpha\nabla u\cdot\nabla|\nabla u|^2 - 2\alpha c|\nabla u|^2 e^u - \alpha\Delta|\nabla u|^2,\\
(\partial_t - \Delta)(\beta e^u) &= \beta ce^u - \beta ce^{2u},\\
(\partial_t - \Delta)\varphi(t) &= \phi_t - \Delta\psi,\\
2\nabla u\cdot\nabla h &= 2\nabla u\cdot\nabla(\Delta u) + 2\alpha\nabla u\cdot\nabla|\nabla u|^2 + 2\beta|\nabla u|^2 e^u + 2\nabla u\cdot\nabla\psi.
\end{aligned}$$

Here we use the Weitzenböck-Bochner formula,

$$\Delta|\nabla u|^2 = 2|\nabla\nabla u|^2 + 2\nabla u\cdot\nabla(\Delta u) + 2\,\mathrm{Ric}(\nabla u, \nabla u),$$

where $\nabla\nabla u$ is the Hessian of $u(x,t)$.

This leads to the equality

$$\begin{aligned}
&(\partial_t - \Delta)h - 2\nabla u\cdot\nabla h\\
&\quad = 2(1-\alpha)|\nabla\nabla u|^2 - ce^u(\Delta u) - |\nabla u|^2 e^u(2\alpha c + 2\beta + c)\\
&\qquad + 2(1-\alpha)\,\mathrm{Ric}(\nabla u, \nabla u) + \beta ce^u - \beta ce^{2u} + \phi_t - \Delta\psi - 2\nabla u\cdot\nabla\psi.
\end{aligned}$$

Using Cauchy–Schwarz $|\nabla\nabla u|^2 \geq (1/n)(\Delta u)^2$ and $\mathrm{Ric} \geq -K$ yields that

$$
\begin{aligned}
(\partial_t - \Delta)h - 2\nabla u \cdot \nabla h \geq 2\frac{(1-\alpha)}{n}(\Delta u)^2 - ce^u(\Delta u) - |\nabla u|^2 e^u(2\alpha c + 2\beta + c)\\
-2(1-\alpha)K|\nabla u|^2 + \beta c e^u - \beta c e^{2u} + \phi_t - \Delta\psi - 2\nabla u \cdot \nabla\psi.
\end{aligned}
$$

Finally, we substitute for Δu:

$$\Delta u = h - \alpha|\nabla u|^2 - \beta e^u - \phi - \psi,$$

to expand and conclude that

$$
\begin{aligned}
&h_t - \Delta h - 2\nabla u \cdot \nabla h\\
&\quad \geq h\Big(\frac{2(1-\alpha)}{n}h - \frac{4(1-\alpha)}{n}(\alpha|\nabla u|^2 + \beta e^u + \phi + \psi) - ce^u\Big)\\
&\qquad + \Big[\frac{2(1-\alpha)}{n}(\alpha^2|\nabla u|^4 + 2\phi\psi) - 2K(1-\alpha)|\nabla u|^2 + \frac{4\alpha(1-\alpha)}{n}\phi|\nabla u|^2\\
&\qquad\qquad + |\nabla u|^2 e^u\Big(\frac{4\alpha\beta(1-\alpha)}{n} - 2\beta - \alpha c - c\Big)\Big]\\
&\qquad + \Big[e^{2u}\Big(\frac{2\beta^2(1-\alpha)}{n}\Big) + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t\Big]\\
&\qquad + \Big[\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi + e^u\psi\Big(c + \frac{4\beta(1-\alpha)}{n}\Big) + \frac{2(1-\alpha)}{n}\psi^2 - \Delta\psi\Big]\\
&\quad = P_1 h + P_2 + P_3 + P_4,
\end{aligned}
$$

as desired. □

We now show that P_2 is nonnegative under the assumptions of Theorem 1.

Lemma 3. *If* $K = 0$ *and assuming that* (i) *and* (ii) *hold, then for any* x, t *where* $\phi(t), \psi(x) \geq 0$ *we have*

$$P_2 \geq 0.$$

Proof. We have assumed that α, $1-\alpha$, ϕ, ψ, $K \geq 0$. Note that

$$\frac{4\alpha\beta(1-\alpha)}{n} - 2\beta - \alpha c - c \geq 0$$

is equivalent to

$$(4\alpha(1-\alpha) - 2n)\beta - cn(\alpha+1) \geq 0,$$

or

$$(-4\alpha(1-\alpha) + 2n)\beta \leq -cn(1+\alpha),$$

which is exactly condition (ii) since $2n \geq 1 \geq 4\alpha(1-\alpha)$. □

Next, we find quantities depending only on ϕ which we will eventually use to guarantee that P_3 is strictly positive.

Lemma 4. *Assume $\alpha < 1$. Define*

$$\mu_1 := \frac{1}{2}c\sqrt{\frac{n}{2(1-\alpha)}}, \qquad \nu_1 = \frac{c+\frac{4\beta(1-\alpha)}{n}}{2\beta}\sqrt{\frac{n}{2(1-\alpha)}},$$

$$\omega_1 = \sqrt{\frac{2(1-\alpha)}{n}},$$

$$P_5(\phi) := -(\mu_1+\nu_1\phi)^2 + (\omega_1\phi)^2 + \phi_t,$$

$$P_{5.1}(\phi) := \Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi + \beta c, \qquad P_{5.2}(\phi) := \frac{2(1-\alpha)}{n}\phi^2 + \phi_t.$$

Then for any (x,t), $P_5 > 0$ implies that $P_3 > 0$. Alternatively, if $P_{5.1} \geq 0$ and $P_{5.2} > 0$, then $P_3 > 0$.

Proof. Recall that

$$P_3(\phi) = e^{2u}\Big(\frac{2\beta^2(1-\alpha)}{n}\Big) + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t.$$

If $P_5 > 0$, then by using $x^2 + 2xy \geq -y^2$, where $x^2 = e^{2u}(2\beta^2(1-a)/n)$, we get

$$\begin{aligned} P_3(\phi) &\geq -\frac{n}{8(1-\alpha)\beta^2}\Big[\beta c + \Big(c + \frac{4(1-\alpha)\beta}{n}\Big)\phi\Big]^2 + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t \\ &= -(\mu_1+\nu_1\phi)^2 + (\omega_1\phi)^2 + \phi_t = P_5(\phi) > 0. \end{aligned}$$

Alternatively, if $P_{5.1} \geq 0$ and $P_{5.2} > 0$, then since $(1-\alpha) > 0$ we can ignore the first term of P_3 and get

$$\begin{aligned} P_3(\phi) &\geq e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t \\ &= e^u P_{5.1} + P_{5.2} > 0. \end{aligned}$$

□

We now find functions $\phi(t)$ such that $P_3(\phi) > 0$. In Lemma 5 we construct $\phi(t)$ in the case that (iii) is true, and in Lemma 6 we construct $\phi(t)$ when (iv) is true.

Lemma 5. *Let μ, ν, ω be any constants such that $\mu \neq 0$, $\nu^2 < \omega^2$ and $\omega > 0$. If for sufficiently small $\varepsilon > 0$ we define*

$$\phi(t) := \frac{\mu\Big(\frac{1}{\nu-(\omega-\varepsilon)}e^{2\mu(\omega-\varepsilon)t} - \frac{1}{\nu+(\omega-\varepsilon)}\Big)}{1-e^{2\mu(\omega-\varepsilon)t}},$$

then

$$-(\mu+\nu\phi)^2 + (\omega\phi)^2 + \phi_t > 0,$$

where $\lim_{t\to 0^+}\phi(t) = \infty$ and $\phi(t) \geq 0$ for all t.

Proof. Choose ε small enough so that $\nu^2 < (\omega-\varepsilon)^2$. We claim that $\phi(t)$ satisfies the following equation:

$$-(\mu+\nu\phi)^2 + [(\omega-\varepsilon)\phi]^2 + \phi_t(t) = 0$$

for all time. This follows from the direct computation below. On the one hand we get that

$$\begin{aligned} -(\mu+\nu\phi)^2 + [(\omega-\varepsilon)\phi]^2 &= \frac{\mu^2(\omega-\varepsilon)^2\Big(\frac{e^{2\mu(\omega-\varepsilon)t}}{\nu-(\omega-\varepsilon)} - \frac{1}{\nu+(\omega-\varepsilon)}\Big)^2}{(1-e^{2\mu(\omega-\varepsilon)t})^2} \\ &\qquad - \left(\mu + \frac{\mu\nu\Big(\frac{e^{2\mu(\omega-\varepsilon)t}}{\nu-(\omega-\varepsilon)} - \frac{1}{\nu+(\omega-\varepsilon)}\Big)}{1-e^{2\mu(\omega-\varepsilon)t}}\right)^2 \\ &= \frac{\mu^2[2(\omega-\varepsilon)(\omega-\varepsilon-\nu)][2(\omega-\varepsilon)(\omega-\varepsilon+\nu)e^{2\mu(\omega-\varepsilon)t}]}{(1-e^{2\mu(\omega-\varepsilon)t})^2(\nu-(\omega-\varepsilon))^2(\nu+(\omega-\varepsilon))^2} \\ &= -\frac{4\mu^2(\omega-\varepsilon)^2e^{2\mu(\omega-\varepsilon)t}}{(\nu+(\omega-\varepsilon))(\nu-(\omega-\varepsilon))(e^{2\mu(\omega-\varphi)t}-1)^2}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \phi_t(t) &= \frac{2\mu^2(\omega-\varepsilon)e^{2\mu(\omega-\varepsilon)t}}{(1-e^{2\mu(\omega-\varepsilon)t})(\nu-(\omega-\varepsilon))} + \frac{2\mu^2(\omega-\varepsilon)e^{2\mu(\omega-\varepsilon)t}\Big(\frac{e^{2\mu(\omega-\varepsilon)t}}{\nu-(\omega-\varepsilon)} - \frac{1}{\nu+(\omega-\varepsilon)}\Big)}{(1-e^{2\mu(\omega-\varepsilon)t})^2} \\ &= \frac{4\mu^2(\omega-\varepsilon)^2e^{2\mu(\omega-\varepsilon)t}}{(\nu+(\omega-\varepsilon))(\nu-(\omega-\varepsilon))(1-e^{2\mu(\omega-\varepsilon)t})^2}. \end{aligned}$$

Therefore it follows that

$$-(\mu+\nu\phi)^2 + [(\omega-\varepsilon)\phi]^2 + \phi_t = 0,$$

and hence

$$-(\mu+\nu\phi)^2 + (\omega\phi)^2 + \phi_t = 2\varepsilon\omega\phi^2 - \varepsilon^2\phi^2 = \phi^2(2\varepsilon\omega - \varepsilon^2).$$

Note that $\nu-(\omega-\varepsilon)$ and $\nu+(\omega-\varepsilon)$ must have different signs since their product is $\nu^2-(\omega-\varepsilon)^2 < 0$; hence $\phi(t) \neq 0$ for all time. It then follows that for sufficiently small ε,

$$-(\mu+\nu\phi)^2 + (\omega\phi)^2 + \phi_t = \phi^2(2\varepsilon\omega - \varepsilon^2) > 0.$$

To show that $\lim_{t\to 0^+}\phi(t) = \infty$, we split $\phi(t)$ into two parts. First, note that

$$\begin{aligned} \lim_{t\to 0^+}\Big(\frac{1}{\nu-(\omega-\varepsilon)}e^{2\mu(\omega-\varepsilon)t} - \frac{1}{\nu+(\omega-\varepsilon)}\Big) &= \frac{1}{\nu-(\omega-\varepsilon)} - \frac{1}{\nu+(\omega-\varepsilon)} \\ &= \frac{2(\omega-\varepsilon)}{\nu^2-(\omega-\varepsilon)^2} < 0. \end{aligned}$$

Further, it is clear that

$$\lim_{t\to 0^+} \frac{\mu}{1-e^{2\mu(\omega-\varepsilon)t}} = -\infty.$$

Combining these two calculations lets us conclude that

$$\lim_{t\to 0^+} \phi(t) = \infty.$$

Finally, since $\phi(t)$ is continuous and starts out positive and $\phi(t) \neq 0$ for any $t > 0$, it follows that $\phi(t) > 0$ for all $t > 0$. □

Remark. We can also compute $\lim\limits_{t\to\infty} \phi(t)$.

If $\mu > 0$ then $e^{2\mu(\omega-\varepsilon)t} \to \infty$ as $t \to \infty$; hence we find that

$$\lim_{t\to\infty} \phi(t) = \frac{\dfrac{\mu}{\nu-(\omega-\varepsilon)}}{-1} = \frac{\mu}{-\nu+(\omega-\varepsilon)}.$$

If $\mu < 0$ then $e^{2\mu(\omega-\varepsilon)t} \to 0$ as $t \to \infty$, which gives us

$$\lim_{t\to\infty} \phi(t) = \frac{-\mu}{\nu+(\omega-\varepsilon)}.$$

Next we deal with the other case.

Lemma 6. *Let μ_1, ν_1, ω_1 be defined as in Lemma 4, and suppose* (iv) *is true* (*i.e.*, (iii) *becomes false*). *Let*

$$T_2 = T_2(\varepsilon) := \frac{n}{2(1-\alpha)(1-\varepsilon)(-\beta c)} \cdot \Bigl(\frac{4\beta(1-\alpha)}{n} + c\Bigr).$$

If for some sufficiently small $\varepsilon > 0$ we define

$$\phi(t) := \begin{cases} \dfrac{n}{2(1-\alpha)(1-\varepsilon)t} & \text{if } t \le T_2, \\ \dfrac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}} & \text{if } t > T_2, \end{cases}$$

then for $t \le T_2$ we get $P_{5.1} \ge 0$ and $P_{5.2} > 0$, and for $t > T_2$ we get $P_5 > 0$. Therefore $P_3(\phi) > 0$ for all t.

In addition, $\lim_{t\to 0^+} \phi(t) = \infty$ and $\phi(t) > 0$ for all t.

Proof. For $\varepsilon < 1$, we have

$$\lim_{t\to 0^+} \phi(t) = \lim_{t\to 0^+} \frac{n}{2(1-\alpha)(1-\varepsilon)t} = \infty.$$

To show that $\phi(t)$ is continuous at T_2, we check its limits from the left and right. The limit from the left is

$$\lim_{t\to T_2^-} \phi(t) = \frac{n}{2(1-\alpha)(1-\varepsilon)T_2} = \frac{-\beta cn}{4\beta(1-\alpha)+cn}.$$

And the limit from the right is

$$\lim_{t\to T_2^+}\phi(t)=\frac{-\mu_1(1+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))}=\frac{-2\mu_1}{2\nu_1}=-\frac{c}{2}\cdot\frac{2\beta n}{(cn+4\beta(1-\alpha))}$$
$$=\frac{-\beta cn}{4\beta(1-\alpha)+cn}.$$

Therefore $\phi(t)$ is continuous.

Next we check that $\phi(t)>0$ for all $t>0$. Note that $\phi(t)$ is continuous, and clearly is positive between 0 and T_2. For $t\geq T_2$, since $\mu_1\neq 0$, it follows that

$$-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)\neq 0,$$

and therefore $\phi(t)\neq 0$ for any $t\geq T_2$. By continuity, it follows that $\phi(t)>0$ for all $t>0$.

Next we show that for $t\leq T_2$ we have $P_{5.1}\geq 0$. That is, we need

$$P_{5.1}=\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi(t)+\beta c\geq 0.$$

First we note that condition (iv) states that $4\beta(1-\alpha)/n+c\geq 0$. Since $\phi(t)$ is decreasing in $t<T_2$, it suffices to check that $P_{5.1}\geq 0$ holds for $t=T_2$:

$$\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi(t)+\beta c\geq\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi(T_2)+\beta c$$
$$=\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\left(\frac{-\beta c}{\frac{4\beta(1-\alpha)}{n}+c}\right)+\beta c=0.$$

Therefore $P_{5.1}\geq 0$ for all $t\leq T_2$.

Now we show that $P_{5.2}>0$ for all $t\leq T_2$. That is, we need

$$P_{5.2}=\frac{2(1-\alpha)}{n}\phi(t)^2+\phi_t(t)>0.$$

We have

$$P_{5.2}=\frac{2(1-\alpha)}{n}\Big[\frac{n}{2(1-\alpha)(1-\varepsilon)t}\Big]^2+\frac{-n}{2(1-\alpha)(1-\varepsilon)t^2}$$
$$=\frac{n}{2(1-\alpha)(1-\varepsilon)^2t^2}-\frac{n}{2(1-\alpha)(1-\varepsilon)t^2}=\frac{\varepsilon n}{2(1-\alpha)(1-\varepsilon)^2t^2}>0.$$

This implies that $P_3(\phi)>0$ for $t\leq T_2$. Next we show that $P_5>0$ for all $t>T_2$. That is, we need that

$$P_5=-(\mu_1+\nu_1\phi)^2+(\omega_1\phi)^2+\phi_t>0$$

for

$$\phi(t)=\frac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}.$$

We first show that for $t > T_2$, $\phi(t)$ satisfies

$$-(\mu_1+\nu_1\phi)^2+[(\omega_1-\varepsilon)\phi]^2+\phi_t=0.$$

Plugging in $\phi(t)$ for $t > T_2$ gives us that

$$\begin{aligned}
&-(\mu_1+\nu_1\phi)^2+[(\omega_1-\varepsilon)\phi]^2\\
&\qquad=-\Big[\mu_1-\frac{\mu_1\nu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}\Big]^2\\
&\qquad\qquad+\Big[(\omega_1-\varepsilon)\frac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}\Big]^2\\
&\qquad=\frac{\mu_1^2(\omega_1-\varepsilon)^2[-(1-e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)})^2+(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)^2]}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}\\
&\qquad=\frac{4\mu_1^2(\omega_1-\varepsilon)^2e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\phi_t(t)=&\frac{-2\mu_1^2(\omega_1-\varepsilon)e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}\\
&-\frac{(\nu_1-(\omega_1-\varepsilon))(2\mu_1(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}[-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)]}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}\\
=&-\frac{4\mu_1^2(\omega_1-\varepsilon)^2e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}.
\end{aligned}$$

Therefore

$$-(\mu_1+\nu_1\phi)^2+[(\omega_1-\varepsilon)\phi]^2+\phi_t=0,$$

and it follows that

$$P_5=-(\mu_1+\nu_1\phi)^2+(\omega_1\phi)^2+\phi_t=(2\varepsilon\omega_1-\varepsilon^2)\phi^2>0$$

for small enough ε. Therefore $P_3(\phi)>0$ for $t>T_2$. □

Remark. Here we observe that

$$\begin{aligned}
\lim_{t\to\infty}\phi(t)&=\lim_{t\to\infty}\frac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}\\
&=\frac{-\mu_1}{\nu_1-(\omega_1-\varepsilon)}=\frac{\mu_1}{-\nu_1+(\omega_1-\varepsilon)},
\end{aligned}$$

which is the same limit as $\phi(t)$ from Lemma 5 since $\mu_1>0$.

Now we are ready to finish the proof of Theorem 1.

Proof of Theorem 1. Let $f : M \times [0, \infty) \to \mathbb{R}$ be a positive solution of $f_t = \Delta f + cf(1-f)$ for $c > 0$, and assume that the following hold:

(i) $0 < \alpha < 1$,

(ii) $\beta \leq \dfrac{-cn(1+\alpha)}{4\alpha^2 - 4\alpha + 2n} < 0$.

Let $u = \log f$, and define

$$h(x,t) := \Delta u + \alpha |\nabla u|^2 + \beta e^u + \varphi,$$

where

$$\varphi = \varphi(x,t) = \phi(t) + \psi(x),$$

and since we are in the closed case we set $\psi(x) = 0$.

With μ_1, ν_1, ω_1, and T_2 as defined in Lemma 4 and Lemma 6, and $\varepsilon > 0$ small enough to satisfy Lemmas 5 and 6, we let

$$\phi(t) = \begin{cases} \dfrac{\mu_1\Big(\dfrac{1}{\nu_1 - (\omega_1 - \varepsilon)} e^{2\mu_1(\omega_1-\varepsilon)t} - \dfrac{1}{\nu_1 + (\omega_1 - \varepsilon)}\Big)}{1 - e^{2\mu_1(\omega_1 - \varepsilon)t}} & \text{if (iii)}, \\ \dfrac{n}{2(1-\alpha)(1-\varepsilon)t} & \text{if (iv) and } t \leq T_2, \\ \dfrac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)} + 1)}{(\nu_1 + (\omega_1 - \varepsilon)) + (\nu_1 - (\omega_1 - \varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}} & \text{if (iv) and } t > T_2. \end{cases}$$

We first show that $h(x,t) > 0$ for all t. Suppose for the sake of a contradiction that $h \leq 0$ somewhere; let t_1 be the first time such that $\min_x h(x,t) = 0$. Since M is closed the minimum is attained, say at the point (x_1, t_1). By Lemmas 5 and 6, $\lim_{t \to 0^+} \phi(t) = \infty$ so it follows that t_1 exists.

By applying Lemma 2, we get that

$$h_t - \Delta h - 2\nabla u \cdot \nabla h \geq P_1 h + P_2 + P_3 + P_4, \tag{4}$$

where $P_1, \ldots, P_4$ are defined as in Lemma 2. Note that in the case (iv), the derivative ϕ_t at $t = T_2$ is considered to be the derivative from the left.

We have $P_1 h = 0$ since $h(x_1, t_1) = 0$. Lemma 3 yields that $P_2 \geq 0$ since $K = 0$, and $P_4 = 0$ since $\psi(x) \equiv 0$.

Since (x_1, t_1) is the first spacetime where $h(x,t) = 0$, the maximum principle yields that $h_t(x_1, t_1) \leq 0$ (where this is a derivative as $t \to t_1^-$), $\Delta h(x_1, t_1) \geq 0$ and $\nabla h(x_1, t_1) = 0$.

Hence (4) yields that

$$0 \geq h_t - \Delta h - 2\nabla u \cdot \nabla h \geq P_1 h + P_2 + P_3 + P_4 \geq P_3. \tag{5}$$

Now we split into cases based on whether (iii) or (iv) holds.

If (iii) is true, since $c > 0$ we have the following inequalities:

$$\frac{4\beta(1-\alpha)}{n} < c + \frac{4\beta(1-\alpha)}{n} < -\frac{4\beta(1-\alpha)}{n},$$
$$\left|c + \frac{4\beta(1-\alpha)}{n}\right| < \left|\frac{4\beta(1-\alpha)}{n}\right|,$$
$$\left(\frac{c+4\beta(1-\alpha)/n}{2\beta}\right)^2 < \left(\frac{2(1-\alpha)}{n}\right)^2,$$
$$\nu_1^2 = \left(\frac{c+4\beta(1-\alpha)/n}{2\beta}\right)^2 \frac{n}{2(1-\alpha)} < \omega_1^2 = \frac{2(1-\alpha)}{n}.$$

Therefore by Lemmas 4 and 5 it follows that $P_3 > 0$, which contradicts (5).

Otherwise, if (iv) is true, it follows from Lemmas 4 and 6 that $P_3 > 0$ again, which still contradicts (5).

This proves that $h(x, t) > 0$ for all x, t. Finally, letting $\varepsilon \to 0$ with

$$T_2\big|_{\varepsilon=0} = \frac{n}{2(1-\alpha)(-\beta c)}\left(\frac{4\beta(1-\alpha)}{n} + c\right),$$

we get that $\phi(t) \to \phi_0(t)$, where

$$\phi_0(t) = \begin{cases} \dfrac{\left(\dfrac{\beta cn}{cn+8\beta(1-\alpha)}\right)e^{-ct} - \beta}{1-e^{-ct}} & \text{if (iii) holds,} \\ \dfrac{n}{2(1-\alpha)t} & \text{if (iv) holds and } t \le T_2\big|_{\varepsilon=0}, \\ \dfrac{-\beta c(e^{c(t-T_2)}+1)}{c+\dfrac{8\beta(1-\alpha)}{n}+ce^{c(t-T_2)}} & \text{if (iv) holds and } t > T_2\big|_{\varepsilon=0}. \end{cases}$$

Therefore $\lim_{\varepsilon\to 0} h(x, t) = \Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0(t) \ge 0$ as desired. □

3. On complete noncompact manifolds

In this section, we study the case in which the manifold is complete but noncompact. The idea is similar to the case when the manifold is compact without boundary. The main technical difficulty here is to ensure that the minimum of the Harnack quantity is attained in a compact region. We first state our main theorem of this section.

Theorem 7. *Let (M^n, g) be an n-dimensional complete (noncompact) Riemannian manifold with nonnegative Ricci curvature. Let $f(x, t) : M \times [0, \infty) \to \mathbb{R}$ be a positive solution of the Fisher–KPP equation $f_t = \Delta f + cf(1 - f)$, where f is C^2 in x and C^1 in t, and $c > 0$ is a constant. Let $u = \log f$. Then we have*

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_1(t) \ge 0, \tag{6}$$

provided the following constraints are satisfied:

(i) $0 < \alpha < 1$,

(ii) $\beta < \dfrac{-cn(1+\alpha)}{2(2\alpha^2-2\alpha+n)} < 0$,

(iii) $\dfrac{-cn(2+\sqrt{2})}{4(1-\alpha)} < \beta < \dfrac{-cn(2-\sqrt{2})}{4(1-\alpha)}$,

where

$$\phi_1(t) = \frac{\mu_2\Big(\dfrac{1}{\nu_2-\omega_2}e^{2\mu_2\omega_2 t} - \dfrac{1}{\mu_2+\omega_2}\Big)}{1-e^{2\mu_2\omega_2 t}},$$

with

$$\mu_2 = \beta c\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}},$$

$$\nu_2 = \Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\cdot\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}}, \qquad \omega_2 = \sqrt{\frac{2(1-\alpha)}{n}}.$$

Technical lemmas. In this subsection, we state and prove some additional lemmas which will be needed in the proof of Theorem 7. Lemma 8 allows us to substitute the sum P_6+P_7 for P_3+P_4; then Lemma 9 bounds P_6 using a new quantity P_8. Lemma 10 allows us to apply Lemma 5 to control P_8. Lemma 11 gives sufficient conditions for bounding P_7. After bounding P_1, we are in a position to prove our theorem.

For any given $\varepsilon' > 0$, let

$$A = A(\varepsilon') := \frac{2\beta^2(1-\alpha)}{n} - \frac{n\Big(c+\dfrac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}.$$

Lemma 8. *Let P_3 and P_4 be as defined in Lemma 2. Define*

$$P_6 := Ae^{2u} + e^u\Big(\frac{4\beta(1-\alpha)\phi}{n} + c\beta + c\phi\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t,$$

$$P_7 := \frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi + \frac{2\varepsilon'}{n}\psi^2 - \Delta\psi.$$

For any $\varepsilon' > 0$ and any (x,t) we have

$$P_3 + P_4 \geq P_6 + P_7.$$

Proof of Lemma 8. Recall that,

$$\begin{aligned} P_3 + P_4 = {}& \frac{2\beta^2(1-\alpha)}{n}e^{2u} + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t \\ & + \frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi - \Delta\psi + e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big) + \frac{2(1-\alpha)}{n}\psi^2. \end{aligned}$$

We write the last two terms as

$$e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha)}{n}\psi^2=e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha-\varepsilon')}{n}\psi^2+\frac{2\varepsilon'}{n}\psi^2.$$

Using $2xy+x^2\geq -y^2$ in the form

$$e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha-\varepsilon')}{n}\psi^2\geq -\frac{n\Big(c+\dfrac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}e^{2u},$$

gives us

$$e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha)}{n}\psi^2\geq \frac{2\varepsilon'}{n}\psi^2-\frac{n\Big(c+\dfrac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}e^{2u}.$$

Applying this inequality then gives

$$\begin{aligned}P_3+P_4&\geq e^{2u}\left(\frac{2\beta^2(1-\alpha)}{n}-\frac{n\Big(c+\dfrac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}\right)+e^u\Big(\frac{4\beta(1-\alpha)\phi}{n}+c\beta+c\phi\Big)\\&\quad+\frac{2(1-\alpha)}{n}\phi^2+\phi_t+\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2-2\nabla u\cdot\nabla\psi+\frac{2\varepsilon'}{n}\psi^2-\Delta\psi,\\&=P_6+P_7,\end{aligned}$$

which finishes the proof. □

Lemma 9. *For* $\mu_1=\beta c/(2\sqrt{A})$, $\nu_1=(4\beta(1-\alpha)/n+c)/(2\sqrt{A})$, *and* $\omega_1=\sqrt{2(1-\alpha)/n}$, *define*

$$P_8(\phi):=-(\mu_1+\nu_1\phi)^2+(\omega_1\phi)^2+\phi_t.$$

If $A>0$, *then* $P_8>0$ *implies* $P_6>0$ *for any* (x,t).

Proof of Lemma 9. Recall that

$$P_6=Ae^{2u}+\Big[\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi+\beta c\Big]e^u+\frac{2(1-\alpha)}{n}\phi^2+\phi_t.$$

Since $A>0$, we use the fact that $x^2+xy\geq -\frac{1}{4}y^2$ in the form

$$Ae^{2u}+\Big[\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi+\beta c\Big]e^u\geq -\frac{\Big[\Big(\dfrac{4\beta(1-\alpha)}{n}+c\Big)\phi+\beta c\Big]^2}{4A}.$$

This gives

$$
\begin{aligned}
P_6 &\geq -\frac{\left[\left(\frac{4\beta(1-\alpha)}{n}+c\right)\phi+\beta c\right]^2}{4A}+\frac{2(1-\alpha)}{n}\phi^2+\phi_t \\
&= -\left[\frac{\beta c}{2\sqrt{A}}+\frac{1}{2\sqrt{A}}\left(\frac{4\beta(1-\alpha)}{n}+c\right)\phi\right]^2+\left(\phi\sqrt{\frac{2(1-\alpha)}{n}}\right)^2+\phi_t
\end{aligned}
$$

as desired. □

Lemma 10. *If condition* (iii) *of Theorem 7 holds, then there always exists some* $\varepsilon' > 0$ *such that* $A > 0$ *and* $\nu_1^2 < \omega_1^2$.

Proof of Lemma 10. We first want to show that $A(\varepsilon') > 0$ for some $\varepsilon' > 0$. We will show that $A(0) > 0$, and since A is a continuous function of ε', this implies that $A(\varepsilon') > 0$ for some $\varepsilon' > 0$.

We have

$$
\begin{aligned}
A(0) &= \frac{2\beta^2(1-\alpha)}{n} - \frac{n\left(c+\frac{4\beta(1-\alpha)}{n}\right)^2}{8(1-\alpha-0)} \\
&= \frac{16\beta^2(1-\alpha)^2-(cn+4\beta(1-\alpha))^2}{8n(1-\alpha)} \\
&= \frac{-c^2n^2-8\beta cn(1-\alpha)}{8n(1-\alpha)}.
\end{aligned}
$$

It follows from (iii) that

$$-8 < -4-2\sqrt{2} < \frac{cn}{\beta(1-\alpha)},$$

which rearranges to give $c^2n^2+8\beta cn(1-\alpha) < 0$. Thus $A(0) > 0$, and so there exists some $\varepsilon' > 0$ such that $A(\varepsilon') > 0$.

Next we want to show that $\nu_1^2 < \omega_1^2$ for some $\varepsilon' > 0$, where

$$\nu_1 = \frac{4\beta(1-\alpha)/n+c}{2\sqrt{A}} \quad \text{and} \quad \omega_1 = \sqrt{\frac{2(1-\alpha)}{n}}.$$

Since ν_1 and ω_1 are continuous functions of ε', if we can show that $\nu_1^2 < \omega_1^2$ for $\varepsilon' = 0$, then it must be that $\nu_1^2 < \omega_1^2$ for some $\varepsilon' > 0$.

When $\varepsilon' = 0$, $\nu_1^2 < \omega_1^2$ is equivalent to

$$\frac{c^2n^2+8\beta cn(1-\alpha)+16(1-\alpha)^2\beta^2}{-c^2n^2-8\beta cn(1-\alpha)} < 1.$$

Restriction (iii) implies

$$-4-2\sqrt{2} < \frac{cn}{\beta(1-\alpha)} < -4+2\sqrt{2},$$

which leads to

$$\frac{c^2n^2}{\beta^2(1-\alpha)^2}+\frac{8cn}{\beta(1-\alpha)}+8<0.$$

This is equivalent to

$$c^2n^2+8\beta cn(1-\alpha)+16(1-\alpha)^2\beta^2<-(c^2n^2+8\beta cn(1-\alpha)),$$

and therefore $\nu_1^2<\omega_1^2$ for $\varepsilon'=0$. □

Lemma 11. *Suppose $R\geq 1$ is a constant and $\rho:M^n\to\mathbb{R}$ is a function that satisfies*

$$\rho(x)\geq 0,\qquad |\nabla\rho(x)|\leq 1,\qquad \Delta\rho\leq\frac{c_1}{\rho},$$

for some constant $c_1>0$. Define

$$\psi(x):=k\frac{R^2+\rho^2}{(R^2-\rho^2)^2}. \tag{7}$$

Then for k sufficiently large, $\psi(x)$ satisfies $P_7>0$.

Proof of Lemma 11. Let

$$\Psi(x):=\frac{R^2+\rho^2}{(R^2-\rho^2)^2},$$

so that $\psi=k\Psi$. We claim that Ψ satisfies

$$|\nabla\Psi|^2\leq 18\Psi^3\quad\text{and}\quad\Delta\Psi\leq c_2\Psi^2, \tag{8}$$

where c_2 depends only on c_1.

Indeed, we can compute

$$\nabla\Psi=\nabla\rho\Big(\frac{6\rho R^2+2\rho^3}{(R^2-\rho^2)^3}\Big),$$
$$|\nabla\Psi|^2\leq 4\rho^2\frac{(3R^2+\rho^2)^2}{(R^2-\rho^2)^6}\leq 18\Psi^3,$$

and

$$\begin{aligned}\Delta\Psi&=\Delta\rho\Big(\frac{6\rho R^2+2\rho^3}{(R^2-\rho^2)^3}\Big)+|\nabla\rho|^2\Big(\frac{6R^4+36\rho^2R^2+6\rho^4}{(R^2-\rho^2)^4}\Big)\\&\leq 6c_1\frac{R^2+\rho^2}{(R^2-\rho^2)^3}+18\frac{(R^2+\rho^2)^2}{(R^2-\rho^2)^4}\\&\leq(6c_1+18)\Psi^2.\end{aligned}$$

Recall that

$$P_7=\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2-2\nabla u\cdot\nabla\psi+\frac{2\varepsilon'}{n}\psi^2-\Delta\psi.$$

Completing the square gives us

$$P_7 \geq \frac{2\varepsilon'}{n}\psi^2 - \Delta\psi - \frac{n}{4\alpha(1-\alpha)\psi}|\nabla\psi|^2.$$

By (8), we know that

$$\frac{\varepsilon'}{n}k^2\Psi^2 \geq \frac{\varepsilon' k}{c_2 n}k\Delta\Psi \quad \text{and} \quad \frac{\varepsilon'}{n}k^2\Psi^2 \geq \frac{\varepsilon' k}{18n}\cdot\frac{k^2|\nabla\Psi|^2}{k\Psi},$$

so if

$$k > \max\Bigl(\frac{c_2 n}{\varepsilon'}, \frac{18n^2}{4\alpha(1-\alpha)\varepsilon'}\Bigr),$$

we immediately obtain $P_7 > 0$. □

Proof of Theorem 7. Fix a point $p \in M$, let $r = r(x) := d(x, p)$, where $d(\cdot,\cdot)$ denotes the geodesic distance in M. We define the Harnack quantity h on the geodesic ball $B_R(p) := \{x \in M \mid d(x, p) < R\}$. The quantity h depends on the positive constants ε, ε', k, R and is defined as follows:

$$\begin{aligned} h(x,t) &= \Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi(t) + \psi(x), \\ \phi = \phi(t) &:= \frac{\mu_2\Bigl(\frac{1}{\nu_2-(\omega_2-\varepsilon)}e^{2\mu_2(\omega_2-\varepsilon)t} - \frac{1}{\nu_2+(\omega_2-\varepsilon)}\Bigr)}{1-e^{2\mu_2(\omega_2-\varepsilon)t}}, \\ \psi = \psi(x) &:= k\frac{R^2+r^2}{(R^2-r^2)^2}, \end{aligned}$$

with μ_2, ν_2, ω_2, and A defined as in Lemma 9 and the paragraph following Theorem 7. Fix $R > 1$. Let ε, ε' and k be positive constants to be chosen later. Note that h is C^1 in t and C^2 in x, except possibly for those x in the cut locus $\mathcal{C}(p)$. We will show that we can choose ε, ε', and k so that $h(x,t) > 0$ for all x, t. Assume for the sake of a contradiction that $h(x,t) \leq 0$ for some x, t.

Let t_1 be the first time t such that $\inf_{x\in B_R(p)} h(x,t) = 0$. Since $\lim_{t\to 0^+} h(t) = \infty$ by Lemma 5, it follows that t_1 exists. Note also that $\psi(x) \to \infty$ as $r = d(x,p)$ approaches R, so the infimum of h is attained inside $B_R(p)$; let (x_1, t_1) be such a point, so that $h(x_1,t_1) = 0$. Now we split into cases based on whether or not x_1 is in the cut locus $\mathcal{C}(p)$.

Case 1: Suppose that $x_1 \notin \mathcal{C}(p)$, so that $\psi(x)$ is twice differentiable at x_1. Then by Lemmas 2 and 3 and 8 we have

$$0 > h_t - \Delta h - 2\nabla h\cdot\nabla u - P_1 h \geq P_2 + P_3 + P_4 \geq P_6 + P_7.$$

By Lemma 10, we can choose $\varepsilon' > 0$ small enough such that $A > 0$ and $\nu^2 < \omega^2$; then, since ϕ is the same as the one defined as in Lemma 5, it follows by Lemmas 5 and 9 that we can choose ε small enough so that $P_6 > 0$.

Note that ψ takes the form of (7), with the distance function $\rho(x)=r(x)=d(x,p)$. We have $r \geq 0$ and $|\nabla r|^2 = 1$; furthermore, by the Laplacian comparison theorem we have $\Delta r \leq (n-1)/r$. Thus we can apply Lemma 11 and choose k sufficiently large such that $P_7 > 0$ as well, which leads to a contradiction.

Case 2: Suppose that $x_1 \in \mathcal{C}(p)$. We apply Calabi's trick. Let $\delta \in (0, d(x_1, p)/2)$ be a positive constant, and let $\gamma(t)$ be any length-minimizing geodesic from p to x_1. Define $p_\delta := \gamma(\delta)$, so that $x_1 \notin \mathcal{C}(p_\delta)$, and define

$$r_\delta(x) := d(x, p_\delta) + \delta, \qquad \psi_\delta(x) := k\frac{R^2 + r_\delta^2}{(R^2 - r_\delta)^2},$$
$$h_\delta := \Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi + \psi_\delta.$$

Note that by the triangle inequality, $r_\delta(x) = d(x, p_\delta) + d(p_\delta, p) \geq r(x)$, with equality at $x = x_1$. Since ψ is an increasing function of r, it follows that $\psi_\delta(x) \geq \psi(x)$ with equality at $x = x_1$. This implies that (x_1, t_1) is still the first time and place where $h_\delta(x,t) = 0$. Furthermore, h_δ is now C^2 at (x_1, t_1) so applying Lemmas 2, 3, 8, 5, and 9 gives that $0 > P_7$.

Note that clearly $r_\delta \geq 0$ and $|\nabla r_\delta| \leq 1$, and at x_1 we get

$$\Delta r_\delta = \Delta(d(x_1, p_\delta)) \leq \frac{n-1}{d(x_1, p_\delta)} = \frac{n-1}{r(x_1)-\delta} \leq \frac{2(n-1)}{r(x_1)},$$

since we assumed that $\delta \leq \frac{1}{2}r(x_1)$. Therefore applying Lemma 11 gets us a contradiction in this case as well.

This shows that $h(x,t) > 0$ for all x, t. Since h varies continuously as a function of $R, \varepsilon, \varepsilon'$, we can take the limit $R \to \infty$ to get $\psi \to 0$. Then by taking $\varepsilon, \varepsilon' \to 0$, we get that $\phi \to \phi_1$ and so

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \frac{\mu_2\Big(\dfrac{1}{\nu_2-\omega_2}e^{2\mu_2\omega_2 t} - \dfrac{1}{\mu_2+\omega_2}\Big)}{1 - e^{2\mu_2\omega_2 t}} \geq 0,$$

with

$$\mu_2 = \beta c\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}},$$
$$\nu_2 = \Big(\frac{4\beta(1-\alpha)}{n} + c\Big)\cdot\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}}, \qquad \omega_2 = \sqrt{\frac{2(1-\alpha)}{n}},$$

which finishes the proof. $\square$

4. Applications

In this section, we derive two applications of our differential Harnack estimates.

Bounds on the wave speed of traveling wave solutions. The first such application shows that our Harnack inequality can be used to prove an interesting fact about traveling wave solutions to Fisher's equation. In particular we look at traveling plane waves, i.e., solutions to (1) of the form

$$f(x,t) = v(z) := v(x + \eta t\hat{a}),$$

for some function $v : \mathbb{R}^n \to \mathbb{R}$ and some wave direction $\hat{a} \in \mathbb{R}^n$, $|\hat{a}| = 1$ and wave speed $\eta > 0$. For $n = 1$, these solutions were first studied by Fisher [1937] (also see [Kolmogorov et al. 1937; Sherratt 1998]) and were considered by him to be a natural model for propagation of mutations. He was able to show that if $n = 1$ and $\lim_{t\to-\infty} f(x,t) = 0$, then it must be that $\eta \geq 2\sqrt{c}$.

We will show a weaker bound that generalizes to higher dimensions.

Theorem 12. *Let $f(x,t) = v(x + \eta t\hat{a})$ be a traveling plane wave solution to* (1), *with wave speed η and wave direction $\hat{a}$. Suppose that*

$$\lim_{\substack{x=k\hat{b},\\ k\to\infty}} v(x) = 0 \quad \textit{for some direction } \hat{b} \in \mathbb{R}^n,\ |\hat{b}| \neq 0. \tag{9}$$

Then

$$\eta \geq \begin{cases} \sqrt{(3-\sqrt{3})c} & \textit{if } n = 1, \\ \sqrt{2c} & \textit{if } n = 2, \\ \sqrt{(7-3\sqrt{3})c} & \textit{if } n = 3. \end{cases}$$

When $n = 1$, $\eta \geq 2\sqrt{c}$ is both a necessary and sufficient condition for the existence of traveling wave solutions. The same condition is sufficient in any higher dimension, but it is not known (at least to us) if it is necessary as well. Our bounds above give a weaker necessary wave speed in dimensions two and three.

Remark. In the proof below we have not used the fact that the traveling wave v approaches 1 in some direction. Although we were ourselves unsuccessful, the authors would like to encourage an attempt to use this additional restriction to obtain a better bound on the wave speed η.

Lemma 13. *For any $v(z)$ and any η that satisfy the conditions of Theorem 12, and for any α, β that satisfy* (i), (ii), *and* (iii) *as in Theorem 7, we have*

$$\eta^2 \geq M' := 4(1-\alpha)[(c - \phi(t)) - (\beta + c)v(z)],$$

for all x, t, where

$$\phi(t) = \frac{\mu\left(\frac{1}{\nu-\omega}e^{2\mu\omega t} - \frac{1}{\nu+\omega}\right)}{(1 - e^{2\mu\omega t})}$$

(*which appears as $\phi_1(t)$ in the statement of Theorem 7*).

Proof. Since Fisher's equation is spherically symmetric, we may assume without loss of generality that $\hat{a} = \hat{x}_1 = (1, 0, 0, \ldots, 0)$. Therefore

$$f(x, t) = v(x_1 + \eta t, x_2, \ldots, x_n) = v(z_1, z_2, \ldots, z_n) = v(\hat{z}).$$

It then follows from (1) that (where $\partial_i := \partial/\partial z_i$)

$$\eta \partial_1 v = \Delta v + cv(1 - v).$$

Combining this with Theorem 7 gives

$$\begin{aligned}\Delta(\log v) + \alpha|\nabla(\log v)|^2 + \beta v + \phi \geq 0,\\ \frac{\Delta v}{v} - (1-\alpha)\frac{|\nabla v|^2}{v^2} + \beta v + \phi \geq 0,\\ \frac{\eta\partial_1 v - cv(1-v)}{v} - (1-\alpha)\frac{|\nabla v|^2}{v^2} + \beta v + \phi \geq 0,\\ (1-\alpha)\frac{\sum_{i=2}^n (\partial_i v)^2}{v^2} + (1-\alpha)\frac{(\partial_1 v)^2}{v^2} - \eta\frac{\partial_1 v}{v} - (\beta + c)v + (c - \phi) \leq 0.\end{aligned}$$

It follows from standard Cauchy–Schwarz that

$$-\frac{\eta^2}{4(1-\alpha)} - (\beta + c)v + (c - \phi) \leq 0,$$

hence $\eta^2 \geq 4(1-\alpha)[(c - \phi) - (\beta + c)v]$, as desired. □

Lemma 14. *Assume that $v(x) \to 0$ along some path, as in* (9). *Then for any $\varepsilon_3 > 0$ there exists (x_3, t_3), possibly depending on n, α, β, and c, such that at (x_3, t_3)*

$$M' > M'' - \tfrac{1}{3}\varepsilon_3,$$

where

$$M'' := 4(1-\alpha)\Big(c - \frac{-\mu}{\nu+\omega}\Big).$$

Proof. Fix $\varepsilon_3 > 0$. Note that

$$\lim_{t\to\infty} \phi(t) = \frac{-\mu}{\nu+\omega}.$$

Choosing $t \geq t_3$ large enough gives

$$\Big|\phi(t_3) - \frac{-\mu}{\nu+\omega}\Big| < \frac{\varepsilon_3}{24(1-\alpha)},$$

so that

$$4(1-\alpha)(c - \phi) > 4(1-\alpha)\Big(c - \frac{-\mu}{\nu+\omega}\Big) - \frac{\varepsilon_3}{6}.$$

Having fixed t_3, we then set $x_3 := -\eta t_3 \hat{a} + \lambda \hat{b}$ with λ sufficiently large. Then by (9) it follows that

$$|v - 0| < \frac{\varepsilon_3}{24(1-\alpha)} \frac{1}{|\beta + c|} \quad \text{and} \quad -4(1-\alpha)(\beta + c)v > 0 - \tfrac{1}{6}\varepsilon_3.$$

Therefore

$$M' = 4(1-\alpha)[(c - \phi) - (\beta + c)\phi] > M'' - \tfrac{1}{3}\varepsilon_3. \qquad \square$$

Remark. Condition (9) can be weakened; it suffices to have $\lim_{z\to\infty} v(z) = 0$ along some path that goes to infinity.

Lemma 15. *If* $n \le 3$, *and* $\beta = -cn(1+\alpha)/(4\alpha^2 - 4\alpha + 2n)$, *and* $0 < \alpha < \alpha_0(\varepsilon_3)$ *is sufficiently close to* 0, *then conditions* (i), (ii), *and* (iii) *are satisfied*, *and* $M'' > M''' - \frac{1}{3}\varepsilon_3$, *where*

$$M''' := M'''(n) = 2c\left(\frac{n-4+2\sqrt{4n-n^2}}{n-2+\sqrt{4n-n^2}}\right).$$

Proof. Conditions (i) and (ii) are clearly satisfied by construction. And note that (iii) is equivalent to

$$-\frac{2+\sqrt{2}}{4} < \frac{\beta(1-\alpha)}{cn} < -\frac{2-\sqrt{2}}{4}.$$

But the quantity in the middle varies continuously with α near $\alpha = 0$, so it suffices to check it at $\alpha = 0$, where we indeed have

$$-\frac{2+\sqrt{2}}{4} < -\frac{1}{2n} < -\frac{2-\sqrt{2}}{4},$$

which holds for all $n \le 3$, so there must exist some α_0 sufficiently small such that (iii) holds for all $\alpha < \alpha_0$.

Next, we compute M'':

$$\begin{aligned}
M'' &= 4(1-\alpha)\left(c - \frac{-\mu}{\nu+\omega}\right) \\
&= 4(1-\alpha)\left(c + \frac{\dfrac{\beta c}{2\sqrt{A}}}{\dfrac{1}{2\sqrt{A}}\left(\dfrac{4\beta(1-\alpha)}{n} + c\right) + \sqrt{\dfrac{2(1-\alpha)}{n}}}\right) \\
&= 4(1-\alpha)\left(c + \frac{\beta c}{\left(c + \dfrac{4\beta(1-\alpha)}{n}\right) + \sqrt{\dfrac{8A(1-\alpha)}{n}}}\right).
\end{aligned}$$

Here $A = A(\varepsilon' = 0)$, so that

$$\frac{8A(1-\alpha)}{n} = \frac{16\beta^2(1-\alpha)^2}{n^2} - \left(c + \frac{4\beta(1-\alpha)}{n}\right)^2 = c^2\left(-1 - \frac{8\beta(1-\alpha)}{cn}\right).$$

This gives

$$M'' = 4(1-\alpha)c\left(1 + \frac{\beta/c}{1+\frac{4\beta(1-\alpha)}{cn}+\sqrt{-1-\frac{8\beta(1-\alpha)}{cn}}}\right).$$

Again, this involves only $(1-\alpha)$ and β, both of which are continuous at $\alpha = 0$, where we have $\beta = -c/2$, so

$$M'' = 4c\left(1 + \frac{-1/2}{1-\frac{2}{n}+\sqrt{-1+\frac{4}{n}}}\right) = 2c\left(2 - \frac{n}{n-2+\sqrt{4n-n^2}}\right) = M'''.$$

Hence for α sufficiently close to 0 we can get $|M'' - M'''| < \varepsilon_3/3$, which gives us the desired conclusion. □

Proof of Theorem 12. Fix a solution $f(x,t) = v(x + \eta t\hat{a})$ of (1) which also satisfies (9), and fix a $\varepsilon_3 > 0$.

Let $\alpha < \alpha_0$ and $\beta = -c/(2(1-\alpha))$, so that (i), (ii), (iii) are satisfied (by Lemma 15). Applying Lemma 13 then gives $\eta^2 \geq M$ for all x, t.

Applying Lemma 14, we find a pair (x_3, t_3) such that $M' > M'' - \varepsilon_3/3$. Then applying Lemma 15 again, we have $M'' > M''' - \varepsilon_3/3$ so that

$$\eta^2 > M''' - \varepsilon_3.$$

However, note that M''' depends only on n. Hence we send $\varepsilon_3 \to 0$, to get that

$$\eta^2 \geq M'''(n) = \begin{cases} c(3-\sqrt{3}), & n = 1,\\ 2c, & n = 2,\\ c(7-3\sqrt{3}), & n = 3,\end{cases}$$

as desired. □

Classical Harnack inequality. In this subsection, we integrate our differential Harnack estimates along a spacetime curve to derive classical Harnack inequalities. We further assume that M is closed, and that $f(x,t) < 1$ for all x, t.

Theorem 16. *Let M be a closed Riemannian manifold with nonnegative Ricci curvature, and $0 < f < 1$ be a bounded positive solution to Fisher's equation. Let α and β satisfy the conditions of Theorem 1. Furthermore, if $\alpha \leq n/4$, then there will always exist β such that $\beta + c \geq 0$ in addition to the constraints of Theorem 1. For such an α and β,*

(i) *if* $8\beta(1-\alpha)+cn<0$, *then we have*

$$\frac{f(x_2,t_2)}{f(x_1,t_1)} \geq \Big(\frac{1-e^{-ct_2}}{1-e^{-ct_1}}\Big)^{\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}} \exp\Big(-\frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}\Big);$$

(ii) *if* $8\beta(1-\alpha)+cn>0,\ t_2>t_1>T_2$, *then we have*

$$\frac{f(x_2,t_2)}{f(x_1,t_1)} \geq \left[\frac{\Big(1+\dfrac{8\beta(1-\alpha)}{cn}\Big)e^{-c(t_2-T_2)}+1}{\Big(1+\dfrac{8\beta(1-\alpha)}{cn}\Big)e^{-c(t_1-T_2)}+1}\right]^{\frac{8\beta^2(1-\alpha)}{c(cn+8\beta(1-\alpha))}} \exp\Big(-\frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}\Big);$$

(iii) *if* $8\beta(1-\alpha)+cn=0,\ t_2>t_1>T_2$, *then we have*

$$\frac{f(x_2,t_2)}{f(x_1,t_1)} \geq \exp\Big[-\frac{\beta}{c}\Big(e^{-c(t_2-T_2)}-e^{-c(t_1-T_2)}\Big)\Big]\exp\Big(-\frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}\Big).$$

Proof of Theorem 16. Let $f(x,t)$ solve $f_t=\Delta f+cf(1-f)$, and $u=\log f$. Fix points (x_1,t_1), (x_2,t_2) and let $\gamma:[t_1,t_2]\to M^n$ be an arbitrary spacetime path connecting them, i.e., $\gamma(t_1)=x_1$, $\gamma(t_2)=x_2$.

Let $v(t):=u(\gamma(t),t)$ be the value of u along γ. We compute

$$v'(t)=u_t+\nabla u\cdot\frac{d\gamma}{dt}.$$

Using the time evolution for $u_t=(\log f)_t=f_t/f$, this is equal to

$$v'(t)=\Delta u+|\nabla u|^2+c(1-e^u)+\nabla u\cdot\frac{d\gamma}{dt}.$$

Applying the Harnack inequality gives

$$v'(t)\geq(1-\alpha)|\nabla u|^2+(c-\phi)-(\beta+c)e^u+\nabla u\cdot\frac{d\gamma}{dt}.$$

By assumption, $f<1$ and $\beta+c\geq 0$. This implies $-(\beta+c)e^u\geq-(\beta+c)$, so defining $\widetilde{\phi}(t)=-\beta-\phi(t)$, we then get

$$\begin{aligned}
v'(t)&\geq(1-\alpha)|\nabla u|^2+(c-\phi)-(\beta+c)+\nabla u\cdot\frac{d\gamma}{dt}\\
&=-\beta-\phi+(1-\alpha)|\nabla u|^2+\nabla u\cdot\frac{d\gamma}{dt}\\
&=\widetilde{\phi}(t)+(1-\alpha)|\nabla u|^2+\nabla u\cdot\frac{d\gamma}{dt},\\
v'(t)&\geq\widetilde{\phi}(t)-\frac{1}{4(1-\alpha)}\Big|\frac{d\gamma}{dt}\Big|^2.
\end{aligned}$$

Integrating in time, we get

$$u(x_2,t_2)-u(x_1,t_1)=v(t_2)-v(t_1)=\int_{t_1}^{t_2} v'(t)\,dt \geq \int_{t_1}^{t_2} \widetilde{\phi}(t)\,dt-\frac{1}{4(1-\alpha)}\int_{t_1}^{t_2}\left|\frac{d\gamma}{dt}\right|^2 dt.$$

Since γ was chosen to be an arbitrary path, we can choose it to be the path minimizing $\int |\gamma'|^2$, which is the minimizing geodesic between the two endpoints. The integral thus becomes

$$\int_{t_1}^{t_2} |\gamma'|^2\,dt = \frac{d(x_1,x_2)^2}{t_2-t_1}.$$

Thus the spacetime Harnack is given by

$$\log\Big(\frac{f(x_2,t_2)}{f(x_1,t_1)}\Big) = u(x_2,t_2)-u(x_1,t_1) \geq \int_{t_1}^{t_2} \widetilde{\phi}(t)\,dt - \frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}.$$

We compute the definite integral, dividing into three cases. First we deal with the case $8\beta(1-\alpha)+cn<0$. In this case we have

$$\phi(t) = \frac{\Big(\frac{\beta cn}{cn+8\beta(1-\alpha)}\Big)e^{-ct}-\beta}{1-e^{-ct}},$$

and

$$\widetilde{\phi}(t) = \Big(\beta e^{-ct} - \frac{\beta cne^{-ct}}{cn+8\beta(1-\alpha)}\Big)\frac{1}{1-e^{-ct}} = \beta\cdot\frac{8\beta(1-\alpha)}{cn+8\beta(1-\alpha)}\cdot\frac{e^{-ct}}{1-e^{-ct}}.$$

Then we can explicitly integrate

$$\int_{t_1}^{t_2} \widetilde{\phi}(t)\,dt = \frac{\beta}{c}\Big(\frac{8\beta(1-\alpha)}{cn+8\beta(1-\alpha)}\Big)\log\Big[\frac{1-e^{-ct_2}}{1-e^{-ct_1}}\Big].$$

Therefore we get that

$$\exp\bigg(\int_{t_1}^{t_2}\widetilde{\phi}(t)\,dt\bigg) = \Big(\frac{1-e^{-ct_2}}{1-e^{-ct_1}}\Big)^{\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}},$$

and the claim follows.

Second, we deal with the case $8\beta(1-\alpha)+cn>0$. Then for $t>T_2$ (recall that T_2 is a constant) we have

$$\phi(t) = \frac{-\beta cne^{c(t-T_2)}-\beta cn}{cn+8\beta(1-\alpha)+cne^{c(t-T_2)}},$$

and so

$$\widetilde{\phi}(t) = -\beta-\phi(t) = \frac{-8\beta^2(1-\alpha)e^{-c(t-T_2)}}{(8\beta(1-\alpha)+cn)e^{-c(t-T_2)}+cn}.$$

If we let $B=-8\beta^2(1-\alpha)$ and $D=cn+8\beta(1-\alpha)$, then we get that

$$\widetilde{\phi}(t)=\frac{Be^{-c(t-T_2)}}{De^{-c(t-T_2)}+cn}.$$

We can integrate

$$\int_{t_1}^{t_2}\widetilde{\phi}(t)\,dt=\Bigl(\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}\Bigr)\log\Bigl(\frac{(8\beta(1-\alpha)+cn)e^{-c(t_2-T_2)}+cn}{(8\beta(1-\alpha)+cn)e^{-c(t_1-T_2)}+cn}\Bigr).$$

Therefore

$$\exp\Bigl(\int_{t_1}^{t_2}\widetilde{\phi}(t)\,dt\Bigr)=\left[\frac{\Bigl(1+8\beta(1-\alpha)/(cn)\Bigr)e^{-c(t_2-T_2)}+1}{\Bigl(1+8\beta(1-\alpha)/(cn)\Bigr)e^{-c(t_1-T_2)}+1}\right]^{\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}}$$

as claimed in the statement of Theorem 16.

In the last case that $8\beta(1-\alpha)+cn=0$, we have

$$\phi(t)=\frac{-\beta e^{c(t-T_2)}-\beta}{e^{c(t-T_2)}},$$

and so

$$\widetilde{\phi}(t)=-\beta-\phi(t)=\frac{\beta}{e^{c(t-T_2)}}.$$

Therefore

$$\exp\Bigl(\int_{t_1}^{t_2}\widetilde{\phi}(t)\,dt\Bigr)=\exp\Bigl[-\frac{\beta}{c}\bigl(e^{-c(t_2-T_2)}-e^{-c(t_1-T_2)}\bigr)\Bigr]$$

as desired.

To finish the proof of our theorem we need to show that we can choose $\beta+c\geq 0$, i.e., $\beta\geq -c$. We have the constraint (ii):

$$\beta\leq\frac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n},$$

so we need to have

$$-c\leq\beta\leq\frac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n}.$$

Note that since $0<\alpha<1$, we have $4\alpha^2-4\alpha+2n\geq -1+2n\geq 1$; thus it remains to choose α so that

$$-(4\alpha^2-4\alpha+2n)\leq -n(1+\alpha),$$

which simplifies to

$$\alpha\leq\tfrac{1}{4}n.$$

This is automatically true if $n\geq 4$, which means we can choose any α we wish, and there will be at least one β satisfying all the constraints including $\beta+c\geq 0$. $\square$

Note that $\lim_{t\to\infty}\phi(t)=-\beta$, and $\lim_{t\to\infty}\widetilde{\phi}(t)=0$. Thus, as $t_1, t_2\to\infty$, the estimate approaches the classical Li–Yau–Harnack [Li and Yau 1986].

Remark. In the compact case we obtain a good bound as t_1 and t_2 get large. In the complete noncompact case, one can still integrate along spacetime curves to obtain an inequality, but the estimate degenerates when time becomes large.

Acknowledgements

Cao's research was partially supported by NSF grant DMS 0904432. Liu, Pendleton, and Ward's research was supported by NSF grant DMS 1156350 through the Research Experience for Undergraduates Program at Cornell University during the summer of 2012. The authors would like to thank Professor Robert Strichartz for his encouragement and Benjamin Fayyazuddin-Ljungberg and Hung Tran for helpful discussions. They also thank the referee for many detailed suggestions which improved the quality of this paper.

References

[Aronson and Bénilan 1979] D. G. Aronson and P. Bénilan, "Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$", *C. R. Acad. Sci. Paris Sér. A-B* **288**:2 (1979), A103–A105. MR Zbl

[Cao 2008] X. Cao, "Differential Harnack estimates for backward heat equations with potentials under the Ricci flow", *J. Funct. Anal.* **255**:4 (2008), 1024–1038. MR Zbl

[Cao and Hamilton 2009] X. Cao and R. S. Hamilton, "Differential Harnack estimates for time-dependent heat equations with potentials", *Geom. Funct. Anal.* **19**:4 (2009), 989–1000. MR Zbl

[Cao et al. 2015] X. Cao, M. Cerenzia, and D. Kazaras, "Harnack estimate for the endangered species equation", *Proc. Amer. Math. Soc.* **143**:10 (2015), 4537–4545. MR Zbl

[Fisher 1937] R. A. Fisher, "The wave of advance of advantageous genes", *Ann. Eugenics* **7**:4 (1937), 355–369. Zbl

[Fujita 1966] H. Fujita, "On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$", *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124. MR Zbl

[Hamilton 1993] R. S. Hamilton, "The Harnack estimate for the Ricci flow", *J. Differential Geom.* **37**:1 (1993), 225–243. MR Zbl

[Hamilton 2011] R. S. Hamilton, "Li–Yau estimates and their Harnack inequalities", pp. 329–362 in *Geometry and analysis, I*, edited by L. Ji, Adv. Lect. Math. **17**, International Press, Somerville, MA, 2011. MR Zbl

[Kolmogorov et al. 1937] A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov, "A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem", *Bull. Univ. Moscou, Sér. Int. A* **1**:6 (1937), 1–26. In Russian; translated in *Selected works of A. N. Kolmogorov*, edited by V. M. Tikhomirov, Math. Appl. (Soviet Series) **25**, Springer, 1991, pp. 242–270. Zbl

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR Zbl

[McKean 1975] H. P. McKean, "Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov", *Comm. Pure Appl. Math.* **28**:3 (1975), 323–331. MR Zbl

[Ó Náraigh and Kamhawi 2013] L. Ó Náraigh and K. Kamhawi, "Multiscale methods and modelling for chemical reactions on oscillating surfaces", *IMA J. Appl. Math.* **78**:3 (2013), 537–565. MR Zbl

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl arXiv

[Sherratt 1998] J. A. Sherratt, "On the transition from initial data to travelling waves in the Fisher–KPP equation", *Dynam. Stability Systems* **13**:2 (1998), 167–174. MR Zbl

[Tuckwell 1988] H. C. Tuckwell, *Introduction to theoretical neurobiology, II: Nonlinear and stochastic theories*, Cambridge Studies in Mathematical Biology **8**, Cambridge University Press, 1988. MR Zbl

Received October 26, 2015. Revised August 30, 2016.

XIAODONG CAO
DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NY 14853
UNITED STATES
cao@math.cornell.edu

BOWEI LIU
DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CA 94305
UNITED STATES
bowei@math.stanford.edu

IAN PENDLETON
DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NY 14853
UNITED STATES
iap26@cornell.edu

ABIGAIL WARD
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF CHICAGO
CHICAGO, IL 60637
UNITED STATES
abigailward@uchicago.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 290 No. 2 October 2017

0030-8730(201710)290:2;1-Z